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LINEAR PARTIAL DIFFERENTIAL EQUATIONS, WITH CONSTANT COEFFICIENTS

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We will derive by a simple method some elementary properties of solutions of systems of linear partial differential equations with constant coefficients. In particular, we will obtain a general theorem on removable singularities. No use will be made of Green's functions or other source functions. Accordingly, our results will be stated for equations in general, although most of them will be of consequence only for equations of elliptic or similar type.

CHAPTER I. DIFFERENTIAL EQUATIONS

For fixed n, we consider an operator of the form

$$\Lambda f \equiv \sum_{r_1+\cdots+r_n \leqq N_0} a_{r_1\cdots r_n} \frac{\partial^{r_1+\cdots+r_n} f}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}, \tag{1}$$

that is a *finite* sum of the form

$$af + b_1 \frac{\partial f}{\partial x_1} + \cdots + b_n \frac{\partial f}{\partial x_n} + c \frac{\partial^2 f}{x_1^2} + \cdots .$$

The coefficients are all constants. The variables $x_1, \cdots, x_n$ are real, the coefficients may be complex. The integer N_0, if it is the smallest possible one, will be called the order of Λf. Whenever an entire system of operators will be introduced, as will be the case in Chapter II, the order N_0 will be the smallest integer admissible for all operators occurring.

If we consider the differential equation

$$\Lambda f = 0, \tag{2}$$

then for $N \geqq N_0$ we will say that $f(x)$ is a *strict* solution of class C^N in an open set D, if $f(x)$ is defined and belongs to differentiability class C^N in D, and if it satisfies the given equation at every point of D.

We will say that $f(x)$ is a *weak* solution of class C^N in D, if it is defined almost everywhere in D and Lebesque integrable in every compact subset of D, and if corresponding to any point x^0 in D there exist a neighborhood $U = U(x^0)$, such that in U, $f(x)$ is a weak limit of strict solutions of class C^N in U. In other words, there exist a sequence of functions $\{f^{(k)}(x)\}$, $k = 1, 2, \cdots$, each of which is defined, and a strict solution of class C^N in U, such that for every bounded measurable function $\psi(x)$ in U, we have

$$\lim_{k\to\infty} \int_U f^{(k)}(x)\psi(x)\, dv_x = \int_U f(x)\psi(x)\, dv_x ,$$

$[dv_x = dx_1 \cdots dx_n]$.

In what follows, the symbol D' will invariably denote an open subset of D whose closure in space is a bounded subset of D. Furthermore, any function $\varphi(x)$ in D which vanishes outside some D' will be called a *testing function*.

We will start with a very simple theorem.

THEOREM 1. *A function $f(x)$ of class C^N in D is a strict solution of* (2), *if and only if we have*

$$\int_D f \cdot \Lambda\varphi \cdot dv_x = 0 \tag{3}$$

for every testing function of class C^N.

PROOF. If f and φ both belong to C^N, and if φ is a testing function, then by Stokes' theorem we have

$$\int_D f \cdot \Lambda\varphi \cdot dv_x = \int_D \varphi \cdot \Lambda f \cdot dv_x .$$

Thus, (3) is equivalent with

$$\int_D \varphi \cdot \Lambda f \cdot dv_x = 0. \tag{4}$$

However, for given f we will have relation (4) holding for all testing functions of any class C^M, if and only if $\Lambda f = 0$, as asserted.

LEMMA 1. *If $\{U\}$ is a covering of D by a system of neighborhoods, if D' is a subset of D as described before, and if $\varphi(x)$ is a testing function of class C^N which vanishes outside D', then there exist a finite number of neighborhoods $U_1, \cdots, U_r$ out of the given covering, and corresponding functions $\varphi_1, \cdots, \varphi_r$ of class C^N, such that φ_ρ vanishes outside U_ρ, $\rho = 1, \cdots, r$, and that*

$$\varphi(x) = \varphi_1(x) + \cdots + \varphi_r(x) \tag{5}$$

in all of D.

This lemma is a familiar tool in the theory of differential equations and of differentiable manifolds.[1]

LEMMA 2. *If $\{U\}$ is a covering of D and if we assume that relation* (3) *holds for every testing function of class C^N which vanishes outside some U, then it holds for all testing functions of class C^N.*

PROOF. Follows from (5).

THEOREM 2. *If $f(x)$ is a weak solution of class C^N, then*

$$\int_D f \cdot \Lambda\varphi \cdot dv_x = 0 \tag{6}$$

holds for all testing functions of class C^N

PROOF. If $f^{(k)}(x)$ approximates weakly to f on U, then by Theorem 1 we have

$$\int_U f^{(k)}(x) \cdot \Lambda\varphi \cdot dv_x = 0 \tag{7}$$

[1] See S. BOCHNER, *Remark on the theorem of Green*, Duke Journal, 3 (1937), 334–338.

for every testing function in U. Letting $k \to \infty$, we obtain (7) for $f(x)$ itself, and by Lemma 2, for D instead of U.

By combining Theorem 2 with one half of Theorem 1, we obtain

THEOREM 3. *If $f(x)$ is a weak solution of class C^N, and if $f(x)$ happens to belong to differentiability class C^M, $M \geqq N_0$, then $f(x)$ is a strict solution of class C^M.*

Our next aim is to invert Theorem 2. This will be done by using so-called h-averages.[2] If the boundary of D' has a distance $\geqq \rho$ from the boundary of D, and if $f(x)$ in D is integrable in every compact subset of D, then for x in D' we can form the h-average

$$f_h(x) = \frac{1}{(2h)^n} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x_1 + t_1, \cdots, x_n + t_n)\, dv_t,$$

if $h < \rho$. If $h < \rho/M$, $M \geqq 1$, we can iterate the process of forming the h-average M times. The M^{th} iterate will be denoted by $f_{h,M}(x)$ and also called the h-average of order M.

THEOREM 4. *If $f(x)$ is integrable and* (6) *holds for $\varphi \in C^N$ and D' is an open subset of D as before, then for every $M \geqq 1$ and h sufficiently small we have relation*

$$\int_{D'} f_{h,M}(x) \cdot \Lambda\varphi \cdot dv_x = 0 \tag{8}$$

for every testing function φ of class C^N in D'.

PROOF. If $\varphi(x)$ is a testing-function in D', then for each sufficiently small $t = (t_1, \cdots, t_n)$, the function $\varphi(x - t)$ is a testing-function in D. Therefore by assumption (6) we have

$$\int_D f(x) \cdot \Lambda\varphi(x - t) dv_x = 0,$$

and by a translation of coordinates we hence obtain

$$\int_{D'} f(x + t) \cdot \Lambda\varphi(x) \cdot dv_x = 0$$

for t sufficiently small. If we integrate this with respect to t, we obtain (8) for $M = 1$ and small h, and by iteration for $M \geqq 1$.

THEOREM 5. *If $f(x)$ is a strict solution of class C^N in D, then in D', $f_{h,M}(x)$ is a strict solution for $M \geqq 1$ and small h.*

PROOFS. Theorems 4 and 1.

THEOREM 6. *If $f(x)$ is a weak solution of* (1) *in D, then in D', $f_{h,M}(x)$ is a strict solution for $M \geqq N_0 + 1$, and small h.*

PROOF. If $f(x)$ is integrable, then $f_h(x)$ is continuous, and $f_{h,M}(x)$ belongs to class C^{M-1}. Now, if $f(x)$ is a weak solution, then by Theorems 2 and 4 we have relation (8). For $M \geqq N_0 + 1$, $f_{h,M}$ is therefore a strict solution by Theorem 1.

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[2] For the role of h-averages in the calculus of varations, see J. W. CALKIN, and C. B. MORREY, *Functions of several variables and absolute continuity*, Duke Journal, 4 (1940), 170-186 and 187-215.

THEOREM 7. *If $f(x)$ is integrable and if* (6) *holds for $\varphi \in C^N$, then for every M, no matter how large, $f(x)$ is a weak solution of* (1) *of class C^M.*

PROOF. By Theorem 4, we have relation (8), and by Theorem 1 it hence follows that $f_{h,M}$ is a strict solution for $M \geqq N_0 + 1$. However, for fixed $M, f_{h,M}(x)$ converges weakly to $f(x)$ in D' as $h \to 0$, and this proves the theorem.

THEOREM 8. *If $f(x)$ is a weak solution of class C^N, then it is also a weak solution of class C^M, for all $M \geqq N$.*

PROOF. Theorems 2 and 7.

We also note

THEOREM 9. *If $f(x)$ is a weak solution in D, then $f_{h,M}$ is a weak solution in D', for $M \geqq 1$.*

Finally we point out a special theorem.

THEOREM 10. *Any weak solution of the Laplace equation*

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

is also (after correction on a null-set) a strict solution, that is, a harmonic function.

PROOF. If a sequence $f^{(k)}(x)$ is weakly convergent, then in particular the norms $\int_{D'} | f^{(k)}(x) | \, dx$ are bounded in k. Now, if $f(x)$ is harmonic in D, then at each point x^0 in D, $f(x^0)$ is equal to the $(n - 1)$-dimensional average over the boundary of the sphere with center at x^0. From this it follows that $f(x^0)$ is also equal to the n-dimensional average over the interior of the sphere. From this it follows that every weakly convergent sequence of harmonic functions in D is boundedly convergent in every compact subset. From the Poisson integral it then follows that the sequence is also uniformly convergent and that so is also the sequence of their partial derivatives of every order. Thus the limit function is likewise harmonic.

CHAPTER II. SYSTEMS OF EQUATIONS

We will next consider systems of equations. If

$$\Lambda^1 f, \cdots, \Lambda^r f \tag{9}$$

are a fixed system of operators of the type considered before, and if

$$L^1 g, \cdots, L^r g$$

is a variable system of such operators, then

$$Lf \equiv L^1(\Lambda^1 f) + L^2(\Lambda^2 f) + \cdots + L^r(\Lambda^r f) \tag{10}$$

is again an operator of this kind. We will say that (10) has been *induced* by the system (9), and we will call it an *induced* operator.

If $f(x)$ is a strict common solution of the system of equations

$$\Lambda^\rho f = 0, \qquad \rho = 1, \cdots, r, \tag{11}$$

and if it has derivatives of sufficiently high order, then it is also a solution of the induced equation

$$Lf = 0. \tag{12}$$

Thus by Theorem 7 we obtain

THEOREM 11. *If $f(x)$ is a common weak solution of the system* (11), *then it is also a weak solution of every induced equation.*

We will now add a second generalization. We will assume that the symbol $f(x)$ represents not one function but a finite set of function

$$f(x) = (f_1(x), \cdots, f_s(x)), \tag{13}$$

the integer s having no arithmetical connection with the dimension n of the space. An operator Λf shall be an expression of the form

$$\Lambda_1 f_1 + \cdots + \Lambda_s f_s,$$

where each $\Lambda_\sigma f_\sigma$ is an operator of our original type. It should be noted that although Λf operates on a vector function of s components, its value is a one-component function. If the notions of integrability, differentiability, weak convergence, h-average, etc. are applied to each component separately; and if the integral

$$\int f \cdot \Lambda\varphi \cdot dv_x$$

which occurs in several theorems is replaced by

$$\int (f_1 \cdot \Lambda_1\varphi + \cdots + f_s \cdot \Lambda_s\varphi) dv_x$$

where $\varphi(x)$ is a one-component function; then previous theorems will also apply to vector-functions, and to systems of equations as well.

APPLICATIONS. As an application of Theorem 11, we consider for $n = 3$ the system of equations

$$\operatorname{div} f = 0, \qquad \operatorname{rot} f = 0, \tag{14}$$

that is the system

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0, \qquad \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0. \tag{15}$$

By the known relation $\Delta f = \operatorname{grad} \operatorname{div} f - \operatorname{rot} \operatorname{rot} f$, the system (14) induces the equations

$$\Delta f_1 = 0, \qquad \Delta f_2 = 0, \qquad \Delta f_3 = 0,$$

and thus by Theorems 10 and 11 we obtain the theorem of H. Weyl.[3]

[3] The method of orthogonal projection in potential theory, Duke Journal, 7 (1940), p. 412.

THEOREM 12. *For $n = 3$, every weak solution of* (14) *has derivatives of all orders.*

Similar to this is

THEOREM 13. *For $n = 2$, a weak solution of the Cauchy-Riemann equations*

$$u_x - v_y = 0, \qquad u_y + v_x = 0 \tag{16}$$

is a strict solution, and thus $u + iv$ is analytic in $x + iy$.

The reason being that equations (16) induce the equations $\Delta u = 0$, $\Delta v = 0$.

Our approach also throws light on the classical theorem of Morera which we will derive in a general set-up. As before, we take any space dimension n, and we introduce for any $s \geqq 1$, expressions

$$\Lambda^i f = \Lambda_1^i f_1 + \cdots + \Lambda_s^i f_s; \qquad i = 1, \cdots, n. \tag{17}$$

We denote by N_0 the precise order of the system and we assume that $f(x)$ belongs to class C^{N_0} in an open set D.

If we will assume that a function f also belongs to C^{N_0+1}, then by Stokes theorem the integral

$$\int_S \sum_{i=1}^n (-1)^{i-1} \Lambda^i(f(x))\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n \tag{18}$$

will have the same value as

$$\int_B \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} \Lambda^i f \right) dv_x , \tag{19}$$

where B is a domain in D, and S is its boundary, the latter being sufficiently smooth. Now, if the surface integral vanishes for all $(n - 1)$-dimensional surfaces S, then f is a strict solution of

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \Lambda^i f = 0. \tag{20}$$

We now claim that the conclusion also holds if f belongs only to C^{N_0}.

THEOREM 14. *If the system* (17) *is of order N_0 and f belongs to C^{N_0} and if* (18) *vanishes over all spheres, then f is a weak solution of* (20).

PROOF. If (18) vanishes for all spheres S, then for fixed S it vanishes for translated spheres $S - (t)$, for sufficiently small t. Or, if we replace x by $x + t$, we see that (18) vanishes for $f(x + t)$ in place of $f(x)$. If we then integrate with respect to t under the integral, we arrive at the vanishing of (18) for $f_h(x)$ instead of $f(x)$. But if $f(x)$ belongs to C^{N_0}, then $f_h(x)$ belongs to C^{N_0+1}, and thus $f_h(x)$ is a strict solution of (20). Therefore, $f(x)$ itself is a weak solution of (20), as asserted.

For instance, if u and v are continuous functions in (x, y), and $\alpha, \beta, \gamma, \delta$ are constants, and

$$\int_S (\alpha u + \beta v)\, dx + (\gamma u + \delta v)\, dy = 0$$

for all closed curves in a domain, then (u, v) are a weak solution of

$$\frac{\partial}{\partial y}(\alpha u + bv) - \frac{\partial}{\partial x}(\gamma u + \delta v) = 0.$$

In particular, if

$$\int_S f\,dz \equiv \int_S (u + iv)(dx + i\,dy) = 0,$$

then we have

$$\frac{\partial}{\partial x}(iu - v) - \frac{\partial}{\partial y}(u + iv) = 0$$

and this equation decomposes into the Cauchy-Riemann equations (16). However, a weak solution of the latter equations is also a strict solution, and in this way we obtain a peculiar new proof for the theorem of Morera.

Another curious little theorem for arbitrary n is as follows.

THEOREM 15. *If $f(x)$ is continuous in D, and if*

$$\int_S f dx_2\, dx_3 \cdots dx_n = 0$$

for every spherical hypersurface, then $(\partial f/\partial x_1)$ exists and is equal to 0.

PROOF. By Theorem 14, $f(x)$ is a weak solution of $(\partial f/\partial x_1) = 0$, and $f_h(x)$ is a strict solution. Thus $f_h(x)$ is constant in x_1, and since in the present case $f_h(x)$ converges uniformly to $f(x)$, the latter function is also constant in x_1.

Finally we note the following theorem.

THEOREM 16. *If it is known that f belongs to C^1 in D, and that*

$$\int_S \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x_i}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n = 0$$

for all spheres S, then f is harmonic.

CHAPTER III. REMOVABLE SINGULARITIES

Let A be an arbitrary bounded measurable set in n-space and for $0 < \epsilon < \infty$ let A_ϵ denote the ϵ-neighborhood of A, that is, the union of all open spheres of radius ϵ with centers at points of A. As shown elsewhere,[1] there is in entire space a function $Q_\epsilon(x)$ of class C^∞ having the following properties: (i) $|Q_\epsilon(x)| \leqq 1$; (ii) $Q_\epsilon(x) = 0$ in $A_{2\epsilon}$; (iii) $Q_\epsilon(x) = 1$ outside $A_{3\epsilon}$; and (iv) for every multi-index $(k_1, \cdots, k_n)$ there exist a constant which is independent of x and ϵ such that

$$\left| \frac{\partial^{k_1+\cdots+k_n} Q_\epsilon(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right| \leqq \frac{C}{\epsilon^{k_1+\cdots+k_n}}.$$

Now, let D be a bounded open set, let D_0 be an open subset, and let A be the difference $D - D_0$. We will look upon A as an "exceptional set" in D. Let

(21) $$\Lambda^\rho f = 0, \qquad \rho = 1, \cdots, r,$$

be a system of equations for the vector function

$$f = (f_1, \cdots, f_s), \tag{22}$$

and let N_0 be the order of the system (21). Let f be a given weak solution of (21) in D_0 and let it be integrable in D_0. We complete the vector function f on the exceptional set by assigning there the values 0, and we are posing the problem of deciding under what conditions the completed function will be a weak solution of (21) in all of D.

If φ is a testing function in D, then

$$\varphi(x) \cdot Q_\epsilon(x)$$

will be a testing function of the same class in D_0. Since f was assumed to be a weak solution of (21) in D_0, we have

$$\int_D (f_1 \cdot \Lambda_1(\varphi Q_\epsilon) + \cdots + f_s \cdot \Lambda_s(\varphi Q_\epsilon))\, dv_x = 0. \tag{23}$$

If we take into consideration that the partial derivatives of φ of order $\leqq N_0$ are bounded in D, and if we make use of all properties of $Q_\epsilon(x)$, it is not hard to see that for x in D we have

$$|\Lambda_\sigma^\rho(\varphi Q_\epsilon) - \Lambda_\sigma^\rho(\varphi)| \leqq \frac{C_0}{\epsilon^{N_0}}, \tag{24}$$

where C_0 is independent of x and ϵ. If we use property (iii) and compare (23) and (24) we next obtain the decisive inequality

$$\left| \int_D (f_1 \cdot \Lambda_1^\rho \varphi + \cdots + f_s \cdot \Lambda_s^\rho \varphi)\, dv_x \right| \leqq \frac{C_0}{\epsilon^{N_0}} \int_{A_{3\epsilon}} (|f_1| + \cdots + |f_s|)\, dv_x .$$

By using Theorem 7 we now arrive at the following theorem.

THEOREM 17. *If a vector function f is a weak solution of a system* (21) *of order N_0 in a bounded open set $D_0 = D - A$, where D is a larger bounded open set and A an exceptional set in it; if B_ϵ is the ϵ-neighborhood of A in D_0 and $v(\epsilon)$ its Lebesque measure; and if we denote by $T(\epsilon)$ the strong average of f over B_ϵ, that is*

$$T(\epsilon) = \frac{1}{v(\epsilon)} \int_{B_\epsilon} (|f_1| + \cdots + |f_s(x)|)\, dv_x ; \tag{25}$$

then adding the values $f \equiv 0$ on A will produce a weak solution of (21) *in all of D provided we have*

$$T(\epsilon) = o\left(\frac{\epsilon^{N_0}}{v(\epsilon)}\right) \tag{26}$$

as $\epsilon \to 0$.

If the set A has (n-dimensional) Lebesque measure 0, it makes no difference by which values we complete f on the points of A, since a weak solution is determined only up to a null-set. If furthermore the system (21) is such that every weak solution is automatically a strict solution, then the given vector f in D_0

will automatically determine its additional values on A, and in this case our theorem is truly a theorem on removable singularities.

If A is a sufficiently smooth pointset of dimension m, $0 \leqq m < n$, then

$$v(\epsilon) = O(\epsilon^{n-m}) \tag{27}$$

and (26) is implied by

$$T(\epsilon) = o(\epsilon^{N_0+m-n}). \tag{28}$$

For $N_0 + m = n$ this is $T(\epsilon) = o(1)$ and thus the requirement is that f shall vanish in the strong average when x approaches the exceptional set A. If $N_0 + m > n$, a stringent mode of vanishing is required, whereas for $N_0 + m < n$ a certain latitude of unboundedness is not precluded *a priori*.

If we take an integer M such that $0 < M < N$, if we introduce the derivatives

$$f_{\sigma,\mu_1\cdots\mu_n} = \frac{\partial^{\mu_1+\cdots+\mu_n} f_\sigma}{\partial x_1^{\mu_1} \cdots \partial x_{x_n}^{\mu_n}} \tag{29}$$

for

$$0 \leqq \mu_1 + \cdots + \mu_n \leqq M,$$

and if f belongs to class C^M, then we can set up the averages

$$T_M(\epsilon) = \frac{1}{v(\epsilon)} \int_{B_\epsilon} \left(\sum_{\sigma=1}^{s} \sum_{\mu} |f_{\sigma,\mu}| \right) dv_x . \tag{30}$$

Very often relation (26) can be replaced by

$$T_M(\epsilon) = o\left(\frac{\epsilon^{N_0-M}}{v(\epsilon)} \right) \tag{31}$$

and similarly (28) by

$$T_M(\epsilon) = o\ (\epsilon^{N_0-M+m-n}). \tag{32}$$

This replacement will certainly be admissible if there exists a system of equations

$$L(f_\sigma\, ; f_{\sigma,\mu}) = 0 \tag{33}$$

of order $N_0 - M$ such that a combination of the systems (33) and (29) will induce the original system (21). In particular for $M = N_0 - 1$ we thus obtain

$$T_{N_0-1}(\epsilon) = o(\epsilon^{m-n+1}),$$

and for a hypersurface A of dimension $n - 1$ this will be fulfilled whenever f and its derivatives of order $\leqq N_0 - 1$ are approaching values 0 as x approaches A.

Chapter IV. A Uniqueness Theorem

We will draw a peculiar conclusion from the last theorem.

Theorem 18. *If D_0 is a bounded domain and if B is an $(n - 1)$-dimensional piece of hypersurface on the boundary of D_0 and B_ϵ is its ϵ-neighborhood in D_0 ; if* *the system* (21) *of order* N_0 *is such that every weak solution is automatically analytic in the real variables* $x_1, \cdots, x_n$; *and if*

$$T(\epsilon) = o\left(\frac{\epsilon^{N_0}}{v(\epsilon)}\right),$$

or more specifically if

$$T(\epsilon) = o(\epsilon^{N_0-1}),$$

then $f(x) \equiv 0$.

If (21) *is inducible from a system of order* $N_0 - M$ *as described in Chapter III, and if*

$$T^M(\epsilon) = o(\epsilon^{N_0-M-1}),$$

then again $f(x) \equiv 0$.

We can take a domain D_1 which borders on B from the outside. If we put $A = B + D_1$, $D = D_0 + A = D + B + D_1$, we may apply Theorem 17 and the conclusion is that $f(x)$, if completed by values 0 in A, is a weak solution in all of D. By our specific assumption, $f(x)$ is analytic in D, and since it vanishes in an open subset D_1, it vanishes identically, as asserted.

Appendix

Operators with non-constant coefficients

Some of our results, notably the theorem on removable singularities, will remain valid if the coefficients a_r which occur in our operators are general functions of $(x_1, \cdots, x_n)$. We introduce an expression of the form

$$\sum_{r_1+\cdots+r_n \leqq N_0} a_{r_1} \cdots {}_{r_n}(x) \frac{\partial^{r_1+\cdots+r_n} f}{\partial_{x_1}^{r_1} \cdots \partial_{x_n}^{r_n}}$$

in which the coefficients $a_r(x)$ have bounded continuous derivatives of order $\leqq N_0$ in the given domain D; or of any larger order N and in some larger domain, if the context will so require. The given operator will be denoted by $\bar{\Lambda} f$; whereas by Λf we will denote its formal adjoint, that is the operator

$$\sum_{r_1+\cdots+r_n \leqq N_0} (-1)^{r_1+\cdots+r_n} \frac{\partial^{r_1+\cdots+r_n}(a_r f)}{\partial_{x_1}^{r_1} \cdots \partial_{x_n}^{r_n}}.$$

If f belongs to C^{N_0} and φ is a testing function of class C^{N_0}, we have

$$\int_D (f \cdot \Lambda\varphi - \varphi \cdot \bar{\Lambda} f)\, dv_x = 0,$$

and we will say that $f(x)$ is a weak solution of the equation

$$\bar{\Lambda} f = 0$$

in D if it is defined almost everywhere and Lebesque integrable in every compact subset of D and if for every testing function φ of class C^{N_0} we have

$$\int_D f \cdot \Lambda\varphi\, dv_x = 0. \tag{34}$$

A similar definition holds for a vector function being a solution of an equation

$$\bar{\Lambda}_1(f_1) + \cdots + \bar{\Lambda}_s(f_s) = 0.$$

In testing relation (34) we may restrict ourselves to functions φ from C^N, for some $N \geqq N_0$, since for every testing function φ of class C^{N_0} there exist a sequence of testing functions $\{\varphi_n\}$, each of class C^N, such that

$$\Lambda\varphi_n \longrightarrow \Lambda\varphi.$$

Furthermore, the fundamental facts about induced equations remain in force, since the formal adjoint of $\bar{L}(\bar{\Lambda}f)$ is $\Lambda(Lf)$ and (34) implies

$$\int_D f\cdot \Lambda(L\varphi)\, dv_x = 0.$$

It must be pointed out that in general no smoothing process in the nature of an h-average will be available for our weak solutions, and we cannot claim that a weak solution is a weak limit of strict solutions. However, the proofs in Chapters III and IV do not depend on the latter facts and we thus may state

THEOREM 19. *Theorems* 17 *and* 18 *are also valid for operators with non-constant coefficients.*

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