We begin with a short historical sketch of nonlinear functional analysis (NFA). During the mid to late 1800s, beginnings of these analytical techniques appeared in the works of the French Poincaré and Boussinesq. However, the first paper properly on NFA appeared around 1926, written by G. D. Birkhoff and Kellogg. During the 1930s, this work was expanded by Schauder and Leray, followed by Levrentief in the 1940s. Finally, during the 1950s, the subject really burgeoned with the work of Browder and many others.

Classical Fixed Point Theorems

The two primary theorems we shall introduce today are the Schauder fixed point theorem and the Picard-Banach fixed point theorem, often known in a slightly different form as the Contraction Mapping Principle.

Vocabulary: Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of open subsets of X such that $\bigcup_\alpha U_\alpha = X$. Then, X is *paracompact* if every open cover of X has a locally finite subcover; that is, for any $x \in X$ there exists a neighborhood V containing x such that $V \bigcap U_\beta = \emptyset$ except for a finite number of U_β.

A further result is that any paracompact space has a *partition of unity*. That is, let X be paracompact, with open cover $\{U_\alpha\}$. Then there exist continuous functions $\phi_\alpha : U_\alpha \to [0,1]$ having $\text{supp}(\phi_\alpha) \subset U_\alpha$ such that for every $x \in X$, $\sum_{\alpha \epsilon I} \phi_\alpha(x) = 1$ where $\phi_\alpha(x) = 0$ except for a finite number of α. Prove this result by applying Urysohn's Lemma.

The next theorem is a generalization of Brouwer's fixed point theorem in finite dimensions.

Theorem (Schauder): Let C be a compact, convex subset of a Banach space X. let $f : C \to C$ be continuous. Then, f has a fixed point.

Proof: (not Schauder's original approach) Let $\varepsilon > 0$. Write $C = \bigcup_{x \in C} B_\varepsilon(x)$. By compactness, take a finite subcover: $C \subset \bigcup_{i=1}^N B_\varepsilon(x_i)$. By paracompactness, we may construct a partition of unity associated to this cover: that is, we have nonnegative $\{\phi_i\}_{i=1}^N$ such that $\sum_{i=1}^N \phi_i(x) = 1 \ \forall x \in C$ and $\text{supp}(\phi_i) \subset B_\varepsilon(x_i)$ for $1 \le i \le N$.

Let $f_\varepsilon : C \to C$ be given by the convex combination of x_i, $f_\varepsilon(x) = \sum_{i=1}^N \phi_i(f(x))x_i$. Notice that f_ε is a finite sum of continuous maps and, thus, is itself continuous. By convexity of C, we have $f_\varepsilon(x) \in C$. In fact, $f_\varepsilon : C \to C_\varepsilon = Co(x_1, \dots, x_N) \subset C$, where C_ε denotes the convex hull.

Now, for $x \in C$, we compute

$$\begin{aligned}
|f(x) - f_\varepsilon(x)| &= \left| \sum_{i=1}^{N} \phi_i(f(x))(f(x) - x_i) \right| \\
&\leq \sum_{i=1}^{N} \phi_i(f(x))\varepsilon \\
&= \varepsilon,
\end{aligned}$$

since $f(x) \in B_\varepsilon(x_i)$ for *some* i (as the B_ε sets form an open cover of C. Thus, we have the fact that $\sup_{x \in C} |f(x) - f_\varepsilon(x)| \leq \varepsilon$.

Now consider $f_\varepsilon|_{C_\varepsilon} : C_\varepsilon \to C_\varepsilon$. By Brouwer's theorem, we know that for each ε there exists a fixed point $x_\varepsilon \in C_\varepsilon$ such that $f_\varepsilon(x_\varepsilon) = x_\varepsilon$.

Notice that we have a net $\{x_\varepsilon\}_{\varepsilon > 0}$. Since C is compact, there must exist a subsequence of indices $\{\varepsilon_n\}_{n=1}^{\infty} \subset R^+$ so that $\varepsilon_n \searrow 0$; hence, x_{ε_n} must converge to some x_∞. Since f is continuous, it follows that $f(x_{\varepsilon_n}) \to f(x_\infty)$. However,

$$f(x_{\varepsilon_n}) = f(x_{\varepsilon_n}) - x_{\varepsilon_n} + x_{\varepsilon_n} = f(x_{\varepsilon_n}) - f_{\varepsilon_n}(x_{\varepsilon_n}) + x_{\varepsilon_n} \to 0 + x_\infty.$$

One might be concerned about uniform convergence in this context. If so, use a norm argument:

$$\begin{aligned}
\|f(x_{\varepsilon_n}) - x_\infty\| &\leq \|f(x_{\varepsilon_n}) - f_{\varepsilon_n}(x_{\varepsilon_n})\| + \|f_{\varepsilon_n}(x_{\varepsilon_n}) - x_\infty\| \\
&\leq \text{small} \quad + \|x_{\varepsilon_n} - x_\infty\|
\end{aligned}$$

but $\|x_{\varepsilon_n} - x_\infty\| \to 0$. Thus, $f(x_\infty) = x_\infty$, which shows that f has a fixed point. $\diamond$

We need a little more notation in order to state the next theorem, familiar from previous studies.

Vocabulary: For $x_1 \in M$, a metric space, the *orbit of x_1 under the map $f: M \to M$* is

$$\mathcal{O}_f(x_1) = \{x_1, f(x_1), f(f(x_1)), f^3(x_1), \ldots, f^n(x_1), \ldots\}.$$

Theorem (Contraction Mapping Principle): Let M be a complete metric space, with metric d. Let $f: M \to M$ be a contraction, i.e.,

$$d(f(x), f(y)) \leq \theta d(x, y) \qquad \text{for a fixed } \theta < 1.$$

Then, there exists a unique $x_0 \in M$ such that $f(x_0) = x_0$ and furthermore, for any $x_1 \in M$, $f^n(x_1) \to x_0$.

Theorem: Let M be a complete, bounded metric space. Suppose there exists a continuous $\phi: R^+ \to R^+$ such that $\phi(0) = 0$ and $\phi(r) < r$ for all $r > 0$. Let $f: M \to M$ be such that $d(f(x), f(y)) \leq \phi(d(x, y))$. Then, f has a unique fixed point $x_\infty \in M$, such that $x_\infty = \lim_{n \to \infty} f^n x_0$ for any $x_0 \in M$.

Notice that without loss of generality, we may assume that ϕ is increasing. This is due to the fact that we may make ϕ monotone for free by defining $\tilde{\phi}(r) = \max_{0 \le s \le r} \phi(s)$.

Proof: Uniqueness. Suppose x and $\bar{x}$ are fixed points with $x \neq \bar{x}$. Then,

$$0 < d(x, \bar{x}) = d(f(x), f(\bar{x})) \le \phi(d(x, \bar{x})) < d(x, \bar{x}),$$

a contradiction unless $x = \bar{x}$.

Existence, by contradiction. Let $x_0 \in M$. Let

$$d_j = \text{diam}\{\mathcal{O}_f(f^j x_0)\} = \sup_{i,k \ge j} d(f^i x_0, f^k x_0).$$

We know that $d_j < \infty$ since M is bounded. Suppose that $d_j \to 0$. Then, $\mathcal{O}_f(x_0)$ is a Cauchy sequence, which perforce converges: $f^j(x_0) \to x_\infty$ as $j \to \infty$. Thus, $f^{j+1}(x_0) \to x_\infty$ but by continuity of f, $f^{j+1}(x_0) = f(f^j(x_0)) \to f(x_\infty)$.

Now,

$$\begin{aligned} d_{j+1} &= \sup_{r,x \ge j} d(f^{r+1}(x_0), f^{s+1}(x_0)) \\ &= \sup_{r,s \ge j} d(f(f^r(x_0)), f(f^s(x_0))) \\ &\le \sup_{r,s \ge j} \phi(d(f^r(x_0), f^s(x_0))) \\ &\le \sup_{r,s \ge j} d(f^r(x_0), f^s(x_0)) = d_j \end{aligned}$$

which implies that $\{d_j\}_{j=1}^\infty$ is a decreasing sequence of real, nonnegative numbers. Hence, there must exist a limit, $d_\infty = \lim_{j \to \infty} d_j$. We want to show that $d_\infty = 0$. Suppose not. Then, by continuity of the function ϕ, $\phi(d_j) \to \phi(d_\infty)$. However, by monotonicity, $\phi(d_j) \ge d_{j+1} \to d_\infty$, which forces $\phi(d_\infty) \ge d_\infty$. This contradicts the condition that $\phi(d_\infty) < d_\infty$, unless $d_\infty = 0$. Therefore, $d_\infty = 0$. $\diamond$

Problem: Use this result to prove the Contraction Mapping Principle stated earlier. The difficulty lies in the fact that the CMP does not require boundedness.

Theorem(Variant of the above): Make the same assumptions about M, ϕ and f except that we require

$$d(f^N(x), f^N(y)) \le \phi(d(x, y)).$$

Then, f has a fixed point in M.
Proof: exercise.
Question: is the fixed point necessarily unique?