SYMMETRY GROUP ANALYSIS OF THE SHALLOW WATER AND SEMI-GEOSTROPHIC EQUATIONS

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Summary

The two-dimensional shallow water equations and their semi-geostrophic approximation that arise in meteorology and oceanography are analysed from the point of view of symmetry groups theory. A complete classification of their associated classical symmetries, potential symmetries, variational symmetries and conservation laws is found. The semi-geostrophic equations are found to lack conservation of angular momentum. We also show how the particle relabelling symmetry can be used to rewrite the semi-geostrophic equations in such a way that a well-defined formal series solution, smooth only in time, may be carried out. We show that such solutions are in the form of an 'infinite linear cascade'.

1. Introduction

Many natural phenomena are described by a system of nonlinear partial differential equations (PDEs), which are often difficult to solve analytically as there is no existing general theory for completely solving nonlinear PDEs.

In the mid-nineteenth century, Sophus Lie introduced a new method for studying differential equations by using their symmetries. For each PDE, or system of PDEs, there is a local group of transformations, called *a symmetry group*, that acts on the space of its independent and dependent variables, with the property that it maps the set of all analytical solutions to itself, and so leaves the form of the equation unchanged. The method of finding the symmetry group associated with the a PDE is called the *classical Lie method*. Moreover, the classical Lie method leads to special types of solutions in terms of solutions of lower-dimensional equations. For a modern description see, for example, (**1**). The classical Lie method is an algorithmic procedure for which many symbolic manipulation programs have became available; cf. (**2** to **4**). Their use became imperative in finding classical symmetries associated with large systems of PDEs.

The *variational symmetry group* of a variational problem is a subgroup of the symmetry group associated with the Euler–Lagrange equations, with the property that it leaves the variational integral unchanged. Knowledge of the variational symmetries leads, by Noether's theorem, to the conservation laws associated with the studied system: for each one-parameter variational symmetry there is an associated conservation law; for details see (**1**, §§4.3, 4.4, 5.3).

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In the past twenty years, the semi-geostrophic equations have become a model for describing atmospheric motions on a synoptic scale, including the presence of fronts (**5** to **7**). In two-dimensional shallow water theory (**8**), a typical particle (more precisely, fluid column) has the Cartesian horizontal coordinates

$$x = x(a, b, t), \quad y = y(a, b, t) \tag{1.1}$$

expressed as functions of the particle labels $(a, b) \in \mathbb{R}^2$ and time $t \in \mathbb{R}^+$. For convenience, each particle is labelled by its position at a reference time $t = 0$; this means the functions in (1.1) are defined such that $x(a, b, 0) = a$ and $y(a, b, 0) = b$. The incompressibility hypothesis requires that

$$\frac{h(a, b, 0)}{h(a, b, t)} = \frac{\partial(x, y)}{\partial(a, b)}, \tag{1.2}$$

where the Jacobian on the right is that of the mapping (1.1). The time derivative of (1.2) following the particle gives the continuity equation. In this paper we assume $h(a, b, 0) = 1$, so that the incompressibility hypothesis becomes

$$h(a, b, t) = 1/(x_a y_b - x_b y_a), \tag{1.3}$$

where subscripts denote partial derivatives. The mapping (1.1) is assumed to be invertible, such that when $a = a(x, y, t)$ and $b = b(x, y, t)$ are inserted into (1.2), then the current depth h is expressed as a function of x, y and t, which represents the Eulerian description.

The equations of the horizontal momentum balance for the flows over a bed which is rotating with position-dependent Coriolis parameter $f = f(y)$ are

$$\ddot{x} + gh_x - f\dot{y} = 0, \quad \ddot{y} + gh_y + f\dot{x} = 0, \tag{1.4}$$

where g is a non-zero constant (representing the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation), a dot denotes the time derivative following a particle, and h_x and h_y are given by

$$h_x = h(y_b h_a - y_a h_b), \quad h_y = h(x_a h_b - x_b h_a).$$

Henceforth we shall assume that f is a constant. It is known that the *shallow water potential vorticity*, defined by

$$\Omega = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right), \tag{1.5}$$

is conserved on particles; see (**9**, **10**) and the references therein.

The semi-geostrophic approximation to (1.4) is the replacement of the true acceleration by the time derivative of the vector

$$u_g = -gh_y/f, \quad v_g = gh_x/f, \tag{1.6}$$

following the particle. This vector field is called the *geostrophic velocity*. Thus, the semi-geostrophic approximation seeks to find motions satisfying

$$\dot{u}_g + gh_x - f\dot{y} = 0, \quad \dot{v}_g + gh_y + f\dot{x} = 0. \tag{1.7}$$

It is also known that the *semi-geostrophic potential vorticity*, given by

$$\Omega^* = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right), \tag{1.8}$$

is conserved on particles (**11**).

In this paper we give a complete classification of the symmetries and the conservation laws associated with the two-dimensional shallow water (SW) equations (1.4) and the semi-geostrophic (SG) equations (1.7). In sections 2 and 3 we determine the Lie point symmetries associated with the SW equations and SG equations, written in their classical forms (2.1) and (3.1), respectively, and in their potential forms (2.5) and (3.2), respectively. In sections 4 and 5, we determine the conservation laws derived from the variational symmetries. The particle relabelling symmetry leads to the conservation of the shallow water potential vorticity Ω (**12**) and semi-geostrophic potential vorticity (**13**), the time invariance leads to the conservation of the energy of the SW equations (**12**) and of SG equations, and the translation invariance to the conservation of linear momentum of the SW and SG equations, respectively. Moreover, the rotation invariance leads to the conservation of the angular momentum of SW equations, but this is lost for the SG equations. Indeed, the SG equations are not suitable for the study of vortex dynamics (**14**).

The particle relabelling symmetry of Lagrangian fluid dynamics is important not only for the conservation of potential vorticity but for geometric studies of the equations. For example, in (**10**) it is shown how the symmetry relates to Hamiltonian properties, while in (**9**) it is shown how the symmetry relates to symplecticity with a view to symplectic integration methods. Here we use the symmetry to reframe the equations in a way which is adapted to the derivation of formal series solutions. The aim is to elucidate the structure of those solutions which are smooth in time, at least for small time. We show that these solutions can be described in terms of an ‘infinite linear cascade’. The method can be used for any equation having the particle relabelling group, and thus is of independent interest. While we find a large class of such solutions, they do not satisfy the physically natural flow property,

$$x(x(a,b,t), y(a,b,t), s) = x(a,b,t+s), \quad y(x(a,b,t), y(a,b,t), s) = y(a,b,t+s),$$

also known as the *integral curve* property. This contrasts with the semi-geostrophic approximation of the two-dimensional Euler equations, which do have such solutions (**15**).

2. Classical and potential symmetries for the shallow water equations

2.1 *Classical symmetries of the shallow water equations*

Substituting the function h defined by (1.3) into (1.4) yields

$$\ddot{x} - \frac{g\left[y_b^2 x_{aa} - 2y_a y_b x_{ab} + y_a^2 x_{bb} - x_b y_b y_{aa} + (x_a y_b + x_b y_a) y_{ab} - x_a y_a y_{bb}\right]}{(x_a y_b - x_b y_a)^3} = f\dot{y}, \tag{2.1a}$$

$$\ddot{y} + \frac{g\left[x_b y_b x_{aa} - (x_a y_b + x_b y_a) x_{ab} + x_a y_a x_{bb} - x_b^2 y_{aa} + 2x_a x_b y_{ab} - x_a^2 y_{bb}\right]}{(x_a y_b - x_b y_a)^3} = -f\dot{x}, \tag{2.1b}$$

which we shall refer to as the *classical form* of the SW equations. To determine the Lie point symmetries, the *classical symmetries*, we consider the one-parameter group of transformations

$$a^* = a + \varepsilon\xi(a, b, t, x, y) + \mathcal{O}(\varepsilon^2), \tag{2.2a}$$

$$b^* = b + \varepsilon\eta(a, b, st, x, y) + \mathcal{O}(\varepsilon^2), \tag{2.2b}$$

$$t^* = t + \varepsilon\theta(a, b, t, x, y) + \mathcal{O}(\varepsilon^2), \tag{2.2c}$$

$$x^* = x + \varepsilon\phi(a, b, t, x, y) + \mathcal{O}(\varepsilon^2), \tag{2.2d}$$

$$y^* = y + \varepsilon\psi(a, b, t, x, y) + \mathcal{O}(\varepsilon^2), \tag{2.2e}$$

where ξ, η, θ, ϕ and ψ are the *infinitesimals* which depend on a, b, t, x and y. Applying the classical Lie method shows that θ is a constant, and

$$\phi(a, b, t, x, y) = Ax - By + C\cos(ft) + D\sin(ft) + E,$$
$$\psi(a, b, t, x, y) = Ay + Bx + D\cos(ft) - C\sin(ft) + F,$$

with A, B, C, D, E and F constants, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ satisfy

$$\xi_a + \eta_b = 4A. \tag{2.3}$$

The infinitesimal generator associated with (2.2) is

$$\mathrm{X} = \xi(a, b)\partial_a + \eta(a, b)\partial_b + \theta\partial_t + \{Ax - By + C\cos(ft) + D\sin(ft) + E\}\,\partial_x$$
$$+ \{Ay + Bx + D\cos(ft) - C\sin(ft) + F\}\,\partial_y,$$

where $\partial_a = \partial/\partial a$, $\partial_b = \partial/\partial b$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$, and so we have the following result.

THEOREM 2.1. *The symmetry group associated with the SW equations* (2.1) *is generated by the following vector fields:*

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = \partial_y, \quad \mathrm{X}_4 = -y\partial_x + x\partial_y,$$
$$\mathrm{X}_5 = \cos(ft)\partial_x - \sin(ft)\partial_y, \quad \mathrm{X}_6 = \sin(ft)\partial_x + \cos(ft)\partial_y,$$
$$\mathrm{X}_0 = \xi(a, b)\partial_a + \eta(a, b)\partial_b + Ax\partial_x + Ay\partial_y,$$

with A an arbitrary constant, and where ξ and η satisfy (2.3).

Therefore the SW equations (2.1) are invariant under translations in t, x and y (vector fields X_1, X_2 and X_3 respectively), rotations in the (x, y)-space (X_4), and helical rotations with respect t in the (x, y)-space (X_5 and X_6). The vector field X_0 with $A = 0$ is the infinitesimal form of the *particle relabelling* symmetry: if a and b are the particle labels and

$$a^* = \phi(a, b), \quad b^* = \psi(a, b) \tag{2.4}$$

is the relabelling, then

$$\phi_a\psi_b - \phi_b\psi_a = 1.$$

A one-parameter family of such transformations yields a flow on (x, y)-space, and then (ξ, η) is the vector field generating that flow. The zero divergence of the vector field corresponds to the area preservation of the flow.

We remark that we are not able to apply the classical Lie symmetry method using currently available software packages to (1.4) and (1.7) as they stand. This is because differentiation with respect to x or y and time t do not commute, as x and y depend on t.

2.2 *Potential symmetries of the shallow water equations*

If we let $u = \dot{x}$ and $\upsilon = \dot{y}$, then the SW equations (1.4) can be written as

$$\dot{x} = u, \tag{2.5a}$$

$$\dot{y} = \upsilon, \tag{2.5b}$$

$$\dot{u} + gh(y_b h_a - y_a h_b) - f\upsilon = 0, \tag{2.5c}$$

$$\dot{\upsilon} + gh(x_a h_b - x_b h_a) + fu = 0, \tag{2.5d}$$

where h is given by (1.3), which we shall refer to as the *potential form* of the SW equations (1.4). To determine the associated Lie point symmetries, the *potential symmetries,* we consider the one-parameter group of transformations

$$\begin{aligned}
a^* &= a + \varepsilon\xi(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
b^* &= b + \varepsilon\eta(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
t^* &= t + \varepsilon\theta(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
x^* &= x + \varepsilon\phi(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
y^* &= y + \varepsilon\psi(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
u^* &= u + \varepsilon\zeta(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),\\
\upsilon^* &= \upsilon + \varepsilon\omega(a, b, t, x, y, u, \upsilon) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where the infinitesimals ξ, η, θ, ϕ, ψ, ζ and ω are functions of a, b, t, x, y, u and υ. Applying the classical Lie method shows that θ is a constant,

$$\phi(a, b, t, x, y, u, \upsilon) = Ax - By + C\cos(ft) + D\sin(ft) + E, \tag{2.6a}$$

$$\psi(a, b, t, x, y, u, \upsilon) = Ay + Bx + D\cos(ft) - C\sin(ft) + F, \tag{2.6b}$$

$$\zeta(a, b, t, x, y, u, \upsilon) = Au - B\upsilon - Cf\sin(ft) + Df\cos(ft), \tag{2.6c}$$

$$\omega(a, b, t, x, y, u, \upsilon) = A\upsilon + Bu - Df\sin(ft) - Cf\cos(ft), \tag{2.6d}$$

with A, B, C, D, E and F constants, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ satisfy the condition (2.3), as previously. The infinitesimal generator is

$$\begin{aligned}
\mathrm{X} = {} & \xi(a, b)\partial_a + \eta(a, b)\partial_b + \theta\partial_t + \{Ax - By + C\cos(ft) + D\sin(ft) + E\}\,\partial_x\\
& + \{Ay + Bx + D\cos(ft) - C\sin(ft) + F\}\,\partial_y\\
& + \{Au - B\upsilon - Cf\sin(ft) + Df\cos(ft)\}\,\partial_u\\
& + \{A\upsilon + Bu - Df\sin(ft) - Cf\cos(ft)\}\,\partial_\upsilon,
\end{aligned} \tag{2.7}$$

and so we obtain the following result.

THEOREM 2.2. *The potential symmetries associated with the SW equations* (2.5) *are generated by the vector fields*

$$\begin{aligned} &X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = -y\partial_x + x\partial_y - v\partial_u + u\partial_v, \\ &X_5 = \cos(ft)\partial_x - \sin(ft)\partial_y - f\sin(ft)\partial_u - f\cos(ft)\partial_v, \\ &X_6 = \sin(ft)\partial_x + \cos(ft)\partial_y + f\cos(ft)\partial_u - f\sin(ft)\partial_v, \\ &X_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + A(x\partial_x + y\partial_y + u\partial_u + v\partial_v), \end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3).

Therefore the potential form of the SW equations (2.5) is invariant under translations in t, x and y (X_1, X_2 and X_3 respectively), rotations in the (x, y, u, v)-space (X_4), helical rotations with respect t in the (x, y, u, v)-space (X_5 and X_6), while the vector field X_0 corresponds to the particle relabelling symmetry (2.4). We remark that the potential symmetries of the SW equations are actually the classical symmetries prolonged to the space of the t-derivatives of the dependent variables x and y.

3. Classical and potential symmetries for semi-geostrophic equations

3.1 *Classical symmetries of the semi-geostrophic equations*

The *classical form* of the SG equations (1.7) is as follows:

$$hh_b\dot{x}_a - hh_a\dot{x}_b + (\dot{h}h_b + h\dot{h}_b)x_a - (\dot{h}h_a + h\dot{h}_a)x_b + fhh_by_a - fhh_ay_b = -f^2\dot{y}/g, \tag{3.1a}$$

$$hh_b\dot{y}_a - hh_a\dot{y}_b + (\dot{h}h_b + h\dot{h}_b)y_a - (\dot{h}h_a + h\dot{h}_a)y_b - fhh_bx_a + fhh_ax_b = f^2\dot{x}/g, \tag{3.1b}$$

with h given by (1.3). This system is obtained after substituting the functions u_g and v_g given by (1.6) into (1.7). The Lie point symmetries of this system of PDEs will be called the *classical symmetries* of the SG equations.

If we consider the one-parameter group of transformations (2.2), then on applying the classical Lie method, we find that θ is a constant,

$$\phi = Ax - By + E, \quad \psi = Ay + Bx + F,$$

where A, B, E, F are constants, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ satisfy (2.3). The infinitesimal generator is

$$X = \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y,$$

and so we have the following result.

THEOREM 3.1. *The vector fields*

$$\begin{aligned} &X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = -y\partial_x + x\partial_y, \\ &X_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + Ax\partial_x + Ay\partial_y, \end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3), *generate the symmetry group associated with the SG equations* (3.1).

Therefore the SG equations (1.7) are invariant under translations in t, x, y and rotations in the (x, y)-space. The relabelling symmetry corresponds to the vector field X_0.

3.2 *Potential symmetries of the semi-geostrophic equations*

The potential form of the SG equations is given by (1.6) and (1.7), that is,

$$u_g = -gh(x_a h_b - x_b h_a)/f, \tag{3.2a}$$

$$v_g = gh(y_b h_a - y_a h_b)/f, \tag{3.2b}$$

$$\dot{u}_g + gh(y_b h_a - y_a h_b) - f\dot{y} = 0, \tag{3.2c}$$

$$\dot{v}_g + gh(x_a h_b - x_b h_a) + f\dot{x} = 0, \tag{3.2d}$$

where f is a non-zero constant and h is given by (1.3). To determine the associated Lie point symmetries, the potential symmetries, we consider the one-parameter group of transformations

$$\begin{aligned}
a^* &= a + \varepsilon\xi(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
b^* &= b + \varepsilon\eta(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
t^* &= t + \varepsilon\theta(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
x^* &= x + \varepsilon\phi(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
y^* &= y + \varepsilon\psi(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
u_g^* &= u_g + \varepsilon\zeta(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),\\
v_g^* &= v_g + \varepsilon\omega(a,b,t,x,y,u_g,v_g) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where the infinitesimals $\xi, \eta, \theta, \phi, \psi, \zeta$ and ω depend a, b, t, x, y, u_g and v_g. Applying the classical Lie method yields

$$\phi(a,b,t,x,y,u_g,v_g) = Ax - By + E, \tag{3.3a}$$

$$\psi(a,b,t,x,y,u_g,v_g) = Ay + Bx + F, \tag{3.3b}$$

$$\zeta(a,b,t,x,y,u_g,v_g) = Au_g - Bv_g, \tag{3.3c}$$

$$\omega(a,b,t,x,y,u_g,v_g) = Av_g + Bu_g, \tag{3.3d}$$

where A, B, E, F and θ are constants, and $\xi = \xi(a,b)$ and $\eta = \eta(a,b)$ satisfy (2.3). The infinitesimal generator is

$$\begin{aligned}
\mathrm{X} = {}& \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y\\
&+ (Au_g - Bv_g)\partial_u + (Av_g + Bu_g)\partial_v,
\end{aligned}\tag{3.4}$$

and so we have the following result.

THEOREM 3.2. *The vector fields*

$$\begin{aligned}
&\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x, \quad \mathrm{X}_3 = \partial_y, \quad \mathrm{X}_4 = -y\partial_x + x\partial_y - v_g\partial_{u_g} + u_g\partial_{v_g},\\
&\mathrm{X}_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + A(x\partial_x + y\partial_y + u_g\partial_{u_g} + v_g\partial_{v_g}),
\end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3), *generate the potential symmetries associated with the SG equations* (3.2).

Therefore (3.2) is invariant under translations in t, x, y and rotations in the (x, y, u_g, v_g)-space. The vector field X_0 corresponds to particle relabelling symmetry (2.4).

4. Variational symmetries and conservation laws for shallow water equations

4.1 *Variational symmetries of the shallow water equations*

A variational symmetry of a system of Euler–Lagrange equations is one that leaves the variational integral invariant. These are important since they yield conservation laws via Noether's theorem. Salmon (**12**) showed that there is a first-order Lagrangian

$$\mathcal{L} = (u - R)\dot{x} + (v + P)\dot{y} - \tfrac{1}{2}(u^2 + v^2 + gh), \tag{4.1}$$

where the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy

$$P_x + R_y = f,$$

with h given by (1.3) and f constant, for which the associated Euler–Lagrange equations represent (up to sign) the SW equations (2.5), that is the potential form of the SW equations.

THEOREM 4.1. *The variational symmetry group of the variational problem associated with the Lagrangian* (4.1) *is generated by the following vector fields*

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{Y}_0 = -S_b(a, b)\partial_a + S_a(a, b)\partial_b, \;\; S_b = -\xi, \;\; S_a = \eta. \tag{4.2}$$

Moreover, if the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy the system

$$B(yR_x - xR_y + P) - ER_x - FR_y = 0, \tag{4.3a}$$

$$B(-yP_x + xP_y + R) + EP_x + FP_y = 0, \tag{4.3b}$$

$$P_x + R_y = f, \tag{4.3c}$$

then the vector field

$$\mathrm{Y} = (E - By)\partial_x + (F + Bx)\partial_y - Bv\partial_u + Bu\partial_v, \tag{4.4}$$

with B, E and F real constants, generates a variational symmetry transformation.

Particular cases of these are as follows.

Case 1. *If $B \neq 0$, then let $\lambda = E/B$, $\gamma = F/B$. Making the transformation $x = -\gamma + r\cos s$, $y = \lambda + r\sin s$, yields*

$$P(r, s) = -\delta(r)\sin s + \left(\tfrac{1}{2}fr + k/r\right)\cos s, \quad R(r, s) = \delta(r)\cos s + \left(\tfrac{1}{2}fr + k/r\right)\sin s, \tag{4.5}$$

and there is a variational symmetry group generated by

$$\mathrm{Y}_1 = -(y - \lambda)\partial_x + (x + \gamma)\partial_y - v\partial_u + u\partial_v. \tag{4.6}$$

Case 2. *If $B = 0$ and $E \neq 0$, then let $\lambda = F/B$. In this case it follows that*

$$P(x, y) = \delta(y - \lambda x), \quad R(x, y) = \lambda\delta(y - \lambda x) - \lambda fx + fy + k, \tag{4.7}$$

where $\delta(x)$ is an arbitrary function, and there is a variational symmetry group generated by

$$\mathrm{Y}_2 = \partial_x + \lambda\partial_y. \tag{4.8}$$

Case 3. *If $B = 0$ and $E = 0$ then*

$$P(x, y) = fx + k, \quad R(x, y) = \delta(x), \tag{4.9}$$

with k an arbitrary real constant and $\delta(x)$ an arbitrary function, and there is a variational symmetry group generated by

$$\mathrm{Y}_3 = \partial_y. \tag{4.10}$$

Proof. Consider the general infinitesimal generator (2.7) of the symmetry group of the SW equations. Since (2.5) is a first-order system of PDEs, the criterion for infinitesimal invariance is

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + \mathcal{L}\nabla \bullet \boldsymbol{\zeta} = 0,$$

where $\boldsymbol{\zeta} = (\xi, \eta, \theta)$ and

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) = \mathrm{X}(\mathcal{L}) + \phi^{[a]}\frac{\partial\mathcal{L}}{\partial x_a} + \phi^{[b]}\frac{\partial\mathcal{L}}{\partial x_b} + \phi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{x}} + \psi^{[a]}\frac{\partial\mathcal{L}}{\partial y_a} + \psi^{[b]}\frac{\partial\mathcal{L}}{\partial y_b} + \psi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{y}} \tag{4.11}$$

is the first-order prolongation of the general infinitesimal generator X given by (2.7) applied to the Lagrangian (4.1), with infinitesimals ϕ, ψ, ζ and ω given by (2.6) and

$$\phi^{[a]} = (A - \xi_a)x_a - By_a - \eta_a x_b, \tag{4.12a}$$

$$\phi^{[b]} = -\xi_b x_a + (A - \eta_b)x_b - By_b, \tag{4.12b}$$

$$\phi^{[t]} = -Cf\sin(ft) + Df\cos(ft) + A\dot{x} - B\dot{y}, \tag{4.12c}$$

$$\psi^{[a]} = Bx_a + (A - \xi_a)y_a - \eta_a y_b, \tag{4.12d}$$

$$\psi^{[b]} = Bx_b - \xi_b y_a + (A - \eta_b)y_b, \tag{4.12e}$$

$$\psi^{[t]} = -Df\sin(ft) - Cf\cos(ft) + B\dot{x} + A\dot{y}. \tag{4.12f}$$

The relation (2.3) implies $\nabla \bullet \boldsymbol{\zeta} = \xi_a + \eta_b = 4A$, and thus we have

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + 4A\mathcal{L} = 0. \tag{4.13}$$

The non-zero partial derivatives of the Lagrangian (4.1) are the following:

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \quad \frac{\partial\mathcal{L}}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \quad \frac{\partial\mathcal{L}}{\partial u} = \dot{x} - u,\\
&\frac{\partial\mathcal{L}}{\partial v} = \dot{y} - v, \quad \frac{\partial\mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2y_b, \quad \frac{\partial\mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2y_a,\\
&\frac{\partial\mathcal{L}}{\partial \dot{x}} = u - R, \quad \frac{\partial\mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2x_b, \quad \frac{\partial\mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2x_a, \quad \frac{\partial\mathcal{L}}{\partial \dot{y}} = v + P.
\end{aligned} \tag{4.14}$$

Substituting (2.6), (4.11), (4.12) and (4.14) into (4.13), and equating to zero the coefficients of the t-derivatives $\dot{x}$, $\dot{y}$ and the free term, it follows $A = C = D = 0$, and the relations (4.3) between the constants B, E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ defining the Lagrangian (4.1). If $A = 0$, then (2.3) implies $\xi_a + \eta_b = 0$, and we can consider a function $S = S(a, b)$ such that $S_b = -\xi$ and $S_a = \eta$. It follows that the variational symmetry generated by the vector fields X_1 and Y_0 is given by (4.2). The compatibility of (4.3) implies the form (4.4) for another variational symmetry.

Let us now discuss the particular cases of (4.3).

Case 1. If $B \neq 0$, then let $\lambda = E/B$ and $\gamma = F/B$, so (4.3) becomes

$$(y-\lambda)R_x - (x+\gamma)R_y + P = 0, \quad (\lambda - y)P_x + (x+\gamma)P_y + R = 0, \quad P_x + R_y = f.$$

Consider the change of variables $x = -\gamma + r\cos s$, $y = \lambda + r\sin s$. Then the general solution is given by (4.5) and so the vector field (4.4) is written as (4.6). For $B = 0$, (4.3) is equivalent to

$$EP_x + FP_y = 0, \quad ER_x + FR_y = 0, \quad P_x + R_y = f. \tag{4.15}$$

Case 2. If $B = 0$ and $E \neq 0$, then (4.15) is written as

$$R_x + \lambda R_y = 0, \quad P_x + \lambda P_y = 0, \quad P_x + R_y = f, \tag{4.16}$$

where $\lambda = F/E$. Integrating this system, we find that (4.7) represents its general solution and, moreover, there is a variational symmetry generated by the vector field (4.8).

Case 3. If $B = 0$ and $E = 0$, then (4.15) turns into

$$FR_y = 0, \quad FP_y = 0, \quad P_x + R_y = f. \tag{4.17}$$

If $F \neq 0$, then we get $R_y = 0$, $P_y = 0$, and so $P_x = f$. Its general solution is (4.9) and we obtain a variational symmetry generated by the vector field (4.10).

In the case $F = 0$, we get that (4.2) is the only vector field that generates a variational symmetry.

DEFINITION 4.2. A *conservation law for the SW equations* is given by a divergence expression

$$D_a(\mathcal{P}_1) + D_b(\mathcal{P}_2) + D_t(\mathcal{P}_3) = 0,$$

where $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is a triple of smooth functions of a, b, t, x, y, u, v and the derivatives of x, y, u, v. The function $\mathcal{P}_3$ is the *conserved density* and the pair $(\mathcal{P}_1, \mathcal{P}_2)$ is the *associated flux vector field*.

THEOREM 4.3. *The conservation laws associated with the SW equations* (2.5), *derived from the variational symmetries are given in Table* 1. *Moreover, the conservation law derived from* Y_0 *is equivalent to* $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$, *where*

$$\widetilde{\mathcal{P}}_1 = \mathcal{P}_1 + D_t(ST_2), \quad \widetilde{\mathcal{P}}_2 = \mathcal{P}_2 - D_t(ST_3), \quad \widetilde{\mathcal{P}}_3 = -S\Omega = P_3 - D_a(ST_2) + D_b(ST_3), \tag{4.18}$$

and Ω *is the shallow water potential vorticity* (1.5).

Proof. If we apply the higher Euler operators to the first-order Lagrangian (4.1), we obtain

$$\mathcal{E}_1^{(1)} = \frac{\partial \mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2 y_b, \quad \mathcal{E}_2^{(1)} = \frac{\partial \mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2 x_b, \quad \mathcal{E}_3^{(1)} = \frac{\partial \mathcal{L}}{\partial u_a} = 0, \quad \mathcal{E}_4^{(1)} = \frac{\partial \mathcal{L}}{\partial v_a} = 0,$$

$$\mathcal{E}_1^{(2)} = \frac{\partial \mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2 y_a, \quad \mathcal{E}_2^{(2)} = \frac{\partial \mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2 x_a, \quad \mathcal{E}_3^{(2)} = \frac{\partial \mathcal{L}}{\partial u_b} = 0, \quad \mathcal{E}_4^{(2)} = \frac{\partial \mathcal{L}}{\partial v_b} = 0,$$

$$\mathcal{E}_1^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{x}} = u - R, \quad \mathcal{E}_2^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{y}} = v + P, \quad \mathcal{E}_3^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{u}} = 0, \quad \mathcal{E}_4^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{v}} = 0.$$

In what follows, we determine the conservation laws derived from the variational symmetries generated by the vector fields (4.2), (4.4), and in particular (4.6), (4.8) and (4.10).

1. For the vector field $\mathrm{X}_1 = \partial_t$, the corresponding characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ has components given by

$$Q_1 = -\dot{x}, \quad Q_2 = -\dot{y}, \quad Q_3 = -\dot{u}, \quad Q_4 = -\dot{v}.$$

Now the conservation law is given by

$$\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = -(\boldsymbol{\mathcal{A}} + \mathcal{L}\boldsymbol{\zeta}), \tag{4.19}$$

where in this case $\boldsymbol{\zeta} = (\xi, \eta, \theta) = (0, 0, 1)$ and where $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with

$$\mathcal{A}_k = \sum_{j=1}^{4} Q_j \frac{\partial \mathcal{L}}{\partial u_k^j}. \tag{4.20}$$

Thus

$$\mathcal{A}_1 = \tfrac{1}{2} g h^2 (\dot{y} x_b - \dot{x} y_b), \quad \mathcal{A}_2 = \tfrac{1}{2} g h^2 (\dot{x} y_a - \dot{y} x_a), \quad \mathcal{A}_3 = \dot{x}(R - u) - \dot{y}(v + P),$$

and so

$$\mathcal{P}_1 = \tfrac{1}{2}(\dot{x} y_b - \dot{y} x_b), \quad \mathcal{P}_2 = \tfrac{1}{2}(\dot{y} x_a - \dot{x} y_a), \quad \mathcal{P}_3 = \tfrac{1}{2}(u^2 + v^2 + gh).$$

Thus, the time invariance leads to the conservation of the energy $\mathcal{P}_3$ of the system (**12**).

Table 1 Symmetries and their associated conservation laws for the SW equations

vector field	$\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$,
$\mathrm{X}_1 = \partial_t$	$(\frac{1}{2} g h^2 (\dot{x} y_b - \dot{y} x_b), \frac{1}{2} g h^2 (\dot{y} x_a - \dot{x} y_a), \frac{1}{2}(u^2 + v^2 + gh))$
$\mathrm{Y}_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b$	$(\xi T_1, \eta T_1, \eta T_2 + \xi T_3)$ $T_1 = \dot{x}(R - u) - \dot{y}(v + P) + \frac{1}{2}(u^2 + v^2) + gh$ $T_2 = x_b(u - R) + y_b(v + P)$ $T_3 = x_a(u - R) + y_a(v + P)$
$\mathrm{Y}_1 = -(y - \lambda)\partial_x + (x + \gamma)\partial_y - v\partial_u + u\partial_v$	$(\frac{1}{2} g h^2 (x_b(x + \gamma) + y_b(y - \lambda))$, $\frac{1}{2} g h^2 (y_a(\lambda - y) - x_a(x + \gamma))$, $(y - \lambda)(u - R) - (x + \gamma)(v + P))$,
$\mathrm{Y}_2 = \partial_x + \lambda\partial_y$	$(\frac{1}{2} g h^2 (\lambda x_b - y_b), \frac{1}{2} g h^2 (y_a - \lambda x_a), R - u - \lambda(v + P))$
$\mathrm{Y}_3 = \partial_y$	$(\frac{1}{2} g h^2 x_b, -\frac{1}{2} g h^2 x_a, -v - P)$

2. The characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ of the vector field $\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$ is given by

$$Q_1 = S_b x_a - S_a x_b, \quad Q_2 = S_b y_a - S_a y_b, \quad Q_3 = S_b u_a - S_a u_b, \quad Q_4 = S_b v_a - S_a v_b.$$

In this case, $\boldsymbol{\zeta} = (-S_b, S_a, 0)$ and from (4.20) we get

$$\mathcal{A}_1 = \tfrac{1}{2} g h S_b, \quad \mathcal{A}_2 = -\tfrac{1}{2} g h S_a,$$

$$\mathcal{A}_3 = S_b[x_a(u-R) + y_a(v+P)] - S_a[x_b(u-R) + y_b(v+P)].$$

Therefore, substituting them into (4.19) one obtains

$$\mathcal{P}_1 = -S_b T_1, \quad \mathcal{P}_2 = S_a T_1, \quad \mathcal{P}_3 = S_a T_2 - S_b T_3,$$

where

$$T_1 = \dot{x}(R-u) - \dot{y}(v+P) + \tfrac{1}{2}(u^2+v^2) + gh,$$

$$T_2 = x_b(u-R) + y_b(v+P), \quad T_3 = x_a(u-R) + y_a(v+P).$$

For the vector field Y_0 we can write an equivalent form of the associated conservation law. Note that the potential vorticity Ω defined by (1.5) satisfies

$$\mathrm{D}_a(T_2) - \mathrm{D}_b(T_3) = \Omega.$$

Define $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ given by (4.18) and let $\boldsymbol{\mathcal{P}}^* = (\mathcal{P}_1^*, \mathcal{P}_2^*, \mathcal{P}_3^*)$ with the components

$$\mathcal{P}_1^* = \mathrm{D}_t(ST_2), \quad \mathcal{P}_2^* = -\mathrm{D}_t(ST_3), \quad \mathcal{P}_3^* = -\mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3).$$

One verifies that $\nabla \bullet \boldsymbol{\mathcal{P}}^* \equiv 0$. It follows that the conservation laws $\boldsymbol{\mathcal{P}}$ and $\widetilde{\boldsymbol{\mathcal{P}}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}^*$ are equivalent. Thus, the particle relabelling symmetry (2.4) leads to the conservation of the potential vorticity Ω (**12**).

3. In the case of the vector field (4.4), the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ is defined by

$$Q_1 = E - By, \quad Q_2 = F + Bx, \quad Q_3 = -Bv, \quad Q_4 = Bu.$$

Because $\boldsymbol{\zeta} = (0,0,0)$, from (4.19) it follows that $\mathcal{P}_j = \mathcal{A}_j$, $j = 1, 2, 3$. From (4.20) we get

$$\mathcal{A}_1 = \tfrac{1}{2} g h^2 [E y_b - F x_b - B(x x_b + y y_b)],$$

$$\mathcal{A}_2 = \tfrac{1}{2} g h^2 [-E y_a + F x_a + B(x x_a + y y_a)],$$

$$\mathcal{A}_3 = E(u-R) + F(v+P) + B[x(v+P) - y(u-R)],$$

and so $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. In particular, for the vector fields Y_1, Y_2 and Y_3 given by (4.6), (4.8) and (4.10), respectively, if one substitutes $B = 1$, $E = \lambda$, $F = \gamma$, and $B = 0$, $E = 1$, $F = \lambda$, and respectively $B = 0$, $E = 0$, $F = 1$ into the above expressions, one obtains the conservation laws contained in Table 1. The rotation invariance leads to the conservation of the angular momentum and the translation invariance to the conservation of linear momentum.

5. Variational symmetries and conservations laws for the semi-geostrophic equations

5.1 *Variational symmetries for the semi-geostrophic equations*

The SW equations (3.2) represent the Euler–Lagrange equations associated with the first-order Lagrangian (**13**)

$$\mathcal{L} = (u_g - R)\dot{x} + (v_g + P)\dot{y} - \tfrac{1}{2}(u_g^2 + v_g^2 + gh) - r\dot{u}_g + p\dot{v}_g, \tag{5.1}$$

where $P(x, y)$, $R(x, y)$, $p(u_g, v_g)$ and $r(u_g, v_g)$ are arbitrary functions satisfying

$$P_x + R_y = f, \quad p_{u_g} + r_{v_g} = 1/f,$$

and h is given by (1.3).

THEOREM 5.1. *The variational symmetry group of the variational problem associated with the Lagrangian* (5.1) *is generated by the vector fields* (4.2). *Moreover, if $P(x, y)$ and $R(x, y)$ are solutions of* (4.15), *then the vector field*

$$\mathrm{Z} = E\partial_x + F\partial_y \tag{5.2}$$

also generates a variational symmetry transformation.

1. *If $E \neq 0$, then let $\lambda = F/E$. It follows that P and R are given by* (4.7). *In this case, there is an additional variational symmetry generated by the vector field* (5.2) *denoted by Z_1 and given by* (4.8).
2. *For $E = 0$, the functions P and R are given by* (4.9). *The vector field $\mathrm{Z} = \mathrm{Z}_2$ defined by* (4.10) *generates the associated additional variational symmetry.*

Proof. Let us consider the infinitesimal generator of the symmetry group associated with the SG equations given by (3.4). The criterion for infinitesimal invariance implies that

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + \mathcal{L}\nabla \bullet \boldsymbol{\zeta} = 0,$$

where $\boldsymbol{\zeta} = (\xi, \eta, \theta)$. Using $\nabla \bullet \boldsymbol{\zeta} = \xi_a + \eta_b = 4A$, the above condition becomes

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + 4A\mathcal{L} = 0. \tag{5.3}$$

The non-zero partial derivatives of the Lagrangian (5.1) are the following:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \quad \frac{\partial \mathcal{L}}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \quad \frac{\partial \mathcal{L}}{\partial u} = \dot{x} - u_g - r_{u_g}\dot{u}_g + p_{u_g}\dot{v}_g,\\
&\frac{\partial \mathcal{L}}{\partial v} = \dot{y} - v - r_{v_g}\dot{u}_g + p_{v_g}\dot{v}_g, \quad \frac{\partial \mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2 y_b, \quad \frac{\partial \mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2 y_a,\\
&\frac{\partial \mathcal{L}}{\partial \dot{x}} = u_g - R, \quad \frac{\partial \mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2 x_b, \quad \frac{\partial \mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2 x_a, \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = v_g + P,\\
&\frac{\partial \mathcal{L}}{\partial \dot{u}_g} = -r, \quad \frac{\partial \mathcal{L}}{\partial \dot{v}_g} = p.
\end{aligned} \tag{5.4}$$

Applying the first-order prolongation of the vector field X to the Lagrangian $\mathcal{L}$ it follows that

$$\begin{aligned}\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) = {} & \phi\frac{\partial\mathcal{L}}{\partial x} + \psi\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial u} + \omega\frac{\partial\mathcal{L}}{\partial v} + \phi^{[a]}\frac{\partial\mathcal{L}}{\partial x_a} + \phi^{[b]}\frac{\partial\mathcal{L}}{\partial u} + \phi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{x}} \\ & + \psi^{[a]}\frac{\partial\mathcal{L}}{\partial y_a} + \psi^{[b]}\frac{\partial\mathcal{L}}{\partial y_b} + \psi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{y}} + \zeta^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{u}_g} + \omega^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{v}_g},\end{aligned} \tag{5.5}$$

where the functions ϕ, ψ, ζ and ω are given by (3.3) and, moreover,

$$\phi^{[a]} = (A - \xi_a)x_a - By_a - \eta_a x_b, \tag{5.6a}$$

$$\phi^{[b]} = -\xi_b x_a + (A - \eta_b)x_b - By_b, \tag{5.6b}$$

$$\phi^{[t]} = A\dot{x} - B\dot{y}, \tag{5.6c}$$

$$\psi^{[a]} = Bx_a + (A - \xi_a)y_a - \eta_a y_b, \tag{5.6d}$$

$$\psi^{[b]} = Bx_b - \xi_b y_a + (A - \eta_b)y_b, \tag{5.6e}$$

$$\psi^{[t]} = B\dot{x} + A\dot{y}, \tag{5.6f}$$

$$\zeta^{[t]} = A\dot{u}_g - B\dot{v}_g, \tag{5.6g}$$

$$\omega^{[t]} = B\dot{u}_g + A\dot{v}_g. \tag{5.6h}$$

Substituting (3.3), (5.5), (5.6) and (5.4) into the relation (5.3), and equating to zero the coefficients of the t-derivatives $\dot{x}$, $\dot{y}$, $\dot{u}_g$ and $\dot{v}_g$ and the free term, we get $A = B = 0$, and the conditions (4.15) between the constants E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ from the Lagrangian (5.1). If $A = 0$, then the condition (2.3) turns into $\xi_a + \eta_b = 0$, and so there is a function $S = S(a, b)$ such that $S_a = \eta$ and $S_b = -\xi$. We obtain that the variational symmetries are generated by the vector fields (4.2). Moreover, if the functions P and R satisfy (4.15), then there is an additional variational symmetry corresponding to the vector field Z given by (5.2).

Some special cases for (4.15) are as follows.

1. If $E \neq 0$, then let $\lambda = F/E$. Hence (4.15) can be written as (4.16) with the general solution given by (4.7). The vector field Z, (5.2), becomes (4.8).
2. If $E = 0$, then (4.15) is written as (4.17).
   (a) If $F \neq 0$ then the general solution is given by (4.9) and the vector field Z, (5.2), becomes (4.10).
   (b) If $F = 0$, there are only variational symmetries generated by the vector fields (5.2).

We remark that the particular cases of Theorem 5.1 correspond to case 2 of Theorem 4.1.

5.2 *Conservation laws for the semi-geostrophic equations*

THEOREM 5.2. *The conservation laws associated with the SG equations are presented in Table* 2, *where* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the associated flux vector fields and* $\mathcal{P}_3$ *the conserved densities. The conservation law deriving from the vector field* Y_0 *is equivalent to that defined by* $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ *with the components*

$$\widetilde{\mathcal{P}}_1 = \mathcal{P}_1 + \mathrm{D}_t(ST_2), \quad \widetilde{\mathcal{P}}_2 = \mathcal{P}_2 - \mathrm{D}_t(ST_3), \quad \widetilde{\mathcal{P}}_3 = -S\Omega^* = \mathcal{P}_3 - \mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3),$$

where

$$T_1 = \dot{x}(R - u_g) - \dot{y}(v_g + P) + \tfrac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u}_g - p\dot{v}_g, \tag{5.7a}$$

$$T_2 = x_b(u_g - R) + y_b(v_g + P) - ru_{g,b} + pv_{g,b}, \tag{5.7b}$$

$$T_3 = x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}, \tag{5.7c}$$

and Ω^* *is the potential vorticity* (1.8).

Proof. In the Appendix we give a brief outline of the method used to calculate conservation laws from variational symmetries (Noether's theorem). This involves the use of the so-called *higher Euler operators* and the *characteristic* of the symmetry. We also define the notion of *equivalence* of conservation laws.

The higher Euler operators applied to the Lagrangian (5.1) are

$$\mathcal{E}_1^{(1)} = \frac{\partial \mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2 y_b, \quad \mathcal{E}_2^{(1)} = \frac{\partial \mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2 x_b, \quad \mathcal{E}_3^{(1)} = \frac{\partial \mathcal{L}}{\partial u_{g,a}} = 0, \quad \mathcal{E}_4^{(1)} = \frac{\partial \mathcal{L}}{\partial v_{g,a}} = 0,$$

$$\mathcal{E}_1^{(2)} = \frac{\partial \mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2 y_a, \quad \mathcal{E}_2^{(2)} = \frac{\partial \mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2 x_a, \quad \mathcal{E}_3^{(2)} = \frac{\partial \mathcal{L}}{\partial u_{g,b}} = 0, \quad \mathcal{E}_4^{(2)} = \frac{\partial \mathcal{L}}{\partial v_{g,b}} = 0,$$

$$\mathcal{E}_1^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{x}} = u_g - R, \quad \mathcal{E}_2^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{y}} = v_g + P, \quad \mathcal{E}_3^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{u}_g} = -r, \quad \mathcal{E}_4^{(3)} = \frac{\partial \mathcal{L}}{\partial \dot{v}_g} = p.$$

For each of the vector fields yielding a variational symmetry (Theorem 5.1), we find the associated conservation law.

1. For the vector field $\mathrm{Y}_1 = \partial_t$, the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ has components given by

$$Q_1 = -\dot{x}, \quad Q_2 = -\dot{y}, \quad Q_3 = -\dot{u}_g, \quad Q_4 = -\dot{v}_g.$$

In this case, $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, given by (4.19), is determined by $\boldsymbol{\zeta} = (0, 0, 1)$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, where

$$\mathcal{A}_1 = \tfrac{1}{2}gh^2(-\dot{x}y_b + \dot{y}x_b), \quad \mathcal{A}_2 = \tfrac{1}{2}gh^2(\dot{x}y_a - \dot{y}x_a),$$
$$\mathcal{A}_3 = -\dot{x}(u_g - R) - \dot{y}(v_g + P) + \dot{u}_g r - \dot{v}_g p.$$

Table 2 Symmetries and their associated conservation laws for the SG equations

vector field	$\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$
$\mathrm{X}_1 = \partial_t$	$(\tfrac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b), \tfrac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a), \tfrac{1}{2}(u_g^2 + v_g^2 + gh))$
$\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$	$(-S_b T_1, S_a T_1, S_a T_2 - S_b T_3)$
$\mathrm{Z}_1 = \partial_x + \lambda\partial_y$	$(\tfrac{1}{2}gh^2(\lambda x_b - y_b), \tfrac{1}{2}gh^2(y_a - \lambda x_a), R - u_g - \lambda(v_g + P))$
$\mathrm{Z}_2 = \partial_y$	$(\tfrac{1}{2}gh^2 x_b, \tfrac{1}{2}gh^2 x_a, -v_g - P)$

On substituting them into (4.19) it follows that

$$\mathcal{P}_1 = \tfrac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b), \quad \mathcal{P}_2 = \tfrac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a), \quad \mathcal{P}_3 = \tfrac{1}{2}(u_g^2 + v_g^2 + gh),$$

which defines the corresponding conservation law. Thus, the time invariance implies the conservation $\mathcal{P}_3$ of the energy of the SG equations.

2. Consider the vector field

$$\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b,$$

for which the characteristic is

$$Q_1 = S_b x_a - S_a x_b, \quad Q_2 = S_b y_a - S_a y_b, \quad Q_3 = S_b u_{g,a} - S_a u_{g,b}, \quad Q_4 = S_b v_{g,a} - S_a v_{g,b}.$$

In the relation (4.19) one substitutes $\boldsymbol{\zeta} = (-S_b, S_a, 0)$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ given by (4.20), namely

$$\mathcal{A}_1 = \tfrac{1}{2}ghS_b, \quad \mathcal{A}_2 = -\tfrac{1}{2}ghS_a,$$

$$\mathcal{A}_3 = S_a[x_b(R - u_g) - y_b(v_g + P) + ru_{g,b} - pv_{g,b}] + S_b[x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}],$$

and it follows that

$$\mathcal{P}_1 = -S_b T_1, \quad \mathcal{P}_2 = S_a T_1, \quad \mathcal{P}_3 = S_a T_2 - S_b T_3,$$

where

$$T_1 = \dot{x}(R - u_g) - \dot{y}(v_g + P) + \tfrac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u}_g - p\dot{v}_g,$$

$$T_2 = x_b(u_g - R) + y_b(v_g + P) - ru_{g,b} + pv_{g,b},$$

$$T_3 = x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}.$$

The conservation law deriving from the vector field Y_0 can be written in an equivalent form. Note that

$$\mathrm{D}_a(T_2) - \mathrm{D}_b(T_3) = \boldsymbol{\Omega}^*,$$

where $\boldsymbol{\Omega}^*$ is the potential vorticity (1.8). Consider $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ defined by

$$\widetilde{\mathcal{P}}_1 = -S_b T_1 + \mathrm{D}_t(ST_2) = P_1 + \mathrm{D}_t(ST_2),$$

$$\widetilde{\mathcal{P}}_2 = S_a T_1 - \mathrm{D}_t(ST_3) = P_2 - \mathrm{D}_t(ST_3),$$

$$\widetilde{\mathcal{P}}_3 = -S\boldsymbol{\Omega}^* = P_3 - \mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3),$$

and

$$\mathcal{P}_1^* = \mathrm{D}_t(ST_2), \quad \mathcal{P}_2^* = -\mathrm{D}_t(ST_3), \quad \mathcal{P}_3^* = -\mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3).$$

Because $\nabla \bullet \boldsymbol{\mathcal{P}}^* = 0$, it follows that $\boldsymbol{\mathcal{P}}$ and $\widetilde{\boldsymbol{\mathcal{P}}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}^*$ are equivalent conservation laws. Conservation of potential vorticity $\boldsymbol{\Omega}^*$ follows (**13**).

3. In the case of the vector field Z given by (5.2), the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ has the components

$$Q_1 = E, \quad Q_2 = F, \quad Q_3 = 0, \quad Q_4 = 0.$$

While $\boldsymbol{\zeta} = (0, 0, 0)$ it follows that $\mathcal{P}_j = \mathcal{A}_j$, $j = 1, 2, 3$. Using (4.20) we get

$$\mathcal{A}_1 = \tfrac{1}{2}gh^2(Ey_b - Fx_b), \quad \mathcal{A}_2 = \tfrac{1}{2}gh^2(Fx_a - Ey_a), \quad \mathcal{A}_3 = E(u_g - R) + F(v_g + P),$$

and in this case, the relation (4.19) implies that

$$\mathcal{P}_1 = \tfrac{1}{2}gh^2(Fx_b - Ey_b), \quad \mathcal{P}_2 = \tfrac{1}{2}gh^2(Ey_a - Fx_a), \quad \mathcal{P}_3 = E(R - u_g) - F(v_g + P).$$

In the particular cases $\mathrm{Z} = \mathrm{Z}_1$ and $\mathrm{Z} = \mathrm{Z}_2$, one substitutes $E = 1$ and $F = \lambda$ and respectively $E = 0$ and $F = 1$ into the above expressions. The corresponding triple $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ leads to the conservation of linear momentum (Table 2).

6. Particle relabelling symmetry for the semi-geostrophic equations

In this section we explore further consequences of the particle relabelling symmetry for the SG equations. As shown above, this is the symmetry responsible for the conservation of potential vorticity.

We first note the invariants, the invariant differential operators and their commutators, and the differential relations, or *syzygies*, between the invariants. These allow us to rewrite the SG equations in a way in which symbolic formal solution mechanisms can be applied effectively. Even so, the application of these series solution methods is far from routine, as formal expansions for the differential operators are also involved.

The result is that solutions of the SG equations can be expressed in terms of solutions of an *infinite linear cascade*; see Theorem 6.3. Finally we ask whether the series solution corresponds to a *physical fluid flow*, that is, do solutions obey the integral curve property, $x(x(a, b, t), y(a, b, t), s) = x(a, b, t + s)$ and $y(x(a, b, t), y(a, b, t), s) = y(a, b, t + s)$. The startling observation is that the formal solution *cannot* correspond to any such flow. We note that this property is not the same as invariance under translation in time, which corresponds rather to the fact that the origin in the time coordinate can be set arbitrarily.

6.1 *Invariants of the particle relabelling symmetry*

Under the particle relabelling symmetry group (2.4), the fluid particle positions satisfy $x^*(a^*, b^*) = x(a, b)$ and $y^*(a^*, b^*) = y(a, b)$, and so in the parlance of group theory, are said to be invariant functions. Since

$$x^*_{a^*}y^*_{b^*} - x^*_{b^*}y^*_{a^*} = \frac{x_a y_b - x_b y_a}{\phi_a \psi_b - \phi_b \psi_a} = x_a y_b - x_b y_a,$$

we have that the function $\Delta = x_a y_b - x_b y_a$ is an invariant of the pseudogroup, as are $u = x_t$ and $v = y_t$.

The invariant differential operators are

$$\partial_x = \frac{y_b \partial_a - y_a \partial_b}{x_a y_b - x_b y_a}, \quad \partial_y = \frac{-x_b \partial_a + x_a \partial_b}{x_a y_b - x_b y_a}, \tag{6.1}$$

where, as above, $\partial_a = \partial/\partial a$ and $\partial_b = \partial/\partial b$ are the usual partial derivatives. The operators (6.1) are the same as those that would be obtained under a standard change of coordinates from (a, b)-space to (x, y)-space.

The systems studied vary in time, t, which is invariant under the pseudogroup, as is the (total) derivative with respect to time. On functions of a, b and t, the derivative with respect to t is simply $\partial_t = \partial/\partial t$. On functions of (x, y), however, since x and y depend on t, it is the total derivative that is used. To distinguish this fact we use the notation D_t to denote total differentiation with respect to time. The notation of f_x for $\partial_x f$ can trap the unwary as D_t and ∂_x do not commute. For this reason we do not use the notation f_x for $\partial_x f$ in expressions where D_t also occurs.

The method of moving frames as applied to this pseudogroup (**16**, **17**) yields the following.

THEOREM 6.1. *The functions* $\Delta = x_a y_b - x_b y_a$, x *and* y *are the fundamental generating differential invariants. Every differential invariant is a function of these and their invariant derivatives. Moreover, the only differential relation between the invariants, or syzygy, is the well-known continuity relation*

$$\mathrm{D}_t \Delta = \Delta(\partial_x \mathrm{D}_t x + \partial_y \mathrm{D}_t y). \tag{6.2}$$

One of the main problems in analysing systems involving both D_t and ∂_x, ∂_y derivatives is that these operators do not commute.

THEOREM 6.2. *The commutation relations between* D_t, ∂_x *and* ∂_y *are*

$$[\partial_x, \partial_y] = 0, \tag{6.3a}$$

$$[\mathrm{D}_t, \partial_x] = -(\partial_x \mathrm{D}_t x)\partial_x - (\partial_x \mathrm{D}_t y)\partial_y, \tag{6.3b}$$

$$[\mathrm{D}_t, \partial_y] = -(\partial_y \mathrm{D}_t x)\partial_x - (\partial_y \mathrm{D}_t y)\partial_y. \tag{6.3c}$$

Proof. We check (6.3b) here:

$$\begin{aligned}
[\mathrm{D}_t, \partial_x] &= \mathrm{D}_t \left(\frac{y_b \partial_a - y_a \partial_b}{x_x y_b - x_b y_a} \right) - \left(\frac{y_b \partial_a - y_a \partial_b}{x_x y_b - x_b y_a} \right) \mathrm{D}_t \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \frac{y_{bt} \partial_a - y_{at} \partial_b}{\Delta} \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \frac{y_{bt}}{\Delta}(x_a \partial_x + y_a \partial_y) - \frac{y_{at}}{\Delta}(x_b \partial_x + y_b \partial_y) \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \left(\frac{y_{bt} x_a - y_{at} x_b}{\Delta} \right) \partial_x + \left(\frac{y_{bt} y_a - y_{at} y_b}{\Delta} \right) \partial_y \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \partial_y (\mathrm{D}_t y) \partial_x - \partial_x (\mathrm{D}_t y) \partial_y \\
&= -\partial_x (\mathrm{D}_t x) \partial_x - \partial_x (\mathrm{D}_t y) \partial_y,
\end{aligned}$$

using the syzygy (6.2). That these make sense can be seen by calculating the commutation relations on x and y.

6.2 *A reformulation of the two-dimensional semi-geostrophic equations*

We write the SG equations (1.6), (1.7) in terms of the invariants and invariant operators given above. Instead of Δ we use h, where

$$h = \frac{1}{\Delta} = \frac{1}{x_a y_b - x_b y_a}.$$

In the notation used in this section, these equations are written as

$$\mathrm{D}_t x = -\frac{g}{f^2}\mathrm{D}_t\partial_x h - \frac{g}{f}\partial_y h, \quad \mathrm{D}_t y = -\frac{g}{f^2}\mathrm{D}_t\partial_y h + \frac{g}{f}\partial_x h. \tag{6.4}$$

The syzygy (6.2) and commutation relations (6.3) hold identically. If we evaluate them on the SG equations, by which we mean backsubstituting for the various quantities, we obtain relations which must be true on solutions of the SG equations, and in fact are equivalent to them.

If we backsubstitute for $\mathrm{D}_t x$ and $\mathrm{D}_t y$ using the SG equations, the syzygy becomes

$$\mathrm{D}_t h = \frac{g}{f^2}(\partial_x \mathrm{D}_t\partial_x h + \partial_y \mathrm{D}_t \partial_y h),$$

and the commutation relations become

$$[\mathrm{D}_t, \nabla] = \frac{g}{f^2}\begin{pmatrix} \partial_x \mathrm{D}_t\partial_x h & \partial_x \mathrm{D}_t \partial_y h \\ \partial_y \mathrm{D}_t\partial_x h & \partial_y \mathrm{D}_t\partial_y h\end{pmatrix}\nabla + \frac{g}{f}\begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy}\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\nabla.$$

If we let these commutation relations act on the function h, and then backsubstitute for $\mathrm{D}_t h$ using the syzygy (6.2) and for $\mathrm{D}_t\partial_x h$ and $\mathrm{D}_t\partial_y h$ from (6.4), we obtain

$$\frac{f^2}{g}u + f h_y = (h(u_x + v_y))_x + u_x h_x + v_x h_y, \tag{6.5a}$$

$$\frac{f^2}{g}v - f h_x = (h(u_x + v_y))_y + u_y h_x + v_y h_y, \tag{6.5b}$$

where we have put $u = \mathrm{D}_t x$ and $v = \mathrm{D}_t y$, a standard notation for these quantities.

We have effected a linear substitution of each of equations (6.4) into one component each of an identity, and it is simple to see that (6.5) are equivalent to the two-dimensional SG equations.

In this notation, the potential vorticity conservation law is (cf. (1.8))

$$\mathrm{D}_t\boldsymbol{\Omega}^* = 0, \quad \text{where} \quad \boldsymbol{\Omega}^* = \frac{1}{h}\left\{ f + \frac{g}{f}\left(h_{xx} + h_{yy}\right) + \frac{g^2}{f^3}\left(h_{xx}h_{yy} - h_{xy}^2\right)\right\}. \tag{6.6}$$

The equations (6.5) are linear in u and v and since they involve only x and y derivatives which commute, we may apply standard completion procedures to this system (**18** to **23**). The result is two compatibility conditions. The important one is easily obtained. If we denote (6.5a) by $\mathcal{A}$ and (6.5b) by $\mathcal{B}$, then a first-order (with respect to u and v) compatibility equation is obtained by calculating $\mathcal{C} = \mathcal{A}_y - \mathcal{B}_x$. This yields

$$u_y\left(\frac{f^2}{g} + h_{xx}\right) - v_x\left(\frac{f^2}{g} + h_{yy}\right) + h_{xy}(v_y - u_x) + f(h_{xx} + h_{yy}) = 0. \tag{6.7}$$

Interestingly, the conservation of potential vorticity, (6.6), is *not* a compatibility condition of (6.5). It turns out that $\mathrm{D}_t\boldsymbol{\Omega}^* = 0$ is an identity on the formal solution of (6.5) that we find, and imposes no further conditions.

6.3 *Formal solution procedures*

One major application of differential system completion procedures is that the calculation of formal series solutions proceeds without the discovery of unsuspected consistency conditions on the coefficients; see, for example, (**22** to **24**). There are, moreover, convergence results which are classical (**25**).

6.3.1 *The naive approach.*

Take power series for x and y in time,

$$x(a,b,t) = a + x_1(a,b)t + x_2(a,b)t^2 + \mathcal{O}(t^3), \tag{6.8a}$$

$$y(a,b,t) = b + y_1(a,b)t + y_2(a,b)t^2 + \mathcal{O}(t^3), \tag{6.8b}$$

which use the standard initial condition $x(a,b,0) = a$, $y(a,b,0) = b$. The corresponding series for $u = \mathrm{D}_t x$, $v = \mathrm{D}_t y$ are simply the time derivatives of those for x and y, while those for h, ∂_x and ∂_y are readily calculated. Indeed, we have

$$h = 1 - (x_{1,a} + y_{1,b})t + \left\{(x_{1,a} + y_{1,b})^2 + x_{1,b}y_{1,a} - x_{1,a}y_{1,b} - x_{2,a} - y_{2,b}\right\} t^2 + \mathcal{O}(t^3) \tag{6.9}$$

and

$$\begin{aligned}\partial_x &= \left\{1 - x_{1,a}t + (x_{1,a}^2 + x_{1,b}y_{1,a} - x_{2,a})t^2 + \mathcal{O}(t^3)\right\}\partial_a \\ &\quad + \left\{-y_{1,a}t + (y_{1,a}x_{1,a} + y_{1,b}y_{1,a} - y_{2,a})t^2 + \mathcal{O}(t^3)\right\}\partial_b, \\ \partial_y &= \left\{-x_{1,b}t + (x_{1,b}x_{1,a} + x_{1,b}y_{1,b} - x_{2,b})t^2 + \mathcal{O}(t^3)\right\}\partial_a \\ &\quad + \left\{1 - y_{1,b}t + (y_{1,b}^2 + y_{1,a}x_{1,b} - y_{2,b})t^2 + \mathcal{O}(t^3)\right\}\partial_b.\end{aligned}$$

These were inserted into (6.5) and (6.7), and coefficients of like powers of t were collected.

We denote the coefficient of t^n in the expansion of $\mathcal{A}$, that is, (6.5a), by a_n, that in $\mathcal{B}$, that is, (6.5b), by b_n, and that in $\mathcal{C}$, that is, (6.7), by c_n. These coefficients must all be zero on solutions. Setting these coefficients equal to zero leads to a system in the x_i, y_i.

6.3.2 *The zeroth level.*

Setting $t = 0$ in the expansions for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ yields

$$c_0 = \frac{f^2}{g}(x_{1,b} - y_{1,a}), \quad a_0 = -\frac{f^2}{g}x_1 + x_{1,aa} + y_{1,ab}, \quad b_0 = -\frac{f^2}{g}y_1 + x_{1,ab} + y_{1,bb}.$$

These must all be zero on a solution of the SG equations. Thus one has, immediately to first order, that

$$x_1 = \alpha_a, \quad y_1 = \alpha_b, \quad \alpha_{aa} + \alpha_{bb} = \frac{f^2}{g}\alpha \tag{6.10}$$

for some function α.

6.3.3 *The first level.* The coefficients are complicated, so we simplify them using the zeroth-order system. In order for this to be done in an effective manner in a computer algebra environment, it is important to use symbolic simplification algorithms (**18** to **23**) that are guaranteed to prevent infinite loops in the reduction process and to obtain a well-defined, indeed unique, result. We obtain

$$2y_{2,a} - 2x_{2,b} + fy_{1,b} + fx_{1,a} = 0, \tag{6.11}$$

and an equation for x_2 of the form

$$x_{2,aa} + x_{2,bb} = \frac{f^2}{g}x_2 + F_2(x_1, y_1).$$

If we take $x_2 = \beta_a + \frac{1}{2}f\alpha_b$ and $y_2 = \beta_b - \frac{1}{2}f\alpha_a$, so that (6.11) is satisfied, and substitute these into a_1 and b_1, we obtain two conditions which are the a and b derivatives of the following condition on β:

$$\beta_{aa} + \beta_{bb} - \frac{f^2}{g}\beta - \frac{f^2}{2g}(\alpha_a^2 + \alpha_b^2) + \alpha_{aa}\alpha_{bb} - \alpha_{ab}^2 - \frac{f^4}{g^2}\alpha^2 = 0. \tag{6.12}$$

Note that we may absorb the constant of integration into β. Algorithms are available which symbolically integrate total derivatives (**26** to **28**).

6.3.4 *The second level.* Continuing as before, we obtain conditions for x_3 and y_3 in terms of x_1, y_1, x_2 and y_2. The first-order compatibility condition is

$$3y_{3,a} - 3x_{3,b} + fy_{2,a} + fx_{2,b} + 3f(x_{1,a}y_{1,b} - x_{1,b}y_{1,a}) = 0,$$

and a condition for x_3 of the form

$$x_{3,aa} + x_{3,ab} = \frac{f^2}{g}x_{3,a} + F_3(x_1, y_1, x_2, y_2)$$

is obtained. Setting $x_3 = \gamma_a + \frac{1}{3}f\beta_b + f\alpha_b\alpha_{aa}$ and $y_3 = \gamma_b - \frac{1}{3}f\beta_a + f\alpha_b\alpha_{ab}$ into a_2 and b_2, we obtain the a and b derivatives of a condition for γ of the form

$$\gamma_{aa} + \gamma_{bb} - \frac{f^2}{g}\gamma = F_3(\beta, \alpha). \tag{6.13}$$

Continuing, we find that the structure of the equations to leading order is the same at the third and fourth levels. One tedious aspect of this process is the need to perform symbolic integrations to find the condition for the function α_n introduced at each order ($\alpha_0 = \alpha$, $\alpha_1 = \beta$, $\alpha_2 = \gamma$). In the proof of the general result, we use a different process to find the condition for the function introduced at each order in one step.

6.3.5 *Results check.* If we substitute

$$x(a, b, t) = a + \alpha_a t + \left(\beta_a + \tfrac{1}{2}f\alpha_b\right)t^2 + \left(\gamma_a + \tfrac{1}{3}f\beta_b + f\alpha_b\alpha_{aa}\right)t^3 + \mathcal{O}(t^4),$$

$$y(a, b, t) = b + \alpha_b t + \left(\beta_b - \tfrac{1}{2}f\alpha_a\right)t^2 + \left(\gamma_b - \tfrac{1}{3}f\beta_a + f\alpha_b\alpha_{ab}\right)t^3 + \mathcal{O}(t^4)$$

into the conservation law for Q and the original SG equations, then they are satisfied to the relevant order provided that (6.10), (6.12) and (6.13) hold. For the sake of completeness, we record the expansions for h, h_x and h_y in terms of α, β etc.:

$$h = 1 - \frac{f^2}{g}\alpha t - \frac{f^2}{2g}(\alpha_a^2 + \alpha_b^2 + 2\beta)t^2 - \frac{f^2}{g}\Big(\beta_a\alpha_a + \beta_b\alpha_b + \tfrac{1}{2} f\alpha_a\alpha_b + \gamma\Big)t^2 + \mathcal{O}(t^3),$$

$$h_x = -\frac{f^2}{g}\alpha_a t - \frac{f^2}{g}\beta_a t^2 - \frac{f^2}{g}(\gamma_a + f\alpha_b\alpha_{aa})t^3 + \mathcal{O}(t^4),$$

$$h_y = -\frac{f^2}{g}\alpha_b t - \frac{f^2}{g}\beta_b t^2 - \frac{f^2}{g}(\gamma_b + f\alpha_b\alpha_{ab})t^3 + \mathcal{O}(t^4).$$

6.3.6 *Proof of the general result.*

THEOREM 6.3. *The two-dimensional SG equations are equivalent to an infinite set of essentially linear equations in a triangular or cascade form. Indeed, if x and y are of the form* (6.8), *then there are functions $p_n(a, b)$, F_n and G_n^i such that*

$$x_{n+1} = \frac{g}{nf^2} p_{n,a} + G_n^1(p_0, p_1, \dots p_{n-1}), \quad y_{n+1} = \frac{g}{nf^2} p_{n,b} + G_n^2(p_0, p_1, \dots p_{n-1}), \tag{6.14}$$

where

$$p_{n,aa} + p_{n,bb} = \frac{f^2}{g} p_n + F_n(p_0, \dots, p_{n-1}). \tag{6.15}$$

Proof. If we define

$$\mathcal{U} = \frac{f^2}{g}u + fh_y - u_x h_x - v_x h_y, \quad \mathcal{V} = \frac{f^2}{g}v - fh_x - u_y h_x - v_y h_y, \tag{6.16}$$

then (6.5) are of the forms

$$\mathcal{U} = \Phi_x, \quad \mathcal{V} = \Phi_y, \tag{6.17}$$

where we have defined Φ by

$$\Phi = h(u_x + v_y). \tag{6.18}$$

The compatibility condition (6.7) is then

$$\mathcal{U}_y - \mathcal{V}_x = 0.$$

By substituting in the definitions of ∂_x and ∂_y, (6.17) can be rewritten as

$$x_a\mathcal{U} + y_a\mathcal{V} = \Phi_a, \quad x_b\mathcal{U} + y_b\mathcal{V} = \Phi_b. \tag{6.19}$$

The recursive expansion method consists of the following steps.

1. Set

$$\Phi = \sum_{j=0}^{\infty} p_j(a, b)t^j, \tag{6.20}$$

where the p_j are functions to be determined.

2. Substitute the expressions for Φ, $\mathcal{U}$ and $\mathcal{V}$ from (6.20) and (6.16) into (6.19), together with the expansions for $u = \mathrm{D}_t x$, $v = \mathrm{D}_t y$ from (6.8) and (6.9), and equate coefficients of powers of t to zero. At each order n we obtain the two equations (6.14) whose form can be seen by leading-order analysis.
3. Substitute (6.14) into (6.18). At each order of t we obtain an equation for p_{n-1} in terms of $p_0, p_1, \ldots, p_{n-2}$. By a leading term analysis, we have that the equation for p_n is of the form (6.15).

6.3.7 *Discussion.* The formal solution provides a mechanism for solving the two-dimensional SG equations in terms of a sequence of boundary-value problems.

6.4 *The cascade in the Legendre transform coordinates*

The numerical solution of two-dimensional SG equations is carried out in Legendre transform coordinates, which we now define. Setting

$$X = x + v_g/f, \quad Y = y - u_g/f,$$

then for solutions of the SG equations we have

$$\mathrm{D}_t X = u_g, \quad \mathrm{D}_t Y = v_g. \tag{6.21}$$

At time $t = 0$, we have $X(a, b, 0) = a$ and $Y(a, b, 0) = b$.

Just as we have the invariants, invariant differential operators and syzygies of the particle relabelling symmetry in the a, b, t, x, y variables, so we have the same information in the a, b, t, X, Y variables. Thus we have the three fundamental invariants $H = 1/(X_a Y_b - X_b Y_a)$, X and Y, and the invariant differential operators

$$\partial_X = \frac{Y_b \partial_a - Y_a \partial_b}{X_a Y_b - X_b Y_a}, \quad \partial_Y = \frac{-X_b \partial_a + X_a \partial_b}{X_a Y_b - X_b Y_a},$$

while the syzygy is

$$\mathrm{D}_t H + H(U_X + V_Y) = 0,$$

where $U = \mathrm{D}_t X$ and $V = \mathrm{D}_t Y$.

In these coordinates, the conservation of potential vorticity takes the form

$$X_a Y_b - X_b Y_a \equiv 1,$$

and so the syzygy takes the form

$$U_X + V_Y = 0.$$

Since $H \equiv 1$, the idea used in the previous section to find a more amenable form of the equations to study, that of evaluating the commutation rules on the invariant h, fails.

The standard procedure is to introduce a potential function Ψ such that

$$U = \mathrm{D}_t X = -\Psi_Y, \quad V = \mathrm{D}_t Y = \Psi_X. \tag{6.22}$$

There are two relations connecting h with Ψ, one from the definitions and one from the conservation of potential vorticity.

THEOREM 6.4. *In terms of the potential function* Ψ, *we have both of*

$$h = \frac{f}{g}\Psi - \frac{1}{2g}(\Psi_X^2 + \Psi_Y^2), \tag{6.23a}$$

$$h^{-1} = 1 - \frac{1}{f}(\Psi_{XX} + \Psi_{YY}) + \frac{1}{f^2}(\Psi_{XX}\Psi_{YY} - \Psi_{XY}^2), \tag{6.23b}$$

and thus

$$1 \equiv \left\{\frac{f}{g}\Psi - \frac{1}{2g}(\Psi_X^2 + \Psi_Y^2)\right\}\left\{1 - \frac{1}{f}(\Psi_{XX} + \Psi_{YY}) + \frac{1}{f^2}(\Psi_{XX}\Psi_{YY} - \Psi_{XY}^2)\right\}. \tag{6.24}$$

A 'cascade' solution to the SG equations can be obtained in terms of the function Ψ as follows. From the fact that $h(a, b, 0) \equiv 1$ and (6.23) we have that $\psi(a, b, 0) \equiv g/f$. Then equations (6.22) can be rewritten as

$$\mathrm{D}_t X = X_b\Psi_a - X_a\Psi_b, \quad \mathrm{D}_t Y = Y_b\Psi_a - Y_a\Psi_b, \tag{6.25}$$

so that the equations for X and Y are the same, but their initial conditions are different; recall $X(a, b, 0) = a$ and $Y(a, b, 0) = b$. Substituting

$$\Psi = \frac{g}{f} + \sum_{j=1}^{\infty} \psi_j t^j$$

into (6.25) and equating coefficients of powers of t to zero, yields a series of equations of the form

$$\begin{aligned}
X_1 &= 0, & Y_1 &= 0,\\
X_2 &= -\tfrac{1}{2}\psi_{1,b}, & Y_2 &= \tfrac{1}{2}\psi_{1,a},\\
&\vdots & &\vdots\\
X_{n+1} &= -\frac{1}{n+1}\psi_{n,b} + \text{l.o.t.}, & Y_{n+1} &= \frac{1}{n+1}\psi_{n,a} + \text{l.o.t.}
\end{aligned}$$

Backsubstituting these into the expansion of (6.24) yields the equations for the ψ_j. We already know $\psi_0 \equiv g/f$ and so

$$\begin{aligned}
\psi_{1,aa} + \psi_{1,bb} &= \frac{f^2}{g}\psi_1,\\
\psi_{2,aa} + \psi_{2,bb} &= \frac{f^2}{g}\psi_2 - \frac{1}{f}(\psi_{1,aa}\psi_{1,bb} - \psi_{1,ab}^2) - \frac{f^3}{g^2}\psi_1^2 - \frac{1}{2f}(\psi_{1,a}^2 + \psi_{1,b}^2),\\
&\vdots\\
\psi_{n,aa} + \psi_{n,bb} &= \frac{f^2}{g}\psi_n + \text{l.o.t.}
\end{aligned}$$

From the first expression for h in (6.23), and noting that $\Phi = -\mathrm{D}_t h$, we may obtain the series for the α_n obtained in the previous section, in terms of the ϕ_j. Using (6.21) we have also that

$$\Psi = \frac{1}{2f}(v_g^2 + u_g^2) + \frac{g}{f}h,$$

and thus we can obtain the functions ψ_j in terms of the α_n.

It is interesting to note that the linear equation governing each cascade is the same. Further, to second order in time, $u_g = u$ and $v_g = v$, presumably reflecting the order of the semi-geostrophic approximation to the SW equations.

6.5 *Cascade solutions are not flows*

Since the evolution of the particles with coordinates (x, y) is supposed to approximate that of a fluid flow, it is interesting to check that the flow condition actually holds, that is,

$$x(x(a,b,t), y(a,b,t), s) = x(a,b,t+s), \quad y(x(a,b,t), y(a,b,t), s) = y(a,b,t+s)$$

are satisfied for all (a, b) and all t and s. Series expansions satisfying these conditions are precisely those that are one-parameter group actions, and are well understood. In fact, the series depends, in a way which can be made precise, on the first-order coefficients only. This is the content of Sophus Lie's theorem that a group action depends only on the infinitesimal generator. In (**29**, p. 27) can be found the conditions that must be satisfied by the second- and higher-order coefficients. In the present case these take the forms

$$x(a,b,t) = a + \alpha_a t + \tfrac{1}{2}t^2(\alpha_a\alpha_{aa} + \alpha_b\alpha_{ab}) + \mathcal{O}(t^3),$$

$$y(a,b,t) = b + \alpha_b t + \tfrac{1}{2}t^2(\alpha_a\alpha_{ab} + \alpha_b\alpha_{bb}) + \mathcal{O}(t^3).$$

It is simple to check that the only solution satisfying both the cascade equations and the flow conditions is the identity solution,

$$x(a,b,t) \equiv a, \quad y(a,b,t) \equiv b,$$

the failure occurring at the second order! Thus while smooth solutions exist, they do not correspond in the obvious physical sense to the motion of a fluid. This is in contrast to the semi-geostrophic approximation of the two-dimensional Euler equations (**15**).

There are several avenues for the further research needed to fully understand this apparent conundrum. One is to find a computationally effective method of understanding the evolution of the solutions in the various function spaces, in which solutions corresponding to fluid flows are known to exist (**30**, **31**).

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APPENDIX A

Variational symmetries and Noether's theorem

In this Appendix we record the formulae used in the paper, together with a brief indication of how Noether's theorem works in computations. Proofs can be found in the text by P. J. Olver (**1**, §§4.3,4.4,5.3).

In the application in this paper, the independent variables are a, b and t, while the dependent variables are x, y, u and v. The infinitesimals of group actions on a, b and t were denoted by ξ, η and θ respectively, while the infinitesimals of group actions on x, y, u and v were denoted by ϕ, ψ, ζ and ω respectively. To ease the exposition, we will denote independent variables by t_i, $i = 1, \ldots, 3$ and dependent variables by u^α, $\alpha = 1, \ldots, 4$. The derivatives of u^α will be denoted by u^α_K, where K is a multi-index of differentiation. The infinitesimal generators for the t_i will be denoted by ξ^i and the infinitesimal generators for the u^α will be denoted by ϕ^α. The infinitesimal generator for u^α_K will be denoted by $\phi^\alpha_{[K]}$. Thus

$$(u^\alpha_K)^* = u^\alpha_K + \epsilon\phi^\alpha_{[K]} + \mathcal{O}(\epsilon^2).$$

Define $\mu = dt_1 \wedge dt_2 \wedge dt_3$ and $\mu^* = dt_1^* \wedge dt_2^* \wedge dt_3^*$. The group action on a Lagrangian volume form is

$$(\mathcal{L}\, d\mu)^* = \mathcal{L}(t_i^*, u^{\alpha *}, u^{\alpha}_K{}^*, \ldots)\, d\mu^* = \mathcal{L}(t_i^*, u^{\alpha *}, u^{\alpha}_K{}^*, \ldots)\, \frac{\partial(t_1^*, t_2^*, t_3^*)}{\partial(t_1, t_2, t_3)}\, d\mu. \tag{A.1}$$

If $\mathcal{L}\, d\mu = (\mathcal{L}\, d\mu)^*$ is true for all elements of the group, then we can differentiate $(\mathcal{L}\, d\mu)^*$ with respect to a group parameter. Suppose ε is a group parameter and that $\varepsilon = 0$ occurs at the identity of the group. Taking $d/d\varepsilon$ at $\varepsilon = 0$ of both sides of (A.1) yields

$$\begin{aligned} 0 &= \sum_{i,\alpha,K} \frac{\partial \mathcal{L}}{\partial t_i}\xi^i + \frac{\partial \mathcal{L}}{\partial u^\alpha}\phi^\alpha + \frac{\partial \mathcal{L}}{\partial u^\alpha_K}\phi^\alpha_{[K]} + \mathcal{L}\nabla \bullet \xi \\ &= \sum_K \mathrm{D}^K \left(\phi^\alpha - \sum_i u^\alpha_i \xi^i\right) \frac{\partial \mathcal{L}}{\partial u^\alpha_K} + \nabla \bullet \left(\mathcal{L}\xi^1, \mathcal{L}\xi^2, \mathcal{L}\xi^3\right). \end{aligned} \tag{A.2}$$

This can be written compactly using the following notation. Defining

$$\mathcal{D}\mathcal{L} = \sum_K \frac{\partial \mathcal{L}}{\partial u^\alpha_K}\, du^\alpha_K \quad \text{and} \quad \mathsf{v}_{\boldsymbol{Q}} = \sum \mathrm{D}^K \left(Q^\alpha\right) \frac{\partial}{\partial u^\alpha_K},$$

where $\boldsymbol{Q}$ is a vector, $\boldsymbol{Q} = (Q^\alpha)$, then (A.2) can be written as

$$\mathsf{v}_{\boldsymbol{Q}} \lrcorner \mathcal{D}(\mathcal{L}) + \nabla \bullet (\mathcal{L}(\xi^1, \xi^2, \xi^3)) = 0, \tag{A.3}$$

where

$$Q^\alpha = \phi^\alpha - \sum_i u^\alpha_i \xi^i.$$

The functions Q^α are called the *characteristic functions* of the symmetry.

The Euler–Lagrange operator also involves the $\mathcal{D}$ operator. In fact,

$$\mathcal{D}(\mathcal{L})\, d\mu = \sum_\alpha E^\alpha(\mathcal{L})\, du^\alpha \wedge d\mu + \nabla \bullet \boldsymbol{A}, \tag{A.4}$$

where $\boldsymbol{A}$ can be explicitly determined.

EXAMPLE. For $\mathcal{L} = \mathcal{L}(x, x_a, x_{aa})$ we have

$$\begin{aligned}
\mathcal{D}(\mathcal{L})\, d\mu &= \frac{\partial \mathcal{L}}{\partial x}\, dx\, d\mu + \frac{\partial \mathcal{L}}{\partial x_a}\, dx_a\, d\mu + \frac{\partial \mathcal{L}}{\partial x_{aa}}\, dx_{aa}\, d\mu \\
&= \left\{ \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{da}\left(\frac{\partial \mathcal{L}}{\partial x_a}\right) + \frac{d^2}{da^2}\left(\frac{\partial \mathcal{L}}{\partial x_{aa}}\right) \right\} dx\, d\mu + \frac{d}{da}\left(\frac{\partial \mathcal{L}}{\partial x_a} dx\, d\mu + 2 \frac{\partial \mathcal{L}}{\partial x_{aa}} dx_a\, d\mu \right) \\
&= E^{[x]}(\mathcal{L})\, dx\, d\mu + \frac{d}{dx}\left(\boldsymbol{A}\right),
\end{aligned}$$

where this defines $\boldsymbol{A}$, and where $d(dx)/da = dx_a$, etc.

Operating on both sides of (A.4) with $\mathsf{v}_{\boldsymbol{Q}} \lrcorner$ and applying the symmetry condition (A.3) yields

$$0 = \sum_\alpha Q^\alpha E^\alpha(\mathcal{L}) + \nabla \bullet \left(\mathsf{v}_{\boldsymbol{Q}} \lrcorner \boldsymbol{A} - \mathcal{L}(\xi^1, \xi^2, \xi^3) \right). \tag{A.5}$$

Equation (A.5) is the essential content of Noether's theorem; it shows how a divergence expression is obtained by the dot product of the Euler–Lagrange equations and the characteristic functions of the symmetry. There remains only the explicit formulae for the vector inside the divergence operator to be given.

The formula for $\boldsymbol{\mathcal{A}} = \mathsf{v}_{\boldsymbol{Q}} \lrcorner \boldsymbol{A}$ is given in terms of the so-called higher Euler operators E^J_α, where J is an index of differentiation. We need some notation regarding the index notation. We write an index of differentiation in concatenated form, so that

$$I = \underbrace{11\ldots1}_{\tilde{\imath}_1 \text{terms}}\underbrace{22\ldots2}_{\tilde{\imath}_2 \text{terms}}\ldots.$$

Then Ik is the multi-index obtained by adding another k to I. Define $\#I = \sum \tilde{\imath}_k$. Given two indices I and J, the expression $I \supset J$ means that $\tilde{\imath}_k \geqslant \tilde{\jmath}_k$ for all k, while the multi-index $K = I \setminus J$ satisfies $\tilde{\kappa}_k = \max\{0, \tilde{\imath}_k - \tilde{\jmath}_k\}$. We define $I! = \tilde{\imath}_1! \tilde{\imath}_2! \cdots \tilde{\imath}_p!$ and, finally, the multinomial coefficient is defined to be

$$\binom{I}{J} = \frac{I!}{J!(I \setminus J)!}.$$

The higher Euler operators are given explicitly by

$$E^J_\alpha(P) = \sum_{I \supset J} \binom{I}{J} (-D)^{I \setminus J} \frac{\partial P}{\partial u^\alpha_I}.$$

We have $\boldsymbol{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and

$$\mathcal{A}_k = \sum_{\alpha} \sum_{\#I>0} \frac{\tilde{i}_k + 1}{\#I + 1} \mathrm{D}^I (Q^\alpha E_\alpha^{Ik}(\mathcal{L})).$$

Finally, if $\boldsymbol{B}$ is a conservation law for the system $\Delta = 0$, that is, if $\nabla \bullet \boldsymbol{B}\big|_{\Delta=0} = 0$, we say that $\boldsymbol{B}$ is *trivial* if either

(i) $\boldsymbol{B} = \nabla \wedge \boldsymbol{C}$ so that $\nabla \bullet \boldsymbol{B} = 0$ is an identity (independent of the system of equations $\Delta = 0$ being studied);

(ii) each component of $\boldsymbol{B}$ is zero on solutions of $\Delta = 0$.

Two conservation laws are said to be *equivalent* if they differ by a trivial conservation law.