G_2 and the "rolling distribution"

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1 The problem

Consider two balls of different size, one rolling along the other, without slipping or spinning. (It may help to think of them as covered by velcro.) The configuration space of this system is a 5-dimensional space $Q \cong \mathrm{SO}_3 \times S^2$ and the no-slip/no-spin condition defines on Q a rank 2 distribution $D \subset TQ$, the "rolling-distribution".

Now D is a non-integrable distribution (unless the balls are of equal size), with an "obvious" 6-dimensional symmetry group $\mathrm{SO}_3 \times \mathrm{SO}_3$. But for balls whose radiii are in the ratio 3:1 something strange happens: the symmetry group increases to G_2, a 14-dimensional Lie group; to be precise, the real non-compact "split form" version of G_2, containing $\mathrm{SO}_3 \times \mathrm{SO}_3$ as a maximal compact subgroup, which is what we mean by "G_2" from now on.

We learned this surprising fact from Robert Bryant. Our purpose is to describe as explicitly as possible how G_2 acts on Q so as to preserve D. In many regards we follow the last section of Bryant's lecture notes [3]. The basic idea is to identify Q with G_2/P where P is a certain 9-dimensional subgroup of G_2.

Some literature and history. Killing uncovered the possible existence of $\mathfrak{g}_2$ as a simple exceptional Lie algebra in 1884. Cartan established its existence in his thesis [5] in 1894. See the introduction to [5] and Cartan's obituary [7] regarding this history. In 1910 in [4], Cartan discovered the realization of G_2 as the symmetry group of a rolling-type distribution. This paper is notoriously difficult. Bryant says "It was, by far, the most elaborate application of his method of equivalence to be fully worked out in his lifetime..." In 1914 Cartan [6] showed that G_2 can be realized as the automorphism group of the octonions. For our split G_2 he used 'split octonions'. Section 3.4 of the 1993 paper [2] describes the rolling of two surfaces along each other in a very clean way and mentions some aspects of G_2 and P. The paper "The Oxford Commemorative Ball Challenge" by Hammersley is a beautifully written paper, accessible at the undergraduate level, concerning rolling a ball on a sphere. (Ref).

Work to do. G_2 is the automorphism group of the split octonions. It should be possible to construct Q, D and the action of G_2 in terms of the algebra of these split octonions.

2 The rolling distribution

Take one ball to be stationary, of radius R, with its center at the origin; roll on it a second ball, of radius 1. The position of the second ball is denoted by a pair $(g, \mathbf{x}) \in Q = \mathrm{SO}_3 \times S^2$, where $\mathbf{x}(1+R)$ is the position of the center of the second ball (i.e. the pt of contact of the two balls is $R\mathbf{x}$) and $g \in \mathrm{SO}_3$ is the rotation of the second ball relative to some initial position.

*** maybe put some picture here ***

Let $(g_t, \mathbf{x}_t) \in Q$ be a rolling motion, $\omega_t \in \mathbb{R}^3 \cong \mathfrak{so}_3$ the angular velocity of the rolling ball relative to its center (i.e. the velocity of a fixed point $\mathbf{P}$ of the rolling ball, moving relative to the center of the rolling ball according to $\mathbf{p}_t = g_t P$, is $\dot{\mathbf{p}} = \dot{g}g^{-1}\mathbf{p} = \omega \times \mathbf{p}$). Then we have

Proposition 1 *A curve $(g_t, \mathbf{x}_t) \in Q$ is tangent to the rolling distribution $D \subset TQ$ iff*

(1) $(R+1)\dot{\mathbf{x}} = \omega \times \mathbf{x}$ (no-slip condition),

(2) $\langle \omega, \mathbf{x} \rangle = 0$ (no-spin condition, i.e. ω need to be tangent to the stationary ball at $R\mathbf{x}$).

Proof. (1) The contact pt between the two balls is $\mathbf{p} = R\mathbf{x}$ on the first ball, $\mathbf{P} = -g^{-1}\mathbf{x}$ on the second ball. For non-slip, their velocities must match: $\dot{\mathbf{p}} = g\dot{\mathbf{P}}$. Now $\dot{\mathbf{p}} = R\dot{\mathbf{x}}$ and

$$\dot{\mathbf{P}} = [-\frac{d}{dt}g^{-1}]\mathbf{x} - g^{-1}\dot{\mathbf{x}} = g^{-1}\dot{g}g^{-1}\mathbf{x} - g^{-1}\dot{\mathbf{x}} = g^{-1}[\omega \times \mathbf{x} - \dot{\mathbf{x}}],$$

hence the condition $\dot{\mathbf{p}} = g\dot{\mathbf{P}}$ is equivalent to $R\dot{\mathbf{x}} = \omega \times \mathbf{x} - \dot{\mathbf{x}}$, from which (1) follows.

(2) Let $\mathbf{P}$ be a point *fixed* on the second ball ($\dot{\mathbf{P}} = 0$). From the point of view of the first ball (stationary), it is seen as $\mathbf{p} = g\mathbf{P} + (R+1)\mathbf{x}$, with velocity

$$\dot{\mathbf{p}} = \dot{g}\mathbf{P} + (R+1)\dot{\mathbf{x}} = \dot{g}g^{-1}[\mathbf{p} - (R+1)\mathbf{x}] + (R+1)\dot{\mathbf{x}} = \omega \times [\mathbf{p} - (R+1)\mathbf{x}] + (R+1)\dot{\mathbf{x}}.$$

Using the no-slip equation, $(R+1)\dot{\mathbf{x}} = \omega \times \mathbf{x}$, we get

$$\dot{\mathbf{p}} = \omega \times [\mathbf{p} - (R+1)\mathbf{x}] + \omega \times \mathbf{x} = \omega \times (\mathbf{p} - R\mathbf{x}).$$

The equation $\dot{\mathbf{p}} = \omega \times (\mathbf{p} - R\mathbf{x})$ means that the motion of the second ball is given, at any given instant during its motion, by a rotation with an axis of rotation (a line) passing through the point of contact $R\mathbf{x}$, in the direction of ω and with angular velocity of magnitude $\|\omega\|$. In particular, if we want no spinning of the second ball around the point of contact of the two balls, ω should have no component orthogonal to the common tangent plane of the two balls, i.e. $\langle \omega, \mathbf{x} \rangle = 0$. □

3 The "obvious" action

There is an action by $K = \mathrm{SO}_3 \times \mathrm{SO}_3$, given by

$$(g, \mathbf{x}) \mapsto (g'gg''^{-1}, g'\mathbf{x}), \quad g', g'' \in \mathrm{SO}_3.$$

One checks easily that K acts transitively on Q preserving the distribution D (and everything else involved).

4 A group theoretic reformulation

Given a Lie group K with Lie algebra $\mathfrak{K}$, the data required for specifying a K-invariant homog distribution (Q, D) is

(1) a subgroup $H \subset K$ with subalgebra $\mathfrak{h} \subset \mathfrak{K}$;

(2) an H-invariant subspace $W \subset \mathfrak{K}/\mathfrak{h}$.

Given such data, one has a K action by left translations on the right H-coset space $Q := K/H$ and $D_{[e]} := W \subset \mathfrak{K}/\mathfrak{h} \cong T_{[e]}(K/H)$ extends uniquely by the K-action to a K-invariant distribution D on K/H.

The group K acts by the adjoint action on the data (H, W) so that K-equivalent pairs $(H, W) \sim (H', W')$ correspond to isomorphic distributions. More generally, there is an obvious notion of equivalence $(K, H, W) \sim (K', H', W')$, corresponding to isomorphic homogeneous distributions.

If we work on the Lie algebra level then the data $(\mathfrak{K}, \mathfrak{h}, W)$ determines (Q, D) only locally, i.e. up to some cover.

Let us determine now the data (K, H, W) corresponding to our rolling-without-spinning distribution.

We have $K = \mathrm{SO}_3 \times \mathrm{SO}_3$, $\dim H = 1$, $\dim W = 2$. We identify as usual the Lie algebra of $K = \mathrm{SO}_3 \times \mathrm{SO}_3$ with $\mathbb{R}^3 \times \mathbb{R}^3$, the set of pairs of angular velocities (ω', ω''), with bracket given by the cross product:

$$[(\omega', \omega''), (\eta', \eta'')] = (\omega' \times \eta', \omega'' \times \eta'').$$

Let us fix a "base point", say $(1, \mathbf{e}_3) \in \mathrm{SO}_3 \times S^2$. The isotropy is the subgroup $H \subset K$ consisting of elements of the form (h, h), where h is a rotation around the $\mathbf{e}_3$ axis, so $H \cong \mathrm{SO}_2$ and $\mathfrak{h} = \mathbb{R}(\mathbf{e}_3, \mathbf{e}_3) \subset \mathbb{R}^3 \times \mathbb{R}^3$. Using the Killing metric on $\mathfrak{K}$ we can identify $\mathfrak{K}/\mathfrak{h} \cong \mathfrak{h}^\perp$, so the plane of the distribution at the base point is given by some 2-plane $W \subset \mathfrak{h}^\perp$. Let us determine explicitly this 2-plane.

Proposition 2 *Under the above identification of* $\mathrm{SO}_3 \times S^2$ *with* K/H*, the rolling distribution on* $\mathrm{SO}_3 \times S^2$ *for radius ratio* R *is given at the base point* $(1, \mathbf{e}_3) \in \mathrm{SO}_3 \times S^2$ *by the 2-plane* $W \subset \mathfrak{h}^\perp \subset \mathbb{R}^3 \times \mathbb{R}^3$ *(the Lie algebra of* K*) defined by the equations*

$$\langle \omega', \mathbf{e}_3 \rangle = \langle \omega'', \mathbf{e}_3 \rangle = 0, \quad R\omega' + \omega'' = 0.$$

PROOF. Since $\mathfrak{h} \subset \mathfrak{K}$ is generated by the vector $(\omega', \omega'') = (\mathbf{e}_3, \mathbf{e}_3)$ and the Killing metric on $\mathfrak{K}$ corresponds to some multiple of the standard metric on $\mathbb{R}^3 \times \mathbb{R}^3$, $\mathfrak{h}^\perp \subset \mathfrak{K}$ is given by the equation $\langle \omega' + \omega'', \mathbf{e}_3 \rangle = 0$.

From the formula for the K-action in §3 we get the infinitesimal action at the base point

$$\omega = \omega' - \omega'', \quad \dot{\mathbf{x}} = \omega' \times \mathbf{e}_3.$$

Substituting these into the rolling-no-spinning conditions at the base point (§2),

$$\langle \omega, \mathbf{e}_3 \rangle = 0, \quad (R+1)\dot{\mathbf{x}} = \omega \times \mathbf{e}_3,$$

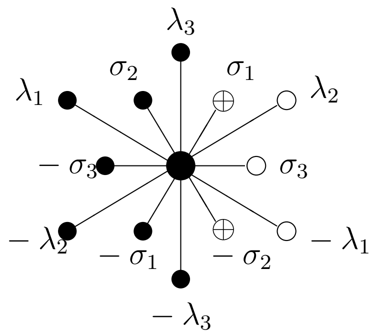
we obtain

$$\langle \omega' - \omega'', \mathbf{e}_3 \rangle = 0, \quad [R\omega' + \omega''] \times \mathbf{e}_3 = 0.$$

Adding the condition of orthogonality to $\mathfrak{h}$, $\langle \omega' + \omega'', \mathbf{e}_3 \rangle = 0$, we obtain the above equations.$\square$

5 G_2

Now consider the root diagram of $\mathfrak{g}_2$, the Lie algebra of G_2.



The root diagram of $\mathfrak{g}_2$.

A reminder of the meaning of the root diagram. The diagram is drawn in the dual of some Cartan subalgebra of $\mathfrak{g}_2$. A Cartan subalgebra of a semi-simple Lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of semi-simple elements, i.e. each $ad(T) \in \mathrm{End}(\mathfrak{g})$, $T \in \mathfrak{t}$, is diagonalizable. In the case of $\mathfrak{g} = \mathfrak{g}_2$, $\mathfrak{t}$ is 2-dimensional, hence the subscript 2 in G_2, the *rank* of the group. The root diagram of $\mathfrak{g}$ encodes the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$, from which one can recover the whole structure of $\mathfrak{g}$.

Example of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. This is a useful example to keep in mind before we proceed with this reminder.It is the set of 3 by 3 traceless real matrices (the Lie algebra of $\mathrm{SL}_3(\mathbb{R})$, the set of 3 by 3 real matrices with determinant=1). Like $\mathfrak{g}_2$, the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ is a non-compact split form of its complexification ($\mathfrak{sl}_3(\mathbb{C})$) and is of rank 2. We take as a Cartan subalgebra the set $\mathfrak{t} \subset \mathfrak{sl}_3(\mathbb{R})$ of traceless diagonal matrices,

$$\mathfrak{t} := \{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} | t_1 + t_2 + t_3 = 0, t_i \in \mathbb{R} \}.$$

Roots and root spaces. The commutativity of the Cartan subalgebra $\mathfrak{t}$ implies that the diagonalizable endomorphisms $ad(T) \in \mathrm{End}(\mathfrak{g})$, $T \in \mathfrak{t}$, are *simultaneously* diagonalizable, resulting in a $\mathfrak{t}$-invariant decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha} \mathfrak{g}_\alpha,$$

where each $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is a 1-dimensional subspace of $\mathfrak{t}$-common eigenvectors called a *root space*. The corresponding eigenvalue depends linearly on the acting element of $\mathfrak{t}$, so is given by a linear functional $\alpha \in \mathfrak{t}^*$, called *root*. Thus

$$[T, X] = \alpha(T)X, \quad T \in \mathfrak{t}, \quad X \in \mathfrak{g}_\alpha.$$

The Killing metric. When we draw the root diagram in $\mathfrak{t}^*$ we use the Killing metric in $\mathfrak{g}$ to determine the size of the roots and the angles between them. The Killing metric in $\mathfrak{g}$ is the inner product $\langle X, Y\rangle = \mathrm{tr}(ad(X)ad(Y))$. It is non-degenerate (this is equivalent to semi-simplicity) and its restriction to $\mathfrak{t}$ is positive definite.

Example: the root diagram of $\mathfrak{sl}_3(\mathbb{R})$. There are 6 roots

$$\alpha_{ij} := t_i - t_j \in \mathfrak{t}^*, \quad i \neq j, \quad i, j \in \{1, 2, 3\},$$

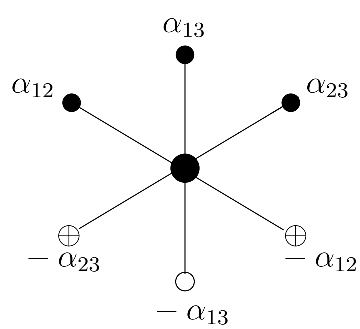
with corresponding root spaces

$$\mathfrak{g}_{\alpha_{ij}} = \mathbb{R}E_{ij},$$

where E_{ij} is the matrix whose ij entry is 1, the rest 0. The corresponding root space decomposition

$$\mathfrak{sl}_3 = \mathfrak{t} \oplus \sum_{i\neq j} \mathfrak{g}_{\alpha_{ij}},$$

is just the decomposition of a matrix as a diagonal matrix plus its off diagonal terms. The metric induced on $\mathfrak{t}$ by the Killing metric is some multiple of the standard euclidean metric, so that $\langle T, T'\rangle = c\sum_i t_i t'_i$ for some $c > 0$.



The root diagram of $\mathfrak{sl}_3$

In the case of $\mathfrak{g}_2$ there are twelve roots and root spaces, as seen in its diagram.

Reading the root diagram. One can read the whole structure of $\mathfrak{g}$ off its root diagram, and some aspects of the structure are very convenient to see in the diagram in a formula-free manner. Here is the key observation. Let α, β be two roots with (non-zero) root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $E_\beta \in \mathfrak{g}_\beta$. That is,

$$[T, E_\alpha] = \alpha(T)E_\alpha, \quad T \in \mathfrak{t},$$

and similarly for β. It then follows immediately from the Jacobi identity that

$$[T,[E_\alpha,E_\beta]]=(\alpha+\beta)(T)[E_\alpha,E_\beta].$$

This means that

(1) if $\alpha+\beta\neq 0$ and is not a root then $[E_\alpha,E_\beta]=0$;

(2) if $\alpha+\beta\neq 0$ and is a root then $[E_\alpha,E_\beta]\in\mathfrak{g}_{\alpha+\beta}$;

(3) if $\alpha+\beta=0$, i.e. $\beta=-\alpha$, then $[E_\alpha,E_\beta]\in\mathfrak{t}$.

This set of 3 conclusions permit us to see at a glance from the diagram a fair amount of the structure of $\mathfrak{g}$. In the last two cases one can further show that $[E_\alpha,E_\beta]$ is non-zero and determine, with some calculations, the actual bracket, as will be illustrated later.

Example: reading the root diagram of $\mathfrak{sl}_3$. Let us consider the subspace $\mathfrak{p}\subset\mathfrak{sl}_3$ spanned by $\mathfrak{t}$ and the root spaces corresponding to the roots marked with dark dots.

The diagram shows that $\mathfrak{p}$ is a 5-dimensional subalgebra, i.e. it is closed under the Lie bracket (there are 4 dark dots, but remember that the thick dot at the origin stands for the 2-dimensional Cartan subalgebra). Indeed, $\mathfrak{p}$ is the subalgebra of upper triangular matrices (including diagonal ones), with corresponding subgroup $P\subset \mathrm{SL}_3$, the subgroup of upper triangular matrices with determinant=1. Consider the quotient space $\mathrm{SL}_3(\mathbb{R})/P$. This can be identified with the space F of flags in $\mathbb{R}^3$, consisting of pairs (l,π), where l is a line and π is a plane, such that $l\subset\pi\subset\mathbb{R}^3$. Now the tangent space to F at some base point with isotropy P is naturally identified with $\mathfrak{sl}_3/\mathfrak{p}$, represented in the root diagram by the remaining three light dots. Two of the light dots are markes with +. The diagram shows that the root spaces corresponding to these roots span a $\mathfrak{p}$-invariant 2-dimensional subspace of $\mathfrak{sl}_3/\mathfrak{p}$ which Lie generates the root space associated with the third light dot. This means that we have on F an $SL_3(\mathbb{R})$-invariant rank 2 contact distribution, i.e. a non-integrable distribution that Lie generates the tangent bundle. It is possible to identify this distribution with the "tautological" contact distribution on F ("the line l moves inside the plane π").

Reading the $\mathfrak{g}_2$ diagram. Now let us draw conclusions in a similar fashion from the $\mathfrak{g}_2$ diagram. Consider the 9-dimensional subspace $\mathfrak{p}\subset\mathfrak{g}_2$ spanned by $\mathfrak{t}$ and the root spaces associated with the roots marked by the dark dots in the diagram. Then the diagram shows that

- $\mathfrak{p}$ is closed under the Lie bracket, i.e. is a subalgebra (a so-called parabolic subalgebra, a subalgebra containing a Cartan subalgebra).
- Let $P\subset G_2$ be the corresponding subgroup. It follows that G_2 has a 5-dim homogeneous space G_2/P, whose tangent space $\mathfrak{g}_2/\mathfrak{p}$ at a point is represented by the remaining 5 light dots.
- Two of the light dots are marked with +. The diagram shows that their root spaces generate a 2-dim $\mathfrak{p}$-invariant subspace of $\mathfrak{g}_2/\mathfrak{p}$, hence a G_2-invariant rank 2 distribution on G_2/P.

- This distribution is not integrable, in fact, it is a distribution of type $(2, 3, 5)$, since the diagram shows that bracketing once gives the light dot marked with σ_3 and bracketing again gives the remaining two light dots.

Now we want to identify G_2/P with $\mathrm{SO}_3 \times S^2$, so that the G_2-invariant rank 2 distribution on G_2/P is identified with the rolling distribution on $\mathrm{SO}_3 \times S^2$ for $R = 3$. The idea is as follows. G_2 contains a maximal compact subgroup K with Lie algebra $\mathfrak{K}$ isomorphic to $\mathfrak{so}_3 \times \mathfrak{so}_3$. When we restrict the G_2 action on G_2/P to K we still get a transitive action, so we get an identification $G_2/P = K/H$, where $H = K \cap P$. We define a Lie algebra isomorphism $\mathfrak{K} \simeq \mathfrak{so}_3 \times \mathfrak{so}_3$ so that $\mathfrak{h} \simeq \mathbb{R}(\mathbf{e}_3, \mathbf{e}_3)$ and the distribution plane $W \subset \mathfrak{h}^\perp$ is mapped onto the rolling distribution plane of §4, with $R = 3$.

Maximal compact subgroup. How can we "see" a maximal compact subgroup of G_2 in its root diagram? Let us look back again at the example of $\mathrm{SL}_3(\mathbb{R})$. Here the maximal compact subgroup is SO_3, with Lie algebra $\mathfrak{so}_3$, the set of 3 by 3 antisymmetric matrices. These are spanned by the vectors $E_{ij} - E_{ji}$, $i > j$. So we see that corresponding to each pair of "antipodal" roots $\pm\alpha_{ij}$ we have one generator of $\mathfrak{K}$, lying in the sum of the two corresponding root spaces.

More generally, for the "split" real form of any semi-simple Lie algebra (such as our $\mathfrak{g}_2$), the situation is similar: we get the Lie algebra $\mathfrak{K}$ of a maximal compact subgroup $K \subset G$ by taking the sum of 1-dimensional subspaces, one such subspace for each pair of antipodal roots $\pm\alpha$, a certain line (1-dimensional subspace) in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. In fact, there is a certain particulary "nice" choice of root vectors $E_\alpha \in \mathfrak{g}_\alpha$ (sometimes called a "Weyl basis"), so that the sought-for line is given by $\mathbb{R}(E_\alpha - E_{-\alpha})$, as in the $\mathfrak{sl}_3$ case.

In the case of $\mathfrak{g}_2$ we thus have that

- $\mathfrak{K}$ is the sum of six 1-dimensional subspaces $\mathfrak{s}_i, \mathfrak{l}_i$, $i = 1, 2, 3$, where $\mathfrak{s}_i$ lies in the sum of the root spaces corresponding to $\pm\sigma_i$, and $\mathfrak{l}_i$ lies in the sum of the root spaces corresponding to $\pm\lambda_i$.
- The isotropy of the K-action, $H = K \cap P \subset K$, is given in the diagram by the vertical segment, $\mathfrak{h} = \mathfrak{l}_3$.
- The distribution plane $W \subset \mathfrak{K}/\mathfrak{h}$ is generated by $\mathfrak{s}_1, \mathfrak{s}_2 \pmod{\mathfrak{h}}$.

We have thus assembled the required ingredients for a "distribution data" $(\mathfrak{K}, \mathfrak{h}, W)$.

6 Identifying $\mathfrak{K} \simeq \mathfrak{so}_3 \oplus \mathfrak{so}_3$

Our task here is to define an isomorphism $\mathfrak{K} \simeq \mathfrak{so}_3 \oplus \mathfrak{so}_3$ that maps $(\mathfrak{K}, \mathfrak{h}, W)$ to the data of §4 with $R = 3$. This entails the decomposition of $\mathfrak{K}$ into the direct sum of two ideals, each of which isomorphic to $\mathfrak{so}_3$. It would have been quite nice and simple if the sought-for decomposition of $\mathfrak{K}$ had been the decomposition into "long" ($\mathfrak{l}_i$) and "short" ($\mathfrak{s}_i$). But this is not the case. For example, the diagram shows that although the $\mathfrak{l}_i$ generate an $\mathfrak{so}_3$ subalgebra of $\mathfrak{K}$, this subalgebra is not an ideal, so is not one of the summands in the decomposition. The $\mathfrak{s}_i$ do not generate even a subalgebra. So we have to work harder, i.e. write down the precise commutation relations.

Proposition 3 *One can pick a basis* $\{S_i, L_i | i = 1,2,3\}$ *of* $\mathfrak{K}$*, with* $S_i \in \mathfrak{s}_i$ *and* $L_i \in \mathfrak{l}_i$*, such that*

$$[L_i, L_j] = \epsilon_{ijk} L_k, \quad [L_i, S_j] = \epsilon_{ijk} S_k, \quad [S_i, S_j] = \epsilon_{ijk}(\frac{3}{4}L_k - S_k),$$

where ϵ_{ijk} *is the "totally antisymmetric tensor on 3 indices" (=1 if ijk is a cyclic permutation of 123, -1 if anticyclic permutation, 0 otherwise).*

The proof of this proposition consists of a simple but tedious calculation, which we cannot "see" in the diagram (we tried). So we are reduced to picking up as nice as possible basis for $\mathfrak{g}_2$ and calculating the corresponding structure constants. Reference: Serre's book on Lie algebras [1]. Details in the next §.

Now set

$$\mathbf{e}'_i := \frac{3L_i + 2S_i}{4}, \quad \mathbf{e}''_i := \frac{L_i - 2S_i}{4}, \quad i = 1,2,3.$$

These 6 vectors form a basis for $\mathfrak{K}$ and satisfy the standard $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ commutation relations

$$[\mathbf{e}'_i, \mathbf{e}'_j] = \epsilon_{ijk} \mathbf{e}'_k, \quad [\mathbf{e}''_i, \mathbf{e}''_j] = \epsilon_{ijk} \mathbf{e}''_k, \quad [\mathbf{e}'_i, \mathbf{e}''_j] = 0,$$

thus establishing a Lie algebra isomorphism $\mathfrak{K} \simeq \mathfrak{so}_3 \oplus \mathfrak{so}_3$.

Corollary 1 *The map* $\mathfrak{K} \to \mathfrak{so}_3 \oplus \mathfrak{so}_3$ *defined by* $\mathbf{e}'_i \mapsto (\mathbf{e}_i, 0)$, $\mathbf{e}''_i \mapsto (0, \mathbf{e}_i)$, $i = 1,2,3,$ *is a Lie algebra isomorphism which maps* $\mathfrak{h} = \mathbb{R}L_3$ *to* $\mathbb{R}(\mathbf{e}_3, \mathbf{e}_3)$ *and the 2-plane in* $\mathfrak{K}$ *generated by* S_1, S_2 *to the 2-plane in* $\mathfrak{so}_3 \times \mathfrak{so}_3$ *defined in the Proposition of §4 for* $R = 3$.

This is easily verified using the previous Proposition. We have thus defined a G_2-action on (some finite cover) of the rolling distribution of two balls of radii ratio 3:1. QED

A note about how we came up with the formulae for $\mathbf{e}'_i, \mathbf{e}''_i$**.** The first thing to observe is that since L_3 generates the isotropy $H = P \cap K$ we should have $L_3 = \mathbf{e}'_3 + \mathbf{e}''_3$. Since everything is symmetric in 1,2,3 we conclude that $L_i = \mathbf{e}'_i + \mathbf{e}''_i$, $i = 1,2,3$. Next since S_3 commutes with L_3 we should have $S_3 = a\mathbf{e}'_3 + b\mathbf{e}''_3$ for some constants a, b, and again by symmetry $S_i = a\mathbf{e}'_i + b\mathbf{e}''_i$, $i = 1,2,3$. Now by using the commutations relations for the $\mathbf{e}'_i, \mathbf{e}''_i$ and the L_i, S_i we get that a, b are roots of the equation $x^2 + x - 3/4 = 0$, i.e. $a = 1/2, b = -3/2$. Hence,

$$L_i = \mathbf{e}'_i + \mathbf{e}''_i, \quad S_i = (\mathbf{e}'_i - 3\mathbf{e}''_i)/2, \quad i = 1,2,3.$$

Inverting these equations we obtain the above equations for $\mathbf{e}'_i, \mathbf{e}''_i$.

A note about the "finite cover" ambiguity. The group $\mathrm{SO}_3 \times \mathrm{SO}_3$ is universally covered 4:1 by $S^3 \times S^3$. So there is not much of an ambiguity and it should be easy to pin down.

7 Details of the proof of the proposition of the previous section

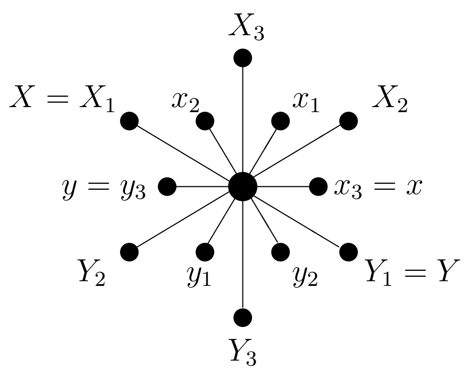
We are following Serre's book [1], page VI-11. $\mathfrak{g}_2$ is Lie generated by the elements x, y, h, X, Y, H, subject to the following relations, which one can read off the root diagram.

$$
\begin{array}{llll}
[x,y] = h, & [h,x] = 2x, & [h,y] = -2y, & \\
[X,Y] = H, & [H,X] = 2X, & [H,Y] = -2Y; & \\
[h,X] = -3X, & [h,Y] = 3Y; & [H,x] = -x, & [H,y] = y; \\
[x,Y] = [X,y] = [h,H] = 0; & & & \\
[ad(x)]^4 X = 0; & [ad(X)]^2 x = 0; & & \\
[ad(y)]^4 Y = 0; & [ad(Y)]^2 x = 0. & &
\end{array}
$$

Taking Lie brackets of the vectors x, y, h, X, Y, H we generate a complete set $\{x_i, X_i, y_i, Y_i | i = 1, 2, 3\}$ of root vectors for $\mathfrak{g}_2$, which, together with the basis h, H for the Cartan subalgebra form a basis for $\mathfrak{g}_2$ as follows:

$$
\begin{array}{llllll}
x_3 = x, & X_1 = X, & x_2 = [x, X_1], & x_1 = [x, x_2], & X_2 = [x, x_1], & X_3 = [X_1, X_2]; \\
y_3 = y, & Y_1 = Y, & y_2 = -[y, Y_1], & y_1 = -[y, y_2], & Y_2 = -[y, y_1], & Y_3 = -[Y_1, Y_2].
\end{array}
$$

We label each root in the diagram with the corresponding root vector.



We end up with a "nice" basis wrt which the structure constants are particulary pleasant; they are integers and have symmetry properties which facilitate greatly the work involved in their determination; you can also apply some elementary $\mathfrak{sl}_2$ representation theory that further facilitate the calculation; it helps to work with the root diagram nearby.

Symmetry properties of the structure constants. Suppose α, β are two roots such that $\alpha + \beta$ is also a root. Let E_α, E_β be the corresponding root vectors, as chosen above. Then $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$, for some non-zero constant $c_{\alpha,\beta} \in \mathbb{Z}$. The nice feature of our base is that the structure constants satisfy

$$c_{-\alpha,-\beta} = -c_{\alpha,\beta}.$$

This cuts in half the amount of work involved, since you need only consider say $\alpha > 0$ (the positive roots are the dark dots in the last root diagram). Combining this with the obvious $c_{\alpha,\beta} = -c_{\beta,\alpha}$ (antisymmetry of Lie bracket) you obtain

$$c_{\alpha,-\beta} = c_{\beta,-\alpha}.$$

This cuts in half again the amount of work.

Proposition 4 *The structure constants of $\mathfrak{g}_2$, with respect to the basis of root vectors $\{x_i, X_i, y_i, Y_i | i = 1, 2, 3\}$ and the Cartan algebra elements $\{h, H\}$ are given as follows. The basis elements are grouped in three sets: positive (three x's and three X's), negative (three y's and three Y's), and Cartan subalgebra elements (h and H).*

- *[Positive, positive]: other then the ones given above, and those which are zero for obvious reasons from the root diagram (sum of roots which is not a root):*

$$[x_1, x_2] = X_3.$$

- *[Positive, negative]:*

$c_{\alpha,\beta}$	y_1	y_2	y_3	Y_1	Y_2	Y_3
x_1	1	4	− 4	0	12	− 12
x_2	4	1	− 3	1	0	3
x_3	− 4	− 3	1	0	− 3	0
X_1	0	1	0	1	0	− 1
X_2	12	0	− 3	0	1	36
X_3	− 12	3	0	− 1	36	1

The 1's on the diagonal stand for the relations $[x_i, y_i] = h_i$, $[X_i, Y_i] = H_i$, where, in terms of our basis $\{h, H\}$ for the Cartan subalgebra,

$$h_1 = 8h + 12H, \quad h_2 = h + 3H, \quad h_3 = h,$$

$$H_1 = H, \quad H_2 = ?, \quad H_3 = ?.$$

- *[Cartan, anything]: this is coded directly by the root diagram:*

  - ad(x) has eigenvalues and eigenvectors

eigenvalue	3	2	1	0	− 1	− 2	− 3
eigenvectors	X_2, Y_1	x_3	x_1, y_2	X_3, Y_3, h, H	x_2, y_1	y_3	X_1, Y_2

  - ad(X) has eigenvalues and eigenvectors

eigenvalue	2	1	0	− 1	− 2
eigenvectors	X_1	X_3, x_2, y_3, Y_2	x_1, y_1, h, H	X_2, x_3, y_2, Y_3	Y_1

Proof. This is elementary, using only the Jacobi identity, but takes time. We will give as a typical example the calculation of $[x_1, x_2]$:

$$\begin{aligned}
[x_1, x_2] &= [x_1, [x, X]] && \text{(by definition of } x_2\text{)} \\
&= [x, [x_1, X]] + [X, [x, x_1]] && \text{(Jacobi identity)} \\
&= [X, [x, x_1]] && \text{(since } [x_1, X] = 0\text{)} \\
&= [X, X_2] = X_3 && \text{(by definitions of } X_2, X_3\text{).}
\end{aligned}$$

The rest of the relations are derived in a similar fashion. □

Now we are ready to define the generators of the Lie algebra of a maximal compact subgroup $K \subset G_2$. Let

$$L_1 = X_1 - Y_1, \quad L_2 = \frac{X_2 - Y_2}{6}, \quad L_3 = \frac{X_3 - Y_3}{6},$$

$$S_1 = \frac{x_1 - y_1}{4}, \quad S_2 = \frac{x_2 - y_2}{2}, \quad S_3 = \frac{x_3 - y_3}{2}.$$

Using the commutation relations of the last Proposition one checks easily that

$$[L_i, L_j] = \epsilon_{ijk} L_k, \quad [L_i, S_j] = \epsilon_{ijk} S_k, \quad [S_i, S_j] = \epsilon_{ijk}(\frac{3}{4} L_k - S_k).$$

Note: the strange-looking coefficients 2,4,6 in the definition of the L_i, S_i are chosen precisely so that we get these pleasing commutation relations.

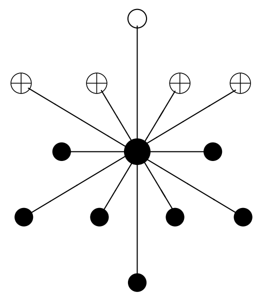
8 The dual fibration. Singular curves.

From the root diagram of $\mathfrak{g}_2$ we see that H, being generated by L_3, acts trivialy on S_3. Write H^* for the circle subgroup generated by S_3. Then H and H^* are commuting circle subgroups of K. It follows that H^* acts (on the right) on K/H defining a circle fibration of the rolling state space, while H acts on K/H^*. We thus have a K-equivariant double fibration:

$$\begin{array}{ccccc}
 & & K & & \\
 & \swarrow & & \searrow & \\
K/H & & & & K/H^* \\
 & \searrow & & \swarrow & \\
 & & K/(H \times H^*) & &
\end{array}$$

where the bottom entry $K/H \times H^*$, is the product of two 2-spheres. XX: you wrote: 'We thus have a G_2-equiv double circle fibration... [this diagram] – I see that G_2 acts on K/H and K/H^*. I do not see that it acts on K. Does it? I have added in the bottom entry, which you had convinced me that it does not act on.

We are curious about the geometry and group theory of the space K/H^*. Go back to the root diagram of G_2 and notice that $H^* = K \cap P^*$, where $P^* \subset G_2$ is the subgroup whose algebra is generated by the black dots in the following drawing



We see from the diagram that $M^* = K/H^* = G_2/P^*$ has a G_2 invariant contact distribution (given by the 4 dots marked with $\oplus$).

What is the rolling interpretation of M^*, its contact structure, and its fibration over $S^2 \times S^2$? We will show that points of M^* correspond to "rolling geodesics" – integral curves to the rolling distribution D formed by rolling one sphere along a great circle on the other sphere. Then points of both spheres trace out great circles. An oriented great circle on a sphere is uniquely determined by its "north" pole, so that the space of geodesics on a sphere forms a "dual sphere" S^2. The fibration $M^* \to S^2 \times S^2$ sends a rolling geodesic to the constituent pairs of geodesics, one for each sphere. The fiber of the fibration describes the relative phase at which these two geodesics are traversed. This description of M^* and its fibration is manifestly K-equivariant. However, it does not indicate the contact structure nor the G_2-symmetry. The contact structure could be described as an embellishment of Hitchin's symplectic reduction description for the symplectic structure on the set of geodesics on a rank 1 symmetric space ([**?**]). However, this viewpoint does not allow us to understand the G_2 symmetry of this contact distribution. We will use a more general viewpoint for understanding M^*, that of "singular curves" for a distribution.

For any $(2,3,5)$ distribution there is a distinguished family of horizontal curves called "singular curves". They correspond in our case to the rolling geodesics. See [3]. In order to describe them we will begin with the situation of a general distribution.

Let D be a distribution on a manifold M. The endpoint map for D sends each D-integral curve $\gamma : [0.1] \to M$, $\dot{\gamma} \in D$ its endpoints $\gamma(0), \gamma(1)$. The space of all D-integral curves has a Hilbert manifold structure in such a way that the endpoint map is a smooth map from this path space to $M \times M$. The critical points of this map are the singular curves. The constant curves are singular, and we exclude them as trivial. See [8]. There is an equivalent definition of nontrivial singular curves as follows. Let $D^\perp \subset T^*Q$ be the bundle of one-forms annihilating D. Restrict the canonical two-form on T^*Q to $D^\perp$. The resulting form may now have a kernel. An absolutely continuous integral curve in $D^\perp$ which is almost every tangent to this kernel and does not intersect the zero section is called a "characteristic" (or "singular extremal") Then every nontrivial singular curve is the projection of some characteristic, and conversely, the projection of a characteristic is a singular curve. Let $Aut(D)$ be the group of diffeomorphisms of M which map D to itself. It is clear from either of these descriptions of characteristics that $Aut(D)$ maps characteristics to characteristics.

Now specialize to the case of a $(2,3,5)$ distribution D on a 5-manifold M. XX -¿ put earlier ?? By definition, for such a distribution the sheaves $D \subset D^2 \subset D^3$ correspond to vector bundles of rank $2,3,5$. Here D^2 is the sheaf generated by D and by Lie brackets

$[X, Y]$ of vector fields X, Y tangent to D while D^3 is generated by D^2 and brackets $[X, W]$, X tangent to D, W to D^2. Equivalently, if X, Y is any local frame for D, then by adding to this frame the bracket $Z = [X, Y]$ we get a local frame for D^2 and by adding to these three the vector fields $U = [X, Z], V = [Y, Z]$ we get a local frame for D^3. The $(2, 3, 5)$ condition is then that X, Y, Z, U, V form a local frame for TM. The $(2, 3, 5)$ condition implies that $D^{\perp} \supset \mathcal{D}^{2\perp} \supset D^{3\perp} = 0$ are subbundles of T^*M of dimensions $3, 2, 0$. One can shows that any characteristic $\chi : I \to D^{\perp}$ for D necessarily lies in $\mathcal{D}^{2\perp}$ and is characteristic there, i.e. is a characteristic curve for D^2, viewed as a distribution in its own right. Conversely, any characteristic for D^2 is automatically a characteristic for D. In other words:

Proposition 5 *Every singular curves for D is the projection to M of a characteristic for $\mathcal{D}^{2\perp}$. And the projection of any characteristic for $\mathcal{D}^{2\perp}$ is a singular curve.*

In our situation of spheres we can understand the topology of $\mathcal{D}^{2\perp}$ easily:

Proposition 6 *In our $3 : 1$ rolling, the 2-plane bundle $\mathcal{D}^{2\perp} \to M$ is isomorphic, as a 2-plane bundle with K-action, to the complex line bundle whose unit circle bundle is $K \to M = K/H$.*

Proof. We only need to check that H acts on $\mathcal{D}^{2\perp}$ by the weight 1 representation. D is spanned by the projection of the span of S_1, S_2 in $\mathfrak{K}$ to TM. Using the K-invariant metric, we see that we can identify $\underset{\sim}{2}p$ with the span of $[S_1, Z], [S_2, Z]$ where $Z = [S_1, S_2]$. Now within $\mathfrak{K}$ we have that H acts on the plane spanned by S_1, S_2 by the weight 1 representation, according to the commutation relations at the end of section 7. And it acts on Z trivially since the determinant of any rotatin is 1. Consequently the H action on the span of $[S_1, Z], [S_2, Z]$ is identical to the H action on the span of S_1, S_2 and so has weight 1. QED

Now, the set of rays through the origin of a vector space can be identified in a canonical way with the unit sphere of that space, relative to any norm. By a ray in a vector bundle we will mean a ray through the origin in any one of its fibers. It follows that we can identify K with the rays in $\mathcal{D}^{2\perp}$. The proposition above now establishes that G_2 acts on K in such a way as to make $K \to K/H$ a G_2-equivariant circle bundle.

We now continue with the general $(2, 3, 5)$ case so as to obtain the invariant contact structure on the space of its singular curves. Let θ denote the restriction to $\mathcal{D}^{2\perp}$ of the canonical one-form $(\Sigma p_i dq^i)$ on T^*M. Then $d\theta$ is a 2-form on $\mathcal{D}^{2\perp}$ whose rank is maximal away from the zero section. Consequently on $\mathcal{D}^{2\perp} \setminus 0$ the kernel of $d\theta$ defines a line field. The integral curves of this line field are the characteristics.

The characteristics satisfy a scaling symmetry. Let E be the 'Euler field' on the vector bundle $\mathcal{D}^{2\perp}$. In terms of fiber coordinates λ_1, λ_2 for $\mathcal{D}^{2\perp}$ we have $E = \lambda_1 \frac{\partial}{\partial \lambda_1} + \lambda_2 \frac{\partial}{\partial \lambda_2}$. Then the following relations hold on the 7-manifold $Y = \mathcal{D}^{2\perp} \setminus 0$

$$L_E \theta = \theta, i_E d\theta = \theta, i_E \theta = 0 \tag{1}$$

where L_E is the Lie derivative with respect to E. As a consequence of the first relation we also have that

$$L_E d\theta = 0.$$

It follows from these relations that the flow of E takes characteristics (being curves γ with $i_{\dot\gamma} d\theta = 0$) to characteristics, without changing their projection since E is vertical relative to the vector bundle fibration $\mathcal{D}^{2\perp} \to M$. Now the orbits of the flow of E are the rays in $\mathcal{D}^{2\perp}$, so that the quotient space Y/E is the unit circle bundle of $\mathcal{D}^{2\perp}$ which is in the rolling case $\cong K$ according to proposition 6. So the quotient space Y/E also carries characteristics.

Consider now the general situation of an odd-dimensional manifold $Y = Y^{2n+1}$ endowed with a non-vanishing one-form θ and vector field E satisfying the above relations. Suppose, as in our situation, that $d\theta$ has maximal rank. Then its kernels form a smooth line field which we denote by Ker. In our example, the integral curves of Ker are the characteristics XX. We can form the following sequence of (local) fibrations:

$$Y^{2n+1} \to^{E} Q^{2n} \to^{ker} N^{2n-1}$$

Q is the quotient of Y by the flow of E. The kernel field pushes down to a line field ker on Q. N is the quotient of Q by this line field. We will show that Q inherits a quasi-contact structure from the data (Y, θ, E) while N inherits a contact structure. Consider the hyperplane field $F = ker(\theta)$ on Y. Then $(Ker) \subset F$ as follows from (1): $L_E\theta = d(i_E\theta) + i_E d\theta = i_E d\theta = \theta$ from which it follows that if $k \in (Ker)$ then $0 = d\theta(E, k) = \theta(k)$. (In particular F is not a contact field.) These same equations (1) imply that $Ker \subset F$ are both invariant under the flow of E and so push down to the quotient Q defining a line field and hyperplane field $ker = \pi_*(Ker) \subset \pi_* F$ on Q. ($\pi_* F$ is a hyperplane field since $E \subset F$.) Here $\pi : Y \to Q$ denotes the quotient map. The maximal rank condition on $d\theta$ implies that $\pi_* F$ is a "quasi-contact" form – the odd-dimensional analogue of a contact distribution. Specifically, let α be any locally defined one form on Q whose vanishing defines $\pi_* F$. Any such α can be expressed as $s^*\theta = \alpha$ where $s : U \subset Q \to Y$ is a smooth local section of the projection $\pi : Y \to Q$. The rank of $d\alpha$ restricted to $\pi_* F$ is independent of the choice of α and is equal to $2n-1$, the rank of $d\theta$ restricted to F. The kernel of $d\alpha$ restricted to $\pi_* F$ is $ker = \pi_*(Ker)$. In the $(2,3,5)$ example the integral curves of this kernel are the projected characteristics. Flowing in the direction of (ker) on any quasi-contact manifold preserves the quasi-contact distribution: $L_k\alpha = 0 \bmod \alpha$ where k is any nonzero vector field tangent to the reduced kernel. Consequently, the quasi-contact field on Q pushes down to the quotient N, of Q by this kernel field. The result is a contact form on N. In our example, every step of the construction is invariant under $Aut(D)$ and hence the contact structure on N is so invariant.

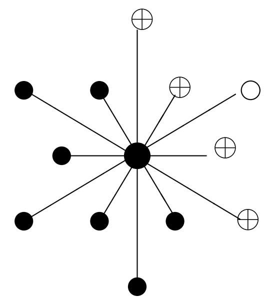
To complete our understanding, it remains to identify the H^*-orbits in the G_2-action with the rolling great circles. H^* is generated by S_3 which corresponds in K to the Lie algebra element $(1/2)(e_3, -3e_3) \in \mathfrak{K}$. This element lies in D, according to proposition 2. so it generates rolling curves. And the projection of the integral curve $exp(S_3)$ to either sphere is..... is .. NO!! ARGGH. The vector $(\omega', \omega'') = (e_3, -3e_3)$ satisfies the first condition of prop 2: $3\omega' + \omega'' = 0$. But it fails the condition $\langle \omega', e_3 \rangle = 0$ and $\langle \omega'', e_3 \rangle = 0$!! It is ORTHOGONAL to rolling curves...

Above was a minor wrong term. RESOLUTION OF APPARENT PARADOX FOLLOWS:

notes to Gil:

Instead of the P^* you marked, take the conjugate P^* indicated below in the root diagram.

The associated $H^* = K \cap P^*$ will now be S_2, and back in the original rolling picture, it will be $(e_2, -3e_2)/2$ the quotient by which yields the great circles.



Another piece of the puzzle. Viewed in K, the subgroup H^* (which I am proposing is generated by $(e_2, -3e_2)$) generating the rolling-along-great-circles and the subgroup H whose quotient yields the rolling state space M^5 DO NOT COMMUTE. Indeed – rolling-along-great circles does not make sense as a circle action in M^5. Why? – through each point of M^5 pass a circle's worth of these great circles, not a single one as would have to be the case if the action of H and H^* commuted.

Question. I know from the last thing I was writing, about realizing K as what I called $PD^{2\perp}$, that K is itself a homogeneous space for G_2. It seems to me that it is the homogeneous space $G_2/(P \cap P^*)$ with P^* my 'rotated' P^* above. At least the dimensions match up, and it admits projections to G_2/P and G_2/P^*.

Answer: Because $\mathfrak{K} \cap \mathfrak{p} \cap \mathfrak{p}^* = 0$, ,the map $K \to G/(P \cap P^*) = N$, defined by restricting to K the G action on the homogeneous space N, is an immersion. By dimension count, it is a local diffeomorphism, hence a covering map. So, up to finite cover, $G_2/P \cap P^*) = K$

Remark. By rotating the roots selected to define P^* (or P) around the root diagram,we obtain a finite family of conjugate P^*'s (or P's) each of which defines a different projection of $N = K$ onto $K/(K \cap P^*)$, (onto $K/K \cap P$)). The existence of this finite set of distinct circle bundle projections reminds me of the three projections in Arnol'ds "triality" article in which he interprets the three quotients of $S^3 = Sp(1)$ by the three subgroups generated by i, by j, and by k in terms of various wavefronts and contact/symplectic geometric constructions involving involutes and evolutes of curves on the sphere.

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