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Similarity: generalizations, applications and open problems

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1 Introduction

This Special Issue of the Journal of Engineering Mathematics focuses on recent theoretical and applied developments in Similarity, including asymptotic/numerical similarity and associated software. In particular, it is devoted to papers by leading researchers who apply similarity analysis to engineering and biological problems (asymptotics involving similarity solutions), develop significant extensions of similarity methods to find and use symmetries and conservation laws of partial differential equations (PDEs), develop and use numerical methods for PDEs based on admitted symmetries and/or conservation laws, and develop symbolic manipulation software to implement similarity analyses. As a consequence, it is hoped that these four distinct groups of similarity researchers will more readily become aware of recent developments and needs in related fields. In turn, this should lead to important and fruitful research directions. These papers arose from presentations at a conference on Similarity: Generalizations, Applications and Open Problems in Vancouver, British Columbia, August 11–15, 2009, sponsored by PIMS (Pacific Institute for the Mathematical Sciences) and AMSI (Australian Mathematical Sciences Institute). This introductory article includes a brief summary of the papers as well as a reference to part of the discussion on open problems during a session of the conference.

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1.1 Similarity

Similarity is concerned with invariance of an aspect of a given problem under continuous symmetries. When a problem is invariant under a continuous transformation, i.e., it has a continuous symmetry, then often the complexity in solving it is reduced by the dimension of the symmetry. A natural scaling invariance of a problem arises from dimensional consistency (dimensional analysis). This is fundamental to modelling (cf. [1–11]). More generally, similarity (symmetry) analysis has become a very sophisticated subject. For a given problem, often one can determine admitted symmetries and conservation laws that are not obvious by inspection. In turn, one can use admitted symmetries and conservation laws for intermediate or long-time asymptotic analysis, as an aid for determining numerical solutions, to obtain particular solutions, to map the problem to a simpler problem (e.g., linearization), etc.

Self-similar solutions for PDEs arise for many real problems from invariance under dimensional analysis. More generally, self-similar solutions arise from invariance under scalings of variables.

When the form of the similarity reduction can be identified a priori (often by means of a conservation law), the result is often termed a self-similar solution of the first kind; when the similarity exponent must be determined as the eigenvalue in an eigenvalue problem for the similarity solution, the latter is termed "second kind" (the most widely studied such examples involving the calculation of travelling-wave speeds).

Still more generally, if a PDE system is invariant under a Lie group of point transformations (point symmetries), one can find, constructively, similarity solutions (invariant solutions) that are invariant under a subgroup of the full group admitted by the PDE system. Similarity solutions arise from solving a reduced system of DEs with fewer independent variables. Similarity solutions can be constructed for specific boundary value problems.

Beginning in the 1980s, and continuing to date, there have been numerous studies analyzing finite-time singularity formation for nonlinear parabolic second-order PDEs (cf. [12–17]) and more recently for their higher-order generalizations (cf. [18–20]). The applications of singularity formation are diverse, including fourth-order lubrication theory models from fluids (cf. [21–23]), chemotactic aggregation governed by systems of nonlinear PDEs (cf. [24,25]), and quenching behaviour in a nonlinear PDE model of a MEMS capacitor (cf. [26]). For such problems, a singularity forms in finite time whereby either the solution or its partial derivatives diverge. Although generally it is not possible to determine analytically the singularity time or point(s) where a solution loses its regularity, it is of considerable interest to characterize analytically the local solution (typically a similarity solution) behaviour near a singularity, and to determine the stability of any locally self-similar solutions. Therefore, in this context, similarity solutions of such PDEs describe local solution behaviour near singularities. A comprehensive survey of rigorous, formal, and numerical approaches to local self-similarity in describing singularity behaviour in a wide class of physically based PDE models, relating primarily to fluid systems, is given in [27]. An extensive bibliography of further problems is given in the bibliography of [28].

Similarity solutions also play an essential role in describing intermediate-asymptotic behaviour for wide classes of PDEs (cf. [5,7]). In particular, important issues arising from the study of similarity solutions include:

- rigorous stability properties of local self-similar solutions for second-order PDE models characterizing singularity formation is rather well-developed (cf. [16,29]) owing to the Sturm–Liouville structure of the linearized operator, leading to a countably discrete spectrum. In contrast, for high-order PDE models exhibiting singularity formation, there can be a countably infinite set of local self-similar solutions whose stability properties must be studied on a case-by-case basis (cf. [18–20,27]);
- the role of quasi-self-similar solutions (which satisfy the governing equations only asymptotically, rather than exactly) in describing ultimate (long-time or close to finite-time singularity) evolutions (cf. [17,30–32]). Remarkably, such solutions arise often in applications (cf. [27,28]) and their analysis requires a synthesis of similarity, dynamical systems, perturbation, and numerical methods;
- rigorous analysis of the above issues (cf. [12–17,29]). In many cases such analysis builds directly on the formal results;

– the development of systematic symmetry-based techniques for identifying candidate intermediate-asymptotic solutions (see, for example, [7]; there is considerable further scope for the application of modern symmetry approaches).

In recent years there have been many developments in extending symmetry-based methods to find similarity solutions and conservation laws for PDE systems. Such advances have focused on how to:

– find and implement further applications for admitted symmetries (cf. [10,11,33–36]);
– find multipliers yielding conservation laws and the resulting fluxes of conservation laws (cf. [37–42]);
– extend the spaces of admitted symmetries and hence construct further similarity solutions (cf. [10,11,33,42–47]);
– extend the applications to construct solutions from admitted symmetries to include admitted "symmetries" arising from generalizations (similarity solutions arising from the nonclassical and other related methods) (cf. [10,42,48–55]);
– efficiently solve the (overdetermined) linear systems of determining equations for symmetries or conservation law multipliers and solve the nonlinear systems of determining equations for the nonclassical and related methods through the development of symbolic manipulation software (cf. [42,56–71]);
– develop numerical schemes that effectively use symmetries and/or conservation laws for ODE's (cf. [72]), for difference equations (cf. [73,74]), and for PDE models that exhibit finite-time singular behaviour (cf. [75–77]).

2 Contents of this issue

The articles in this issue can be grouped into the following rough classification.

2.1 Self-similarity

As mentioned previously, self-similar solutions arise naturally as particular solutions of PDE systems from dimensional analysis and, more generally, from invariance of PDE systems under scalings of variables. Usually, such solutions do not globally satisfy imposed boundary conditions. However, through delicate analysis, one can often show that a self-similar solution holds asymptotically in certain identified domains (near singularities, large time, small time, intermediate, etc).

2.1.1 Stability and dynamics of self-similarity in evolution equations

In [28], a methodology is presented for studying linear stability for self-similar solutions. It is shown that self-similar phenomena can be studied through use of many ideas arising in the study of dynamical systems. In particular, there is a discussion of the role of symmetries in the context of stability of self-similar dynamics. For blow-up solutions, it is demonstrated that symmetries give rise to positive eigenvalues associated with rescaling symmetries as opposed to instability. Moreover, it is shown how such a stability analysis can identify a unique (and observable) stable solution from a countable infinity of self-similar solutions. It is argued that linearization methods, combined with careful analysis of associated symmetries, provide a powerful tool for analyzing stability of self-similar solutions.

2.1.2 Thin-film rupture for large slip

In [78], there is a study of the rupture of thin liquid films on hydrophobic substrates, assuming large slip at the liquid/solid interface. Using a strong-slip lubrication model, it is shown that the rupture can pass through up to three self-similar regimes with different dominant balances and different scaling exponents. For one of the regimes, there is a self-similar solution of the second kind, and the similarity (scaling) exponent is determined by solving a boundary-value problem for a corresponding reduced nonlinear ODE. Moreover, in this regime it is shown that the self-similar solution blows up after a finite time.

2.1.3 Self-similarity in particle-laden flows at constant volume

In [79], there is consideration of constant volume thin-film slurries on an incline. Clear fluids in this geometry are known to have a front position that moves according to a $t^{1/3}$ scaling law, based on self-similar solution analysis [80]. By comparing theory with experiments, it is shown that the $t^{1/3}$ scaling law persists, to leading order, for slurry flows with particle settling.

2.1.4 Asymptotic analysis of extinction behaviour in fast nonlinear diffusion

In [81], there is a summary of the range of applications that arise for the equation of fast nonlinear diffusion. There is a discussion of relevant similarity solutions and intermediate-asymptotic behaviour arising from the invariance of the fast nonlinear diffusion equations under translations in space and time and a two-parameter family of scalings. In turn, this yields a one-parameter family of self-similar solutions for consideration to obtain relevant asymptotic solutions (in terms of determining appropriate values in different regimes for the free scaling parameter) through delicate analysis in terms of posed data.

2.2 Applications of symmetry methods

In the past, symmetries and conservation laws used in applications and the uses themselves were often obvious by inspectional analysis. However, in recent years, with the development of more sophisticated symmetry methods and especially with the help of ever-improving software, one is able to find nontrivial uses of symmetries and conservation laws in applications.

2.2.1 Temperature-dependent surface diffusion near a grain boundary

For a linear partial differential equation, an admitted point symmetry (Lie symmetry) leads algorithmically, through a corresponding separation of variables, to solutions that are much more general than the usual similarity solutions directly obtained by symmetry reduction. This allows one to formally solve free-boundary problems that have approximate symmetries at early times. In [82], this method is applied to a complicated practical boundary-value problem for fourth-order surface diffusion near a grain boundary at changing temperature.

2.2.2 Relevance of symmetry methods in mechanics of materials

In [83], the interest and relevance of symmetry methods as a predictive and systematic methodology in the continuum mechanics of materials are analyzed, relying on a classification of the inherent aspects in terms of direct, extended direct and inverse methods. In particular, the direct problem of finding invariants associated with a given material's constitutive law, including dissipation is considered as well as the inverse problem of constructing a material's constitutive law from invariance under a given Lie group of transformations.

2.2.3 Higher-order symmetries and conservation laws of the G-equation for premixed combustion and resulting numerical schemes

In [84], it is shown that the set of computable local symmetries of the G-equation for flame front propagation of premixed combustion is considerably extended if higher-order symmetries are considered. In particular, it is shown that the G-equation admits an infinite number of higher-order symmetries for an arbitrary velocity field. Geometrical and kinematic interpretations of the symmetries are given. In the case of constant flow velocity, the direct method (cf. [37–39]) is used to derive an infinite set of local conservation laws of the G-equation. The derived infinite sets of local symmetries and conservation laws are used to develop novel numerical schemes to perform calculations in practical applications involving the G-equation.

2.2.4 Hidden symmetries and reductions for ideal magnetohydrodynamics equilibria

In [85], hidden symmetries are derived for the axially symmetric steady-state solutions of the ideal magnetohydrodynamic equations. A reduction of these equations to a scalar second-order partial differential equation is obtained. Applications of the hidden symmetries yield large families of exact axially symmetric MHD equilibria.

2.3 Construction of conservation laws

The direct method for constructing local conservation laws of PDEs is applicable to wide classes of PDE systems (cf. [37–39,42]). Within this method, one seeks multipliers such that the linear combination of the PDEs in a given PDE system with the multipliers will yield a divergence expression. After local conservation law multipliers are found, one constructs the fluxes of the corresponding conservation law.

2.3.1 Computation of fluxes of conservation laws

In the review paper [86], there is a discussion of various methods for flux computation, including a comparison of these methods as well as illustrations by examples. There is also a presentation of the implementation of these methods in symbolic software.

2.4 Extending the spaces of admitted symmetries

2.4.1 Potential systems for PDEs having several conservation laws

Knowledge of the symmetry properties of the modelling PDE system of a physical process can be very useful for understanding the behaviour of solutions, e.g., the group invariance property of a PDE system allows one to generate new solutions from known ones (cf. [85]), to construct conservation laws (cf. [86]), and find wide classes of exact invariant solutions. For many nonlinear systems, invariant solutions are the only known solutions and can be used as testing solutions for numerical and other approximate solutions. Hence knowledge of a new symmetry can be of great importance for a given PDE system. In [87], another way of constructing potential systems is illustrated that can lead to finding new potential symmetries (cf. [10,42,44,46,47,55]) of a given PDE system.

2.5 Symmetry-classification algorithms

Recent developments in solving overdetermined systems of differential equations and their implementation in terms of various symbolic software packages have allowed one to solve seemingly intractable problems of symmetry classification.

2.5.1 An algorithm for the complete symmetry classification of differential equations based on Wu's method

In [88], an algorithm is presented which gives a new application of Wu's method (differential characteristic set algorithm) for the complete symmetry classification of differential equations containing arbitrary parameters/functions. As illustrative examples, complete potential symmetry classifications of linear and nonlinear wave equations with an arbitrary function as well as classical and nonclassical symmetries of parametric Burgers' equations are presented.

2.5.2 Algorithmic symmetry classification with invariance

In [89], symmetry classification for a system of differential equations is achieved algorithmically by applying a differential reduction and completion (DRC) algorithm (cf. [63]) to the linear infinitesimal determining equations of the system. It is shown that the invariance of the classification under the action of the equivalence group can be tested algorithmically knowing only the determining equations of the equivalence group. The method is implemented in Maple.

2.6 Numerical methods involving symmetries and/or conservation laws

2.6.1 How to adaptively resolve evolutionary singularities in differential equations with symmetry

In [90], the theory of self-similar blow-up in evolutionary differential equations is reviewed and a moving mesh method is presented to simulate numerically such phenomena. The method exploits the evolving symmetries in such problems to guide the adaptivity in both time and space. This is shown to provide an efficient and reliable way to simulate self-similar singularity formulation. This enables one to capture dynamics where the behaviour is exactly or asymptotically self-similar. It turns out that this method is simple to program and extends the utility of naive finite difference discretization methods. The focus is on the practical implementation with examples drawn from applications.

2.6.2 A rarefaction-tracking method for hyperbolic conservation laws

In [91], a numerical method is presented for one-dimensional scalar conservation laws that combines the method of characteristics, local similarity solutions, and particle management. The solution is approximated by local similarity solutions. While traditional approaches use shocks, the presented method uses rarefaction and compression waves. Although shocks are not explicitly tracked, they can be located accurately. The method is exhibited by numerical examples. Specific applications are outlined as well as extensions of the method.

2.7 Other topics

2.7.1 Invariance and first integrals of continuous and discrete Hamiltonian equations

In [92], there is consideration of the relationship between symmetries and first integrals (conservation laws) for both continuous and discrete Hamiltonian equations. It is shown that canonical Hamiltonian equations can be obtained through a variational principle from an action functional. The invariance properties of the functional are considered as is done in the Lagrangian formalism. The well-known Noether identity is rewritten in terms of the Hamiltonian and symmetry operators. This is shown to provide a simple and clear way to construct first integrals of Hamiltonian equations without integration. The discrete analog of this identity for discrete Hamiltonian equations can be used to construct conservative finite-difference schemes in the Hamiltonian framework that are important in numerical implementations.

2.7.2 Coarsening dynamics of slipping droplets

In [93], the late-phase dewetting process of nanoscopic thin polymer films on hydrophobized substrate is studied, using some recent lubrication models that take account of large slippage at the polymer–substrate interface.

2.7.3 Ultrasound detection of externally induced microthrombi cloud formation: a theoretical study

In [94], a mathematical model is presented for the formation of microaggregates (microthrombi) of fibrin polymers in blood flow, induced by an external source.

2.8 Open problems

During the Vancouver conference, there was a two-part session, led by George Bluman, on open problems relating to the Similarity areas 2.1–2.6. The tape of the first part, which includes presentations on open problems by Andrew Bernoff, George Bluman, Philip Broadbridge, Chris Budd and Jean-François Ganghoffer, is available at the following website address: http://new.pims.math.ca/pix/video/similarity4.mov. Unfortunately, the tape of the shorter second part is unavailable.

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