LIE ALGEBROIDS AND CARTAN'S METHOD OF EQUIVALENCE

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Abstract. Cartan's method of equivalence constructs the local invariants of geometric structures, realizing them as obstructions to symmetry; these invariants generalize the familiar curvature invariant of a Riemannian structure. Here Cartan's method is recast in the language of Lie algebroids. The resulting formalism is fully invariant, for it is not only coordinate-free but *model*-free.

Details are developed for transitive finite-type geometric structures but rudiments of the theory include *intransitive* structures (intransitive symmetry deformations). Detailed illustrations include subriemannian contact structures and conformal geometry.

c'est la dissymétrie qui crée le phénomène — Pierre Curie, 1894.

Contents

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1. Introduction

This paper concerns the symmetries (isometries) of geometric structures, defined on smooth manifolds. Of particular interest are invariants arising as the local obstructions to symmetry. In this context, the quintessential analytical tool is Élie Cartan's 'method of equivalence' which we recast in the language of Lie algebroids. To do so is exceedingly natural, from both theoretical and computational viewpoints.

Context and scope. Cartan's method of equivalence makes its appearance in a famous paper of 1910 [5]; see also [6, 11, 2, 18, 12, 17]. The manifold of problems solved by the method include many at the heart of differential geometry, yet the method remains a tool of specialists.

Even experts would appear to concede the unfriendly nature of the method's current theoretical basis, the following assessment being typical: 'The subject is so rich that a worker in the field is torn between the devil of the general theory and the angel of geometrical applications' [2].

Despite this view, the vision underpinning Cartan's method is simply this:

Geometric structures are symmetries deformed by curvature.

Here we are speaking of locally rigid geometric structures, i.e., those of *finite type.* This includes, for example, Riemannian or conformal structures, but not symplectic or complex ones. (Structures of infinite-type and the related Cartan-Kähler theory are not discussed here; see, e.g., [12].) Disappointingly, this elegant vision is mostly obscured in attempts to transform it into a computationally effective formalism.

Lie algebroids are an attractive framework because they are an ideal marriage between Lie algebras (read 'infinitesimal symmetries') and tangent bundles (the basis of coordinate free-calculus), which they simultaneously generalize. Cartan's own student Ehresmann introduced their 'global' predecessor, Lie *groupoids*, as well as the subsequently more popular (but less invariant) principal bundles. See Ehresmann's papers of the 1950's collected in [9]. Principal bundles are the basis of the *G-structure* approach to Cartan's method (see below). For more on groupoids versus principal bundles, see the historical notes in [14]. Recently the Lie algebroid point of view has led to a beautiful and deep generalization of Lie's Third Theorem [8].

Finite-type structures may be described as those structures having, at most, finite-dimensional symmetry. More precisely, they are the structures admitting an associated *Cartan algebroid*. By definition, this is a Lie algebroid (and, in particular, a vector bundle) equipped with a linear connection that is suitably compatible with the algebroid structure. The curvature of this connection becomes the geometric structure's local symmetry obstruction. Cartan algebroids were introduced in [1] and are described further in Sections 2 and 4 below.

Intransitivity. It is widely recognized that Lie algebroids are well suited to the study of intransitive phenomena. Roughly speaking, a structure is *intransitive* if it possesses, as a fundamental invariant, a non-trivial, and possibly singular, foliation of the underlying space M ('non-trivial' meaning one or more leaves of positive codimension). A basic example is the symplectic stratification of a Poisson structure (which is not, however, of finite type). Another is the foliation by level sets of the energy function $\frac{1}{2}\|V\|^2$ associated with a vector field V on a Riemannian manifold M (which is).

A structure of finite-type is intransitive when it deforms an intransitive symmetry; the leaves of the invariant foliation are then 'deformed orbits.' Some finite-type structures frequently treated as transitive are actually *intransitive* in this sense, when invariantly formulated (the associated Cartan algebroid should be intransitive). For example, one expects conformal structures to be intransitive for generic values of the Weyl curvature. Other examples abound.

The basic formalism described here applies to intransitive geometric structures. However, a lack of transitivity at certain steps in Cartan's method can cause problems. This limitation, and how it might be overcome, is discussed further at the end of Sect. 2.

No models. Not only is the new approach coordinate-free, it is even *model*-free. An example of a model is $\mathbb{R}^n$, equipped with its standard metric, relevant to Riemannian structures. Models are spaces which already possess a large (usually maximal) and self-evident measure of symmetry. The traditional coordinate-free implementation of Cartan's method identifies a geometric structure with a *G-structure* (see, e.g., [20, 13]), this being a bundle of frames over the underlying manifold M. For a Riemannian structure, for example, this is the bundle of orthonormal frames. A 'frame' is an isomorphism between $\mathbb{R}^n$ and a tangent space $T_m M$, and in the G-structure approach $\mathbb{R}^n$ is generally the implicit model.

In one version of Cartan's method one attempts to construct a *Cartan geometry* (see, e.g., [18, 19]), either by starting with a G-structure, or using local 'gauges.' Alternatively, one can construct a sort of infinitesimal version of a Cartan geometry, known as an *adjoint tractor bundle* [4]. In either construction the explicit model is a quotient of Lie groups G/H. All these formalisms are innately *transitive* because the underlying model is transitive.

In any case, fixing a model means sacrificing some measure of invariance. (A roughly analogous loss of invariance occurs in fixing a point to define the fundamental group of a topological space, an invariance that is recovered by switching to the fundamental *groupoid*.) The Lie algebroid approach dispenses with models by replacing the frames of a G-structure on M with *infinitesimal*, moving, *relative* frames. By a relative frame we mean an isomorphism between different tangent spaces of the *same* space M.

Curvature. In a model-free approach 'curvature' assumes a subtly altered meaning. Generally, curvature has referred to the local deviation from the underlying model (typically $\mathbb{R}^n$) which is fixed. In the Lie algebroid approach all potential models are created equal, and curvature merely measures the local deviation from *some* maximally symmetric space. From this point of view, Euclidean space, hyperbolic space, and spheres, are all 'flat' Riemannian manifolds. If Cartan's method determines that a space is flat in the weaker sense, then the particular flat space one has at hand is simply part of the method's output. This is in contrast to model-based approaches where 'model mutation' (or something similar) might be needed to detect all possible cases [18].

An infinitesimal theory. In the Lie algebroid approach described here the basic object on which Cartan's method operates is called an *infinitesimal geometric structure*. For a concrete structure, such as a tensor or differential operator, this is just a geometrization of the PDE's you solve to find the infinitesimal isometries of the structure (the vector fields with flows leaving the structure invariant); more abstractly, it is the isotropy subalgebroid of the structure for certain natural jet-bundle 'representations.' The basic idea is explained in Sect. 2. Sect. 5 goes into detail and gives examples.

Classical G-structures and Cartan geometries are *global* objects, rather than infinitesimal ones, and this makes them cumbersome in computations, a fact noted in [4]. Nevertheless, the global point of view is illuminating and some aspects in the present setting are sketched in Appendix A.

Applications and prospects. This paper does not attempt substantially novel applications of Cartan's method of equivalence. We do expect the new approach to be especially helpful in dealing with intransitive — perhaps even singular — structures. And we believe it is as easy as existing approaches to implement in practice. Readers can draw their own conclusions from detailed applications to subriemannian contact three-manifolds (Sect. 10) and conformal geometry (Sect. 12). In the case of subriemannian contact structures, we explicitly compute the fundamental invariant differential operators, the general theory of these being addressed in Sect. 6.

Some general results on 'first-order' structures, which include Riemannian structures, appear in Sect. 9 (Theorems 9.4 and 9.5). Affine structures furnish another simple illustration, appearing in Sect. 11.

Notation. We use $\mathrm{Alt}^k(V) \cong \Lambda^k(V^*)$ and $\mathrm{Sym}^k(V) \cong \mathsf{S}^k(V^*)$ to denote the spaces of $\mathbb{R}$-valued alternating and symmetric k-linear maps on a vector space V. Similar notation applies to the tensor algebra of a vector bundle E over M. If σ is a section of E, then this is indicated by writing $\sigma \in \Gamma(E)$ or $\sigma \subset E$. Thus $\sigma \subset \mathrm{Alt}^2(TM) \otimes E$ means σ is an E-valued differential two-form on M.

All constructions in this paper are made in the C^∞ category.

2. Résumé of the Lie algebroid approach

This section describes the main ingredients of the Lie algebroid approach to Cartan's method, focusing on the familiar example of Riemannian structures. Subsequent sections furnish details.

2.1. **First-order infinitesimal symmetry.** Let M be a smooth n-dimensional manifold. We typically study a geometric structure on M by replacing it at each point with its first-order infinitesimal symmetries. Differential operators can be analyzed similarly but symmetries of higher order will apply, as we illustrate in 5.7.

As an example, consider a Riemannian metric σ on M. Call the 1-jet of a vector field V on M, evaluated at some point $m \in M$, a *1-symmetry* of σ, whenever σ has, at $m \in M$, vanishing Lie derivative along V. Then the collection of all 1-symmetries is a subbundle $\mathfrak{g} \subset J^1(TM)$ and V is a genuine infinitesimal symmetry (Killing field) if and only if its first-order prolongation J^1V is a section of $\mathfrak{g}$. The sections of $J^1(TM)$ of the form J^1V for some V are called *holonomic*. Whence:

(1) *The Killing fields of σ are in one-to-one correspondence with the holonomic sections of $\mathfrak{g}$.*

Moreover, as σ can be recovered from $\mathfrak{g}$, up to a constant factor (see Proposition 5.3), essentially nothing is lost by restricting attention to $\mathfrak{g}$.

Our analysis of $\mathfrak{g}$ takes as its point of departure the observation that $\mathfrak{g}$ is a *Lie algebroid* over M (see definition below), as is the tangent bundle TM, and its first jet $J^1(TM)$. In fact, adopting the natural language associated with these objects, we have:

(2) *The bundle $\mathfrak{g} \subset J^1(TM)$ of 1-symmetries of a Riemannian metric σ is the isotropy subalgebroid of σ under the representation of $J^1(TM)$ on $\mathrm{Sym}^2(TM)$ determined by the adjoint representation of $J^1(TM)$ on TM.*

The terms 'isotropy' and 'adjoint representation' are natural generalizations to Lie algebroids of familiar Lie algebra notions. Lie algebroid representations are intimately linked with Cartan's infinitesimal moving frames. Consequently, they are an appropriate setting for the study of linear connections. All this is reviewed in Sect. 3.

2.2. **Lie algebroids.** Recall that a *Lie algebroid* over M consists of a vector bundle $\mathfrak{g}$ over M, a Lie bracket $[\,\cdot\,,\,\cdot\,]$ on the space of sections $\Gamma(\mathfrak{g})$, together with a vector bundle morphism $\#\colon \mathfrak{g} \to TM$, called the *anchor*. One requires that the bracket satisfy a Leibnitz identity,

$$[X, fY] = f[X, Y] + df(\#X)Y,$$

where f is an arbitrary smooth function. The tangent bundle TM, equipped with the Jacobi-Lie bracket on vector fields, and identity map as anchor, is a Lie algebroid; in the general case one demands that the induced map of sections $\#\colon \Gamma(\mathfrak{g}) \to \Gamma(TM)$ be a Lie algebra homomorphism.

A Lie algebroid is *transitive* if its anchor is surjective. A subbundle $\mathfrak{h} \subset \mathfrak{g}$ is a *subalgebroid* if $\Gamma(\mathfrak{h}) \subset \Gamma(\mathfrak{g})$ is a subalgebra. Lie *algebras* are simply the Lie algebroids over a single point.

2.3. **Jet bundles.** If $\mathfrak{g}$ is any Lie algebroid over M with anchor $\#$, then the vector bundle $J^k\mathfrak{g}$ of k-jets of sections of $\mathfrak{g}$ is another Lie algebroid. Its anchor is the composite

$$J^k\mathfrak{g} \to \mathfrak{g} \xrightarrow{\#} TM.$$

Its bracket is uniquely determined by requiring that kth-order prolongation

$$J^k\colon \Gamma(\mathfrak{g}) \to \Gamma(J^k\mathfrak{g})$$

be a morphism of Lie algebroids (but see also 3.5 below). One may view J^k as a functor from the category of Lie algebroids to itself. See [7].

As this paper will demonstrate, many naturally occurring Lie algebroids in differential geometry can be constructed from TM using just two operations: application of the prolongation functor $J^1(\,\cdot\,)$; and passage to a subalgebroid.

2.4. **Action algebroids.** When a Lie algebroid is equipped with a suitable connection, then we may view it as an infinitesimal symmetry deformed by curvature. We explain this now and in 2.5 below.

Let $\mathfrak{g}_0$ be a finite-dimensional Lie algebra with bracket $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}_0}$ acting infinitesimally, smoothly, and from the right, on a manifold M. We may regard such actions as abstract infinitesimal symmetries. The trivial bundle $\mathfrak{g}_0 \times M$ over M has a natural Lie algebroid structure. This is the associated *action* (or *transformation*) *algebroid.*

The anchor of this Lie algebroid is the 'action map' $\#\colon \mathfrak{g}_0 \times M \to TM$ sending (ξ, m) to the infinitesimal generator associated with $\xi \in \mathfrak{g}_0$, evaluated at $m \in M$. The Lie bracket on $\Gamma(\mathfrak{g}_0 \times M)$ is a natural extension of the bracket on $\mathfrak{g}_0$, regarded as the subspace $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g}_0 \times M)$ of constant sections. To describe it, let $\tau(\,\cdot\,,\,\cdot\,)$ denote the naive extension of this bracket, i.e., the one that is bilinear with respect to all smooth functions (and consequently *not* Leibnitz):

$$\tau(X,Y)(m) := [X(m), Y(m)]_{\mathfrak{g}_0}; \qquad X, Y \in \Gamma(\mathfrak{g}_0 \times M).$$

And let ∇ denote the canonical flat connection on $\mathfrak{g}_0 \times M$. Then the Lie bracket on $\Gamma(\mathfrak{g}_0 \times M)$ is defined by

$$[X,Y] := \nabla_{\#X} Y - \nabla_{\#Y} X + \tau(X,Y). \tag{1}$$

2.5. **Cartan algebroids.** Once again, let $\mathfrak{g} \subset J^1(TM)$ denote the bundle of 1-symmetries of a Riemannian metric σ. Then we claim there exists a natural linear connection $\nabla^{(1)}$ on $\mathfrak{g}$ with the following properties:

(1) *A section of $X \subset \mathfrak{g}$ is holonomic if and only if it is $\nabla^{(1)}$-parallel.*
(2) *$\nabla^{(1)}$ is compatible with the Lie algebroid structure on $\mathfrak{g}$.*

The proof of (1) rests on Cartan's method. Together with 2.1(1) it implies:

(3) *The curvature of $\nabla^{(1)}$ is the sole local obstruction to symmetry, i.e., to the existence, locally, of Killing fields.*

A connection satisfying property (2) — made precise in Sect. 4 — is called a *Cartan connection* on $\mathfrak{g}$. A Lie algebroid $\mathfrak{g}$ equipped with a Cartan connection is a *Cartan algebroid.*

An action algebroid $\mathfrak{g}_0 \times M$, equipped with its canonical flat connection, is the most basic example of a Cartan algebroid. Conversely, an arbitrary Cartan algebroid $\mathfrak{g}$ has the property that its parallel sections constitute a finite-dimensional subalgebra $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g})$ acting infinitesimally on the underlying manifold M. This determines an action algebroid which is in fact identifiable with a subalgebroid of $\mathfrak{g}$. This 'symmetric part' of $\mathfrak{g}$ coincides with $\mathfrak{g}$ exactly when the Cartan connection on $\mathfrak{g}$ is flat (Theorem 4.5, Sect. 4). Thus Cartan algebroids are infinitesimal symmetries deformed by curvature.

We recall that a classical Cartan connection is flat precisely when the associated Cartan geometry is locally isomorphic to the underlying model G/H (on which the group G is acting transitively) [18, Theorem **5**.5.3]. Thus Cartan algebroids may be viewed as infinitesimal, model-free (and possibly intransitive) versions of Cartan geometries.

Following is a concrete description of $\nabla^{(1)}$. It's existence is sketched in 2.9 below, with details being deferred until 9.6.

The Levi-Cevita connection ∇ associated with σ amounts to a splitting of a canonical exact sequence,

$$0 \to T^*M \otimes TM \hookrightarrow J^1(TM) \to TM \to 0,$$

which leads to an identification $\mathfrak{g} \cong TM \oplus \mathfrak{h}$, where $\mathfrak{h} \subset T^*M \otimes TM$ denotes the $\mathfrak{o}(n)$-bundle of all σ-skew-symmetric tangent space endomorphisms. This bundle is ∇-invariant. Then,

$$\nabla^{(1)}_U(V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\nabla_U \phi + \operatorname{curv} \nabla\,(U, V)).$$

With the help of Bianchi's identity, $d_\nabla \operatorname{curv} \nabla = 0$, one computes

$$\operatorname{curv} \nabla^{(1)}(U_1, U_2)(V \oplus \phi) = 0 \oplus (-(\nabla_V \operatorname{curv} \nabla + \phi \cdot \operatorname{curv} \nabla)(U_1, U_2)),$$

implying that $\nabla^{(1)}$ is flat if and only if $\operatorname{curv} \nabla$ is both ∇-parallel and $\mathfrak{h}$-invariant. As is well known, this condition is equivalent to pure, constant scalar curvature, or, equivalently to constant sectional curvature. Whence from (3) one recovers the standard criterion for maximal local homogeneity of a Riemannian manifold.

2.6. **Infinitesimal geometric structures and their symmetries.**
Infinitesimal geometric structures, as defined below, generalize the vector bundle of 1-symmetries of a Riemannian metric described above. The notion is rather general, as we shall see in Sect. 5, which contains many examples, both transitive and intransitive. Infinitesimal geometric structures simultaneously generalize Cartan algebroids and (infinitesimal) G-structures. Infinitesimal geometric structures are this paper's central object of study.

Definition. Let $\mathfrak{t}$ be any Lie algebroid over M (the tangent bundle TM in the simplest case). Then an *infinitesimal geometric structure* on $\mathfrak{t}$ is any subalgebroid $\mathfrak{g} \subset J^1\mathfrak{t}$.

Analogous to the Killing fields of a Riemannian structure are the *symmetries* of $\mathfrak{g}$. These are the sections V of $\mathfrak{t}$ whose prolongations J^1V are sections of $\mathfrak{g}$. The *structure kernel* of $\mathfrak{g}$ is the kernel $\mathfrak{h}$ of the restriction

$$a\colon \mathfrak{g} \to \mathfrak{t}$$

of the projection $J^1\mathfrak{t} \to \mathfrak{t}$, this morphism being the anchor of $\mathfrak{g}$ when $\mathfrak{t} = TM$. This is the infinitesimal analogue of the structure group G of a G-structure. The structure kernel of $\mathfrak{g}$ is a subalgebroid whenever it has constant rank.

The *image* of $\mathfrak{g}$ is the image of a. If $a\colon \mathfrak{g} \to \mathfrak{t}$ is surjective, we call $\mathfrak{g}$ a *surjective* infinitesimal geometric structure. If $\mathfrak{t} = TM$, then 'surjective' is synonymous with 'transitive.' For example, the bundle $\mathfrak{g} \subset J^1(TM)$ of 1-symmetries of a Riemannian metric is surjective (see 5.3 for a proof). Every Poisson structure has an associated infinitesimal geometric structure $\mathfrak{g} \subset T^*M$ which *is* surjective but generally *not* transitive; see 5.6. For a simple example of an infinitesimal geometric structure failing to be surjective, see 5.4.

The infinitesimal geometric structures corresponding to G-structures are *always* surjective; see 5.9.

Evidently, the symmetries of $\mathfrak{g}$ are in one-to-one correspondence with the *holonomic* sections of $\mathfrak{g} \subset J^1\mathfrak{t}$ — those sections that are prolongations of something. Symmetries are necessarily sections of the image of $\mathfrak{g}$ and are closed under Lie algebroid bracket.

> *The guiding goal of this paper is to characterize the symmetries of infinitesimal geometric structures, and in particular to determine all local obstructions to their existence.*

Since locally equivalent infinitesimal geometric structures must have corresponding local obstructions to symmetry, this addresses the local 'Equivalence Problem.'

Note that the notions of structure kernel, image, surjectivity, and symmetry all make sense for *arbitrary* subsets $\mathfrak{g} \subset J^1\mathfrak{t}$. No further comment will accompany their application in this extended context.

2.7. **Cartan connections via generators.** Cartan's method of equivalence attacks the obstruction problem by attempting to reduce it to a special case solved in the theorem below. To formulate the result, recall that the linear connections ∇ on a vector bundle $\mathfrak{t}$ are in one-to-one correspondence with the splittings $s\colon \mathfrak{t} \to J^1\mathfrak{t}$ of a natural exact sequence

$$0 \to T^*M \otimes \mathfrak{t} \hookrightarrow J^1\mathfrak{t} \to \mathfrak{t} \to 0.$$

For details, see 3.3. We call ∇ a *generator* of $\mathfrak{g} \subset J^1\mathfrak{t}$ if $s(\mathfrak{t}_1) \subset \mathfrak{g}$, where $\mathfrak{t}_1 \subset \mathfrak{t}$ denotes the image of $\mathfrak{g}$. Generators, discussed further in Sect. 6, are certain 'preferred' connections, which generally exist but need not be unique. For example, the Levi-Cevita connection is a generator for the bundle $\mathfrak{g} \subset J^1(TM)$ of 1-symmetries of a Riemannian metric σ, but so is any other connection whose 'dual' leaves σ invariant; see 5.3. Generators are indispensable in explicit computations.

In Sect. 6 we establish the following straightforward but crucial result:

Theorem. *Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure and assume the projection $a\colon \mathfrak{g} \to \mathfrak{t}$ has constant rank. Then $\mathfrak{g}$ has a unique generator ∇ if and only if it is surjective and has structure kernel $\mathfrak{h} = 0$. In that case ∇ is a Cartan connection on $\mathfrak{t}$ whose parallel sections are precisely the symmetries of $\mathfrak{g}$.*

In particular, $\operatorname{curv}\nabla$ then becomes the sole local obstruction to symmetry.

2.8. **Cartan's method.** Cartan's method of equivalence is an algorithm with two fundamental operations: *reduction* and *prolongation*. In the present context they may be roughly described as follows.

Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure and consider the equation

$$J^1V \subset \mathfrak{g}$$

whose solutions $V \subset \mathfrak{t}$ are the symmetries of $\mathfrak{g}$; here $\subset$ means 'is a section of.' In coordinates this equation is a system of first-order partial differential equations, linear in the derivatives. Reduction amounts to lowering the number of dependent and independent variables by identifying some necessary algebraic constraint. Prolongation means increasing the number of variables by introducing derivatives of the independent variables as new independent variables; new equations are added to account for the equality of mixed partial derivatives.

By applying a sufficient number of reductions and prolongations one attempts to transform $\mathfrak{g} \subset J^1\mathfrak{t}$ into an infinitesimal geometric structure $\mathfrak{g}' \subset J^1\mathfrak{t}'$ satisfying the hypotheses of the above theorem, and having symmetries in natural one-to-one correspondence with those of $\mathfrak{g}$. A specific algorithm is offered in 2.13 below.

2.9. **Prolongation.** In coordinate-free language, the *prolongation* of an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ is a natural 'lift' of $\mathfrak{g}$ to a subset $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$. This lift is itself an infinitesimal geometric structure on $\mathfrak{g}$, whenever it has constant rank. Most importantly, there is a one-to-one correspondence between symmetries of $\mathfrak{g}$ and symmetries of $\mathfrak{g}^{(1)}$, furnished by prolongation of sections,

$$V \mapsto J^1V : \Gamma(\mathfrak{t}) \to \Gamma(\mathfrak{g}).$$

This is *provided* $\mathfrak{t}$ is *transitive*. Otherwise prolongation may lead to spurious symmetry (see Proposition 8.2).

If, for example, $\mathfrak{g} \subset J^1(TM)$ is the bundle of 1-symmetries of a Riemannian metric σ, then one can show that $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is surjective with a trivial structure kernel. Replacing $\mathfrak{g}$ and $\mathfrak{t}$ in Theorem 2.7 with $\mathfrak{g}^{(1)}$ and $\mathfrak{g}$, one establishes establishes the existence of the Cartan connection $\nabla^{(1)}$ in 2.5 above, as the unique generator of $\mathfrak{g}^{(1)}$. (For details, see 5.3.)

The general definition of $\mathfrak{g}^{(1)}$ outlined now will be clearer after the description of Lie algebroid adjoint representations (Sect. 3) and isotropy (Sect. 5). A detailed definition of prolongation is repeated in Sect. 8. An in-depth analysis of the procedure is postponed until Sect. 11.

Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure and view the restriction $a : \mathfrak{g} \to \mathfrak{t}$ of the projection $J^1\mathfrak{t} \to \mathfrak{t}$ as a section of $\mathfrak{g}^* \otimes \mathfrak{t}$. Then a has a naturally defined exterior derivative da called the *torsion* of $\mathfrak{g}$, this being a certain section of $\mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t}$. We shall see that both vector bundles $\mathfrak{g}^* \otimes \mathfrak{t}$ and $\mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t}$ are natural representations of the Lie algebroid $J^1\mathfrak{g}$. By definition, $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is the joint isotropy of a and da under these representations.

2.10. **Reduction.** Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure. By a *reduction* of $\mathfrak{g}$ we shall mean any subalgebroid $\mathfrak{g}' \subset \mathfrak{g}$ with the same symmetries as $\mathfrak{g}$; it suffices to check that symmetries of $\mathfrak{g}$ are symmetries of $\mathfrak{g}'$. In contrast to prolongation, there is no unique way to construct reductions. Notice, however, that if $\mathfrak{g}' \subset \mathfrak{g}$ is a reduction and $\mathfrak{g}'' \subset \mathfrak{g}$ merely a subalgebroid satisfying $\mathfrak{g}' \subset \mathfrak{g}'' \subset \mathfrak{g}$, then $\mathfrak{g}''$ is automatically a reduction of $\mathfrak{g}$ also; we say $\mathfrak{g}''$ is a *cruder* reduction than $\mathfrak{g}'$.

We now describe the most important reduction techniques: elementary reduction and Θ-reduction.

2.11. **Elementary reduction.** Returning to Cartan's method described above, we emphasize that transitivity is not a hypothesis of Theorem 2.7 (and that Cartan algebroids can be *intransitive*). Rather, one requires surjectivity. If an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ is *not* surjective, we may attempt to make it so by passing to the *elementary reduction* $\mathfrak{g}_1$ of $\mathfrak{g}$. By definition,

$$\mathfrak{g}_1 := \mathfrak{g} \cap J^1\mathfrak{t}_1,$$

where $\mathfrak{t}_1 \subset \mathfrak{t}$ denotes the image of $\mathfrak{g}$. Assuming $\mathfrak{t}_1 \subset \mathfrak{t}$ and $\mathfrak{g}_1 \subset \mathfrak{g}$ have constant rank, they are subalgebroids. In particular, $\mathfrak{g}_1 \subset J^1\mathfrak{t}_1$ becomes an infinitesimal geometric structure. Moreover, one easily proves:

Proposition. *If the elementary reduction $\mathfrak{g}_1$ of $\mathfrak{g}$ has constant rank then it is a reduction of $\mathfrak{g}$ in the sense above. One has $\mathfrak{g}_1 = \mathfrak{g}$ if and only if $\mathfrak{g}$ is surjective.*

Because surjectivity is built into the definition of G-structures, elementary reduction never appears in that context. Elementary reduction is described further in Sect. 7, together with a cruder alternative called *image reduction*.

2.12. **Θ-reduction.** If an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ is already surjective but has a non-trivial structure kernel, then, with a view to applying Theorem 2.7, one can try to shrink the kernel by prolonging (see above). However, the prolongation $\mathfrak{g}^{(1)}$ generally fails to be surjective itself. While one might attempt to correct this by turning to the elementary reduction of $\mathfrak{g}^{(1)}$, there is an alternative that is computationally more attractive. One anticipates the lack of surjectivity of $\mathfrak{g}^{(1)}$ by first replacing $\mathfrak{g}$ by its *Θ-reduction*. By definition, this is the image of $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$, i.e., the set $\mathfrak{g}_1^{(1)} := p(\mathfrak{g}^{(1)}) \subset \mathfrak{g}$, where $p\colon J^1\mathfrak{g} \to \mathfrak{g}$ is the natural projection. The point is that Θ-reductions can be computed without directly computing the larger space $\mathfrak{g}^{(1)}$. In Sect. 9 we prove:

Proposition. *If $\mathfrak{g}_1^{(1)}$ and $\mathfrak{g}^{(1)}$ have constant rank, then the Θ-reduction $\mathfrak{g}_1^{(1)}$ of $\mathfrak{g}$ is a reduction of $\mathfrak{g}$ in the sense of 2.10. One has $\mathfrak{g}_1^{(1)} = \mathfrak{g}$ if and only if $\mathfrak{g}^{(1)}$ is surjective.*

Classically, 'reduction' has usually meant *reduction by torsion*, a notion we define in 8.4 for arbitrary surjective infinitesimal geometric structures. If $\mathfrak{t} = TM$, then reduction by torsion coincides with Θ-reduction. More generally, however, torsion reduction is a cruder reduction technique. In some cases Θ-reduction is not only more efficient but also more computationally convenient; conformal geometry (Sect. 12) appears to be a case in point. While Θ-reductions are always defined, we have restricted its detailed analysis to the case of surjective structures over *transitive* Lie algebroids. This analysis appears in Sect. 11.

2.13. **A specific algorithm and its limitations.** Before describing a specific algorithm, we define two auxiliary procedures.

By Proposition 2.11, the following procedure, which we shall call surjectify $\mathfrak{g}$, forces $\mathfrak{g}$ to be surjective:

```
do while g is not surjective
    replace g with g_1 (elementary reduction)
end do.
```

Next, we let strongly surjectify $\mathfrak{g}$ denote the following procedure making $\mathfrak{g}$ and $\mathfrak{g}^{(1)}$ simultaneously surjective (by Propositions 2.11 and 2.12):

```
do while g^(1) is not surjective
    surjectify g
    replace g with g_1^(1) (Θ-reduction)
    surjectify g
end do.
```

By construction, Θ-reduction need be described for surjective structures only.

We claim that it is possible to implement Cartan's Method using elementary reduction and prolongation alone. In practice, however, the following algorithm is easier to apply:

```
surjectify 𝔤
repeat until stop encountered
    if 𝔥 = 0 apply Theorem 2.7 and stop
    strongly surjectify 𝔤
    if 𝔥 = 0 apply Theorem 2.7 and stop
    replace 𝔤 with 𝔤^(1) (prolongation)
end repeat.
```

Notice that prolongation is delayed as long as possible. We now describe the ways in which the above algorithm can fail.

Firstly, an execution of surjectify $\mathfrak{g}$ or strongly surjectify $\mathfrak{g}$ could fail because $\mathfrak{g}$, at some iteration of these procedures' do-while loops, loses the constancy of its rank. While prolongation of $\mathfrak{g}$ might resolve this kind of singularity (by recovering rank constancy), this requires a prolongation theory for 'variable rank Lie algebroids' (or Lie pseudoalgebras) which we do not provide here. Similarly, $\mathfrak{g}$ might lose rank constancy at some iteration of the repeat-until loop of the main algorithm.

Even if all singularities are successfully resolved, it may happen that no stop is ever encountered in the repeat-until loop, which then becomes perpetual. In that case we deem $\mathfrak{g}$ to be of *infinite type* and must turn to Cartan-Kähler theory to proceed.

Another possibility for failure concerns Θ-reduction. In practice, it is complicated to implement without making the added assumption that the base algebroid $\mathfrak{t}$ of $\mathfrak{g} \subset J^1\mathfrak{t}$ is *transitive*. Similarly, prolongation is unhelpful if $\mathfrak{t}$ is intransitive because of the problem of spurious symmetry alluded to in 2.9. One way to proceed in these cases would be to *restrict* the infinitesimal geometric structure $\mathfrak{g}$ to each orbit[1] of $\mathfrak{t}$. This restriction will sit over a transitive base (this being a Lie algebroid over the orbit) but is not simply the pullback in the category of Lie algebroids, for one wants an infinitesimal geometric structure over the base, not merely a Lie algebroid. Also, one needs to understand how conclusions regarding the restricted structure combine with 'transverse' information to solve the original problem. Fortunately, a splitting theory for Lie algebroids exists [10] and this possibly reduces the transverse problem to the case of an isolated singularity (zero-dimensional orbit). None of this is explored here.

If the Cartan algorithm above succeeds it ends in an application of Theorem 2.7, this delivering a Cartan algebroid whose parallel sections are in natural one-to-one correspondence with the symmetries of $\mathfrak{g}$. We will say that $\mathfrak{g}$ is of *finite type*, or *admits an associated Cartan algebroid.*

3. Representations and their deformations

This section is a rapid review of the theory of connections, from the point of view of Lie algebroid representations, our main purpose being to establish notation and terminology. Of particular relevance will be the *adjoint* representation of $J^1\mathfrak{g}$ on

[1] An *orbit* of a Lie algebroid over M is a leaf of the foliation integrating the image of the anchor, this being an integrable, possibly singular, distribution on M.

a Lie algebroid $\mathfrak{g}$, and a concrete description of the bracket on $J^1\mathfrak{g}$. We introduce *associated* connections, which are developed further in Sect. 6.

We hereafter assume the reader has a working knowledge of Lie groupoids and algebroids, referring her to [3] or [15] for background.

3.1. **Frames.** The 'frames' in Cartan's 'method of moving frames' depend upon a Euclidean model of the space under consideration. The corresponding model-free notion is that of a *relative frame*. In a vector bundle E over M (e.g., $E = TM$), this is just a vector space isomorphism between two fibers of E.

The collection of all relative frames constitutes a Lie groupoid $\mathrm{GL}(E)$ generalizing the automorphism group of a vector space (where M is a point). A *representation* of a groupoid G, on the vector bundle E, is just a groupoid morphism $G \to \mathrm{GL}(E)$. Linear connections may be interpreted as infinitesimal representations 'deformed by curvature.' This point of view leads simultaneously to the more general notion of a $\mathfrak{g}$-*connection*, defined in 3.2 below. Proposition 3.2 gives an 'analytic' characterization of these connections, generalizing the usual one, which the reader may take as an alternative definition.

In the model-free setting, an 'infinitesimal moving frame' is an element of the Lie algebroid $\mathfrak{gl}(E)$ of $\mathrm{GL}(E)$. This Lie algebroid may be viewed as a subbundle of $\mathrm{Hom}(J^1E, E) \cong (J^1E)^* \otimes E$. See, e.g., [15] — where $\mathfrak{gl}(E)$ is denoted $\mathcal{D}(E)$ — or [1]. Its sections, when accordingly viewed as first-order differential operators, correspond precisely to the *derivations* of E, namely to those $\mathbb{R}$-linear $D\colon \Gamma(E) \to \Gamma(E)$ for which there exists a vector field V giving the rule

$$D(f\sigma) = fD\sigma + df(V)\sigma,$$

which is to hold for all sections σ of E and all functions f on M; one has $V = \#D$, where $\#$ is the anchor of $\mathfrak{gl}(E)$. The kernel of this anchor is the Lie algebra bundle $E^* \otimes E \subset (J^1E)^* \otimes E$. The Lie algebroid bracket on $\mathfrak{gl}(E)$ is the operator commutator bracket; it extends the standard bracket on (sections of) $E^* \otimes E$:

$$[X, Y]_{\mathfrak{gl}(E)}\, \sigma = XY\sigma - YX\sigma.$$

3.2. **Lie algebroid representations and $\mathfrak{g}$-connections.** A *representation* of a Lie algebroid $\mathfrak{g}$ on E is a morphism $\mathfrak{g} \to \mathfrak{gl}(E)$ of Lie algebroids. When M is a single point one recovers the usual representations of a Lie *algebra*. Deforming the representation notion we arrive at the following:

Definition. Let $\mathfrak{g}$ be any Lie algebroid over M. A $\mathfrak{g}$-*connection* on a vector bundle E over M is a vector bundle morphism

$$\nabla\colon \mathfrak{g} \to \mathfrak{gl}(E)$$

that is *not* required to be a Lie algebroid morphism, but *is* nevertheless required to respect the anchors, $\#\nabla = \nabla\#$.

Suppose X is a section of $\mathfrak{g}$. When the section $\nabla(X)$ of $\mathfrak{gl}(E) \subset \mathrm{Hom}(J^1E, E)$ is to be viewed as a differential operator, we instead write ∇_X, i.e., $\nabla_X V := \nabla(X)(J^1V)$. In view of the preceding characterization of the sections of $\mathfrak{gl}(E)$ as derivations, we have the Leibnitz identity

$$\nabla_X(f\sigma) = f\nabla_X\sigma + df(\#X)\sigma; \qquad X \subset \mathfrak{g}, \sigma \subset E.$$

Conversely:

Proposition. *Every vector bundle morphism $\nabla\colon \mathfrak{g} \to \mathrm{Hom}(J^1E, E)$ that is Leibnitz in the above sense is a $\mathfrak{g}$-connection.*

If ∇ is a $\mathfrak{g}$-connection, then the formula

$$\operatorname{curv}\nabla\,(X,Y) := [\nabla(X), \nabla(Y)]_{\mathfrak{gl}(E)} - \nabla(\,[X,Y]_{\mathfrak{g}}),$$

defining the Lie algebroid curvature of the map $\nabla\colon \mathfrak{g} \to \mathfrak{gl}(E)$ takes a familiar form:

$$\operatorname{curv}\nabla\,(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]_{\mathfrak{g}}} Z.$$

The $\mathfrak{g}$-connection ∇ is a $\mathfrak{g}$-representation when $\operatorname{curv}\nabla = 0$.

Example. If $\mathfrak{g}$ is a Lie algebroid and $E \subset \mathfrak{g}$ is a vector subbundle contained in the kernel of its anchor then a canonical representation ρ of $\mathfrak{g}$ on E is well defined by $\rho_X Y := [X,Y]_{\mathfrak{g}}$. Important cases in point are the kernel of the anchor itself, and the structure kernel of an infinitesimal geometric structure, when these have constant rank.

3.3. **Linear connections.** Using the language of the preceding discussion, a linear connection ∇ on E is just a TM-connection on E, becoming a TM-representation when ∇ is flat. It is an elementary fact that the linear connections on E are in one-to-one correspondence with the splittings $s\colon E \to J^1E$ of the exact sequence

$$0 \to T^*M \otimes E \hookrightarrow J^1E \to E \to 0.$$

Here $T^*M \otimes E \hookrightarrow J^1E$ is the inclusion which, as a map on sections, sends[2] $df \otimes \sigma$ to $fJ^1\sigma - J^1(f\sigma)$. The splitting associated with a linear connection ∇ on E is given by

$$s\sigma = J^1\sigma + \nabla\sigma; \qquad \sigma \subset E. \tag{1}$$

Here $\nabla\sigma \subset T^*M \otimes E$ is defined by $(\nabla\sigma)(V) := \nabla_V\sigma$.

Of particular interest is the case where E is a Lie algebroid, discussed in 3.6 and Sect. 4.

3.4. **The adjoint representation.** If $\mathfrak{g}$ is a Lie algebroid over M, then so is the associated bundle $J^1\mathfrak{g}$ of 1-jets of sections of $\mathfrak{g}$ (see 2.3). A god-given $J^1\mathfrak{g}$-connection on $\mathfrak{g}$, denoted $\mathrm{ad}^{\mathfrak{g}}$, is well defined by

$$\mathrm{ad}^{\mathfrak{g}}_{J^1X}\, Y = [X,Y]_{\mathfrak{g}}.$$

Using the identity

$$[J^1X, J^1Y]_{J^1\mathfrak{g}} = J^1[X,Y]_{\mathfrak{g}}, \tag{1}$$

one shows that $\mathrm{ad}^{\mathfrak{g}}$ is even a representation of $J^1\mathfrak{g}$ on $\mathfrak{g}$. It is called the *adjoint representation* and generalizes the Lie *algebra* representations of the same name.

We note that

$$\mathrm{ad}^{\mathfrak{g}}_{\phi}\, X = \phi(\#X); \qquad X \subset \mathfrak{g}, \tag{2}$$

for all sections $\phi \subset T^*M \otimes \mathfrak{g} \subset J^1\mathfrak{g}$. If $a\colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebroids, then one has the identity

$$a\Big(\,\mathrm{ad}^{\mathfrak{g}}_{\xi}\, X\Big) = \mathrm{ad}^{\mathfrak{h}}_{(J^1a)\xi}(aX); \qquad \xi \subset J^1\mathfrak{g},\ X \subset \mathfrak{g}. \tag{3}$$

[2]An opposite sign convention is adopted in our paper [1] and elsewhere.

3.5. **The bracket on $J^1(\cdot)$ of a Lie algebroid.** Recall that the bracket on $J^1\mathfrak{g}$ is implicitly defined by the requirement 3.4(1). With the help of the adjoint representation, we now describe this bracket concretely.

Although the exact sequence

$$0 \to T^*M \otimes \mathfrak{g} \hookrightarrow J^1\mathfrak{g} \to \mathfrak{g} \to 0 \tag{1}$$

possesses no canonical splitting, the corresponding exact sequence of section spaces,

$$0 \to \Gamma(T^*M \otimes \mathfrak{g}) \hookrightarrow \Gamma(J^1\mathfrak{g}) \to \Gamma(\mathfrak{g}) \to 0,$$

is split by $J^1 \colon \Gamma(\mathfrak{g}) \to \Gamma(J^1\mathfrak{g})$, delivering a canonical identification

$$\Gamma(J^1\mathfrak{g}) \cong \Gamma(\mathfrak{g}) \oplus \Gamma(T^*M \otimes \mathfrak{g}).$$

Under this identification, the Lie algebra $\Gamma(J^1\mathfrak{g})$ is a semidirect product described in the proposition below.

In addition to having the adjoint representation of $J^1\mathfrak{g}$ on $\mathfrak{g}$, we have a representation of $J^1\mathfrak{g}$ on TM, given by the composite

$$J^1\mathfrak{g} \xrightarrow{J^1\#} J^1(TM) \xrightarrow{\mathrm{ad}^{TM}} \mathfrak{gl}(TM), \tag{2}$$

$$\text{i.e.,} \quad J^1X \cdot V = [\#X, V]; \qquad X \subset \mathfrak{g},\ V \subset TM.$$

So we can construct a natural representation of $J^1\mathfrak{g}$ on $T^*M \otimes \mathfrak{g}$; it is given by

$$(J^1X \cdot \phi)V = [X, \phi(V)]_{\mathfrak{g}} - \phi([\#X, V]_{TM}); \qquad V \subset TM. \tag{3}$$

On the other hand, $T^*M \otimes \mathfrak{g}$ is the structure kernel of $J^1\mathfrak{g}$ so that $J^1\mathfrak{g}$ acts on $T^*M \otimes \mathfrak{g}$ via bracket; see the example in 3.2. A consequence of the following result is that these two representations coincide.

Proposition. *The subalgebroid $T^*M \otimes \mathfrak{g} \subset J^1\mathfrak{g}$ inherits the bracket*

$$[\phi_1, \phi_2]_{T^*M\otimes\mathfrak{g}}(V) = \phi_1(\#\phi_2 V) - \phi_2(\#\phi_1 V); \qquad V \subset TM, \tag{4}$$

where $\#$ denotes the anchor of $\mathfrak{g}$. Sections of $J^1\mathfrak{g}$ are of the form $J^1X + \phi$ for uniquely determined sections $X \subset \mathfrak{g}$ and $\phi \subset T^*M \otimes \mathfrak{g}$. The bracket on $J^1\mathfrak{g}$ is given by

$$[J^1X + \phi, J^1Y + \psi]_{J^1\mathfrak{g}} = J^1[X, Y]_{\mathfrak{g}} + J^1X \cdot \psi - J^1Y \cdot \phi + [\phi, \psi]_{T^*M\otimes\mathfrak{g}},$$

where the action of $J^1\mathfrak{g}$ on $T^*M \otimes \mathfrak{g}$ is defined by (3)*.*

To prove the proposition one uses 3.4(1) and the fact that sections of $T^*M \otimes \mathfrak{g}$ are finitely generated by those of the form $df \otimes X = f J^1X - J^1(fX)$.

3.6. **Dual connections, torsion and associated connections.** Let $\mathfrak{g}$ be a Lie algebroid and ∇ a $\mathfrak{g}$-connection on itself. We define the *dual* of ∇ to be the $\mathfrak{g}$-connection ∇^* on $\mathfrak{g}$ defined by

$$\nabla^*_X Y := \nabla_Y X + [X, Y].$$

One has 'duality' in the sense that $\nabla^{**} = \nabla$.

The *torsion* of ∇ is the section, $\operatorname{tor}\nabla$, of $\operatorname{Alt}^2(\mathfrak{g}) \otimes \mathfrak{g}$ measuring the difference between ∇ and its dual:

$$\operatorname{tor}\nabla\,(X, Y) := \nabla_X Y - \nabla^*_X Y = \nabla_X Y - \nabla_Y X - [X, Y].$$

The torsion or curvature of ∇ can be expressed in terms of the torsion and curvature of ∇^* (and, by duality, vice versa):

Proposition. *Let ∇ be a $\mathfrak{g}$-connection on $\mathfrak{g}$, and ∇^* its dual. Then:*

$$\operatorname{tor}\nabla = -\operatorname{tor}\nabla^* \tag{1}$$

$$\begin{cases} \operatorname{curv}\nabla\,(X,Y)Z = (\nabla^*_Z \operatorname{tor}\nabla^*)(X,Y) - \operatorname{curv}\nabla^*\,(Z,X)Y \\ \qquad\qquad + \operatorname{curv}\nabla^*\,(Z,Y)X; \qquad X,Y,Z \subset \mathfrak{g}. \end{cases} \tag{2}$$

We now introduce two important connections generalizing the dual of a linear connection on M. They are examples of *associated connections*, defined more generally in 6.3. Associated connections will be ubiquitous in our study of infinitesimal geometric structures.

Let ∇ be an arbitrary *linear* (i.e., TM-) connection on a Lie algebroid $\mathfrak{g}$. The *associated $\mathfrak{g}$-connection* on $\mathfrak{g}$ is defined by

$$\bar\nabla_X Y = \nabla_{\#Y} X + [X,Y]_{\mathfrak{g}}; \qquad X,\, Y \subset \mathfrak{g}.$$

The *associated $\mathfrak{g}$-connection* on TM is defined by

$$\bar\nabla_X V = \#\nabla_V X + [\#X, V]_{TM}; \qquad X \subset \mathfrak{g},\, V \subset TM.$$

The anchor $\#\colon \mathfrak{g} \to TM$ is equivariant with respect to these connections: $\bar\nabla_X \#Y = \#\bar\nabla_X Y$. If $\mathfrak{g} = TM$ both connections coincide with the dual of ∇.

4. Cartan algebroids

This section summarizes some results of [1] where Cartan algebroids were introduced. We also define the *cocurvature* of an arbitrary linear connection on a Lie algebroid, a notion to reappear in the context of generators of infinitesimal geometric structures. Some further remarks linking *flat* Cartan algebroids with Lie pseudogroups appear in Appendix A.

4.1. **Cartan connections.** Let ∇ be a linear connection on a Lie algebroid $\mathfrak{g}$. Then ∇ is a *Cartan connection* if the corresponding splitting $s_\nabla = J^1\sigma + \nabla\sigma\colon \mathfrak{g} \to J^1\mathfrak{g}$ of the exact sequence of Lie algebroids,

$$0 \to T^*M \otimes \mathfrak{g} \hookrightarrow J^1\mathfrak{g} \to \mathfrak{g} \to 0,$$

is a Lie algebroid morphism. A *Cartan algebroid* is a Lie algebroid equipped with a Cartan connection. A *morphism* of Cartan algebroids is simply a connection-preserving morphism of the underlying Lie algebroids.

It follows immediately from the definition that the $\mathfrak{g}$-connections $\bar\nabla$ on $\mathfrak{g}$ and TM, associated with a Cartan connection ∇, are flat i.e, are $\mathfrak{g}$-representations. In particular, every Cartan algebroid $\mathfrak{g}$ has a canonical self-representation. For a converse statement, see the corollary below.

4.2. **Cocurvature.** Associated with an arbitrary linear connection ∇ on $\mathfrak{g}$ is what we call its *cocurvature*. This is a section of $\operatorname{Alt}^2(\mathfrak{g}) \otimes T^*M \otimes \mathfrak{g}$, denoted $\operatorname{cocurv}\nabla$, measuring the compatibility of ∇ with the bracket on $\mathfrak{g}$:

$$\begin{aligned} \operatorname{cocurv}\nabla\,(X,Y)V := {} & \nabla_V[X,Y] - [\nabla_V X, Y] - [X, \nabla_V Y] \\ & + \nabla_{\bar\nabla_X V} Y - \nabla_{\bar\nabla_Y V} X. \end{aligned} \tag{1}$$

Here $\bar\nabla$ denotes the associated $\mathfrak{g}$-connection on TM; see 3.6 above. The name 'cocurvature' comes from (5) below.

Define s_∇ as above. Then the description of the bracket on $J^1\mathfrak{g}$ in 3.5 establishes the result (2) below. The other conclusions of the following proposition are then straightforward.

Proposition.

(2) $-\operatorname{cocurv}\nabla(X,Y) = \operatorname{curv} s_\nabla(X,Y) := [s_\nabla X, s_\nabla Y]_{J^1\mathfrak{g}} - s_\nabla[X,Y]_\mathfrak{g}$.

(3) ∇ *is a Cartan connection if and only if* $\operatorname{cocurv}\nabla = 0$.

(4) *For any sections* $X, Y, Z \subset \mathfrak{g}$ *and* $V \subset TM$ *one has*

$$\operatorname{cocurv}\nabla(X,Y)\#Z = -\operatorname{curv}\bar{\nabla}(X,Y)Z,$$
$$\#\operatorname{cocurv}\nabla(X,Y)V = -\operatorname{curv}\bar{\nabla}(\#X,\#Y)V,$$

where $\bar{\nabla}$ *denotes the associated* $\mathfrak{g}$*-connection on* $\mathfrak{g}$ *in the first formula, and on* TM *in the second.*

(5) *In particular, if* $\mathfrak{g} = TM$*, then*

$$\operatorname{cocurv}\nabla = -\operatorname{curv}\bar{\nabla},$$

where $\bar{\nabla}$ *denotes the dual linear connection on* TM*.*

As simple consequences of (4) we have:

Corollary.

(6) *Suppose* $\mathfrak{g}$ *is transitive. Then* ∇ *is a Cartan connection on* $\mathfrak{g}$ *if and only if the associated* $\mathfrak{g}$*-connection* $\bar{\nabla}$ *on* $\mathfrak{g}$ *is flat.*

(7) *Suppose* $\mathfrak{g}$ *has an injective anchor. Then* ∇ *is a Cartan connection if and only if the associated* $\mathfrak{g}$*-connection* $\bar{\nabla}$ *on* TM *is flat.*

Although we shall make no use of the fact here, it is worth remarking that a Cartan connection ∇ on a transitive Lie algebroid $\mathfrak{g}$ is *uniquely* determined by the corresponding self-representation $\bar{\nabla}$; see [1, Proposition 6.1].

4.3. **Basic examples of Cartan algebroids.** We now list some elementary examples of Cartan algebroids. Example (7) explains the choice of name 'Cartan algebroid.'

(1) Every action algebroid $\mathfrak{g}_0 \times M$, equipped with its canonical flat connection ∇, is a Cartan algebroid. Locally this is the only flat example. See 4.5 below.

(2) As we sketch in Appendix A, every Lie pseudogroup of transformations in M has a flat Cartan algebroid as its 'infinitesimalization.'

(3) According to Proposition 4.2(5) a linear connection ∇ on TM is a Cartan connection if and only if its dual ∇^* is flat, i.e., is an infinitesimal parallelism on M. By duality, every Cartan connection on TM arises as the dual of some infinitesimal parallelism. See also 5.5.

(4) If M is a Lie group, then the flat linear connection ∇ on TM corresponding to left (or right) trivialization of TM is, as a special case of (3), a Cartan connection on TM.

(5) The distribution D tangent to some regular foliation $\mathcal{F}$ on M is a subalgebroid of TM. According to Corollary 4.2(7), a linear connection ∇ on D is a Cartan connection if and only if the D-connection $\bar{\nabla}$ on TM, defined by $\bar{\nabla}_U V = \nabla_V U + [U,V]$, $U \subset D$, $V \subset TM$, is flat.

(6) In 5.7 elementary arguments will show that every *torsion-free* linear connection ∇ on TM determines a Cartan connection on $J^1(TM)$. The parallel sections of this Cartan connection are the prolonged infinitesimal isometries of ∇.

(7) Let M be a classical Cartan geometry modeled on some homogeneous space G_0/H_0 (see, e.g., [18]). If $\pi\colon P \to M$ denotes the associated principal H_0-bundle, then according to [1], the vector bundle TP/H_0 is a Lie algebroid

supporting a Cartan connection determined by the classical Cartan connection on P.

4.4. **The symmetric part of a Cartan algebroid.** An arbitrary Cartan algebroid $\mathfrak{g}$ has a canonical subalgebroid isomorphic to an action Lie algebroid. Indeed, let ∇ denote the Cartan connection and let $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g})$ be the subspace of ∇-parallel sections, which is finite-dimensional. Then vanishing cocurvature ensures that $\mathfrak{g}_0 \subset \Gamma(\mathfrak{g})$ is a Lie subalgebra, and we obtain an action of $\mathfrak{g}_0$ on M given by

$$\begin{aligned} \mathfrak{g}_0 \times M &\to TM \\ (X, m) &\mapsto \#X(m). \end{aligned}$$

Equipping the action algebroid $\mathfrak{g}_0 \times M$ with its canonical flat connection, we obtain a morphism of Cartan algebroids,

$$\begin{aligned} \mathfrak{g}_0 \times M &\to \mathfrak{g} \\ (X, m) &\mapsto X(m). \end{aligned} \tag{1}$$

Assuming M is connected, this morphism is injective because ∇-parallel sections vanishing at a point vanish everywhere. We call the image of the monomorphism (1) the *symmetric part* of $\mathfrak{g}$.

4.5. **Curvature as the local obstruction to symmetry.** A Cartan algebroid $\mathfrak{g}$ is *globally flat* if it is isomorphic to an action algebroid $\mathfrak{g}_0 \times M$, equipped with its canonical flat connection — or, equivalently, if it coincides with its symmetric part. We call $\mathfrak{g}$ *flat* if every point of M has an open neighborhood U on which the restriction $\mathfrak{g}|_U$ is globally flat.[3]

The following theorem shows that a Cartan algebroid may be viewed as an infinitesimal symmetry deformed by curvature.

Theorem ([1])**.** *Let $\mathfrak{g}$ be a Cartan algebroid with Cartan connection ∇, defined over a connected manifold M. Then $\mathfrak{g}$ is flat if and only if* $\operatorname{curv} \nabla = 0$. *When M is simply-connected, flatness already implies global flatness.*

In the globally flat case the bracket on the Lie algebra $\mathfrak{g}_0$ of ∇-parallel sections is given by

$$[\xi, \eta]_{\mathfrak{g}_0} = \operatorname{tor} \bar{\nabla}(\xi, \eta) = 0, \tag{1}$$

where $\bar{\nabla}$ denotes the associated representation of $\mathfrak{g}$ on itself:

$$\bar{\nabla}_X Y = \nabla_{\#Y} X + [X, Y].$$

Proof. The necessity of vanishing curvature is immediate. To establish the assertions in the first paragraph it suffices to show that 4.4(1) is an isomorphism whenever $\operatorname{curv} \nabla = 0$ and M is simply-connected. Indeed, in that case ∇ determines a trivialization of the bundle $\mathfrak{g}$ in which constant sections correspond to the ∇-parallel sections of $\mathfrak{g}$ — that is, to elements of $\mathfrak{g}_0$. In particular, $\mathfrak{g}_0 \times M$ and $\mathfrak{g}$ will have the same rank, implying the monomorphism 4.4(1) is an isomorphism.

The formula (1) holds in the globally flat case because it holds for any action algebroid, as is readily established. (In 2.4 one has $\operatorname{tor} \bar{\nabla} = \tau$.) ☐

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[3]In [1] we used *symmetric* and *locally symmetric* in place of *globally flat* and *flat*, respectively.

Example. Every Lie group possesses a dual pair of flat linear connections ∇, ∇^* corresponding to the left and right trivializations of the tangent bundle (see 4.3(4) above). Conversely, whenever a simply-connected manifold M supports a linear connection ∇ on TM such that ∇ and its dual ∇^* are simultaneously flat, then ∇ is a flat Cartan connection on TM and the theorem above delivers an isomorphism $TM \cong \mathfrak{g}_0 \times M$, where $\mathfrak{g}_0$ is the Lie algebra of ∇-parallel vector fields. This isomorphism amounts to a $\mathfrak{g}_0$-valued Mauer-Cartan form on M, integrating to a Lie group structure under a suitable completeness hypothesis. The Lie bracket on $\mathfrak{g}_0$ is given by $[U, V] = -\operatorname{tor}\nabla(U, V)$.

For the application of Theorem 4.5 to examples 4.3(5) and 4.3(7), see [1].

5. Examples of infinitesimal geometric structures

This section describes several examples of infinitesimal geometric structures, including those associated with Riemannian structures, Poisson structures, affine structures, projective structures, arbitrary G-structures, and Cartan algebroids. From our description of affine structures, it will be clear how one may associate an infinitesimal geometric structure with an arbitrary (but suitably regular) differential operator on M. Since every classical Cartan geometry has an associated Cartan algebroid (see 4.3(7) above) every such geometry has an associated infinitesimal geometric structure as well. A simple example of a non-surjective infinitesimal geometric structure is given in 5.4.

Conformal structures and subriemannian contact structures are described separately in Sections 12 and 10.

5.1. **Isotropy.** Most infinitesimal geometric structures occurring in nature are best understood as isotropy (or *joint* isotropy) subalgebroids, of certain jet-bundle representations. In the case of Riemannian geometry it is the isotropy of the Riemannian metric $\sigma \subset \mathrm{Sym}^2(TM)$, i.e., of a *section* of some vector bundle; in conformal geometry, it is the isotropy of a rank-one *subbundle* $\langle\sigma\rangle \subset \mathrm{Sym}^2(TM)$; in projective geometry, it is the isotropy of an *affine* subbundle of $J^1(TM)^* \otimes T^*M \otimes TM$. The following definition of isotropy is general enough to cover all these possibilities.

Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(E)$ denote some representation of a Lie algebroid $\mathfrak{g}$. Let $\Sigma \subset E$ denote any affine subbundle of E (a single section of E in the simplest case) and $\Sigma_0 \subset E$ the corresponding vector subbundle parallel to Σ (resp., the zero section). Then the *isotropy* of Σ is the collection of all elements $x \in \mathfrak{g}$ for which

$$\sigma \subset \Sigma \implies \rho_x\sigma \in \Sigma_0,$$

for arbitrary local sections $\sigma \subset E$; here $\rho_x\sigma := \rho(x)(J^1\sigma(m))$ where $m \in M$ is the base point of x.

The isotropy of Σ is a subset of $\mathfrak{g}$ intersecting fibers in subspaces whose dimensions may vary, i.e., is a 'variable-rank subbundle'; sections of this bundle are closed under the bracket of $\mathfrak{g}$. When this rank is constant the isotropy is a bona fide subbundle and consequently a subalgebroid, called the *isotropy subalgebroid* of Σ. A section X of $\mathfrak{g}$ is then a section of the isotropy if and only if $\rho_X\sigma$ is a section of Σ_0 for all sections $\sigma \subset \Sigma$.

5.2. **On structure kernels.** Let $\mathfrak{h}$ denote its structure kernel of an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$. Since $J^1\mathfrak{t} \to \mathfrak{t}$ has kernel $T^*M \otimes \mathfrak{t}$, this kernel contains $\mathfrak{h}$. If $\mathfrak{h}$ has constant rank, then $\mathfrak{h} \subset T^*M \otimes \mathfrak{t}$ is a subalgebroid. Since $T^*M \otimes \mathfrak{t}$ is totally intransitive, this simply means $\mathfrak{h}(m)$ is a subalgebra of $T^*_m M \otimes \mathfrak{t}(m)$ at each point $m \in M$, which is also true in the variable rank case. In classical parlance, each fiber of $\mathfrak{h}$ is a *tableau* [2]. Recall from Proposition 3.5 that the bracket on $T^*M \otimes \mathfrak{t}$ is given by

$$[\phi_1, \phi_2]_{T^*M\otimes\mathfrak{t}}(U) = \phi_1(\#\phi_2 U) - \phi_2(\#\phi_1 U); \qquad U \subset TM.$$

5.3. **Riemannian structures.** The adjoint representation of $J^1(TM)$ on TM determines a representation of $J^1(TM)$ on $\mathrm{Sym}^2(TM)$: A section $X \subset J^1(TM)$ acts on a section $\sigma \subset \mathrm{Sym}^2(TM)$ according to

$$(1) \qquad (X \cdot \sigma)(V_1, V_2) := \mathcal{L}_{\#X}(\sigma(V_1, V_2)) - \sigma(\mathrm{ad}^{TM}_X V_1, V_2) - \sigma(V_1, \mathrm{ad}^{TM}_X V_2).$$

Since this simply means $J^1V \cdot \sigma = \mathcal{L}_V \sigma$, the isotropy $\mathfrak{g} \subset J^1(TM)$ of a Riemannian metric $\sigma \subset \mathrm{Sym}^2(TM)$ is indeed its associated bundle of 1-symmetries, as described in 2.1.

According to (1), the subalgebroid $T^*M \otimes TM \subset J^1(TM)$ acts on $\mathrm{Sym}^2(TM)$ via

$$(\phi \cdot \sigma)(V_1, V_2) = -\sigma(\phi V_1, V_2) - \sigma(V_1, \phi V_2); \qquad \phi \subset T^*M \otimes TM.$$

The structure kernel of $\mathfrak{g}$ is the isotropy $\mathfrak{h} \subset T^*M \otimes TM$ of σ under this restricted representation. So $\mathfrak{h}$ is the bundle of σ-skew-symmetric endomorphisms of tangent spaces, a Lie algebra bundle modeled on $\mathfrak{o}(n)$, $n := \dim M$. In particular, all conformally equivalent metrics give the same structure kernel.

One way to see that $\mathfrak{g}$ is surjective (i.e., transitive) is to apply Lemma B.1 to the morphism $X \mapsto X \cdot \sigma \colon J^1(TM) \to \mathrm{Sym}^2(TM)$, whose kernel is $\mathfrak{g}$. On account of the surjectivity of the restriction $\phi \mapsto \phi \cdot \sigma \colon T^*M \otimes TM \to \mathrm{Sym}^2(TM)$ of this morphism, the lemma delivers an exact sequence

$$(2) \qquad 0 \to \mathfrak{h} \to \mathfrak{g} \to TM \to 0.$$

Thus $\mathfrak{g} \subset J^1(TM)$ is surjective and has constant rank (making it a subalgebroid and thus an infinitesimal geometric structure).

The lemma just applied is very useful in determining the image and structure kernel of infinitesimal geometric structures defined by isotropy. A less trivial application of Lemma B.1 appears in 5.4 below. Minimal comment will accompany subsequent applications.

The symmetries of $\mathfrak{g}$ (in the sense of 2.6) are the vector fields along which σ has vanishing Lie derivative, i.e., its Killing fields. A generator of $\mathfrak{g}$ is a linear connection ∇ on TM such that σ is $\bar{\nabla}$-parallel, where $\bar{\nabla}$ denotes the dual of ∇. The Levi-Cevita connection is thus the unique torsion-free generator of $\mathfrak{g}$.

From $\mathfrak{g}$ one can recover the metric σ up to a positive *constant* (not merely its conformal class). In the simply-connected case, slightly more is true:

Proposition. *Let $\mathfrak{h} \subset T^*M \otimes TM$ denote the $\mathfrak{o}(n)$-bundle associated with an arbitrary conformal structure. Then on simply-connected open subsets of M, every surjective infinitesimal geometric structure $\mathfrak{g} \subset J^1(TM)$ having structure kernel $\mathfrak{h}$ is the isotropy subalgebroid of some Riemannian structure σ. This structure is uniquely determined up to a constant.*

Proof. Suppose $\mathfrak{g} \subset J^1(TM)$ has structure kernel $\mathfrak{h}$ and let $L \subset \mathrm{Sym}^2(TM)$ be the line bundle determined by the conformal structure. That is, L is the bundle of $\mathfrak{h}$-invariant elements of $\mathrm{Sym}^2(TM)$. The non-vanishing elements of L are either positive or negative definite. By Lemma B.2, L has a non-vanishing $\mathfrak{g}$-invariant section σ, unique up to constant. Changing the sign of σ if necessary, we obtain the sought after metric. □

The application of Cartan's method to Riemannian structures is given in 9.6.

In analogy with the Riemannian case, the isotropy subset $\mathfrak{g} \subset J^1(TM)$ of an arbitrary tensor on M is an infinitesimal geometric structure, whenever this isotropy has constant rank. Moreover, in important cases (e.g., complex and symplectic structures), this structure encodes all useful information (i.e., some analogue of the preceding proposition applies.)

For structures defined by more than one tensor one considers the *joint* isotropy, defined by intersecting the individual isotropies. An example is almost Hermitian structures. Here is another:

5.4. **Vector fields on Riemannian manifolds.** Let V be a *nowhere vanishing* vector field on a Riemannian manifold M with metric σ. The vector fields on M that are simultaneously infinitesimal isometries of $\sigma \subset \mathrm{Sym}^2(TM)$ and $V \subset TM$ are the symmetries of the joint isotropy of σ and V, with respect to representations of $J^1(TM)$ on $\mathrm{Sym}^2(TM)$ and TM respectively. Denoting the isotropy of σ alone by $\mathfrak{g} \subset J^1(TM)$ as above, $\mathfrak{g}$ acts on TM by restricted adjoint action and the joint isotropy is the isotropy $\mathfrak{g}_V \subset \mathfrak{g}$ of V.

The structure kernel of $\mathfrak{g}_V$ is the $\mathfrak{o}(n-1)$-bundle of skew-symmetric tangent space endomorphisms infinitesimally fixing V (mapping V to 0).

Proposition. *The image of $\mathfrak{g}_V$ is the distribution $D \subset TM$ tangent to the level sets of $\frac{1}{2}\|V\|^2$.*

In particular, $\mathfrak{g}$ has constant rank (is an infinitesimal geometric structure) if and only if V has constant length or $\frac{1}{2}\|V\|^2$ is a free of critical points; $\mathfrak{g}_V$ is surjective (transitive) in former case only.

Proof. Applying Lemma B.1 to the morphism $X \mapsto X \cdot V \colon \mathfrak{g} \to TM$, of which $\mathfrak{g}_V$ is the kernel, we deduce that D is the kernel of the morphism Θ making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\#} & TM \\ {\scriptstyle X \mapsto X \cdot V}\big\downarrow & & \big\downarrow{\scriptstyle \Theta} \\ TM & \longrightarrow & TM/V^{\perp} \end{array} .$$

Let ∇ be any generator of $\mathfrak{g}$ (e.g., the Levi-Cevita connection) and $s \colon TM \to \mathfrak{g}$ the corresponding splitting of (2) above. Then $\Theta(U) = sU \cdot V \mod V^{\perp} = \bar{\nabla}_U V \mod V^{\perp}$, where $\bar{\nabla}$ is the dual connection. Or, identifying $TM/V^{\perp}$ with the trivial line bundle $\mathbb{R} \times M$, using V and the metric, we have $\Theta(U) = \sigma(V, \bar{\nabla}_U V) = \frac{1}{2}\bar{\nabla}_U(\sigma(V,V)) = d(\frac{1}{2}\|V\|^2)(U)$. Here we have used $\bar{\nabla}\sigma = 0$, which holds because ∇ is a generator. □

5.5. **Parallelism.** The simplest non-trivial example of an infinitesimal geometric structure is a transitive infinitesimal geometric structure $\mathfrak{g} \subset J^1(TM)$ having trivial structure kernel. In other words, $\mathfrak{g}$ is a subalgebroid of $J^1(TM)$ mapped isomorphically onto TM by the anchor $\# \colon J^1(TM) \to TM$. According to Theorem 2.7, $\mathfrak{g}$ has a unique generator ∇ that is a Cartan connection on TM. From our observations in 4.3(3), the dual connection $\bar{\nabla}$ is flat, i.e., an infinitesimal parallelism on M, and conversely all infinitesimal parallelisms arise in its way.

When M is simply-connected the Lie algebroid morphism $\bar{\nabla}\colon TM \to \mathfrak{gl}(TM)$ integrates to a Lie groupoid morphism $M \times M \to \mathrm{GL}(TM)$, i.e., to an *absolute* parallelism on M (a trivialization of the tangent bundle).

The symmetries of $\mathfrak{g}$ are the ∇-parallel vector fields. When $\bar{\nabla}$ comes from an absolute parallelism, viewed as some non-degenerate vector-valued 1-form ω on M, then $\mathfrak{g} \subset J^1(TM)$ is the isotropy subalgebroid of ω, and a symmetry of $\mathfrak{g}$ is a vector field along which ω has vanishing Lie derivative.

5.6. **Poisson structures.** In general, the isotropy $\mathfrak{g} \subset J^1(TM)$ of a Poisson tensor on M fails to have constant rank, and so fails to be an infinitesimal geometric structure on TM. Nevertheless, one can define an infinitesimal geometric structure on the *cotangent* bundle T^*M, which the Poisson tensor makes into a Lie algebroid (see below). Although not transitive, this structure *is* surjective.

Let ω be a symplectic structure on M and let $\#\colon T^*M \to TM$ denote the inverse of $v \mapsto \omega(v, \cdot)$. Since $\#$ is an isomorphism, there is a unique bracket on $\Gamma(T^*M)$ making T^*M into a Lie algebroid with anchor $\#$. This bracket is given by

$$[\alpha, \beta]_{T^*M} = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha + d(\Pi(\alpha, \beta)), \qquad \alpha, \beta \in \Gamma(T^*M), \tag{1}$$

where $\mathcal{L}$ denotes Lie derivative and Π is the Poisson tensor. This tensor is defined by $\Pi(\alpha, \beta) := \omega(\#\alpha, \#\beta)$ and so satisfies

$$\langle \alpha, \#\beta \rangle = \Pi(\alpha, \beta) \qquad \alpha, \beta \in \Gamma(T^*M). \tag{2}$$

More generally, (1) defines a Lie algebroid structure on T^*M for *any* Poisson manifold (M, Π), with anchor $\#$ defined by (2). The symplectic leaves of Π are precisely the orbits of the Lie algebroid T^*M.

An infinitesimal isometry of a Poisson manifold (M, Π) is a vector field V on M such that $\mathcal{L}_V\Pi = 0$. Poisson manifolds have an abundance of infinitesimal isometries. In particular, every closed 1-form α on M determines an infinitesimal isometry $\#\alpha$ tangent to the symplectic leaves known as a *local Hamiltonian vector field*, or a *Hamiltonian vector field* if α is exact.

It is not too difficult to establish the following result; see [1] for some details:

Proposition. *Let $\mathfrak{g} \subset J^1(T^*M)$ denote the kernel of the vector bundle morphism $J^1(T^*M) \to \mathrm{Alt}^2(TM)$ whose corresponding map on sections sends $J^1\alpha$ to $d\alpha$. Then $\mathfrak{g}$ is a surjective infinitesimal geometric structure on T^*M, with structure kernel $\mathrm{Sym}^2(TM)$, whose symmetries are the closed one-forms on M.*

A linear connection ∇ on T^*M is a generator of $\mathfrak{g}$ if and only if the corresponding linear connection on TM is torsion free. Such a generator is a Cartan connection on T^*M if and only if

$$\operatorname{curv} \nabla\,(V, \#\alpha)\beta - \operatorname{curv} \nabla\,(V, \#\beta)\alpha - (\nabla_V(\nabla\Pi))(\alpha, \beta) = 0,$$

for all sections $\alpha, \beta \subset T^*M; V \subset TM$.

If M is the dual of a Lie algebra, equipped with its Lie-Poisson structure (see, e.g., [16, §10.1]), then the canonical flat linear connection ∇ on $T^*M \cong M \times M^*$ is an example of a Cartan connection as described in the proposition. Up to certain momentum map equivariance obstructions, this is locally the only flat example [1, Corollary 3.4].

5.7. **Affine structures.** Any suitably non-degenerate, kth-order, linear, differential operator on M, defines an infinitesimal geometric structure $\mathfrak{g} \subset J^{k+1}(TM) \subset J^1(J^k(TM))$. As a simple example, which will suffice to illustrate the general principle, we consider an affine structure on M, i.e., an arbitrary linear connection ∇ on M, in which case $k = 1$. The relevant non-degeneracy condition is that the isotropy of the torsion of ∇ should have constant rank; see below.

View an affine structure ∇ as a section of $J^1(TM)^* \otimes T^*M \otimes TM$, via

$$\nabla(J^1W, V) := \nabla_V W; \qquad V, W \subset TM.$$

In order to associate a natural isotropy subalgebroid with ∇, we begin with two observations. First, $J^1(J^1(TM))$ acts on $J^1(TM)^* \otimes T^*M \otimes TM$ because $J^1(J^1(TM))$ acts on $J^1(TM)$ via adjoint action, and on TM via the composite

$$J^1(J^1(TM)) \xrightarrow{p} J^1(TM) \xrightarrow{\mathrm{ad}^{TM}} \mathfrak{gl}(TM),$$
$$\text{i.e.,} \quad J^1X \cdot W = \mathrm{ad}^{TM}_{pX}\, W; \qquad X \subset J^1(TM),\ W \subset TM.$$

Secondly, $J^2(TM)$ may be identified with a subalgebroid of $J^1(J^1(TM))$ via the canonical embedding $J^2(TM) \hookrightarrow J^1(J^1(TM))$ whose corresponding map on sections sends J^2V to $J^1(J^1V)$. Combining the two observations, we obtain a natural action of $J^2(TM)$ on $J^1(TM)^* \otimes T^*M \otimes TM$.

Proposition. *Let* $\mathfrak{g} \subset J^2(TM)$ *denote the isotropy of* $\nabla \subset J^1(TM)^* \otimes T^*M \otimes TM$*, and* $\mathfrak{t} \subset J^1(TM)$ *the isotropy of* $\mathrm{tor}\,\nabla \subset \mathrm{Alt}^2(TM) \otimes TM$*. Then:*

(1) *The symmetries of* $\mathfrak{g}$ *are the prolonged infinitesimal isometries of* ∇*.*
(2) *The image of* $\mathfrak{g} \subset J^1(J^1(TM))$ *is* $\mathfrak{t}$ *and* $\mathfrak{g}$ *has trivial structure kernel.*

In particular, (2) implies that $\mathfrak{g} \subset J^2(TM)$ has constant rank (and is therefore an infinitesimal geometric structure on $J^1(TM)$) if and only if $\mathfrak{t} \subset J^1(TM)$ has constant rank.

Now $\mathfrak{t} = J^1(TM)$ if and only if ∇ is torsion-free (because id_{TM} is a section of $T^*M \otimes TM \subset J^1(TM)$). On account of (2), $\mathfrak{g}$ is surjective if and only if ∇ is torsion-free. Applying Theorem 2.7, we obtain:

Corollary. *If* $\mathrm{tor}\,\nabla = 0$*, then the unique generator* $\nabla^{(1)}$ *of* $\mathfrak{g}$ *is a Cartan connection on* $J^1(TM)$ *whose parallel sections are the prolonged infinitesimal isometries of* ∇*.*

From an explicit formula for $\nabla^{(1)}$ one may completely characterize the obstructions to the existence of infinitesimal isometries; see 11.5.

Proof of proposition. Recall that a vector field U on M is an *infinitesimal isometry* of ∇ if

$$(3) \qquad [U, \nabla_V W] - \nabla_{[U,V]} W - \nabla_V [U, W] = 0; \qquad V, W \subset TM,$$

a condition that is second-order in U. Unravelling the definition of the representations defined above, we may write this condition as

$$J^1(J^1(U)) \cdot \nabla = 0.$$

It easily follows that J^1U is a symmetry of $\mathfrak{g}$ whenever U is an infinitesimal isometry of ∇.

Suppose, conversely, that $X \subset J^1(TM)$ is a symmetry of $\mathfrak{g}$, i.e., that J^1X lies in $\mathfrak{g}$. This means:

$$J^1X \subset J^2(TM), \tag{4}$$

$$\text{and } \; J^1X \cdot \nabla = 0. \tag{5}$$

It is well known that (4) is equivalent to $X \subset J^1(TM)$ being holonomic (see, e.g., the example concluding 8.2). So $X = J^1U$, where U is an infinitesimal isometry, on account of (5), which reads $J^1(J^1U) \cdot \nabla = 0$. This completes the proof of (1).

Let ξ be any section of $J^2(TM)$. It is easy to check that the section $\xi \cdot \nabla \subset J^1(TM)^* \otimes T^*M \otimes TM$ is tensorial, i.e., drops to some section $(\xi \cdot \nabla)^\vee \subset T^*M \otimes T^*M \otimes TM$. Noting that $\mathfrak{g}$ is then the kernel of the morphism

$$\begin{gathered} \xi \mapsto (\xi \cdot \nabla)^\vee \\ J^2(TM) \to T^*M \otimes T^*M \otimes TM, \end{gathered}$$

whose domain $J^2(TM)$ fits into an exact sequence

$$0 \to \mathrm{Sym}^2(TM) \otimes TM \hookrightarrow J^2(TM) \to J^1(TM) \to 0,$$

one shows, by applying Lemma B.1, that $\mathfrak{g}$ fits into a corresponding exact sequence

$$0 \to 0 \to \mathfrak{g} \xrightarrow{b} \mathfrak{t} \to 0.$$

Here b is the restriction of the canonical projection $J^2(TM) \to J^1(TM)$. This establishes (2). $\square$

5.8. **Projective structures.** Recall that two linear connections ∇, ∇' are *projectively equivalent* if their geodesics coincide as unparameterized curves. Equivalently their difference $\nabla - \nabla'$, which may be viewed as a section of

$$T^*M \otimes T^*M \otimes TM \subset J^1(TM)^* \otimes T^*M \otimes TM,$$

should take its values in the subbundle $\Sigma_0 := (\mathrm{Alt}^2(TM) \otimes TM) \oplus j_{\mathrm{S}}(T^*M)$ of

$$T^*M \otimes T^*M \otimes TM \cong \Big(\mathrm{Alt}^2(TM) \otimes TM \Big) \oplus \Big(\mathrm{Sym}^2(TM) \otimes TM \Big).$$

Here $j_{\mathrm{S}} \colon T^*M \to \mathrm{Sym}^2(TM) \otimes TM$ is the embedding defined by

$$j_{\mathrm{S}}(\alpha)(V_1, V_2) := \alpha(V_1)V_2 + \alpha(V_2)V_1.$$

A *projective structure* is a projective equivalence class of linear connections; since Σ_0 contains $\mathrm{Alt}^2(TM) \otimes TM$, each such class has a torsion-free representative ∇.

Let ∇ be a torsion-free linear connection on TM and $\langle \nabla \rangle$ the corresponding projective structure. To specify the structure it suffices to specify the affine subbundle

$$\nabla + \Sigma_0 \subset J^1(TM)^* \otimes T^*M \otimes TM,$$

which we denote by $\langle \nabla \rangle$ also. As explained in 5.7 above, we have $J^2(TM) \subset J^1(J^1(TM))$ acting on $J^1(TM)^* \otimes T^*M \otimes TM$ and can therefore define the isotropy $\mathfrak{g} \subset J^2(TM)$ of $\langle \nabla \rangle$; see 5.1. Arguing as in the proof of Lemma 5.7(1), one shows that the symmetries of $\mathfrak{g}$ are the prolonged infinitesimal isometries of $\langle \nabla \rangle$.

It is not hard to see that $\mathfrak{g}$ has $j_{\mathrm{S}}(T^*M) \cong T^*M$ as structure kernel and in fact that

$$\mathfrak{g} = \mathfrak{g}_\nabla \oplus j_{\mathrm{S}}(T^*M),$$

where $\mathfrak{g}_\nabla \subset J^2(TM)$ denotes the isotropy of ∇ (denoted $\mathfrak{g}$ in 5.7). The Cartan connection $\nabla^{(1)}$ on $J^1(TM)$ in Corollary 5.7 is a generator of $\mathfrak{g}_\nabla$ and consequently a generator of $\mathfrak{g}$ as well. An explicit formula appears in 11.5.

5.9. ***G*-structures.** Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$, where n is the dimension of M. A *G-structure* on M is a G-reduction P of the bundle of (absolute) frames on M; see, e.g., [13]. In particular, P is a principal G-bundle, so that $\mathfrak{g} := TP/G$ is a transitive Lie algebroid over M, and the associated vector bundles of P are representations of $\mathfrak{g}$; see, e.g., [15]. As P is a frame bundle, TM will be such a representation (see also below). That is, we have a Lie algebroid morphism

$$\mathfrak{g} \to \mathfrak{gl}(TM) \overset{\mathrm{ad}}{\cong} J^1(TM).$$

This turns out to be injective, identifying $\mathfrak{g}$ with a subalgebroid of $J^1(TM)$. This infinitesimal geometric structure on TM is surjective because $\mathfrak{g}$ is transitive.

The representation of $\mathfrak{g}$ on TM may be described as follows. Identify sections X of $\mathfrak{g} := TP/G$ with G-invariant vector fields on P, and use the tautological 1-form on P to identify sections V of TM with G-invariant $\mathbb{R}^n$-valued functions on P. Then $X \cdot V := \mathcal{L}_X V$, where $\mathcal{L}$ denotes Lie derivative on P.

5.10. **Cartan algebroids as infinitesimal geometric structures.** We have seen that all surjective infinitesimal geometric structures with trivial structure kernel define Cartan algebroids (Theorem 2.7). Conversely, if $\mathfrak{t}$ is a Cartan algebroid with Cartan connection ∇, then $\mathfrak{g} := s_\nabla(\mathfrak{t}) \subset J^1\mathfrak{t}$ is a surjective infinitesimal geometric structure generated by ∇ with trivial structure kernel. Here $s_\nabla \colon \mathfrak{t} \to J^1\mathfrak{t}$ is the splitting of

$$0 \to T^*M \otimes \mathfrak{t} \hookrightarrow J^1\mathfrak{t} \to \mathfrak{t} \to 0$$

determined by ∇.

6. Generators and associated differential operators

Picking a generator for an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ allows us to identify $\mathfrak{g}$ with the direct sum $\mathfrak{t}_1 \oplus \mathfrak{h}$ of its image $\mathfrak{t}_1$ and its structure kernel $\mathfrak{h}$. This greatly facilitates computations. Generators are also the appropriate connections for which to develop all the usual formalisms of differential geometry: covariant differentiation, covariant exterior differentiation, Bianchi identities, etc. (It will be natural, however, to use the more encompassing descriptor 'associated' in place of 'covariant.') By virtue of 5.10, we obtain formalism for Cartan algebroids as a special case.

The present section is rather formal in nature. In 6.1 we address the existence and uniqueness of generators and prove the theorem in 2.7, where generators were defined. In 6.2 we see how information about $\mathfrak{g}$ is encoded in $\mathfrak{t}_1$, $\mathfrak{h}$, and ∇. The basic 'algebraic' invariants of an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ are the vector bundles occurring as representations of $\mathfrak{g}$. Associated with these representations, and a choice of generator ∇, are the *associated connections* and *associated differential operators*, described in 6.3. The latter generalize the divergence, gradient, etc. of Riemannian geometry when ∇ is the Levi-Cevita connection. In Sect. 10 we describe these objects for subriemannian contact three-manifolds. In principle, any invariant differential operator in a transitive geometry may be expressed in terms of associated differential operators. In 6.4 we describe the *associated exterior derivative*, and in 6.5 analogues of the classical Bianchi identities.

6.1. **Basic properties of generators.** Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure, with structure kernel $\mathfrak{h}$, and image $\mathfrak{t}_1 \subset \mathfrak{t}$. The projection $\mathfrak{g} \xrightarrow{a} \mathfrak{t}$ is of constant rank if and only if $\mathfrak{h} \subset T^*M \otimes \mathfrak{t}$ and $\mathfrak{t} \subset \mathfrak{t}_1$ are subalgebroids.

Proposition. *If* $\mathfrak{g} \xrightarrow{a} \mathfrak{t}$ *has constant rank then:*

(1) $\mathfrak{g}$ *admits a generator* ∇.
(2) ∇ *is unique if and only if* $\mathfrak{g}$ *is surjective and* $\mathfrak{h} = 0$.
(3) *Every* ∇*-parallel section of* $\mathfrak{t}_1$ *is a symmetry of* $\mathfrak{g}$.

Proof of proposition and Theorem 2.7. The constant rank hypothesis means that

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{a} \mathfrak{t}_1 \longrightarrow 0$$

is an exact sequence of vector bundles. Assuming M is paracompact, it possesses a splitting $s\colon \mathfrak{t}_1 \to \mathfrak{g}$ which can be extended to a splitting $s\colon \mathfrak{t} \to J^1\mathfrak{t}$ of

$$0 \to T^*M \otimes \mathfrak{t} \hookrightarrow J^1\mathfrak{t} \to \mathfrak{t} \to 0. \tag{4}$$

To prove (1), let ∇ be the corresponding linear connection on $\mathfrak{t}$.

Conclusion (2) follows readily from the correspondence between connections on $\mathfrak{t}$ and splitting of (4). To prove (3), let $s\colon \mathfrak{t} \to J^1\mathfrak{t}$ be the splitting corresponding to a generator ∇, i.e., $sV = J^1V + \nabla V$. Then if $V \subset \mathfrak{t}_1$ is ∇-parallel then $J^1V = sV$. Since sV lies in $\mathfrak{g}$, by the definition of generators, we conclude V is a symmetry.

Assume ∇ is a generator and $\mathfrak{h} = 0$ Suppose that $V \subset \mathfrak{t}$ is a symmetry, i.e., $J^1V = sV - \nabla V$ is a section of $\mathfrak{g}$. Then $sV \subset \mathfrak{g}$ because ∇ is a generator, implying $\nabla V \subset \mathfrak{g}$. So $\nabla V \subset (T^*M \otimes \mathfrak{t}) \cap \mathfrak{g} = \mathfrak{h} = 0$. Symmetries are thus ∇-parallel. This, together with (3), establishes Theorem 2.7. □

In the remainder of this section it is tacitly assumed that all infinitesimal geometric structures have constant rank in the sense above.

6.2. **Reconstructing geometric structures from generators.** Knowing the structure kernel $\mathfrak{h}$, image $\mathfrak{t}_1$, and a generator ∇ of an infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ determines it completely: the splitting determined by the generator determines a vector bundle isomorphism $\mathfrak{g} \cong \mathfrak{t}_1 \oplus \mathfrak{h}$ and the induced Lie algebroid structure on $\mathfrak{t}_1 \oplus \mathfrak{h}$ can be explicitly written down; see (4) below.

It is not difficult to characterize those linear connections ∇ on $\mathfrak{t}$ occurring as generators of infinitesimal geometric structures. Let $\mathfrak{t}$ be an arbitrary Lie algebroid, $\mathfrak{h}$ a subalgebroid of $T^*M \otimes \mathfrak{t} \subset J^1\mathfrak{t}$, and ∇ an arbitrary linear connection on $\mathfrak{t}$. Let $\mathfrak{t}_1 \subset \mathfrak{t}$ be an arbitrary subalgebroid. We define a $\mathfrak{t}_1$-connection $\bar\nabla$ on $T^*M \otimes \mathfrak{t}$ in the obvious way:

$$(\bar\nabla_V \phi)(U) = \bar\nabla_V(\phi(U)) - \phi(\bar\nabla_V(U)); \quad V \subset \mathfrak{t}_1,\, \phi \subset \mathfrak{h},\, U \subset TM. \tag{1}$$

Here $\bar\nabla$ on the right-hand side denotes the associated $\mathfrak{t}$-connections on $\mathfrak{t}$ and TM respectively, as defined in 3.6. From the formula $s_\nabla V = J^1V + \nabla V$ and the characterization of cocurvature 4.2(2), one readily obtains:

Proposition. *A linear connection* ∇ *on a Lie algebroid* $\mathfrak{t}$ *is a generator of some reduced infinitesimal geometric structure* $\mathfrak{g} \subset \mathfrak{t}$ *with structure kernel* $\mathfrak{h}$ *and image* $\mathfrak{t}_1 \subset \mathfrak{t}$ *if and only if:*

(2) $\mathfrak{h} \subset T^*M \otimes \mathfrak{t}$ *is* $\bar\nabla$*-invariant, i.e.,* $\bar\nabla_V \phi \subset \mathfrak{h}$ *for all sections* $V \subset \mathfrak{t}_1$ *and* $\phi \subset \mathfrak{h}$*; and*
(3) $\operatorname{cocurv} \nabla\, (V_1, V_2) \subset \mathfrak{h}$ *for all sections* $V_1, V_2 \subset \mathfrak{t}_1$.

If $\mathfrak{g} \subset J^1\mathfrak{t}$ *is such an infinitesimal geometric structure, then the Lie algebroid structure of* $\mathfrak{g} \cong \mathfrak{t}_1 \oplus \mathfrak{h}$ *is given by*

$$(4) \qquad \begin{cases} \#(V \oplus \phi) = \#V \\ [V_1 \oplus \phi_1, V_2 \oplus \phi_2] = \\ [V_1, V_2]_{\mathfrak{t}_1} \oplus ([\phi_1, \phi_2]_{\mathfrak{h}} + \bar{\nabla}_{V_1}\phi_2 - \bar{\nabla}_{V_2}\phi_1 - \operatorname{cocurv}\nabla(V_1, V_2)). \end{cases}$$

We recall that cocurvature was defined in 4.2.

6.3. **Associated connections and differential operators.** Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure with structure kernel $\mathfrak{h}$, image $\mathfrak{t}_1 \subset \mathfrak{t}$, and ∇ a generator of $\mathfrak{g}$. Then for each representation E of $\mathfrak{g}$, i.e., for each Lie algebroid morphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(E)$, we have an *associated* $\mathfrak{t}_1$-connection $\bar{\nabla}$ on E (an associated $\mathfrak{t}$-connection if $\mathfrak{g}$ is surjective). By definition, this is the composite $\mathfrak{t}_1 \xrightarrow{s_\nabla} \mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(E)$, where $s_\nabla\colon \mathfrak{t} \to J^1\mathfrak{t}$ is the splitting of 6.1(4) corresponding to ∇.

Examples.

(1) Taking $\mathfrak{g} := J^1\mathfrak{t}$ and $\rho = \operatorname{ad}^{\mathfrak{t}}$, we obtain

$$\bar{\nabla}_U V = \operatorname{ad}^{\mathfrak{t}}_{s_\nabla U} V = \operatorname{ad}^{\mathfrak{t}}_{J^1 U} V + \operatorname{ad}^{\mathfrak{t}}_{\nabla U} V,$$
$$\text{i.e.,} \quad \bar{\nabla}_U V = \nabla_{\# V} U + [U, V]_{\mathfrak{t}}; \quad U, V \subset \mathfrak{t}.$$

This is the associated $\mathfrak{t}$-connection on $\mathfrak{t}$ defined already in 3.6.

(2) Let $\mathfrak{g} := J^1\mathfrak{t}$ act on TM via the composite

$$J^1\mathfrak{t} \xrightarrow{J^1\#} J^1(TM) \xrightarrow{\operatorname{ad}^{TM}} \mathfrak{gl}(TM).$$

Then we similarly compute

$$\bar{\nabla}_U W = \#\nabla_W U + [\#U, W]_{TM}; \qquad U \subset \mathfrak{t},\, W \subset TM.$$

This is the associated $\mathfrak{t}$-connection on TM defined in 3.6.

(3) An arbitrary infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$ acts on its structure kernel $\mathfrak{h}$ via bracket: $\rho_X Y := [X, Y]_{\mathfrak{g}}$. The associated $\mathfrak{t}_1$-connection on $\mathfrak{h}$ is simply the connection $\bar{\nabla}$ appearing in Proposition 6.2 and satisfying 6.2(1).

(4) If $\mathfrak{g} \subset J^1\mathfrak{t}$ is an infinitesimal geometric structure and ∇ a linear connection on $\mathfrak{g}$ (rather than $\mathfrak{t}$) then ∇ generates $J^1\mathfrak{g}$, which acts on $\mathfrak{t}$ via the composite

$$J^1\mathfrak{g} \xrightarrow{J^1 a} J^1\mathfrak{t} \xrightarrow{\operatorname{ad}^{\mathfrak{t}}} \mathfrak{gl}(\mathfrak{t}),$$

where $a\colon \mathfrak{g} \to \mathfrak{t}$ is the projection. The associated $\mathfrak{g}$-connection $\bar{\nabla}$ on $\mathfrak{t}$ is given by $\bar{\nabla}_X W = a\nabla_{\# W} X + [aX, W]_{\mathfrak{t}}$. For an application, see 11.1.

Let a $\mathfrak{g}$-*tensor* be any section $\sigma \subset E$ of a $\mathfrak{g}$-representation E. Then, by the definition of the associated connections, we have:

Proposition. *A* $\mathfrak{g}$*-tensor is* $\mathfrak{g}$*-invariant if and only if it is simultaneously* $\mathfrak{h}$*-invariant and* $\bar{\nabla}$*-parallel.*

The *associated derivative* of a $\mathfrak{g}$-tensor $\sigma \in \Gamma(E)$ is defined to be $\bar{\nabla}\sigma \in \Gamma(\mathfrak{t}_1^* \otimes E)$, where $\bar{\nabla}$ is the $\mathfrak{t}_1$-associated connection on E. Assume $\mathfrak{g}$ is *image-reduced*; by this we mean that $\mathfrak{t}_1 \subset \mathfrak{t}$ is invariant under the adjoint action of $\mathfrak{g} \subset J^1\mathfrak{t}$ (true if $\mathfrak{g}$ is surjective). Then $\mathfrak{t}_1$ is a $\mathfrak{g}$-representation, implying $\bar{\nabla}\sigma$ is another $\mathfrak{g}$-tensor. That is, the $\mathfrak{g}$-tensors will be closed under associated derivative. In particular, the derivative can be iterated to obtain higher order differential operators.

Additionally supposing that all $\mathfrak{g}$-representations may be direct-sum decomposed into $\mathfrak{g}$-representations coming from some collection E_i $(i \in I)$ of irreducible ones, we have

$$\mathfrak{t}_1^* \otimes E_i \cong E_{n_{i1}} \oplus E_{n_{i2}} \oplus E_{n_{i3}} \oplus \cdots \quad \text{(finitely many non-zero terms)},$$

for some $n_{ij} \in I$, and obtain a corresponding decomposition,

$$\bar{\nabla}|\Gamma(E_i) = \partial_{i1} \oplus \partial_{i2} \oplus \partial_{i3} \oplus \cdots \ .$$

We call the differential operators $\partial_{ij} : \Gamma(E_i) \to \Gamma(E_{n_{ij}})$ $(i, j \in I)$ the *associated differential operators*; all differential operators which can be constructed algebraically out of associated connections $\bar{\nabla}$ are combinations of these basic ones.

If there is a *canonical* way in which to choose the generator ∇ then the associated differential operators become *invariant* differential operators associated with the infinitesimal geometric structure $\mathfrak{g}$. Significant cases in point are:

(5) The case where $\mathfrak{t}$ is a Cartan algebroid discussed in 5.10. Here $\mathfrak{g}$-representations are just $\mathfrak{t}$-representations because $\mathfrak{g} \cong \mathfrak{t}$.
(6) The case where the generator ∇ of $\mathfrak{g}$ is *unique*, i.e., Theorem 2.7 applies, reducing the situation to case (5) above.
(7) The case where torsion $\operatorname{tor} \bar{\nabla}$ has a natural 'normalization'; see 8.3.

For invariant differential operators associated with subriemannian contact 3-manifolds, see Sect. 10.

6.4. **The associated exterior derivative.** Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure with structure kernel $\mathfrak{h}$. Then a *differential form of type* $\mathfrak{g}$ *and degree* k is a section $\theta \subset \operatorname{Alt}^k(\mathfrak{t}_1) \otimes E$, where $\mathfrak{t}_1 \subset \mathfrak{t}$ is the image of $\mathfrak{g}$ and E some $\mathfrak{g}$-representation. (We use $\mathfrak{t}_1$, rather than $\mathfrak{t}$, to ensure (2) below.) The *exterior derivative* $d_{\bar{\nabla}}\theta \subset \operatorname{Alt}^k(\mathfrak{t}_1) \otimes E$ of θ is defined in the obvious way. For example,

$$d_{\bar{\nabla}}\theta\,(U_1) := \bar{\nabla}_{U_1}\theta, \quad \text{for } k = 0,$$
$$\text{and } d_{\bar{\nabla}}\theta\,(U_1, U_2) := \bar{\nabla}_{U_1}(\theta(U_2)) - \bar{\nabla}_{U_2}(\theta(U_1)) - \theta(\,[U_1, U_2]_{\mathfrak{t}_1}), \quad \text{for } k = 1.$$

Wedge products of $\mathfrak{g}$-type differential forms are likewise defined by familiar formulas.

The principal invariants of the generator ∇ are the $\mathfrak{g}$-type differential forms $T \subset \operatorname{Alt}^2(\mathfrak{t}_1) \otimes \mathfrak{t}$ and $\Omega \subset \operatorname{Alt}^2(\mathfrak{t}_1) \otimes \mathfrak{h}$, obtained by restricting $\operatorname{tor} \bar{\nabla} \subset \operatorname{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}$ and $-\operatorname{cocurv} \nabla \subset \operatorname{Alt}^2(\mathfrak{t}) \otimes T^*M \otimes \mathfrak{t}$ to $\mathfrak{t}_1$ (note carefully the minus sign).

Proposition.

(1) *If θ_1 and θ_2 are $\mathfrak{g}$-type differential forms, then*

$$d_{\bar{\nabla}}(\theta_1 \wedge \theta_2) = d_{\bar{\nabla}}\theta_1 \wedge \theta_2 + (-1)^k\, \theta_1 \wedge d_{\bar{\nabla}}\theta_2,$$

where k is the degree of θ_1.

(2) *For any $\mathfrak{g}$-type differential form θ, we have*

$$d_{\bar{\nabla}}^2\theta = \Omega \wedge \theta.$$

Here the wedge implies a contraction $\phi \otimes \sigma \mapsto \phi \cdot \sigma : \mathfrak{h} \otimes E \to E$, defined by the representation of $\mathfrak{h}$ on E.

Proof of (2). The general case can easily be reduced to the $k = 0$ case that we prove now. Letting $s\colon \mathfrak{t} \to J^1\mathfrak{t}$ denote the splitting of 6.1(4) determined by ∇, we compute, for arbitrary $U_1, U_2 \subset \mathfrak{t}_1$,

$$\begin{aligned} d^2_{\bar\nabla}\theta\,(U_1,U_2) &= \bar\nabla_{U_1}\bar\nabla_{U_2}\theta - \bar\nabla_{U_2}\bar\nabla_{U_1}\theta - \bar\nabla_{[U_1,U_2]_{\mathfrak{t}_1}}\theta \\ &= sU_1\cdot(sU_2\cdot\theta) - sU_2\cdot(sU_1\cdot\theta) - s[U_1,U_2]_{\mathfrak{t}_1}\cdot\theta \\ &= (sU_1\cdot(sU_2\cdot\theta) - sU_2\cdot(sU_1\cdot\theta) - [sU_1,sU_2]_{\mathfrak{g}}\cdot\theta) \\ &\quad - \operatorname{cocurv}\nabla\,(U_1,U_2)\cdot\theta, \quad \text{by 4.1(2)} \\ &= 0 + \Omega(U_1,U_2)\cdot\theta. \end{aligned}$$

□

6.5. **Bianchi identities.** Generalizing the classical situation, the Bianchi identities below exhibit certain algebraic and differential dependencies between T and Ω, rooted in the equality of mixed partial derivatives. First, since $T = d_{\bar\nabla}i$, where $i \subset \mathfrak{t}_1^* \otimes \mathfrak{t}$ denotes the inclusion $\mathfrak{t}_1 \subset \mathfrak{t}$, We deduce from (2) above,

$$d_{\bar\nabla}T = \Omega \wedge i. \tag{1}$$

Next, assume $\mathfrak{g}$ admits a representation E for which the restricted representation $\mathfrak{h} \to \mathfrak{gl}(E)$ is faithful (injective), and let $\theta \subset E$ be a section, viewed as a $\mathfrak{g}$-type differential form of degree zero. Then, combining parts (1) and (2) of the preceding proposition, we obtain $d^3_{\bar\nabla}\theta = d_{\bar\nabla}\Omega \wedge \theta + \Omega \wedge d_{\bar\nabla}\theta$. Applying part (2) again, we conclude that $d_{\bar\nabla}\Omega \wedge \theta = 0$. Since θ is arbitrary and $\mathfrak{h}$ acts faithfully on E, we obtain

$$d_{\bar\nabla}\Omega = 0. \tag{2}$$

A little manipulation allows us to write (1) and (2) in the form

$$\begin{aligned} &(\bar\nabla_{U_3}T)(U_1,U_2) + T(T(U_1,U_2),U_3) \\ &\qquad + \Omega(U_1,U_2)\#U_3 + \text{1-2-3-cyclic terms} = 0 \qquad \text{(Bianchi I)}, \end{aligned} \tag{3}$$

$$\begin{aligned} &(\bar\nabla_{U_3}\Omega)(U_1,U_2) + \Omega(T(U_1,U_2),U_3) \\ &\qquad + \text{1-2-3-cyclic terms} = 0 \qquad \text{(Bianchi II)}. \end{aligned} \tag{4}$$

If ∇ is the Cartan Connection on some Lie algebroid $\mathfrak{t}$ and $\mathfrak{g} \subset J^1\mathfrak{t}$ the corresponding infinitesimal geometric structure (see 5.10) then $T = \operatorname{tor}\bar\nabla$ and $\Omega = 0$. Bianchi I becomes

$$(\bar\nabla_{U_3}\operatorname{tor}\bar\nabla)(U_1,U_2) + \operatorname{tor}\bar\nabla(\operatorname{tor}\bar\nabla(U_1,U_2),U_3) + \text{1-2-3-cyclic terms} = 0.$$

7. Elementary reduction and image reduction `(under construction)`

8. Torsion and prolongation

8.1. **Torsion.** Let $\mathfrak{g}$ be an arbitrary infinitesimal geometric structure on a Lie algebroid $\mathfrak{t}$. Regard the restriction $a\colon \mathfrak{g} \to \mathfrak{t}$ of $J^1\mathfrak{t} \to \mathfrak{t}$ as a $\mathfrak{t}$-valued $\mathfrak{g}$-form of degree one (the *tautological 1-form*). The adjoint representation of $J^1\mathfrak{t}$ on $\mathfrak{t}$ restricts to a representation of $\mathfrak{g}$ on $\mathfrak{t}$. So the exterior derivative da is a well defined $\mathfrak{t}$-valued $\mathfrak{g}$-form, of degree two. This is the *torsion* of the structure. Explicitly,

$$da(X_1,X_2) = \operatorname{ad}^{\mathfrak{t}}_{X_1}(aX_2) - \operatorname{ad}^{\mathfrak{t}}_{X_2}(aX_1) - [aX_1,aX_2]_{\mathfrak{t}}.$$

Now a and da are sections of $\mathfrak{g}^* \otimes \mathfrak{t}$ and $\mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t}$, respectively. We get representations of $J^1\mathfrak{g}$ on these spaces by taking $J^1\mathfrak{g}$ to act on $\mathfrak{g}$ via adjoint action, and on $\mathfrak{t}$ via the composite

$$J^1\mathfrak{g} \xrightarrow{p} \mathfrak{g} \hookrightarrow J^1\mathfrak{t} \xrightarrow{\mathrm{ad}^{\mathfrak{t}}} \mathfrak{gl}(\mathfrak{t}),$$
$$\text{i.e.,} \quad J^1X \cdot W = \mathrm{ad}^{\mathfrak{t}}_{pX} W; \qquad X \subset \mathfrak{g},\, W \subset \mathfrak{t}.$$

Here $p\colon J^1\mathfrak{g} \to \mathfrak{g}$ denotes the canonical projection. We can accordingly define subsets of $J^1\mathfrak{g}$,

$$(J^1\mathfrak{g})_a := \text{isotropy of } a,$$
$$(J^1\mathfrak{g})_{da} := \text{isotropy of } da.$$

8.2. **Prolongation.** We define the *prolongation* of $\mathfrak{g}$ to be the set

$$\mathfrak{g}^{(1)} := (J^1\mathfrak{g})_a \cap (J^1\mathfrak{g})_{da} \subset J^1\mathfrak{g}.$$

Sections of $\mathfrak{g}^{(1)}$ are evidently closed under the bracket of $\mathfrak{g}$. If $\mathfrak{g}^{(1)}$ has constant rank then it is an infinitesimal geometric structure on $\mathfrak{g}$. According to the following proposition, the symmetries of $\mathfrak{g}$ and $\mathfrak{g}^{(1)}$ are in natural one-to-one correspondence if $\mathfrak{g}$ is transitive; more generally however, prolongation introduces new symmetries.

Proposition. *Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure, with structure kernel $\mathfrak{h}$. Then for any symmetry $V \subset \mathfrak{t}$ of $\mathfrak{g}$, the prolonged section $J^1V \subset \mathfrak{g}$ is a symmetry of $\mathfrak{g}^{(1)}$. That is, we have a natural injection*

(1) $$\Big\{\text{symmetries of } \mathfrak{g}\Big\} \overset{J^1}{\hookrightarrow} \Big\{\text{symmetries of } \mathfrak{g}^{(1)}\Big\}.$$

Moreover:

(2) *If $\mathfrak{g}$ is transitive, then* (1) *is an isomorphism.*
(3) *If $\mathfrak{g}$ is intransitive, then every section of $\mathfrak{h} \subset T^*M \otimes \mathfrak{t}$ annihilating tangent vectors in the image of the anchor $\#\colon \mathfrak{g} \to TM$ is a symmetry of $\mathfrak{g}^{(1)}$* not *in the image of* (1).

We now formulate and prove a 'Prolongation Lemma' of which the above proposition is a straightforward corollary. We begin by defining the *deviation* of a section $X \subset \mathfrak{g} \subset J^1\mathfrak{t}$:

$$\operatorname{dev} X := J^1(aX) - X.$$

Once again $a\colon \mathfrak{g} \to \mathfrak{t}$ denotes the restriction of $J^1\mathfrak{t} \to \mathfrak{t}$. As $\operatorname{dev} X$ is a section of the kernel of $J^1\mathfrak{t} \to \mathfrak{t}$, we are entitled to view it as a section of $T^*M \otimes \mathfrak{t}$. Clearly, it vanishes if and only if X is holonomic.

Lemma (Prolongation Lemma)**.** Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure and let $X \subset \mathfrak{g}$ be an arbitrary section. Then $J^1X \subset \mathfrak{g}^{(1)}$ if and only if $\operatorname{dev} X(\#Y) = 0$ for all vectors $Y \in \mathfrak{g}$. In particular, if $\mathfrak{g}$ is transitive, then $J^1X \subset \mathfrak{g}^{(1)}$ if and only if X is holonomic.

Example. Identify the second jet bundle $J^2(TM)$ with its image under the canonical embedding $J^2(TM) \hookrightarrow J^1(J^1(TM))$ sending a section J^2V to $J^1(J^1V)$. Then one can show that $(J^1(TM))^{(1)} = J^2(TM)$. Since $J^1(TM)$ is transitive, the Prolongation Lemma implies that $X \subset J^1(TM)$ is holonomic if and only if J^1X lies in $J^2(TM)$.

Proof of lemma. It is a consequence of the readily verified identity $J^1X \cdot da = d(J^1X \cdot a)$ that $J^1X \subset (J^1\mathfrak{g})_a$ already implies $J^1X \subset (J^1\mathfrak{g})_{da}$. Hence $J^1X \subset (J^1\mathfrak{g})_a$ if and only if $J^1X \subset \mathfrak{g}^{(1)}$. It consequently suffices to establish the lemma with $\mathfrak{g}^{(1)}$ replaced with $(J^1\mathfrak{g})_a$. This restated lemma follows from the following fact:

$$\xi \in (J^1\mathfrak{g})_a \iff p\xi = (J^1a)\xi + \phi \text{ for some } \phi \in A \otimes \mathfrak{t}. \tag{4}$$

Here $\xi \in J^1\mathfrak{g}$ is arbitrary, $p\colon J^1\mathfrak{g} \to \mathfrak{g}$ denotes the canonical projection, $J^1a\colon J^1\mathfrak{g} \to J^1\mathfrak{t}$ is the morphism which, as a map on sections, sends J^1X to $J^1(aX)$, and $A \subset T^*M$ denotes the annihilator of the image of the anchor $\#\colon \mathfrak{g} \to TM$, vanishing when $\mathfrak{g}$ is transitive.

To prove (4), begin by observing that

$$(\xi \cdot a)(Y) = \mathrm{ad}^{\mathfrak{t}}_{p\xi}(aY) - a(\mathrm{ad}^{\mathfrak{g}}_{\xi} Y),$$

where Y is an arbitrary element of $\mathfrak{g}$, extended to a local section of $\mathfrak{g}$ to make the right-hand side well defined. Since $a\colon \mathfrak{g} \to \mathfrak{t}$ is a Lie algebroid morphism, the identity 3.4(3) gives us $a(\mathrm{ad}^{\mathfrak{g}}_{\xi} Y) = \mathrm{ad}^{\mathfrak{t}}_{(J^1a)\xi}(aY)$, and so

$$(\xi \cdot a)(Y) = \mathrm{ad}^{\mathfrak{t}}_{p\xi - (J^1a)\xi}(aY).$$

Because $p\xi - (J^1a)\xi$ is a section of the kernel of $J^1\mathfrak{t} \to \mathfrak{t}$, we may view it as a section of $T^*M \otimes \mathfrak{t}$ and, applying 3.4(2), obtain

$$(\xi \cdot a)(Y) = -(p\xi - (J^1a)\xi)(\#Y).$$

Since $Y \in \mathfrak{g}$ is arbitrary, we conclude that

$$\xi \in (J^1\mathfrak{g})_a \iff p\xi - (J^1a)\xi \in A \otimes \mathfrak{t},$$

which proves (4). □

The remainder of the section describes, in Lie algebroid language, some classical constructions related to torsion. We will require the additional assumption that $\mathfrak{g}$ is *surjective*. We continue to denote the structure kernel of $\mathfrak{g}$ by $\mathfrak{h}$.

8.3. **Normalizing torsion and the upper coboundary morphism.** Identify $\mathfrak{g}$ with $\mathfrak{t} \oplus \mathfrak{h}$ by choosing a generator ∇ of $\mathfrak{g}$. Then we obtain a corresponding identification,

$$\mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t} \cong \Big(\mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}\Big) \oplus \Big(\mathfrak{t}^* \otimes \mathfrak{h}^* \otimes \mathfrak{t}\Big) \oplus \Big(\mathrm{Alt}^2(\mathfrak{h}) \otimes \mathfrak{t}\Big),$$

and a corresponding splitting of the torsion

$$da = \mathrm{tor}\,\bar\nabla \oplus \mathrm{ev} \oplus 0.$$

Here $\bar\nabla$ denotes the associated $\mathfrak{t}$-connection on $\mathfrak{t}$ and ev is just evaluation, $\mathrm{ev}(V \otimes \phi) := \phi(V)$. Notice that $\mathrm{tor}\,\bar\nabla$ is the only component of da depending on the choice of generator. Given two generators ∇^1 and ∇^2, their difference $\nabla^2 - \nabla^1$ may be viewed as a section of $\mathfrak{t}^* \otimes \mathfrak{h}$ and one readily computes

$$\mathrm{tor}\,\bar\nabla^2 = \mathrm{tor}\,\bar\nabla^1 + \Delta(\nabla^2 - \nabla^1), \tag{1}$$

where Δ denotes the *upper coboundary morphism*, defined as the composite

$$\mathfrak{t}^* \otimes \mathfrak{h} \hookrightarrow \mathfrak{t}^* \otimes T^*M \otimes \mathfrak{t} \xrightarrow{\mathrm{id} \otimes \#^* \otimes \mathrm{id}} \mathfrak{t}^* \otimes \mathfrak{t}^* \otimes \mathfrak{t} \xrightarrow{A \otimes \mathrm{id}} \mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}.$$

Here $\#^*\colon T^*M \to \mathfrak{t}^*$ is the dual of the anchor $\#\colon \mathfrak{t} \to TM$ and $A(\alpha \otimes \beta) := \alpha \wedge \beta$. This morphism is one version of Spencer's coboundary morphism (see, e.g., [12]). For another, see 9.1. As an elementary consequence of (1) above, we obtain:

Proposition. *If $C \subset \mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}$ is a complement for the image of Δ, then there exists a generator ∇ such that* $\operatorname{tor} \bar{\nabla} \subset C$*. If Δ is injective, then this generator is unique.*

Note that there is no need to require that C be $\mathfrak{g}$-invariant.

8.4. **Intrinsic torsion and torsion reduction.** Mimicking a classical construction, we define the *torsion bundle*,

$$H(\mathfrak{g}) := \frac{\mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}}{\operatorname{im} \Delta},$$

and call the image τ of $\operatorname{tor} \bar{\nabla}$, under the map $\Gamma(\mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}) \to \Gamma(H(\mathfrak{g}))$, induced by the projection $\mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t} \to H(\mathfrak{g})$, the *intrinsic torsion* of $\mathfrak{g}$. By (1), τ is independent of the choice of generator, i.e., is an invariant of $\mathfrak{g}$. Since Δ is $\mathfrak{g}$-equivariant, $H(\mathfrak{g})$ is a $\mathfrak{g}$-representation whenever it is a bona fide vector bundle (has constant rank). In that case we can define the isotropy $\mathfrak{g}_\tau \subset \mathfrak{g}$ of τ. This is the *torsion reduction* of $\mathfrak{g}$ and is indeed a reduction, as we show in 9.3.

9. Θ-reduction

Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be an infinitesimal geometric structure with structure kernel $\mathfrak{h}$ and let $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ be its prolongation. Recall that the Θ*-reduction* of $\mathfrak{g}$ is simply the image of $\mathfrak{g}^{(1)}$. We will denote it by $\mathfrak{g}_1^{(1)}$. If $\mathfrak{g}^{(1)}$ and $\mathfrak{g}_1^{(1)}$ have constant rank then $\mathfrak{g}_1^{(1)}$ is evidently a subalgebroid of $\mathfrak{g}$. Proposition 8.2 shows that the symmetries of $\mathfrak{g}$ are automatically symmetries of $\mathfrak{g}_1^{(1)} \subset \mathfrak{g}$. So $\mathfrak{g}_1^{(1)}$ is indeed a reduction, as claimed in Proposition 2.12.

For our purposes it will suffice to suppose that $\mathfrak{g}$ is surjective; see 2.13. For simplicity, however, we strengthen this requirement:

Assumption. In this section $\mathfrak{g} \subset J^1\mathfrak{t}$ is a surjective infinitesimal geometric structure over a *transitive* Lie algebroid $\mathfrak{t}$. In particular, $\mathfrak{g}$ is transitive. Being a surjective, $\mathfrak{g}$ has a structure kernel $\mathfrak{h}$ of constant rank and $\mathfrak{g}$ admits generators (Proposition 6.1(1)).

Our chief objective is a characterization of the Θ-reduction $\mathfrak{g}_1^{(1)}$ that does not require an explicit knowledge of $\mathfrak{g}^{(1)}$.

9.1. **The lower coboundary morphism.** As in torsion reduction, a 'coboundary morphism' plays a central role in Θ-reduction. However, unless $\mathfrak{t} = TM$, the upper coboundary morphism Δ, defined in 8.3, is not the appropriate one. Rather, we need the *lower coboundary morphism* δ, defined as the composite

$$T^*M \otimes \mathfrak{h} \hookrightarrow T^*M \otimes T^*M \otimes \mathfrak{t} \xrightarrow{A \otimes \mathrm{id}} \mathrm{Alt}^2(TM) \otimes \mathfrak{t},$$

where $A(\alpha \otimes \beta) := \alpha \wedge \beta$. This morphism is also a morphism of $\mathfrak{g}$-representations.

As we assume $\mathfrak{t}$ is transitive, we may, by dualizing the anchor map $\#\colon \mathfrak{t} \to TM$, regard T^*M as a subbundle of $\mathfrak{t}^*$, and obtain natural inclusions

$$T^*M \otimes \mathfrak{h} \hookrightarrow \mathfrak{t}^* \otimes \mathfrak{h}$$
$$\mathrm{Alt}^2(TM) \otimes \mathfrak{t} \hookrightarrow \mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}.$$

With this understanding, we may regard $\delta\colon T^*M \otimes \mathfrak{h} \to \mathrm{Alt}^2(TM) \otimes \mathfrak{t}$ as the restriction of the upper coboundary morphism $\Delta\colon \mathfrak{t}^* \otimes \mathfrak{h} \to \mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}$ defined in 8.3.

The analogue of the torsion bundle $H(\mathfrak{g})$ defined in 8.3 is the (variable-rank) bundle

$$h(\mathfrak{g}) := \frac{\mathrm{Alt}^2(TM) \otimes \mathfrak{t}}{\mathrm{im}\,\delta}.$$

Whenever $h(\mathfrak{g})$ is a genuine vector bundle (has constant rank), it is a $\mathfrak{g}$-representation. There is evidently a natural morphism $\psi\colon h(\mathfrak{g}) \to H(\mathfrak{g})$ making the following diagram commute:

(1)
$$\begin{array}{ccc} \mathrm{Alt}^2(TM) \otimes \mathfrak{t} & \xrightarrow{/\,\mathrm{im}\,\delta} & h(\mathfrak{g}) \\ {\scriptstyle \text{inclusion}}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ \mathrm{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t} & \xrightarrow[/\,\mathrm{im}\,\Delta]{} & H(\mathfrak{g}) \end{array} .$$

Note that ψ need not be injective.

9.2. **The characterization of $\mathfrak{g}_1^{(1)}$.** The significance of the bundle $h(\mathfrak{g})$ is the existence of a natural morphism $\Theta\colon \mathfrak{g} \to h(\mathfrak{g})$ such that $\mathfrak{g}_1^{(1)} = \ker\Theta$. (This is the origin of our terminology ‘Θ-reduction.’) Recalling that $\mathfrak{g}^{(1)} := (J^1\mathfrak{g})_a \cap (J^1\mathfrak{g})_{da}$, our construction of Θ begins with the following observation:

(1) The isotropy $(J^1\mathfrak{g})_a \subset J^1\mathfrak{g}$ is a surjective infinitesimal geometric structure with structure kernel $T^*M \otimes \mathfrak{h}$.

Proof. It is not difficult to see that $(J^1\mathfrak{g})_a$ is the kernel of

$$\xi \mapsto p\xi - (J^1a)\xi$$
$$J^1\mathfrak{g} \to T^*M \otimes \mathfrak{t},$$

where $p\colon J^1\mathfrak{g} \to \mathfrak{g}$ is the projection. This follows from the transitivity of $\mathfrak{g}$ and, e.g., 8.2(4). One establishes (1) by applying Lemma B.1 to this morphism. □

Now $\mathfrak{g}^{(1)}$ is the kernel of the morphism $\xi \mapsto \xi \cdot da\colon (J^1\mathfrak{g})_a \to \mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t}$. However, as we establish in 9.7:

(2) For any $\xi \in (J^1\mathfrak{g})_a$, the element $\xi \cdot da \in \mathrm{Alt}^2(\mathfrak{g}) \otimes \mathfrak{t}$ is tensorial — i.e., drops to an element $(\xi \cdot da)^\vee \in \mathrm{Alt}^2(TM) \otimes \mathfrak{t}$.

This means we may regard $\mathfrak{g}^{(1)}$ as the kernel of a morphism

$$(J^1\mathfrak{g})_a \xrightarrow{\theta} \mathrm{Alt}^2(TM) \otimes \mathfrak{t}$$
$$\xi \mapsto (\xi \cdot da)^\vee.$$

According to (1), the domain of θ fits into an exact sequence,

$$0 \to T^*M \otimes \mathfrak{h} \hookrightarrow (J^1\mathfrak{g})_a \to \mathfrak{g} \to 0.$$

Applying Lemma B.1 to the morphism θ, we obtain a corresponding exact sequence,

$$0 \to \ker\delta \hookrightarrow \mathfrak{g}^{(1)} \xrightarrow{a^{(1)}} \ker\Theta \to 0,$$

where Θ is the unique morphism making the following diagram commute:

$$\begin{array}{ccc} (J^1\mathfrak{g})_a & \longrightarrow & \mathfrak{g} \\ \big\downarrow \theta & & \big\downarrow \Theta \\ \operatorname{Alt}^2(TM)\otimes\mathfrak{t} & \xrightarrow{/\operatorname{im}\delta} & h(\mathfrak{g}) \end{array} . \tag{3}$$

Summarizing:

Proposition. *If $\mathfrak{g} \subset J^1\mathfrak{t}$ is surjective and $\mathfrak{t}$ is transitive, then there is a natural morphism $\Theta\colon \mathfrak{g} \to h(\mathfrak{g})$, constructed above, such that*

$$0 \to \ker\delta \hookrightarrow \mathfrak{g}^{(1)} \xrightarrow{a^{(1)}} \mathfrak{g} \xrightarrow{\Theta} h(\mathfrak{g})$$

is exact. In particular, the structure kernel of $\mathfrak{g}^{(1)}$ is the kernel of the lower coboundary morphism δ, while the image $\mathfrak{g}_1^{(1)}$ of $\mathfrak{g}^{(1)}$ (the Θ-reduction of $\mathfrak{g}$) is the kernel of Θ. If $\ker\delta$ and $\ker\Theta$ have constant rank then $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is an infinitesimal geometric structure.

Remark. In concrete calculations it is often useful to think of $\ker\delta$ as the collection of all $\sigma \in \operatorname{Sym}^2(TM)\otimes\mathfrak{t}$ such that $\sigma(U,\,\cdot\,) \subset T^*M\otimes\mathfrak{t}$ lies in $\mathfrak{h}$ for all $U \in TM$.

9.3. **The relationship with torsion.** By construction Θ is an *invariant* of $\mathfrak{g}$. However, the construction of Θ given here is not immediately useful in computations. In Sect. 11 we describe $\Theta\colon \mathfrak{g} \to h(\mathfrak{g})$ explicitly in terms of a generator ∇ of $\mathfrak{g}$. As a byproduct, we will obtain a proof of the following link between Θ-reduction and intrinsic torsion:

Theorem. *If $\mathfrak{g} \subset J^1\mathfrak{t}$ is surjective, $\mathfrak{t}$ is transitive, and $H(\mathfrak{g})$ has constant rank, then $\psi(\Theta(X)) = X\cdot\tau$.*

Here ψ is the natural morphism $\psi\colon h(\mathfrak{g}) \to H(\mathfrak{g})$ defined in 9.1, and $\tau \subset H(\mathfrak{g})$ is the intrinsic torsion, defined in 8.4. As a corollary we obtain the following result showing that torsion reduction is generally cruder than Θ-reduction:

Corollary. *Suppose that the torsion bundle $H(\mathfrak{g})$ has constant rank, so that the torsion reduction $\mathfrak{g}_\tau$ of $\mathfrak{g}$ is well-defined. Then $\mathfrak{g}_1^{(1)} \subset \mathfrak{g}_\tau$. In particular, if the lower coboundary morphism δ has constant rank, and both $\mathfrak{g}_\tau$ and $\mathfrak{g}_1^{(1)}$ have constant rank, then $\mathfrak{g}_\tau$ is a reduction of $\mathfrak{g}$ in the sense of 2.10. If $\mathfrak{t} = TM$ then Θ-reduction and torsion reduction coincide.*

Here the rank hypotheses and Proposition 9.2 ensure that $\mathfrak{g}^{(1)}$ has constant rank, so that Proposition 2.12 applies. However, the result presumably holds with a constant rank hypothesis on $\mathfrak{g}_\tau$ alone.

9.4. **Structures both surjective and Θ-reduced.** We say that $\mathfrak{g} \subset J^1\mathfrak{t}$ is Θ-*reduced* if it coincides with its Θ-reduction.

Theorem. *Let $\mathfrak{g} \subset J^1\mathfrak{t}$ be a surjective infinitesimal geometric structure on a transitive Lie algebroid $\mathfrak{t}$. Assume that $\mathfrak{g}$ is Θ-reduced (equivalently, that the map Θ defined above vanishes). Assume that the associated lower coboundary morphism δ*

is injective. Then $\mathfrak{g}$ has an associated Cartan algebroid, namely $\mathfrak{g}$ itself, equipped with a canonical Cartan connection $\nabla^{(1)}$. The $\nabla^{(1)}$-parallel sections of $\mathfrak{g}$ coincide with the prolonged symmetries of $\mathfrak{g}$.

Proof. Proposition 2.12 implies the prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is surjective. In addition, $\mathfrak{g}^{(1)}$ has trivial structure kernel, because we suppose δ is injective (Proposition 9.2). Applying Theorem 2.7 to the infinitesimal geometric structure $\mathfrak{g}^{(1)}$, we obtain a Cartan connection $\nabla^{(1)}$ on $\mathfrak{g}$ whose parallel sections are the symmetries of $\mathfrak{g}^{(1)}$. These are nothing but the *prolonged* symmetries of $\mathfrak{g}$, by Proposition 8.2. □

In Proposition 11.1 we characterize $\nabla^{(1)}$ as the unique 'natural' connection on $\mathfrak{g}$ whose curvature $\operatorname{curv} \nabla^{(1)} \subset \operatorname{Alt}^2(TM) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ takes values in $\mathfrak{h} \subset \mathfrak{g}$. A general formula expressing $\nabla^{(1)}$ in terms of a generator of $\mathfrak{g}$ will appear in 11.2. We preview this formula now in the special case $\mathfrak{t} = TM$.

9.5. **The special case $\mathfrak{t} = TM$.** When $\mathfrak{t} = TM$, Θ-reduction and torsion reduction are the same thing, as are the upper and lower coboundary morphisms, δ and Δ. We now rewrite the above theorem accordingly, adding explicit information about the Cartan connection that we establish later in 11.4.

Here $\bar{\nabla}$ will denote the dual of ∇, i.e., $\bar{\nabla}_U V = \nabla_V U + [U, V]$. We call $\mathfrak{g} \subset J^1(TM)$ *reductive* if Δ has constant rank and if the image of Δ admits a $\mathfrak{g}$-invariant complement C. We call the generator ∇ *normal* if $\operatorname{tor} \bar{\nabla} \subset C$ for some such C. Proposition 8.3 guarantees the existence of normal generators when $\mathfrak{g}$ is reductive.

We call an arbitrary structure $\mathfrak{g} \subset \mathfrak{t}$ *torsion-reduced* if $\mathfrak{g} = \mathfrak{g}_\tau$, i.e., if the intrinsic torsion τ is $\mathfrak{g}$-invariant.

Theorem. *Let $\mathfrak{g} \subset J^1(TM)$ be a transitive, torsion-reduced infinitesimal geometric structure, and suppose that the associated upper coboundary morphism Δ is injective. Then $\mathfrak{g}$ has an associated Cartan algebroid, namely $\mathfrak{g}$ itself, equipped with a Cartan connection $\nabla^{(1)}$ described below. The $\nabla^{(1)}$-parallel sections of $\mathfrak{g}$ coincide with the prolonged symmetries of $\mathfrak{g}$.*

Choose a generator ∇ for $\mathfrak{g}$ (equivalently, choose a complement for the image of Δ; see Proposition 8.3). Then:

(1) *The equation*

$$\Delta(\epsilon(V \oplus \phi)) = \bar{\nabla}_V \operatorname{tor} \bar{\nabla} + \phi \cdot \operatorname{tor} \bar{\nabla} \quad (V \subset TM,\ \phi \subset \mathfrak{h} \text{ arbitrary})$$

has a unique solution morphism $\epsilon \colon TM \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h}$.

(2) *Identifying $\mathfrak{g}$ with $TM \oplus \mathfrak{h}$ using the generator, we have*

$$\nabla^{(1)}_U (V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\bar{\nabla}_U \phi + \epsilon(V \oplus \phi) U + \operatorname{curv} \bar{\nabla}(U, V)).$$

(3) *If $\mathfrak{g}$ is reductive, then $\epsilon = 0$ for any normal generator ∇.*

Assuming $\mathfrak{g}$ is reductive, obstructions to symmetry are particularly simple to describe, as is the symmetry Lie algebra $\mathfrak{g}_0$ in the globally flat ('maximally symmetric') case. Indeed, in the reductive case one computes, with the help of 6.2(4) and the Bianchi identity $d_{\bar{\nabla}} \operatorname{curv} \bar{\nabla} = 0$,

$$\operatorname{curv} \nabla^{(1)}(U_1, U_2)(V \oplus \phi) = 0 \oplus \Big(-(\bar{\nabla}_V \operatorname{curv} \bar{\nabla} + \phi \cdot \operatorname{curv} \bar{\nabla})(U_1, U_2) \Big),$$

$$\operatorname{tor} \overline{\nabla^{(1)}}(V_1 \oplus \phi_1, V_2 \oplus \phi_2) =$$
$$(\operatorname{tor} \bar{\nabla}(V_1, V_2) + \phi_1(V_2) - \phi_2(V_1)) \oplus ([\phi_1, \phi_2]_{\mathfrak{h}} - \operatorname{curv} \bar{\nabla}(V_1, V_2)).$$

Here $\overline{\nabla^{(1)}}$ denotes the representation of $\mathfrak{g}$ on itself associated with the Cartan connection $\nabla^{(1)}$ on $\mathfrak{g}$. Applying Theorem 4.5:

Corollary. *Let $\mathfrak{g} \subset J^1(TM)$ be a reductive infinitesimal geometric structure satisfying the hypotheses of the above theorem. Let $\mathcal{U} \subset M$ be an arbitrary open set and $\mathfrak{g}_0$ be the Lie algebra of all symmetries of $\mathfrak{g}|\mathcal{U}$. Then* $\dim \mathfrak{g}_0 \leqslant \operatorname{rank} \mathfrak{g}$. *If $\mathcal{U}$ is simply-connected and ∇ is a generator of $\mathfrak{g}|\mathcal{U}$, then equality holds if and only if* $\operatorname{curv} \bar{\nabla}$ *is both $\mathfrak{h}$-invariant and $\bar{\nabla}$-parallel. In that case $\mathfrak{g}_0$ is naturally isomorphic to $T_mM \oplus \mathfrak{h}(m)$ ($m \in \mathcal{U}$ arbitrary) with Lie bracket given by*

$$\begin{aligned}&[V_1 \oplus \phi_1, V_2 \oplus \phi_2] \\ &\qquad = (\operatorname{tor} \bar{\nabla}(V_1, V_2) + \phi_1(V_2) - \phi_2(V_1)) \oplus ([\phi_1, \phi_2]_{\mathfrak{h}} + \operatorname{curv} \bar{\nabla}(V_1, V_2)).\end{aligned}$$

9.6. **The symmetries of Riemannian structures.** Let $\mathfrak{g} \subset J^1(TM)$ denote the bundle of 1-symmetries of a Riemannian metric σ, as described in detail in 5.3. The upper coboundary morphism for $\mathfrak{g}$ is a map

$$T^*M \otimes \mathfrak{h} \xrightarrow{\Delta} \operatorname{Alt}^2(TM) \otimes TM,$$

where $\mathfrak{h} \subset T^*M \otimes TM$ is the $\mathfrak{o}(n)$-bundle of all skew-symmetric tangent space endomorphisms. This morphism is well known to be an isomorphism. For example, using the metric σ to make various identifications, we may view Δ as the map

$$\begin{aligned}T^*M \otimes \operatorname{Alt}^2(TM) &\to \operatorname{Alt}^2(TM) \otimes T^*M \\ \alpha \otimes (\beta_1 \wedge \beta_2) &\mapsto (\alpha \wedge \beta_1) \otimes \beta_2 - (\alpha \wedge \beta_2) \otimes \beta_1,\end{aligned}$$

which has explicit inverse

$$(\beta_1 \wedge \beta_2) \otimes \alpha \mapsto \frac{1}{2}\Big(\beta_1 \otimes (\beta_2 \wedge \alpha) - \beta_2 \otimes (\beta_1 \wedge \alpha) - \alpha \otimes (\beta_1 \wedge \beta_2) \Big).$$

Since Δ is an isomorphism $\mathfrak{g}$ has, by Proposition 8.3, a unique torsion-free generator ∇; this is the Levi-Cevita connection. The intrinsic torsion vanishes because $H(\mathfrak{g}) = 0$, making $\mathfrak{g}$ torsion-reduced. Also, $\mathfrak{g}$ is trivially reductive. Applying Theorem 9.5, we obtain the Cartan connection $\nabla^{(1)}$ and related claims given in 2.5. (We have $\bar{\nabla} = \nabla$ since $\operatorname{tor} \nabla = 0$.)

According to Corollary 9.5, we are in the maximally symmetric case when $\operatorname{curv} \nabla$ is both $\mathfrak{h}$-invariant and ∇-parallel. According to well-known representation-theoretic analysis of the curvature module, this happens if and only if

$$\operatorname{curv} \nabla(V_1, V_2) = s\Big(\sigma(V_1) \otimes V_2 - \sigma(V_2) \otimes V_1 \Big); \qquad V_1, V_2 \in TM,$$

for some constant $s \in \mathbb{R}$ (the scalar curvature). The Lie algebra $\mathfrak{g}_0$ of symmetries described in the corollary is then isomorphic to the Lie algebra of infinitesimal isometries of Euclidean space, hyperbolic space, or the sphere, according to whether $s = 0$, $s < 0$, or $s > 0$.

9.7. **Technical details.** We end this section with the proof of 9.2(2).

First note that for arbitrary sections $X, Y \subset \mathfrak{g}$, we may write

$$\begin{aligned}da(X, Y) &= \operatorname{ad}^{\mathfrak{t}}_X(aY) - \operatorname{ad}^{\mathfrak{t}}_{Y - J^1(aY)}(aX) \\ &= \operatorname{ad}^{\mathfrak{t}}_X(aY) + (Y - J^1(aY))(\#X),\end{aligned}$$

where we view $Y - J^1(aY)$ as a section of $T^*M \otimes \mathfrak{t}$ (the kernel of $J^1\mathfrak{t} \to \mathfrak{t}$). In particular,

$$\#X = 0 \implies da(X,Y) = \mathrm{ad}^{\mathfrak{t}}_X(aY). \tag{1}$$

Now let $\xi \subset (J^1\mathfrak{g})_a$ and $X, Y \subset \mathfrak{g}$ be arbitrary sections. Supposing $\#X = 0$, we also obtain $\#(\xi \cdot X) = 0$, and, with the help of (1), compute

$$\begin{aligned}(\xi \cdot da)(X,Y) &= \xi \cdot (da(X,Y)) - da(\xi \cdot X, Y) - da(X, \xi \cdot Y)\\ &= \mathrm{ad}^{\mathfrak{t}}_{p\xi}\,\mathrm{ad}^{\mathfrak{t}}_X(aY) - \mathrm{ad}^{\mathfrak{t}}_{\mathrm{ad}^{\mathfrak{g}}_\xi X}(aY) - \mathrm{ad}^{\mathfrak{t}}_X(a\,\mathrm{ad}^{\mathfrak{g}}_\xi Y)\\ &= \mathrm{ad}^{\mathfrak{t}}_X\,\mathrm{ad}^{\mathfrak{t}}_{p\xi}(aY) + \mathrm{ad}^{\mathfrak{t}}_{[p\xi,X]_{\mathfrak{g}}}(aY) - \mathrm{ad}^{\mathfrak{t}}_{\mathrm{ad}^{\mathfrak{g}}_\xi X}(aY) - \mathrm{ad}^{\mathfrak{t}}_X\,\mathrm{ad}^{\mathfrak{t}}_{(J^1a)\xi}(aY).\end{aligned}$$

Because $\mathfrak{g}$ is transitive, $\xi \subset (J^1\mathfrak{g})_a$ means $p\xi = (J^1a)\xi$ (by, e.g., 8.2(4)). So the first and last terms above cancel, giving us

$$(\xi \cdot da)(X,Y) = \mathrm{ad}^{\mathfrak{t}}_{[p\xi,X]_{\mathfrak{g}} - \mathrm{ad}^{\mathfrak{g}}_\xi X}(aY).$$

But

$$[p\xi, X]_{\mathfrak{g}} - \mathrm{ad}^{\mathfrak{g}}_\xi X = \mathrm{ad}^{\mathfrak{g}}_{J^1(p\xi)-\xi} X = -(J^1(p\xi) - \xi)(\#X),$$

where we are viewing $J^1(p\xi) - \xi$ as a section of $T^*M \otimes \mathfrak{g}$ (the kernel of $J^1\mathfrak{g} \xrightarrow{p} \mathfrak{g}$). Since $\#X = 0$, by assumption, we deduce that $(\xi \cdot da)(X,Y) = 0$.

10. Subriemannian contact three-manifolds `(under construction)`

11. Advanced prolongation theory

This section describes in detail the prolongation $\mathfrak{g}^{(1)}$ of a *surjective* infinitesimal geometric structure $\mathfrak{g} \subset J^1\mathfrak{t}$. This description, obtained by 'prolonging' a generator of $\mathfrak{g}$ to a generator of $\mathfrak{g}^{(1)}$, is concrete enough to permit computations in examples. The central result, Theorem 11.2, is the probably the most widely applicable in such calculations. This theorem is used to prove several theoretical results stated earlier; some special cases are also considered.

We assume throughout that $\mathfrak{t}$ is a *transitive* Lie algebroid. For basic implications, see Sect. 9 under 'Assumption.' We continue to denote the structure kernel of $\mathfrak{g}$ by $\mathfrak{h}$, and the associated lower coboundary morphism, defined in 9.1, by δ.

11.1. **Natural connections.** Call a linear connection D on $\mathfrak{g} \subset J^1\mathfrak{t}$ *natural* if the associated $\mathfrak{g}$-connection on $\mathfrak{t}$ (see Example 6.3(4)) is the adjoint representation of $\mathfrak{g} \subset J^1\mathfrak{t}$ on $\mathfrak{t}$; in symbols, if

$$aD_{\#V}X + [aX, V]_{\mathfrak{t}} = \mathrm{ad}^{\mathfrak{t}}_X V; \qquad V \subset \mathfrak{t},\ X \subset \mathfrak{g}.$$

Here $a\colon \mathfrak{g} \to \mathfrak{t}$ is the restricted projection $J^1\mathfrak{t} \to \mathfrak{t}$. The following proposition is not immediately useful in computations but is a natural intermediate result. It stands between the rather abstract Proposition 9.2 and the computationally useful Theorem 11.2 given later.

Proposition. *Let D be any natural connection on $\mathfrak{g}$ and consider the morphism,*

$$\mathfrak{g} \xrightarrow{\dot\Theta} \mathrm{Alt}^2(TM) \otimes \mathfrak{t}$$
$$\dot\Theta(X)(U_1, U_2) := a(\mathrm{curv}\, D\,(U_1, U_2)X),$$

where a is the projection $\mathfrak{g} \to \mathfrak{t}$. Then:

(1) *The morphism* $\Theta\colon \mathfrak{g} \to h(\mathfrak{g})$, *defined in 9.2, coincides with the composite*

$$\mathfrak{g} \xrightarrow{\dot\Theta} \operatorname{Alt}^2(TM) \otimes \mathfrak{t} \xrightarrow{/\operatorname{im}\delta} h(\mathfrak{g}).$$

Moreover, if $\ker\delta$ *and* $\ker\Theta$ *have constant rank (so that* $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ *is an infinitesimal geometric structure, by Proposition 9.2) then:*

(2) *There exist natural connections on* $\mathfrak{g}$ *generating* $\mathfrak{g}^{(1)}$, *with all generators being natural if* $\Theta = 0$.

(3) *A natural connection* D *on* $\mathfrak{g}$ *generates* $\mathfrak{g}^{(1)}$ *if and only if*

$$\operatorname{curv} D(\,\cdot\,,\,\cdot\,)X \in \operatorname{Alt}^2(\mathfrak{g}) \otimes \mathfrak{h} \qquad \textit{for all } X \in \ker\Theta.$$

Corollary. *The Cartan connection* $\nabla^{(1)}$ *in Theorem 9.4 is the unique natural connection on* $\mathfrak{g}$ *such that* $\operatorname{curv}\nabla^{(1)}$ *takes values in the subbundle*

$$\operatorname{Alt}^2(\mathfrak{g}) \otimes \mathfrak{g}^* \otimes \mathfrak{h} \subset \operatorname{Alt}^2(\mathfrak{g}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}.$$

This corollary holds because $D := \nabla^{(1)}$ is the unique generator of $\mathfrak{g}^{(1)}$ under the hypotheses of Theorem 9.4.

The proposition's proof rests on parts (4) and (5) of the following technical result proven in 11.6.

Lemma.

(4) A linear connection D on $\mathfrak{g}$ is natural if and only if it is a generator of the isotropy $(J^1\mathfrak{g})_a \subset J^1\mathfrak{g}$ of a.

(5) If D is a natural connection on $\mathfrak{g}$, then

$$(sX \cdot da)^\vee(U_1, U_2) = a(\operatorname{curv} D(U_1, U_2)X); \qquad X \subset \mathfrak{g}.$$

Here $s\colon \mathfrak{g} \to J^1\mathfrak{g}$ denotes the splitting of the exact sequence,

$$0 \to T^*M \otimes \mathfrak{g} \to J^1\mathfrak{g} \to \mathfrak{g} \to 0.$$

corresponding to D, and $(sX \cdot da)^\vee$ is the 'reduction' of $sX \cdot da$ as per 9.2(2).

(6) If ∇ is a connection on $\mathfrak{t}$ generating $\mathfrak{g}$ and $\nabla^{\mathfrak{h}}$ is any linear connection on $\mathfrak{h}$, then, identifying $\mathfrak{g}$ with $\mathfrak{t} \oplus \mathfrak{h}$ using the generator ∇, every natural connection on $\mathfrak{g}$ is of the form

$$D_U(V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\nabla^{\mathfrak{h}}_U \phi + \epsilon(V \oplus \phi)U),$$

for some vector bundle morphism $\epsilon\colon \mathfrak{t} \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h}$.

Proof of proposition. By (4), D generates $(J^1\mathfrak{g})_a$. Recalling that $(J^1\mathfrak{g})_a$ is surjective (see 9.2(1)), let $s\colon \mathfrak{g} \to (J^1\mathfrak{g})_a$ denote the splitting of the exact sequence,

(7) $$0 \to T^*M \otimes \mathfrak{h} \hookrightarrow (J^1\mathfrak{g})_a \to \mathfrak{g} \to 0,$$

determined by the generator D. By the commutativity of the diagram 9.2(3) defining Θ, we have

$$\Theta(X) = \theta(sX) \bmod \operatorname{im}\delta = (sX \cdot da)^\vee \bmod \operatorname{im}\delta.$$

Invoking (5), we prove (1). Part (2) of the proposition follows from (4) and the following facts: (i) $(J^1\mathfrak{g})_a$ is surjective, (ii) $(J^1\mathfrak{g})_a \subset \mathfrak{g}^{(1)}$, and (iii) $\mathfrak{g}^{(1)}$ is surjective when $\Theta = 0$ (by Proposition 9.2). Conclusion (3) is just a consequence of (1) and the definition of $\dot\Theta$. □

11.2. **Prolonging a generator.** We now show how to construct a generator for the prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ given a generator of $\mathfrak{g}$. Fix a generator ∇ of $\mathfrak{g}$; recall that this is a certain linear connection on $\mathfrak{t}$. Also fix an arbitrary linear connection $\nabla^{\mathfrak{h}}$ on $\mathfrak{h}$. (In practice, there is usually a preferred choice, and if $\mathfrak{t} = TM$ this is always true; see 11.4. No canonical choice exists in general, however).

With ∇ and $\nabla^{\mathfrak{h}}$ fixed, there is vector bundle morphism

$$\mathfrak{t} \oplus \mathfrak{h} \xrightarrow{\tilde{\Theta}} \mathrm{Alt}^2(TM) \otimes \mathfrak{t}$$

well defined by its action on sections via

$$\tilde{\Theta}\,(V \oplus \phi) := \operatorname{curv} \nabla(\,\cdot\,,\,\cdot\,)V + d_\nabla \phi - \delta(\nabla^{\mathfrak{h}} \phi). \tag{1}$$

Here

$$\begin{aligned} d_\nabla \phi\,(U_1, U_2) &:= \nabla_{U_1}(\phi(U_2)) - \nabla_{U_2}(\phi(U_1)) - \phi([U_1, U_2]), \\ (\nabla^{\mathfrak{h}} \phi)U &:= \nabla^{\mathfrak{h}}_U \phi. \end{aligned}$$

Theorem (Prolonging a generator of $\mathfrak{g} \subset J^1\mathfrak{t}$)**.** *Let $\mathfrak{g}$ be a surjective infinitesimal geometric structure on a transitive Lie algebroid $\mathfrak{t}$, with $\mathfrak{h}$, ∇, $\nabla^{\mathfrak{h}}$, and $\tilde{\Theta}$ as above. Use ∇ to identify $\mathfrak{g}$ with $\mathfrak{t} \oplus \mathfrak{h}$. Then:*

(2) *The composite morphism,*

$$\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{h} \xrightarrow{\tilde{\Theta}} \mathrm{Alt}^2(TM) \otimes \mathfrak{t} \xrightarrow{/\operatorname{im}\delta} h(\mathfrak{g}),$$

coincides with the morphism $\Theta \colon \mathfrak{g} \to h(\mathfrak{g})$ defined in 9.2.

(3) $\ker \Theta \subset \mathfrak{t} \oplus \mathfrak{h}$ *is precisely the set of all $V \oplus \phi$ for which the generator equation,*

$$\delta(\epsilon) = \tilde{\Theta}(V \oplus \phi),$$

admits a solution $\epsilon \in T^*M \otimes \mathfrak{h}$.

(4) *Assuming* $\ker \delta$ *and* $\ker \Theta$ *have constant rank (so that $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is an infinitesimal geometric structure, by Proposition 9.2) a linear connection $\nabla^{(1)}$ on $\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{h}$ generating $\mathfrak{g}^{(1)}$ is given by*

$$\boxed{\nabla^{(1)}_U (V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\nabla^{\mathfrak{h}}_U \phi + \epsilon(V \oplus \phi)U),}$$

where $\epsilon \colon \mathfrak{t} \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h}$ is any of the vector bundle morphisms for which $\epsilon := \epsilon(V \oplus \phi)$ solves the generator equation defined in (3) *above, for each $V \oplus \phi$ in* $\ker \Theta$*. If $\mathfrak{g}^{(1)}$ is surjective (i.e., $\Theta = 0$) then every generator is of the above form.*

Proof. Let D denote the general form of a natural connection on $\mathfrak{g}$ given in (6), with $\epsilon \colon \mathfrak{t} \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h}$ completely arbitrary. If $\dot{\Theta}$ is the morphism defined in Proposition 11.1, then one computes

$$\dot{\Theta}(V \oplus \phi)(U_1, U_2) = a\Big(\operatorname{curv} D\,(U_1, U_2)(V \oplus \phi)\Big) = \tilde{\Theta}(V \oplus \phi) - \delta(\epsilon(V \oplus \phi)),$$

where $\tilde{\Theta}$ is the morphism defined by (1). Conclusion (2) of the theorem now follows from Proposition 11.1(1). Conclusion (3) is just a consequence of (2). One obtains (4) by taking $\nabla^{(1)} := D$; choosing ϵ as described guarantees the curvature condition in Proposition 11.1(3). If $\Theta = 0$ then every generator of $\mathfrak{g}^{(1)}$ is of the stated form because every generator is natural (Proposition 11.1(2)). $\square$

11.3. **Torsion revisited.** The preceding theorem will allow us to prove Theorem 9.3 relating Θ-reduction and torsion reduction. The appropriate argument also leads to an alternative form of Theorem 11.2 when $\mathfrak{t} = TM$. See 11.4 below.

Although the definition of $\tilde{\Theta}$ in (1) above is explicit, it depends on a choice of linear connection $\nabla^{\mathfrak{h}}$ on $\mathfrak{h}$. Here is an implicit formula depending only on a choice of generator ∇ for $\mathfrak{g}$:

Proposition. *Let $\bar{\nabla}$ denote the $\mathfrak{t}$-connection on $\mathfrak{t}$ associated with a generator ∇ of $\mathfrak{g} \subset J^1\mathfrak{t}$. Then for arbitrary sections $X \subset \mathfrak{g}$ and $U_1, U_2 \subset \mathfrak{t}$, one has*

$$\tilde{\Theta}(X)(\#U_1, \#U_2) = \Big(X \cdot \operatorname{tor} \bar{\nabla} + \Delta(\operatorname{cocurv} \nabla(aX, \,\cdot\,)) \Big)(U_1, U_2),$$

where Δ is the upper coboundary morphism and $a\colon \mathfrak{g} \to \mathfrak{t}$ the projection.

Note that $\operatorname{cocurv} \nabla(aX, \,\cdot\,)$ is a section of $T^*M \otimes \mathfrak{h}$, by Proposition 6.2(3).

Proof. Since $\mathfrak{t}$ is assumed to be transitive, there exists a linear connection $\nabla^{\mathfrak{h}}$ on $\mathfrak{h}$ such that $\nabla^{\mathfrak{h}}_{\#U} = \bar{\nabla}_U$ for all $U \in \mathfrak{t}$. Here $\bar{\nabla}$ denotes the associated $\mathfrak{t}$-connection on $\mathfrak{h}$ discussed in 6.3(3). After a little manipulation, we obtain

$$(1) \qquad (d_{\nabla}\phi - \delta(\nabla^{\mathfrak{h}}\phi))(\#U_1, \#U_2) = (\phi \cdot \operatorname{tor} \bar{\nabla})(U_1, U_2); \quad U_1, U_2 \subset \mathfrak{t}.$$

Note that $T^*M \otimes \mathfrak{t}$ (of which ϕ is a section) acts on $\operatorname{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}$ as a subalgebroid of $J^1\mathfrak{t}$ (which acts on $\mathfrak{t}$ via adjoint action).

Replace $\mathfrak{g}$ in Proposition 3.6 with $\mathfrak{t}$ and replace ∇ there by the composite

$$\mathfrak{t} \xrightarrow{\#} TM \xrightarrow{\nabla} \mathfrak{gl}(\mathfrak{t}).$$

Then part (2) of that proposition delivers the formula,

$$\begin{aligned} \operatorname{curv} \nabla\, (\#U_1, \#U_2)V = (\bar{\nabla}_V \operatorname{tor} \bar{\nabla})(U_1, U_2) &- \operatorname{curv} \bar{\nabla}\, (V, U_1)U_2 \\ &+ \operatorname{curv} \bar{\nabla}\, (V, U_2)U; \qquad U_1, U_2, V \subset \mathfrak{t}. \end{aligned}$$

Applying Proposition 4.2(4), we may rewrite this as,

$$(2) \qquad \operatorname{curv} \nabla(\#U_1, \#U_2)V = (\bar{\nabla}_V \operatorname{tor} \bar{\nabla} + \Delta(\operatorname{cocurv} \nabla(V, \,\cdot\,))\,)(U_1, U_2).$$

Substituting (1) and (2) into the definition 11.2(1) of $\tilde{\Theta}$ gives,

$$\tilde{\Theta}(V \oplus \phi)(\#U_1, \#U_2) = \Big(\bar{\nabla}_V \operatorname{tor} \bar{\nabla} + \phi \cdot \operatorname{tor} \bar{\nabla} + \Delta(\operatorname{cocurv} \nabla(V, \,\cdot\,)) \Big)(U_1, U_2).$$

Under the identification $\mathfrak{g} \cong \mathfrak{t} \oplus \mathfrak{h}$ determined by the generator ∇, we obtain the stated formula. □

Proof of Theorem 9.3. By Theorem 11.2(2) and the commutativity of 9.1(1), we have,

$$\psi(\Theta(X)) = i(\tilde{\Theta}(X)) \bmod \operatorname{im} \Delta,$$

where $i\colon \operatorname{Alt}^2(TM) \otimes \mathfrak{t} \to \operatorname{Alt}^2(\mathfrak{t}) \otimes \mathfrak{t}$ denotes inclusion. The proposition above then gives

$$\psi(\Theta(X)) = X \cdot \operatorname{tor} \bar{\nabla} \bmod \operatorname{im} \Delta = X \cdot \tau.$$

□

11.4. **The special case $\mathfrak{g} \subset J^1(TM)$.** We now specialize Theorem 11.2 to the case $\mathfrak{t} = TM$. As an application, we complete the proof of Theorem 9.5, the last unproven assertion of preceding sections.

Let $\mathfrak{g} \subset J^1(TM)$ be an infinitesimal geometric structure with structure kernel $\mathfrak{h}$ and generator ∇. Let $\bar{\nabla}$ denote the dual connection, i.e., $\bar{\nabla}_U V = \nabla_V U + [U, V]$ and define

$$TM \oplus \mathfrak{h} \xrightarrow{\tilde{\tilde{\Theta}}} \mathrm{Alt}^2(TM) \otimes TM$$
$$\tilde{\tilde{\Theta}}(V \oplus \phi) := \bar{\nabla}_V \operatorname{tor} \bar{\nabla} + \phi \cdot \operatorname{tor} \bar{\nabla} = (V \oplus \phi) \cdot \operatorname{tor} \bar{\nabla}.$$

Theorem (Prolonging a generator of $\mathfrak{g} \subset J^1(TM)$)**.** *With $\mathfrak{g}$, $\mathfrak{h}$, ∇, and $\tilde{\tilde{\Theta}}$ as above, we have:*

(1) *The composite morphism,*

$$\mathfrak{g} \cong TM \oplus \mathfrak{h} \xrightarrow{\tilde{\tilde{\Theta}}} \mathrm{Alt}^2(TM) \otimes TM \xrightarrow{\operatorname{im} \Delta} H(\mathfrak{g})$$

coincides with the morphism $\Theta\colon \mathfrak{g} \to h(\mathfrak{g}) = H(\mathfrak{g})$ defined in 9.2.

(2) $\ker \Theta \subset \mathfrak{g} \cong TM \oplus \mathfrak{h}$ *is precisely the set of all $V \oplus \phi$ for which the generator equation,*

$$\Delta(\epsilon) = \tilde{\tilde{\Theta}}(V \oplus \phi),$$

admits a solution $\epsilon \in T^*M \otimes \mathfrak{h}$.

(3) *Assume* $\ker \Delta$ *and* $\ker \Theta$ *have constant rank, so that $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is an infinitesimal geometric structure (by Proposition 9.2) and that $H(\mathfrak{g})$ is a $\mathfrak{g}$-representation (has constant rank). Then $\Theta(X) = X \cdot \tau$, where $\tau \subset H(\mathfrak{g})$ is the intrinsic torsion. Also, a linear connection $\nabla^{(1)}$ on $\mathfrak{g} \cong TM \oplus \mathfrak{h}$ generating $\mathfrak{g}^{(1)}$ is given by*

$$\boxed{\nabla^{(1)}_U (V \oplus \phi) = (\nabla_U V + \phi(U)) \oplus (\bar{\nabla}_U \phi + \epsilon(V \oplus \phi)U + \operatorname{curv} \bar{\nabla}(U, V)),}$$

where $\epsilon\colon TM \oplus \mathfrak{h} \to T^*M \otimes \mathfrak{h}$ is any of the vector bundle morphisms for which $\epsilon := \epsilon(V \oplus \phi)$ solves the generator equation defined in (2) *above, for each $V \oplus \phi$ in* $\ker \Theta$*. If $\mathfrak{g}^{(1)}$ is surjective (i.e., $\Theta = 0$) then every generator is of the above form.*

Note. The ϵ's solving the generator equation above, and the generator equation of Theorem 11.2, are different.

Proof. In 11.2 above take $\mathfrak{t} = TM$ and let $\nabla^{\mathfrak{h}}$ be the TM-connection on $\mathfrak{h}$ associated with the generator ∇ (given by 6.2(1) with $\mathfrak{t} = \mathfrak{t}_1 = TM$). Then Proposition 11.3 gives

$$\tilde{\Theta}(V \oplus \phi) = \tilde{\tilde{\Theta}}(V \oplus \phi) + \Delta(\operatorname{cocurv} \nabla(V, \cdot\,).$$

Noting that $\operatorname{cocurv} \nabla = -\operatorname{curv} \bar{\nabla}$ and $\delta = \Delta$ (because $\mathfrak{t} = TM$) one obtains the stated results as a special case of Theorem 11.2. □

Proof of Theorem 9.5. The hypothesis that $\mathfrak{g}$ be torsion-reduced means $\Theta = 0$. So the generator equation defined in (2) has a solution for all $V \oplus \phi \in TM \oplus \mathfrak{h}$. The solution is unique because Δ is injective, by hypothesis. Part (1) of 9.5 follows. The Cartan connection on $\mathfrak{g}$ in Theorem 9.5 is the unique generator of $\mathfrak{g}^{(1)}$; conclusion (3) above implies that it has the form given in part (2) of 9.5.

Suppose $\mathfrak{g}$ is reductive and let ∇ be a normal generator. Then $\operatorname{tor}\bar\nabla \subset C$ for some $\mathfrak{g}$-invariant complement $C \subset \operatorname{Alt}^2(TM)\otimes TM$ of the image of Δ. In particular,

$$\bar\nabla_V \operatorname{tor}\bar\nabla + \phi\cdot\operatorname{tor}\bar\nabla = (V\oplus\phi)\cdot\operatorname{tor}\bar\nabla$$

is a section of C. However, this last also lies in $\operatorname{im}\Delta$ because its image under the projection

$$\operatorname{Alt}^2(TM)\otimes TM \xrightarrow{\operatorname{im}\Delta} H(\mathfrak{g})$$

is $(V\oplus\phi)\cdot\tau$, which vanishes because $\mathfrak{g}$ is torsion-reduced. We conclude that $\bar\nabla_V \operatorname{tor}\bar\nabla + \phi\cdot\operatorname{tor}\bar\nabla = 0$. Part (3) of 9.5 follows. □

11.5. **Symmetries of torsion-free affine structures.** We now offer a simple application of Theorem 11.4. Let ∇ be a torsion-free linear connection on TM. Then ∇ is a generator of $J^1(TM)$. Taking $\mathfrak{g} = J^1(TM)$, the generator equation in 11.4(2) reads $\Delta(\epsilon)=0$, with trivial solution $\epsilon = 0$. Theorem 11.4(3) delivers the following generator $\nabla^{(1)}$ for $J^2(TM) = (J^1(TM))^{(1)} = \mathfrak{g}^{(1)}$:

$$\nabla^{(1)}(V\oplus\phi) = (\nabla_U V + \phi(U))\oplus(\nabla_U\phi + \operatorname{curv}\nabla(U,V)).$$

Here we are identifying $\mathfrak{g} = J^1(TM)$ with $TM\oplus(T^*M\otimes TM)$ using ∇.

Now let $\mathfrak{g}\subset J^2(TM)$ instead denote the isotropy subalgebroid of ∇, as described in 5.7. Then it is not too difficult to check that $\nabla^{(1)}$ is even a generator for $\mathfrak{g}$. In this regard, a helpful formula, readily derived, is

$$\nabla^{(1)}(J^1V)(U_1,U_2) = 0\oplus((J^2V)\cdot\nabla)^\vee(U_1,U_2),$$

for any section $V\subset TM$.

The generator $\nabla^{(1)}$ is necessarily the Cartan connection on $J^1(TM)$ referred to in Proposition 5.7. Its curvature is given by

$$\operatorname{curv}\nabla^{(1)}(U_1,U_2)(V\oplus\phi) = 0\oplus\Big(-(\nabla_V\operatorname{curv}\nabla + \phi\cdot\operatorname{curv}\nabla)(U_1,U_2)\Big).$$

In particular, $\operatorname{curv}\nabla^{(1)}$ vanishes if and only if $\operatorname{curv}\nabla$ is ∇-parallel and $T^*M\otimes TM$-invariant. But as id_{TM} is a section of $T^*M\otimes TM$, this happens if and only if $\operatorname{curv}\nabla = 0$. In that case we obtain

$$\operatorname{tor}\overline{\nabla^{(1)}}(V_1\oplus\phi_1, V_2\oplus\phi_2) = (\,\phi_1(V_2)-\phi_2(V_1)\,)\oplus[\phi_1,\phi_2]_{T^*M\otimes TM},$$

where $\overline{\nabla^{(1)}}$ denotes the representation of $J^1(TM)$ on itself associated with the Cartan connection $\nabla^{(1)}$ on $J^1(TM)$. Applying Theorem 4.5, we recover the following classical result:

Proposition. *Let ∇ be a torsion-free linear connection on TM and $\mathfrak{g}_0$ the Lie algebra of infinitesimal isometries of ∇ on some open set $\mathcal{U}\subset M$. Then* $\dim\mathfrak{g}_0 \leqslant \operatorname{rank} J^1(TM) = n(n+1)$, $n=\dim M$. *If $\mathcal{U}$ is simply-connected then equality holds if and only if* $\operatorname{curv}\nabla = 0$, *in which case $\mathfrak{g}_0$ is naturally isomorphic to the semidirect product $T_mM\oplus(T_m^*M\otimes T_mM)$, $m\in U$.*

11.6. **Technical details.** We conclude the section with the proof of Lemma 11.1.

Keeping in mind the observation 9.2(1), and its proof, it is not difficult to established 11.1(4). One readily checks that the connection in 11.1(6) is natural and, with the help of 9.2(1) and 11.1(4) that all possibilities are covered.

It remains to prove 11.1(5). Let D be a natural connection on $\mathfrak{g}$ and $s\colon\mathfrak{g}\to J^1\mathfrak{g}$ the corresponding splitting. Denote the associated $\mathfrak{g}$-connection on $\mathfrak{g}$ by $\bar D$, i.e.,

(1) $$\bar D_X Y = D_{\#Y}X + [X,Y]_{\mathfrak{g}};\qquad X,Y\subset\mathfrak{g}.$$

Then $sX \cdot Y = \bar{D}_X Y$ and naturality means $a\bar{D}_X Y = \mathrm{ad}^{\mathfrak{t}}_X(aY)$; $X, Y \subset \mathfrak{g}$. For arbitrary sections $Y_1, Y_2 \subset \mathfrak{g}$ we compute

$$\begin{aligned}
(sX \cdot da)^\vee(\#Y_1, \#Y_2) &= (sX \cdot da)(Y_1, Y_2) \\
&= \mathrm{ad}^{\mathfrak{t}}_X(da(Y_1, Y_2)) - da(\bar{D}_X Y_1, Y_2) - da(Y_1, \bar{D}_X Y_2) \\
&= \mathrm{ad}^{\mathfrak{t}}_X \mathrm{ad}^{\mathfrak{t}}_{Y_1}(aY_2) - \mathrm{ad}^{\mathfrak{t}}_X \mathrm{ad}^{\mathfrak{t}}_{Y_2}(aY_1) - \mathrm{ad}^{\mathfrak{t}}_X[aY_1, aY_2]_{\mathfrak{t}} \\
&\quad - \mathrm{ad}^{\mathfrak{t}}_{\bar{D}_X Y_1}(aY_2) + \mathrm{ad}^{\mathfrak{t}}_{Y_2}\mathrm{ad}^{\mathfrak{t}}_X(aY_1) + [a\bar{D}_X Y_1, aY_2]_{\mathfrak{t}} \\
&\quad - \mathrm{ad}^{\mathfrak{t}}_{Y_1}\mathrm{ad}^{\mathfrak{t}}_X(Y_2) + \mathrm{ad}^{\mathfrak{t}}_{\bar{D}_X Y_2}(aY_1) + [aY_1, a\bar{D}_X Y_2]_{\mathfrak{t}} \\
&= \mathrm{ad}^{\mathfrak{t}}_{\bar{D}_X Y_2 + [Y_2, X]_{\mathfrak{g}}}(aY_1) - \mathrm{ad}^{\mathfrak{t}}_{\bar{D}_X Y_1 + [Y_1, X]_{\mathfrak{g}}}(aY_2) \\
&\quad - \mathrm{ad}^{\mathfrak{t}}_X(a[Y_1, Y_2]_{\mathfrak{t}}) + a[\bar{D}_X Y_1, Y_2]_{\mathfrak{t}} + a[Y_1, \bar{D}_X Y_2]_{\mathfrak{t}} \\
&= a\Big(\bar{D}_{\bar{D}_X Y_2 + [Y_2, X]_{\mathfrak{g}}} Y_1 - \bar{D}_{\bar{D}_X Y_1 + [Y_1, X]_{\mathfrak{g}}} Y_2 \\
&\qquad - \bar{D}_X[Y_1, Y_2]_{\mathfrak{g}} + [\bar{D}_X Y_1, Y_2]_{\mathfrak{g}} + [Y_1, \bar{D}_X Y_2]_{\mathfrak{g}} \Big).
\end{aligned}$$

Proceeding with several applications of (1), we obtain:

$$\begin{aligned}
\text{term in parentheses} &= \bar{D}_{D_{\#Y_2} X} Y_1 - \bar{D}_{D_{\#Y_1} X} Y_2 \\
&\quad - \bar{D}_X[Y_1, Y_2]_{\mathfrak{g}} + [\bar{D}_X Y_1, Y_2]_{\mathfrak{g}} + [Y_1, \bar{D}_X Y_2]_{\mathfrak{g}} \\
&= D_{\#Y_1} D_{\#Y_2} X + [D_{\#Y_2} X, Y_1]_{\mathfrak{g}} - D_{\#Y_2} D_{\#Y_1} X - [D_{\#Y_1} X, Y_2]_{\mathfrak{g}} \\
&\quad - D_{[\#Y_1, \#Y_2]_{TM}} X - [X, [Y_1, Y_2]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&\quad + [D_{\#Y_1} X + [X, Y_1]_{\mathfrak{g}}, Y_2]_{\mathfrak{g}} + [Y_1, D_{\#Y_2} X + [X, Y_2]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= D_{\#Y_1} D_{\#Y_2} X - D_{\#Y_2} D_{\#Y_1} X - D_{[\#Y_1, \#Y_2]_{TM}} X = \operatorname{curv} D\,(\#Y_1, \#Y_2)X.
\end{aligned}$$

Jacobi's identity has been applied in the second-last equality. We deduce that

$$(sX \cdot da)^\vee(\#Y_1, \#Y_2) = a(\operatorname{curv} D\,(\#Y_1, \#Y_2)X).$$

Since $\mathfrak{g}$ is transitive (because $\mathfrak{t}$ is transitive and $\mathfrak{g}$ is surjective), the claim in 11.1(5) follows.

12. Conformal structures

In this section we turn to the application of Cartan's method to conformal structures. Our results are summarized in Theorems 12.3, 12.4 and 12.5 below.

12.1. **The Lie algebroid setting.** Let σ be a Riemannian metric on a smooth connected manifold M, with $\dim M \geqslant 3$. Let $\langle\sigma\rangle$ denote its conformal class, viewed as the one-dimensional subbundle of $\mathrm{Sym}^2(TM)$ generated by the section σ.

Let $\mathfrak{g}_\sigma \subset J^1(TM)$ denote the isotropy of $\sigma \subset \mathrm{Sym}^2(TM)$ (denoted $\mathfrak{g}$ in 5.3). Let $\mathfrak{g} \subset J^1(TM)$ denote the isotropy of $\langle\sigma\rangle \subset \mathrm{Sym}^2(TM)$. According to 5.1, this means that the 1-jet of a vector field V at $m \in M$ lies in $\mathfrak{g}$ if and only if $(\mathcal{L}_V\sigma)(m) \in \langle\sigma\rangle(m)$, were $\mathcal{L}$ denotes Lie derivative. Evidently, the symmetries of $\mathfrak{g}$ are the infinitesimal isometries of the conformal structure $\langle\sigma\rangle$, henceforth called the *conformal Killing fields*.

The rank of $\mathfrak{g}$ is constant (see below), making it an infinitesimal geometric structure on TM. Observe that $\mathfrak{g}$ is surjective because $\mathfrak{g}_\sigma$ is surjective and $\mathfrak{g}_\sigma \subset \mathfrak{g}$. If $\mathfrak{g}_\sigma$ and $\mathfrak{g}$ have respectively $\mathfrak{h}_\sigma$ and $\mathfrak{h}$ as structure kernels, then

$$\mathfrak{h} = \mathfrak{h}_\sigma \oplus \langle \mathrm{id}_{TM}\rangle, \text{ implying } \mathfrak{g} = \mathfrak{g}_\sigma \oplus \langle \mathrm{id}_{TM}\rangle.$$

Here $\langle \mathrm{id}_{TM} \rangle$ denotes the one-dimensional subbundle of $T^*M \otimes TM$ generated by the identity section id_{TM}. If vector bundles E_1 and E_2 are representations of $\mathfrak{g}$, then a vector bundle morphism $\phi\colon E_1 \to E_2$ is a morphism of $\mathfrak{g}$-representations if and only if it is a morphism of $\mathfrak{g}_\sigma$-representations commuting with the action of $\mathrm{id}_{TM} \subset \mathfrak{g}$. In particular, this applies to $\alpha \otimes V \mapsto (\alpha \otimes V)^{\mathrm{t}} := \sigma(V) \otimes \sigma^{-1}(\alpha)$ (the transpose involution of $T^*M \otimes TM$) and to the epimorphism, skew: $T^*M \otimes TM \to \mathfrak{h}_\sigma$, defined by $\mathrm{skew}(\phi) := (\phi - \phi^{\mathrm{t}})$. These morphisms and $\mathfrak{h}_\sigma$ depend only on the conformal class of σ.

Because the Levi-Cevita connection ∇ associated with σ generates $\mathfrak{g}_\sigma$ (see 9.6), it also generates $\mathfrak{g} \supset \mathfrak{g}_\sigma$.

12.2. **Classical ingredients.** From elementary classical theory we know that the curvature of the Levi-Cevita connection ∇ takes values in a proper subbundle $E_{\mathrm{Weyl}} \oplus E_{\mathrm{Ricci}} \subset \mathrm{Alt}^2(TM) \otimes \mathfrak{h}_\sigma$, with E_{Ricci} being (for $\dim M \geqslant 3$) the isomorphic image of $\mathrm{Sym}^2(TM)$ under the monomorphism of $\mathfrak{g}$-representations

$$T^*M \otimes T^*M \xrightarrow{\mathrm{coRicci}} \mathrm{Alt}^2(TM) \otimes \mathfrak{h}_\sigma$$
$$\mathrm{coRicci}(\Phi)(V_1, V_2) := \mathrm{skew}(\Phi V_1 \otimes V_2 - \Phi V_2 \otimes V_1).$$

$E_{\mathrm{Weyl}} \subset \mathrm{Alt}^2(TM) \otimes \mathfrak{h}_\sigma$ is the intersection of the kernels of the so-called Bianchi and Ricci morphisms; see, e.g., [18, p. 230]. Whence,

$$\mathrm{curv}\, \nabla = W + \mathrm{coRicci}(R) \tag{1}$$

for uniquely determined sections $W \subset E_{\mathrm{Weyl}}$ and $R \subset \mathrm{Sym}^2(TM)$. These are the *Weyl* and *modified Ricci* curvatures of σ. Both E_{Weyl} and E_{Ricci} are $\mathfrak{g}$-representations of $\mathrm{Alt}^2(TM) \otimes_{\mathfrak{h}_\sigma}$ and in particular we may speak of the isotropy $\mathfrak{g}_W \subset \mathfrak{g}$ of W.

Also of significance will be the *Cotton-York tensor*. This is the associated exterior derivative $d_\nabla R$ of $R \subset \mathrm{Sym}^2(TM) \subset T^*M \otimes T^*M$, viewed as a T^*M-value one-form on M:

$$d_\nabla R(U_1, U_2) := \nabla_{U_1}(R(U_2)) - \nabla_{U_2}(R(U_1)) - R([U_1, U_2]).$$

Alternatively, by torsion-freeness, $d_\nabla R$ is the image of ∇R under the composite

$$T^*M \otimes \mathrm{Sym}^2(TM) \hookrightarrow T^*M \otimes T^*M \otimes T^*M \to \mathrm{Alt}^2(TM) \otimes T^*M$$
$$\alpha \otimes \beta \otimes \gamma \mapsto \alpha \wedge \beta \otimes \gamma.$$

Bianchi's second identity 6.5(4) for the generator ∇ enforces a relationship between the Cotton-York tensor $d_\nabla R$, and the derivative ∇W. In particular, it is well known that $W = 0$ implies the vanishing of $d_\nabla R$ in all dimensions except three, where the values of $d_\nabla R$ are merely restricted to a certain six-dimensional subbundle of $\mathrm{Alt}^2(TM) \otimes T^*M$.

12.3. **Results I: Conformal invariance of W.**

Theorem. *The Weyl curvature W is an invariant of $\mathfrak{g}$ and therefore a conformal invariant.*

Classically, the conformal invariance of W required an explicit check. In our proof of this theorem conformal invariance is simply a manifestation of the invariance of the morphism Θ defined in general in 9.2, as applied to the prolongation of $\mathfrak{g}$ (rather than to $\mathfrak{g}$ itself). This is invariant by construction.

12.4. **Results II: The $W = 0$ case.** Our second theorem likewise expresses, in invariant Lie algebroid language, results that are essentially classical:

Theorem. *Suppose $W = 0$. Then $\mathfrak{g}$ has an associated Cartan algebroid, namely its prolongation $\mathfrak{g}^{(1)} \subset J^2(TM)$, which is surjective and transitive. Denoting the Cartan connection on $\mathfrak{g}^{(1)}$ by $\nabla^{(2)}$, we have:*

(1) *The $\nabla^{(2)}$-parallel sections of $\mathfrak{g}^{(1)} \subset J^2(TM)$ are precisely the twice-prolonged conformal Killing fields.*
(2) *Each metric σ in the conformal class determines natural isomorphisms*
$$\mathfrak{g}^{(1)} \cong \mathfrak{g} \oplus T^*M, \qquad \mathfrak{g} \cong TM \oplus \mathfrak{h},$$
and an associated explicit formula for $\nabla^{(2)}$ (see 12.10(1) and 12.8(1) below).
(3) *If $n \geq 4$, then $\nabla^{(2)}$ is automatically flat. If $n = 3$, then $\nabla^{(2)}$ is flat if and only if $d_\nabla R = 0$. In particular, the Lie algebra $\mathfrak{g}_0$ of all conformal Killing fields over any simply-connected open set $\mathcal{U} \subset M$ satisfies*
$$\dim \mathfrak{g}_0 \leq \operatorname{rank} \mathfrak{g}^{(1)} = \frac{1}{2}(n+1)(n+2),$$
with equality holding if and only if $n \geq 4$ or $d_\nabla R = 0$.

12.5. **Results III: The $W \neq 0$ case and intransitivity.** In the general case, our application of Cartan's method is incomplete. We do, however, establish the following preliminary result:

Theorem. *The isotropy $\mathfrak{g}_W \subset \mathfrak{g}$ of W is a reduction of $\mathfrak{g}$ whenever it has constant rank.*

There is the possibility that $\mathfrak{g}_W \subset J^1(TM)$ (and hence the Cartan algebroid associated with $\langle \sigma \rangle$) fails to be transitive. Indeed, applying Lemma B.1 to the morphism $X \mapsto W \cdot X \colon \mathfrak{g} \to E_{\mathrm{Weyl}}$, it is not difficult to see that $\mathfrak{g}$ has image $D \subset TM$ (a possibly singular distribution on M), where D is the kernel of the morphism
$$\begin{aligned} TM &\to E_{\mathrm{Weyl}}/(\mathfrak{h} \cdot W) \\ V &\mapsto \nabla_V W \bmod \mathfrak{h} \cdot W. \end{aligned}$$
Recall here that $\mathfrak{h} \subset T^*M \otimes TM$ is the vector bundle whose fiber $\mathfrak{h}(m)$ over m is the Lie algebra of all infinitesimally conformal endomorphisms of $T_m M$. This Lie algebra acts on $E_{\mathrm{Weyl}}(m)$ and
$$\mathfrak{h} \cdot W = \bigcup_{m \in M} \{\phi \cdot W(m) \mid \phi \in \mathfrak{h}(m)\}.$$
For 'generic' W the above morphism is not likely to vanish, even when conditions placed on ∇W by Bianchi's second identity are taken into account.

The remainder of the paper is devoted to the proofs of the three preceding theorems.

12.6. **The torsion reduction of $\mathfrak{g}$.** Since $\mathfrak{g} \subset J^1(TM)$, Θ-reduction is the same as torsion reduction. To compute it, we turn to the upper (=lower) coboundary morphism for $\mathfrak{g}$,
$$T^*M \otimes \mathfrak{h} \xrightarrow{\Delta} \mathrm{Alt}^2(TM) \otimes TM.$$
Its restriction to $T^*M \otimes \mathfrak{h}_\sigma$ is nothing but the upper coboundary morphism for $\mathfrak{g}_\sigma$. Since the latter is an isomorphism (see 9.6) the former is surjective. In particular,

$H(\mathfrak{g}) = 0$, implying $\mathfrak{g}$ is already torsion-reduced. Theorem 9.5 does not apply, however, because Δ has non-trivial kernel. Indeed, counting dimensions, we have

$$\text{rank}(\ker \Delta) = \text{rank}(TM). \tag{1}$$

12.7. **The first prolongation $\mathfrak{g}^{(1)}$.** Since $\mathfrak{g}$ is torsion-reduced (and hence Θ-reduced) the prolongation $\mathfrak{g}^{(1)}$ is surjective (Proposition 2.12). By Proposition 9.2, its structure kernel $\mathfrak{h}^{(1)}$ is $\ker \delta = \ker \Delta$. Define a map

$$T^*M \xrightarrow{i} \text{Sym}^2(TM) \otimes TM$$
$$i(\alpha) := j_{\text{S}}(\alpha) - \sigma \otimes \sigma^{-1}(\alpha),$$

where $j_{\text{S}} \colon T^*M \to \text{Sym}^2(TM) \otimes TM$ is the canonical morphism defined by

$$j_{\text{S}}(\alpha)(V_1, V_2) = \alpha(V_1)V_2 + \alpha(V_2)V_1.$$

Then i is a monomorphism of $\mathfrak{g}$-representations ($\dim M \geqslant 2$). Since

$$i(\alpha)V = \text{skew}(\alpha \otimes V) + \alpha(V)\text{id}_{TM}; \qquad V \subset TM,$$

we have $i(T^*M) \subset T^*M \otimes \mathfrak{h}$. Therefore

$$i(T^*M) \subset (T^*M \otimes \mathfrak{h}) \cap (\text{Sym}^2(TM) \otimes TM) = \ker \Delta = \mathfrak{h}^{(1)}.$$

Invoking 12.6(1), we conclude that $\mathfrak{h}^{(1)} = i(T^*M)$.

12.8. **A generator for $\mathfrak{g}^{(1)}$.** To obtain a generator for $\mathfrak{g}^{(1)}$ we apply Theorem 11.4. Recalling that the Levi-Cevita connection ∇ generates $\mathfrak{g}$, we compute $\tilde{\tilde{\Theta}} = 0$. We may therefore take $\epsilon = 0$ in 11.4 and, using ∇ to identify $\mathfrak{g}$ with $TM \oplus \mathfrak{h}$, obtain

$$\nabla^{(1)}_U(V \oplus \phi) := (\nabla_U V + \phi(U)) \oplus (\nabla_U \phi + \text{curv}\, \nabla(U, V)). \tag{1}$$

We compute, with the help of Bianchi's second identity,

$$\text{curv}\, \nabla^{(1)}(U_1, U_2)(V \oplus \phi) = 0 \oplus \Big(- (\nabla_V \,\text{curv}\, \nabla + \phi \cdot \text{curv}\, \nabla)(U_1, U_2) \Big).$$

This formula may also be written

$$\text{curv}\, \nabla^{(1)}(U_1, U_2)X = -(X \cdot \text{curv}\, \nabla)(U_1, U_2) \subset \mathfrak{h}_\sigma; \quad X \subset \mathfrak{g}. \tag{2}$$

12.9. **The Θ-reduction of $\mathfrak{g}^{(1)}$.** Since $\mathfrak{g}^{(1)} \subset J^1\mathfrak{g}$ is surjective and $\mathfrak{g}$ is transitive, the Θ-reduction of $\mathfrak{g}^{(1)}$ is the kernel of a morphism $\mathfrak{g}^{(1)} \to h(\mathfrak{g}^{(1)})$, which we denote by $\Theta^{(1)}$, to distinguish it from the corresponding morphism $\Theta \colon \mathfrak{g} \to h(\mathfrak{g})$ for $\mathfrak{g}$; see 9.2. The definition of $h(\mathfrak{g}^{(1)})$ depends on the lower coboundary morphism for $\mathfrak{g}^{(1)}$, which we denote by

$$T^*M \otimes \mathfrak{h}^{(1)} \xrightarrow{\delta^{(1)}} \text{Alt}^2(TM) \otimes \mathfrak{g}.$$

Identifying $\mathfrak{h}^{(1)}$ with T^*M as described above, one shows that $\delta^{(1)}$ is the map

$$\alpha \otimes \beta \mapsto \text{coRicci}(\alpha \otimes \beta) + (\alpha \wedge \beta) \otimes \text{id}_{TM}.$$

Note that the first term on the right belongs to $\text{Alt}^2(TM) \otimes \mathfrak{h}_\sigma$ and the second to $\text{Alt}^2(TM) \otimes \langle \text{id}_{TM} \rangle$. In particular, the image of $\delta^{(1)}$ lies entirely within $\text{Alt}^2(TM) \otimes \mathfrak{h}$.

Since coRicci is injective ($n \geqslant 3$) we have $\ker \delta^{(1)} = 0$. Therefore the second prolongation $\mathfrak{g}^{(2)} := (\mathfrak{g}^{(1)})^{(1)}$ of $\mathfrak{g}$ has trivial structure kernel (Proposition 9.2). In particular, $h(\mathfrak{g}^{(1)}) := (\text{Alt}^2(TM) \otimes \mathfrak{g}) / \operatorname{im} \delta^{(1)}$ has constant rank.

Next, we observe that the composite morphism of $\mathfrak{g}$-representations,

$$E_{\mathrm{Weyl}} \hookrightarrow \mathrm{Alt}^2(TM) \otimes \mathfrak{h}_\sigma \hookrightarrow \mathrm{Alt}^2(TM) \otimes \mathfrak{g} \xrightarrow{/\operatorname{im}\delta^{(1)}} h(\mathfrak{g}^{(1)})$$

is injective. This follows from the description of $\delta^{(1)}$ above, and

$$E_{\mathrm{Weyl}} \cap E_{\mathrm{Ricci}} = 0,$$
$$\text{where} \quad E_{\mathrm{Ricci}} = \mathrm{coRicci}(\mathrm{Sym}^2(TM)).$$

Identifying E_{Weyl} with the corresponding $\mathfrak{g}$-subrepresentation of $h(\mathfrak{g}^{(1)})$, we have:

Proposition. *The following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}^{(1)} & \xrightarrow{\Theta^{(1)}} & h(\mathfrak{g}^{(1)}) \\ {\scriptstyle\text{projection}}\big\downarrow & & \big\uparrow{\scriptstyle\text{inclusion}} \\ \mathfrak{g} & \xrightarrow[X \mapsto -X\cdot W]{} & E_{\mathrm{Weyl}} \end{array}.$$

Using the fact that $\mathfrak{g}^{(1)}$ and $\Theta^{(1)}$ are invariants of $\mathfrak{g}$, together with the fact that id_{TM} is a section of $\mathfrak{g}$, one deduces Theorem 12.3. The proposition also shows that the Θ-reduction of $\mathfrak{g}^{(1)}$ (the kernel of $\Theta^{(1)}$) is the preimage of $\mathfrak{g}_W$ under the projection $\mathfrak{g}^{(1)} \to \mathfrak{g}$. Theorem 12.5 is a consequence.

Proof of proposition. We will apply part (2) of Theorem 11.2, with the roles of $\mathfrak{g}, \mathfrak{t}, \mathfrak{h}, \delta, \Theta, \tilde{\Theta}, \mathfrak{g}^{(1)}$ in the theorem being played by $\mathfrak{g}^{(1)}, \mathfrak{g}, \mathfrak{h}^{(1)}, \delta^{(1)}, \Theta^{(1)}, \tilde{\Theta}^{(1)}, \mathfrak{g}^{(2)}$.

Our first task is to choose a connection $\nabla^{\mathfrak{h}^{(1)}}$ on $\mathfrak{h}$. The Levi-Cevita connection ∇ on TM determines a linear connection on $T^*M \otimes (T^*M \otimes TM)$ and one has a chain of inclusions

$$\mathfrak{h}^{(1)} \subset T^*M \otimes \mathfrak{h} \subset T^*M \otimes (T^*M \otimes TM),$$

which we claim are ∇-invariant. The ∇-invariance of $\mathfrak{h} \subset T^*M \otimes TM$ follows from Proposition 6.2(2). So the second inclusion is indeed ∇-invariant. Because ∇ generates $\mathfrak{g}$ and because

$$\Delta \colon T^*M \otimes \mathfrak{h} \to \mathrm{Alt}^2(TM) \otimes TM$$

is $\mathfrak{g}$-equivariant, it follows that Δ is $\bar{\nabla}$-equivariant (by Proposition 6.3). But ∇ is torsion free, meaning $\bar{\nabla}$-invariance is the same as ∇-invariance. So the kernel $\mathfrak{h}^{(1)} \subset T^*M \otimes \mathfrak{h}$ of Δ must be ∇-invariant, as claimed.

We choose $\nabla^{\mathfrak{h}^{(1)}}$ to be the connection that $\mathfrak{h}^{(1)}$ inherits from $T^*M \otimes \mathfrak{h}$ as a ∇-invariant subbundle. Appealing to 12.8(1) and the fact that $\mathfrak{h}^{(1)} \subset \mathrm{Sym}^2(TM) \otimes TM$, one can show that

$$\Big(d_{\nabla^{(1)}}\phi - \delta^{(1)}(\nabla^{\mathfrak{h}^{(1)}}\phi) \Big)(U_1, U_2) = \phi(\operatorname{tor}\nabla(U_1, U_2)) = 0, \tag{1}$$

for all sections $\phi \subset \mathfrak{h}^{(1)} \subset T^*M \otimes \mathfrak{h} \subset T^*M \otimes \mathfrak{g}$ and $U_1, U_2 \subset TM$.

In the present context 11.2(1) reads

$$\tilde{\Theta}^{(1)}(X \oplus \phi) := \operatorname{curv}\nabla^{(1)}(\,\cdot\,,\,\cdot\,)X - d_{\nabla^{(1)}}\phi + \delta^{(1)}(\nabla^{\mathfrak{h}^{(1)}}\phi).$$

From 12.8(2) and (1) above one obtains

$$\tilde{\Theta}^{(1)}(X \oplus \phi) = -X \cdot \operatorname{curv}\nabla = -X \cdot W + \mathrm{coRicci}(X \cdot R), \tag{2}$$

for arbitrary sections $X \subset \mathfrak{g}$ and $\phi \subset \mathfrak{h}^{(1)}$. Since $\Theta^{(1)}\colon \mathfrak{g}^{(1)} \to h(\mathfrak{g}^{(1)})$ is the composite

$$\mathfrak{g}^{(1)} \cong \mathfrak{g} \oplus \mathfrak{h}^{(1)} \xrightarrow{\tilde{\Theta}^{(1)}} \mathrm{Alt}^2(TM) \otimes \mathfrak{g} \xrightarrow{/\operatorname{im}\delta^{(1)}} h(\mathfrak{g}^{(1)}),$$

and because the image of $\delta^{(1)}$ contains the image of coRicci, this completes the proof. □

12.10. **The $W = 0$ case.** By the above, $\mathfrak{g}^{(1)}$ is Θ-reduced if and only if $\mathfrak{g}_W = \mathfrak{g}$. But as $\mathrm{id}_{TM} \subset T^*M \otimes TM \subset J^1(TM)$ is a section of $\mathfrak{g}$, this is clearly equivalent to the vanishing of W.

Assuming $W = 0$, Theorem 9.4 applies (with $\mathfrak{g}$ and $\mathfrak{t}$ in the theorem replaced with $\mathfrak{g}^{(1)}$ and $\mathfrak{g}$), establishing our claim that $\mathfrak{g}^{(1)}$ is the Cartan algebroid associated with $\mathfrak{g}$ when $W = 0$. To compute the Cartan connection $\nabla^{(2)}$ on $\mathfrak{g}^{(1)}$ we will now apply parts (3) and (4) of Theorem 11.2.

In the present context, the generator equation defined in 11.2(3) reads

$$\delta^{(1)}(\epsilon) + \mathrm{coRicci}(X \cdot R) = 0.$$

We have used (2) above. Referring to the description of $\delta^{(1)}$ in 12.9, we see that a solution is given by $\epsilon = -X \cdot R$. Using $\nabla^{(1)}$ to identify $\mathfrak{g}^{(1)}$ with $\mathfrak{g} \oplus \mathfrak{h}^{(1)}$, and keeping in mind the identification $\mathfrak{h}^{(1)} \cong T^*M$ implicit above, we deduce from 11.2(4),

$$(1)\quad \nabla^{(2)}_U (X \oplus \alpha) = \Big(\nabla^{(1)}_U X + \mathrm{skew}(\alpha \otimes U) + \alpha(U)\mathrm{id}_{TM}\Big) \oplus (\,\nabla_U \alpha + (X \cdot R)U\,),$$

for arbitrary sections $X \subset \mathfrak{g}$ and $\alpha \subset T^*M$.

We claim

$$(2)\qquad \operatorname{curv} \nabla^{(2)}(U_1, U_2)(X \oplus \alpha) = (X \cdot d_\nabla R)(U_1, U_2),$$

where $d_\nabla R$ is the Cotton-York tensor, defined in 12.2. In particular, whenever $W = 0$, the tensor $d_\nabla R$ is a conformal invariant which vanishes if and only if $\nabla^{(2)}$ is flat, i.e., if and only if the Cartan algebroid $\mathfrak{g}^{(1)}$ is flat (Theorem 4.5). This completes the proof of Theorem 12.4.

Proof of (2). Since $W = 0$ we have $\operatorname{curv} \nabla = \mathrm{coRicci}(R)$ and, with a little effort, one computes

$$\begin{aligned}\operatorname{curv} \nabla^{(2)}(U_1, U_2)(X \oplus \alpha) = &-((\nabla^{(1)}_{U_1} X) \cdot R)U_2 - \nabla_{U_1}((X \cdot R)U_2)\\ &+ ((\nabla^{(1)}_{U_2} X) \cdot R)U_1 + \nabla_{U_2}((X \cdot R)U_1)\\ &- (X \cdot R)([U_1, U_2]).\end{aligned}$$

Equation (2) now follows from the readily verified identities

$$(\nabla^{(1)}_U X) \cdot V = \nabla_U (X \cdot V) - X \cdot (\nabla_U V) + \nabla_{X \cdot U} V \ \text{ for } X \subset \mathfrak{g},\ V \subset TM;$$

$$(\nabla^{(1)}_U X) \cdot \alpha = \nabla_U (X \cdot \alpha) - X \cdot (\nabla_U \alpha) + \nabla_{X \cdot U} \alpha \ \text{ for } X \subset \mathfrak{g},\ \alpha \subset T^*M.$$

One also makes use of the fact that $\operatorname{tor} \nabla = 0$. □

Appendix A. Cartan groupoids and Lie pseudogroups

We now explain how *flat* Cartan algebroids may be viewed as infinitesimal versions of Lie pseudogroups; and conversely, how Lie pseudogroups integrate flat Cartan algebroids. As a byproduct of this discussion, we are led to define *Cartan groupoids*. These are the global versions of Cartan algebroids and may be viewed as deformations of Lie pseudogroups. Flat Cartan groupoids appear in [22] where they are called 'groupoid etalifications.'

A.1. **Lie pseudogroups via pseudoactions.** Let us explain, in invariant groupoid language, what it means for a pseudogroup to be a *Lie* pseudogroup. For the classical description see, e.g., [21].

A group of transformations in a smooth manifold M is a Lie group of transformations if it arises from the (smooth) action of some abstract Lie group. Analogously, we declare an arbitrary pseudogroup of transformations in M to be a *Lie pseudogroup* if it is arises from the pseudoaction of some Lie *groupoid*. It remains to explain what we mean by pseudoactions and the pseudogroups of transformations they define. We shall understand all constructions to be made in the smooth category.

Let G be a Lie groupoid over M. Call an immersed submanifold $\Sigma \subset G$ a *pseudotransformation* if the restrictions to Σ of the groupoid's source and target maps are local diffeomorphisms. In other words, each point of Σ should have an open neighborhood in Σ that is a (smooth) local bisection of G. For example, the pseudotransformations of the pair groupoid $M \times M$ are the local transformations in M taking possibly multiple values.

A *pseudoaction* of G on M is any foliation $\mathcal{F}$ on G such that:

(1) The leaves of $\mathcal{F}$ are pseudotransformations.
(2) $\mathcal{F}$ is *multiplicatively closed*.

To define what is meant in (2) let $\hat{\mathcal{F}}$ denote the collection of those subsets of G that are simultaneously an open subset of some leaf of $\mathcal{F}$, and a local bisection. Let $\hat{G}$ denote the collection of *all* local bisections of G, this being a groupoid over the power set of M. Then condition (2) is the requirement that $\hat{\mathcal{F}} \subset \hat{G}$ be a subgroupoid.

Given a pseudoaction $\mathcal{F}$ of G on M, each element of $\hat{\mathcal{F}}$ defines a local diffeomorphism in M and, by (2), the collection of all such local diffeomorphisms constitutes a pseudogroup of transformations in M. For example, if G is an action groupoid $G = G_0 \times M$, then the canonical horizontal foliation $\mathcal{F}$ furnishes us with the usual pseudogroup of transformations associated with the prescribed action of the Lie group G_0.

A.2. **The flat Cartan algebroid associated with a Lie pseudogroup.** Let $\mathcal{G}$ be a Lie pseudogroup of transformations in M. Then $\mathcal{G}$ is generated by the pseudoaction $\mathcal{F}$ of some Lie groupoid G over M. Define $\hat{\mathcal{F}}$ as in A.1 above. Then each point $g \in G$ lies in some bisection $b \in \hat{\mathcal{F}}$ and all such bisections have the same one-jet at g. Thus $\mathcal{F}$ defines a map $D_{\mathcal{F}}\colon G \to J^1G$ into the Lie groupoid of all one-jets of bisections of G. This map, which is a right inverse for the natural projection $J^1G \to G$, is a groupoid morphism because $\mathcal{F}$ is multiplicatively closed.

An arbitrary groupoid morphism $D\colon G \to J^1G$ furnishing a right inverse for $J^1G \to G$ is what we call a *Cartan connection* on G. These connections may be viewed as certain 'multiplicatively closed' distributions on G. The connection D is Frobenius integrable precisely when it comes from a pseudoaction $\mathcal{F}$ as above, in which case D is simply the tangent distribution. A Lie groupoid equipped with a (possibly non-integrable) Cartan connection is a *Cartan groupoid*. Thus Cartan groupoids are deformed Lie pseudogroups.

Differentiating a Cartan connection $D\colon G \to J^1G$, we obtain a splitting $\mathfrak{g} \to J^1\mathfrak{g}$ for the exact sequence of Lie algebroids

$$0 \to T^*M \otimes \mathfrak{g} \hookrightarrow J^1\mathfrak{g} \to \mathfrak{g} \to 0.$$

This splitting will be a morphism of Lie algebroids, i.e., amounts to a Cartan connection ∇ on $\mathfrak{g}$, as defined in 4.1. If $D = D_{\mathcal{F}}$ for some pseudoaction $\mathcal{F}$ as above, then D is Frobenius integrable and we claim that this implies $\operatorname{curv}\nabla = 0$, so that $\mathfrak{g}$ is flat (Theorem 4.5); $\mathfrak{g}$ is then a flat Cartan algebroid associated with the pseudogroup $\mathcal{G}$ generated by $\mathcal{F}$.

A.3. **The Lie pseudogroup integrating a flat Cartan algebroid.** Let $\mathfrak{g}$ be a Cartan algebroid over M with Cartan connection ∇ and assume $\mathfrak{g}$ is the Lie algebroid of some Lie groupoid G. For simplicity, suppose G has connected source-fibers. The connection ∇ determines a Lie algebroid morphism $\mathfrak{g} \to J^1\mathfrak{g}$ splitting the exact sequence A.2(A.2). By the groupoid version of Lie's Second Theorem, this morphism integrates to a groupoid morphism $D\colon G \to J^1G$, i.e., to a Cartan connection on the Lie groupoid G. Supposing $\mathfrak{g}$ is flat, we have $\operatorname{curv}\nabla = 0$ (Theorem 4.5), and we claim this guarantees that D is Frobenius integrable. The integrating foliation $\mathcal{F}$ is a pseudoaction generating a Lie pseudogroup $\mathcal{G}$ of transformations in M.

For each locally defined ∇-parallel section $X \subset \mathfrak{g}$, the vector field $\#X \subset TM$ integrates to a one-parameter family of local transformations belonging to $\mathcal{G}$. Conversely each transformation in the pseudogroup $\mathcal{G}$ — or at least each transformation 'close' to the identity — arises as the time-one map associated with such a vector field. In this sense $\mathcal{G}$ integrates the flat Cartan algebroid $\mathfrak{g}$.

Appendix B. Some ancillary results

B.1. **On morphisms whose domain sits in a short exact sequence.** The proof of the following is a straightforward diagram chase.

Lemma (Algebraic Lemma)**.** In the category of vector spaces, or of vector bundles over M, let $\theta\colon B \to B_1$ be an arbitrary morphism, B_0 its kernel, and suppose B occurs in some exact sequence

$$0 \to A \hookrightarrow B \to C \to 0.$$

Then the sequence

$$0 \to A_0 \hookrightarrow B_0 \to C_0 \to 0$$

is also exact, where $A_0 \subset A$ and $C_0 \subset C$ are defined in (1) below, and where $B_0 \to C_0$ is the restriction of $B \to C$.

(1) Let A_0 and A_1 denote, respectively, the kernel and image of the composite morphism $A \hookrightarrow B \xrightarrow{\theta} B_1$. Put $C_1 := B_1/A_1$ to obtain an exact sequence

$$0 \to A_1 \hookrightarrow B_1 \to C_1 \to 0.$$

Then there exists a unique morphism $\Theta\colon C \to C_1$ such that

$$\begin{array}{ccc} B & \longrightarrow & C \\ \big\downarrow \theta & & \big\downarrow \Theta \\ B_1 & \longrightarrow & C_1 \end{array}$$

is commutative. Define C_0 to be the kernel of Θ.

B.2. **Gluing Lie algebroid 'point' invariants into global invariants.** Let $\mathfrak{g}$ be a transitive Lie algebroid over M and $\mathfrak{h} \subset \mathfrak{g}$ the kernel of its anchor. Let E be a $\mathfrak{g}$-representation. Each fiber $\mathfrak{h}(m)$ of $\mathfrak{h}$ is a Lie algebra acting on the vector space $E(m)$. The following lemma furnishes conditions under which the the existence of $\mathfrak{h}(m)$-invariant elements $\sigma(m) \in E(m)$, for each $m \in M$, implies the existence of *global* $\mathfrak{g}$-invariant sections $\sigma \subset E$. For applications, see 5.3 and **??**.

Lemma (Extension Lemma)**.** Suppose that M is simply-connected and that the set $E^{\mathfrak{h}} \subset E$ of $\mathfrak{h}$-invariant elements has constant rank $r > 0$. Then E possesses a non-vanishing $\mathfrak{g}$-invariant section σ. If $r = 1$, then σ is unique up to constant.

Proof. Noting that $Y \subset \mathfrak{h}$ implies $[X,Y]_{\mathfrak{g}} \subset \mathfrak{h}$, the identity

$$Y \cdot (X \cdot \sigma) = X \cdot (Y \cdot \sigma) - [X,Y]_{\mathfrak{g}} \cdot \sigma; \qquad X \subset \mathfrak{g},\, Y \subset \mathfrak{h},\, \sigma \subset E,$$

shows that the rank-r subbundle $E^{\mathfrak{h}} \subset E$ is $\mathfrak{g}$-invariant. Because $\mathfrak{h}$ acts trivially on $E^{\mathfrak{h}}$, the representation $\mathfrak{g} \to \mathfrak{gl}(E^{\mathfrak{h}})$ factors through the anchor, delivering a representation $TM \to \mathfrak{gl}(E^{\mathfrak{h}})$, i.e., a *flat* linear connection D on $E^{\mathfrak{h}}$. One takes σ to be any non-vanishing D-parallel section of $E^{\mathfrak{h}}$, whose existence is guaranteed by flatness and the simple-connectivity of M. The uniqueness claim is clear. □

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