Symmetries and integrability of a fourth-order Euler–Bernoulli beam equation

Ashfaque H. Bokhari,[1] F. M. Mahomed,[2,a)] and F. D. Zaman[1]
[1]*Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia*
[2]*Centre for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, South Africa*

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The complete symmetry group classification of the fourth-order Euler–Bernoulli ordinary differential equation, where the elastic modulus and the area moment of inertia are constants and the applied load is a function of the normal displacement, is obtained. We perform the Lie and Noether symmetry analysis of this problem. In the Lie analysis, the principal Lie algebra which is one dimensional extends in four cases, viz. the linear, exponential, general power law, and a negative fractional power law. It is further shown that two cases arise in the Noether classification with respect to the standard Lagrangian. That is, the linear case for which the Noether algebra dimension is one less than the Lie algebra dimension as well as the negative fractional power law. In the latter case the Noether algebra is three dimensional and is isomorphic to the Lie algebra which is $\mathrm{sl}(2,\mathbb{R})$. This exceptional case, although admitting the nonsolvable algebra $\mathrm{sl}(2,\mathbb{R})$, remarkably allows for a two-parameter family of exact solutions via the Noether integrals. The Lie reduction gives a second-order ordinary differential equation which has nonlocal symmetry.
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I. INTRODUCTION

The Euler–Bernoulli beam equation that describes the relationship between the applied load and the deflection in the beam is the fourth-order differential equation (DE) (see, e.g., Refs. 1 and 2),

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)=f, \tag{1}$$

where E is the elastic modulus, I the area moment of inertia, and f the applied load. Equation (1) has been studied extensively for E and I constant and f depending on the independent variable x. In the case of centripetal force distribution, the load is of the form $f=\mu\omega^2 y$, where μ is the linear mass density and ω the angular frequency. This motivates one to consider the load as a general function of y and thus consider the equation

$$\frac{d^4y}{dx^4}=f(y), \tag{2}$$

where E and I are assumed to be constant.

We now provide a brief survey of the literature on the algebraic properties of scalar lower-order ordinary DEs (ODEs) to provide further motivation for our study.

a)Electronic mail: fazal.mahomed@wits.ac.za.

Second-order ODEs are of great importance in mathematical analysis and in several applications. They arise in relativity, continuum mechanics, and various other fields. Significant contributions have been made to second-order ODEs from various viewpoints. This is due to these equations naturally arising in several applications such as classical mechanics. Their algebraic properties have been studied by many researchers (a sample, apart from the works of Lie,[3] is given by the contributions Prince and Eliezer,[4] Sen,[5] Damianou and Sophocleaus,[6] Gorringe and Leach,[7] Wafo and Mahomed,[8] Naeem and Mahomed,[9] and Mahomed and Qadir[10]). Some fundamental equations of classical mechanical systems are the free particle, the oscillator systems, Henon–Heiles, and the Kepler problem with and without drag laws. These have enjoyed considerable attention over several decades (see, e.g., Refs. 4 and 7 and Leach and Gorringe[11]).

Third-order ODEs have also enjoyed some interest since the initial works of Lie[3] on the subject. Their canonical forms and integrability were investigated (see Mahomed and Leach[12] and Ibragimov and Nucci[13]). Linearization criteria were also studied. These include the works of Chern,[14] Mahomed and Leach,[15] Grebot,[16] Neut *et al.*,[17] Euler *et al.*,[18] and Ibragimov and Meleshko[19] (see also Mahomed,[20] for a review).

Scalar fourth-order ODEs have been studied to some extent in Lie's works. He gave an implicit classification included in his general classification scheme. The explicit canonical forms for third-order ODEs admitting four point symmetries were given in Cerquetelli *et al.*[21] The algebraic criteria for linearization were given in Ref. 15. Recently Ibragimov and Meleshko[22] gave invariant criteria for linearization for such equations.

There has been some work on the algebraic properties of the beam equation (1) in the case when the Eq. (1) is a partial DE due to Özkaya and Pakdemirli[23] and equivalent characterizations using symmetries when the fourth-order partial DE has two functions given by Wafo Soh.[24]

In this work for the first time the Euler–Bernoulli fourth-order ODE (2) is studied from the symmetry standpoint. We perform a complete Lie symmetry as well as Noether, with respect to the standard Lagrangian, classification of the beam equation (2). In this analysis an exceptional power law model $y^{-5/3}$ arises which allows for integration in terms of a two-parameter family of solutions.

The paper is organized as follows. In Sec. II we present the complete Lie classification up to equivalence point transformations. In Sec. III we obtain all the Noether point symmetries. Section IV deals with the Noether integrals and Lie reductions. Finally in Sec. V we present a discussion of the results obtained.

II. COMPLETE LIE POINT SYMMETRY CLASSIFICATION

We commence with equivalence transformations (see, e.g., Torrisi *et al.*,[25] for computations of these for a diffusion system) which are essential for simplifying the determining equation and for obtaining disjoint classes. Equivalence transformations of the Eq. (2) are point transformations in the (x,y) space of independent and dependent variables of this equation which leaves invariant the family (2). That is, the equivalence transformations map any Eq. (2) with arbitrary function f into the same family (2) with, in general, another function f. Equivalence transformations of the Eq. (2) are straightforward to obtain. These are

$$\bar{x} = a_1 x + a_1, \quad \bar{y} = b_1 y + b_2,$$

$$\bar{f} = \frac{b_1}{a_1^4} f, \quad a_1 b_1 \neq 0, \tag{3}$$

where a_i and b_i are constants. These are used to simplify the classifying relation in the determining equation for the infinitesimal generators of symmetry.

The generators of a symmetry group of Eq. (2) is

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{4}$$

The fourth prolongation of the generator X is

$$X^{[4]} = X + \sum_{i=1}^{4} \zeta_i \frac{\partial}{\partial y^{(i)}}, \tag{5}$$

where

$$\zeta_1 = D_x(\eta) - y' D_x(\xi),$$

$$\zeta_i = D_x(\zeta_{i-1}) - y^{(i)} D_x(\xi), \quad i = 2,3,4, \tag{6}$$

where $y^{(i)}=d^i y/dx^i$ and D_x is the total derivative operator. The determining equations for the symmetry is given by

$$X^{[4]}(y^{(4)} - f(y))|_{(2)} = 0. \tag{7}$$

This gives rise to

$$\xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x), \quad \alpha'' = 0, \quad 3\xi'' - 2\alpha' = 0, \tag{8}$$

$$-f'(\alpha y + \beta) + f(\alpha - 4\xi') + \beta^{(4)} = 0. \tag{9}$$

If f is arbitrary in y, then system (8) and (9) easily yields

$$\xi = c_1, \quad \eta = 0. \tag{10}$$

Hence for arbitrary $f(y)$, Eq. (2) has symmetry generator,

$$X_1 = \frac{\partial}{\partial x}. \tag{11}$$

Thus the principal algebra of Eq. (2) is one dimensional and is spanned by (11). Now we consider all possibilities of $f(y)$, up to equivalence transformations, for which an extension of the principal algebra occurs. We utilize the equivalence transformations (3). Since f depends only on y, it is possible for Eq. (9) to be satisfied when the coefficients vanish [this amounts to (10)] or are proportional to a function of x. From (9) we therefore require that $f(y)$ satisfies

$$-f'(ay + b) + cf + d = 0, \tag{12}$$

where a, b, c, and d are constants not all zero. If $a=b=c=d=0$, one obtains the principal algebra. This relation (12) is the classifying relation which is simplified by means of the equivalence transformations (3). We find that the following cases arise for which an extension of the principal Lie algebra is possible. We note that the log function cases do not provide extensions and hence are not given below.

(I) f is linear. This linear case is of interest as the Lie algebra is not unique in dimension as for the scalar linear second-order ODEs (Ref. 15) and they have other interesting properties when compared with the corresponding Noether algebra which we pursue in Sec. III. There are three subcases.

(i) $f=0$. The principal algebra extends by 7. Hence the Lie algebra is eight dimensional. It is spanned by (11) and

$$X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial y}, \quad X_5 = x\frac{\partial}{\partial y},$$

$$X_6 = x^2\frac{\partial}{\partial x}, \quad X_7 = x^3\frac{\partial}{\partial y}, \quad X_8 = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}, \tag{13}$$

(ii) $f=1$. Again the algebra is eight dimensional and is spanned by (11) and the operators

$$X_2 = \frac{\partial}{\partial y}, \quad X_3 = \left(y - \frac{x^4}{24}\right)\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial y}, \quad X_5 = x^2\frac{\partial}{\partial y},$$

$$X_6 = x^3\frac{\partial}{\partial y}, \quad X_7 = 6x\frac{\partial}{\partial x} + x^4\frac{\partial}{\partial y}, \quad X_8 = \frac{x^2}{3}\frac{\partial}{\partial x} + \left(xy + \frac{x^5}{72}\right)\frac{\partial}{\partial y}, \tag{14}$$

(iii) There are two subcases.

(a) $f=y$. The principal algebra extends by 5 and we have

$$X_2 = y\frac{\partial}{\partial y}, \quad X_3 = \exp x\frac{\partial}{\partial y}, \quad X_4 = \exp(-x)\frac{\partial}{\partial y}, \quad X_5 = \sin x\frac{\partial}{\partial y}, \quad X_6 = \cos x\frac{\partial}{\partial y}. \tag{15}$$

Therefore, the Lie algebra is six dimensional.

(b) $f=-y$. The principal algebra extends by 5 and one has

$$X_2 = y\frac{\partial}{\partial y}, \quad X_3 = \exp\frac{x}{\sqrt{2}}\sin\frac{x}{\sqrt{2}}\frac{\partial}{\partial y}, \quad X_4 = \exp\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}}\frac{\partial}{\partial y}, \quad X_5 = \exp\left(-\frac{x}{\sqrt{2}}\right)\sin\frac{x}{\sqrt{2}}\frac{\partial}{\partial y},$$

$$X_6 = \exp\left(-\frac{x}{\sqrt{2}}\right)\cos\frac{x}{\sqrt{2}}\frac{\partial}{\partial y}, \tag{16}$$

The Lie algebra is six dimensional here too.

(II) f is an exponential function. It is of the form $f=\delta \exp y$, $\delta=\pm 1$.

The principal algebra extends by 1 with additional operator,

$$X_2 = x\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}. \tag{17}$$

Thus the Lie algebra is two dimensional.

(III) f is a general power law. In this case we have $f=\delta y^\sigma$, $\delta=\pm 1, \sigma \neq 0, 1, -5/3$.

The principal algebra extends by 1. We have

$$X_2 = (1-\sigma)x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y}. \tag{18}$$

The Lie algebra is two dimensional as well.

(IV) f is a negative fractional power law. We have $f=\delta y^{-5/3}, \delta=\pm 1$.

The principal algebra extends by 2. Here we have the $\mathrm{sl}(2,\mathbb{R})$ symmetry algebra. That is, (11) and the generators,

$$X_2 = x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}, \tag{19}$$

span the Lie algebra $\mathrm{sl}(2,\mathbb{R})$. This case is of great importance for integrability which we investigate in Sec. IV. It also arises in the Noether classification in Sec. III.

In the next section we investigate Noether point symmetries for our Eq. (2).

III. COMPLETE NOETHER CLASSIFICATION

There has been much work done on the Noether classification for scalar second-order ODEs (see Kara *et al.*[26]), whereas Noether symmetries of fourth-order equations are scarce in the literature. Here we perform a Noether point symmetry classification of Eq. (2) with respect to the standard Lagrangian.

The standard Lagrangian for Eq. (2) is

$$L=\frac{1}{2}(y'')^2-\int f(y)dy. \tag{20}$$

The determining equation (see, e.g., Kara and Mahomed[27] for the general formula) for the Noether point symmetries corresponding to L in (20) is

$$X^{[2]}L+LD_x(\xi)=D_x(B), \tag{21}$$

where X is the generator of Noether symmetry and is of the form (2) and B is the gauge term which in this case is a function of x, y, and y' as L has the second derivative term y''.

The solution of Eq. (21) results in

$$\xi'''(x)=0, \quad \eta=\tfrac{3}{2}\xi' y+\alpha(x), \tag{22}$$

$$-f\left(\frac{3}{2}\xi' y+\alpha\right)-\xi'\int f(y)dy+\alpha^{(4)}y-\gamma'(x)=0. \tag{23}$$

The gauge function B is given by

$$B=\xi'' y'^2+\alpha'' y'-\alpha''' y+\gamma(x). \tag{24}$$

It is clear that if $f(y)$ is arbitrary, system (22) and (23) implies that there is one Noether point symmetry generator given by (11). Therefore, in the generic case we have a one-dimensional Noether algebra which is the same as the Lie algebra for arbitrary f. The other cases for which more than one Noether symmetry results are listed below.

(NI) f is linear. This case also occurs in the Lie classification. However, it is important to comment on these subcases again in order to compare the dimensionality of the Noether algebra with the Lie algebra. There is a difference from what transpires for scalar second-order ODEs (see Mahomed *et al.*[26]).

(i) $f=0$. We have all the Lie symmetry generators in (13) except for X_3 and X_4 which combine as $X_3+3X_4/2$ in order to be Noether. Hence we obtain a seven-dimensional Noether algebra which is a subalgebra of the eight-dimensional Lie algebra. Here we have one less dimension whereas for second-order linear ODEs one has eight and five dimensions for the Lie and Noether algebras, respectively (see Ref. 26).

(ii) $f=1$. Here too we deduce that all the Lie generators in (14) except for X_3 and X_7 which has the combination $3X_3/2+X_7/6$ to be Noether. Again the same remarks as in (i) apply.

(iii) $f=\delta y, \delta=\pm 1$. In this case the homogeneity symmetry generator is no longer Noether. We obtain a five-dimensional Noether algebra which is a subalgebra of the six-dimensional Lie algebra spanned by the generators (15) and (16), respectively. These submaximal algebra do not arise for second-order ODEs.[15,26]

Thus, a study of the Lie and Noether symmetries for scalar fourth-order linear ODEs is quite interesting in its own right.

(NII) f is a negative power law $f=\delta y^{-5/3}$, $\delta=\pm 1$. The Lie algebra is isomorphic to the Noether algebra and it is the nonsolvable algebra $\mathrm{sl}(2,\mathbb{R})$. This case is exceptional for another reason as well. The integrability of the ODE for this f is quite interesting as we see in Sec. IV. We consider both the Lie and Noether routes for reductions.

It is worth mentioning that the general power law and exponential function cases do not provide further Noether point symmetries. They are part of the generic case.

In summary of this section, there are two cases for which the Noether point symmetries extend beyond translations. These are (NI) and (NII) as given above. Both these cases having interesting properties as remarked. The exceptional case (NII) will be further analyzed in Sec. IV.

IV. REDUCTIONS AND INTEGRABILITY

The integrability of the maximal Lie and Noether cases, i.e., the linear cases are trivial. The other cases that arise in both the Lie and Noether classifications are nontrivial.

We first discuss the Lie reductions for the generic, exponential function, general power law, and negative fractional power law in this order.

In the generic case when f is an arbitrary function of y the Lie reduction gives rise to the third-order ODE,

$$v\frac{d}{du}\left(v\frac{d}{du}\left(v\frac{dv}{du}\right)\right)=f(u), \tag{25}$$

where $u=y$ and $v=y'$ are invariants of the translation group generated by X_1 as given in (11). One cannot, in general, proceed further in the absence of further symmetries.

For the exponential function, $f=\delta \exp y$, $\delta=\pm 1$, Eq. (2) admits the generators of symmetry X_1 and X_2 given by (11) and (17). The Lie algebra for this case has $[X_1,X_2]=X_1$ which gives a solvable algebra. Thus, one can reduce the order of the ODE (2) with this f twice. A basis of invariants is given by

$$u=y'^4\exp(-y), \quad v=y'^{-2}y''. \tag{26}$$

The second-order ODE after reduction is

$$(4uv-u)\frac{dr}{du}+3rv=\delta u^{-1}, \quad \delta=\pm 1, \tag{27}$$

where

$$r=(4uv-u)\frac{dv}{du}+2v^2. \tag{28}$$

For the general power law, $f=\delta y^{\sigma}$, $\delta=\pm\sigma\neq 01,-5/3$, the ODE (2) admits two generators of symmetry X_1 and X_2 given by (11) and (18). The Lie algebra has commutation relation $[X_1,X_2]=(1-\sigma)X_1$. There arise two subcases here.

If $\sigma\neq -3$, a basis of invariants for this solvable group is

$$u=y^{-(3+\sigma)/4}y', \quad v=y'^{-2(1+\sigma)/(3+\sigma)}y''. \tag{29}$$

The reduced ODE is the second-order equation,

$$\left(uv-\frac{3+\sigma}{4}u^{(7+\sigma)/(3+\sigma)}\right)\frac{dr}{du}+\frac{3\sigma+1}{3+\sigma}rv=\delta u^{-4\sigma/(3+\sigma)}, \quad \delta=\pm 1, \tag{30}$$

where

$$r=\left(uv-\frac{3+\sigma}{4}\right)\frac{dv}{du}+\frac{2(1+\sigma)}{3+\sigma}v^2. \tag{31}$$

If $\sigma=-3$, a basis of invariants for the solvable group is

$$u=y', \quad v=yy'^{-1}y''. \tag{32}$$

In this subcase the reduced equation is the second-order ODE,

$$v\frac{dr}{du}+2rv-2r=u^{-3}, \tag{33}$$

where

$$r=v\frac{dv}{du}+v^2u^{-1}-v. \tag{34}$$

The Lie reductions for the general power law and exponential function, which do not arise in the Noether classification, give at most second-order ODEs and one cannot proceed further in the absence of further symmetries or deeper insights.

Now we comment on the Lie reduction for $f=\delta y^{-5/3}$, $\delta=\pm 1$. The ODE (2) for this negative fractional power law admits the $\mathrm{sl}(2,\mathbb{R})$ symmetry algebra which is nonsolvable. So one can at most perform a double reduction in order by use of the Lie point symmetries. Basis of invariants for the two-dimensional ideal (generated by X_1 and X_2) of this symmetry group is given by

$$u=y'y^{-1/3}, \quad v=y'y'', \tag{35}$$

by means of which this fourth-order ODE reduces to

$$\left(uv-\frac{1}{3}u^4\right)\frac{dr}{du}-3rv=\delta u^5, \quad \delta=\pm 1, \tag{36}$$

where

$$r=\left(uv-\frac{1}{3}u^4\right)\frac{dv}{du}-v^2. \tag{37}$$

This Lie reduced second-order ODE (35) has nonlocal symmetry,

$$X_3=\exp 2\int v\left(uv-\frac{1}{3}u^4\right)^{-1}du\left(3u^{-2}\frac{\partial}{\partial u}+(3u^{-3}v+4)\frac{\partial}{\partial v}\right). \tag{38}$$

We are unable to further reduce this ODE by use of this symmetry generator.

We now investigate the Noether integrals for Eq. (2). The only interesting case is the negative power law $f=\delta y^{-5/3}$, $\delta=\pm 1$ for which Eq. (2) has $\mathrm{sl}(2,\mathbb{R})$ symmetry algebra. This is listed as (IV) and (NII) in Secs. II and III as they arise in both the Lie and Noether analyses. We examine both the generic case and this exceptional negative power law case below.

In order to examine Noether integrals we invoke the Noether theorem (see, e.g., Ref. 27) adapted to the second-order Lagrangian (19).

If X as in (4) is a generator of Noether point symmetry with respect to the Lagrangian L given by (19), then

$$I=B-\xi L-W\frac{\delta L}{\delta y'}-D_xW\frac{\delta L}{\delta y''} \tag{39}$$

is a first integral corresponding to X, where $W=\eta-y'\xi$ is the characteristic function, $\delta L/\delta y'=-y'''$, $\delta L/\delta y''=y''$, and B is the gauge function given by (23).

We first consider the generic case for which $X_1=\partial/\partial x$. The formula (39) provides the first integral,

$$I=\frac{1}{2}(y'')^2+\int f(y)dy-y'y'''. \tag{40}$$

Thus the reduced ODE is

$$\frac{1}{2}(y'')^2+\int f(y)dy-y'y'''=c, \tag{41}$$

where c is a constant. It is well known that one can use the same symmetry to affect another reduction in order. We thus have the second-order ODE for f arbitrary as

$$-\frac{1}{2}v^2\left(\frac{dv}{du}\right)^2-v^3\frac{d^2v}{du^2}+\int f(u)du=c, \tag{42}$$

where $u=y$ and $v=y'$. One needs further symmetry to affect further reductions. Thus in the generic case one has at most a second-order ODE.

We now investigate the integrals for the exceptional case $f=\delta y^{-5/3}$, $\delta=\pm 1$. The generators for this case are listed in (IV) and (NII).

For X_1 as in (11) we have from (39) that the first integral is

$$I_1=\tfrac{1}{2}(y'')^2-\tfrac{3}{2}\delta y^{-2/3}-y'y'''. \tag{43}$$

The invocation of (39) for X_2 results in the first integral,

$$I_2=\tfrac{1}{2}x(y'')^2-\tfrac{3}{2}x\delta y^{-2/3}-xy'y'''+\tfrac{3}{2}yy'''-\tfrac{1}{2}y'y''. \tag{44}$$

Finally, for X_3 (39) yields

$$I_3=\tfrac{1}{2}x^2(y'')^2-\tfrac{3}{2}x^2\delta y^{-2/3}-x^2y'y'''+3xyy'''-xy'y''-3yy''+2y'^2. \tag{45}$$

We see a pattern in these integrals in that

$$I_2=xI_1+\tfrac{3}{2}yy'''-\tfrac{1}{2}y'y'', \tag{46}$$

$$I_3=x^2I_1+3xyy'''-xy'y''-3yy''+2y'^2. \tag{47}$$

Thus if we set $I_1=c_1$, $I_2=c_2$, and $I_3=c_3$, where c_is are constants, we obtain the reduced second-order ODE,

$$3yy''-2y'^2+x^2c_1-2xc_2+c_3=0. \tag{48}$$

We can immediately integrate this ODE when $c_1=c_2=0$ to obtain

$$\int\frac{dy}{\pm\sqrt{\frac{1}{2}c_3+c_4y^{4/3}}}=x+c_5, \tag{49}$$

where c_4 and c_5 are constants. Now c_3 and c_4 are not independent and are related by $27\delta+2c_3c_4=0$. Indeed this is seen by the substitution of the solution (49) into the ODE (2) for this f. For all the cs zero one does not obtain a solution.

Thus, the Lie reductions do not provide a transparent way for integrability for this exceptional case as the Noether point symmetries via their integrals do. This is done quite elegantly in the Noether formalism as we have shown above.

V. CONCLUDING REMARKS

We have performed the complete Lie and Noether, with respect to the standard Lagrangian, group classifications of the fourth-order Euler–Bernoulli beam ODE (2). Apart from the linear case the only integrable case arose from the negative fractional power law $f=\delta y^{-5/3}$, $\delta=\pm 1$. The centripetal force distribution for which f is proportional to the normal displacement y gives rise to submaximal Lie and Noether point symmetry algebras. It is also physical to have negative powers for the normal displacement but it is intriguing that the negative fractional power law $y^{-5/3}$ occurs. This exceptional case which admits the nonsolvable algebra $\mathrm{sl}(2,\mathbb{R})$ remarkably allows for a two-parameter family of exact solutions via the Noether integrals. The Lie reductions resulted in a second-order equation with nonlocal symmetry. Second-order ODEs occur in the Lie reductions for the exponential function and general power cases as well. The generic case in the Lie reduction gives a third-order ODE.

Thus scalar nth-order equations $(n=2,3,4)$ admitting $\mathrm{sl}(2,\mathbb{R})$ symmetry are of vital importance in analysis as well as in several applications. In fact, $y^{(4)}=\delta y^{-5/3}$ is the simplest fourth-order equation that admits the nonsolvable algebra $\mathrm{sl}(2,\mathbb{R})$ and it arises in the study of an important physical problem—the Euler–Bernoulli beam equation.

It certainly would be of great interest and benefit to further analyze the nonlocal symmetry properties of this fractional power law beam equation.

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