CARTAN CONNECTIONS IN FOLIATED BUNDLES

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1. Introduction. Let M be a smooth connected manifold of dimension n and let $\mathfrak{F}$ be a smooth codimension q foliation of M. Let G be a Lie group and let H be a closed subgroup of G such that G/H has dimension q. Let $\pi: P \to M$ be a foliated principal H-bundle in the sense of [10]. We define a Cartan connection in P as a certain type of one-form ω on P with values in the Lie algebra of G. This generalizes the notion of a Cartan connection in an ordinary principal bundle and provides a unified setting for the study of Riemannian, conformal, and projective foliations as well as other types of geometric structures for foliations.

THEOREM 1. *Let ω be a complete Cartan connection in P. Then all the leaves of $\mathfrak{F}$ have the same universal cover. In particular, if $\mathfrak{F}$ has a compact leaf with finite fundamental group, then all the leaves of $\mathfrak{F}$ are compact with finite fundamental group.*

As a corollary to Theorem 1, we will obtain the stability theorem of B. Reinhart [23] for Riemannian foliations.

THEOREM 2. *Let ω be a complete flat Cartan connection in P. Let $p: \tilde{M} \to M$ be the universal cover of M and let $(G/H)^{\sim}$ be the universal cover of G/H. There is a locally trivial fiber bundle $\tilde{M} \to (G/H)^{\sim}$ whose fibers are the leaves of $p^{-1}(\mathfrak{F})$.*

As a corollary to Theorem 2, we will obtain the structure theorem of G. Reeb [22] for codimension one foliations of a compact manifold defined by a nonsingular closed one-form.

We consider projective and conformal foliations from the point of view of Cartan connections in foliated bundles and we obtain:

THEOREM 3. *Let $\mathfrak{F}$ be a complete projective or conformal foliation of codimension q ($q \geqslant 2$ in the projective case, $q \geqslant 3$ in the conformal case). If $\mathfrak{F}$ has a compact leaf with finite holonomy group, then all the leaves of $\mathfrak{F}$ are compact with finite holonomy group.*

THEOREM 4. *Let $\mathfrak{F}$ be a complete flat projective or conformal foliation of codimension q ($q \geqslant 2$ in the projective case, $q \geqslant 3$ in the conformal case) of a connected manifold M. Then the universal cover of M fibers over S^q, the fibers being the leaves of the lifted foliation.*

We give a class of examples of complete Cartan connections in foliated bundles (a class which includes the generalized Roussaire foliations) as well as an example in the incomplete case.

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2. Foliated bundles. We briefly recall some generalities on foliated bundles (cf. [10], [11], [17], [18], [6]).

Let M be a smooth connected manifold of dimension n and let $\mathfrak{F}$ be a smooth codimension q foliation of M. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the tangent bundle of $\mathfrak{F}$. Let H be a Lie group and let $\pi: P \to M$ be a smooth principal H-bundle. We say $\pi: P \to M$ is a foliated bundle if there is a foliation $\tilde{\mathfrak{F}}$ of P satisfying

(i) $\tilde{\mathfrak{F}}$ is H-invariant,

(ii) $\tilde{E}_u \cap V_u = \{0\}$ for all $u \in P$,

(iii) $\pi_{*_u}(\tilde{E}_u) = E_{\pi(u)}$ for all $u \in P$,

where $\tilde{E} \subset T(P)$ is the tangent bundle of $\tilde{\mathfrak{F}}$ and $V \subset T(P)$ is the bundle of vertical vectors. If K is a Lie subgroup of H, then a reduction $P_0 \subset P$ to a principal K-bundle is said to be foliated if P_0 is a union of leaves of $\tilde{\mathfrak{F}}$.

EXAMPLE. Let $Q = T(M)/E$ be the normal bundle of $\mathfrak{F}$. The frame bundle $F(Q)$ of Q is a foliated $\mathrm{GL}(q, \mathbf{R})$-bundle. A foliated reduction of $F(Q)$ to a Lie subgroup $K \subset \mathrm{GL}(q, \mathbf{R})$ is a transverse K-structure for $\mathfrak{F}$. Riemannian (respectively, conformal) foliations correspond to the case where K is the orthogonal (respectively, conformal) group. More generally, the bundle $P^r(M, \mathfrak{F})$ of transverse r-frames is a foliated $G^r(q)$-bundle where $G^r(q)$ is the group of r-frames at $0 \in \mathbf{R}^q$. A conformal or projective structure for $\mathfrak{F}$ is a foliated reduction of $P^2(M, \mathfrak{F})$ to an appropriate Lie subgroup of $G^2(q)$.

3. Cartan connections in foliated bundles. Let $\mathfrak{F}$ be a codimension q foliation of M. Let G be a Lie group and let $H \subset G$ be a closed subgroup with $\mathrm{dimension}(G/H) = q$. Let $\pi: P \to M$ be a foliated principal H-bundle. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h}$ be the Lie algebra of H. For each $A \in \mathfrak{h}$, let A^* be the corresponding fundamental vector field on P.

DEFINITION. A Cartan connection in the foliated bundle $\pi: P \to M$ is a $\mathfrak{g}$-valued one-form ω on P satisfying

(i) $\omega(A^*) = A$ for all $A \in \mathfrak{h}$,

(ii) $(R_a)^*\omega = \mathrm{ad}(a^{-1})\omega$ for all $a \in H$ where R_a denotes the right translation by a acting on P and $\mathrm{ad}(a^{-1})$ is the adjoint action of a^{-1} on $\mathfrak{g}$,

(iii) For each $u \in P$, $\omega_u: T_u(P) \to \mathfrak{g}$ is onto and $\omega_u(\tilde{E}_u) = 0$,

(iv) $L_X\omega = 0$ for all $X \in \Gamma(\tilde{E})$ where $\Gamma(\tilde{E})$ denotes the smooth sections of $\tilde{E}$ and L_X is the Lie derivative.

REMARK. If $\dim \mathfrak{F} = 0$, then ω is a Cartan connection in the sense of Ehresmann [7]. See also [15] and [16]. Note that $\omega_u: \tilde{Q}_u \to \mathfrak{g}$ is an isomorphism where $\tilde{Q}$ is the normal bundle of $\tilde{\mathfrak{F}}$.

Following Molino [19] we say that a section $\tilde{Y} \in \Gamma(\tilde{Q})$ is complete if there exists a complete vector field Y on P which projects to $\tilde{Y}$ under the natural projection $T(P) \to \tilde{Q}$.

DEFINITION. We say ω is complete if each section $\tilde{Y}$ of $\tilde{Q}$ such that $\omega(\tilde{Y})$ is constant is complete.

REMARK. If dim $\mathfrak{F}=0$, this reduces to the notion of completeness given in [12] and [13].

EXAMPLE. Let $\tilde{\omega}$ be a connection in $F(Q)$ which is basic in the sense of [10] (transversely projectable in the terminology of [17]). Let θ be the canonical $\mathbf{R}^q$-valued one-form on $F(Q)$. Then $\omega=\theta+\tilde{\omega}$ is a Cartan connection in $F(Q)$ and ω is complete if and only if $\tilde{\omega}$ is complete in the sense of [19].

THEOREM 3.1. *If ω is complete, then all the leaves of $\tilde{\mathfrak{F}}$ are diffeomorphic.*

Proof. Let Y be a smooth vector field on P such that $\omega(Y)$ is constant. Then for any $X\in\Gamma(\tilde{E})$ we have

$$-\tfrac{1}{2}\omega[X,Y]=d\omega(X,Y)=(L_X\,\omega)(Y)-d(\omega(X))(Y)=0,$$

and so $[X,Y]\in\Gamma(\tilde{E})$. Let $X_1,\dots,X_r$ be a basis of $\mathfrak{g}$. Let $\tilde{Y}_1,\dots,\tilde{Y}_r\in\Gamma(\tilde{Q})$ be the unique sections satisfying $\omega(\tilde{Y}_i)=X_i$, $i=1,\dots,r$. Let $Y_1,\dots,Y_r$ be complete vector fields on P which project to $\tilde{Y}_1,\dots,\tilde{Y}_r$ under the natural projection $T(P)\to\tilde{Q}$. For $i=1,\dots,r$ let ϕ_t^i, $t\in\mathbf{R}$, be the flow generated by Y_i. Since $[X,Y_i]\in\Gamma(\tilde{E})$ for all $X\in\Gamma(\tilde{E})$, it follows that ϕ_t^i preserves $\tilde{\mathfrak{F}}$. Moreover, the group generated by the diffeomorphisms ϕ_t^i acts transitively on the set of leaves of each connected component of P ([5], [20]). Finally, we get from one component of P to another by a suitable element of H. □

Since the leaves of $\tilde{\mathfrak{F}}$ are coverings of the leaves of $\mathfrak{F}$, Theorem 1 follows from Theorem 3.1.

As a corollary to Theorem 1, we obtain the Reinhart Stability Theorem [23]:

COROLLARY 3.2. *Let $\mathfrak{F}$ be a Riemannian foliation of a compact connected manifold M. Then all the leaves of $\mathfrak{F}$ have the same universal covering space.*

Proof. Let $O(Q)$ be the bundle of orthonormal frames of the normal bundle Q of $\mathfrak{F}$. Let θ be the canonical $\mathbf{R}^q$-valued one-form on $O(Q)$ and let $\tilde{\omega}$ be the unique torsion-free basic connection on $O(Q)$. Then $\omega=\theta+\tilde{\omega}$ is a Cartan connection in the foliated bundle $O(Q)\to M$. Since $O(Q)$ is compact, ω is complete and so by Theorem 1 all the leaves of $\mathfrak{F}$ have the same universal cover.

PROPOSITION 3.3. *Let M and M' be manifolds with* $\dim M=\dim M'=\dim G/H$. *Let $\pi'\colon P'\to M'$ and $\pi\colon P\to M$ be principal H-bundles. Let ω' and ω be Cartan connections in P' and P, respectively. Let $f\colon M'\to M$ be a connection-preserving local diffeomorphism. If ω' is complete, then f is onto, f is a covering map, and ω is complete.*

Proof. The hypothesis on f means that we have a bundle homomorphism $\bar{f}\colon P'\to P$ satisfying $\bar{f}^*\omega=\omega'$. For each $A\in\mathfrak{h}$, let $\omega'^{-1}(A)$ (respectively, $\omega^{-1}(A)$) be the unique vector field on P' (respectively, P) such that $\omega'(\omega'^{-1}(A))=A$ (respectively, $\omega(\omega^{-1}(A))=A$). Let ∇' (respectively, ∇) be the linear connection on P' (respectively, P) such that the vector fields $\omega'^{-1}(A)$ (respectively, $\omega^{-1}(A)$) are parallel along every curve. Since the geodesics of ∇' are the integral curves of the vector fields $\omega'^{-1}(A)$, it follows that ∇' is complete. Moreover, $\bar{f}^{-1}(\nabla)=\nabla'$. Hence ∇ is complete and $\bar{f}$ is a covering map [9]. In particular, the vector fields $\omega^{-1}(A)$ are complete and so ω is complete.

Let $x_0 \in M$ and let $\sigma: [0,1] \to M$ be a curve with $\sigma(0)=x_0$. Let $y_0 \in f^{-1}\{x_0\}$. Let $z_0 \in \pi'^{-1}\{y_0\}$, and let $w_0=\bar{f}(z_0)$. There is a curve $\tau: [0,1] \to P$ such that $\pi \circ \tau = \sigma$, $\tau(0)=w_0$ and a curve $\rho: [0,1] \to P'$ such that $\bar{f} \circ \rho = \tau$, $\rho(0)=z_0$. Then $\tilde{\sigma} = \pi' \circ \rho$ is a curve in M' satisfying $\tilde{\sigma}(0)=y_0$, $f \circ \tilde{\sigma} = \sigma$ and so f is a covering projection ([17], [24]). □

Let N be a connected manifold with $\dim N = \dim G/H$, let $p: Q \to N$ be a principal H-bundle, and let $\bar{\omega}$ be a Cartan connection in the bundle $p: Q \to N$. Let M be a connected manifold and let $f: M \to N$ be a submersion. Let $\pi: P \to M$ be the pull-back of Q under f and let $F: P \to Q$ be the map such that $p \circ F = f \circ \pi$. Then $\pi: P \to M$ is a foliated bundle with respect to the foliations $\mathfrak{F}$ of M and $\tilde{\mathfrak{F}}$ of P defined by the submersions f and F, respectively. Let $\omega = F^*\bar{\omega}$. Then ω is a Cartan connection in the foliated bundle $\pi: P \to M$.

THEOREM 3.4. *If ω is complete, then f is a locally trivial fiber bundle and $\bar{\omega}$ is also complete.*

Proof. Let $X_1, \ldots, X_q, X_{q+1}, \ldots, X_r$ be a basis of $\mathfrak{g}$ such that $X_{q+1}, \ldots, X_r$ is a basis of $\mathfrak{h}$. For each $i=1,\ldots,q$ let $\tilde{Y}_i$ be the section of the normal bundle of $\tilde{\mathfrak{F}}$ satisfying $\omega(\tilde{Y}_i)=X_i$ and let Y_i be a complete vector field on P which projects $\tilde{Y}_i$. For each $i=q+1,\ldots,r$ let Y_i be the fundamental vector field on P corresponding to X_i. For each $i=1,\ldots,r$ let ϕ_t^i, $t \in \mathbf{R}$, be the flow generated by Y_i. Since $\omega(Y_i)$ is constant, the diffeomorphisms ϕ_t^i send leaves to leaves.

Let $y_0 \in P$ and let $\tilde{L}$ be the leaf of $\tilde{\mathfrak{F}}$ through y_0. Define $\Phi: \mathbf{R}^l \times \tilde{L} \to P$ by

$$\Phi(t_{q+1}, \ldots, t_r, t_1, \ldots, t_q) = \phi_{t_{q+1}}^{q+1} \circ \cdots \circ \phi_{t_r}^{r} \circ \phi_{t_1}^{1} \circ \cdots \circ \phi_{t_q}^{q}(y).$$

Since the leaves of $\tilde{\mathfrak{F}}$ are closed, there is a neighborhood V of 0 in $\mathbf{R}^r$ such that $\Phi: V \times \tilde{L} \to U$ is a diffeomorphism where U is an open saturated set in P [20]. We may assume V is of the form $V_1 \times V_2$ where V_1 is a neighborhood of 0 in $\mathbf{R}^{r-q}$ and V_2 is a neighborhood of 0 in $\mathbf{R}^q$. Let $L = \pi(\tilde{L})$. Since $Y_{q+1}, \ldots, Y_r$ are vertical, Φ induces a smooth map $\Psi: V_2 \times L \to M$ such that $\pi \circ \Phi = \Psi \circ (\tau \times \pi)$ where $\tau: V \to V_2$ is projection onto the second factor. By shrinking V_2 if necessary, we may assume that Ψ is a diffeomorphism. Thus $\pi(U)$ is an open saturated set in M and Ψ maps the foliation of $V_2 \times L$ by leaves of the form $\{t\} \times L$, $t \in V_2$, diffeomorphically to $\mathfrak{F}$. Let C be a compact neighborhood of 0 in $\mathbf{R}^q$, $C \subset V_2$. Then $\Psi(C \times L)$ is a closed saturated neighborhood of L in M and so the leaf space $M/\mathfrak{F}$ is regular. Since the leaves of $\mathfrak{F}$ are closed, $M/\mathfrak{F}$ is Hausdorff. Thus $M/\mathfrak{F}$ is a smooth Hausdorff manifold and the natural projection $M \to M/\mathfrak{F}$ is a locally trivial fiber bundle.

The principal H-bundle $\pi: P \to M$ induces a principal H-bundle $\bar{\pi}: P/\tilde{\mathfrak{F}} \to M/\mathfrak{F}$ and ω induces a complete Cartan connection $\tilde{\omega}$ in this bundle such that the map $\bar{f}: M/\mathfrak{F} \to N$ induced by f is a connection-preserving local diffeomorphism. By Proposition 3.3, $\bar{\omega}$ is complete and $\bar{f}$ is a covering map. Hence f is a locally trivial fiber bundle. □

DEFINITION. The curvature of ω is the $\mathfrak{g}$-valued 2-form Ω on P defined by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. We say ω is flat if $\Omega = 0$.

We now prove Theorem 2. The general idea is to use the flatness of ω to produce a G/H-cocyle defining $\mathfrak{F}$ such that the transition functions are G-translations of G/H. Then a monodromy argument yields a submersion $g: \tilde{M} \to G/H$ defining $p^{-1}(\mathfrak{F})$. We show g is a fiber bundle by applying Theorem 3.4.

Since $d\omega = -\frac{1}{2}[\omega, \omega]$, we have that for each $u \in P$ there is a neighborhood $\tilde{U}$ of u and a submersion $\tilde{f}: \tilde{U} \to G$ defining $\tilde{\mathfrak{F}}/\tilde{U}$ and satisfying $\tilde{f}^*\omega_0 = \omega$ where ω_0 is the Maurer-Cartan form on G [8]. Let $U = \pi(\tilde{U})$, let $p = \pi(u)$ and let $s: U \to P$ be a section such that $s(U) \subset \tilde{U}$, $s(p) = u$. Define $\tilde{g}: \tilde{U}H \to G$ as follows. For each $y \in \tilde{U}H$ there is a unique $h \in H$ such that $y = s(\pi(y))h$. Set $\tilde{g}(y) = \tilde{f}(s(\pi(y)))h$. Then $\tilde{g}$ is H-equivariant and $\tilde{g} = \tilde{f}$ on $\tilde{U}$. Moreover, $\tilde{g}$ is a submersion defining $\tilde{\mathfrak{F}}/\tilde{U}H$ and $\tilde{g}^*\omega_0 = \omega$. It follows that $\tilde{g}$ induces a smooth submersion $g: U \to G/H$ defining $\mathfrak{F}/U$ such that $\rho \circ \tilde{g} = g \circ \pi$ where $\rho: G \to G/H$ is the natural projection. Let $\{\tilde{U}_\alpha\}_{\alpha \in A}$ be an open cover of P and for each $\alpha \in A$ let $\tilde{g}_\alpha: \tilde{U}_\alpha H \to G$, $g_\alpha: U_\alpha \to G/H$ be as above. If $U_\alpha \cap U_\beta \neq \emptyset$, there exists $g_{\alpha\beta} \in G$ such that $\tilde{g}_\alpha = g_{\alpha\beta}\tilde{g}_\beta$. Hence $g_\alpha = g_{\alpha\beta}g_\beta$ on $U_\alpha \cap U_\beta$. The proof of the following lemma is straightforward.

LEMMA 3.5. *Let ω be a Cartan connection in the foliated bundle $\pi: P \to M$. Let $p: \tilde{M} \to M$ be the universal cover of M. Let $\lambda: p^*(P) \to \tilde{M}$ be the pull-back of P and let $\bar{p}: p^*(P) \to P$ be the map such that $\pi \circ \bar{p} = p \circ \lambda$. Then $\lambda: p^*(P) \to \tilde{M}$ is a foliated bundle and $\bar{p}^*\omega$ is a Cartan connection. If ω is complete, then $\bar{p}^*\omega$ is complete.*

By a monodromy argument we obtain maps $\tilde{g}: p^*(P) \to G$ and $g: \tilde{M} \to G/H$ such that g is a submersion defining $p^{-1}(\mathfrak{F})$ and $\bar{p}^*\omega = \tilde{g}^*\omega_0$ (cf. [1] and [2]). By Lemma 3.5 $\tilde{g}^*\omega_0$ is complete, and hence by Theorem 3.4 g is a locally trivial fiber bundle. Finally, g lifts to a bundle map $\tilde{M} \to (G/H)^\sim$. □

COROLLARY 3.6 (G. Reeb [22]). *Let $\mathfrak{F}$ be a codimension one foliation of a compact manifold M defined by a nonsingular closed one-form. Then the universal cover of M is a product $L \times \mathbf{R}$ and the lifted foliation is the product foliation.*

Proof. The closed one-form defines a complete flat Cartan connection in the foliated bundle $\mathrm{id}: M \to M$ where $G = \mathbf{R}$, $H = \{0\}$. Since $\mathbf{R}$ is contractible, the bundle $\tilde{M} \to \mathbf{R}$ is trivial and so $\tilde{M} \cong L \times \mathbf{R}$. □

EXAMPLE. Let G be a Lie group and $H \subset G$ a closed subgroup. Let $\mathfrak{F}_0$ be the foliation of G defined by the natural projection $f: G \to G/H$. Let $\rho: f^*(G) \to G$ be the pull-back under f of the principal H-bundle $f: G \to G/H$ and let $\bar{f}: f^*(G) \to G$ be the map such that $f \circ \bar{f} = f \circ \rho$. Let ω_0 be the Maurer-Cartan form on G and let $\bar{\omega} = \bar{f}^*\omega_0$. Then $\bar{\omega}$ is a Cartan connection in the foliated bundle $\rho: f^*(G) \to G$ and an elementary argument shows that $\bar{\omega}$ is complete. Let Γ be a discrete subgroup of G. Then $\mathfrak{F}_0$ passes to a foliation $\mathfrak{F}$ of $M = \Gamma\backslash G$. The left action of Γ on $G \times G$ given by $(\gamma, g_1, g_2) \mapsto (\gamma g_1, \gamma g_2)$ preserves $f^*(G)$ and since all the structure of the foliated H-bundle $\rho: f^*(G) \to G$ is preserved by the left action of Γ, we obtain a foliated H-bundle $\pi: P \to M$ where $P = \Gamma\backslash f^*(G)$. Clearly

$\bar{\omega}$ projects to a $\mathfrak{g}$-valued one-form ω on P and ω is a complete flat Cartan connection in the foliated bundle $\pi: P \to M$. We remark that the Roussarie example [4] is the case where $G = \mathrm{SL}(2, \mathbf{R})$, H is the two-dimensional affine group, and M is the unit tangent bundle of a compact connected Riemann surface of genus $\geqslant 2$. The generalized Roussarie example is the case where $G = \mathrm{SL}(q+1, \mathbf{R})$ and $G/H \cong \mathbf{R}P^q$.

EXAMPLE. Define $f: \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = e^y \sin 2\pi x$. The codimension one foliation of $\mathbf{R}^2$ defined by f passes to a codimension one foliation $\mathfrak{F}$ of the torus T^2. There is an $\mathbf{R}$-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ defining $\mathfrak{F}$ such that each $g_{\alpha\beta}$ is the restriction of an affine transformation of $\mathbf{R}$. Let $F(\mathbf{R})$ be the frame bundle of $\mathbf{R}$. Let $\tilde{\theta}$ be the canonical one-form on $F(\mathbf{R})$ and let $\tilde{\omega}$ be the connection form on $F(\mathbf{R})$ corresponding to the canonical linear connection of $\mathbf{R}$. Then $\bar{\omega} = \tilde{\theta} + \tilde{\omega}$ is a flat Cartan connection in $F(\mathbf{R})$. Since $\bar{\omega}$ is preserved by the maps $g_{\alpha\beta}$, it induces a flat Cartan connection ω in the frame bundle of the normal bundle of $\mathfrak{F}$. Observe that $f: \mathbf{R}^2 \to \mathbf{R}$ is not a locally trivial fiber bundle and ω is not complete.

4. Projective and conformal foliations. In this section we prove Theorems 3 and 4 by considering the normal projective (respectively, conformal) connection in the projective (respectively, conformal) normal bundle of a projective (respectively, conformal) foliation.

Let $\mathfrak{F}$ be a smooth codimension q foliation of a connected manifold M. Let $\mathrm{CO}(q)$ be the conformal group, $\mathrm{CO}(q) = \{A \in \mathrm{GL}(q, \mathbf{R}) : {}^tAA = cI,\ c \in \mathbf{R},\ c > 0\}$. We say $\mathfrak{F}$ is a conformal foliation if the frame bundle $F(Q)$ of the normal bundle Q of $\mathfrak{F}$ admits a foliated reduction to a principal $\mathrm{CO}(q)$-bundle $P \subset F(Q)$ (cf. [25], [21], [3]).

A projective foliation cannot be described via a foliated reduction of $F(Q)$. In order to treat conformal and projective foliations in a unified manner we consider the bundle of transverse 2-frames.

Let U and V be neighborhoods of 0 in $\mathbf{R}^q$ and let $f: U \to M$, $g: V \to M$ be smooth maps transverse to $\mathfrak{F}$ with $f(0) = g(0) = x$. Let W be a neighborhood of x and let $F: W \to \mathbf{R}^q$ be a submersion constant along the leaves of $\mathfrak{F}/W$. We say that f and g define the same transverse r-frame at x if $F \circ f$ and $F \circ g$ have the same partial derivatives up to order r at 0. This definition is independent of the choice of the submersion F. Let $j_x^r(f)$ denote the transverse r-frame determined by f and let $P^r(M, \mathfrak{F})$ be the set of transverse r-frames on M. Then $\pi_r: P^r(M, \mathfrak{F}) \to M$, $\pi_r(j_x^r(f)) = x$, is a principal bundle over M with group $G^r(q)$ where $G^r(q)$ is the group of r-frames at $0 \in \mathbf{R}^q$. The right action of $G^r(q)$ on $P^r(M, \mathfrak{F})$ is given by $j_x^r(f) j_0^r(g) = j_x^r(f \circ g)$, for $j_x^r(f) \in P^r(M, \mathfrak{F})$, $j_0^r(g) \in G^r(q)$. Clearly, $P^1(M, \mathfrak{F})$ is the bundle $F(Q)$ of linear frames of the normal bundle of $\mathfrak{F}$ with group $G^1(q) = \mathrm{GL}(q, \mathbf{R})$.

Let W be an open set in M and let $F: W \to \mathbf{R}^q$ be a submersion constant along the leaves of $\mathfrak{F}/W$. Then F induces a submersion $F^{(r)}: P^r(M, \mathfrak{F})/W \to P^r(\mathbf{R}^q)$ by $F^{(r)}(j_x^r(f)) = j_{F(x)}^r(F \circ f)$ where $P^r(\mathbf{R}^q)$ is the bundle of r-frames of $\mathbf{R}^q$. Hence if $\{(W_\alpha, F_\alpha, g_{\alpha\beta})\}_{\alpha,\beta \in A}$ is an $\mathbf{R}^q$-cocycle defining $\mathfrak{F}$, then

$$\{(\pi_r^{-1}(W_\alpha), F_\alpha^{(r)}, g_{\alpha\beta}^{(r)})\}_{\alpha,\beta \in A}$$

is a $P^r(\mathbf{R}^q)$-cocycle on $P^r(M,\mathfrak{F})$ and hence defines a foliation $\mathfrak{F}^{(r)}$ of $P^r(M,\mathfrak{F})$ which makes $\pi_r: P^r(M,\mathfrak{F}) \to M$ a foliated bundle.

(A) *Projective model.* Let $G = \mathrm{PGL}(q,\mathbf{R}) = \mathrm{SL}(q+1,\mathbf{R})/\text{center}$. Then $\mathbf{R}P^q \cong G/H$. The Lie algebra $\mathfrak{g}$ of G is $\mathrm{sl}(q+1,\mathbf{R})$ and is graded as $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ where $\mathfrak{g}_{-1} \cong \mathbf{R}^q$, $\mathfrak{g}_0 \cong \mathrm{gl}(q,\mathbf{R})$, $\mathfrak{g}_1 \cong (\mathbf{R}^q)^*$ (cf. [15]).

(B) *Conformal model.* Let $G = O(q+1,1)$. Then $S^q \cong G/H$. The Lie algebra $\mathfrak{g}$ of G is $o(q+1,1)$ and is graded as $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ where $\mathfrak{g}_{-1} \cong \mathbf{R}^q$, $\mathfrak{g}_0 \cong \mathrm{co}(q)$, $\mathfrak{g}_1 \cong (\mathbf{R}^q)^*$ (cf. [15]).

Let G/H be as in (A) or (B). Then the mapping $\mathbf{R}^q = \mathfrak{g}_{-1} \xrightarrow{\exp} G \to G/H$ gives a diffeomorphism from a neighborhood of 0 in $\mathbf{R}^q$ onto a neighborhood of H in G/H. Let $a \in H$. Since a is a transformation of G/H fixing H we may regard a as a local diffeomorphism of $\mathbf{R}^q$ fixing 0. Let $j_0^2(a) \in G^2(q)$ be the 2-frame determined by a. Since the homomorphism $a \mapsto j_0^2(a)$ is one-one, we may regard H as a subgroup of $G^2(q)$ ([15], [16]).

Let $\mathfrak{F}$ be a smooth codimension q foliation of a connected manifold M. Let $\pi_2: P^2(M,\mathfrak{F}) \to M$ be the bundle of transverse 2-frames. We say $\mathfrak{F}$ is a projective (respectively, conformal) foliation if $P^2(M,\mathfrak{F})$ admits a foliated reduction to the group H of (A) (respectively, (B)).

The foliation $\mathfrak{F}$ may be defined by an N-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ where

(i) N is a (not necessarily connected) q-dimensional manifold with a projective (respectively, conformal) structure $\bar{P} \subset P^2(N)$,

(ii) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M,

(iii) $f_\alpha: U_\alpha \to N$ is a submersion defining $\mathfrak{F}/U_\alpha$,

(iv) $g_{\alpha\beta}: f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ is a projective (respectively, conformal) transformation satisfying $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $U_\alpha \cap U_\beta$.

Let $\mathfrak{a}(q,\mathbf{R}) = \mathbf{R}^q + \mathrm{gl}(q,\mathbf{R})$ be the Lie algebra of the group of affine transformations of $\mathbf{R}^q$. Let $\theta = (\theta^i, \theta^i_j)$ be the canonical $\mathfrak{a}(q,\mathbf{R})$-valued one-form on $P^2(N)$ ([14], [15]). Let $(\bar{\omega}^i, \bar{\omega}^i_j)$ be the restriction to P of θ. The following theorem is well-known; see, e.g., [15] and [16].

THEOREM 4.1. *Let $q \geqslant 2$ in the projective case and $q \geqslant 3$ in the conformal case. There is a unique Cartan connection $\bar{\omega} = (\bar{\omega}^i, \bar{\omega}^i_j, \bar{\omega}_j)$ in $\bar{P}$ such that the curvature $\bar{\Omega} = (0, \bar{\Omega}^i_j, \bar{\Omega}_j)$ satisfies $\sum K^i_{jil} = 0$ where $\bar{\Omega}^i_j = \frac{1}{2}\sum K^i_{jkl}\bar{\omega}^k \wedge \bar{\omega}^l$.*

This connection $\bar{\omega}$ is called the normal projective (respectively, conformal) connection.

From now on we assume $q \geqslant 2$ in the projective case and $q \geqslant 3$ in the conformal case.

Since the maps $g_{\alpha\beta}$ are projective (respectively, conformal) transformations, the maps $g^{(2)}_{\alpha\beta}$ preserve $\bar{P}$ and hence the restriction of $\mathfrak{F}^{(2)}$ to P will be defined by the $\bar{P}$-cocycle $\{(\pi_2^{-1}(U_\alpha), f^{(2)}_\alpha, g^{(2)}_{\alpha\beta})\}_{\alpha,\beta\in A}$. Since the normal projective (respectively, conformal) connection is unique, we have $g^{(2)*}_{\alpha\beta}\bar{\omega} = \bar{\omega}$. Let ω be the $\mathfrak{g}$-valued one-form on P given by $\omega/\pi_2^{-1}(U_\alpha) = f^{(2)*}_\alpha\bar{\omega}$. Then ω is a well-defined Cartan connection in the foliated bundle $\pi_2: P \to M$ depending only on the projective (respectively, conformal) structure $P \subset P^2(M,\mathfrak{F})$. We call ω the normal projective (respectively, conformal) connection in the projective (respectively, conformal) normal bundle P of $\mathfrak{F}$.

DEFINITION. We say $\mathfrak{F}$ is a complete projective (respectively, conformal) foliation if ω is complete. We say $\mathfrak{F}$ is a flat projective (respectively, conformal) foliation if ω is flat.

To prove Theorem 3, let $G(q)$ be the group of germs of local diffeomorphisms of $\mathbf{R}^q$ fixing 0 and let $\pi^r: G(q) \to G^r(q)$ be the natural projection. Let L be a leaf of $\mathfrak{F}$ and let $x_0 \in L$. Let $H: \pi_1(L, x_0) \to G(q)$ be the holonomy homomorphism and let $H^r(L, x_0)$ be the infinitesimal holonomy group of order r of L at x_0; that is, $H^r(L, x_0) = \text{image}(\pi^r \circ H) \subset G^r(q)$. Let L^r be a leaf of $\mathfrak{F}^{(r)}$ such that $\pi_r(L^r) = L$. Then $\pi_r: L^r \to L$ is a regular covering whose group of covering transformations is isomorphic to $H^r(L, x_0)$.

Since the germ at a point of a projective (respectively, conformal) transformation of a q-dimensional manifold with $q \geqslant 2$ (respectively, $q \geqslant 3$) is determined by the 2-jet of the transformation at that point, it follows that $H^2(L, x_0)$ is isomorphic to the germinal holonomy group $H(L, x_0)$ of L at x_0.

Suppose L_0 is a compact leaf with finite holonomy group. Let L_0^2 be a leaf of $\mathfrak{F}^{(2)}$ such that $\pi_2(L_0^2) = L_0$. Then L_0^2 is compact. By Theorem 3.1, all the leaves of $\mathfrak{F}^{(2)}$ are compact. Hence all the leaves of $\mathfrak{F}$ are compact with finite holonomy group, proving Theorem 3. □

Theorem 4 follows from Theorem 2 and the fact that the universal cover of the G/H in (A) or (B) is S^q.

EXAMPLE. Let G/H be as in (A) (respectively, (B)). Let $a \in G$. Then the composition $\mathbf{R}^q = \mathfrak{g}_{-1} \xrightarrow{\exp} G \xrightarrow{f} G/H \xrightarrow{a} G/H$ determines a 2-frame $j^2_{aH}(a)$ at $aH \in G/H$. The set $\{j^2_{aH}(a) : a \in G\} \subset P^2(G/H)$ defines a projective (respectively, conformal) structure on G/H which can be identified with the bundle $f: G \to G/H$. The Maurer-Cartan form ω_0 of G is the normal projective (respectively, conformal) connection. Let Γ be a discrete subgroup of G and let ω be the Cartan connection in the foliated bundle $\pi: \Gamma \backslash f^*(G) = P \to M = \Gamma \backslash G$ constructed in the first example in Section 3. Then $\pi: P \to M$ is a projective (respectively, conformal) structure for the foliation $\mathfrak{F}$ of M and ω is the normal projective (respectively, conformal) connection in the projective (respectively, conformal) normal bundle P of $\mathfrak{F}$. Thus $\mathfrak{F}$ is a complete flat projective (respectively, conformal) foliation. In particular, the generalized Roussarie example is a complete flat projective foliation.

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