ON SOME EQUIVALENCE PROBLEMS FOR DIFFERENTIAL EQUATIONS

A. V. Bocharov,♠ V. V. Sokolov,♡ and S. I. Svinolupov◊

International Erwin Schrödinger Institute for Mathematical Physics, Pasteurgasse 4/7, A-1090 Wien, Austria.

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1. Introduction

From an intuitive viewpoint nonlinear differential equations that are integrable by the modern (nonclassical) methods are differential equations reducible to linear equations by transformations of some specific type. Besides if one analyzes a list of known integrable equations one may easily find out that there are quite few genuinely different integrable equations. Many integrable equations are reduced to different (simpler) forms by appropriate changes of variables.

Therefore, the following problem seems to be rather important:

Find out whether a specific equation can be reduced by some transformation to a certain model equation.

Certainly, to make a plausible problem formulation out of this we must specify a class of transformations to be used.

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♠Wolfram Research Inc.
♡ International Erwin Schrödinger Institute for Mathematical Physics
◊ Mathematical Institute, Ufa Center of Russian Academy

A classical choice would be to take point transformations or contact transformations. With this choice the corresponding problem is called the **Cartan's equivalence problem**.

The principal result of the present paper is a set of formulae that have been made as effective as possible so that in the simplest cases a decision whether or not a certain equation is equivalent to a linear may be made algorithmically.

We may think of these formulae and of their prospective generalizations as a possible theoretical foundation for building a computer expert system on nonlinear differential equations.

Since available Computer Algebraic Systems (such as Mathematica, Axiom, Reduce, Maple...) facilitate straightforward algebraic and differential operations with symbolic expressions but do not provide much help with the inverse and more sophisticated operations, it is desirable to reduce each algorithm in the equivalence theory to routine checking of appropriate differential identities.

We believe that it is possible to reach this high level of efficiency in many reasonable classes of equivalence problems.

2. Point transform linearizability of the second order ordinary differential equation

Let us show using the undergoing classical example ([1], [2]) how effective the answer may be in a simple case.

Problem 1. *Find necessary and sufficient conditions for an ordinary differential equation of the form*

$$\frac{d^2y}{dx^2} = F(x, y, \frac{dy}{dx}) \tag{1}$$

to be reduced by a transformation of the form

$$\tilde{x} = \phi(x, y), \quad \tilde{y} = \psi(x, y) \tag{2}$$

to the equation

$$\frac{d^2\tilde{y}}{d\tilde{x}^2} = 0. \tag{3}$$

Solution.

(1) The equation (1) must be of the form

$$\frac{d^2y}{dx^2} = a(x, y)\big(\frac{dy}{dx}\big)^3 + 3b(x, y)\big(\frac{dy}{dx}\big)^2 + 3c(x, y)\frac{dy}{dx} + d(x, y); \tag{4}$$

(2) Both of the following differential polynomials depending on the coefficients a, b, c, d of the expression (4) must be identically zero:

$$K = \alpha_y + \gamma_x + a\beta + 2b\alpha + c\gamma, \tag{5}$$

$$M = \alpha_x + \beta_y - b\beta - 2c\alpha - d\gamma$$

where

$$\begin{aligned} \alpha &= b_x - c_y + ad - bc, \\ \beta &= d_y - c_x + 2c^2 - 2bd, \\ \gamma &= b_y - a_x + 2b^2 - 2ac \end{aligned} \tag{6}$$

Assuming the condition for the (5) holds (and thus the equation is proven to be linearizable), the next question would be how easy it is to find the functions ϕ and ψ giving the actual linearizing transformation. E.g. is it easier than to solve the original equation directly?

In a sense, it is, since it turns out that the required ϕ and ψ satisfy an overdetermined system of **linear** differential equations.

Here is the precise formulation of this fact.

Proposition 1. *Transformation (2) linearizes the equation (1) if ϕ and ψ are an arbitrary pair of functionally independent solutions of the system*

$$\begin{aligned} X_{xx} &= 2pX_x + cX_x - dX_y, \\ X_{xy} &= qX_x + pX_y + bX_x - cX_y, \\ X_{yy} &= 2qX_y + aX_x - bX_y. \end{aligned} \tag{7}$$

where $p = -S_x/S$, $q = -S_y/S$ and where S stands for an arbitrary nonzero solution of the system

$$\begin{aligned} S_{xx} - cS_x + dS_y - \beta S &= 0, \\ S_{yy} - aS_x + bS_y - \gamma S &= 0, \\ S_{xy} - bS_x + cS_y + \alpha S &= 0. \end{aligned} \tag{8}$$

Remark 1.

Provided conditions (5) hold, system (7), (8) is compatible in the following sense.

Define S, S_x, S_y, X, X_x, X_y arbitrarily at a generic point (x_0, y_0) as initial conditions for the system. Then there is a unique solution (S, X) of the system (7), (8) with these initial conditions. In terms of the original unknowns ϕ and ψ it is to be observed that the required linearizing transformation is defined by 8 parameters (e.g. by values of $\phi, \phi_x, \phi_y, \psi, \psi_x, \psi_y, p, q$ at a generic point (x_0, y_0)).

This degree of arbitrariness is to be expected. Indeed, if a certain transformation (2) linearizes equation (1), then its composition with any point symmetry of the equation (3) does the same. It is well known however (see [3]) that symmetries of the latter constitute the 8-parameter Lie group isomorphic to the $SL(3)$. ■

Remark 2.

It is easy to derive from system (7), (8) some relevant ordinary differential equations in each of the variables x and y. Consider, for example, system (8). If $a = 0$ then the second equation is ordinary. Assume that $a \neq 0$. Then it follows from this equation that

$$S_x = \frac{1}{a}S_{yy} + \frac{b}{a}S_y - \frac{\gamma}{a}S.$$

Substituting this expression for S_x to the third equation, we find that S satisfies the following third-order liner ordinary differential equation:

$$S_{yyy} - \frac{a_y}{a} S_{yy} - \Big(\frac{ba_y - ab_y}{a} + b^2 - ac + \gamma\Big) S_y - \Big(\frac{\gamma_y a - a_y \gamma}{a} - b\gamma - a\alpha\Big) S = 0$$

In a similar way we get an ordinary differential equation in x. ∎

It is true that to solve the resulting equations is sometimes more difficult than to integrate the original equation. Here is however a restricted version of linearization problem which can really be solved in quadratures:

Restrict ourselves to transformations of the form

$$\tilde{x} = x, \qquad \tilde{y} = \psi(x, y). \tag{9}$$

Proposition 2.

(1) *Equation (4) is reduced to a linear equation of the form*

$$y_{xx} = 3f(x)y_x + g(x)y + h(x) \tag{10}$$

by a transformation of the form (9) if and only if

$$a = 0, \qquad 2b_x - c_y = 0, \qquad M_y = 0, \tag{11}$$

where

$$M = 2d_y - 6bd - 3c_x + 9/2c^2.$$

(2) *If (11) is valid, then equation (4) is reduced by the transformation (9), where $\psi(x, y)$ is an arbitrary solution of the equation*

$$\psi_{yy} + 3b\psi_y = 0$$

depending on y effectively[1], *to equation (10) with*

$$f = \frac{2}{3}\frac{\psi_{yx}}{\psi_y} + c, \qquad g = \frac{1}{2}M + \frac{3}{2}f_x - \frac{9}{4}f^2, \qquad h = \psi_{xx} - 3f\psi_x + d\psi_y - gu. \tag{12}$$

∎

Remark 3.

It is easy to check that, due to (11) the functions f, g, h do not depend on y. Thus the task of linearizing w.r.t. to the transformation group (9) is solved in quadratures. ∎

Remark 4.

It is known [3] that using a point transformation of the form

$$\tilde{x} = a(x), \qquad \tilde{y} = b(x)y + c(x)$$

any equation of the form (10) may be brought to (3). However, in general, it can not be done in quadratures [4]. ∎

Note that the reduction of (4) to (10) while investigating a specific nonlinear equation means a considerable progress, because for linear equations the issue of integrability in quadratures is very well understood and perfectly algorithmized.

Remark 5.

With the help of an original code written in muMATH we have performed a complete testing of the equations of the form (4) from the Kamke reference book [5]. It turned out that more than one third of them are linearized using the algorithm described in the Proposition 2. ∎

[1] It is obvious that such a solution is obtainable in quadratures

3. Contact linearizability of equations of the third order

One of the results of the present paper is a list of conditions for reducibility of a third-order ordinary differential equation to the equation $y''' = 0$ by a contact transformation. Though the problem of contact linearizability for third order ODE have been considered in several papers (see, for example, [6]) we could not find explicit formulas solving this problem at the algorithmic level.

Let us first recall some basic facts about contact transformations.

The most popular example of a proper contact transformation is the Legendre transformation

$$\tilde{x} = y_1, \qquad \tilde{y} = y - xy_1,$$

where $y_1 = \dfrac{dy}{dx}$ etc.

Using the chain rule for differentiation, it is not difficult to find out how derivatives are affected by this transformation. In particular,

$$\tilde{y}_1 = -x, \qquad \tilde{y}_2 = -\frac{1}{y_2}.$$

A generic contact transformation in the case of one independent variable is a transformation of the form:

$$\tilde{x} = \phi(x, y, y_1), \qquad \tilde{y} = \psi(x, y, y_1), \tag{13}$$

where ϕ and ψ are any functions, satisfying the contactness condition

$$\frac{\partial \phi}{\partial y_1}\Big(y_1 \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial x}\Big) = \frac{\partial \psi}{\partial y_1}\Big(y_1 \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\Big). \tag{14}$$

The contactness condition means exactly that $\tilde{y}_1$ does not depend on y_2:

$$\tilde{y}_1 = \frac{y_2 \dfrac{\partial \psi}{\partial y_1} + y_1 \dfrac{\partial \psi}{\partial y} + \dfrac{\partial \psi}{\partial x}}{y_2 \dfrac{\partial \phi}{\partial y_1} + y_1 \dfrac{\partial \phi}{\partial y} + \dfrac{\partial \phi}{\partial x}} = \chi(x, y, y_1).$$

Of course, the functions ψ, ϕ and χ must be functionally independent.

In case when ϕ and ψ do not depend on y_1, condition (14) holds automatically. This means that the point transformation (2) is a special case of a contact transformation.

In virtue of results by Bäcklund [7], (13) is the most general form of invertible local transformation for the case of one dependent and one independent variable.

Theorem 1. *An equation*

$$\frac{d^3 y}{dx^3} = F\Big(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}\Big) \tag{15}$$

is reduced by a contact transformation (13) to

$$\frac{d^3 \tilde{y}}{d\tilde{x}^3} = 0 \tag{16}$$

if and only if

(1) *it has the form*

$$\frac{d^3y}{dx^3} = a(x,y,\frac{dy}{dx})(\frac{d^2y}{dx^2})^3+3b(x,y,\frac{dy}{dx})(\frac{d^2y}{dx^2})^2+3c(x,y,\frac{dy}{dx})\frac{d^2y}{dx^2}+d(x,y,\frac{dy}{dx}); \quad (17)$$

(2) *the coefficients a, b, c and d of equation (17) satisfy the following differential identities:*

$$\begin{aligned} &L = 0, \quad N = 0, \quad 3H - K = 0, \quad 3F - M = 0, \\ &h = 0, \quad G = 0, \quad f = 0, \quad \lambda = 0, \quad \mu = 0, \end{aligned} \quad (18)$$

where

$$\begin{aligned} K &= \partial_1(\alpha) + \partial(\gamma) + a\beta + 2b\alpha + c\gamma, \\ M &= \partial(\alpha) + \partial_1(\beta) - b\beta - 2c\alpha - d\gamma, \\ L &= \partial(\beta) - 2\partial_y(d) - 2c\beta - 2d\alpha, \\ N &= \partial_1(\gamma) - 2\partial_y(a) + 2b\gamma + 2a\alpha, \\ F &= \partial(\alpha) + 2\partial_y(c) + b\beta - d\gamma, \\ H &= \partial_1(\alpha) + 2\partial_y(b) - c\gamma + a\beta, \\ Q &= \partial(K) + \partial_1(M) + 3/2\alpha^2 - 3/2\beta\gamma, \\ G &= -3\partial_y(\alpha) - \partial_1(M) + \partial(K) + 2bM + 2cK, \\ f &= 3\partial_y(\beta) - 2\partial(M) + 2cM + 2dK, \\ h &= 3\partial_y(\gamma) + 2\partial_1(K) + 2aM + 2bK, \\ \lambda &= 2\partial(Q) - 2\partial_y(M) + 2\alpha M - 2\beta K, \\ \mu &= 2\partial_1(Q) + 2\partial_y(K) + 2\alpha K - 2\gamma M, \\ \alpha &= \partial(b) - \partial_1(c) + ad - bc, \\ \beta &= \partial_1(d) - \partial(c) - 2bd + 2c^2, \\ \gamma &= \partial_1(b) - \partial(a) - 2ac + 2b^2. \end{aligned}$$

Here ∂, ∂_y and ∂_1 stand for the $\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y}$, $\frac{\partial}{\partial y}$ *and* $\frac{\partial}{\partial y_1}$ *respectively.*

Example 1.
Let us figure out when an equation of the from

$$\frac{d^3y}{dx^3} = f(x)\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + h(x)y + s(x); \quad (19)$$

is transformable to equation (16).

Substituting the right hand side of the equation into the (18) we find out that all of them, except $L = 0$, hold trivially. The condition $L = 0$ is then equivalent to the relation

$$h = -\frac{1}{6}f'' + \frac{1}{3}ff' - \frac{1}{3}fg - \frac{2}{27}f^3 + \frac{1}{2}g'.$$

■

Example 2.

Consider an equation of the form

$$\frac{d^3y}{dx^3} = c\Big(\frac{d^2y}{dx^2}\Big)^2 \Big/ \frac{dy}{dx} \tag{20}$$

with a constant parameter c in the right-hand side. The only nontrivial linearization condition for equation (20) is the relation $N = 0$. In this particular case it is equivalent to the equation

$$2c^3 - 9c^2 + 9c = 0.$$

It follows that the nonlinear equation (20) is transformable into equation (16) if and only if $c = 3/2$ or $c = 3$.

4. Further Equivalence Problems

Here we would like to discuss some unsolved or partially solved Cartan equivalence problems.

4.1. Ordinary differential equations.

We consider the following problem to be important:

Problem 2. *Find necessary and sufficient condition for equation (1) to be transformable by a point transformation (2) to one of the 6 Painlevé transcendental equations*[2].

The importance of this problem is in particular due to the fact that the nowadays fashionable Painlevé integrability test ([8], [9]) has one basic drawback: it is not invariant with respect to variable changes of the form (2).

In the paper [10] the problem of reducibility of the equation (4) to the first or second Painlevé transcendental equations has been considered.

Recall that a standard form of these equations is:

$$\frac{d^2y}{dx^2} = y^2 + x, \tag{21}$$

$$\frac{d^2y}{dx^2} = y^3 + xy + a \tag{22}$$

The reducing transformations were selected within the smaller group of transformations of the form

$$\tilde{x} = \phi(x), \qquad \tilde{y} = \psi(x, y) \tag{23}$$

The equivalence problem has been solved in [10] for the equation (21), and a solution for the equation (22) has been suggested that appears to be incorrect for various reasons. In particular the authors of [10] make a wrong assertion that equations of the from (22) with different values of a are equivalent to each other.

Below we generalize the results of [10] concerning equation (21) giving criteria for reducing equation (4) to equation (21) by a general point transformation of the form (2).

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[2]It is easy to see that an equation, transformable to one of the Painlevé's, must have the form (4).

Lemma. *Any equation of the form (4) with $M = 0$ is reduced to an equation with $K = 0$ by the substitution $\tilde{x} = y$, $\tilde{y} = x$. Here K and M are differential expressions defined by (5).*

Proposition 2. *If two equations of the form (4), both with $K = 0$, are related by a transformation of the form (2) then this transformation must actually have the form (23).*

This proposition is very important in the sequel, because the first Painlevé equation (21) has the property $K = 0$. Therefore if the source equation (4) has either $K = 0$ or $M = 0$, we easily fall into the context of the [10].

Theorem 2.

(1) *Assume that for an equation (4) both $K \neq 0$ and $M \neq 0$. Then the equation is reduced to the first Painlevé (21) if and only if the following identities and inequations hold:*

$$k = 0, \quad m = 0, \quad F_x = H_y, \quad R = 0, \quad S_1 \neq 0, \quad S_3 \neq 0, \quad S_4 \neq 0,$$

where

$$\begin{aligned}
k &= \frac{1}{3}KK_x + \frac{4}{3}KM_y - MK_y - aM^2 - 2bKM - cM^2,\\
m &= \frac{1}{3}MM_y + \frac{4}{3}MK_x - KM_x + bM^2 + 2cKM + dK^2,\\
F &= \frac{K_y}{K} + a\frac{M}{K} + b,\\
H &= \frac{M_x}{M} - d\frac{K}{M} - c,\\
S_1 &= \frac{2}{5M^2}\big(H_x - \frac{1}{5}H^2 - cH + dF + 5\beta\big),\\
S_2 &= (S_1)_x + \frac{6}{5}HS_1,\\
S_3 &= 4(\frac{S_2}{M})_x/M + \frac{36}{5}HS_2/M^2 + 8d/M^3 - S_1^2,\\
S_4 &= 2(S_3)_x/M + \frac{24}{5}S_3H/M.\\
R &= (S_4)_x + 3HS_4.
\end{aligned}$$

Here the expression for β defined in the same way as in the Theorem 1.

(2) *The required transformation is given by the explicit formulae*

$$\tilde{x} = S_3 S_4^{-4/5}, \qquad \tilde{y} = S_1 S_4^{-2/5}$$

Remark 6.

Note that the this result is much more effective then in the case of the Problem 1. The required transformation is built in a straightforward manner using the coefficients of the original equation (4).

The reason for such high efficiency is that the first Painlevé equation (21) does not have point symmetries and therefore the reducing transformation is unique (see Remark 1). The situation with the rest five Painlevé transcendental equations in this respect is the same. ■

Along with the Painlevé equations, integrable from the classical viewpoint are those equations of the form (1) that admit a 2-dimensional Lie group of point symmetries [11, p. 200]. Consequently, the following problem is of major practical interest

Problem 3. *Find necessary and sufficient conditions for a given equation of the from (1) to be point-equivalent to one of the model equations, admitting a 2-dimensional Lie group of point symmetries.*

Remark 7.

A list of model equations, admitting 2-dimensional symmetry groups can be found, for example, in [12]. The Problem 3 restricted to transformations of the form (23) was considered in [13]. The results of the latter work would be of much practical use should they be completed by providing recipes for building the required equivalence transformations in quadratures. ■

4.2. Partial differential equations.

From the viewpoint of possible computer implementations, it would be important to obtain criteria of linearizability of simpler partial differential equations.

Let us point out the following generalization of the Problem 1 (that may come unexpected for the reader).

Problem 4. *Find criteria of reducibility of the equation*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{24}$$

to the linear ordinary differential equation

$$\tilde{u}_{\tilde{x}\tilde{x}} = 0$$

it by a point transformation

$$\tilde{x} = \phi(x, y, u), \qquad \tilde{y} = \psi(x, y, u), \qquad \tilde{u} = \chi(x, y, u) \tag{25}$$

Remark 8.

We have come across a number of research papers dealing with formal properties of specific equations from the (24) class in the context when they are actually reducible to (3) by transformations (25) (in a nontrivial way, the circumstance being concealed from the authors). ■

Let us list some of the simplest results relevant to the partial differential equations.

Theorem 3.

(1) *Equation (24) is reduced to the ordinary differential equation $\tilde{u}_{\tilde{x}\tilde{x}} = 0$ by a point transformation*

$$\tilde{x} = x, \qquad \tilde{y} = y, \qquad \tilde{u} = \chi(x, y, u)$$

if and only if it has the form

$$u_{xy} = A(x,y,u)u_x u_y + B(x,y,u)u_x + C(x,y,u)u_y + D(x,y,u)$$

where the coefficients A, B, C, D are related by the following identities:

$$C_u = A_x, \qquad B_u = A_y, \qquad B_x = C_y, \qquad D_u - B_x - AD + BC = 0.$$

(2) *The function χ, defining the transformation, is available in quadratures as a solution of the following compatible system:*

$$\begin{aligned} &(ln(\chi_u))_x = -C, \\ &(ln(\chi_u))_y = -B, \\ &(ln(\chi_u))_u = -A, \\ &\chi_{xy} = -\chi_u D. \end{aligned}$$

Theorem 4. *For the equation*

$$u_t = A(t,x,u)u_{xx} + F(t,x,u,u_x) \tag{26}$$

to reduce via a contact transformation to the equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} \tag{27}$$

it is necessary and sufficient that

(1) *equation (26) is of the form*

$$u_t = A(t,x,u)u_{xx} + B(t,x,u)u_x^2 + C(t,x,u)u_x + D(t,x,u);$$

(2) *the following identities hold:*

$$A_u = 0, \tag{28}$$

$$C_u/A - 2(B/A)_x = 0, \tag{29}$$

$$(2C_x - 4D_u - 2CA_x/A + (C^2 - 4BD)/A)_u - 4(B/A)_t = 0, \tag{30}$$

$$(K_3)_x = 0,$$

where

$$K_3 = 2A^{1/2}(K_2)_x + A_{tt}A^{-1} - \frac{3}{2}A_t^2 A^{-2}$$

$$K_2 = A^{1/2}(CA^{-1})_t + A^{1/2}(\frac{1}{2}A_{xx} - \frac{3}{8}A_x^2 A^{-1} - A_t A^{-1} - 2K_1)_x,$$

$$K_1 = \frac{1}{2}C_x - D_u - \frac{1}{2}CA_x A^{-1} + \frac{1}{4}(C^2 - 4BD)A^{-1}.$$

Remark 9.

In fact, it turns out that the required reducing contact transformation is necessarily a point transformation of the following special form:

$$\bar{t} = \chi(t), \qquad \bar{x} = \phi(t,x), \qquad \bar{u} = \psi(t,x,u).$$

Equations for the required functions χ, ϕ and ψ may be found in the paper [14].

■

On the whole, situation here is much similar to the one described in section 1. That is the problem of reducing the source equation to (27) cannot generally be solved in quadratures. However, an important step for reducing it to a more general linear equation

$$v_t = \alpha(t,x)v_{xx} + \beta(t,x)v_x + \gamma(t,x)v + \delta(t,x)$$

is done in quadratures provided the conditions (28)-(30) are satisfied.

5. A few words concerning the techniques used

There are several formally different ways to solve equivalence problems for differential equations.

The most popularized are approaches related to the theory of invariants as well is to the G-structure theory (cf.[15], [16]).

Our approach however is based on a more or less straightforward step-by-step (often a computer-aided) study of the overdetermined partial differential system defining the required transformation. In the process of bringing this overdetermined system to "passive from" (cf.[17], [18]), we get a number of compatibility conditions. These conditions are differential identities for the coefficients of the source equation.

Let us, for example, present an overdetermined defining system relevant to the problem 1.

It is easily seen that a transformation of the form (2) would imply the following transformation of the second derivative:

$$\begin{aligned}\frac{d^2\tilde{y}}{d\tilde{x}^2} =& J\frac{d^2y}{dx^2}\big(\phi_y\frac{dy}{dx}+\phi_x\big)^{-3} + \big(\psi_{yy}(\frac{dy}{dx})^2 + 2\psi_{yx}\frac{dy}{dx}+\psi_{xx}\big)\big(\phi_y\frac{dy}{dx}+\phi_x\big)^{-2} - \\ &\big(\psi_y\frac{dy}{dx}+\psi_x\big)\big(\phi_{yy}(\frac{dy}{dx})^2+2\phi_{yx}\frac{dy}{dx}+\phi_{xx}\big)\big(\phi_y\frac{dy}{dx}+\phi_x\big)^{-3},\end{aligned}$$

where

$$J = \psi_y\phi_x - \psi_x\phi_y \tag{32}$$

is the Jacobian of the transformation (2).

Substituting the above expressions into equation (3), we get the equation of the form (4) the coefficients of which are defined by the formulae:

$$\begin{aligned}a &= J^{-1}(\psi_y\phi_{yy} - \phi_y\psi_{yy}),\\ b &= J^{-1}(\frac{1}{3}\psi_x\phi_{yy} - \frac{1}{3}\phi_x\psi_{yy} + \frac{2}{3}\psi_y\phi_{yx} - \frac{2}{3}\phi_y\psi_{yx})\\ c &= J^{-1}(\frac{1}{3}\psi_y\phi_{xx} - \frac{1}{3}\phi_y\psi_{xx} + \frac{2}{3}\psi_x\phi_{xy} - \frac{2}{3}\phi_x\psi_{xy}\\ d &= J^{-1}(\psi_x\phi_{xx} - \phi_x\psi_{xx}).\end{aligned} \tag{33}$$

To solve the Problem 1, we should consider the four equations (33) as an overdetermined system of nonlinear partial differential equations for the unknown functions ϕ and ψ. The system turns out to be equivalent to the set of relations (5), (7) and (8). Note that the S function of the system (8) is nothing else $J^{-1/3}$.

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CORRESPONDENCE TO: S.I.SVINOLUPOV, MATHEMATICAL INSTITUTE OF UFA CENTER OF RUSSIAN ACADEMY OF SCIENCES, CHERNYSHEVSKY STR. 112, 450000, UFA, RUSSIA

E-mail address: ken@nkc.bashkiria.su