Symmetry Group Analysis of the Shallow Water and Semi-Geostrophic Equations

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Abstract

The two-dimensional shallow water and semi-geostrophic partial differential equation systems that arise in meteorology and oceanography are analyzed from the point of view of symmetry groups theory, and a complete classification of their associated classical symmetries, potential symmetries, variational symmetries and conservation laws are found. In the second part we show how the particle relabelling symmetry can be used to rewrite the semi-geostrophic equations in such a way that a well-defined formal series solution, smooth only in time, may be carried out. We show that such solutions are in the form of an "infinite linear cascade".

1 Introduction

Many natural phenomena are described by a system of nonlinear partial differential equations (PDEs), which are often difficult to solve analytically as there is no existing general theory for completely solving nonlinear PDEs.

In the mid-nineteenth century, Sophus Lie introduced a new method for studying differential equations by using their symmetries. For each PDE, or system of PDEs, there is a local group of transformations, called *symmetry group*, that acts on the space of its independent and dependent variables, with the property that it maps the set of all analytical solutions to itself, and so leaves the form of the equation unchanged. The method of finding the symmetry group associated with the a PDE is called *classical Lie method*. Moreover, the classical Lie method leads to special type of solutions in terms of solutions of lower dimensional equations. For a modern description see, for example, [13]. The classical Lie method is an algorithmic procedure for which many symbolic manipulation programs have became available (cf. [4]). Their use became imperative in finding classical symmetries associated with large systems of PDEs.

The *variational symmetry group* of a variational problem is a subgroup of the symmetry group associated with the Euler-Lagrange equations, with the property that it leaves the variational integral unchanged. Knowledge of the variational symmetries lead, by Noether's theorem, to the conservation laws associated with the the studied system: for each one-parameter variational symmetry there is an associated conservation law, for details see [13, SS4.3, 4.4, 5.3].

In the past twenty years, the semi-geostrophic equations have become a model for describing atmospheric motions on a synoptic scale, including the presence of fronts [20, 22]. In the two-dimensional shallow water theory [23] a typical particle (more precisely, fluid column) has the Cartesian horizontal coordinates

$$x = x(a, b, t), \qquad y = y(a, b, t), \tag{1.1}$$

expressed as functions of the particle labels $(a, b) \in \mathbb{R}^2$ and time $t \in \mathbb{R}^+$. For convenience, each particle is labelled by its position at a reference time $t = 0$, this means the functions in (1.1) are defined such that $x(a, b, 0) = a$ and $y(a, b, 0) = b$. The incompressibility hypothesis requires

$$\frac{h(a, b, 0)}{h(a, b, t)} = \frac{\partial(x, y)}{\partial(a, b)}, \tag{1.2}$$

where the Jacobian on the right is that of the mapping (1.1). The time derivative of (1.2) following the particle gives the continuity equation. In this paper we assume $h(a, b, 0) = 1$, so that the incompressibility hypothesis becomes

$$h(a, b, t) = \frac{1}{x_a y_b - x_b y_a}, \tag{1.3}$$

where subscripts denote partial derivatives. The mapping (1.1) is assumed to be invertible, such that when $a = a(x, y, t)$ and $b = b(x, y, t)$ are inserted into (1.2), then the current depth h is expressed as function of x, y and t, which represents the Eulerian description.

The equations of the horizontal momentum balance for the flows over a bed which is rotating with position dependent Coriolis parameter $f = f(y)$ are

$$\ddot{x} + gh_x - f\dot{y} = 0, \tag{1.4a}$$
$$\ddot{y} + gh_y + f\dot{x} = 0, \tag{1.4b}$$

where g is a nonzero constant (representing the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation), dot denotes the time derivative following a particle, and h_x and h_y are given by

$$h_x = h(y_b h_a - y_a h_b), \tag{1.5a}$$
$$h_y = h(x_a h_b - x_b h_a). \tag{1.5b}$$

Henceforth we shall assume that f is a constant. It is known that the *shallow-water potential vorticity* defined by

$$\Omega = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right), \tag{1.6}$$

is conserved on particles (see [2, 14] and the references therein).

The semi-geostrophic approximation to equations (1.4) is the replacement of the true acceleration by the time derivative of the vector

$$u_g = -\frac{g}{f}h_y, \qquad v_g = \frac{g}{f}h_x, \tag{1.7}$$

following the particle. This vector field is a notional velocity and is called *geostrophic velocity*. Thus, the semi-geostrophic approximation seeks to find motions satisfying

$$\dot{u}_g + gh_x - f\dot{y} = 0, \tag{1.8a}$$
$$\dot{v}_g + gh_y + f\dot{x} = 0. \tag{1.8b}$$

It is also known that the *semi-geostrophic potential vorticity*, given by

$$\Omega^* = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right), \tag{1.9}$$

is conserved on particles [21].

In this paper we give a complete classification of the symmetries and the conservation laws associated with the two-dimensional shallow water (SW) equations (1.4) and the semi-geostrophic (SG) equations (1.8). In §§2 and 3 we determine the Lie point symmetries associated with the SW equations and SG equations, written in their classical forms (2.1) and (3.1), respectively, and in their potential forms (2.4) and (3.2), respectively. In §§4 and 5, we determine the conservation laws deriving from the variational symmetries. The particle relabelling symmetry leads to the conservation of the shallow water potential vorticity Ω [24] and semi-geostrophic potential vorticity [8], the time invariance leads to the conservation of the energy of the SW equations [24] and of SG equations, and the translation invariance to the conservation of linear momentum of the SW and SG equations, respectively. Moreover, the rotation invariance leads to the conservation of the angular momentum of SW equations.

The particle relabelling symmetry of Lagrangian fluid dynamics is important for the analysis of the equations. For example, in [14] it is shown how the symmetry relates to Hamiltonian properties, while in [2] it is shown how the symmetry relates to symplecticity with a view to symplectic integration methods. Here we use the symmetry to reframe the equations in a way which is adapted to, amongst other things, formal series solutions. The aim is to elucidate the structure of these solutions which are smooth in time, with repsect to the particle labels, at least for small time. We show that these solutions form an "infinite linear cascade". The method can be used for any equation having the particle relabelling group and thus is of independent interest.

2 Classical and potential symmetries for the shallow water equations

2.1 Classical symmetries of the shallow water equations

Substituting the function h defined by (1.3) into (1.4), yields

$$\ddot{x} - \frac{g}{(x_a y_b - x_b y_a)^3}\left[y_b^2 x_{aa} - 2y_a y_b x_{ab} + y_a^2 x_{bb} - x_b y_b y_{aa} + (x_a y_b + x_b y_a) y_{ab} - x_a y_a y_{bb}\right] = f\dot{y}, \tag{2.1a}$$

$$\ddot{y} + \frac{g}{(x_a y_b - x_b y_a)^3}\left[x_b y_b x_{aa} - (x_a y_b + x_b y_a) x_{ab} + x_a y_a x_{bb} - x_b^2 y_{aa} + 2x_a x_b y_{ab} - x_a^2 y_{bb}\right] = -f\dot{x}, \tag{2.1b}$$

which we shall refer to as the *classical form* of the SW equations. To determine the Lie point symmetries, the *classical symmetries*, we consider the one-parameter group of transformations

$$a^* = a + \varepsilon\xi(a,b,t,x,y) + \mathcal{O}(\varepsilon^2), \tag{2.2a}$$
$$b^* = b + \varepsilon\eta(a,b,st,x,y) + \mathcal{O}(\varepsilon^2), \tag{2.2b}$$
$$t^* = t + \varepsilon\theta(a,b,t,x,y) + \mathcal{O}(\varepsilon^2), \tag{2.2c}$$
$$x^* = x + \varepsilon\phi(a,b,t,x,y) + \mathcal{O}(\varepsilon^2), \tag{2.2d}$$
$$y^* = y + \varepsilon\psi(a,b,t,x,y) + \mathcal{O}(\varepsilon^2), \tag{2.2e}$$

where ξ, η, θ, ϕ and ψ are the *infinitesimals* which depend on a, b, t, x and y. Applying the classical Lie method shows that θ is a constant, and

$$\phi(a,b,t,x,y) = Ax - By + C\cos(ft) + D\sin(ft) + E,$$
$$\psi(a,b,t,x,y) = Ay + Bx + D\cos(ft) - C\sin(ft) + F,$$

with A, B, C, D, E and F constants, and that $\xi = \xi(a,b)$ and $\eta = \eta(a,b)$ satisfy

$$\xi_a + \eta_b = 4A. \tag{2.3}$$

The infinitesimal generator associated with (2.2) is

$$\begin{aligned}\mathrm{X} = \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + \{Ax - By + C\cos(ft) + D\sin(ft) + E\}\,\partial_x \\ + \{Ay + Bx + D\cos(ft) - C\sin(ft) + F\}\,\partial_y.\end{aligned}$$

where $\partial_a = \partial/\partial a$, $\partial_b = \partial/\partial b$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$, and so we have the following result.

Theorem 2.1. *The symmetry group associated with the SW equations (2.1) is generated by the following vector fields*

$$\begin{aligned}&\mathrm{X}_1 = \partial_t, \qquad \mathrm{X}_2 = \partial_x, \qquad \mathrm{X}_3 = \partial_y, \qquad \mathrm{X}_4 = -y\partial_x + x\partial_y,\\ &\mathrm{X}_5 = \cos(ft)\partial_x - \sin(ft)\partial_y, \qquad \mathrm{X}_6 = \sin(ft)\partial_x + \cos(ft)\partial_y,\\ &\mathrm{X}_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + Ax\partial_x + Ay\partial_y,\end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3).

Therefore the SW equations (2.1) are invariant under translations in t, x and y (vector fields X_1, X_2 and X_3 respectively), rotations in the (x,y)-space (X_4), helical rotations with respect t in the (x,y)-space (X_5 and X_6), while the vector field X_0 corresponds to the particle relabelling symmetry.

We remark that we are not able to apply the classical Lie symmetry method to equations (1.4) and (1.8) since $\partial/\partial x$ and D_t do not commute.

2.2 Potential symmetries of the shallow water equations

If we let $u = \dot{x}$ and $v = \dot{y}$, then the SW equations (1.4) can be written as

$$\dot{x} = u, \tag{2.4a}$$
$$\dot{y} = v, \tag{2.4b}$$
$$\dot{u} + gh(y_b h_a - y_a h_b) - fv = 0, \tag{2.4c}$$
$$\dot{v} + gh(x_a h_b - x_b h_a) + fu = 0, \tag{2.4d}$$

where h is given by (1.3), which we shall refer to as the *potential form* of the SW equations (1.4). To determine the associated Lie point symmetries, the *potential symmetries*, we consider the one-parameter group of transformations

$$\begin{aligned}
a^* &= a + \varepsilon\xi(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
b^* &= b + \varepsilon\eta(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
t^* &= t + \varepsilon\theta(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
x^* &= x + \varepsilon\phi(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
y^* &= y + \varepsilon\psi(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
u^* &= u + \varepsilon\zeta(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),\\
v^* &= v + \varepsilon\omega(a,b,t,x,y,u,v) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where the infinitesimals ξ, η, θ, ϕ, ψ, ζ and ω are functions of a, b, t, x, y, u and v. Applying the classical Lie method shows that θ is a constant,

$$\phi(a,b,t,x,y,u,v) = Ax - By + C\cos(ft) + D\sin(ft) + E, \tag{2.5a}$$
$$\psi(a,b,t,x,y,u,v) = Ay + Bx + D\cos(ft) - C\sin(ft) + F, \tag{2.5b}$$
$$\zeta(a,b,t,x,y,u,v) = Au - Bv - Cf\sin(ft) + Df\cos(ft), \tag{2.5c}$$
$$\omega(a,b,t,x,y,u,v) = Av + Bu - Df\sin(ft) - Cf\cos(ft), \tag{2.5d}$$

with A, B, C, D, E and F constants, and $\xi = \xi(a,b)$ and $\eta = \eta(a,b)$ satisfy the condition (2.3), as previously. The infinitesimal generator is

$$\begin{aligned}
\mathrm{X} = \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + \{Ax - By + C\cos(ft) + D\sin(ft) + E\}\,\partial_x&\\
+ \{Ay + Bx + D\cos(ft) - C\sin(ft) + F\}\,\partial_y + \{Au - Bv - Cf\sin(ft) + Df\cos(ft)\}\,\partial_u&\\
+ \{Av + Bu - Df\sin(ft) - Cf\cos(ft)\}\,\partial_v.&
\end{aligned} \tag{2.6}$$

and so we obtain the following result.

Theorem 2.2. *The potential symmetries associated with the SW equations (2.4) are generated by the vector fields*

$$\begin{aligned}
&\mathrm{X}_1 = \partial_t, \qquad \mathrm{X}_2 = \partial_x, \qquad \mathrm{X}_3 = \partial_y, \qquad \mathrm{X}_4 = -y\partial_x + x\partial_y - v\partial_u + u\partial_v,\\
&\mathrm{X}_5 = \cos(ft)\partial_x - \sin(ft)\partial_y - f\sin(ft)\partial_u - f\cos(ft)\partial_v,\\
&\mathrm{X}_6 = \sin(ft)\partial_x + \cos(ft)\partial_y + f\cos(ft)\partial_u - f\sin(ft)\partial_v,\\
&\mathrm{X}_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + A(x\partial_x + y\partial_y + u\partial_u + v\partial_v),
\end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3).

Therefore the potential form of the SW equations (2.4) are invariant under translations in t, x and y (X_1, X_2 and X_3 respectively), rotations in the (x,y,u,v)-space (X_4), helical rotations with respect t in the (x,y,u,v)-space (X_5 and X_6), while the vector field X_0 corresponds to the particle relabelling symmetry. We remark that the potential symmetries of the SW equations are actually its classical symmetries prolonged to the space of the t-derivatives of the dependent variables x and y.

3 Classical and potential symmetries for semi-geostrophic equations

3.1 Classical symmetries of the semi-geostrophic equations

The *classical form* of the SG equations (1.8) are

$$hh_b\dot{x}_a - hh_a\dot{x}_b + (\dot{h}h_b + h\dot{h_b})x_a - (\dot{h}h_a + h\dot{h_a})x_b + fhh_by_a - fhh_ay_b = -\frac{f^2}{g}\dot{y}, \tag{3.1a}$$
$$hh_b\dot{y}_a - hh_a\dot{y}_b + (\dot{h}h_b + h\dot{h_b})y_a - (\dot{h}h_a + h\dot{h_a})y_b - fhh_bx_a + fhh_ax_b = \frac{f^2}{g}\dot{x}, \tag{3.1b}$$

with h given by (1.3). This system is obtained after substituting the functions u_g and v_g given by (1.7) into (1.8). The Lie point symmetries of this system of PDEs will be called the *classical symmetries* of the SG equations.

If we consider the one-parameter group of transformations (2.2), then on applying the classical Lie method, we find that θ is a constant,

$$\phi = Ax - By + E, \qquad \psi = Ay + Bx + F,$$

where A, B, E, F are constants, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ satisfy (2.3). The infinitesimal generator is

$$\mathrm{X} = \xi(a, b)\partial_a + \eta(a, b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y,$$

and so we have the following result.

Theorem 3.1. *The vector fields*

$$\begin{aligned}
&\mathrm{X}_1 = \partial_t, \qquad \mathrm{X}_2 = \partial_x, \qquad \mathrm{X}_3 = \partial_y, \qquad \mathrm{X}_4 = -y\partial_x + x\partial_y, \\
&\mathrm{X}_0 = \xi(a, b)\partial_a + \eta(a, b)\partial_b + Ax\partial_x + Ay\partial_y,
\end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3), generate the symmetry group associated with the SG equations (3.1).

Therefore the SG equations (1.8) are invariant under translations in t, x, y and rotations in the (x, y)-space. The relabelling symmetry corresponds to the vector field X_0.

3.2 Potential symmetries of the semi-geostrophic equations

The potential form of the SG equations is given by (1.7) and (1.8), i.e.

$$u_g = -\frac{g}{f}h(x_a h_b - x_b h_a), \tag{3.2a}$$

$$v_g = \frac{g}{f}h(y_b h_a - y_a h_b), \tag{3.2b}$$

$$\dot{u}_g + gh(y_b h_a - y_a h_b) - f\dot{y} = 0, \tag{3.2c}$$

$$\dot{v}_g + gh(x_a h_b - x_b h_a) + f\dot{x} = 0, \tag{3.2d}$$

where f is a nonzero constant and h is given by (1.3). To determine the associated Lie point symmetries, the potential symmetries, we consider the one-parameter group of transformations

$$\begin{aligned}
a^* &= a + \varepsilon\xi(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
b^* &= b + \varepsilon\eta(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
t^* &= t + \varepsilon\theta(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
x^* &= x + \varepsilon\phi(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
y^* &= y + \varepsilon\psi(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
u_g^* &= u_g + \varepsilon\zeta(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2), \\
v_g^* &= v_g + \varepsilon\omega(a, b, t, x, y, u_g, v_g) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where the infinitesimals ξ, η, θ, ϕ, ψ, ζ and ω depend a, b, t, x, y, u_g and v_g. Applying the classical Lie method yields

$$\phi(a, b, t, x, y, u_g, v_g) = Ax - By + E, \tag{3.3a}$$

$$\psi(a, b, t, x, y, u_g, v_g) = Ay + Bx + F, \tag{3.3b}$$

$$\zeta(a, b, t, x, y, u_g, v_g) = Au_g - Bv_g, \tag{3.3c}$$

$$\omega(a, b, t, x, y, u_g, v_g) = Av_g + Bu_g, \tag{3.3d}$$

where A, B, E, F and θ are constants, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ satisfy (2.3). The infinitesimal generator is

$$\begin{aligned}
\mathrm{X} = \xi(a, b)\partial_a + \eta(a, b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y \\
+ (Au_g - Bv_g)\,\partial_u + (Av_g + Bu_g)\,\partial_v,
\end{aligned} \tag{3.4}$$

and so we have the following result.

Theorem 3.2. *The vector fields*

$$\begin{aligned}
&\mathrm{X}_1 = \partial_t, \qquad \mathrm{X}_2 = \partial_x, \qquad \mathrm{X}_3 = \partial_y, \qquad \mathrm{X}_4 = -y\partial_x + x\partial_y - v_g\partial_{u_g} + u_g\partial_{v_g}, \\
&\mathrm{X}_0 = \xi(a, b)\partial_a + \eta(a, b)\partial_b + A(x\partial_x + y\partial_y + u_g\partial_{u_g} + v_g\partial_{v_g}),
\end{aligned}$$

with A an arbitrary constant, and where ξ and η satisfy (2.3), generate the potential symmetries associated with the SG equations (3.2).

Therefore the system (3.2) is invariant under translations in t, x, y and rotations in the (x, y, u_g, v_g)-space. The vector field X_0 corresponds to particle relabelling symmetry.

4 Variational symmetries and conservation laws for shallow water equations

4.1 Variational symmetries of the shallow water equations

A variational symmetry of a system of Euler-Lagrange equations is one that leaves the variational integral invariant. These are important since they yield conservation laws via Noether's Theorem. Salmon [24] showed that there is a first order Lagrangian

$$\mathcal{L} = (u - R)\dot{x} + (v + P)\dot{y} - \tfrac{1}{2}\left(u^2 + v^2 + gh\right), \tag{4.1}$$

where the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy

$$P_x + R_y = f,$$

with h given by (1.3) and f constant, for which the associated Euler-Lagrange equations represent (up to a sign) the SW equations (2.4), that is, the potential form of the SW equations. In the appendix we give brief details of the derivation of variational symmetries and of the calculations involved in using Noether's theorem to obtain conservation laws.

Theorem 4.1. *The variational symmetry group of the variational problem associated with the the Lagrangian (4.1) is generated by the following vector fields*

$$\mathrm{X}_1 = \partial_t, \qquad \mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b, \qquad S_b = -\xi, \qquad S_a = \eta. \tag{4.2}$$

Moreover, if the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy the system

$$B(yR_x - xR_y + P) - ER_x - FR_y = 0, \tag{4.3a}$$
$$B(-yP_x + xP_y + R) + EP_x + FP_y = 0, \tag{4.3b}$$
$$P_x + R_y = f, \tag{4.3c}$$

then the vector field

$$\mathrm{Y} = (E - By)\partial_x + (F + Bx)\partial_y - Bv\partial_u + Bu\partial_v, \tag{4.4}$$

with B, E and F real constants, generates a variational symmetry transformation.

Particular cases of these are as follows.

<u>*Case 1*</u>*. If $B \neq 0$, then let $\lambda = E/B$, $\gamma = F/B$. Making the transformation*

$$x = -\gamma + r\cos s, \qquad y = \lambda + r\sin s,$$

yields

$$P(r,s) = -\delta(r)\sin s + \left(\tfrac{1}{2}fr + k/r\right)\cos s, \qquad R(r,s) = \delta(r)\cos s + \left(\tfrac{1}{2}fr + k/r\right)\sin s, \tag{4.5}$$

and there is a variational symmetry group generated by

$$\mathrm{Y}_1 = -(y - \lambda)\partial_x + (x + \gamma)\partial_y - v\partial_u + u\partial_v. \tag{4.6}$$

<u>*Case 2*</u>*. If $B = 0$ and $E \neq 0$, then let $\lambda = F/B$. In this case it follows*

$$P(x,y) = \delta(y - \lambda x), \qquad R(x,y) = \lambda\delta(y - \lambda x) - \lambda f x + f y + k, \tag{4.7}$$

where $\delta(x)$ is an arbitrary function, and there is a variational symmetry group generated by

$$\mathrm{Y}_2 = \partial_x + \lambda\partial_y. \tag{4.8}$$

<u>*Case 3*</u>*. If $B = 0$ and $E = 0$ then*

$$P(x,y) = fx + k, \qquad R(x,y) = \delta(x), \tag{4.9}$$

with k an arbitrary real constant and $\delta(x)$ an arbitrary function, and there is a variational symmetry group generated by

$$\mathrm{Y}_3 = \partial_y. \tag{4.10}$$

Proof. Consider the general infinitesimal generator (2.6) of the symmetry group of the SW equations. Since the system (2.4) is a first order system of PDEs, the criterion for infinitesimal invariance is

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + \mathcal{L}\nabla\bullet\boldsymbol{\zeta} = 0,$$

where $\boldsymbol{\zeta} = (\xi, \eta, \theta)$ and

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) = \mathrm{X}(\mathcal{L}) + \phi^{[a]}\frac{\partial \mathcal{L}}{\partial x_a} + \phi^{[b]}\frac{\partial \mathcal{L}}{\partial x_b} + \phi^{[t]}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \psi^{[a]}\frac{\partial \mathcal{L}}{\partial y_a} + \psi^{[b]}\frac{\partial \mathcal{L}}{\partial y_b} + \psi^{[t]}\frac{\partial \mathcal{L}}{\partial \dot{y}} \tag{4.11}$$

is the first order prolongation of the general infinitesimal generator X given by (2.6) applied to the Lagrangian (4.1), with infinitesimals ϕ, ψ, ζ and ω given by (2.5) and

$$\phi^{[a]} = (A - \xi_a)x_a - By_a - \eta_a x_b, \tag{4.12a}$$
$$\phi^{[b]} = -\xi_b x_a + (A - \eta_b)x_b - By_b, \tag{4.12b}$$
$$\phi^{[t]} = -Cf\sin(ft) + Df\cos(ft) + A\dot{x} - B\dot{y}, \tag{4.12c}$$
$$\psi^{[a]} = Bx_a + (A - \xi_a)y_a - \eta_a y_b, \tag{4.12d}$$
$$\psi^{[b]} = Bx_b - \xi_b y_a + (A - \eta_b)y_b, \tag{4.12e}$$
$$\psi^{[t]} = -Df\sin(ft) - Cf\cos(ft) + B\dot{x} + A\dot{y}. \tag{4.12f}$$

The relation (2.3) implies $\nabla\bullet\boldsymbol{\zeta} = \xi_a + \eta_b = 4A$, and thus we have

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + 4A\mathcal{L} = 0. \tag{4.13}$$

The nonzero partial derivatives of the Lagrangian (4.1) are the following

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \qquad \frac{\partial \mathcal{L}}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \qquad \frac{\partial \mathcal{L}}{\partial u} = \dot{x} - u,\\
&\frac{\partial \mathcal{L}}{\partial v} = \dot{y} - v, \qquad \frac{\partial \mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2 y_b, \qquad \frac{\partial \mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2 y_a,\\
&\frac{\partial \mathcal{L}}{\partial \dot{x}} = u - R, \qquad \frac{\partial \mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2 x_b, \qquad \frac{\partial \mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2 x_a \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = v + P.
\end{aligned} \tag{4.14}$$

Substituting (2.5), (4.12), (4.14) and (4.11) into (4.13), and equalizing to zero the coefficients of the t-derivatives $\dot{x}$, $\dot{y}$ and the free term, it follows $A = C = D = 0$, and the relations (4.3) between the constants B, E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ defining the Lagrangian (4.1). If $A = 0$, then (2.3) implies $\xi_a + \eta_b = 0$, and we can consider a function $S = S(a, b)$ such that $S_b = -\xi$ and $S_a = \eta$. It follows that the variational symmetry are generated by the vector fields X_1 and Y_0 given by (4.2). The compatibility of the system (4.3) implies the form (4.4) for another variational symmetry.

Let us discuss now the particular cases of the system (4.3).

Case 1. If $B \neq 0$, then let $\lambda = E/B$ and $\gamma = F/B$, so the system (4.3) becomes

$$(y - \lambda)R_x - (x + \gamma)R_y + P = 0, \qquad (\lambda - y)P_x + (x + \gamma)P_y + R = 0, \qquad P_x + R_y = f. \tag{4.15}$$

Consider the change of variables $x = -\gamma + r\cos s$, $y = \lambda + r\sin s$. Then the general solution is given by (4.5) and so the vector field (4.4) is written as (4.6). For $B = 0$, the system (4.3) is equivalent to

$$EP_x + FP_y = 0, \qquad ER_x + FR_y = 0, \qquad P_x + R_y = f. \tag{4.16}$$

Case 2. If $B = 0$ and $E \neq 0$, then the system (4.16) is written as

$$R_x + \lambda R_y = 0, \qquad P_x + \lambda P_y = 0, \qquad P_x + R_y = f, \tag{4.17}$$

where $\lambda = F/E$. Integrating this system, we find that (4.7) represents its general solution, and moreover, there is a variational symmetry generated by the vector field (4.8).

Case 3. If $B = 0$ and $E = 0$, then the system (4.16) turns into

$$FR_y = 0, \qquad FP_y = 0, \qquad P_x + R_y = f. \tag{4.18}$$

If $F \neq 0$, then we get $R_y = 0$, $P_y = 0$, and so $P_x = f$. Its general solution is (4.9) and we obtain a variational symmetry generated by the vector field (4.10).

In the case $F = 0$, we get that (4.2) is the only vector field that generates a variational symmetry. ■

Definition 4.2. A *conservation law for the SW equations*, is given by the divergence expression

$$\mathrm{D}_a(\mathcal{P}_1) + \mathrm{D}_b(\mathcal{P}_2) + \mathrm{D}_t(\mathcal{P}_3) = 0, \tag{4.19}$$

where $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is a triple of smooth functions of a, b, t, x, y, u, v and the derivatives of x, y, u, v. The function $\mathcal{P}_3$ is the *conserved density* and the pair $(\mathcal{P}_1, \mathcal{P}_2)$ is the *associated flux vector field.*

Theorem 4.3. *The conservation laws associated with the SW equations (2.4), that derive from its variational symmetries, are given in Table 4.1. Moreover, the conservation law deriving from* Y_0 *is equivalent to* $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$*, where*

$$\widetilde{\mathcal{P}}_1 = \mathcal{P}_1 + \mathrm{D}_t(ST_2), \qquad \widetilde{\mathcal{P}}_2 = \mathcal{P}_2 - \mathrm{D}_t(ST_3), \qquad \widetilde{\mathcal{P}}_3 = -S\Omega = P_3 - \mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3), \tag{4.20}$$

and Ω *is the shallow water potential vorticity (1.6).*

vector field	$\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$,
$\mathrm{X}_1 = \partial_t$	$\left(\frac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b), \frac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a), \frac{1}{2}(u^2+v^2+gh)\right)$
$\mathrm{Y}_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b$	$(\xi T_1, \eta T_1, \eta T_2 + \xi T_3)$ $T_1 = \dot{x}(R-u) - \dot{y}(v+P) + \frac{1}{2}(u^2+v^2) + gh$ $T_2 = x_b(u-R) + y_b(v+P)$ $T_3 = x_a(u-R) + y_a(v+P)$
$\mathrm{Y}_1 = -(y-\lambda)\partial_x + (x+\gamma)\partial_y - v\partial_u + u\partial_v$	$\left(\frac{1}{2}gh^2\left(x_b(x+\gamma) + y_b(y-\lambda)\right), \frac{1}{2}gh^2\left(y_a(\lambda-y) - x_a(x+\gamma)\right),\right.$ $\left.(y-\lambda)(u-R) - (x+\gamma)(v+P)\right),$
$\mathrm{Y}_2 = \partial_x + \lambda\partial_y$	$\left(\frac{1}{2}gh^2(\lambda x_b - y_b), \frac{1}{2}gh^2(y_a - \lambda x_a), R - u - \lambda(v+P)\right)$
$\mathrm{Y}_3 = \partial_y$	$\left(\frac{1}{2}gh^2x_b, -\frac{1}{2}gh^2x_a, -v-P\right)$

Table 4.1: Symmetries and their associated conservation laws for the SW equations.

Proof. If we apply the higher Euler operators (see Appendix) to the first order Lagrangian (4.1), we obtain

$$\begin{aligned}
&\mathcal{E}_1^{(1)} = \frac{\partial\mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2y_b, &&\mathcal{E}_2^{(1)} = \frac{\partial\mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2x_b, &&\mathcal{E}_3^{(1)} = \frac{\partial\mathcal{L}}{\partial u_a} = 0, &&\mathcal{E}_4^{(1)} = \frac{\partial\mathcal{L}}{\partial v_a} = 0,\\
&\mathcal{E}_1^{(2)} = \frac{\partial\mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2y_a, &&\mathcal{E}_2^{(2)} = \frac{\partial\mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2x_a, &&\mathcal{E}_3^{(2)} = \frac{\partial\mathcal{L}}{\partial u_b} = 0, &&\mathcal{E}_4^{(2)} = \frac{\partial\mathcal{L}}{\partial v_b} = 0,\\
&\mathcal{E}_1^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{x}} = u - R, &&\mathcal{E}_2^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{y}} = v + P, &&\mathcal{E}_3^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{u}} = 0, &&\mathcal{E}_4^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{v}} = 0.
\end{aligned}$$

In what follows, we determine the conservation laws deriving form the variational symmetries generated by the vector fields (4.2), (4.4), and in particular (4.6), (4.8) and (4.10).

1. For the vector field $\mathrm{X}_1 = \partial_t$, the corresponding characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ (see Appendix) has components given by

   $$Q_1 = -\dot{x}, \quad Q_2 = -\dot{y}, \quad Q_3 = -\dot{u}, \quad Q_4 = -\dot{v}.$$

   Now the conservation law is given by

   $$\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = -(\boldsymbol{\mathcal{A}} + \mathcal{L}\boldsymbol{\zeta}), \tag{4.21}$$

   where in this case $\boldsymbol{\zeta} = (\xi, \eta, \theta) = (0, 0, 1)$ and where $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with

   $$\mathcal{A}_k = \sum_{j=1}^{4} Q_j \mathcal{E}_j^{(k)}(\mathcal{L}), \tag{4.22}$$

   with

   $$\mathcal{E}_j^{(k)}(\mathcal{L}) = \frac{\partial\mathcal{L}}{\partial u_k^j}. \tag{4.23}$$

   Thus

   $$\mathcal{A}_1 = \tfrac{1}{2}gh^2(\dot{y}x_b - \dot{x}y_b), \qquad \mathcal{A}_2 = \tfrac{1}{2}gh^2(\dot{x}y_a - \dot{y}x_a), \qquad \mathcal{A}_3 = \dot{x}(R-u) - \dot{y}(v+P).$$

and so

$$\mathcal{P}_1 = \tfrac{1}{2}(\dot{x}y_b - \dot{y}x_b), \qquad \mathcal{P}_2 = \tfrac{1}{2}(\dot{y}x_a - \dot{x}y_a), \qquad \mathcal{P}_3 = \tfrac{1}{2}(u^2 + v^2 + gh),$$

Thus, the time invariance leads to the conservation of the energy $\mathcal{P}_3$ of the system [24].

2. The characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ of the vector field

$$\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$$

is given by

$$Q_1 = S_b x_a - S_a x_b, \qquad Q_2 = S_b y_a - S_a y_b, \qquad Q_3 = S_b u_a - S_a u_b, \qquad Q_4 = S_b v_a - S_a v_b.$$

In this case, $\boldsymbol{\zeta} = (-S_b, S_a, 0)$ and from (4.22) we get

$$\mathcal{A}_1 = \tfrac{1}{2}ghS_b, \qquad \mathcal{A}_2 = -\tfrac{1}{2}ghS_a, \qquad \mathcal{A}_3 = S_b[x_a(u-R) + y_a(v+P)] - S_a[x_b(u-R) + y_b(v+P)]$$

Therefore, substituting them into (4.21) one obtains

$$\mathcal{P}_1 = -S_bT_1, \qquad \mathcal{P}_2 = S_aT_1, \qquad \mathcal{P}_3 = S_aT_2 - S_bT_3,$$

where

$$\begin{aligned} &T_1 = \dot{x}(R-u) - \dot{y}(v+P) + \tfrac{1}{2}(u^2+v^2) + gh, \\ &T_2 = x_b(u-R) + y_b(v+P), \qquad T_3 = x_a(u-R) + y_a(v+P). \end{aligned}$$

For the vector field Y_0 we can write an equivalent form of the associated conservation law (see the Appendix for the definition of equivalence). Note that the potential vorticity Ω defined by (1.6) satisfies

$$\mathrm{D}_a(T_2) - \mathrm{D}_b(T_3) = \Omega.$$

Define $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ given by (4.20) and let $\boldsymbol{\mathcal{P}}^* = (\mathcal{P}_1^*, \mathcal{P}_2^*, \mathcal{P}_3^*)$ with the components

$$\mathcal{P}_1^* = \mathrm{D}_t(ST_2), \qquad \mathcal{P}_2^* = -\mathrm{D}_t(ST_3), \qquad \mathcal{P}_3^* = -\mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3).$$

One verifies that $\nabla\bullet\boldsymbol{\mathcal{P}}^* \equiv 0$. It follows that the conservation laws $\boldsymbol{\mathcal{P}}$ and $\widetilde{\boldsymbol{\mathcal{P}}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}^*$ are equivalent (see Appendix).

Thus, the particle relabelling symmetry leads to the conservation of the potential vorticity Ω [24].

3. In the case of the vector field (4.4), the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ is defined by

$$Q_1 = E - By, \qquad Q_2 = F + Bx, \qquad Q_3 = -Bv, \qquad Q_4 = Bu.$$

Because $\boldsymbol{\zeta} = (0,0,0)$, from (4.21) it follows $\mathcal{P}_j = \mathcal{A}_j$, $j = 1,2,3$. From (4.22) we get

$$\begin{aligned} \mathcal{A}_1 &= \tfrac{1}{2}gh^2[Ey_b - Fx_b - B(xx_b + yy_b)], \\ \mathcal{A}_2 &= \tfrac{1}{2}gh^2[-Ey_a + Fx_a + B(xx_a + yy_a)], \\ \mathcal{A}_3 &= E(u-R) + F(v+P) + B[x(v+P) - y(u-R)], \end{aligned}$$

and so $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. In particular, for the vector fields Y_1, Y_2 and Y_3 given by (4.6), (4.8) and (4.10), respectively, if one substitutes $B = 1$, $E = \lambda$, $F = \gamma$, and $B = 0$, $E = 1$, $F = \lambda$, and respectively $B = 0$, $E = 0$, $F = 1$ into the above expressions, one obtains the conservation laws contained in Table 4.1. The rotation invariance leads to the conservation of the angular momentum and the translation invariance to the conservation of linear momentum.

5 Variational symmetries and conservations laws for semi-geostrophic equations

5.1 Variational symmetries for the semi-geostrophic equations

The SW equations (3.2) represent the Euler-Lagrange equations associated with the the first order Lagrangian [8]

$$\mathcal{L} = (u_g - R)\dot{x} + (v_g + P)\dot{y} - \tfrac{1}{2}(u_g^2 + v_g^2 + gh) - r\dot{u}_g + p\dot{v}_g, \tag{5.1}$$

where $P(x, y)$, $R(x, y)$, $p(u_g, v_g)$ and $r(u_g, v_g)$ are arbitrary functions satisfying

$$P_x + R_y = f, \qquad p_{u_g} + r_{v_g} = 1/f, \tag{5.2}$$

and h is given by (1.3).

Theorem 5.1. *The variational symmetry group of the variational problem associated with the Lagrangian (5.1) is generated by the vector fields (4.2). Moreover, if $P(x, y)$ and $R(x, y)$ are solutions of the system (4.16), then the vector field*

$$\mathrm{Z} = E\partial_x + F\partial_y \tag{5.3}$$

also generates a variational symmetry transformation.

1. *If $E \neq 0$, then let $\lambda = F/E$. It follows that P and R are given by (4.7). In this case, there is an additional variational symmetry generated by the vector field (5.3) denoted by Z_1 and given by (4.8).*
2. *For $E = 0$, then the functions P and R are given by (4.9). The vector field $\mathrm{Z} = \mathrm{Z}_2$ defined by (4.10) generates the associated additional variational symmetry.*

Proof. Let us consider the infinitesimal generator of the symmetry group associated with the SG equations given by (3.4). The criterion for infinitesimal invariance implies

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + \mathcal{L}\nabla\bullet\boldsymbol{\zeta} = 0,$$

where $\boldsymbol{\zeta} = (\xi, \eta, \theta)$. Using $\nabla\bullet\boldsymbol{\zeta} = \xi_a + \eta_b = 4A$, the above condition becomes

$$\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) + 4A\mathcal{L} = 0. \tag{5.4}$$

The nonzero partial derivatives of the Lagrangian (5.1) are the following

$$\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \qquad \frac{\partial\mathcal{L}}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \qquad \frac{\partial\mathcal{L}}{\partial u} = \dot{x} - u_g - r_{u_g}\dot{u}_g + p_{u_g}\dot{v}_g,\\
&\frac{\partial\mathcal{L}}{\partial v} = \dot{y} - v - r_{v_g}\dot{u}_g + p_{v_g}\dot{v}_g, \qquad \frac{\partial\mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2 y_b, \qquad \frac{\partial\mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2 y_a,\\
&\frac{\partial\mathcal{L}}{\partial \dot{x}} = u_g - R, \qquad \frac{\partial\mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2 x_b, \qquad \frac{\partial\mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2 x_a \qquad \frac{\partial\mathcal{L}}{\partial \dot{y}} = v_g + P,\\
&\frac{\partial\mathcal{L}}{\partial \dot{u}_g} = -r, \qquad \frac{\partial\mathcal{L}}{\partial \dot{v}_g} = p.
\end{aligned} \tag{5.5}$$

Applying the first order prolongation of the vector field X to the Lagrangian $\mathcal{L}$ it follows

$$\begin{aligned}
\mathrm{pr}^{(1)}\mathrm{X}(\mathcal{L}) = \phi\frac{\partial\mathcal{L}}{\partial x} + \psi\frac{\partial\mathcal{L}}{\partial y} + \zeta\frac{\partial\mathcal{L}}{\partial u} + \omega\frac{\partial\mathcal{L}}{\partial v} + \phi^{[a]}\frac{\partial\mathcal{L}}{\partial x_a} + \phi^{[b]}\frac{\partial\mathcal{L}}{\partial u} + \phi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{x}}\\
+ \psi^{[a]}\frac{\partial\mathcal{L}}{\partial y_a} + \psi^{[b]}\frac{\partial\mathcal{L}}{\partial y_b} + \psi^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{y}} + \zeta^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{u}_g} + \omega^{[t]}\frac{\partial\mathcal{L}}{\partial \dot{v}_g},
\end{aligned} \tag{5.6}$$

where the functions ϕ, ψ, ζ and ω are given by (3.3), and moreover

$$\phi^{[a]} = (A - \xi_a)x_a - By_a - \eta_a x_b, \tag{5.7a}$$

$$\phi^{[b]} = -\xi_b x_a + (A - \eta_b)x_b - By_b, \tag{5.7b}$$

$$\phi^{[t]} = A\dot{x} - B\dot{y}, \tag{5.7c}$$

$$\psi^{[a]} = Bx_a + (A - \xi_a)y_a - \eta_a y_b, \tag{5.7d}$$

$$\psi^{[b]} = Bx_b - \xi_b y_a + (A - \eta_b)y_b, \tag{5.7e}$$

$$\psi^{[t]} = B\dot{x} + A\dot{y}, \tag{5.7f}$$

$$\zeta^{[t]} = A\dot{u}_g - B\dot{v}_g, \tag{5.7g}$$

$$\omega^{[t]} = B\dot{u}_g + A\dot{v}_g. \tag{5.7h}$$

Substituting (3.3), (5.6), (5.7) and (5.5) into the relation (5.4), and equalizing to zero the coefficients of the t-derivatives $\dot{x}$, $\dot{y}$, $\dot{u}_g$ and $\dot{v}_g$ and the free term, we get $A = B = 0$, and the conditions (4.16) between the constants E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ from the Lagrangian (5.1). If $A = 0$, then the condition (2.3) turns into $\xi_a + \eta_b = 0$, and so there is a function $S = S(a, b)$ such that $S_a = \eta$, and $S_b = -\xi$. We obtain that the variational symmetries are generated by the vector fields (4.2). Moreover, if the functions P and R satisfy (4.16), then there is an additional variational symmetry corresponding to the vector field Z given by (5.3).

Some special cases for the system (4.16) are as follows.

1. If $E \neq 0$, then let $\lambda = F/E$. Hence the system (4.16) can be written as (4.17) with the general solution given by (4.7). The vector field Z, (5.3), becomes (4.8).
2. If $E = 0$, then the system (4.16) is written as (4.18).
   (a) If $F \neq 0$ then the general solution is given by (4.9) and the vector field Z, (5.3), becomes (4.10).
   (b) If $F = 0$, there are only variational symmetry generated by the vector fields (5.3).

We remark that the particular cases of the Theorem 5.1 correspond to case 2 of Theorem 4.1.

5.2 Conservation laws for the semi-geostrophic equations

Theorem 5.2. *The conservation laws associated with the SG equations, are presented in Table 5.1, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the associated flux vector fields and $\mathcal{P}_3$ the conserved densities. The conservation law deriving from the vector field Y_0 is equivalent to that defined by $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ with the components*

$$\widetilde{\mathcal{P}}_1 = \mathcal{P}_1 + \mathrm{D}_t(ST_2), \qquad \widetilde{\mathcal{P}}_2 = \mathcal{P}_2 - \mathrm{D}_t(ST_3), \qquad \widetilde{\mathcal{P}}_3 = -S\Omega^* = \mathcal{P}_3 - \mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3), \tag{5.8}$$

where

$$T_1 = \dot{x}(R - u_g) - \dot{y}(v_g + P) + \tfrac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u}_g - p\dot{v}_g, \tag{5.9a}$$
$$T_2 = x_b(u_g - R) + y_b(v_g + P) - ru_{g,b} + pv_{g,b}, \tag{5.9b}$$
$$T_3 = x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}, \tag{5.9c}$$

and Ω^* is the potential vorticity (1.9).

vector field	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
$\mathrm{X}_1 = \partial_t$	$\frac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b)$	$\frac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a)$	$\frac{1}{2}(u_g^2 + v_g^2 + gh)$
$\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$	$-S_bT_1$	S_aT_1	$S_aT_2 - S_bT_3$
$\mathrm{Z}_1 = \partial_x + \lambda\partial_y$	$\frac{1}{2}gh^2(\lambda x_b - y_b)$	$\frac{1}{2}gh^2(y_a - \lambda x_a)$	$R - u_g - \lambda(v_g + P)$
$\mathrm{Z}_2 = \partial_y$	$\frac{1}{2}gh^2x_b$	$\frac{1}{2}gh^2x_a$	$-v_g - P$

Table 5.1: Symmetries and their associated conservation laws for the SG equations.

Proof. Here

$$\begin{aligned}
&\mathcal{E}_1^{(1)} = \frac{\partial\mathcal{L}}{\partial x_a} = \tfrac{1}{2}gh^2y_b, &&\mathcal{E}_2^{(1)} = \frac{\partial\mathcal{L}}{\partial y_a} = -\tfrac{1}{2}gh^2x_b, &&\mathcal{E}_3^{(1)} = \frac{\partial\mathcal{L}}{\partial u_{g,a}} = 0, &&\mathcal{E}_4^{(1)} = \frac{\partial\mathcal{L}}{\partial v_{g,a}} = 0,\\
&\mathcal{E}_1^{(2)} = \frac{\partial\mathcal{L}}{\partial x_b} = -\tfrac{1}{2}gh^2y_a, &&\mathcal{E}_2^{(2)} = \frac{\partial\mathcal{L}}{\partial y_b} = \tfrac{1}{2}gh^2x_a, &&\mathcal{E}_3^{(2)} = \frac{\partial\mathcal{L}}{\partial u_{g,b}} = 0, &&\mathcal{E}_4^{(2)} = \frac{\partial\mathcal{L}}{\partial v_{g,b}} = 0,\\
&\mathcal{E}_1^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{x}} = u_g - R, &&\mathcal{E}_2^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{y}} = v_g + P, &&\mathcal{E}_3^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{u}_g} = -r, &&\mathcal{E}_4^{(3)} = \frac{\partial\mathcal{L}}{\partial \dot{v}_g} = p,
\end{aligned}$$

are the higher Euler operators applied to the Lagrangian (5.1). For each of the vector fields yielding a variational symmetry (Theorem 5.1), we find the associated conservation law.

1. For the vector field $\mathrm{Y}_1 = \partial_t$, the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$, has components given by

$$Q_1 = -\dot{x}, \qquad Q_2 = -\dot{y}, \qquad Q_3 = -\dot{u}_g, \qquad Q_4 = -\dot{v}_g.$$

   In this case, $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$, given by (4.21), is determined by $\boldsymbol{\zeta} = (0, 0, 1)$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, where

$$\mathcal{A}_1 = \tfrac{1}{2}gh^2(-\dot{x}y_b + \dot{y}x_b), \qquad \mathcal{A}_2 = \tfrac{1}{2}gh^2(\dot{x}y_a - \dot{y}x_a), \qquad \mathcal{A}_3 = -\dot{x}(u_g - R) - \dot{y}(v_g + P) + \dot{u}_g r - \dot{v}_g p.$$

   Substituting them into (4.21) it follows that

$$\mathcal{P}_1 = \tfrac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b), \qquad \mathcal{P}_2 = \tfrac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a), \qquad \mathcal{P}_3 = \tfrac{1}{2}(u_g^2 + v_g^2 + gh),$$

   which defines the corresponding conservation law. Thus, the time invariance implies the conservation $\mathcal{P}_3$ of the energy of the SG equations.

2. Consider the vector field

$$\mathrm{Y}_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b,$$

   for which the characteristic is

$$Q_1 = S_b x_a - S_a x_b, \qquad Q_2 = S_b y_a - S_a y_b, \qquad Q_3 = S_b u_{g,a} - S_a u_{g,b}, \qquad Q_4 = S_b v_{g,a} - S_a v_{g,b}.$$

   In the relation (4.21) one substitutes $\boldsymbol{\zeta} = (-S_b, S_a, 0)$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ given by (4.22), namely

$$\begin{aligned}
&\mathcal{A}_1 = \tfrac{1}{2}ghS_b, \qquad \mathcal{A}_2 = -\tfrac{1}{2}ghS_a,\\
&\mathcal{A}_3 = S_a[x_b(R - u_g) - y_b(v_g + P) + ru_{g,b} - pv_{g,b}] + S_b[x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}],
\end{aligned}$$

   and it follows

$$\mathcal{P}_1 = -S_b T_1, \qquad \mathcal{P}_2 = S_a T_1, \qquad \mathcal{P}_3 = S_a T_2 - S_b T_3,$$

   where

$$\begin{aligned}
T_1 &= \dot{x}(R - u_g) - \dot{y}(v_g + P) + \tfrac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u}_g - p\dot{v}_g,\\
T_2 &= x_b(u_g - R) + y_b(v_g + P) - ru_{g,b} + pv_{g,b},\\
T_3 &= x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}.
\end{aligned}$$

   The conservation law deriving from the vector field Y_0 can be written in an equivalent form. Note that

$$\mathrm{D}_a(T_2) - \mathrm{D}_b(T_3) = \Omega^*,$$

   where Ω^* is the potential vorticity (1.9). Consider $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$ defined by

$$\begin{aligned}
\widetilde{\mathcal{P}}_1 &= -S_b T_1 + \mathrm{D}_t(ST_2) = P_1 + \mathrm{D}_t(ST_2),\\
\widetilde{\mathcal{P}}_2 &= S_a T_1 - \mathrm{D}_t(ST_3) = P_2 - \mathrm{D}_t(ST_3),\\
\widetilde{\mathcal{P}}_3 &= -S\Omega^* = P_3 - \mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3),
\end{aligned}$$

   and

$$\mathcal{P}_1^* = \mathrm{D}_t(ST_2), \qquad \mathcal{P}_2^* = -\mathrm{D}_t(ST_3), \qquad \mathcal{P}_3^* = -\mathrm{D}_a(ST_2) + \mathrm{D}_b(ST_3).$$

   Because $\nabla \bullet \boldsymbol{\mathcal{P}}^* = 0$, it follows that $\boldsymbol{\mathcal{P}}$ and $\widetilde{\boldsymbol{\mathcal{P}}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}^*$ are equivalent conservation laws. Thus, the conservation law defines by $\widetilde{\boldsymbol{\mathcal{P}}} = (\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3)$, derives from the particle relabelling symmetry, and implies the conservation of the potential vorticity Ω^* [8].

3. In the case of the vector field Z given by (5.3), the characteristic $\boldsymbol{Q} = (Q_1, Q_2, Q_3, Q_4)$ has the components

$$Q_1 = E, \quad Q_2 = F, \quad Q_3 = 0, \quad Q_4 = 0.$$

   While $\boldsymbol{\zeta} = (0, 0, 0)$ it follows $\mathcal{P}_j = \mathcal{A}_j$, $j = 1, 2, 3$. Using (4.22) we get

$$\mathcal{A}_1 = \tfrac{1}{2}gh^2(Ey_b - Fx_b), \qquad \mathcal{A}_2 = \tfrac{1}{2}gh^2(Fx_a - Ey_a), \qquad \mathcal{A}_3 = E(u_g - R) + F(v_g + P),$$

   and in this case, the relation (4.21) implies

$$\mathcal{P}_1 = \tfrac{1}{2}gh^2(Fx_b - Ey_b), \qquad \mathcal{P}_2 = \tfrac{1}{2}gh^2(Ey_a - Fx_a), \qquad \mathcal{P}_3 = E(R - u_g) - F(v_g + P).$$

   In the particular cases $\mathrm{Z} = \mathrm{Z}_1$ and $\mathrm{Z} = \mathrm{Z}_2$, one substitutes $E = 1$ and $F = \lambda$ and respectively, $E = 0$ and $F = 1$ into the above expressions. The corresponding triple $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ leads to the conservation of linear momentum (Table 5.1).

6 Particle relabelling symmetry for the semi-geostrophic equations

In this section we explore further consequences of the particle relabelling symmetry for the SG equations. This is the symmetry responsible for the conservation of potential vorticity.

We first note the invariants, the invariant differential operators and their commutators, and the differential relations, or identities, between the invariants. These allow us to rewrite the SG equations in a way in which symbolic formal solution mechanisms can be applied effectively. Even so, the application of these series solution methods is far from routine, as formal expansions for the differential operators are also involved.

The result is an *infinite linear cascade*; the reasons for this name will become clear in the sequel. Finally we ask whether the series solution corresponds to a *fluid flow*, that is, do solutions obey the one-parameter condition,

$$x(x(a,b,t),y(a,b,t),s) = x(a,b,t+s), \qquad y(x(a,b,t),y(a,b,t),s) = y(a,b,t+s).$$

The startling observation is that the formal solution *cannot* correspond to any flow. We note that this property is *not* the same as invariance under translation in time, which corresponds rather to the fact that the origin in the time co-ordinate can be set arbitrarily.

6.1 Invariants of the particle relabelling symmetry

The particle relabelling symmetry group is the pseudogroup of co-ordinate transformations with constant jacobian equal to one. If a and b are the particle labels and

$$a^* = \phi(a,b), \qquad b^* = \psi(a,b),$$

is the relabelling, then we require that

$$\phi_a\psi_b - \phi_b\psi_a = 1.$$

The associated infinitesimal symmetry generator is

$$\mathrm{X} = \xi(a,b)\partial_a + \eta(a,b)\partial_b,$$

where the (otherwise arbitrary) functions ξ and η satisfy

$$\xi_a + \eta_b = 0.$$

The fluid particle positions satisfy

$$x^*(a^*,b^*) = x(a,b), \qquad y^*(a^*,b^*) = y(a,b),$$

and so in the parlance of group theory, are said to be invariant functions. Since

$$x^*_{a^*}y^*_{b^*} - x^*_{b^*}y^*_{a^*} = \frac{x_ay_b - x_by_a}{\phi_a\psi_b - \phi_b\psi_a} = x_ay_b - x_by_a,$$

we have that the function $\Delta = x_ay_b - x_by_a$ is an invariant of the pseudogroup, as are $u = x_t$ and $v = y_t$.

The invariant differential operators are

$$\partial_x = \frac{y_b\partial_a - y_a\partial_b}{x_ay_b - x_by_a}, \qquad \partial_y = \frac{-x_b\partial_a + x_a\partial_b}{x_ay_b - x_by_a}, \tag{6.1}$$

where, as above, $\partial_a = \partial/\partial a$ and $\partial_b = \partial/\partial b$ are the usual partial derivatives. The operators (6.1) are the same as those that would be obtained under a standard change of co-ordinates from (a,b) space to (x,y) space.

The systems studied vary in time, t, which is invariant under the pseudogroup, as is the (total) derivative with respect to time. On functions of a, b and t, the derivative with respect to t is simply $\partial_t = \partial/\partial t$. On functions of (x,y) however, since x and y depend on t, it is the **total** derivative that is used. To distinguish this fact we use the notation D_t to denote total differentiation with respect to time. The notation of f_x for $\partial_x f$ can trap the unwary as D_t and ∂_x do not commute. For this reason we do not use the notation f_x for $\partial_x f$ in expressions where D_t also occurs.

The method of moving frames as applied to this pseudogroup [15, 16] yields the following :

Theorem 6.1. *The functions $\Delta = x_ay_b - x_by_a$, x and y are the fundamental generating differential invariants. Every differential invariant is a function of these and their invariant derivatives. Moreover, the only differential relation between the invariants, or* syzygy, *is the well known continuity relation*

$$\mathrm{D}_t\Delta = \Delta(\partial_x\mathrm{D}_tx + \partial_y\mathrm{D}_ty). \tag{6.2}$$

One of the main problems in analysing systems involving both D_t and ∂_x, ∂_y derivatives is that these operators do not commute.

Theorem 6.2. *The commutation relations between* D_t, ∂_x *and* ∂_y *are*

$$[\partial_x, \partial_y] = 0, \tag{6.3a}$$

$$[\mathrm{D}_t, \partial_x] = -(\partial_x \mathrm{D}_t x)\partial_x - (\partial_x \mathrm{D}_t y)\partial_y, \tag{6.3b}$$

$$[\mathrm{D}_t, \partial_y] = -(\partial_y \mathrm{D}_t x)\partial_x - (\partial_y \mathrm{D}_t y)\partial_y. \tag{6.3c}$$

Proof. We check equation (6.3b) here.

$$\begin{aligned}
[\mathrm{D}_t, \partial_x] &= \mathrm{D}_t \left(\frac{y_b \partial_a - y_a \partial_b}{x_x y_b - x_b y_a} \right) - \left(\frac{y_b \partial_a - y_a \partial_b}{x_x y_b - x_b y_a} \right) \mathrm{D}_t \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \frac{y_{bt} \partial_a - y_{at} \partial_b}{\Delta} \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \frac{y_{bt}}{\Delta}(x_a \partial_x + y_a \partial_y) - \frac{y_{at}}{\Delta}(x_b \partial_x + y_b \partial_y) \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \left(\frac{y_{bt} x_a - y_{at} x_b}{\Delta} \right) \partial_x + \left(\frac{y_{bt} y_a - y_{at} y_b}{\Delta} \right) \partial_y \\
&= -\frac{\mathrm{D}_t \Delta}{\Delta} \partial_x + \partial_y (\mathrm{D}_t y) \partial_x - \partial_x (\mathrm{D}_t y) \partial_y \\
&= -\partial_x (\mathrm{D}_t x) \partial_x - \partial_x (\mathrm{D}_t y) \partial_y,
\end{aligned}$$

using the syzygy (6.2). That these make sense can be seen by calculating the commutation relations on x and y.

6.2 A reformulation of the two dimensional semi-geostrophic equations

We write the SG equations (1.7), (1.8) in terms of the invariants and invariant operators given above. Instead of Δ we use h where

$$h = \frac{1}{\Delta} = \frac{1}{x_a y_b - x_b y_a}. \tag{6.4}$$

In the notation we use in this section, these equations are written as

$$\mathrm{D}_t x = -\frac{g}{f^2} \mathrm{D}_t \partial_x h - \frac{g}{f} \partial_y h, \qquad \mathrm{D}_t y = -\frac{g}{f^2} \mathrm{D}_t \partial_y h + \frac{g}{f} \partial_x h. \tag{6.5}$$

The syzygy and commutation relations hold identically. If we evaluate them on the SG equations, by which we mean back-substituting for the various quantities, we obtain relations which must be true on solutions of the SG equations, and in fact are equivalent to them. That this is a useful thing to do has been noted by several authors [11, 25].

If we backsubstitute for $\mathrm{D}_t x$ and $\mathrm{D}_t y$ using the SG equations, the syzygy becomes

$$\mathrm{D}_t h = \frac{g}{f^2} (\partial_x \mathrm{D}_t \partial_x h + \partial_y \mathrm{D}_t \partial_y h), \tag{6.6}$$

and the commutation relations become

$$[\mathrm{D}_t, \nabla] = \frac{g}{f^2} \begin{pmatrix} \partial_x \mathrm{D}_t \partial_x h & \partial_x \mathrm{D}_t \partial_y h \\ \partial_y \mathrm{D}_t \partial_x h & \partial_y \mathrm{D}_t \partial_y h \end{pmatrix} \nabla + \frac{g}{f} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla. \tag{6.7}$$

If we let these commutation relations act on the function h, and then backsubstitute for $\mathrm{D}_t h$ using the syzygy (6.2) and for $\mathrm{D}_t \partial_x h$ and $\mathrm{D}_t \partial_y h$ from (6.5), we obtain

$$\frac{f^2}{g} u + f h_y = (h(u_x + v_y))_x + u_x h_x + v_x h_y, \tag{6.8a}$$

$$\frac{f^2}{g} v - f h_x = (h(u_x + v_y))_y + u_y h_x + v_y h_y, \tag{6.8b}$$

where we have put $u = \mathrm{D}_t x$ and $v = \mathrm{D}_t y$, a standard notation for these quantities.

We have effected a linear substitution of each of the equations (6.5) into one component each of an identity, and it is simple to see that (6.8) are equivalent to the two dimensional semi-geostrophic equations.

In this notation, the potential vorticity conservation law is

$$\mathrm{D}_t Q = 0, \qquad \text{where} \quad Q = \frac{1}{h}\left\{ f + \frac{g}{f}\left(h_{xx} + h_{yy}\right) + \frac{g^2}{f^3}\left(h_{xx}h_{yy} - h_{xy}^2\right)\right\}. \tag{6.9}$$

The equations (6.8) are linear in u and v and since they involve only x and y derivatives which commute, we may apply standard completion procedures to this system [5, 6, 9, 10, 17, 18]. The result is two compatibility conditions. The important one is easily obtained. If we denote the first equation (6.8a) by $\mathcal{A}$ and the second equation (6.8b) by $\mathcal{B}$, then a first order (with respect to u and v) compatibility equation is obtained by calculating $\mathcal{C} = \mathcal{A}_y - \mathcal{B}_x$. This yields

$$u_y\left(\frac{f^2}{g} + h_{xx}\right) - v_x\left(\frac{f^2}{g} + h_{yy}\right) + h_{xy}(v_y - u_x) + f(h_{xx} + h_{yy}) = 0. \tag{6.10}$$

The other compatibility condition is complicated and as we don't actually use it in the sequel, we don't record it in full. It has the form

$$z_1 u_x + z_2 u + z_3 + z_4 v_{xx} + z_5 v_{xy} + z_6 v_{yy} + z_7 v_x + z_8 v_y = 0$$

where the z_i are expressions in h and its derivatives.

Interestingly, the conservation of potential vorticity, equation (6.9), is *not* a compatibility condition of the system (6.8). It turns out that $\mathrm{D}_t Q = 0$ is an identity on the formal solution of (6.8) that we find, and imposes no further conditions.

6.3 Formal solution procedures

One major application of differential system completion procedures is that the calculation of formal series solutions proceeds without the discovery of unsuspected consistency conditions on the coefficients (see, for example, [17, 18]). There are, moreover, convergence results which are classical [19].

6.3.1 The naive approach

In order to understand the nature of the system (6.8,6.10), a formal solution procedure was carried out. To this end, series for x and y were taken to be

$$x(a,b,t) = a + x_1(a,b)t + x_2(a,b)t^2 + \mathcal{O}(t^3), \tag{6.11a}$$

$$y(a,b,t) = b + y_1(a,b)t + y_2(a,b)t^2 + \mathcal{O}(t^3), \tag{6.11b}$$

which uses the standard initial condition $x(a,b,0) = a$, $y(a,b,0) = b$. The corresponding series for $u = \mathrm{D}_t x$, $v = \mathrm{D}_t y$ are simply the time derivative of those for x and y, while those for h, ∂_x, and ∂_y are readily calculated. Indeed, we have

$$h = 1 - (x_{1,a} + y_{1,b})t + \left\{(x_{1,a} + y_{1,b})^2 + x_{1,b}y_{1,a} - x_{1,a}y_{1,b} - x_{2,a} - y_{2,b}\right\}t^2 + \mathcal{O}(t^3), \tag{6.12}$$

and

$$\partial_x = \left\{1 - x_{1,a}t + (x_{1,a}^2 + x_{1,b}y_{1,a} - x_{2,a})t^2 + \mathcal{O}(t^3)\right\}\partial_a \tag{6.13a}$$

$$+ \left\{-y_{1,a}t + (y_{1,a}x_{1,a} + y_{1,b}y_{1,a} - y_{2,a})t^2 + \mathcal{O}(t^3)\right\}\partial_b, \tag{6.13b}$$

$$\partial_y = \left\{-x_{1,b}t + (x_{1,b}x_{1,a} + x_{1,b}y_{1,b} - x_{2,b})t^2 + \mathcal{O}(t^3)\right\}\partial_a \tag{6.13c}$$

$$+ \left\{1 - y_{1,b}t + (y_{1,b}^2 + y_{1,a}x_{1,b} - y_{2,b})t^2 + \mathcal{O}(t^3)\right\}\partial_b. \tag{6.13d}$$

These were inserted into (6.8) and (6.10), and coefficients of like powers of t were collected.

We denote the coefficient of t^n in the expansion of $\mathcal{A}$, i.e. (6.8a), by a_n, that in $\mathcal{B}$, i.e. (6.8b), by b_n, and that in $\mathcal{C}$, i.e. (6.10), by c_n. These coefficients must all be zero on solutions. Setting these coefficients equal to zero leads to a system in the x_i, y_i.

The zeroth level

Setting $t = 0$ in the expansions for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ yields

$$c_0 = \frac{f^2}{g}(x_{1,b} - y_{1,a}), \qquad a_0 = -\frac{f^2}{g}x_1 + x_{1,aa} + y_{1,ab}, \qquad b_0 = -\frac{f^2}{g}y_1 + x_{1,ab} + y_{1,bb}.$$

These must all be zero on a solution of the SG equations. Thus one has immediately to first order that

$$x_1 = \alpha_a, \qquad y_1 = \alpha_b, \qquad \alpha_{aa} + \alpha_{bb} = \frac{f^2}{g}\alpha, \tag{6.14}$$

for some function α.

The first level

The coefficients are complicated, so we simplify these using the zeroth order system. In order for this to be done in an effective manner, we use standard differential procedures: first calculate a characteristic set of the above system and then perform a differential reduction of a_1, b_1 and c_1, all according to some fixed termordering. This is necessary to prevent infinite loops in the reduction process and to obtain a well-defined, indeed unique, result. We have used a termordering governed by total degree followed by elimination of y_2. Then a characteristic set of the new coefficients can be calculated. When this is done, and spurious factors in the nonzero constants f and g removed, we obtain

$$2y_{2,a} - 2x_{2,b} + fy_{1,b} + fx_{1,a} = 0, \tag{6.15}$$

and an equation for x_2 of the form

$$x_{2,aa} + x_{2,bb} = \frac{f^2}{g}x_2 + F_2(x_1, y_1).$$

We remark that a termordering which eliminates x_2 yields a condition whose leading terms are $y_{2,aa} + y_{2,bb} - (f^2/g)y_2$.

If we take $x_2 = \beta_a + \frac{1}{2}f\alpha_b$ and $y_2 = \beta_b - \frac{1}{2}f\alpha_a$, so that (6.15) is satisfied and set these into a_1 and b_1, we obtain two conditions which are the a and b derivatives of the following condition on β,

$$\beta_{aa} + \beta_{bb} - \frac{f^2}{g}\beta - \frac{f^2}{2g}(\alpha_a^2 + \alpha_b^2) + \alpha_{aa}\alpha_{bb} - \alpha_{ab}^2 - \frac{f^4}{g^2}\alpha^2 = 0. \tag{6.16}$$

Note we may absorb the constant of integration into β. Algorithms are available which symbolically integrate total derivatives [26, 27, 28].

The second level

Continuing as before, we obtain conditions for x_3 and y_3 in terms of x_1, y_1, x_2 and y_2. The first order compatibility condition is

$$3y_{3,a} - 3x_{3,b} + fy_{2,a} + fx_{2,b} + 3f(x_{1,a}y_{1,b} - x_{1,b}y_{1,a}) = 0,$$

and a condition for x_3 of the form

$$x_{3,aa} + x_{3,ab} = \frac{f^2}{g}x_{3,a} + F_3(x_1, y_1, x_2, y_2),$$

is obtained. Setting

$$x_3 = \gamma_a + \tfrac{1}{3}f\beta_b + f\alpha_b\alpha_{aa}, \qquad y_3 = \gamma_b - \tfrac{1}{3}f\beta_a + f\alpha_b\alpha_{ab}, \tag{6.17}$$

into a_2 and b_2, we obtain the a and b derivatives of the condition for γ,

$$\gamma_{aa} + \gamma_{bb} - \frac{f^2}{g}\gamma = \text{complicated in } \beta \text{ and } \alpha \tag{6.18}$$

Continuing, we find that the structure of the equations is the same at the third and fourth levels. One tedious aspect of this process is the need to perform symbolic integrations to find the condition for the function α_n introduced at each order ($\alpha_0 = \alpha$, $\alpha_1 = \beta$, $\alpha_2 = \gamma$). In the proof of the general result, we use a different process to find the condition for the function introduced at each order in one step.

Check results

If we substitute

$$x(a,b,t) = a + \alpha_a t + (\beta_a + \tfrac{1}{2}f\alpha_b)t^2 + (\gamma_a + \tfrac{1}{3}f\beta_b + f\alpha_b\alpha_{aa})t^3 + \mathcal{O}(t^4),$$
$$y(a,b,t) = b + \alpha_b t + (\beta_b - \tfrac{1}{2}f\alpha_a)t^2 + (\gamma_b - \tfrac{1}{3}f\beta_a + f\alpha_b\alpha_{ab})t^3 + \mathcal{O}(t^4),$$

into the conservation law for Q and the original SG equations, then they are satisfied to the relevant order provided that (6.14), (6.16) and (6.18) hold.

6.3.2 Proof of the general result

Theorem 6.3. *The two-dimensional semi-geostrophic equations are equivalent to an infinite set of essentially linear equations in a triangular or cascade form.*

Proof. We first pull apart the equations (6.8). Define

$$\mathcal{U} = \frac{f^2}{g}u + fh_y - u_x h_x - v_x h_y, \qquad \mathcal{V} = \frac{f^2}{g}v - fh_x - u_y h_x - v_y h_y, \tag{6.19}$$

Then the equations (6.8) are of the form

$$\mathcal{U} = \Phi_x, \qquad \mathcal{V} = \Phi_y, \tag{6.20}$$

where we have defined Φ by

$$\Phi = h(u_x + v_y). \tag{6.21}$$

The compatibility condition (6.10) is then

$$\mathcal{U}_y - \mathcal{V}_x = 0. \tag{6.22}$$

By substituting in the definitions of ∂_x and ∂_y, the equations (6.20) can be rewritten as

$$x_a\mathcal{U} + y_a\mathcal{V} = \Phi_a, \qquad x_b\mathcal{U} + y_b\mathcal{V} = \Phi_b. \tag{6.23}$$

The recursive expansion method consists of the following steps.

1. Set
$$\Phi = \sum_{j=0}^{\infty} p_j(a,b)t^j, \tag{6.24}$$
where the p_j are functions to be determined.

2. Substitute the expressions for Φ, $\mathcal{U}$ and $\mathcal{V}$ from (6.24) and (6.19) into (6.23), together with the expansions for $u = \mathrm{D}_t x$, $v = \mathrm{D}_t y$ from (6.11) and (6.12), and equate coefficients of powers of t to zero. At each order n we obtain two equations of the form
$$x_{n+1} = \frac{g}{nf^2}p_{n,a} + G_n^1(p_0, p_1, \dots p_{n-1}), \qquad y_{n+1} = \frac{g}{nf^2}p_{n,b} + G_n^2(p_0, p_1, \dots p_{n-1}), \tag{6.25}$$
which can be seen by a leading term analysis.

3. Substitute (6.25) into (6.21). At each order of t we obtain an equation for p_{n-1} in terms of $p_0, p_1, \dots, p_{n-2}$. By a leading term analysis, we have that the equation for p_n is of the form
$$p_{n,aa} + p_{n,bb} = \frac{f^2}{g}p_n + F_n(p_0, \dots, p_{n-1})$$

6.3.3 Discussion

The formal solution provides a mechanism for solving the two-dimensional SG equations equations in terms of a sequence of boundary value problems.

For the sake of interest, the expansions for h, h_x and h_y in terms of α, β etc. are

$$h = 1 - \frac{f^2}{g}\alpha t - \frac{f^2}{2g}(\alpha_a^2 + \alpha_b^2 + 2\beta)t^2 - \frac{f^2}{g}(\beta_a\alpha_a + \beta_b\alpha_b + \tfrac{1}{2}f\alpha_a\alpha_b + \gamma)t^2 + \mathcal{O}(t^3), \tag{6.26a}$$

$$h_x = -\frac{f^2}{g}\alpha_a t - \frac{f^2}{g}\beta_a t^2 - \frac{f^2}{g}(\gamma_a + f\alpha_b\alpha_{aa})t^3 + \mathcal{O}(t^4), \tag{6.26b}$$

$$h_y = -\frac{f^2}{g}\alpha_b t - \frac{f^2}{g}\beta_b t^2 - \frac{f^2}{g}(\gamma_b + f\alpha_b\alpha_{ab})t^3 + \mathcal{O}(t^4). \tag{6.26c}$$

6.4 The cascade in the Legendre transform co-ordinates

The numerical solution of two-dimensional SG equations is done in Legendre transform co-ordinates, which we now define. Setting

$$X = x + \frac{1}{f}v_g, \qquad Y = y - \frac{1}{f}u_g. \tag{6.27}$$

Then for solutions of the SG equations, we have

$$\mathrm{D}_t X = u_g, \qquad \mathrm{D}_t Y = v_g. \tag{6.28}$$

At time $t = 0$, we have

$$X(a,b,0) = a, \qquad Y(a,b,0) = b. \tag{6.29}$$

Just as we have the moving frame data for the a, b, t, x, y variables, so we have the moving frame data for the a, b, t, X, Y variables. Thus we have the three fundamental invariants

$$H = 1/(X_a Y_b - X_b Y_a), \qquad X, \qquad Y,$$

and the invariant differential operators,

$$\partial_X = \frac{Y_b \partial_a - Y_a \partial_b}{X_a Y_b - X_b Y_a}, \qquad \partial_Y = \frac{-X_b \partial_a + X_a \partial_b}{X_a Y_b - X_b Y_a},$$

while the syzygy is

$$\mathrm{D}_t H + H(U_X + V_Y) = 0, \tag{6.30}$$

where $U = \mathrm{D}_t X$ and $V = \mathrm{D}_t Y$.

In these co-ordinates, the conservation of potential vorticity takes the form

$$X_a Y_b - X_b Y_a \equiv 1, \tag{6.31}$$

and so the syzygy takes the form

$$U_X + V_Y = 0. \tag{6.32}$$

Since $H \equiv 1$, the idea used in the previous section to find a more amenable form of the equations to study, that of evaluating the commutation rules on the invariant h, fails.

One standardly introduces a potential function Ψ such that

$$U = \mathrm{D}_t X = -\Psi_Y, \qquad V = \mathrm{D}_t Y = \Psi_X. \tag{6.33}$$

Theorem 6.4. *In terms of the potential function Ψ, we have both*

$$h = \frac{f}{g}\Psi - \frac{1}{2g}(\Psi_X^2 + \Psi_Y^2), \tag{6.34a}$$

$$h^{-1} = 1 - \frac{1}{f}(\Psi_{XX} + \Psi_{YY}) + \frac{1}{f^2}(\Psi_{XX}\Psi_{YY} - \Psi_{XY}^2), \tag{6.34b}$$

and thus

$$1 \equiv \left\{\frac{f}{g}\Psi - \frac{1}{2g}(\Psi_X^2 + \Psi_Y^2)\right\}\left\{1 - \frac{1}{f}(\Psi_{XX} + \Psi_{YY}) + \frac{1}{f^2}(\Psi_{XX}\Psi_{YY} - \Psi_{XY}^2)\right\}. \tag{6.35}$$

From the fact that $h(a,b,0) \equiv 1$ and (6.34) we have that

$$\psi(a,b,0) \equiv g/f.$$

Then equations (6.33) can be rewritten as

$$\mathrm{D}_t X = X_b \Psi_a - X_a \Psi_b, \qquad \mathrm{D}_t Y = Y_b \Psi_a - Y_a \Psi_b, \tag{6.36}$$

so that the equations for X and Y are the same, but their initial conditions are different; recall $X(a,b,0) = a$ and $Y(a,b,0) = b$. Substituting

$$\Psi = \frac{g}{f} + \sum_1^\infty \psi_j t^j \tag{6.37}$$

into (6.36) and equating coefficients of powers of t to zero, yields a series of equations of the form,

$$\begin{array}{ll} X_1 = 0, & Y_1 = 0, \\ X_2 = -\frac{1}{2}\psi_{1,b}, & Y_2 = \frac{1}{2}\psi_{1,a}, \\ \quad\vdots & \quad\vdots \\ X_{n+1} = -\dfrac{1}{n+1}\psi_{n,b} + \text{l.o.t.s}, & Y_{n+1} = \dfrac{1}{n+1}\psi_{n,a} + \text{l.o.t.s.} \end{array} \tag{6.38}$$

Backsubstituting these into the expansion of (6.35) yields the equations for the ψ_j. We already know $\psi_0 \equiv g/f$ and then one obtains,

$$\begin{aligned} \psi_{1,aa} + \psi_{1,bb} &= \frac{f^2}{g}\psi_1 \\ \psi_{2,aa} + \psi_{2,bb} &= \frac{f^2}{g}\psi_2 - \frac{1}{f}(\psi_{1,aa}\psi_{1,bb} - \psi_{1,ab}^2) - \frac{f^3}{g^2}\psi_1^2 - \frac{1}{2f}(\psi_{1,a}^2 + \psi_{1,b}^2) \\ &\vdots \\ \psi_{n,aa} + \psi_{n,bb} &= \frac{f^2}{g}\psi_n + \text{l.o.t.s} \end{aligned} \tag{6.39}$$

From the first expression for h in (6.34), and noting that $\Phi = -\mathrm{D}_t h$, we may obtain the series for the α_n obtained in the previous section, in terms of the ϕ_j. Using (6.28) we have also that

$$\Psi = \frac{1}{2f}(v_g^2 + u_g^2) + \frac{g}{f}h$$

and thus we can obtain the functions ψ_j in terms of the α_n.

It is interesting to note that the linear equation governing each cascade is the same. Further, to second order in time, $u_g = u$ and $v_g = v$, presumably reflecting the order of the semi-geostrophic approximation to the shallow water wave equations.

6.5 Cascade solutions are not flows

It is interesting to compare the series solutions obtained above with those satisfying the conditions for a flow. By a flow, we mean that

$$x(x(a,b,t), y(a,b,t), s) = x(a,b,t+s), \qquad y(x(a,b,t), y(a,b,t), s) = y(a,b,t+s)$$

is satisfied for all (a,b) and all t and s. Series expansions satisfying these conditions are precisely those that are one-parameter group actions, and are well understood. In fact, the series depends, in a way which can be made precise, on the first order coefficients only. This is the content of Sophus Lie's theorem that a group action depends only on the infinitesimal generator. In [1, p. 27] can be found the conditions that must be satisfied by the second and higher order coefficients. In the present case these take the form,

$$\begin{aligned} x(a,b,t) &= a + \alpha_a t + \tfrac{1}{2}t^2(\alpha_a\alpha_{aa} + \alpha_b\alpha_{ab}) + \mathcal{O}(t^3), \\ y(a,b,t) &= b + \alpha_b t + \tfrac{1}{2}t^2(\alpha_a\alpha_{ab} + \alpha_b\alpha_{bb}) + \mathcal{O}(t^3). \end{aligned}$$

It is simple to check that the only solution satisfying both the cascade equations and the flow conditions is the identity solution,

$$x(a,b,t) \equiv a, \qquad y(a,b,t) \equiv b.$$

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Appendix: Variational symmetries and Noether's Theorem

In this Appendix we record the formulae used in the paper, together with a brief indication of how Noether's Theorem works in computations. Proofs and a full explanation can be found in Olver §§4.3,4.4,5.3.

In the application in this paper, the independent variables are a, b and t, while the dependent variables are x, y, u and v. The infinitesimals of group actions on a, b and t were denoted by ξ, η and θ respectively, while the infinitesimals of group actions on x, y, u and v were denoted by ϕ, ψ, ζ and ω respectively. *To ease the exposition, we will denote independent variables by t_i, $i = 1, \dots 3$ and dependent variables by u^α, $\alpha = 1, \dots 4$. The derivatives of u^α will be denoted by u^α_K where K is a multi-index of differentiation. The infinitesimal generators for the t_i will be denoted as ξ^i and the infinitesimal generators for the u^α will be denoted by ϕ^α. The infinitesimal generator for u^α_K will be denoted by $\phi^\alpha_{[K]}$. Thus,*

$$(u^\alpha_K)^* = u^\alpha_K + \epsilon\phi^\alpha_{[K]} + \mathcal{O}(\epsilon^2).$$

Define

$$\mu = \mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \mathrm{d}t_3, \qquad \mu^* = \mathrm{d}t_1^* \wedge \mathrm{d}t_2^* \wedge \mathrm{d}t_3^*.$$

The group action on a Lagrangian volume form is

$$\begin{aligned}(\mathcal{L}\,\mathrm{d}\mu)^* &= \mathcal{L}(t_i^*, u^{\alpha *}, u^\alpha_K{}^*, \dots)\,\mathrm{d}\mu^* \\ &= \mathcal{L}(t_i^*, u^{\alpha *}, u^\alpha_K{}^*, \dots)\,\frac{\partial(t_1^*, t_2^*, t_3^*)}{\partial(t_1, t_2, t_3)}\,\mathrm{d}\mu.\end{aligned} \tag{A.1}$$

If $\mathcal{L}\,\mathrm{d}\mu = (\mathcal{L}\,\mathrm{d}\mu)^*$ is true for all elements of the group, then we can differentiate $(\mathcal{L}\,\mathrm{d}\mu)^*$ with respect to a group parameter. Suppose ε is a group parameter and that $\varepsilon = 0$ occurs at the identity of the group. Taking $\dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}$ at $\varepsilon = 0$ of both sides of equation (A.1) yields,

$$\begin{aligned}0 &= \sum_{i,\alpha,K} \frac{\partial\mathcal{L}}{\partial t_i}\xi^i + \frac{\partial\mathcal{L}}{\partial u^\alpha}\phi^\alpha + \frac{\partial\mathcal{L}}{\partial u^\alpha_K}\phi^\alpha_{[K]} + \mathcal{L}\nabla{\bullet}\xi \\ &= \sum_K \mathrm{D}^K\left(\phi^\alpha - \sum_i u^\alpha_i\xi^i\right)\frac{\partial\mathcal{L}}{\partial u^\alpha_K} + \nabla{\bullet}\,\left(\mathcal{L}\xi^1, \mathcal{L}\xi^2, \mathcal{L}\xi^3\right)\end{aligned} \tag{A.2}$$

This can be written compactly using the following notation. Defining

$$\mathcal{D}\mathcal{L} = \sum_K \frac{\partial\mathcal{L}}{\partial u^\alpha_K}\,\mathrm{d}u^\alpha_K,$$

and

$$\mathbf{v}_{\boldsymbol{Q}} = \sum \mathrm{D}^K\left(Q^\alpha\right)\frac{\partial}{\partial u^\alpha_K},$$

where $\boldsymbol{Q}$ is a vector, $\boldsymbol{Q} = (Q^\alpha)$, then equation (A.2) can be written as

$$\mathbf{v}_{\boldsymbol{Q}} \lrcorner\, \mathcal{D}(\mathcal{L}) + \nabla{\bullet}(\mathcal{L}(\xi^1, \xi^2, \xi^3)) = 0, \tag{A.3}$$

where

$$Q^\alpha = \phi^\alpha - \sum_i u^\alpha_i\xi^i.$$

The functions Q^α are called the *characteristic functions* of the symmetry.

The Euler Lagrange operator also involves the $\mathcal{D}$ operator. In fact,

$$\mathcal{D}(\mathcal{L})\,\mathrm{d}\mu = \sum_\alpha E^\alpha(\mathcal{L})\,\mathrm{d}u^\alpha \wedge \mathrm{d}\mu + \nabla{\bullet}\,\boldsymbol{A}, \tag{A.4}$$

where $\boldsymbol{A}$ can be explicitly determined.

Example. For $\mathcal{L} = \mathcal{L}(x, x_a, x_{aa})$ we have

$$\begin{aligned}\mathcal{D}(\mathcal{L})\,\mathrm{d}\mu &= \frac{\partial\mathcal{L}}{\partial x}\,\mathrm{d}x\,\mathrm{d}\mu + \frac{\partial\mathcal{L}}{\partial x_a}\,\mathrm{d}x_a\,\mathrm{d}\mu + \frac{\partial\mathcal{L}}{\partial x_{aa}}\,\mathrm{d}x_{aa}\,\mathrm{d}\mu \\ &= \left\{\frac{\partial\mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\partial\mathcal{L}}{\partial x_a}\right) + \frac{\mathrm{d}^2}{\mathrm{d}a^2}\left(\frac{\partial\mathcal{L}}{\partial x_{aa}}\right)\right\}\mathrm{d}x\,\mathrm{d}\mu + \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\partial\mathcal{L}}{\partial x_a}\mathrm{d}x\,\mathrm{d}\mu + 2\frac{\partial\mathcal{L}}{\partial x_{aa}}\mathrm{d}x_a\,\mathrm{d}\mu\right) \\ &= E^{[x]}(\mathcal{L})\,\mathrm{d}x\,\mathrm{d}\mu + \frac{\mathrm{d}}{\mathrm{d}x}\left(\boldsymbol{A}\right)\end{aligned}$$

where this defines $\boldsymbol{A}$, and where $\dfrac{\mathrm{d}}{\mathrm{d}a}(\mathrm{d}x) = \mathrm{d}x_a$ etc.

Operating on both sides of equation (A.4) with $\mathbf{v}_{\boldsymbol{Q}} \lrcorner$ and applying the symmetry condition, equation (A.3), yields

$$0 = \sum_{\alpha} Q^{\alpha} E^{\alpha}(\mathcal{L}) + \nabla\bullet \left(\mathbf{v}_{\boldsymbol{Q}} \lrcorner \boldsymbol{A} - \mathcal{L}(\xi^1, \xi^2, \xi^3) \right). \tag{A.5}$$

Equation (A.5) is the essential content of Noether's theorem; it shows how a divergence expression is obtained by the dot product of the Euler-Lagrange equations and the characteristic functions of the symmetry. There remains only the explicit formulae for the vector inside the divergence operator to be given.

The formula for $\boldsymbol{\mathcal{A}} = \mathbf{v}_{\boldsymbol{Q}} \lrcorner \boldsymbol{A}$ is given in terms of the so-called higher Euler operators, E^J_α where J is an index of differentiation. We need some notation regarding the index notation. If we write an index of differentiation in "concatenated form", so that

$$I = \underbrace{11\ldots1}_{\tilde{\imath}_1 \text{terms}} \underbrace{22\ldots2}_{\tilde{\imath}_2 \text{terms}} \ldots$$

then define

$$\#I = \sum \tilde{\imath}_k.$$

Moreover, given two indices I and J, the expressions $I \supset J$ and $I \setminus J$ have the obvious meaning.

The higher Euler operators are given explicitly by

$$E^J_\alpha(P) = \sum_{I \supset J} \binom{I}{J} (-D)^{I \setminus J} \frac{\partial P}{\partial u^\alpha_I}.$$

We have $\boldsymbol{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ and

$$\mathcal{A}_k = \sum_{\alpha} \sum_{\#I > 0} \frac{\tilde{\imath}_k + 1}{\#I + 1} \mathrm{D}^I \left(Q^\alpha E^{Ik}_\alpha(\mathcal{L}) \right).$$

Finally, if $\boldsymbol{B}$ is a conservation law for the system $\Delta = 0$, that is, if

$$\nabla\bullet \boldsymbol{B}\big|_{\Delta=0} = 0,$$

we say $\boldsymbol{B}$ is *trivial* if either

(i) $\boldsymbol{B} = \nabla \wedge \boldsymbol{C}$ so that $\nabla\bullet \boldsymbol{B} = 0$ is an identity (independent of the system of equations $\Delta = 0$ being studied)

(ii) each component of $\boldsymbol{B}$ is zero on solutions of $\Delta = 0$.

Two conservation laws are equivalent if they differ by a trivial conservation law.

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