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Differential Operators on Homogeneous Spaces. I

Irreducibility of the Associated Variety for Annihilators of Induced Modules

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Contents

Summary. In this paper, we extend recent work of one of us [Br] to investigate an old problem of the other one [B2]. Given a connected semisimple complex Lie-group G with Lie-algebra $\mathfrak{g}$, we study the representation ψ_X: $U(\mathfrak{g}) \to D(X)$ of the enveloping algebra of $\mathfrak{g}$ by global differential operators on a complete homogeneous space $X = G/P$. It turns out that the kernel I_X of ψ_X is the annihilator of a generalized *Verma-module*. On the other hand, we study the associated graded ideal $\operatorname{gr} I_X$, and relate it to the geometry of a generalized *Springer-resolution*, that is a map π_X: $T^*(X) \to \mathfrak{g}$ of the cotangent-bundle of X onto a nilpotent variety in $\mathfrak{g}$, as studied e.g. in [BM 1]. We prove, for instance, that $\operatorname{gr} I_X$ is prime if and only if π_X is birational with normal image. In general, we show that $\sqrt{\operatorname{gr} I_X}$ is prime. Equivalently, the associated variety of I_X in $\mathfrak{g}$ is irreducible: In fact, it is the closure of the *Richardson-orbit* determined by P. For the homogeneous space $Y = G/(P, P)$, we prove that the analogous ideal I_Y has for associated variety the closure of the *Dixmier-sheet* determined by P. From this main result, we derive as a corollary, that for any module induced from a finite-dimensional Lie P-module the associated variety of the annihilator is irreducible, proving an old conjecture [B2], 2.5. Finally, we give some applications to the study of associated varieties of primitive ideals.

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Remark. The present paper is designed to be the first – and introductory – one in a series of three. In part two we intend to expose a theory of "*relative enveloping algebras*" associated with a principal homogeneous fibre-bundle, and to apply this to induction of infinite-dimensional modules. In part three we intend to deal with the classification of primitive ideals, using the notion of a *characteristic variety* of a primitive ideal in the cotangent-bundle of the flag-variety, which maps onto the *associated variety* under the Springer-resolution map. One of our purposes is to contribute to a better understanding of the classification of primitive ideals, as achieved by Joseph and Barbasch-Vogan in various recent papers. The classification, as carried out in [BaV1], [BaV2], is still somewhat tedious and mysterious, although the final result is most beautiful. Our intention is to pave the path for a direct, purely geometrical explanation.

The subdivision of the material in three papers corresponds to three essentially different levels of difficulty and complexity of methods, each of which is – hopefully – interesting enough in its own right. On each level, we shall reobtain the main applications of the previous level as special cases.

Introduction

The *homogeneous spaces* considered in this paper are algebraic varieties X with a transitive action of some connected algebraic group G, such as the affine or projective complex n-space, or Grassmannians, or flag varieties. The *differential operators* considered are linear with algebraic coefficients. Our first aim is to study the ring $D(X)$ of all such operators which are globally defined on X. This is a class of noncommutative Noetherian domains arising most naturally from geometry and analysis, and so should deserve some attention in algebra. For affine space with coordinates $x_1, \ldots, x_n$ for example, $D(X)$ is the Weyl algebra generated by all partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ and multiplications $x_1, \ldots, x_n$; moreover the Heisenberg commutation relations are known to provide a set of defining relations. For projective n-space with homogeneous coordinates $x_0, \ldots, x_n$ for example, we shall see that $D(X)$ is generated by the operators $x_i \frac{\partial}{\partial x_j}$ for $0 \leqq i, j \leqq n$, and we also know how to give a list of defining relations, which includes the obvious Lie commutator relations, as well as $x_0 \frac{\partial}{\partial x_0} + \ldots + x_n \frac{\partial}{\partial x_n} = 0$, the Euler differential equation for homogeneous polynomials. Strangely enough, we could not even find this example in the literature, whereas the case of affine space, that is to say the Weyl algebra, has been subject to extensive research, of course (see e.g. [Bj]).

For a general homogeneous space X, we start from the observation that the Lie algebra $\mathfrak{g}$ of our group G acts by vector fields on X which indeed *are* globally defined on all of X, and we are able to prove that *they even generate* $D(X)$, *whenever the variety* X *is complete* (Theorem 3.8). This extends recent work of Brylinski [Br], who treated the case of a flag variety (that is the variety of all Borel subgroups in a semisimple group), which is basic for [Br Ka]. For the present generalization, we have to use a difficult theorem of Berline-Duflo [CD] on $\mathfrak{g}$-finite endomorphisms of induced modules. However, for the subsequent main goals of this paper, we only need a much weaker result on $D(X)$, which turns out to be an easy consequence of the completeness of X: We only need that $D(X)$ is a module of finite type over the subalgebra generated by the vector fields arising from $\mathfrak{g}$.

As to the problem of finding also defining relations for $D(X)$, this is reduced to a question about certain primitive ideals in the universal enveloping algebra $U(\mathfrak{g})$, for which a highly developed theory is available by now. In fact, the representation of $\mathfrak{g}$ by vector fields on X extends to a homomorphism ψ_X of $U(\mathfrak{g})$ to $D(X)$, and *we identify its kernel I_X as being the annihilator of the $\mathfrak{g}$-module induced from a certain one-dimensional representation of an isotropy subalgebra* (see 3.6). This result is even true if X is not complete, so if $U(\mathfrak{g})$ might not surject onto $D(X)$. It generalizes work of Kempf and Brylinski [Ke2], [Br] about the case of flag varieties, in which case the induced modules mentioned above are so called *Verma modules*, that is a certain class of infinite-dimensional representations of $\mathfrak{g}$, for which a highly developed theory is available (see e.g. [J], [Br Ka], [BeBe]).

Another essential tool for our study of $D(X)$ in terms of $U(\mathfrak{g})$ is the so-called *momentum map* π_X, as introduced by Souriau [So], and independently by Kostant [Ko1], which has been extensively used by mathematical physicists since then [A Ma]. This is a canonical map $\pi_X\colon T^*(X)\to\mathfrak{g}^*$ from the cotangent bundle of X into the dual space of $\mathfrak{g}$, which is closely related to our homomorphism $\psi_X\colon U(\mathfrak{g})\to D(X)$, as follows: Viewing the principal symbol of a differential operator on X as a function on the cotangent bundle of X, as usual, the associated graded map of ψ_X becomes a homomorphism of the symmetric algebra $S(\mathfrak{g})$ into the global regular functions on $T^*(X)$, and hence gives rise to a map from $T^*(X)$ to $\mathfrak{g}^*$. This map is π_X.

Let us henceforth assume in this introduction that G is semisimple, and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form. Then the image of the moment map of the flag variety, for example, is the variety of all nilpotent elements of $\mathfrak{g}$, and the map itself is nothing else but the well known *Springer resolution* of singularities, as studied extensively since long by algebraic group theorists (see e.g. [S1], [S2], [St], [BM1], [BM2] or 6.2). More generally, the map π_X is proper, whenever X is complete, and then the image of π_X is the closure of a certain adjoint orbit of nilpotent elements, in fact, of the so called *Richardson orbit* determined by the parabolic subgroup P of G which is the isotropy group of X. We prove then that *this orbit closure coincides with the associated variety of the ideal I_X* of $U(\mathfrak{g})$ (see 4.4). Here the associated variety of an ideal J in $U(\mathfrak{g})$ is defined as the set of zeros in $\mathfrak{g}^*$ of the associated graded ideal $\operatorname{gr} J$ of $S(\mathfrak{g})$. In particular, *the nilradical $\sqrt{\operatorname{gr} I_X}$ is a prime ideal.* Furthermore, we prove that $\operatorname{gr} I_X$ *itself is prime if and only if π_X is birational with normal image* (Theorem 5.6), which is e.g. always satisfied if G is SL_n (Kraft-Procesi [KP1]). In general, π_X is frequently not birational, i.e. its mapping degree is bigger than one, and we show that this degree gives a measure for the deviation of $\operatorname{gr} I_X$ from being prime (Theorem 5.8). This number has many interesting alternative interpretations, e.g. in terms of multiplicity functions as in [BK2] (7.2(c) and (d)), and its numerical value is explicitly known for all cases (see 2.6).

So far, our first aim has been to use $U(\mathfrak{g})$ as a tool for the study of $D(X)$. Let us now take a different point of view, interchanging the rôles of $U(\mathfrak{g})$ and $D(X)$: We now study ideals of the enveloping algebra of our fixed complex semisimple Lie algebra $\mathfrak{g}$, using rings of differential operators on various homogeneous G-spaces as a tool. Our main goal is to prove (see 4.7) the following old conjecture of Borho [B2], 2.5: *Let I be an ideal of $U(\mathfrak{g})$ which is "induced" from a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$*, that is to say I is the annihilator of a $\mathfrak{g}$-module induced from *a finite dimensional* $\mathfrak{p}$-module. *Then the associated variety of I in $\mathfrak{g}$ is irreducible.* This implies that the variety coincides with the closure of the Richardson orbit determined by the parabolic subgroup P with Lie algebra $\mathfrak{p}$, (see [B1]). Previously, the conjecture had been proved by Joseph for the case $G=\mathrm{SL}_n$ only, by completing investigations of Borho and Kraft-Procesi, see [Jo2]. The crucial ingredient for this old approach was the normality of closures of orbits [KP1], plus the connectedness of centralizers of nilpotent elements. The former is a beautiful but difficult fact about SL_n, and both are false in general. As the reader will realize, these difficulties will completely disappear in our present approach: Neither normality nor connectedness properties matter for our arguments.

Let us next sketch the *proof* of the conjecture. Recall first our above result on the associated variety of the ideal I_X, the kernel of the representation of $U(\mathfrak{g})$ by differential operators on the complete homogeneous space $X=G/P$. This already proves our conjecture for the (very special) case of inducing a certain *particular onedimensional* $\mathfrak{p}$-module. The crucial step is to extend this to the case of inducing an *arbitrary onedimensional* $\mathfrak{p}$-module. Using the method of tensoring with finite-dimensional representations, the general case is finally reduced to this onedimensional case.

To deal with this case, we first work on the homogeneous space $Y=G/(P,P)$, which is never complete, unless it reduces to a point $(P=G)$. Nevertheless, we are still able to prove the following result, which is our main theorem:

Theorem (4.6). *The associated variety of I_Y is the image of the moment map π_Y and hence irreducible; in fact, it is the closure of the Diximier sheet* [B4] *determined by P.*

In the easy case where P is a Borel subgroup, for example, (P,P) is a maximal unipotent subgroup, the image of π_Y is all of $\mathfrak{g}$, and π_Y itself is essentially *Grothendieck's simultaneous resolution.* Our proof of the theorem depends on the observation that $U(\mathfrak{g})$ acts by differential operators which commute with the right action of P on Y, and that the subring $D(Y)^P$ of such operators is of finite type as a $U(\mathfrak{g})$-module (while $D(Y)$ is generally far from being so). The geometric fact granting this is that Y is a principal homogeneous fibre bundle over the *complete* homogeneous space $X=G/P$.

To finish our sketch of proof for the conjecture, we note that all ideals I in question (induced from any one-dimensional $\mathfrak{p}$-module) contain the ideal I_Y. We conclude that the associated variety of I is contained in the closure of the Dixmier sheet, and also in the cone of nilpotent elements. Since the intersection of these two sets is just the closure of the Richardson orbit determined by P, we are done. (At least with the difficult inclusion claimed in the conjecture. But the other inclusion was known before.)

Finally, we turn to the study of *primitive ideals* in $U(\mathfrak{g})$. Conjecturally, their associated varieties are always irreducible, and hence closures of nilpotent orbits (cf. [B2], 2.9). This was proved for $G = \mathrm{SL}_n$ by Joseph [Jo2], and is in fact an easy corollary to the result on induced ideals (see 4.9). In the present paper, we prove this conjecture for all classical groups and all primitive ideals of integral central characters (Theorem 6.5). Unfortunately, we have no direct geometric approach to the associated variety of a primitive ideal, except for the case of induced ideals. So we employ the classification theory of primitive ideals, as achieved by Joseph and Barbasch-Vogan (cf. 6.4 or [BaV2]), to deduce the result in the general case from the result in the induced case.

Let us give a rough idea of the proof. This deduction is necessarily somewhat involved, since the classification of primitive ideals (with trivial central character, say) is in terms of Springer's correspondence between nilpotent orbits and Weyl group representations. Moreover we need both Joseph's relation from a primitive ideal to a Weyl group representation as well as Barbasch-Vogan's relation from a primitive ideal to a nilpotent orbit, using "wave front sets". Barbasch and Vogan [BaV1], [BaV2] have verified that (1) this "triangle correspondence" commutes, that (2) the nilpotent orbit corresponding to a primitive ideal is contained in its associated variety, with equality of dimensions, and that (3) the orbits which occur are exactly the "*special orbits*" in the sense of Lusztig [L], (which include the Richardson orbits). It is left to prove that the associated variety is equal to the closure of the corresponding special orbit. If the special orbit is even Richardson, this equality follows from our theorem on induced ideals, using a lemma of Joseph (see 6.7). If not, then we use a representation of the closure of the special orbit as an *intersection* of closures of Richardson orbits, and we use primitive ideals *contained* in the given one, which correspond to these Richardson orbits, in order to derive the desired equality also for this special orbit (see 6.8). In order to make this method work in general, we have to know that (1) there are "enough" intersections of closures of Richardson orbits, and that (2) there are "enough" inclusions of primitive ideals. But these ingredients for the proof are fortunately provided by two results of Kempken [Kk] and Spaltenstein [Sp2] about special orbits in classical groups, one of which translates into (2) by a theorem of Vogan (6.10). (Let us note that (1) fails for exceptional groups.) We are grateful to Gisela Kempken, Nicolas Spaltenstein, and David Vogan for kindly explaining to us their results, which have not yet been published. Using [S3], [BM2], these results admit most interesting geometrical reformulations in terms of the geometry of the moment map of a complete homogeneous space (see 6.12, 6.13), underlining their independent interest. For more details, more back-ground, some examples, and an outlook the interested reader is refered to Chap. 6.

We conclude with a few remarks concerning the exposition. In the first chapters, we have tried to convince the experts in enveloping algebras and representation theory that it is both natural and useful to study differential operators on homogeneous spaces in connection with modules over (and ideals in) enveloping algebras. The rather little machinery needed to establish this connection is exposed in the first three chapters. In Chap. 1, we have collected some generalities about differential operators, while in Chap. 2, the group action is introduced, and the general notion of a moment map is explained. These two chapters contain essentially no original results, but they will hopefully be useful in explaining various relations to (or in) the previous literature, and in preparing for the subsequent chapters. In Chap. 3, we explain the relation between $D(X)$ and induced modules, extending ideas of Kempf [Ke1], [Ke2]. The determination of the associated variety of an induced ideal is given in Chap. 4, while Chap. 5 is dedicated to the more delicate study of the associated graded ideal (of an induced ideal), which mainly amounts to a careful comparison of the natural filtrations on $D(X)$ and $U(\mathfrak{g})$. We hope that these two chapters are readable not only for group and ring theorists, but also for geometers and PDE-experts, who might be interested in these algebraic applications. The long Chap. 6, however, heavily depends on many deep results in various connected fields, and is mainly written for experts in semisimple groups and enveloping algebras. However, we have spent some care on summarizing, or at least fully stating, most of the results which are used, and so we hope that e.g. readers mainly interested in Weyl group representations, or in nilpotent orbits might still manage to read this chapter with some benefit, provided that they are ready to believe the statements quoted from various places in the literature.

§ 1. Some Terminology on Differential Operators

1.1. Our algebraic varieties and algebraic groups are defined over some fixed algebraically closed field k of characteristic O. Our varieties, morphisms, group actions, vector-bundles, sheaves are understood to be algebraic, and to refer to the Zariski-topology, if not otherwise stated.

For a vector-bundle E with base X we denote by E_X the fibre at a point $x \in X$, and by $\Gamma(E)$ the sheaf of sections of E. For a sheaf F of groups, rings, ... on X, we denote by F_x the stalk at $x \in X$, and by $\Gamma(X, F)$ the group, ring, ... of global sections. For a nonsingular variety X, $T(X)$ is the tangent - and $T^*(X)$ the cotangent - bundle of X, $T(X)_x$ is the tangent-space at a point $x \in X$, and $\Gamma(T(X))$ is the sheaf of algebraic vector-fields on X. Moreover, $\mathcal{O}_X$ resp. $\mathcal{D}_X$ is the sheaf of algebraic functions resp. of algebraic *differential operators* on X, $\mathcal{O}_{X,x}$ resp. $\mathcal{D}_{X,x}$ is the local ring of functions resp. the ring of differential operators defined in a neighbourhood of x, and $R(X) := \Gamma(X, \mathcal{O}_X)$ resp. $D(X) := \Gamma(X, \mathcal{D}_X)$ is the ring of global regular functions resp. global differential operators on X. For example, if X is an affine space, then $R(X)$ is a polynomial ring $k[x_1, \ldots, x_n]$, and $D(X)$ is a *Weyl-algebra* $k\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]$, a non-commutative ring which is simple noetherian. In general, $\mathcal{D}_X$ is a sheaf of simple noetherian rings: In fact, each stalk $\mathcal{D}_{X,x}$ is "essentially" a localized Weyl-algebra. (It is a left module of finite rank over a subalgebra of this type.) For further details, we refer to [Bj]. If $x_1, \dots, x_n$ is a transcendence basis for the field $K(X)$ of rational functions on X, then the ring of differential operators on X with rational coefficients

$$D(X)\,K(X) = K(X)\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$$

is generated by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$, and is a left and right module of finite rank over

$$k(x_1, \dots, x_n)\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right],$$

a *localized* Weyl-algebra.

1.2. The sheaf $\mathcal{D}_X$ is filtered by the subsheaves $\mathcal{D}_X(m)$ of differential operators of degree $\leqq m$. For example, $\mathcal{D}_X(O)$ is $\mathcal{O}_X$, and $\mathcal{D}_X(1)/\mathcal{D}_X(0)$ is the sheaf of vector-fields $\Gamma(T(X))$. In general, $\mathcal{D}_X(m)/\mathcal{D}_X(m-1)$ is the sheaf of sections of the m-th symmetric power of the tangent-bundle, $\Gamma(S^m(T(X)))$, which we consider in an obvious way as consisting of functions on $T^*(X)$: in fact as the regular functions which are homogeneous polynomials of degree m on the fibres. Consequently, the *associated graded sheaf* $\operatorname{gr}\mathcal{D}_X$, defined as

$$\operatorname{gr}\mathcal{D}_X := \bigoplus_{m \geqq 0} \mathcal{D}_X(m)/\mathcal{D}_X(m-1),$$

identifies with the direct image of the sheaf of functions on the cotangent-bundle $p\colon T^*(X) \to X$:

$$\operatorname{gr}\mathcal{D}_X \cong \bigoplus_{m \geqq 0} \Gamma(S^m(T(X))) = p_* \mathcal{O}_{T^*(X)}.$$

1.3. The non-commutative ring $D(X) = \Gamma(X, \mathcal{D}_X)$ of global differential operators on X is filtered by the subspaces $D_m(X) = \Gamma(X, \mathcal{D}_X(m))$ of global differential operators of degree $\leqq m$. For example, $D_0(X)$ is $R(X)$, and $D_1(X)/D_0(X)$ is the $R(X)$-module of global vector-fields on X. In general, the image of an operator P of degree m in $D_m(X)/D_{m-1}(X)$, the so-called *principal symbol* of P, is a global function on $T^*(X)$ which is homogeneous of degree m on the fibres, in view of the identifications made in 1.2. Consequently, the associated graded ring

$$\operatorname{gr} D(X) := \bigoplus_{m \geqq 0} D_m(X)/D_{m-1}(X)$$

is considered as a subring of the ring $R(T^*(X))$ of global regular functions on the cotangent bundle. Are the two rings even equal? We give the following result on this question, which, however, will not be used for the proofs of our main results.

1.4. Lemma. *If X is a complete homogeneous space, then*

$$\operatorname{gr} D(X) = R(T^*(X)).$$

Proof. In this case, it follows from a result of R. Elkik [El], see also [BK2], Lemma A.2, that the cohomology groups $H^i(T^*(X), \mathcal{O}_{T^*(X)})$ vanish in all dimensions $i > 0$.

Since $p\colon T^*(X)\to X$ is an affine morphism, we have

$$H^i(T^*(X),\mathcal{O}_{T^*(X)})=H^i(X,p_*\mathcal{O}_{T^*(X)})=H^i(X,\operatorname{gr}\mathcal{D}_X)$$

(see 1.2). So the cohomology groups $H^i(X,\mathcal{D}_X(m)/\mathcal{D}_X(m-1))$ vanish for all $i>0$, $m\geqq 0$. An easy induction on m then shows that $H^i(X,\mathcal{D}_X(m))=0$ for $i>0$. It follows that the map from $D_m(X)=\Gamma(X,\mathcal{D}_X(m))$ to $\Gamma(X,\mathcal{D}_X(m)/\mathcal{D}_X(m-1))$ is surjective, with kernel $D_{m-1}(X)$. Since $R(T^*(X))$ is the direct sum of the groups $\Gamma(X,\mathcal{D}_X(m)/\mathcal{D}_X(m-1))$, this gives the lemma. Q.e.d.

1.5. We fix a base point x in our nonsingular variety X of dimension $\dim X=n$. Consider the highest cohomology group of X with coefficients in $\mathcal{O}_X$ and support in x [Ha], which may be defined as follows:

$$H^n_x(X,\mathcal{O}_X):=\varinjlim_i \operatorname{Ext}^n_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1},\mathcal{O}_{X,x}),$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. We want to exhibit a natural structure of a $D(X)$-module on this group.

Let us first give a more concrete description of $H^n_x(X,\mathcal{O}_x)$. It follows from the definition of this group, that it is isomorphic to $H^n_x(\hat{X}_x,\mathcal{O}_{\hat{X},x})$, where $\hat{X}_x$ is the spectrum of the completion of the local ring $\mathcal{O}_{X,x}$. Since $\hat{X}_x$ is isomorphic to $\hat{Y}_0$, where $Y=\mathbb{A}^n$ is an affine space with origin 0, we loose no generality if we assume that $X=\mathbb{A}^n$ and $x=0$. Choose coordinates $(z_1,\dots,z_n)$ at x. Using the Čech-complex for the covering of $X\setminus\{x\}$ by the open subsets $(z_i\neq 0)$, one may derive the following identification of $H^n_x(X,\mathcal{O}_X)$:

$$H^n_x(X,\mathcal{O}_X)\cong\{\textit{germs of rational functions regular on } \bigcap_i (z_i\neq 0)\} \bmod.$$

$$\sum_j\{\textit{those which are regular on } \bigcap_{i\neq j}(z_i\neq 0)\}.$$

A basis for this k-vectorspace is given by the rational functions $\frac{1}{z^\alpha}$, where $z^\alpha=z_1^{\alpha_1}\dots z_n^{\alpha_n}$ is a monomial with exponents $\alpha_1,\dots,\alpha_n$ all $\geqq 1$. With this description, the structure of $H^n_x(X,\mathcal{O}_X)$ as a $D(X)$-module becomes obvious: For instance, $\frac{\partial}{\partial z_i}\frac{1}{z^\alpha}=-\frac{\alpha_i}{z_i}\frac{1}{z^\alpha}$. This makes evident that the module is cyclic, generated by $\frac{1}{z_1\dots z_n}$, even as a module over the (commutative) ring $k\left[\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right]$ of differential operators with constant coefficients. If we call $\alpha_1+\dots+\alpha_n-n$ the degree of $\frac{1}{z^\alpha}$, then $H^n_x(X,\mathcal{O}_X)$ is a graded free cyclic $k\left[\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right]$-module, the generator being of degree 0. Let us denote M_m the span of all $\frac{1}{z^\alpha}$ of degree $\leqq m$, for $m=0,1,2,\dots$. Then M_m is an $\mathcal{O}_{X,x}$-submodule of $M=H^n_x(X,\mathcal{O}_x)$, which is also characterized as the annihilator of $\mathfrak{m}_x^{m+1}$, and which coincides with the image of $\operatorname{Ext}^n_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\mathfrak{m}_x^{m+1}\mathcal{O}_{X,x})$ in $H^n_x(X,\mathcal{O}_X)$ in terms of the abstract definition given above. Now it is obvious that $D_m(X)M_0=M_m$, and that the subspaces $O = M_{-1} \subset M_0 \subset M_1 \subset \dots$ form a filtration of the $D(X)$-module M such that

$$\operatorname{gr} M := \bigoplus_{m \geq 0} M_m/M_{m-1} \cong k\left[\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right] \cong \bigoplus_{m \geq 0} S^m(T(X)_x) = R(T^*(X)_x),$$

as a graded vector-space.

1.6. Assume that the variety X is connected.

Lemma. a) *$H^n_x(X, \mathcal{O}_X)$ is a faithful cyclic $D(X)$-module M.*

b) *M is generated by a one-dimensional $\mathcal{O}_{X,x}$-submodule $M_0 \cong \Lambda^n(T(X)_x)$ isomorphic to the n-th exterior power of the tangent-space at x.*

c) *M is filtered by finite-dimensional $\mathcal{O}_{X,x}$-submodules*

$$M_m = \{\eta \in M \mid \mathfrak{m}_x^{m+1} \eta = 0\} = D_m(X) M_0 \qquad (m = 0, 1, 2, \dots).$$

d) *$\operatorname{gr} M$ is isomorphic to the ring of polynomial functions on the cotangent-space at x (as a graded vector-space).*

All statements of the lemma follow from the discussion in 1.5 in terms of local affine coordinates. That M has to be faithful as a $D(X)$-module is also a consequence of 1.1, since all stalks $\mathcal{D}_{X,x}$ are simple, and $D(X)$ injects into $\mathcal{D}_{X,x}$. - However, let us give now an intrinsic description of the $D(X)$-action on $M = H^n_x(X, \mathcal{O}_X)$, which exhibits the independence of the choice of coordinates.

For this purpose, consider the sheaf Ω_X of algebraic differential forms of order n on X. The stalk $\Omega_{X,x}$ is a right $\mathcal{D}_{X,x}$-module in an obvious way. In terms of coordinates as in 1.5, the n-form $dz_1 \wedge \dots \wedge dz_n$ is a free generator for $\Omega_{X,x}$ as an $\mathcal{O}_{X,x}$-module, and is annihilated by the derivatives $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. Now there are canonical linear maps

$$\Omega_{X,x} \otimes H^n_x(X, \mathcal{O}_X) \xrightarrow{\mu} H^n_x(X, \Omega_X) \xrightarrow{\text{res}} k.$$

Their composition $\operatorname{Res} = \operatorname{res} \cdot \mu$ gives a non-degenerate pairing, which induces a non-degenerate pairing

$$\Omega_{X,x}/\Omega_{X,x}\mathfrak{m}_x^{m+1} \otimes M_m \to k,$$

for each $m \geq 0$. Hence this identifies M_m with the dual vectorspace $(\Omega_{X,x}/\Omega_{X,x}\mathfrak{m}_x^{m+1})^*$ for each m, and $M = H^n_x(X, \mathcal{O}_X)$ itself with the topoligical dual of $\Omega_{X,x}$ with respect to the Krull-topology. Now the $D(X)$-action on M is intrinsically given by

$$\operatorname{Res}(\omega \otimes P\eta) = \operatorname{Res}(\omega P \otimes \eta) \qquad (*)$$

for all forms $\omega \in \Omega_{X,x}$, operators $P \in D(X)$, and classes $\eta \in M$.

Let us verify that this definition agrees with the previous one, which used a local coordinate system. An element η of $H^n_x(X, \mathcal{O}_X)$ is represented by some function f, regular outside $z_1 z_2 \dots z_n = 0$. It suffices to prove (*) for the case when $P = v$ is a vector field. Recall that then $\omega \cdot v = -\theta_v(\omega)$, where θ_v denotes Lie-derivation of differential forms with respect to v. So we have to prove

$$\operatorname{Res}(\omega \otimes v\eta + \theta_v(\omega) \otimes \eta) = 0,$$

which also means that

$$\mathrm{Res}(v(f)\cdot\omega+f\cdot\theta_v(\omega))=0,$$

where f is viewed as a meromorphic function. We have $v(f)\cdot\omega+f\cdot\theta_v(\omega)=\theta_v(f\cdot\omega)$ since θ_v is a derivation and $v(f)=\theta_v(f)$. So it suffices to prove that $\mathrm{Res}\cdot\theta_v$ annihilates sections of Ω_X, which are regular outside $z_1 z_2 \dots z_n=0$. If i_v denotes interior product with v, then $\theta_v=d\cdot i_v+i_v\cdot d$ coincides on Ω_X with $d\cdot i_v$. But it is clear that Res annihilates any form which is exact. This follows from

$$\mathrm{Res}\left(\sum_\alpha f_\alpha(z)\frac{dz_1\dots dz_n}{z_1^{\alpha_1}\dots z_n^{\alpha_n}}\right)=f_{(1,\dots,1)}(0),$$

and proves our claim.

§ 2. The Moment Map of a Homogeneous Space

2.1. Let G be a connected linear algebraic group with Lie-algebra $\mathfrak{g}$. We shall always assume that our non-singular variety X is a *G-space*. This means that there is given a morphism $G\times X\to X$ satisfying the axioms of a group action. The space is called *homogeneous* resp. *quasi-homogeneous* if the action is transitive resp. transitive on a dense subset. For each point $x\in X$ the morphism $G\to X$ sending $g\in G$ to gx gives rise to a linear map $\mathfrak{g}\to T(X)_x$, the *tangent-map* at x, and to a dual map $T^*(X)\to\mathfrak{g}^*$, the *cotangent-map* at x. Clearly, for X to be quasi-homogeneous, it is necessary and sufficient that the tangent map at a generic point x is *surjective* (resp. the cotangent map at x is injective).

2.2. The *isotropy-group* of X at a point x is the closed subgroup G_x of all $g\in G$ such that $gx=x$. Its Lie-algebra, the *isotropy-algebra* of X at x, is denoted $\mathfrak{g}_x$. This is also the kernel of the tangent map at x. The image of the cotangent map at x is $\mathfrak{g}_x^\perp$, the subspace in $\mathfrak{g}^*$ of linear forms which vanish on $\mathfrak{g}_x$. In the case where X is homogeneous, the isotropy-groups resp. -algebras of X at its various points are all conjugate to a fixed one, H resp. $\mathfrak{h}$, say. The choice of a base point $x\in X$ is equivalent to the choice of an identification of X with the coset space G/H. This implies an identification of $\mathfrak{g}/\mathfrak{h}$ with the tangent- and of $\mathfrak{h}^\perp$ with the cotangent-space of X at x.

2.3. The collection of tangent-maps $\mathfrak{g}\to T(X)_x$ at the various points x of our G-space X gives rise to an algebraic map $\mathfrak{g}\times X\to T(X)$ into the tangent-bundle, and to a dual map $T^*(X)\to\mathfrak{g}^*\times X$, compatible with the projections to the base space X. We shall refer to these maps as the *tangent-* resp. *cotangent-map* of X. Composing the cotangent-map with the map forgetting the base point in X, we obtain a canonical map $T^*(X)\to\mathfrak{g}^*$ from the cotangent-bundle into the dual of the Lie-algebra, denoted π or π_X. Following J.M. Souriau [So], let us call it the *moment(um) map* of the G-space X. – For example, if G happens to be the group of translations in real 3-space X, then π is the map attaching to a mass point moving within X its *linear momentum*. The same notion of a moment map has independently been introduced by Kostant [Ko1] (without using the word), and has since then been extensively used in theoretical mechanics, see e.g. [A Ma], [KKS], [Ko 2].

A very prominent *example* for this notion is provided by Springer's resolution of singularities for the variety of nilpotent elements of a semisimple Lie-algebra [S 1], [S 2]. Here G is semisimple, and $\mathfrak{g}^*$ is understood to be identified with $\mathfrak{g}$ by means of the Killing form whenever we find it convenient. With this convention, we may point out here the following fact, which will become clear in the sequel, and which will be discussed in more generality and more detail in 2.6: *Springer's resolution is the moment map of the flag variety.*[1]

2.4. Let us determine the *image of the moment map* π in general. Since the cotangent-map maps $T^*(X)$ onto the union of all fibres $\mathfrak{g}_x^\perp \times \{x\}$ for the various points $x \in X$, we observe that this image is just

$$\pi(T^*(X)) = \bigcup_{x \in X} \mathfrak{g}_x^\perp \subset \mathfrak{g}^*.$$

This is clearly a *union of G-orbits* under the coadjoint action of G in $\mathfrak{g}^*$, because $\mathfrak{g}_{gx}^\perp = g\,\mathfrak{g}_x^\perp$ for all $g \in G$.

In particular, if X is *homogeneous* with isotropy-group H of Lie-algebra $\mathfrak{h}$, we find that the image of the moment map is the *union of all G-conjugates of* $\mathfrak{h}^\perp$, which we simply denote $G\mathfrak{h}^\perp$. Moreover, the tangent-map $\mathfrak{g} \times X \to T(X)$ and the cotangent-map $T^*(X) \to \mathfrak{g}^* \times X$ in this case are homomorphisms of homogeneous vector-bundles with base X. The kernel of the first one has fibre $\mathfrak{h}$ and is called the *isotropy-bundle*, the image of the second one has fibre $\mathfrak{h}^\perp$ and is called the *co-isotropy-bundle*. This has an alternative description as an associated fibre-bundle $G \times^H \mathfrak{h}^\perp$, which may be defined as the quotient of $G \times \mathfrak{h}^\perp$ by the free H-action $(g, z) \mapsto (gh^{-1}, hz)$ for all $g \in G$, $z \in \mathfrak{h}^\perp$, $h \in H$ (cf. [BK 2], §7). With this description of the co-isotropy-bundle, the moment map is just the canonical map $G \times^H \mathfrak{h}^\perp \to G\mathfrak{h}^\perp$ given by $(g, z) \mapsto gz$. This description puts in evidence that G *acts also on the total space* of $T^*(X) \cong G \times^H \mathfrak{h}^\perp$ such that the moment map is G-equivariant. We summarize our observations:

Proposition. *For a homogeneous G-space X with isotropy-group H of Lie-algebra* $\mathfrak{h}$ *the moment map* π: $T^*(X) \to \mathfrak{g}^*$ *has image* $G\mathfrak{h}^\perp$ *and identifies with the canonical G-equivariant map* $G \times^H \mathfrak{h}^\perp \to G\mathfrak{h}^\perp$.

2.5. The moment-map determines an algebra-homomorphism $S(\mathfrak{g}) \to R(T^*(X))$ of the symmetric algebra $S(\mathfrak{g}) = R(\mathfrak{g}^*)$ into the regular functions on the cotangent-bundle. In particular, $R(T^*(X))$ is viewed as an $S(\mathfrak{g})$-module by means of the moment-map.

Proposition. *If the homogeneous space X is complete, then its moment map is proper, and its image is closed in* $\mathfrak{g}^*$. *Moreover* $R(T^*(X))$ *is a finitely generated* $S(\mathfrak{g})$*-module.*

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[1] *Remark* added in proof. In a recent preprint of Ginsburg [Gi], which we received after distribution of our first version of this paper, the same observation is made, and is taken as a basis for a discussion of fixed point varieties in flag varieties, repeating and clarifying work of Steinberg [St].

In fact, $T^*(X)$ identifies with a closed subvariety of $\mathfrak{g}^* \times X$, since X is homogeneous. The projection-map $\mathfrak{g}^* \times X \to \mathfrak{g}^*$ forgetting the base point in X is proper, since X is complete. Since the moment-map π is the restriction of this map, π is also proper, and maps the closed subvariety $T^*(X)$ onto a closed subvariety of $\mathfrak{g}^*$. The final statement of the proposition is a general property of proper morphisms, see [EGA] III, Théorème 3.2.1.

Remark. In view of 2.4, this is essentially [BK 2], 7.9.

2.6. Let us discuss now in more detail the case where G is *semisimple* and X is a *generalized flag manifold.* This means X is homogeneous and complete. In this case, the isotropy-group P and its Lie-algebra $\mathfrak{p}$ are *parabolic*, and X identifies with the variety of conjugates of P. The moment map π is exactly the "*generalized Springer resolution*" in the terminology of [BM 1], § 7, which has been studied extensively in the literature – although in various disguises. The co-isotropyspace $\mathfrak{p}^\perp$ identifies with the nilradical of $\mathfrak{p}$ (convention 2.3), and contains a dense P-orbit (Richardson [R]), generated by y, say. This gives rise to a dense G-orbit in $G \times^P \mathfrak{p}^\perp = T^*(X)$, which is mapped onto a dense G-orbit in $G\mathfrak{p}^\perp$, the moment-map π: $G \times^P \mathfrak{p}^\perp \to G\mathfrak{p}^\perp$ being a covering of degree $[G_y : P_y]$, a finite number, on the dense orbit. This number is referred to as the *degree* $\deg \pi = [G_y : P_y]$ of the moment map, and the nilpotent G-orbit of which $\operatorname{Im}\pi = G\mathfrak{p}^\perp$ is the closure is called the *Richardson-orbit* determined by P. These orbits and degrees are explicitly known in all cases (Hesselink [He], Elashvili-Panov [EP]). The degrees are powers of 2 for classical, and divisors of 120 for exceptional groups (Alekse'evski [Al], Mizuno [Mi], Shoji [Sh]). Quite frequently, the degree is 1, or equivalently, *π is birational.* In this case, the moment map is actually a *resolution of singularities* for the closure of the corresponding Richardson-orbit. For instance, this is always the case if p is a Borel-subgroup: Then $G\mathfrak{p}^\perp$ is the set of all nilpotent elements and π is Springer's resolution (cf. 2.3). The fibres of this map are identical with the "fixed point varieties", extensively studied by Spaltenstein, Springer, and Steinberg [Sp], [S 2], [St], and since then by many others. We shall come back to this topic in 6.2 and 6.13.

2.7. *Example* (notation 2.6). The group $G = \mathrm{SL}_n$ enjoys three particularly nice properties, all of which fail in general:

(1) *Each nilpotent orbit of* $\mathfrak{g}$ *is a Richardson-orbit for a suitable choice of* P.

(2) *The moment map* π *is always birational.*

(3) *The image of* π *is always normal* (Kraft-Procesi [KP 1]).

2.8. We shall need the following generalization of Proposition 2.5 to certain homogeneous spaces which are not necessarily complete, but are homogeneous fibre-bundles with complete base.

Proposition. *Let X be a homogeneous G-space with isotropy-group H normalized by a parabolic subgroup P, $H \subset P \subset G$. Then*

a) *X is a principal A-bundle for the group $A = P/H$ with complete base $X/A = G/P$.*

b) *$T^*(X)$ is a principal A-bundle with base $T^*(X)/A$, say.*

c) *The map* $\bar{\pi}$: $T^*(X)/A \to \mathfrak{g}^*$ *induced from the moment map is proper.*

d) $R(T^*(X)/A) = R(T^*(X))^A$, *the ring of A-invariant functions on* $T^*(X)$, *is a finitely generated* $S(\mathfrak{g})$*-submodule of* $R(T^*(X))$.

Proof. Since P normalizes H, we have a right P-action on $G/H = X$, which commutes with the left G-action, and is free as an action of $P/H = A$. Since P is parabolic, the variety $G/P = X/A$ is complete. Let us view $T^*(X)$ as $G \times^H \mathfrak{h}^\perp$ as in 2.4. Since P normalizes H, it stabilizes also $\mathfrak{h}^\perp$, and we may form the vector-bundle $G \times^P \mathfrak{h}^\perp$. The obvious map $G \times^H \mathfrak{h}^\perp \to G \times^P \mathfrak{h}^\perp$ is clearly a principal A-bundle. This exhibits a free A-action on $T^*(X)$, and the existence of a good quotient-variety $T^*(X)/A = G \times^P \mathfrak{h}^\perp$. Viewing the moment map π: $T^*(X) \to \mathfrak{g}^*$ as the canonical map $G \times^H \mathfrak{h}^\perp \to G\mathfrak{h}^\perp$ as in 2.4, we see that the induced map $\bar{\pi}$: $T^*(X)/A \to \mathfrak{g}^*$ is the canonical map $G \times^P \mathfrak{h}^\perp \to G\mathfrak{h}^\perp$. Now c) and d) follow from the completeness of G/P by the same arguments as used in 2.5. Q.e.d.

2.9. *Example.* Proposition 2.8 applies to the situation where X is the affine space $\mathbb{A}^{n+1}$ minus origin, considered as a principal homogeneous $\mathbb{G}_m$-bundle over projective n-space.

More generally, we consider any homogeneous space X with isotropy-group $H = (P, P)$, the commutator-subgroup of a parabolic $P \subset G$ with Lie-algebra $\mathfrak{p}$. In this case, the co-isotropy-space $\mathfrak{h}^\perp = [\mathfrak{p}, \mathfrak{p}]^\perp$ identifies with the solvable radical of $\mathfrak{p}$, and the image of the moment map, $G\mathfrak{h}^\perp$, is *the closure of a sheet* S, in the sense of [B 4], that is to say a maximal irreducible subvariety of $\mathfrak{g}$ consisting of G-orbits of a fixed dimension. In fact, S is the so-called *Dixmier-sheet* determined by P (see loc. cit.). Now 2.8 applies with $A = P/(P, P)$, a torus, and the map $\bar{\pi}$: $T^*(X)/A \to \bar{S}$ in this case is the map known as the "*simultaneous resolution*" for (the fibres of the adjoint quotient of) $\bar{S}$. – For instance, if P is a Borel-subgroup, then H is a maximal unipotent subgroup of G, $\mathfrak{h}^\perp$ is a Borel-subalgebra, $G\mathfrak{h}^\perp$ is all of $\mathfrak{g}$, and $\bar{\pi}$ is the well-known *simultaneous resolution* for $\mathfrak{g}$, first introduced by Grothendieck (cf. e.g. [St]).

Remark. As a subvectorbundle of the trivial bundle $(X/A) \times \mathfrak{g}^*$ on $X/A = G/P$, $T^*(X)/A$ is the orthogonal of $T^*(G/P)$, viewed as a subvectorbundle of $(G/P) \times \mathfrak{g}^*$, if we identify $\mathfrak{g}^*$ with its dual. In some recent unpublished work, Kashiwara proved that Fourier transformation maps Harish-Chandra's system of P.D.E. for invariant eigendistributions on $\mathfrak{g}$ to the holonomic $\mathcal{D}_{\mathfrak{g}^*}$-module with regular singularities corresponding to the perverse sheaf $\mathcal{A}^\cdot$ studied in [BM 1]. It seems clear that this could be rephrased more topologically, using the concept of "topological Fourier transform" in the sense of Sato-Kashiwara-Kawai-Malgrange: up to a shift, the Fourier transform of the constant sheaf on $T^*(X)/A$, extended by zero to $(X/A) \times \mathfrak{g}^*$, is the constant sheaf on $T^*(X/A)$, extended similarly; also this Fourier transformation on this pair of dual trivial vector bundles over the proper variety $X/A = G/P$ does commute with the operation of projecting to the second factor.

§3. Homogeneous Spaces and Induced Representations

3.1. Let X be a nonsingular G-space. The G-action on X induces a group-homomorphism $G \to \operatorname{Aut} K(X)$ of G into the *automorphisms* of the field $K(X)$ of rational functions on X. It determines a unique Lie-homomorphism $\mathfrak{g} \to \operatorname{Der} K(X)$ of the Lie-algebra of G into the *derivations* of $K(X)$. In fact, $\mathfrak{g}$ acts even by global vector-fields on X. In other words, we have a representation of $\mathfrak{g}$ by global differential operators of degree 1 on X. This extends uniquely to an algebra-homomorphism $U(\mathfrak{g}) \to D(X)$ of the *universal enveloping algebra* of $\mathfrak{g}$ into the global differential operators of any degree on X. This homomorphism will be called the *operator-representation* of $U(\mathfrak{g})$ (or $\mathfrak{g}$) on X, denoted ψ or ψ_X. - For instance, if $X = G$ and G acts by *left* translation, then the operator-representation ψ provides the well-known isomorphism identifying $\mathfrak{g}$ with the Lie-algebra $(\operatorname{Der} K(G))$ of *right-invariant vector-fields* on G, resp. $U(\mathfrak{g})$ with the algebra $D(G)$ of *right-invariant differential operators* on G.

3.2. The operator-representation ψ_X: $U(\mathfrak{g}) \to D(X)$ should be thought of as a "refined (non-commutative) version" of the "moment-representation" ϕ_X: $S(\mathfrak{g}) \to R(T^*(X))$ studied in the previous section (cf. 2.5). In fact, ϕ_X may be reobtained from ψ_X. Recall that $D(X)$ is filtered by the $D_m(X)$ and similarly $U(\mathfrak{g})$ is filtered by subspaces $U_m(\mathfrak{g})$. Of course, ψ_X sends $U_m(\mathfrak{g})$ to $D_m(X)$; the associated map on the graded rings sends $S(\mathfrak{g}) = \bigoplus_m U_m(\mathfrak{g})/U_{m-1}(\mathfrak{g})$ to $\operatorname{gr} D(X)$, which is naturally a subring of $R(T^*(X))$. The morphism from $S(\mathfrak{g})$ to $R(T^*(X))$ obtained by composition is precisely ϕ_X.

3.3. The image of the operator-representation, the subalgebra $\psi_X(U(\mathfrak{g}))$ of $D(X)$, is equipped with *two filtrations:* Since both algebras $U(\mathfrak{g})$ and $D(X)$ carry natural filtrations, by subspaces denoted $U_m(\mathfrak{g})$ resp. $D_m(X)$ for $m = -1, 0, 1, 2, \ldots$, we may filter $\psi_X(U(\mathfrak{g}))$ either by the subspaces $\psi_X(U_m(\mathfrak{g}))$, or by $\psi_X(U(\mathfrak{g})) \cap D_m(X)$. We shall refer to the first filtration as the *natural* one, or the G-filtration, while the second is called the *operator*-filtration (*induced from* $D(X)$), or the X-filtration. For example, if $X = G$ then ψ_G maps $U_m(\mathfrak{g})$ isomorphically onto $D_m(G)$ for all m (notation as in 3.1), so both filtrations coincide in this case. In general, $U_m(\mathfrak{g})$ is of course represented by operators of degree $\leqq m$ on X. However, even if $U(\mathfrak{g}) \to D(X)$ is surjective, $U_1(\mathfrak{g}) \to D_1(X)$ need not be, see 3.10 for an example.

Consequently, it will be a crucial point in our analysis to make a careful distinction between the two filtrations. For a *complete* homogeneous space X, we shall prove later (5.6) that the two filtrations coincide if and only if the moment map is birational with normal image. In view of 2.6, this shows that the problem of identifying the two filtrations is closely related to the question: Which of the nilpotent orbits in $\mathfrak{g}$ have *normal closures*? This is very delicate and has been answered so far only for the classical groups (Kraft-Procesi [KP 1], [KP 2]).

3.4. Now let X be a nonsingular quasi-homogeneous G-space of dimension n with $x \in X$ in the dense orbit. Then the tangent map $\mathfrak{g} \to T(X)_x$ is surjective at x, and $T(X)_x$ becomes an n-dimensional $\mathfrak{g}_x$-module isomorphic to $\mathfrak{g}/\mathfrak{g}_x$ with respect to the adjoint action. Hence the n-th exterior power $\Lambda^n(T(X)_x)$ is a one-dimensional $\mathfrak{g}_x$-module, on which $\mathfrak{g}_x$ acts by some linear form, denoted by $\lambda_x \in \mathfrak{g}_x^*$. We may easily compute it: Using Dixmier's notation [Di], 5.2.1 we find

$$\lambda_x(h) = \operatorname{tr}_{\mathfrak{g}/\mathfrak{g}_x} \operatorname{ad}_{\mathfrak{g}}(h) \qquad \text{for all } h \in \mathfrak{g}_x,$$

or

$$\lambda_x = 2\theta_{\mathfrak{g},\mathfrak{g}_x}.$$

If λ is any linear form on a Lie-algebra, k_λ will denote the corresponding one-dimensional module.

3.5. Each $D(X)$-module is also viewed as a $\mathfrak{g}$-module, by means of the operator-representation of $\mathfrak{g}$ on X. In particular, the cohomology-group $H_x^n(X, \mathcal{O}_X)$ discussed in 1.6 is a $\mathfrak{g}$-module. Let us identify this $\mathfrak{g}$-module with an *induced* module:

Proposition (Notation 3.4). *As a* $\mathfrak{g}$-*module*,

$$H_x^n(X, \mathcal{O}_X) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} k_{\lambda_x}.$$

Proof. Recall the filtration of $M = H_x^n(X, \mathcal{O}_X)$ by subspaces (cf. 1.6)

$$M_i \cong (\Omega_{X,x}/\Omega_{X,x}\mathfrak{m}_x^{i+1})^* \qquad \text{for } i = -1, 0, 1, \ldots$$

which are obviously $\mathfrak{g}_x$-submodules. For instance, ($M_{-1} = 0$ and)

$$M_0 \cong \Lambda^n(T(X)_x) \cong k_{\lambda_x}.$$

By the universal property of induced representations, the $\mathfrak{g}_x$-homomorphism $k_{\lambda_x} \cong M_0 \subset M$ extends to a $\mathfrak{g}$-homomorphism of the induced module $N := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} k_{\lambda_x}$ into M. To prove that this is actually an isomorphism, consider the filtration of N by the subspaces $N_i := U_i(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} k_{\lambda_x}$. Since $U_i(\mathfrak{g})$ maps into $D_i(X)$ (3.3), we see that N_i maps into $M_i = D_i(X) M_0$, hence a homomorphism $\operatorname{gr} N \to \operatorname{gr} M$ of the associated graded modules. In each degree i we have vector-spaces

$$\begin{aligned} M_i/M_{i-1} &\cong (\Omega_{X,x}\mathfrak{m}_x^i/\Omega_{X,x}\mathfrak{m}_x^{i+1})^* \cong (\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1})^* \\ &\cong S^i(T(X)_x) \text{ resp. } N_i/N_{i-1} \cong S^i(\mathfrak{g}/\mathfrak{g}_x), \end{aligned}$$

and the map $\operatorname{gr} N \to \operatorname{gr} M$ comes from the tangent map $\mathfrak{g} \to T(X)_x$ at x. We conclude that $M \cong N$ iff $\operatorname{gr} M \cong \operatorname{gr} N$ iff $\mathfrak{g}/\mathfrak{g}_x \cong T(X)_x$ iff x generates a dense orbit in X. Q.e.d.

3.6. The kernel of the operator-representation ψ_X: $U(\mathfrak{g}) \to D(X)$ is a two-sided ideal, which we denote by $I_X := \ker \psi_X$. As an application of 3.5, we can identify I_X as the annihilator of an induced module:

Corollary. $I_X := \ker \psi_X = \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_x)} k_{\lambda_x}$.

In fact, since $H_x^n(X, \mathcal{O}_X)$ is faithful as a $D(X)$-module by 1.6, it is also faithful as a $U(\mathfrak{g})/I_X$-module. Hence the isomorphism in 3.5 gives the result.

3.7. For example, if G is semisimple and X a *generalized flag variety* (cf. 2.6), then Proposition 3.5 says that $H_x^n(X, \mathcal{O}_X)$ is a *generalized Verma module.*

In order to discuss this case in more detail, let us introduce some *notations*: $P = G_x$ is the (parabolic) isotropy-subgroup, $\mathfrak{p} = \mathfrak{g}_x$ its Lie-algebra, $\mathfrak{b} \subset \mathfrak{p}$ a Borel-subalgebra, $\mathfrak{t} \subset \mathfrak{b}$ a Cartan-subalgebra, R^+ the system of positive roots occuring in $\mathfrak{b}$, R_P^+ the subsystem of positive roots occuring in $\mathfrak{b}/\mathfrak{p}^\perp$, and ρ resp. ρ_P is half the sum of positive roots in R^+ resp. R_P^+. Furthermore, we denote by

$$M_P(\lambda)=M_{\mathfrak{p}}(\lambda):=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}k_\lambda$$

the "generalized Verma module" of highest weight $\lambda\in\mathfrak{t}^*$ (a basic reference for such modules is Jantzen [J]), and by $I_P(\lambda)$ or $I_{\mathfrak{p}}(\lambda)$ its annihilator. With these notations, we find that $\lambda_x=2\rho_P-2\rho$, and Proposition 3.5 gives:

Corollary. $H_x^n(X,\mathcal{O}_X)\cong M_P(2\rho_P-2\rho)$,

$$I_X=I_P(2\rho_P-2\rho).$$

Remark. In particular, *if X is the (ordinary) flag variety*, that is $\mathfrak{p}=\mathfrak{b}$, *then $H_x^n(X,\mathcal{O}_X)$ is an (ordinary) Verma module*, the one of highest weight -2ρ (denoted $M(-\rho)$ in the conventions of [Di], 7.1). This result appears in the work of Kempf [Ke 2]. The second author has studied this particular situation in more detail in [Br]. There $H_x^n(X,\mathcal{O}_X)$ appears actually as a "dual" of a Verma module, which is no contradiction, since this particular Verma module is "self-dual". The following theorem extends the main result of [Br] to generalized flag varieties.

3.8. Theorem. $D(X)\cong U(\mathfrak{g})/I_X$ *for any complete homogeneous space X.*

Proof. The operator representation ψ_X has kernel I_X by 3.6. It is left to prove surjectivity. A k-linear endomorphism of a generalized Verma module $M_P(\lambda)$ (Notation 3.6) is called $\mathfrak{g}$-*finite*, if it generates a finite-dimensional linear subspace of $\mathrm{End}_k M_P(\lambda)$ closed under commutation with elements of $\mathfrak{g}$. Now let $\lambda=2\rho_P-2\rho$. Then $D(X)$ acts on $M_P(\lambda)$ (3.5) by endomorphisms which are obviously $\mathfrak{g}$-finite. From the work of N. Conze-Berline and M. Duflo [CD], (2.12, 4.7, 6.3, see also [Jo 1], Theorem 4.5), we can conclude that in the present case all $\mathfrak{g}$-finite endomorphisms of $M_P(\lambda)$ are induced by an element of $U(\mathfrak{g})$. This implies now that $U(\mathfrak{g})$ surjects onto $D(X)$. Q.e.d.

3.9. *Remark.* As in [Br], this result may be extended to:

$$D^\mu(X)\cong U(\mathfrak{g})/I_P(\mu+2\rho_P-2\rho),$$

where $D^\mu(X)$ denotes differential operators with "twisted" coefficients, i.e. with coefficients in the line-bundle associated with a character of the group P, of weight μ.

3.10. *Example.* a) Let X be projective n-space, and G the projective linear group. Let $x_0,\dots,x_n$ denote homogeneous coordinates. Then a matrix unit e_{ij} in the Lie-algebra acts by the differential operator $x_i\dfrac{\partial}{\partial x_j}$ on the functions on X. Now Theorem 3.8 gives that $D(X)$ is the algebra generated by these operators. Note that they satisfy the relation

$$\sum_j x_j\frac{\partial}{\partial x_j}=0$$

by Euler's differential equation for homogeneous polynomials. It is not difficult to find generating relations for the $x_i \frac{\partial}{\partial x_j}$ by computing generators of the ideal I_X in $U(\mathfrak{g})$.

b) Now let X be the projective space of dimension $n=2m-1$ as before, but consider it as the space of lines in a symplectic vector-space, that is to say: take $G=\mathrm{Sp}_{2m}$. As we have seen in a), $D(X)$ is generated by the $4m^2$ independent operators $x_i \frac{\partial}{\partial x_j}$ of degree ≤ 1. But now only $\dim \mathfrak{sp}_{2m}=2m^2+m$ of them represent an element of the Lie-algebra (a *symplectic* matrix). Since ψ_X: $U(\mathfrak{g})\to D(X)$ is nevertheless surjective by Theorem 3.8, we see that the remaining global vector-fields are represented by elements of order >1 from $U(\mathfrak{g})$. This shows that the operator-filtration on $\psi_X U(\mathfrak{g})\cong U(\mathfrak{g})/I_X$ differs from the natural filtration. (This example has been pointed out to us by R. Elkik.)

3.11. We shall need another special case of 3.5:

Corollary (Notation 3.7). *The operator-representation of* $U(\mathfrak{g})$ *on the homogeneous space* $Y=G/(P,P)$ *has kernel*

$$I_Y = \operatorname{Ann} U(\mathfrak{g}) \otimes_{U([\mathfrak{p},\mathfrak{p}])} k_0 = \bigcap_{\lambda\in\Lambda} I_P(\lambda)$$

where $\Lambda=[\mathfrak{p},\mathfrak{p}]^{\perp}\cap\mathfrak{t}$ *is the set of all characters* (one-dimensional representations) *of* $\mathfrak{p}$.

Proof. In the present case, the isotropy-algebra is $[\mathfrak{p},\mathfrak{p}]$, which has no nontrivial characters at all. Hence $\lambda_x=0$ in 3.5, and 3.6 gives our first equation. For the second equation, we refer to [BJ], Lemma 3.9a). Q.e.d.

Remark. Let us mention that Gelfand-Kirillov [GKi] and Sapovalov [Š] have made a more detailed study of the representation of $U(\mathfrak{g})$ by differential operators on the "basic affine space", that is to say on $Y=G/(B,B)$ for a Borel subgroup B. (In this case, $I_Y=0$, and so $U(\mathfrak{g})$ embeds into $D(Y)$.)

§4. Associated Varieties of Induced Ideals

4.1. If M is a $\mathfrak{g}$-module filtered by vector-spaces $0=M_{-1}\subset M_0\subset M_1\subset M_2\ldots$ such that $\mathfrak{g}M_i\subset M_{i+1}$ for all i, then the *associated graded module* $\operatorname{gr} M = \bigoplus_{i\geq 0} M_i/M_{i-1}$ is a graded $S(\mathfrak{g})$-module. The filtration of M is called *good*, if $\operatorname{gr} M$ is of finite type as a $S(\mathfrak{g})$-module. A good filtration on M exists, if and only if M is of finite type as a $U(\mathfrak{g})$-module. If such is the case, then the *Bernstein-variety of* M, denoted $V(M)$ or $V_{\mathfrak{g}}(M)$, is defined as the support of the $S(\mathfrak{g})$-module $\operatorname{gr} M$ in $\mathfrak{g}^*$. In other words, we define $V(M)=\mathscr{V}(\operatorname{Ann}\operatorname{gr} M)$, where $\mathscr{V}(\ldots)$ denotes the set of zeros in $\mathfrak{g}^*$ for any set … of polynomials on $\mathfrak{g}^*$. This definition refers to a good filtration chosen on M, but is actually independent of the choice (Bernstein [Be], see also [Bj] for the material of this section).

Lemma. $V(E\otimes M)=V(M)$ *for any finite-dimensional* $\mathfrak{g}$*-module* E.

Proof. Given a good filtration on M by M_i, we obtain a good filtration on $E\otimes M$ by $E\otimes M_i$. With these filtrations, $\mathrm{gr}(E\otimes M)$ as an $S(\mathfrak{g})$-module is a sum of copies of $\mathrm{gr}\, M$. Hence the lemma. Q.e.d.

4.2. The *associated variety* of a left ideal I in $U(\mathfrak{g})$ is defined as the zero-set $\mathscr{V}(\mathrm{gr}\, I)\subset\mathfrak{g}^*$ of the associated graded ideal $\mathrm{gr}\, I$ with respect to the natural filtration. Obviously, we have

$$\mathscr{V}(\mathrm{gr}\, I)=V(U(\mathfrak{g})/I).$$

If I is a two-sided ideal, then I and $\mathrm{gr}\, I$ are invariant under the adjoint G-action. Consequently, the associated variety is then a *union of G-orbits.*

Lemma. *Let E be a finitely generated module for a subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$. *Let M be the induced* $\mathfrak{g}$*-module. Let* r: $\mathfrak{g}^*\to\mathfrak{h}^*$ *denote restriction.* Then

$$\mathscr{V}(\mathrm{gr\, Ann}\, M)\supset G r^{-1}(V_{\mathfrak{h}}(E))\supset G\mathfrak{h}^{\perp}.$$

Proof. Since this associated variety is G-stable as noted above, the lemma will follow from $V(U(\mathfrak{g})/\mathrm{Ann}\, M)\supset V(M)=r^{-1}(V_{\mathfrak{h}}(E))\supset\mathfrak{h}^{\perp}$. Here the first inclusion comes from the fact that M is a finite sum of homomorphic images of $U(\mathfrak{g})/\mathrm{Ann}\, M$ (4.1), and the last one from the trivial fact that associated varieties are homogeneous, so $V_{\mathfrak{h}}(E)$ contains O, and $r^{-1}(O)=\mathfrak{h}^{\perp}$. The equation in the middle is obtained by using a good filtration of type $U_i(\mathfrak{g})\otimes E_i$ and observing that $\mathrm{gr}\, M\cong S(\mathfrak{g})\otimes S(\mathfrak{h})\,\mathrm{gr}\, E$ has support $r^{-1}(V_{\mathfrak{h}}(E))$. Q.e.d.

4.3. Recall that I_X denotes the kernel of the operator-representation ψ: $U(\mathfrak{g})\to D(X)$ for our G-space X. Let us now relate the associated variety of I_X to the image of the moment map π: $T^*(X)\to\mathfrak{g}^*$.

Proposition. *Let X be a homogeneous G-space. Then we have*

$$(*)\qquad \mathscr{V}(\mathrm{gr}\, I_X)\supset\overline{\pi(T^*(X))},$$

and equality holds whenever $D(X)$ *induces a good filtration on* $\psi(U(\mathfrak{g}))$.

Proof. The ideal I_X is the annihilator of some module induced from the isotropy-algebra $\mathfrak{h}$ of X, by 3.5. Hence its associated variety contains $G\mathfrak{h}^{\perp}$, by 4.2. But $G\mathfrak{h}^{\perp}$ is the image of the moment map, by 2.4. This proves the inclusion $(*)$. – Now consider $M=\psi(U(\mathfrak{g}))\cong U(\mathfrak{g})/I_X$ as a $\mathfrak{g}$-submodule of $D(X)$, and assume that the operator-filtration on M *is good.* Then the associated graded module $\mathrm{gr}\, M$ with respect to this filtration has support $V(M)=\mathscr{V}(\mathrm{gr}\, I_X)$ by 4.1, 4.2. On the other hand, it is an $S(\mathfrak{g})$-submodule of $\mathrm{gr}\, D(X)$, and hence of $R(T^*(X))$ (1.3), *the support of which is* $\overline{\pi T^*(X))}=\overline{G\mathfrak{h}^{\perp}}$. This gives the inclusion converse to $(*)$. Q.e.d.

4.4. *Example:* The proposition applies for instance to the computation of $\mathscr{V}(\mathrm{gr}\, I_X)$ for the case that $X=G/P$ is *complete.*

In fact, in this case we know that even $R(T^*(X))$ itself is finitely generated as an $S(\mathfrak{g})$-module by 2.5, so $\mathrm{gr}\, M$ (as in the proof) is automatically finitely generated. Hence the operator-filtration is good in this case. (Note that we even have $\mathrm{gr}\, M=\mathrm{gr}\, D(X)=R(T^*(X))$ in this case by 3.8 and 1.4, but these are deeper results which are not needed here.) – In conclusion, the proposition gives us the associated variety of $I_P(\lambda) = \operatorname{Ann} M_P(\lambda)$ for a *particular* weight λ (cf. 3.7). However, we want to prove the same result for an arbitrary λ, and this requires some additional effort.

4.5. Lemma. *Let X be a homogeneous G-space with isotropy-group H normalized by a parabolic subgroup P of G. Then $D(X)$ induces a good filtration on $U(\mathfrak{g})/I_X$.*

Proof. The left G-action on $X = G/H$ commutes with the action of $A := P/H$ by right translations on X. So the vector-fields by which $\mathfrak{g}$ acts on X must be invariant under this A-action. Consequently, $M := \psi(U(\mathfrak{g}))$ must be contained in $D(X)^A$, the ring of A-invariant differential operators, which is an $U(\mathfrak{g})$-submodule in $D(X)$. Now we have

$$\operatorname{gr} M \subset \operatorname{gr} D(X)^A \subset R(T^*(X))^A$$

with respect to the operator-filtration, since this is preserved by A. Now Proposition 2.8d) gives that $R(T^*(X))^A$ and hence $\operatorname{gr} M$ are $S(\mathfrak{g})$-modules of finite type. This proves the lemma. Q.e.d.

4.6. We are ready to prove our main result on associated varieties.

Theorem. *Let $\mathfrak{p}$ be a parabolic subalgebra of a semisimple Lie-algebra $\mathfrak{g}$. Let*

$$M = U(\mathfrak{g}) \otimes_{U([\mathfrak{p},\mathfrak{p}])} E$$

be the $\mathfrak{g}$-module induced from some finite-dimensional $[\mathfrak{p},\mathfrak{p}]$-module E. Then the associated variety of the annihilator of M is the closure of the Dixmier-sheet determined by $\mathfrak{p}$,

$$\mathscr{V}(\operatorname{gr} \operatorname{Ann} M) = G[\mathfrak{p},\mathfrak{p}]^{\perp}.$$

In particular, this variety is irreducible.

Proof. a) Let us first consider the case where $E = k_0$ is the trivial representation of $[\mathfrak{p},\mathfrak{p}]$. Then $\operatorname{Ann} M = I_X$ is the kernel of the operator-representation of $U(\mathfrak{g})$ on the homogeneous space $X = G/(P,P)$, by 3.11. Here P is the parabolic subgroup with Lie-algebra $\mathfrak{p}$. Since it normalizes (P,P), the operator-filtration on $U(\mathfrak{g})/I_X$ is *good*, by 4.5. Hence $\mathscr{V}(\operatorname{gr} I_X)$ equals the image of the moment map, by 4.3. But this is $G[\mathfrak{p},\mathfrak{p}]^{\perp}$ by 2.4, or also the closure of the Dixmier-sheet determined by $\mathfrak{p}$, see 2.9.

b) Now let E be any finite-dimensional $[\mathfrak{p},\mathfrak{p}]$-module. By devissage, we may first reduce to the case E irreducible. Then it occurs as a subquotient of some finite-dimensional $\mathfrak{g}$-module.

So we may assume without restriction that E is a $\mathfrak{g}$-module, and even a simple one, of highest weight μ, say (Notation 3.7). Let M_μ denote our module $U(\mathfrak{g}) \otimes_{U([\mathfrak{p},\mathfrak{p}])} E$, and let M_0 denote the corresponding module for $\mu = 0$. It is not hard to see that M_μ occurs as a subquotient in $E \otimes M_0$. This implies an inclusion of the annihilators, and hence for their associated varieties:

$$\mathscr{V}(\operatorname{gr} \operatorname{Ann} M_\mu) \subset \mathscr{V}(\operatorname{gr} \operatorname{Ann} E \otimes M_0) = \mathscr{V}(\operatorname{gr} \operatorname{Ann} M_0): \qquad (*)$$

Here the last equation follows from Lemma 4.1 by means of the inclusion

$$U(\mathfrak{g})/\mathrm{Ann}(E \otimes M_0) \hookrightarrow (U(\mathfrak{g})/\mathrm{Ann}\, E) \otimes (U(\mathfrak{g})/\mathrm{Ann}\, M_0):$$

Now the right-hand side in (*) is an irreducible variety by a), so in order to prove equality, it is enough to prove equality of dimensions in (*), or equivalently, the equality of GK-dimensions

$$d(U(\mathfrak{g})/\mathrm{Ann}\, M_\mu) = d(U(\mathfrak{g})/\mathrm{Ann}\, M_0).$$

For this equality, see for example [BJ], 3.10 plus 3.9a). Q.e.d.

4.7. Now we prove an old conjecture of the first author, as stated in [B 2], conjecture 2.5 (cf. also [B 1]).

Corollary. *Let $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ be the $\mathfrak{g}$-module induced from a finite-dimensional $\mathfrak{p}$-module E. Then the associated variety of the annihilator of M is the closure of the Richardson-orbit determined by $\mathfrak{p}$,*

$$\mathscr{V}(\mathrm{gr}\, \mathrm{Ann}\, M) = \overline{G\mathfrak{p}^\perp}.$$

In particular, this variety is irreducible.

Proof. It is known that in this case $\mathscr{V}(\mathrm{gr}\, \mathrm{Ann}\, M)$ is contained in the set $\mathscr{N}$ of nilpotent elements in $\mathfrak{g} = \mathfrak{g}^*$. Let us briefly recall the reason (cf. [BK 1], proof of 7.1): Since M has a central character, it is annihilated by a maximal ideal $\mathfrak{m}$ of the center of $U(\mathfrak{g})$. So $\mathrm{gr}\, \mathrm{Ann}\, M$ contains the symbols of all elements in $\mathfrak{m}$, which are easily seen to generate the G-invariant polynomials on $\mathfrak{g}$, and so – by a theorem of Kostant – define $\mathscr{N}$ as their variety of common zeros. We conclude that $\mathscr{V}(\mathrm{gr}\, \mathrm{Ann}\, M) \subset \mathscr{N}$.

On the other hand, $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ is clearly a quotient of the module $U(\mathfrak{g}) \otimes_{U([\mathfrak{p},\mathfrak{p}])} E$, and so the associated variety of its annihilator must be contained in $\overline{G[\mathfrak{p},\mathfrak{p}]^\perp}$ by Theorem 4.6. Now we conclude that

$$\mathscr{V}(\mathrm{gr}\, \mathrm{Ann}\, M) \subset \mathscr{N} \cap [\mathfrak{p},\mathfrak{p}]^\perp = \overline{G\mathfrak{p}^\perp}$$

(an argument from [BJ], 5.17) where $\overline{G\mathfrak{p}^\perp}$ is the closure of the Richardson-orbit determined by $\mathfrak{p}$ (2.6). The converse inclusion was already known (4.2 or [B 1]). Q.e.d.

4.8. Let I be an ideal of $U(\mathfrak{g})$ *induced* from $\mathfrak{p}$. By definition, this means that $I = \mathrm{Ann}\, M$ is of the type considered in 4.7. Let $P_1, \ldots, P_r$ denote the prime ideals containing I which are minimal over I, so $P_1 \cap \ldots \cap P_r = \sqrt{I}$. By a result of Joseph-Small [JoS], these ideals have all the same GK-dimension $d(U(\mathfrak{g})/P_i) = d(U(\mathfrak{g})/I)$ for $i = 1, \ldots, r$. Following Joseph [Jo 3], let us call these ideals "*almost induced*" from $\mathfrak{p}$.

Lemma. *If an ideal of $U(\mathfrak{g})$ is almost induced from $\mathfrak{p}$, then its associated variety is $\overline{G\mathfrak{p}^\perp}$.*

Proof. In the above notation, we have $\mathscr{V}(\mathrm{gr}\, P_i) \subset \mathscr{V}(\mathrm{gr}\, I) = \overline{G\mathfrak{p}^\perp}$ by 4.7, with equality of dimensions by what has been stated before the lemma. Now the irreducibility of $\overline{G\mathfrak{p}^\perp}$ implies the lemma. Q.e.d.

4.9. Conjecture ($\mathfrak{g}$ semisimple). *The associated variety of a primitive ideal in $U(\mathfrak{g})$ is always irreducible.*

This goes back to [B 2], 2.9. We can prove it now for ideals with integral central character (§6). Let us note already at this point:

Corollary (Joseph [Jo 2]). *The conjecture is true for* $\mathfrak{g}=\mathfrak{sl}_n$.

In fact, in this case all primitive ideals are almost induced (as noted e.g. in [Jo 3], 10.3), so 4.8 does always apply and gives the corollary.

Comment. Joseph's proof [Jo 2] uses a method from [B 3], which depends on the nice properties of the $\mathfrak{sl}_n$-case listed in 2.7 (birationality of the moment map, and normality of its image). In particular, it depends on the Kraft-Procesi normality-theorem [KP 1]. Note that our present method of proof is completely independent of those properties. This makes it work more generally and even simpler (overcoming two of the three difficulties discussed in [Jo 2], 3.4).

§5. Complementary Results on Associated Graded Ideals

5.1. In this section, X is always a homogeneous G-space, and $I_X=\ker\psi_X$ resp. $U=\psi_X(U(\mathfrak{g}))$ denote the kernel resp. image of the operator-representation ψ_X: $U(\mathfrak{g})\to D(X)$. While the last section dealt with the study of the associated variety $\mathscr{V}(\operatorname{gr} I_X)$, which is equivalent to the study of the radical ideal $\sqrt{\operatorname{gr} I_X}$, we are now going to study more refined questions about the ideal $\operatorname{gr} I_X$ itself.

With respect to the *natural filtration* on $U\cong U(\mathfrak{g})/I_X$ we denote by $\deg u$ the degree of $u\in U$, by U_n the subspace of elements of degree $\leqq n$, and by $\sigma(u)=\sigma_n(u)$ the *symbol* of an element of degree n (that is σ_n is the canonical map from U_n to U_n/U_{n-1}); finally $\operatorname{gr} U\cong S(\mathfrak{g})/\operatorname{gr} I_X$ denotes the associated graded algebra and $S(\mathfrak{g})$-module. With respect to the *operator-filtration* on U, the notations $\deg' u$, U'_n, $\sigma'(u)=\sigma'_n(u)$ and $\operatorname{gr}' U$ have the analogous meaning. Recall (3.3) that $U_n\subset U'_n$ for all n, and hence $\deg u\geqq\deg' u$ for all $u\in U$.

Definition. The non-negative integer $\delta(u):=\deg u-\deg' u$ is called the *delay* of u. If the numbers $\delta(u)$ are bounded for $u\in U$, we say that U (or the operator filtration) has *bounded delay*.

Lemma. *We have $\delta(u^m)\leqq m\,\delta(u)$ for all $m\geqq 1$, $0\neq u\in U$, with equality if and only if $\sigma(u)^m\neq 0$.*

Proof. Since $\operatorname{gr}' U$ is a subring of $R(T^*(X))$ (1.3), it has no zero-divisors. In particular, $\sigma'(u)^m\neq 0$, which implies $\sigma'(u^m)=\sigma'(u)^m$ and $\deg'(u^m)=m\deg' u$. Combined with the general fact that $\deg(u^m)\leqq m\deg u$, this gives already $\delta(u^m)\leqq m\,\delta(u)$. Now $\deg(u^m)=m\deg u$ resp. $\deg(u^m)<m\deg u$ holds according to whether $\sigma(u)^m\neq 0$ resp. $=0$. This gives the last part of the lemma.

5.2. Lemma. *If the symbol $\sigma(u)$ of an element $0\neq u\in U$ is nilpotent, then its delay is positive.*

Proof. If $\delta(u)=0$, then $\delta(u^m)=0$ for all $m\geqq 1$ by the previous lemma. And this implies $\sigma(u)^m\neq 0$, again by that lemma. Hence $\sigma(u)$ would not be nilpotent. Q.e.d.

Remark. The converse of Lemma 5.2 is also true, i.e. positive delay implies nilpotent symbol, provided that the operator-filtration is good. This will become clear in the proof of 5.3.

5.3. Proposition. *The operator-filtration has bounded delay if and only if it is good. Moreover, if the delay is bounded by δ, then*

$$\sqrt{\operatorname{gr} I_X^{\delta+1}} \subset \operatorname{gr} I_X.$$

Proof. In the terminology of Bernstein [Be], 1.3, our statement that the operator-filtration "has bounded delay" (in fact: $U'_m \subset U_{m+\delta}$ for all m) means that it is "equivalent to the standard-filtration". Now it is a well-known fact that any two good filtrations are equivalent in that sense (see e.g. [Bj], 1.3.5), and the converse is similar. – Now suppose $\delta(u) \leqq \delta$ for all $u \in U$. If $\sigma(u)$ is nilpotent, then $\delta(u) \geqq 1$ by 5.2, but $m\delta(u) = \delta(u^m) \leqq \delta$ as long as $\sigma(u)^m \neq 0$, by 5.1. Hence $\sigma(u)^m = 0$ as soon as $m > \delta/\delta(u)$. In particular $\sigma(u)^{\delta+1} = 0$. This implies the proposition. Q.e.d.

5.4. Proposition. *If the operator-filtration on $U \cong U(\mathfrak{g})/I_X$ is good, then the following statements are equivalent:*

(i) $\sqrt{\operatorname{gr} I_X} = \operatorname{gr} I_X$.

(ii) *The operator-filtration coincides with the natural one.*

(iii) $\operatorname{gr} I_X$ *is prime.*

Proof. (i) means that $\operatorname{gr} U$ has no nilpotent elements $\neq 0$. Hence $\sigma(u)^m \neq 0$ for all $m \geqq 1$, $0 \neq u \in U$. Then u^m has delay $\delta(u^m) = m\delta(u)$ by 5.1. This can only be bounded if $\delta(u) = 0$. Hence the operator-filtration to be good implies that all elements of U have delay zero. But this means that the operator-filtration coincides with the natural one, or (i) $\Rightarrow$ (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are trivial: For instance, (ii) means that $\operatorname{gr} U = \operatorname{gr}' U$. Since we know that the annihilator of $\operatorname{gr}' U$, a subalgebra of $R(T^*(X))$, is prime, while the annihilator of $\operatorname{gr} U$ is $\operatorname{gr} I_X$, we conclude that $\operatorname{gr} I_X$ is prime. Q.e.d.

5.5. Let us recall (and rephrase) some results of H. Kraft and the first author [BK2]. Let G be semisimple, and Y a quasi-homogeneous G-space. Let Ω denote the set of (equivalence classes of) irreducible finite-dimensional representations of G. If G acts on A linearly, then $m_\omega(A)$ denotes the multiplicity of $\omega \in \Omega$ in A. The numbers $m_\omega(Y) := m_\omega(R(Y))$ are finite for all $\omega \in \Omega$, in fact: $m_\omega(Y) \leqq m_\omega(G) = \dim \omega$. They are called the *$G$-multiplicities of Y.*

Theorem [BK2]. *Let $S \subset \mathfrak{g}$ be the Dixmier sheet determined by the parabolic subgroup P with Lie-algebra $\mathfrak{p}$. Then:*

a) The following two statements are equivalent:

(i) *All G-orbits in S have the same G-multiplicities.*

(ii) *The moment-map $\pi_{G/P}$ is birational.*

b) The following two statements are equivalent:

(i) *All closures in $\mathfrak{g}$ of G-orbits in S have the same G-multiplicities.*

(ii) *The moment-map $\pi_{G/P}$ is birational with normal image.*

Proof. a) If $x \in \mathfrak{p}^\perp$ generates the dense orbit in $G\mathfrak{p}^\perp$, then $\pi_{G/P}$ birational is equivalent to $[G_x : P_x] = 1$, cf. 2.6. Now (ii) $\Rightarrow$ (i) follows from [BK2], Theorem A2, while the converse is obvious from [BK2], Theorem 7.2.

b) Let $Z \subset S$ be an orbit. Then $\bar{Z}$ is non-singular in codimension 1. (In fact: All orbit-dimensions are even, and $\bar{Z}$ is a finite union of orbits, so $\bar{Z} \setminus Z$ has codimension $\geqq 2$.) This implies that $R(Z)$ is the integral closure of $R(\bar{Z})$ ([BK2], 3.7). So $R(Z) = R(\bar{Z})$ if and only if $\bar{Z}$ is normal. But $R(Z) = R(\bar{Z})$ means that $m_\omega(Z) = m_\omega(\bar{Z})$ for all $\omega \in \Omega$. Now it is left to observe that normality of $G\mathfrak{p}^\perp$ implies normality of $\bar{Z}$ for all orbits $Z \subset S$. This follows from [BK2], Theorem 6.3. (Note that in the arguments used there, it suffices to replace the assumption "G_x connected" by the slightly weaker assumption "$\pi_{G/P}$ birational".) Now it is clear that b) is a consequence of a). Q.e.d.

5.6. Theorem. *Let X be a complete homogeneous G-space with isotropy-algebra $\mathfrak{p}$. Then the following statements are equivalent:*

(i) *The operator- and the natural filtration on $U(\mathfrak{g})/I_X$ coincide.*

(ii) $\operatorname{gr} I_X$ *is prime.*

(iii) $R(G\mathfrak{p}^\perp)$ *and* $R(T^*(X))$ *are isomorphic as G-modules.*

(iv) *The moment map π_X is birational with normal image.*

(v) *For all 1-dimensional $\mathfrak{p}$-modules E,*

$$\operatorname{gr} \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E = \mathscr{I}(G\mathfrak{p}^\perp)$$

is the prime-ideal of functions on $\mathfrak{g}$ which vanish on $G\mathfrak{p}^\perp$.

Proof. (i) $\Leftrightarrow$ (ii) follows from 5.4, since the operator-filtration is good by 4.5.

(ii) $\Rightarrow$ (iii) If $\operatorname{gr} I_X$ is prime, then $\operatorname{gr} I_X = \mathscr{I}(G\mathfrak{p}^\perp) =: p$ by §4. Hence (ii) implies G-module-isomorphisms

$$R(G\mathfrak{p}^\perp) \cong S(\mathfrak{g})/p = \operatorname{gr} U(\mathfrak{g})/I_X$$
$$\cong \operatorname{gr}' U(\mathfrak{g})/I_X = \operatorname{gr}' D(X) = R(T^*(X))$$

where the last two equations come from 3.8 resp. 1.4.

(iii) $\Rightarrow$ (iv) The moment-map π_X: $T^*(X) \to G\mathfrak{p}^\perp$ defines an embedding $R(G\mathfrak{p}^\perp) \hookrightarrow R(T^*(X))$. In view of the finiteness of the G-multiplicities of $G\mathfrak{p}^\perp$ and $T^*(X)$ (cf. 5.5), (iii) implies that this embedding must be an isomorphism. In particular, π_X is birational, and since $T^*(X)$ is smooth, $R(G\mathfrak{p}^\perp) \cong R(T^*(X))$ is integrally closed, so $G\mathfrak{p}^\perp$ is normal.

(iv) $\Rightarrow$ (v) The previous argument is easily reversed to give that (iv) implies (Notation 5.5):

$$m_\omega(G\mathfrak{p}^\perp) = m_\omega(T^*(X)) \qquad \text{for all } \omega \in \Omega. \tag{1}$$

Now let $I = \operatorname{Ann} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ as considered in (v). It follows from the work of Conze-Berline and Duflo [CD] (cf. [B3], 3.4(1)), that

$$m_\omega(U(\mathfrak{g})/I) \leqq m_\omega(R(T^*(X))) \qquad \text{for all } \omega \in \Omega. \tag{2}$$

But since $\operatorname{gr} I \subset \mathscr{I}(G\mathfrak{p}^\perp)$ by 4.7 (even by 4.2) we obtain

$$m_\omega(G\mathfrak{p}^\perp) = m_\omega(S(\mathfrak{g})/\mathscr{I}(G\mathfrak{p}^\perp)) \leqq m_\omega(S(\mathfrak{g})/\operatorname{gr} I) = m_\omega(U(\mathfrak{g})/I). \tag{3}$$

Now (1) and (2) force equality in (3), and this implies the equality $\operatorname{gr} I = \mathscr{I}(G\mathfrak{p}^{\perp})$. - Finally, (v) $\Rightarrow$ (ii) since I_X is one of the ideals considered in (v), see §3. Q.e.d.

5.7. *Comments.* Part (ii) $\Rightarrow$ (v) of the theorem is essentially an old result of the first author, stated first (without proof) in [B2], Theorem 2.6. Our method of proof used above for this part is essentially the old one, as exposed e.g. in [B3], Satz 3.4, and in [Jo2], Theorem 3.2.

In the special cases, where the *irreducibility* of the associated variety (Theorem 4.7) had been proved already before the present paper, the method of proof was actually this one, so it went via the - stronger - *primality* statement (5.6(ii)) for the associated graded ideal. It was a proori clear that this method could not work in general, since 5.6 (v) will not hold in general, as was observed already in [B1], 3.3: If, for example, $\mathfrak{g}$ is of type B_2 resp. G_2, and if the weights $\neq 0$ of $\mathfrak{p}/\mathfrak{p}^{\perp}$ are long roots, *then* $\operatorname{gr} I_X$ *is not prime*, but is only *primary* with multiplicity 2 resp. 3. Let us now (define and) compute the multiplicity of $\operatorname{gr} I_X$ in general.

5.8. For any prime ideal p of $S(\mathfrak{g})$, and any $S(\mathfrak{g})$-module N, the *p-multiplicity* $\operatorname{mtp}_p(N)$ of N is defined as the *length* of

$$M_p = S(\mathfrak{g})_p \otimes_{S(\mathfrak{g})} M$$

as an $S(\mathfrak{g})_p$-module. By abuse of language, the p-multiplicity of an *ideal* J in $S(\mathfrak{g})$ is the p-multiplicity of $S(\mathfrak{g})/J$. This is a finite positive number if and only if $\sqrt{J} = p$. For a p-primary ideal J for instance the p-multiplicity is a finite number; this number is 1 if and only if J is prime (i.e. $J = p$).

We are going to prove:

Theorem. *Let $X = G/P$ be a complete homogeneous space. Let I_X be the kernel of the operator representation, and $p = \mathscr{I}(G\mathfrak{p}^{\perp}) = \sqrt{\operatorname{gr} I_X}$, as in 4.7. Then the p-multiplicity of $\operatorname{gr} I_X$ is equal to the degree of the moment-map π_X:*

$$\operatorname{mtp}_p(S(\mathfrak{g})/\operatorname{gr} I_X) = \deg \pi_X.$$

5.9. If M is a $U(\mathfrak{g})$- or an $S(\mathfrak{g})$-module $\neq 0$ with a good filtration by vector-spaces $M_i (i \geqq -1)$ as in 4.1, then the dimensions $\dim M_i$ are given by the well-known *Hilbert-Samuel-polynomial* $q_M(T)$, that is $q_M(i) = \dim M_i$ at least for large i. The leading term of this polynomial has the form $\dfrac{e(M)}{d(M)!} T^{d(M)}$. This defines two *positive integers* $d(M)$, $e(M)$, called the (Gelfand-Kirillov-)*dimension* resp. the (Bernstein-)*multiplicity* of M (cf. [Be]). Both numbers $d(M)$, $e(M)$ turn out to be *independent* of the choice of a good filtration on M. While $d(M)$ is already determined by the variety $V(M)$ - as its dimension, $e(M)$ is not, but contains additional information on M. - It is clear that we have

$$d(M) = d(\operatorname{gr} M), \quad \text{and} \quad e(M) = e(\operatorname{gr} M).$$

5.10. Lemma. *Let M be a finitely generated $S(\mathfrak{g})$-module. Let $P=\sqrt{\operatorname{Ann} M}$ be prime. Then the p-multiplicity of M is given by*

$$\operatorname{mtp}_p(M)=e(M):\ e(S(\mathfrak{g})/p).$$

Proof. There exists a chain of submodules $0=N_0\subset N_1\subset\ldots\subset N_r=M$ and primes $p_1,\ldots,p_r$ such that

$$\bar{N}_i:=N_i/N_{i-1}\cong S(\mathfrak{g})/p_i \qquad \text{for } i=1,\ldots,r.$$

It is obvious that $\operatorname{mtp}_p(M)$ is the number of i such that $p_i=p$. For such i, we have $d(\bar{N}_i)=d(S(\mathfrak{g})/p)=d(M)$, while for the other i, we have $p_i\supsetneqq p$ and hence $d(\bar{N}_i)=d(S(\mathfrak{g})/p_i)<d(M)$. From the additivity properties of the Hilbert-Samuel-polynomial we conclude then that

$$\begin{aligned} e(M)&=\sum_{i,\,p_i=p} e(\bar{N}_i)=\#\{i\,|\,p_i=p\}\, e(S(\mathfrak{g})/p)\\ &=\operatorname{mtp}_p(M)\cdot e(S(\mathfrak{g})/p). \quad \text{Q.e.d.}\end{aligned}$$

5.11. Lemma. *Let B an integral $S(\mathfrak{g})$-algebra, which is finitely generated as an $S(\mathfrak{g})$-module. Let $A\cong S(\mathfrak{g})/p$ be the image of $S(\mathfrak{g})$ in B, and let $Q(B)$, $Q(A)$ denote the quotient-fields. Then the degree of the field-extension is given by*

$$[Q(B):Q(A)]=\operatorname{mtp}_p(B)=e(B):e(A).$$

Proof. Let $b_1,\ldots,b_d\in B$ be a $Q(A)$-basis for $Q(B)$. Then $M=Ab_1+\ldots+Ab_d$ is a free A-module of rank d, while B/M is an A-torsionmodule and has dimension $d(B/M)<d(A)$. We conclude that $e(B)=e(M)=d\cdot e(A)$. It is also clear that $d=\operatorname{mtp}_p(B)$. Q.e.d.

Proof of Theorem 5.8. The moment map π defines an embedding of $A:=S(\mathfrak{g})/p$ into $B:=R(T^*(X))$, where $p=\mathscr{I}(G\mathfrak{p}^\perp)$ is the ideal of functions that vanish on $G\mathfrak{p}^\perp$. From 5.11 we conclude that the degree of π is given by

$$\deg\pi=e(B):\ e(A). \tag{1}$$

On the other hand, we have

$$e(U(\mathfrak{g})/I_X)=e(D(X))=e(\operatorname{gr} D(X))=e(R(T^*(X))=e(B) \tag{2}$$

by 3.8 resp. 5.9 resp. 1.4. Again by 5.9, we have

$$e(S(\mathfrak{g})/\operatorname{gr} I_X)=e(U(\mathfrak{g})/I_X). \tag{3}$$

Since $p=\sqrt{\operatorname{gr} I_X}$ by 4.7, we may compute the p-multiplicity of $\operatorname{gr} I_X$ by Lemma 5.10:

$$\operatorname{mtp}_p(S(\mathfrak{g})/\operatorname{gr} I_X)=e(S(\mathfrak{g})/\operatorname{gr} I_X):e(S(\mathfrak{g})/p). \tag{4}$$

In view of (2), (3) this number is $e(B)$: $e(A)$, or $\deg\pi$ by (1). Hence (4) gives the theorem. Q.e.d.

5.12. Corollary (Notation 5.8). *The following are equivalent:*

(i) *The ideal* $\operatorname{gr} I_X$ *is reduced at all points of the dense orbit in* $G\mathfrak{p}^\perp$.

(ii) *The moment map is birational.*

(iii) *All G-orbits in the Dixmier-sheet determined by P have the same G-multiplicities.*

Proof. By the theorem, (ii) is equivalent to $\mathrm{mtp}_p(S(\mathfrak{g})/\mathrm{gr}\, I_X)=1$, and this means that $\mathrm{gr}\, I_X$ is reduced at a *generic* point of $G\mathfrak{p}$. But now $S(\mathfrak{g})/\mathrm{gr}\, I_X$ is a scheme on which G acts by automorphism. Hence $\mathrm{gr}\, I_X$ must be reduced at all points G-conjugate to a generic point, so (ii) and (iii) comes from [BK 2], as explained already in 5.5 a). Q.e.d.

Problems. 1. Assume $G\mathfrak{p}^{\perp}$ is normal. Is it true then that $\mathrm{gr}\, I_X$ is primary?

2. Compute the maximal delay of a vector-field or any operator in $D(X)$, for instance in Example 3.10, and give a geometric interpretation.

§6. Applications to the Study of Primitive Ideals

6.1. In this last chapter, $\mathfrak{g}$ is always complex semisimple, and is identified with $\mathfrak{g}^*$ (convention 2.3). We consider primitive ideals J in the enveloping algebra of $\mathfrak{g}$. An old outstanding conjecture says that their associated varieties $\mathscr{V}(\mathrm{gr}\, J)$ should be always irreducible, and consequently should be equal to the closure of some nilpotent orbit $\mathcal{O}_x$ in $\mathfrak{g}$,

$$\mathscr{V}(\mathrm{gr}\, J)=\bar{\mathcal{O}}_x \tag{$*$}$$

(see [B2], 2.9). We have seen in Chap. 4, why this conjecture is true for $\mathfrak{g}=\mathfrak{sl}_n$ (see 4.9, or [Jo2]). We are going to establish it now for the case where $\mathfrak{g}$ is classical and J has integral central character (see 6.5). It will be essential for our proof that we first make the conjecture more precise by specifying a candidate $\Omega(J)$ for the nilpotent orbit $\mathcal{O}_x$ in ($*$). The map Ω from primitive ideals to nilpotent orbits is obtained by composing *Springer's correspondence* from Weyl group representations to nilpotent orbits with *Joseph's correspondence* from primitive ideals to Weyl group representations. Let us begin with recalling these two basic notions in 6.2 resp. 6.4.

6.2. *Springer's Correspondence* [S2], [BM 1], [BM 2], [AvL]

In this chapter, the variety of all Borel subgroups of G is denoted $\mathscr{B}$. Recall that its moment map $\pi=\pi_{\mathscr{B}}$ identifies with Springer's resolution of $\mathcal{N}$, the variety of all nilpotent elements in $\mathfrak{g}=\mathfrak{g}^*$ (2.6). For any nilpotent orbit $\mathcal{O}_x$ with base point $x\in\mathcal{N}$, we denote by $C(x)=G_x/G_x^0$ the group of connected components of the centralizer of x, by $\mathscr{B}_x=\{B\in\mathscr{B}\,|\,x\in\mathrm{Lie}\,B\}\cong\pi^{-1}x$ the "Springer fibre" at x, by d_x the dimension of $\pi^{-1}x$, and by $V_x=H^{2d_x}(\pi^{-1}x,\mathbb{Q})$ the highest cohomology group of $\pi^{-1}x$. There is a $\mathbb{Q}$-linear action of the Weyl group W on V_x, which commutes with the $C(x)$-action. We denote by $\rho_x=\rho_{(x,1)}$ the representation of W on the $C(x)$-invariants $V_x^{C(x)}$ (which are $\neq 0$, since $C(x)$ acts by a permutation representation). It turns out that ρ_x is irreducible, and characterizes the orbit $\mathcal{O}_x$ (Springer). In other words, we have an injective map $\mathcal{O}_x\mapsto\rho_x$ from the set $\mathcal{N}/G$ of nilpotent orbits into the set $W^\wedge$ of (equivalence classes of) irreducible representations of W. Let us mention that the inverse map $\rho_x \mapsto \mathcal{O}_x$ extends to a surjection

$$W^\wedge \to \mathcal{N}/G,$$

which maps any $\sigma \in W^\wedge$ to the (unique!) orbit $\mathcal{O}_x$ such that σ occurs in V_x. This map is $\rho_{(x,\phi)} \mapsto \mathcal{O}_x$ in the notation of the [BM1], to which we refer for more details and references.

6.3. *Some Comments on Special Orbits*

Let us mention at this point that Lusztig [L] has defined a certain subset S_W in $W^\wedge$, called the set of *special representations*, which turns out to be very important for our purposes. It consists entirely of representations of the form $\rho_x = \rho_{(x,1)}$, and so is in bijection with a certain set of nilpotent orbits, called the *special orbits*. For example, all Richardson orbits are special. Consequently, for $G = \mathrm{SL}_n$ all nilpotent orbits are special, whereas e.g. $G = \mathrm{SO}_{13}$ has 35 nilpotent, 26 special and 24 Richardson orbits. Unfortunately, the notions of special representations or orbits are not yet well understood in general. Lusztig's definition [L] is rather technical. One would prefer a characterization of special orbits in geometrical terms, but this is still an open problem. However, they have been explicitly listed for exceptional G, and they have been explicitly determined in terms of combinatorics for classical G, see [L], [AvL]. We shall suggest below (6.12) some nice geometrical properties of special orbits in terms of [BM2], which have been verified for G classical in purely combinatorial terms by G. Kempken [Kk]. In the present paper the proof of our main result on associated varieties will depend on this result from [Kk], and Lusztig's definition of special orbits will be used only implicitly through this reference.

6.4. *Joseph's Correspondence*

We consider the set $\mathcal{X}_0$ of *primitive ideals* with central character equal to that of the trivial representation. We have a surjective map $W \to \mathcal{X}_0$, which is denoted $w \mapsto I_w$, and which results from well known theorems of Duflo [D] and Harish Chandra as follows. Using the notations of 3.7, we denote by

$$M_w = M_{\mathfrak{b}}(w(-\rho) - \rho)$$

the Verma module with highest weight $w(-\rho)-\rho$, and by L_w its unique simple quotient. Then I_w is defined as

$$I_w = \mathrm{Ann}_{U(\mathfrak{g})} L_w.$$

To each primitive ideal I_w, Joseph [Jo4] attaches a polynomial function p_w on $\mathfrak{t}^*$, homogeneous of degree

$$a_w := \dim \mathfrak{b}^\perp - \dim V(L_w),$$

which gives the values of the Goldie ranks of primitive ideals obtained from I_w by the "translation principle" (in the sense of [BJ]). Up to some nonzero scalar factor c_w, Joseph's Goldie rank polynomial p_w may be computed from the formal character of L_w as follows. In an appropriate Grothendieck group, we may write

$$L_w = \sum_{w' \in W} a_{w',w} M_{w'}$$

with integer coefficients $a_{w',w}$. Then

$$p_w(\mu) = c_w \sum_{w' \in W} a_{w',w} f(w'\mu)^{a_w} \quad \text{for all } \mu \in \mathfrak{t}^*$$

where f is a dominant regular element in $\mathfrak{t}$, considered as a linear form on $\mathfrak{t}^*$. The significance of Joseph's polynomials p_w for our purposes is that they *classify* the primitive ideals, i.e.

$$I_w = I_{w'} \quad \text{iff} \quad k p_w = k p_{w'} \quad (\text{for all } w, w' \in W).$$

Moreover, the cyclic W submodule generated by p_w turns out to be irreducible. This defines a map $\mathscr{X}_0 \to W^\wedge$, attaching to each primitive ideal I_w an irreducible W representation denoted $\sigma(I_w) = \sigma(w)$, called its *Joseph representation* (or Joseph's Goldie rank representation determined by I_w). We refer to this map as *Joseph's correspondence*. Its fibres define a partition of $\mathscr{X}_0$ into disjoint subsets corresponding to non equivalent representations. These subsets are called *clans* of primitive ideals[1]. The cardinality of such a clan is given by the dimension of the corresponding representation: In fact, the Goldie rank polynomials $p_{w'}$ of the various primitive ideals $I_{w'}$ corresponding to a representation $\cong \sigma(w)$ form a *basis* for $\sigma(w)$. After these basic results of Joseph, it is only left to determine the list of all representations $\sigma(w)(w \in W)$. It turns out that this subset of $W^\wedge$ coincides with the set S_w of "*special representations*", as defined by Lusztig (cf. 6.3). This result was verified by Barbasch-Vogan for all cases ([Ba V2], Theorem 2.29). It completes the classification of primitive ideals in $\mathscr{X}_0$: The total number for instance is given by

$$\# \mathscr{X}_0 = \sum_{\sigma \in S_W} \dim \sigma.$$

For G of type E_8 for example, this means that there are exactly 101796 primitive ideals with trivial central character.

6.5. *Statement of the Main Theorem*

Let us now compose Joseph's correspondence $\mathscr{X}_0 \to W^\wedge$ with Springer's correspondence $W^\wedge \to \mathscr{N}/G$ to obtain a map Ω from primitive ideals to nilpotent orbits. This map Ω attaches to the primitive ideal $I_w (w \in W)$ the orbit

$$\Omega(I_w) = \mathcal{O}_x$$

corresponding to the representation

$$\sigma(w) = \rho_x.$$

This is our desired candidate for the associated variety of I_w (cf. [B2], [Jo3], 7.4).

[1] This is suggested by terminology in abstract ring theory [Ja], [Mü]; detailed explanations will be given else where, since it does not matter here logically

Conjecture. *We have* $\mathscr{V}(\operatorname{gr} J)=\overline{\Omega(J)}$ *for all* $J\in\mathscr{X}_0$.

We are going to prove:

Theorem. *The conjecture is true for G classical.*

Remarks. a) It follows from the work of Barbasch and Vogan, that the inclusion $\mathscr{V}(\operatorname{gr} J)\supset\overline{\Omega(J)}$ is always true, with equality of dimension. (See [BaV3], Theorem 4.1, and [BaV1], Theorems 2ff. and 17; for a summary see also [BaV2], Theorem 4.8.)

b) Barbasch and Vogan use for this work the (non-algebraic) notion of "wave front sets", attached to certain representations by a procedure of Howe. This is used to define a G-invariant closed set $\mathscr{W}(J)$ satisfying

$$\overline{\Omega(J)}\subset\mathscr{W}(J)\subset\mathscr{V}(\operatorname{gr} J),$$

with equality of dimensions. Moreover, for G classical they verify in all cases that the left inclusion is an equality.

c) For the proof of our theorem, we shall use from this work of Barbasch-Vogan not the inclusion, but the equality of dimension stated in a).

In fact, it will suffice to know the following inequality:

Lemma. $\dim\mathscr{V}(\operatorname{gr} J)\leq\dim\overline{\Omega(J)}$ *for all* $J\in\mathscr{X}_0$.

This can be proved independently of the work of Barbasch-Vogan as follows. Retaining the notation introduced above, let $w\in W$ such that $J=I_w$, and let

$$\Omega(J)=\mathcal{O}_x \qquad \text{resp. } \sigma(J)=\sigma(w)=\rho_x$$

be the corresponding nilpotent orbit resp. Weyl group representation. Any irreducible representation η of W can be realized by homogeneous polynomials on the Cartan subalgebra $\mathfrak{t}$; hence we may attach to it as an important numerical invariant the smallest degree i for which η occurs with positive multiplicity in $S(\mathfrak{t})_i$. We call it the *smallest polynomial degree*, $\operatorname{spd}(\eta)$. The point of the proof is that the spd of a Springer representation relates to the dimension of the corresponding nilpotent orbit, while the spd of a Joseph representation relates in a similar way to the GK-dimension of the corresponding primitive ideal.

In more detail, we have as a general fact about the Springer correspondence that (Notation 6.2)

$$\operatorname{spd}(\rho_{(x,\varphi)})\geq\operatorname{spd}(\rho_x)=d_x=\dim\pi^{-1}x=\tfrac{1}{2}(\dim\mathscr{N}-\dim\mathcal{O}_x). \tag{1}$$

Here the first two relations follow from [BM1], Corollaire 4, and the last one from Steinberg [St]. On the other hand, we have from the work of Joseph on his correspondence that

$$\operatorname{spd}(\sigma(w))=\deg p_w=a_w=\dim\mathfrak{b}^{\perp}-\dim V(L_w)=\tfrac{1}{2}(\dim\mathscr{N}-\dim\mathscr{V}(\operatorname{gr} J)). \tag{2}$$

Here the first equation is contained in [Jo4], Theorem 5.5 of I, the next two were mentioned already in 6.4, and the last one is the well known fact about

GK-dimension that $\dim V(L_w)=\frac{1}{2}\dim \mathcal{V}(\operatorname{gr} I_w)$, combined with the triviality $\dim \mathcal{N}=2\dim \mathfrak{b}^{\perp}$. Combination of (1) and (2) gives $\dim \overline{\mathcal{O}_x}=\dim \mathcal{V}(\operatorname{gr} J)$, that is the desired equality of dimensions. Moreover, even without assuming the information that the Joseph representation $\sigma(w)$ takes the form ρ_x, i.e. the form $\rho_{(x,\varphi)}$ with $\varphi=1$, we still conclude from (1) and (2) the inequality stated in the lemma.

d) As pointed out to us by the reviewer, an alternative definition of the map Ω, in terms of analysis, follows from Hotta's report [H] of recent work of Kashiwara.

6.6. *Example.* Let $P\subset G$ be a parabolic subgroup, and R_p^+ the corresponding subsystem of positive roots (notation as in 3.7). Let w_P denote the longest element of the Weyl subgroup $W(P)$ in W determined by P. Consider the primitive ideal I_{w_P}. We claim that it is *induced* from $\mathfrak{p}=\operatorname{Lie} P$. In fact, the induced module $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}k_{w_P(-\rho)-\rho}$ is simple, say by [J], 1.17, or also by [CD], [W] (this is now also geometrically clear by [Br Ka], since the corresponding Schubert variety $\bar{X}_{w_P}$ is smooth). Hence this induced module is $=L_{w_P}$, and its annihilator is I_{w_P}.

Now we know from [Jo3], Theorem 10.5, that Joseph's Goldie rank polynomial of I_{w_P} is given up to a constant $c\neq 0$ by

$$p_{w_P}=c\prod\alpha, \quad \text{the product extended over } \alpha\in R_P^+.$$

This is a polynomial function on $\mathfrak{t}$, homogeneous of degree $\# R_P^+$, which transforms under $W(P)$ by the sign-character, denoted $\varepsilon_{W(P)}$. The cyclic W-module generated by this module in $S(\mathfrak{t}^*)$ is irreducible by an easy argument of Macdonald [Md]. Hence the Joseph representation $\sigma(w_P)$ corresponding to I_{w_P} is the "Macdonald representation" determined by P (or the representation obtained from $\varepsilon_{W(P)}$ by "truncated induction", in the terminology of [Ba V2]). By a result of Lusztig (see [HS], 1.4), this representation corresponds to the Richardson orbit determined by P under Springer's correspondence

$$\sigma(w_P)=\rho_y, \quad \text{where} \quad \bar{\mathcal{O}}_y=G\mathfrak{p}^{\perp}.$$

In conclusion, we obtain that $\Omega(I_{w_P})=\mathcal{O}_y$ is the Richardson orbit determined by P. On the other hand, since I_{w_P} is induced form $\mathfrak{p}$, we know from 4.7 that

$$\mathcal{V}(\operatorname{gr} I_{w_P})=G\mathfrak{p}^{\perp}=\overline{\Omega(I_{w_P})}$$

is in fact true.

6.7. Recall that the fibres of Ω (or of σ) are called *clans* of primitive ideals (6.4). We shall use the result of Joseph [Jo2], Lemma 2.6, that

$$\mathcal{V}(\operatorname{gr} J)=\mathcal{V}(\operatorname{gr} J') \quad \text{whenever} \quad \Omega(J)=\Omega(J'),$$

or in other words, that the associated variety is constant on clans.

Proposition. *Conjecture* 6.5 *holds whenever* $\Omega(J)$ *is Richardson.*

In fact, let $\overline{\Omega(J)}=G\mathfrak{p}^{\perp}$ be the closure of the Richardson orbit determined by P. We have seen in 6.6, that $\mathcal{V}(\operatorname{gr} J')=G\mathfrak{p}^{\perp}=\overline{\Omega(J')}$ is true for a particular primitive ideal $J' = I_{w_P}$ in the same clan as J. Hence the constancy on clans gives $\mathscr{V}(\operatorname{gr} J) = G\mathfrak{p}^{\perp} = \overline{\Omega(J)}$.

6.8. *Proof of Theorem* 6.5 (*reduction to* 6.9)

Let $J \in \mathscr{X}_0$ be given. From Barbasch-Vogan (see 6.4), we know that $\Omega(J) = \mathcal{O}_x$ is *special*. Since we assume G classical, each special orbit is either Richardson, or is given by an intersection of two Richardson orbit (closure)s, that is to say

$$\bar{\mathcal{O}}_x = \bar{\mathcal{O}}_{y_1} \cap \bar{\mathcal{O}}_{y_2}, \text{ with } \mathcal{O}_{y_i} \text{ Richardson } (i = 1, 2).$$

This fact was pointed out to us by N. Spaltenstein; for a proof see G. Kempken [Kk], 6.6. In the first case, we are done (6.7). In the second case, we chose (for $i = 1, 2$) primitive ideals J_i and J_i' such that $\Omega(J_i) = \mathcal{O}_{y_i}$, $\Omega(J_i') = \mathcal{O}_x$, and $J_i \subset J_i'$. We shall prove in 6.9–6.11 that this choice is possible. Using 6.7, we conclude that

$$\mathscr{V}(\operatorname{gr} J) = \mathscr{V}(\operatorname{gr} J_i') \subset \mathscr{V}(\operatorname{gr} J_i) = \overline{\Omega(J_i)} = \bar{\mathcal{O}}_{y_i}$$

for $i = 1, 2$, and hence that

$$\mathscr{V}(\operatorname{gr} J) \subset \bar{\mathcal{O}}_{y_1} \cap \bar{\mathcal{O}}_{y_2} = \bar{\mathcal{O}}_x = \overline{\Omega(J)}.$$

Since $\overline{\Omega(J)}$ is irreducible, the desired equality (6.5) follows from the equality of dimensions (cf. Remark 6.5c)).

Comments. This type of argument, which we shall refer to as the "inclusion method", was first applied by Barbasch and Vogan in some special situations in the course of their analogous considerations on "wave front sets" (see the proof of Lemma 24 in [BaV1]). However, Barbasch and Vogan had to use in addition a different type of argument, in both [BaV1], [BaV2], which we shall call the "induction method". This is based on their result that wave front sets behave well under induction of (even infinite dimensional) representations of Levi quotients of parabolic subalgebras. We are able to prove the analogous result on associated varieties (generalizing our Corollary 4.7 to infinite dimensional E). However, this needs more advanced methods than those developed here so far. We have to work with sheaves of "relative enveloping algebras". So we decided to treat this in a separate paper. However, let us mention already at this point that the induction method will suffice to prove conjecture 6.5 for the exceptional groups, because – as was kindly pointed out to us by N. Spaltenstein – for these groups all special orbits except one in type E_8 turn out to be either induced from a special orbit in a proper Levi subalgebra, or else are comparable with all other orbits. (The single exception in type E_8 can be settled by the inclusion method.)

6.9. *Ordering of Clans*

Let us denote by $\mathscr{C}_x$ the clan of primitive ideals corresponding to the nilpotent orbit $\mathcal{O}_x$, that is put

$$\mathscr{C}_x = \Omega^{-1} \mathcal{O}_x = \{J \in \mathscr{X}_0 \mid \Omega(J) = \mathcal{O}_x\}.$$

Recall the main result of Barbasch-Vogan stated in 6.4, which says that $\mathscr{C}_x \neq \emptyset$ *iff* $\mathcal{O}_x$ *is special.* We are interested in the following question: Given two special orbits $\mathcal{O}_x$, $\mathcal{O}_y$ such that $\bar{\mathcal{O}}_x \subset \bar{\mathcal{O}}_y$, do there exist $J_x \in \mathscr{C}_x$ and $J_y \in \mathscr{C}_y$ such that $J_x \supset J_y$? Let us write $\mathscr{C}_x \leqq \mathscr{C}_y$ if the answer is positive.

Conjecture. For $\mathcal{O}_x$, $\mathcal{O}_y$ *special, we have* $\mathscr{C}_x \leqq \mathscr{C}_y$ *iff* $\bar{\mathcal{O}}_x \subset \bar{\mathcal{O}}_y$.

In other words, the bijection between special orbits and clans of primitive ideals should be an order isomorphism. Of this conjecture, we can prove at least the following special case, which suffices to complete our argument in 6.8:

Proposition. *Let G classical with* $\mathcal{O}_x$ *special and* $\mathcal{O}_y$ *Richardson. Then* $\bar{\mathcal{O}}_x \subset \bar{\mathcal{O}}_y$ *implies* $\mathscr{C}_x \leqq \mathscr{C}_y$.

In fact, this is a corollary to the following results of D. Vogan and G. Kempken.

6.10. Let $P \subset G$ be a parabolic subgroup. Then we denote by $\varepsilon_{W(P)}$ the sign-character of the corresponding Weyl subgroup $W(P)$ in W. The induced representation of W is denoted $\varepsilon_{W(P)}^W$.

Proposition. *Let* $\mathcal{O}_y$ *be the Richardson orbit determined by P. Let* $\mathcal{O}_x \subset \bar{\mathcal{O}}_y$ *be a special orbit. Then*

a) (D. Vogan) $\mathscr{C}_x \leqq \mathscr{C}_y$ *iff the Springer representation* ρ_x *occurs in* $\varepsilon_{W(P)}^W$ *with multiplicity* $\operatorname{mtp}(\rho_x, \varepsilon_{W(P)}^W) \neq 0$.

b) (G. Kempken) [Kk], 6.7) *For G classical,* $\operatorname{mtp}(\rho_x, \varepsilon_{W(P)}^W) \neq 0$ *does always hold.*

We are grateful to David Vogan for pointing out to us his unpublished result a), and for kindly explaining to us in a letter, how it may be derived from the work of Joseph [Jo4]. However, for 6.9 we need here only the "if" part of a), and this follows also from the following more precise interpretation of the multiplicity $\operatorname{mtp}(\rho_x, \varepsilon_{W(P)}^W)$ in terms of numbers of primitive ideals, which we shall derive directly from [BaV2]. We shall do this next, in Sect. 6.11, in order to complete our proof of Theorem 6.5. Afterwards, we are going to discuss in 6.12 some reinterpretations in geometrical terms of these most interesting multiplicities.

6.11. *Conclusion of the Proof of Theorem* 6.5

We retain the notations of 6.10, and we recall from 6.6 that the clan $\mathscr{C}_y = \Omega^{-1}\mathcal{O}_y$ contains the primitive ideal I_{w_P}, which is *induced* from $\mathfrak{p} = \operatorname{Lie} P$. For the problem posed in 6.9, let us now make the particular choice $J_y = I_{w_P}$, and let us count the number of choices for J_x in the clan $\mathscr{C}_x$ which solve our problem, i.e. which contain J_y. We claim that this number of choices is given by

Proposition.

$$\#\{J \mid J \in \mathscr{C}_x, J \supset I_{w_P}\} = \operatorname{mtp}(\rho_x, \varepsilon_{W(P)}^W). \tag{1}$$

Proof. Let us first rewrite formula (1) in the terminology of Barbasch and Vogan [BaV2], following Joseph's work in [Jo5]. The *left cell* resp. *left cone* of an element $w \in W$ is the subset of W defined by

$$\mathscr{C}_w^L = \{w' \in W \mid I_w = I_{w'}\}$$

resp.

$$\bar{\mathscr{C}}_w^L = \{w' \in W \mid I_{w'} \supset I_w\},$$

see [Ba V2], Definition 2.10. Hence (1) may be rewritten

$$\#(\mathscr{C}_x \cap \bar{\mathscr{C}}_{w_P}^L) = \mathrm{mtp}(\rho_x, \varepsilon_{W(P)}^W). \tag{2}$$

Next the *left cone representation* resp. *the left cell representation* of an element $w \in W$ is defined to be the left W submodule of the group ring $\mathbb{C}[W]$ given by

$$\bar{V}_w^L = \bigoplus_{w'} \mathbb{C}\, L(w'\lambda), \qquad \text{the sum over } w' \in \bar{\mathscr{C}}_w^L$$

resp.

$$V_w^L = \bar{V}_w^L / \bigoplus_{w'} \mathbb{C}\, L(w'\lambda), \qquad \text{the sum over } w' \in \bar{\mathscr{C}}_w^L \setminus \mathscr{C}_w^L.$$

Here the group ring $\mathbb{C}[W]$ is identified with the set of formal complex linear combinations of the various $L(w'\lambda)$ for $w' \in W$ (as in loc. cit., Definition 2.8). It follows from [Ba V2], Proposition 3.15b), that the left cone representation of $w = w_P$ is given by

$$\bar{V}_w^L \cong \varepsilon_{W(P)}^W, \tag{3}$$

since $\varepsilon_{W(P)}$ is the representation of $W(P)$ corresponding to the zero orbit in a Levi subalgebra of $\mathfrak{p}$.

On the other hand, the left cone representation $\bar{V}_w^L$ is (up to isomorphism) the direct sum of all left cell representations $V_{w'}^L$, the summation ranging over all left cells $\mathscr{C}_{w'}$ contained in the left cone $\bar{\mathscr{C}}_{w_P}^L$. What we have to count is the number of those summands $V_{w'}^L$, such that $I_{w'}$ belongs to $\mathscr{C}_x$, or equivalently, such that $\sigma(w') = \rho_x$, because this is the number of left cells counted on the left hand side of (2). But from [Ba V2], Corollaries 2.15, 2.16 we see that each left cell representation $V_{w'}^L$ contains the Joseph representation $\sigma(w')$ with multiplicity 1, and contains no Joseph representations other than $\sigma(w')$. In conclusion, the desired number of left cells can be found just by counting how often ρ_x occurs in the left cone representation $\bar{V}_{w_P}^L$:

$$\#(\mathscr{C}_x \cap \bar{\mathscr{C}}_{w_P}^L) = \mathrm{mtp}(\rho_x, \bar{V}_{w_P}^L). \tag{4}$$

Combining (4) with (3) gives (2) and hence (1). This completes the proof of the proposition, and also the proof of Theorem 6.5.

6.12. *Reinterpretations in Terms of Geometry of Springer's Resolution*

Although we have finished now the proof of our main goal (Theorem 6.5), let us add here some geometrical background information from [S3], [BM2], which may throw some light onto the rather technical intermediate results which occured in the course of our proof. We denote by $\mathscr{P}$ the variety of all conjugates of our fixed parabolic subgroup P, and we call $\phi: \mathscr{B} \to \mathscr{P}$ the map sending a Borel subgroup onto the unique conjugate of P containing it. A

subvariety of $\mathscr{B}$ is called *of type P*, if it is a union of fibres of ϕ. Recall the notations of 6.2.

Proposition. *Let $\mathcal{O}_x$ be any nilpotent orbit. Then the number of $C(x)$ orbits of type P components of the Springer fibre $\mathscr{B}_x$ is given by* $\mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)})$.

Proof. This is only a reformulation of Springer's result [S3], Corollary 4.5, see also [BM2], 3.5. In fact, consider the moment map $\pi_{\mathscr{P}}$ of $\mathscr{P}$. In the terminology of [BM2], its fibres are the Spaltenstein varieties

$$\mathscr{P}^0_x := \{P' \in \mathscr{P} \mid x \in (\mathrm{Lie}\, P')^\perp\} \cong \pi^{-1}_{\mathscr{P}}(x),$$

and its image is (by 2.6) the closure of the Richardson orbit $\mathcal{O}_y$ determined by P. The dimension of $\mathscr{P}^0_x$ is $\leqq d_x - d_y$, and its $d_x - d_y$-dimensional irreducible components are exactly the images of the type P components of $\mathscr{B}_x$ under the map ϕ (which has fibres of dimension $d_y = \dim \mathscr{B} - \dim \mathscr{P}$). Hence to count components in $\mathscr{B}_x$ (which is known to be equidimensional) of type P amounts to the same as to count components in $\mathscr{P}^0_x$ of dimension $d_x - d_y$. A similar statement holds for $C(x)$ orbits. Now the proposition follows from Corollary 3.5b) in [BM2], which can be expressed by the following formula:

$$\dim H^{2d_x - 2d_y}(\mathscr{P}^0_x, \mathbb{Q})^{C(x)} = \mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}).$$

6.13. Corollary. *Let $\mathcal{O}_x$ be a special orbit, and $\mathcal{O}_y$ the Richardson orbit determined by P. Then the following statements are equivalent, and imply $\bar{\mathcal{O}}_x \subset \bar{\mathcal{O}}_y$:*

(i) *$\mathscr{B}_x$ has a component of type P.*

(ii) *$\mathscr{P}^0_x$ has dimension equal to $d_x - d_y \geqq 0$.*

(iii) $\mathrm{mtp}(\rho_x, \varepsilon^W_{W(P)}) \neq 0$.

(iv) $\mathscr{C}_x \leqq \mathscr{C}_y$.

Proof. The equivalence of (i), (ii) and (iii) is clear from 6.12, even without the assumption $\mathcal{O}_x$ special. The equivalence of (iii) and (iv) is Vogan's Proposition 6.10a). Finally, statement (ii) includes $\mathscr{P}^0_x$ is not empty, and this means that x belongs to the image of $\pi_{\mathscr{P}}$, which is $\bar{\mathcal{O}}_y$ (2.6). So (ii) implies $x \in \bar{\mathcal{O}}_y$, hence $\bar{\mathcal{O}}_x \subset \bar{\mathcal{O}}_y$. Q.e.d.

Comments. Another equivalent formulation for (ii) in the terminology of [BM2] is that

(v) *$\mathcal{O}_x$ is relevant for the moment map of $\mathscr{P}$.*

Recall (6.9) that we conjecture that the equivalent conditions, (i) to (v) *are always satisfied* for $\mathcal{O}_x$ a special orbit, and that G. Kempken has proved this conjecture for classical G by verifying condition (iii). This was verified also independently by N. Spaltenstein (private communication).[2] It should be possible to check it also for exceptional G using the tables of Alvis [Al].[3]

[2] According to Spaltenstein [Sp2], for classical G even the following converse is true: If (iii) holds for *all* parabolic subgroups P such that the (given) nilpotent orbit $\mathcal{O}_x$ is contained in the closure of the Richardson orbit determined by P, then $\mathcal{O}_x$ is necessarily special. In view of the equivalence of (iii) with (v), this gives a *geometric characterization* for special orbits of classical groups.

[3] *Added in Proof.* Meanwhile, this has also been done by Spaltenstein (private communication)

6.14. *Some Additional Remarks on the Classification of Primitive Ideals*

a) Joseph's main result on the classification of primitive ideals (cf. 6.4) is expressed by the equality $\#\mathscr{C}_x = \dim \rho_x$, i.e. the number of primitive ideals corresponding to a given nilpotent orbit is given by the dimension of the corresponding Springer representation. For $G = \mathrm{SL}_n$, this was known for some time as "Jantzen's conjecture" cf. [B2], 5.9 and [Jo1], [Jo2].

Observe that this result is contained in 6.11(1) as the special case where $P = B$ is a Borel subgroup. In fact, in this case $W(B)$ is 1, so $\varepsilon^W_{W(B)}$ is the regular representation of W, and the regular representation contains ρ_x with multiplicity $\dim \rho_x$. On the other hand, $w_B = 1$, so I_{w_B} is the minimal ideal in $\mathscr{X}_0$, or in other words, the left cone $\mathscr{C}^L_{w_B}$ is all of $\mathscr{X}_0$. So 6.11(1) for $P = B$ reduces just to Joseph's result $\#\mathscr{C}_x = \dim \rho_x$.

b) Similarly, Proposition 6.12 reduces for $P = B$ to the well known fact that the same number $\dim \rho_x$ does also give the number of $C(x)$-orbits of irreducible components of $\mathscr{B}_x$. It is easy to check that this is also equal to the number of irreducible components of $\pi^{-1}\mathcal{O}_x$, the preimage of the nilpotent orbit $\mathcal{O}_x$ under Springer's resolution [BM2].

c) More generally, combining 6.11(1) with 6.12 gives the following remarkable fact:

Corollary. *The number of primitive ideals J containing the induced ideal I_{w_P}, which correspond to a specified special orbit $\mathcal{O}_x$, is equal to the number of irreducible components of the preimage $\pi_{\mathscr{P}}^{-1}\mathcal{O}_x$ of this orbit under the moment map of $\mathscr{P}$.*

It was this remarkable coincidence of numbers, which suggested to us that there might be some direct relation between primitive ideals with orbit $\mathcal{O}_x$ on one hand, and the components of $\pi^{-1}\mathcal{O}_x$ on the other hand. Meanwhile, we have found such a direct relation, which at least for $G = \mathrm{SL}_n$ gives a satisfactory geometrical explanation for the striking coincidences of numbers stated above. In fact, we know how to define for a primitive ideal $J \in \mathscr{X}_0$ a *characteristic variety* in the cotangent bundle $T^*(\mathscr{B})$, which maps onto the associated variety, and which establishes such a relation (at least for $G = \mathrm{SL}_n$; in general it seems that one has to consider rather "characteristic cycles" – with multiplicities). We intend to develop these new ideas, which should contribute to a geometrical explanation for the results on classification of primitive ideals, in a subsequent paper.

6.15. *Example* SO_{13}

Let us conclude with illustrating the results of the present chapter by some example. We chose $G = \mathrm{SO}_{13}$, of type B_6, since it is the first example which is delicate enough to involve the type of problems considered in 6.9ff.

The Lie algebra $\mathfrak{g} = \mathfrak{so}_{13}$ has 35 nilpotent orbits, of dimensions up to 72. Only 26 of them turn out to be special, and are listed in diagram 1, which shows their inclusion relations. Hence $\mathscr{X}_0$ splits into 26 clans of primitive ideals J. Let us verify Theorem 6.5, the irreducibility of $\mathscr{V}(\mathrm{gr}\, J)$, for this case explicitly, clan by clan. Since 24 of the 26 special orbits turn out to be even Richardson, the corresponding clans are easily settled, using 4.7 (see 6.6–6.7). The remaining two special orbits (encircled in the diagram) have dimensions 50 resp. 36.

Consider for instance the 36-dimensional special orbit $\mathcal{O}_x$ which is not Richardson, and let us make explicit how this case is settled by the inclusion method (as explained in 6.8). Let P_1 resp. P_2 be parabolic subgroups with Levi groups of type A_5 resp. $B_4 \times A_1$. They determine Richardson orbits $\mathcal{O}_{y_1}$ resp. $\mathcal{O}_{y_2}$ of dimension 42 resp. 38 such that $\bar{\mathcal{O}}_x = \bar{\mathcal{O}}_{y_1} \cap \bar{\mathcal{O}}_{y_2}$. Now the inclusion method (6.8) will work to give $\mathcal{V}(\mathrm{gr} J) = \bar{\mathcal{O}}_x$ for J in $\mathscr{C}_x$, as soon as we can

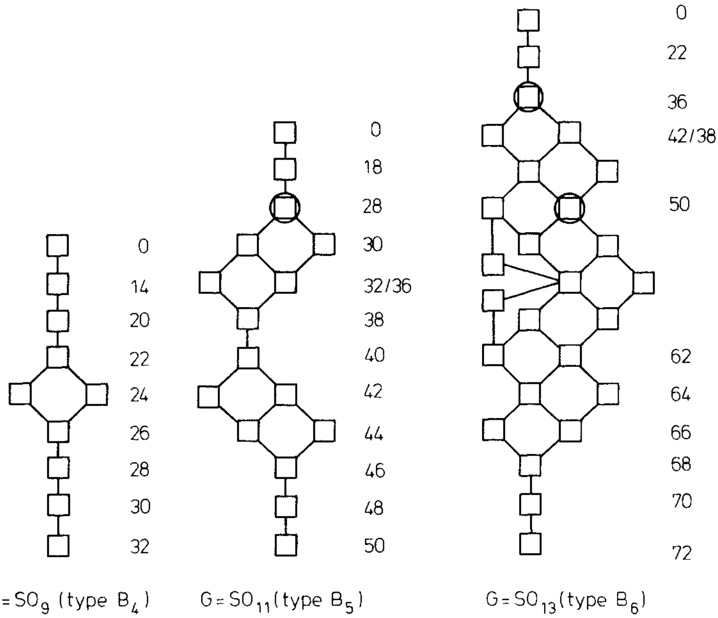

Diagram 1. *Inclusion diagram for the set of special orbits* in some classical groups, or also the order-diagram for the set of clans of primitive ideals. The numbers on the right are the orbit dimensions (resp. GK-dimensions of primitive ideals). Most orbits shown are Richardson; the three exceptions are encircled

verify $\mathscr{C}_x \leqq \mathscr{C}_{y_i}$ for $i=1,2$. This is done in general by Kempken and Vogan (6.10); for the particular case at hand, one may also proceed as follows: Recall that there is an order-reversing involution $\mathscr{X}_0 \mapsto \mathscr{X}_0$, which is denoted by $J \mapsto J'$, and which is given by $I'_w = I_{ww_G}$, see [BaV2], 2.24. Then the clans $\mathscr{C}'_x, \mathscr{C}'_{y_1}$ resp. $\mathscr{C}'_{y_2}$ correspond to Richardson orbits of dimensions 68, 66 resp. 66, determined by parabolics of Levi types $A_1 \times A_1$ resp. $A_1 \times A_1 \times A_1$ resp. A_2. For these it is easy to verify $\mathscr{C}'_x \geqq \mathscr{C}'_{y_i}$ for $i=1,2$, and the desired result follows by applying the order reversing involution.

For the only special orbit which is left to be considered now, the 50-dimensional one, the inclusion method works similarly. Let us mention that in this case (but not in the preceding one), we may also apply the "induction method" (cf. comments to 6.8) as follows: This particular orbit is induced from a 28-dimensional one in $\mathfrak{so}_{11}$. Since all the other nilpotent orbits in $\mathfrak{so}_{11}$ are comparable with this one, the irreducibility Theorem (6.5) is trivial for the corresponding clan in case $\mathfrak{so}_{11}$. Hence the desired result in case $\mathfrak{so}_{13}$ will follow from the general result (to be proved elsewhere) that irreducibility of associated varieties is preserved under parabolic induction.

PILE

Et maintenant notre noble navire s'élance à toute vitesse sur les sombres lames de la mer du Nord.

PÈRE UBU

Mer farouche et inhospitalière qui baigne le pays appelé Germanie, ainsi nommé parce que les habitants de ce pays sont tous cousins germains.

MÈRE UBU

Voilà ce que j'appelle de l'érudition. On dit ce pays fort beau.

PÈRE UBU

Ah! messieurs! si beau qu'il soit il ne vaut pas la Pologne. S'il n'y avait pas de Pologne, il n'y aurait pas de Polonais!

(Alfred Jarry, *Ubu Roi*)

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