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NONHOLONOMIC MECHANICS

Conservation Laws, Hierarchy of Dynamics and Explicit Integration of Nonholonomic Systems

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Abstract—This paper can be regarded as a continuation of our previous work [1, 2] on the hierarchy of the dynamical behavior of nonholonomic systems. We consider different mechanical systems with nonholonomic constraints; in particular, we examine the existence of tensor invariants (laws of conservation) and their connection with the behavior of a system. Considerable attention is given to the possibility of conformally Hamiltonian representation of the equations of motion, which is mainly used for the integration of the considered systems.

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1. EQUATIONS OF MOTION OF NONHOLONOMIC SYSTEMS

1.1. Nonholonomic Constraints and Ferrers Equations

Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ be generalized coordinates on the configuration space of a system $\mathcal{M}$ (the space of states etc.). In the general case, the equations of (nonholonomic) constraints have the form

$$f_\mu(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0, \quad \mu = 1, \ldots, m < n. \tag{1.1}$$

Roughly speaking, the main distinction of nonholonomic constraints from holonomic is the fact that tay cannot be represented in the finite (integral) form

$$F_\mu(\boldsymbol{q}, t) = 0, \quad \mu = 1, \ldots, \overline{m} < n.$$

A criterion that allows one to verify the nonholonomy of constraints 1.1 can be obtained by using the Frobenius theorem. A more detailed discussion of this problem in the application to a concrete example of a nonholonomic system will be presented below.

As a rule, constraints considered in the nonholonomic mechanics are linear in generalized velocities and do not depend explicitly on time:

$$f_\mu(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_k a_{\mu k}(\boldsymbol{q})\dot{q}_k + b_\mu(\boldsymbol{q}) = 0. \tag{1.2}$$

Exactly these constraints are realized in substantial problems. However, Appell and Hamel stated a rather artificial example of a nonlinear nonholonomic constraint. In what follows, we consider only constraints of the form (1.2).

The Ferrers equations are the historically first form of the equations of nonholonomic mechanics[1)] with undetermined multipliers (1872) [3]

$$\frac{d}{dt}\Big(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\Big) - \frac{\partial T}{\partial \boldsymbol{q}} = \boldsymbol{Q} + \sum_\mu \lambda_\mu \frac{\partial f_\mu}{\partial \dot{\boldsymbol{q}}}, \tag{1.3}$$

where $\frac{\partial T}{\partial \boldsymbol{q}} = \Big(\frac{\partial T}{\partial q_1}, \ldots, \frac{\partial T}{\partial q_n}\Big)$, $\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = \Big(\frac{\partial T}{\partial \dot{q}_1}, \ldots, \frac{\partial T}{\partial \dot{q}_n}\Big)$. In Eqs. (1.3), T is the kinetic energy, $\boldsymbol{Q}$ is the generalized force, and λ_μ are undetermined multipliers that can be uniquely found from the constraint conditions (1.2).

For example, if the kinetic energy is a homogeneous quadratic form of generalized velocities

$$T = \frac{1}{2}(\dot{\boldsymbol{q}}, \mathbf{A}(\boldsymbol{q})\dot{\boldsymbol{q}}), \tag{1.4}$$

where $\mathbf{A}(\dot{\boldsymbol{q}})$ is a symmetric matrix, then the undetermined multipliers satisfy the system of linear equations of the form

$$\sum_\nu \left(\frac{\partial f_\mu}{\partial \dot{\boldsymbol{q}}}, \mathbf{A}^{-1}\frac{\partial f_\nu}{\partial \dot{\boldsymbol{q}}}\right)\lambda_\nu = \left(\mathbf{A}^{-1}\frac{\partial f_\mu}{\partial \dot{\boldsymbol{q}}}, \dot{\mathbf{A}}\dot{\boldsymbol{q}} - \frac{\partial T}{\partial \boldsymbol{q}} - \boldsymbol{Q}\right) - \left(\dot{\boldsymbol{q}}, \frac{\partial f_\mu}{\partial \boldsymbol{q}}\right), \quad \mu = 1 \ldots m. \tag{1.5}$$

Here and in what follows, $(\cdot, \cdot)$ is the inner product of vectors.

Solving this system, we obtain the undetermined multipliers as functions of generalized coordinates and velocities: $\lambda_\mu = \lambda_\mu(\dot{\boldsymbol{q}}, \boldsymbol{q})$. Substituting these multipliers in Eqs. (1.3), we obtain a closed system; moreover, Eqs. (1.2) define an invariant variety of this system.

Remark 1. The proof of formula (1.5) consists of the differentiation of Eqs. (1.2):

$$\left(\frac{\partial f_\mu}{\partial \boldsymbol{q}}, \dot{\boldsymbol{q}}\right) + \left(\frac{\partial f_\mu}{\partial \dot{\boldsymbol{q}}}, \ddot{\boldsymbol{q}}\right) = 0$$

and the substitution of accelerations $\ddot{\boldsymbol{q}}$ from Eqs. (1.3).

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1) Ferrers considered only Cartesian coordinates in deriving the equations.

Ferrers also eliminated the undetermined multipliers and obtained an analogue of Lagrange equations of motion [3]. Except for the Ferrers equations, nonholonomic mechanics also deals with equations of Appell, Chaplygin, Maggi, Volterra, Voronets, Boltzmann–Hamel (the Boltzmann–Hamel equations are sometimes called the Euler–Lagrange equations); see, e.g., [4, 5] and the references therein. All these forms are related to different ways of elimination of undetermined multipliers and are rarely used in practice. In the derivation of concrete equations of motion, for example, for rolling bodies, one usually uses general dynamical equations containing reactions or universal equations of the form (1.3).

Remark 2. As an interesting fact, we note that S.A. Chaplygin himself, who obtained the general form of the equations of nonholonomic mechanics, which are called now the Chaplygin equations, did not use these equations but relied on the general principles of mechanics. On the other hand, P.V. Voronets, a noted Kiev mechanician, who also obtained general dynamical nonholonomic equations (which, by the way, have very cumbersome form), repeatedly used these equations in the study of concrete systems [6–11]. In many cases, this leads to unreasonable complication in the analysis.

If the forces acting on a system are potential, then Eqs. (1.3) can be represented in the form

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^{\cdot} - \frac{\partial L}{\partial \boldsymbol{q}} = \sum_\mu \lambda_\mu \frac{\partial f_\mu}{\partial \dot{\boldsymbol{q}}}, \quad L = T - U, \tag{1.6}$$

where L is the Lagrange function of the system without constraints and T and U are the kinetic and potential energies, respectively.

1.2. Poincaré–Suslov Equations

We present another form of the equations of motion, which is convenient for the derivation of equations of nonholonomic systems that appear in practical examples.

For the configuration space $\mathcal{M}$, we define the non-coordinate basis of vector fields by the formula

$$\boldsymbol{E}_i = \sum_j G_{ji}(\boldsymbol{q})\boldsymbol{e}_i, \quad \boldsymbol{e}_i = \frac{\partial}{\partial \boldsymbol{q}_i}, \quad i = 1, \ldots, n. \tag{1.7}$$

Using (1.7), we obtain for velocities, constraints, and the Lagrange function of the system

$$\dot{q}_i = \sum_j G_{ij}(\boldsymbol{q}) w_j, \quad f_\mu(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \hat{f}_\mu(\boldsymbol{q}, \boldsymbol{w}) = 0,$$

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \hat{L}(\boldsymbol{q}, \boldsymbol{w}),$$

where $\boldsymbol{w} = (w_1, \ldots, w_n)$ are the components of the velocity of the system in the basis $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_n$.The quantities w_i are sometimes called the Poincaré parameters.

In the general case, the commutators of vector fields (1.7) are represented in the form

$$[\boldsymbol{E}_i, \boldsymbol{E}_j] = \sum_k c_{ij}^k(\boldsymbol{q})\boldsymbol{E}_k. \tag{1.8}$$

Proposition 1. *The equations of motion of the nonholonomic system* (1.6) *are equivalent to the (first-order) system of the form*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \hat{L}}{\partial w_i}\right) - \displaystyle\sum_{i,k} c_{ji}^k(\boldsymbol{q}) \frac{\partial \hat{L}}{\partial w_k} w_j - \boldsymbol{E}_i(\hat{L}) = \sum_\mu \hat{\lambda}_\mu \frac{\partial f_\mu}{\partial w_i}, \\ \dot{\boldsymbol{q}}_i = \displaystyle\sum_j G_{ij}(\boldsymbol{q}_i) w_j, \end{cases} \tag{1.9}$$

where $\hat{\lambda}_\mu$ are undetermined multipliers and $\boldsymbol{E}_i(\hat{L}) = \sum\limits_j G_{ji}(\boldsymbol{q}) \frac{\partial \hat{L}}{\partial q_j}$.

The proof can be found, e.g., in [12].

Remark 3. Equations (1.9) are the most general form of the D'Alembert–Lagrange equations of the second kind; if we set $G_{ij} = \delta_{ij}$ in (1.7) we obtain usual Eqs. (1.6).

1.3. A Heavy Body on the Plane without Slipping and Spinning (Rubber Body)

We demonstrate the derivation of the equations of a nonholonomic system by the example of the motion of a heavy rigid body, which touches an immovable horizontal plane at one point such that at the touching point, the conditions of absence of slipping and spinning hold.

Mechanical realizations of such constraints by various hinges and pendants are discussed in details in [13]; namely, we assume that the body is covered by a layer of soft rubber, which provides the appropriate contact with the plane. For brevity, in what follows, this system will be called the *rubber body on the plane*. In addition, we will consider the rolling of bodies, when only slipping is absent (i.e., the spinning is allowed by constraints); this system will be called the *smooth, or marble body* (the term is suggested in [14]).

Configuration space and kinematic relations. The configuration space of the system considered is the product $\mathcal{M} = \mathbb{R}^3 \otimes SO(3)$, here the first factor describes the position of the center of mass of the body and the second — the orientation of the body. We introduce the following two coordinate systems (see Fig. 1):

- *immovable system* $OXYZ$, whose origin O is located at some point of the plane and the OZ-axis is perpendicular to the plane;
- *movable system* $Cxyz$, whose origin C is located at the center of mass of the body and the axes are directed along the principal axes of inertia of the body.

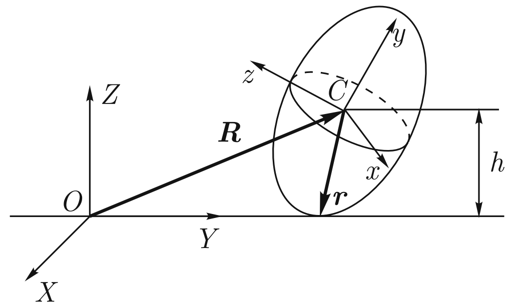

Fig. 1. Body on a plane.

Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be the projections of the orths of the immovable space (i.e., the unit vectors of $OXYZ$) on the movable axes $Cxyz$ and $\boldsymbol{R} = (R_1, R_2, R_3)$ be the coordinates of the center of mass of the body in the system $OXYZ$. Introducing the orthogonal matrix

$$\boldsymbol{Q} = \begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{Vmatrix} \in SO(3), \tag{1.10}$$

we see that the pair $(\boldsymbol{R}, \mathbf{Q}) \in \mathbb{R}^3 \otimes SO(3)$ uniquely determines the position of the body.

Remark 4. Hence, we have defined some coordinates on the configuration space, although $\alpha_i, \beta_i, \gamma_i$ define redundant coordinates on $SO(3)$.

Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ be projections of the angular velocity and the velocity of the center of mass of the body on the movable axes $Oxyz$, respectively. In the corresponding basis of the vector fields (1.7) we have

$$\begin{gathered}\boldsymbol{\omega} = \sum_k \omega_k \xi_k, \quad \boldsymbol{\xi}_k = -\sum_{i,j} \varepsilon_{kij} \left(\alpha_i \frac{\partial}{\partial \alpha_j} + \beta_i \frac{\partial}{\partial \beta_j} + \gamma_i \frac{\partial}{\partial \gamma_j} \right), \\ \boldsymbol{v} = \sum_k v_k \zeta_k, \quad \boldsymbol{\zeta}_k = \alpha_i \frac{\partial}{\partial R_1} + \beta_i \frac{\partial}{\partial R_2} + \gamma_i \frac{\partial}{\partial R_3},\end{gathered} \tag{1.11}$$

the corresponding commutators (1.8) have the form

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \varepsilon_{ijk} \boldsymbol{\xi}_k, \quad [\boldsymbol{\xi}_i, \boldsymbol{\zeta}_j] = \varepsilon_{ijk} \boldsymbol{\zeta}_k, \quad [\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j] = 0. \tag{1.12}$$

Moreover, the following kinematic relations hold:

$$\dot{\mathbf{Q}} = \tilde{\boldsymbol{\omega}} \mathbf{Q}, \quad \dot{\boldsymbol{R}} = \mathbf{Q}^{-1} \boldsymbol{v}, \tag{1.13}$$

where $\tilde{\boldsymbol{\omega}} = \|\tilde{\omega}_{ij}\|$ is the skew-symmetric matrix whose entries are identified with the angular velocities by the usual rule

$$\tilde{\omega}_{ij} = \varepsilon_{ijk} \omega_k.$$

Equations of constraints. Since the slipping at the contact points is absent, its velocity vanishes, i.e.,

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0, \tag{1.14}$$

where $\boldsymbol{r}$ is the vector joining the center of mass with the contact point, also projected on the movable axes.

If the body is elsewhere convex and $f(\boldsymbol{r}) = 0$ is the equation of its surface, then the vector $\boldsymbol{r}$ can be uniquely expressed through the vector $\boldsymbol{\gamma}$ by the inverting the Gauss mapping

$$\boldsymbol{\gamma} = -\frac{\nabla f(\boldsymbol{r})}{|\nabla f(\boldsymbol{r})|}.$$

(This equation expresses the fact that the normal of the surface of the body coincides with the normal of the plane.) In the sequel, we assume that $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\gamma})$ is a given function.

The absence of the spinning means that the projection of the angular velocity on the normal of the plane vanishes:

$$(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0. \tag{1.15}$$

Therefore, the equations of constraints are defined by relations (1.14) and (1.15); these relations imply that the constraints depend only on the vectors $\boldsymbol{v}$, $\boldsymbol{\omega}$, and $\boldsymbol{\gamma}$.

Note that this simple form of the equations of constraints is due to the special choice of the basis of vector fields (1.11); for example, in the Euler angles the constraints have a cumbersome form.

The Lagrange function of a free rigid body has the form

$$L = T - U.$$

The kinetic energy in the coordinate system $Cxyz$ has the form

$$T = \frac{1}{2} m \boldsymbol{v}^2 + \frac{1}{2} (\boldsymbol{\omega}, \mathbf{I} \boldsymbol{\omega}),$$

where m is the mass of the body and $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the tensor of inertia.

The potential energy in the gravity field can be represented in the form (see Fig. 1)

$$U = mgh = -mg(\boldsymbol{r}, \boldsymbol{\gamma}), \tag{1.16}$$

where g is the free fall acceleration. Since $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\gamma})$, the potential U depends only on $\boldsymbol{\gamma}$ (i.e., $U = U(\boldsymbol{\gamma})$).

Equations of motion with undetermined multipliers. Writing Eqs. (1.9) taking (1.12) and (1.13) into account, we obtain the system, which describes motions of a heavy rigid body on the plane with constraints (1.14) and (1.15):

$$\begin{gathered} m\dot{\boldsymbol{v}} = m\boldsymbol{v}\times\boldsymbol{\omega}+\boldsymbol{\lambda}, \quad \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega}\times\boldsymbol{\omega} - \frac{\partial U}{\partial \boldsymbol{\gamma}}\times\boldsymbol{\gamma}+\boldsymbol{r}\times\boldsymbol{\lambda}+\lambda_0\boldsymbol{\gamma}, \\ \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\times\boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta}\times\boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}, \\ \dot{R}_1 = (\boldsymbol{\alpha},\boldsymbol{v}), \quad \dot{R}_2 = (\boldsymbol{\beta},\boldsymbol{v}), \quad \dot{R}_3 = (\boldsymbol{\gamma},\boldsymbol{v}), \end{gathered} \tag{1.17}$$

where $\lambda_0, \boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3)$ are undetermined multipliers.

Elimination of multipliers and reduction. We differentiate constraint (1.14) and find $\boldsymbol{\lambda}$ from the first Eq. (1.17):

$$\boldsymbol{\lambda} = m\dot{\boldsymbol{\omega}}\times\boldsymbol{r}+m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}+m(\boldsymbol{r}\times\boldsymbol{\omega})\times\boldsymbol{\omega}.$$

Substituting $\boldsymbol{\lambda}$ just found in the second Eq. (1.17) and taking into account the fact that it is independent of $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{R}$, we obtain the closed system, which describes the evolution of the vectors $\boldsymbol{\omega},\boldsymbol{\gamma}$ in the form

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega}\times\boldsymbol{\omega} - m\boldsymbol{r}\times(\boldsymbol{\omega}\times\dot{\boldsymbol{r}})+\boldsymbol{\gamma}\times\frac{\partial U}{\partial\boldsymbol{\gamma}}+\lambda_0\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}, \tag{1.18}$$

where $\tilde{\mathbf{I}} = \mathbf{I}+m\boldsymbol{r}^2\mathbf{E}-m\boldsymbol{r}\otimes\boldsymbol{r}$ is the tensor of inertia with respect to the contact point satisfying the relation $\tilde{\mathbf{I}}\boldsymbol{\omega} = \mathbf{I}\boldsymbol{\omega}+m\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ and the undetermined multiplier $\lambda_0 = \lambda_0(\boldsymbol{\omega},\boldsymbol{\lambda})$ can be found from the constraint equation (1.15):

$$\lambda_0 = -\frac{\left(\tilde{\mathbf{I}}^{-1}\boldsymbol{\gamma}, \tilde{\mathbf{I}}\boldsymbol{\omega}\times\boldsymbol{\omega} - m\boldsymbol{r}\times(\boldsymbol{\omega}\times\dot{\boldsymbol{r}})+\boldsymbol{\gamma}\times\frac{\partial U}{\partial\boldsymbol{\gamma}}\right)}{(\boldsymbol{\gamma},\tilde{\mathbf{I}}^{-1}\boldsymbol{\gamma})}. \tag{1.19}$$

For given $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ (that are found from Eq. (1.18)), we can find $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{R}$ from (1.17) by quadratures.

It is easy to show that if one parametrizes the position of the body (i.e., the vectors$\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$) by the Euler angles θ,φ,ψ, Eqs. (1.18) describe only the evolution of the angles θ and φ. In other words, in the transition from the general system (1.17) to system (1.18), in addition to the elimination of undetermined multipliers, the reduction with respect to the precession angle ψ and the motion of the center of mass $\boldsymbol{R}$ also occur.

In the sequel, in the study of the dynamics of rubber bodies and similar systems, we, as a rule, restrict ourselves to the analysis of the reduced system, which describes the behavior of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$.

1.4. Equations of Motion of Some Systems in Nonholonomic Mechanics

In this section, we present (without derivation) the equations of motion of nonholonomic systems, which can be studied below: the problem on the rolling of a Chaplygin ball, the Veselova system, and the problem on the motion of a rubber body on a sphere. Note that the equations of motion of the first two systems can be obtained as particular cases of system (1.18).

Chaplygin ball [15] is the system describing the rolling without slipping (but with spinning) of a ball on the horizontal plane. Setting in this case $\lambda_0 = 0$ and $\boldsymbol{r} = -a\boldsymbol{\gamma}$ in (1.17), where a is the radius of the ball, we obtain ther equations, which describe the evolution of $\boldsymbol{\omega},\boldsymbol{\gamma}$:

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega}\times\boldsymbol{\omega} - ma^2\boldsymbol{\gamma}\times(\boldsymbol{\omega}\times\dot{\boldsymbol{\gamma}})+\boldsymbol{\gamma}\times\frac{\partial U}{\partial\boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}, \tag{1.20}$$

where, in our case, $\tilde{\mathbf{I}} = \mathbf{I}+ma^2\mathbf{E}-ma^2\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}$. $\tilde{\mathbf{I}} = \mathbf{I}+ma^2\mathbf{E}-ma^2\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}$.

We see that one can assume that only forces depending on the normal of the plane act on the ball.

Veselova's system [16] describes the motion of a rigid body with a fixed point under constraint (1.15).

Setting $\boldsymbol{r} = 0$ and $\boldsymbol{\lambda} = 0$ in (1.17), we have

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}} + \lambda_0 \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{1.21}$$

where

$$\lambda_0 = -\frac{\left(\mathbf{I}^{-1}\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}\right)}{(\mathbf{I}^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})}. \tag{1.22}$$

Rubber body on a sphere [14]. If a rubber body moves on the sphere of radius a, then the equations presented in the previous section must be modified. However, it can be shown that if we denote by $\boldsymbol{\gamma}$ the normal of the sphere at the contact point (projected on the movable axes, see Fig. 2), then the equations describing the evolution of the vector $\boldsymbol{\omega}$ remain the same as in system (1.18). In this case, the equations describing the motion of the vector $\boldsymbol{\gamma}$ can be obtained from the condition that the velocities of the contact point on the sphere and on the body coincide. Finally, we obtain

$$\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\omega} - m\boldsymbol{r} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}} + \lambda_0 \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \frac{1}{a}\dot{\boldsymbol{r}}. \tag{1.23}$$

The tensor $\tilde{\mathbf{I}}$, the vector $(\boldsymbol{\gamma})$, and the undetermined multiplier λ_0 are defined by the same relations as on the plane. As $a \to \infty$, system (1.23) becomes system (1.18) on the plane. We see that Eqs. (1.23) constitute a closed system only in the case where the potential of external forces depends only on the normal of the sphere $\boldsymbol{\gamma}$.

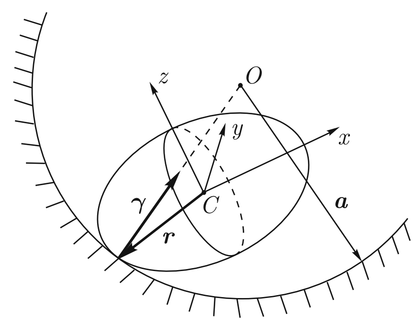

Fig. 2. Body on a sphere.

2. NONHOLONOMY OF CONSTRAINTS AND SOME ISOMORPHISMS

In this section, based on an example, we consider a method of proving the nonholonomy of constraints by using the Frobenius theorem and its generalizations. We also present examples of nonholonomic systems in rigid-body dynamics that are equivalent to some holonomic systems in the dynamics of systems of particles and, in concluding the section, specify an isomorphism of two nonholonomic systems.

2.1. Frobenius Theorem and Its Generalizations and Nonholonomy of Equations of Constraints

The verification of the nonholonomy of constraints by using the Frobenius theorem and its generalizations is considered here on the example of linear and homogeneous in velocities constraints that are given in the coordinates w_i for the non-coordinate basis of vector fields $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_n$ (1.7):

$$\sum_i \hat{a}_{\mu i}(\boldsymbol{q}) w_i = 0, \quad \mu = 1, \ldots, m. \tag{2.1}$$

We restrict ourselves to the description of possible (not the most general) situations without detailed proofs; proofs and generalizations can be obtained by standard methods of differential geometry. Assuming that all constraints (2.1) are independent, we perform the following two operations.

1) Find $k = n - m$ linearly independent solutions of system (2.1) $\boldsymbol{w}^{(s)} = (w_1^{(s)}, \dots, w_n^{(s)})$ can construct (in the explicit form) the basis of admissible vector fields on $\mathcal{M}$:

$$\boldsymbol{\eta}_s = \sum_i w_i^{(s)} \boldsymbol{E}_i, \quad s = 1, \dots, k. \tag{2.2}$$

Hence we have constructed a distribution on $\mathcal{M}$ defined by constraints (2.1). (2.1).

2) At any point $\boldsymbol{q} \in \mathcal{M}$, we construct the linear span $\mathcal{L}$ of the vector fields $\boldsymbol{\eta}_s$ and their various commutators $[\boldsymbol{\eta}_i, \boldsymbol{\eta}_j], \dots, [[\boldsymbol{\eta}_i, \boldsymbol{\eta}_j], \boldsymbol{\eta}_k], \dots$ etc. and calculate its dimension $\dim \mathcal{L}$.

Depending on $\dim \mathcal{L}$ the following situations are possible:

- *if* $\dim \mathcal{L} = \dim \mathcal{M}$*, then the system of constraints* (2.1) *is completely holonomic* and, by the Rashevsky–Chow theorem [17, 18], for any pair of points in $\mathcal{M}$, there exists an admissible path joining these points;
- *if* $\dim \mathcal{L} = k$*, then the system of constraints* (2.1) *is holonomic* and, by the Frobenius theorem, constraints can be represented in the form $F_1(\boldsymbol{q}) = 0, \dots, F_k(\boldsymbol{q}) = 0$;
- *if* $k < \dim \mathcal{L} < \dim \mathcal{M}$*, then the system is nonholonomic,* but, by the Frobenius theorem, there exist $\overline{k} = \dim \mathcal{M} - \dim \mathcal{L}$ holonomic constraints that define the submanifold $\overline{\mathcal{M}} = \{q | F_1(\boldsymbol{q}) = 0, \dots, F_k(\boldsymbol{q}) = 0\}$; the restriction to this manifold, we obtain a completely nonholonomic system.

By the example of a ball on the horizontal plane, we show that the constraints defined by system (1.14), (1.15) are nonholonomic. Let b be the radius of the ball; then the vector $\boldsymbol{r}(\boldsymbol{\gamma})$ can be explicitly represented in the form

$$\boldsymbol{r} = -b\boldsymbol{\gamma}.$$

1) Using the fields $\boldsymbol{\xi}_i, \boldsymbol{\zeta}_i$ (1.11) as a basis of vector fields on the configuration space $\mathbb{R}^3 \otimes SO(3)$, we represent the basis of admissible vector fields (2.2) in the form

$$\begin{aligned} \boldsymbol{\eta}_1 &= \boldsymbol{\xi}_1 - b\frac{\gamma_1\gamma_2}{\gamma_3}\boldsymbol{\zeta}_1 + b\frac{\gamma_1^2 + \gamma_3^2}{\gamma_3}\boldsymbol{\zeta}_2 - b\gamma_2\boldsymbol{\zeta}_3, \\ \boldsymbol{\eta}_2 &= \boldsymbol{\xi}_2 - b\frac{\gamma_2^2 + \gamma_3^2}{\gamma_3}\boldsymbol{\zeta}_1 + b\frac{\gamma_1\gamma_2}{\gamma_3}\boldsymbol{\zeta}_2 + b\gamma_1\boldsymbol{\zeta}_3. \end{aligned} \tag{2.3}$$

As was said above, the coefficients of the vector fields $\boldsymbol{\xi}_i, \boldsymbol{\eta}_i$ are linearly independent solutions of system (1.14), (1.15).

2) It is easy to show by a direct computation (by using a system of analytic computations, for example, Maple or Mathematica), that the dimension of the linear span $\mathcal{L}$ of fields (2.3) and their commutators is

$$\dim \mathcal{L} = 5.$$

Therefore, the system of constraints (1.14), (1.15) is nonholonomic and admits a unique holonomic constraint, which in our case has the form

$$R_3 = b = \text{const}, \tag{2.4}$$

i.e., the center of the ball moves in a horizontal plane.

Remark 5. In the general case of an arbitrary (convex) body, the holonomic constraint (2.4) is generalized as follows:

$$R_3 + (\boldsymbol{\gamma}, \boldsymbol{r}(\boldsymbol{\gamma})) = 0.$$

A geometric study of nonholonomic constraints for a rubber ball on a sphere can be found in [19, 20].

2.2. A Homogeneous Smooth (Marble) Ball on an Absolutely Rough Plane

In this case, we introduce an immovable coordinate system $OXYZ$, whose OZ-axis is perpendicular to the plane (see Fig. 3). Let $\mathbf{I} = \mu\mathbf{E}$ be the spherical moment of inertia, b be the radius of the ball, $\boldsymbol{R} = (X, Y, Z)$ be the position vector of the ball, $\boldsymbol{v}$ be the velocity of its center, and $\boldsymbol{\omega}$ be the angular velocity.

As was shown above, the condition of the absence of slipping (absolute roughness) can be expressed by saying that the velocity of the contact point vanishes

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0, \tag{2.5}$$

where $\boldsymbol{r} = -b\boldsymbol{e}_z$ is the vector joining the center of the ball with the contact point (see Fig. 3).

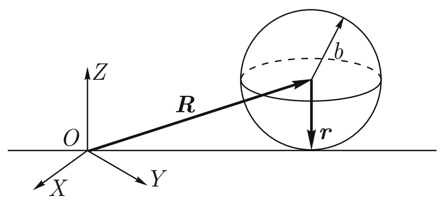

Fig. 3. Ball on a plane.

Proposition 2. *Assume that forces acting on the ball can be replaced by the resultant force* $\boldsymbol{F}$ *applied to the center. Then the center of the ball moves in a horizontal plane as a particle of mass* $\tilde{m} = m + \frac{\mu}{b^2}$ *under the law*

$$\tilde{m}\ddot{X} = F_X, \quad \tilde{m}\ddot{Y} = F_Y.$$

Proof. Taking into account the fact that the tensor of inertia is spherical, we see that the equations of motion of the ball in the immovable axes $OXYZ$ have the form

$$m\dot{\boldsymbol{v}} = \boldsymbol{N} + \boldsymbol{F}, \quad \mu\dot{\boldsymbol{\omega}} = \boldsymbol{r} \times \boldsymbol{N}, \quad \dot{\boldsymbol{R}} = \boldsymbol{v}, \tag{2.6}$$

where $\boldsymbol{N}$ is the reaction of constraint (2.5).

Differentiating constraint (2.5) and taking into account the relation $\dot{\boldsymbol{r}} = 0$ and the first two equations (2.6), we obtain

$$\mu\dot{\boldsymbol{v}} = \mu\boldsymbol{r} \times \dot{\boldsymbol{\omega}} = \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{N}) = \boldsymbol{r} \times (\boldsymbol{r} \times (m\dot{\boldsymbol{v}} - \boldsymbol{F})).$$

Regrouping the terms, we finally find

$$\left(m + \frac{\mu}{b^2}\right)\dot{\boldsymbol{v}} = \boldsymbol{F} - (\boldsymbol{F}, \boldsymbol{e}_z)\boldsymbol{e}_z.$$

□

2.3. Homogeneous Rubber Ball on (Convex) Surface

As above, we introduce the immovable coordinate system $OXYZ$ and consider the equation of the surface passing through the center of the ball (see Fig. 4):

$$f(\boldsymbol{R}) = 0, \tag{2.7}$$

where $\boldsymbol{R}$ is the position vector of the center of the ball.

It is clear that surface (2.7) is equidistant to the support surface and their normals coincide and have the same direction as the vector $\boldsymbol{r}$ joining the center of the ball with the contact point:

$$\boldsymbol{r} = -b\boldsymbol{n}, \quad \boldsymbol{n} = \frac{\nabla f(\boldsymbol{R})}{|\nabla f(\boldsymbol{R})|},$$

where b is the radius of the ball.

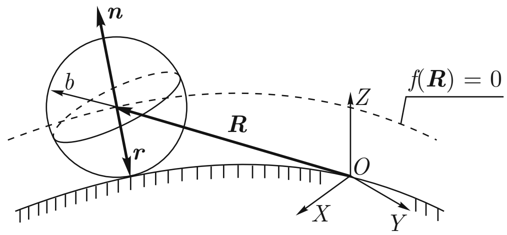

Fig. 4. Ball on an arbitrary surface.

The constraints that express the conditions of the absence of slipping and spinning can be represented in the form

$$\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r} = 0, \quad (\boldsymbol{\omega}, \boldsymbol{r}) = 0, \tag{2.8}$$

where $\boldsymbol{v} = \dot{\boldsymbol{R}}$ is the velocity of the center of the ball and $\boldsymbol{\omega}$ is its angular velocity.

Proposition 3. *Assume that forces acting on the ball can be replaced by the resultant force $\boldsymbol{F}$ applied to the center. Then the center of the ball moves along the surface* (2.7) *as a particle of mass $\tilde{m} = m + \frac{\mu}{b^2}$:*

$$\tilde{m}\dot{\boldsymbol{v}} = \boldsymbol{F} + \lambda\boldsymbol{n}, \quad \lambda = -(\boldsymbol{F}, \boldsymbol{n})\boldsymbol{n} - (\tilde{m}\boldsymbol{v}, \dot{\boldsymbol{n}})\boldsymbol{n}. \tag{2.9}$$

Remark 6. Equations (2.9) are the equations of motion of a particle on a surface with undetermined Lagrange multipliers.

Proof. The equations of motion of the ball under constraints (2.8) in the immovable coordinate system have the form

$$m\dot{\boldsymbol{v}} = \boldsymbol{N} + \boldsymbol{F}, \quad \mu\dot{\boldsymbol{\omega}} = \boldsymbol{r} \times \boldsymbol{N} + N_0\boldsymbol{r}, \quad \dot{\boldsymbol{R}} = \boldsymbol{v}, \tag{2.10}$$

where $\boldsymbol{N}$ and N_0 are undetermined multipliers (reactions of constraints).

Using the equations of motion and constraints, it is easy to obtain the relations

$$\boldsymbol{N} = m\dot{\boldsymbol{v}} - \boldsymbol{F}, \quad b^2\boldsymbol{\omega} = \boldsymbol{v} \times \boldsymbol{r}, \quad (\dot{\boldsymbol{r}}, \boldsymbol{r}) = (\boldsymbol{v}, \boldsymbol{r}) = 0, \quad (\dot{\boldsymbol{v}}, \boldsymbol{r}) = (\boldsymbol{v}, \dot{\boldsymbol{r}}) = 0. \tag{2.11}$$

Differentiating the first constraint (2.8) we have

$$\mu\dot{\boldsymbol{v}} = \mu\boldsymbol{r} \times \dot{\boldsymbol{\omega}} + \mu\dot{\boldsymbol{r}} \times \boldsymbol{\omega} = \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{N}) + \mu\dot{\boldsymbol{r}} \times \boldsymbol{\omega},$$

Substituting $\boldsymbol{N}$ and $\boldsymbol{\omega}$ in this equation from (2.11) we have

$$\mu\dot{\boldsymbol{v}} = \boldsymbol{r} \times (\boldsymbol{r} \times m\dot{\boldsymbol{v}}) - \boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{F}) - \mu(\boldsymbol{v}, \dot{\boldsymbol{r}})\boldsymbol{r}.$$

Taking into account the last relation (2.11) and regrouping terms in this equation, we obtain (2.9). □

If there are no external forces ($\boldsymbol{F} = 0$), the center of mass of the ball (in the absence of spinning) moves along a geodesic.

2.4. The Chaplygin Ball and the Veselova Problem

Since the tensor of inertia is not spherical in this case, it is more convenient to use the movable coordinate system related to the principal axes. In this case, we must set in the equations of motion (1.18) obtained above

$$\boldsymbol{r} = -b\gamma,$$

where b is the radius of the ball.

Using the constraint equations (1.14) and (1.15) and the second equation (1.18), we obtain

$$(\dot{\boldsymbol{\omega}}, \boldsymbol{\gamma}) = (\dot{\boldsymbol{\gamma}}, \boldsymbol{\gamma}) = 0, \quad \boldsymbol{r} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) = 0,$$

as consequently

$$\tilde{\mathbf{I}}\boldsymbol{\omega} = \mathbf{J}\boldsymbol{\omega}, \quad \tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \mathbf{J}\dot{\boldsymbol{\omega}}, \quad \mathbf{J} = \mathbf{I} + mb^2\mathbf{E}.$$

Finally, if external forces are potential (with potential depending only on $\boldsymbol{\gamma}$), we can represent the equations of motion in the form

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}} + \lambda_0 \boldsymbol{\gamma},$$

$$\lambda_0 = -\frac{\left(\mathbf{J}^{-1}\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma}}\right)}{(\mathbf{J}^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})}.$$

Comparing these equations with (1.21), we see that *the Chaplygin ball without spinning is equivalent to the Veselova system* (in which one must replace $\mathbf{I} \to \mathbf{J}$).

2.5. Discussion

1. In his unpublished paper [21] Chaplygin tried to integrate the equations of motion of dynamically non-symmetric ball on a plane when both the velocity at the point of contact and the projection of the angular velocity onto a ball-fixed axis are zero. This additional nonholonomic constraint is analogous to the Suslov constraint [22]. Not being aware of [22], Chaplygin also proposed to implement this system as a torsion-free thread. This implementation is not quite correct, and in [21] neither a compact form of equations nor any significant conclusions were obtained (maybe it is for this reason that Chaplygin decided not to publish this work). This problem is still not studied. In [13] we propose a new mechanical interpretation of the Chaplygin constraint. In some sense, the Suslov constraint is reciprocal to the Veselova constraint that appears in the rubber rolling system.

The examples considered in this section allow one to state the following conclusions.

1) The fact that a given set of constraints is non-holonomic has a comparatively simple geometry behind it and has no direct relation to mechanical system on which these constraints are imposed.

2) The holonomy of a dynamical system under nonholonomic constraints is a dynamical effect (it will be shown below that this effect is connected with certain conservation laws. As shown above, one and the same constraint can lead to a holonomic or nonholonomic dynamical system.

Remark 7. Holonomic dynamical systems with a potential or a generalized potential are called Lagrangian systems. Upon the Legendre transformation they turn into Hamiltonian systems whose study makes one of the major brunches of dynamics.

2. As we saw, the motion of a homogeneous ball on a surface in the absence of slipping and spinning is described by the holonomic system (2.9); based on this fact, some authors (see V.F. Zhuravlev [23]) conclude that "nonholonomic setting of the problem of rolling motion of rigid bodies on a rough plane is principally unsuitable: the no-slip condition can be assumed only for $u \equiv 0$ (the absence of spinning), but in this case the kinematic condition is integrable, that is, the constraint is holonomic"

The last example in this section demonstrates that this statement is not correct.

3. CONSERVATION LAWS IN NONHOLONOMIC MECHANICS

Let us consider in greater detail the conservation laws and corresponding (tensor) invariants that are encountered within therefore realm of nonholonomic mechanics. The equations of motion can be written in standard form as a system of first-order ordinary differential equations solved for the derivatives

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \tag{3.1}$$

where $\boldsymbol{v}(\boldsymbol{x})$ is a vector field in the phase space given by (1.6) or (1.9). Generally, in the problems under consideration the components of $\boldsymbol{v}(\boldsymbol{x})$ are analytic functions, which is assumed to be the case in what follows.

3.1. Integrals and Symmetry Fields. Conservation of Energy Law

Most trivial invariants of the system (3.1) are first integrals and symmetry fields. Recall that a function $F(\boldsymbol{x})$ is a first integral if

$$\dot{F} = (\nabla F, \boldsymbol{v}(\boldsymbol{x})) = 0,$$

a vector field $\boldsymbol{u}(\boldsymbol{x})$ is a symmetry field if

$$[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})] = 0, \tag{3.2}$$

where $[\cdot,\cdot]$ is the Lie bracket in the phase space.

Under comparatively general assumptions, the conservation of energy law is obeyed by equations from nonholonomic mechanics. For the Poincaré-Suslov equation (1.9) the following theorem is valid.

Theorem 1. *Suppose that the Lagrangian function $\hat{L}(\boldsymbol{q}, \boldsymbol{w})$ and the constraints are independent of time and the constraints are homogeneous functions of w_i; then the system of equations* (1.9) *preserves the energy*

$$\mathcal{E} = \sum w_i \frac{\partial \hat{L}}{\partial w_i} - \hat{L}. \tag{3.3}$$

Proof. Differentiating (3.3) with respect to time and using (1.7) and (1.9), we get

$$\dot{\mathcal{E}} = \sum_{\mu,i} \lambda_\mu \frac{\partial \hat{f}_\mu}{\partial w_i} w_i,$$

which by virtue of the constraints and the Euler's theorem

$$\frac{\partial \hat{f}_\mu}{\partial w_i} w_i = æ_\mu \hat{f}_\mu,$$

($æ_\mu$ is the degree of homogeneity of $f_\mu(\boldsymbol{q}, \boldsymbol{w})$) gives $\dot{\mathcal{E}} = 0$. □

For the equation in local coordinates (1.6), the energy integral reads

$$\mathcal{E} = (\dot{\boldsymbol{q}}, \frac{\partial L}{\partial \dot{\boldsymbol{q}}}) - L.$$

For the constraints (1.2) the homogeneity condition is equivalent to

$$b_\mu(\boldsymbol{q}) = 0, \quad \mu = 1 \ldots m.$$

We can see that these conditions are fulfilled in the examples considered above for a rubber body (1.14), (1.15) and Chaplygin's and Veselova's problems.

3.2. Invariant Measure

As it has been shown above, nonholonomic systems are conservative systems from the standpoint of conservation of energy. At the same time, it is customary to call a system *conservative* if it preserves the phase volume. In invariant (coordinate-free) form this property can be formulated using the notion of invariant measure.

A function $\rho(\boldsymbol{x}) > 0$ is called *the density of an invariant measure* of the system (3.1) if is satisfies the Liouville equation

$$\div(\rho(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})) = 0. \tag{3.4}$$

In this case, the volume of any domain Γ_t in the phase space (calculated with the help of the density $\rho(\boldsymbol{x})$) is preserved by the phase flow of (3.1)

$$\int\limits_{\Gamma_t} \rho(\boldsymbol{x}) d^n\boldsymbol{x} = \text{const}.$$

In [24, 25] invariant measure is also referred to as *integral invariant*.

Nonholonomic mechanics is populated with systems both with (a Chaplygin ball, a body of revolution on a plane, etc.) and without (Celtic stones, Suslov's problem) an invariant measure [1, 2, 13, 26–28, etc.].

Obstructions to the existence of invariant measure are discussed in [24], and the effects that the absence of invariant measure can cause are considered in [26].

3.3. Poisson Structure. Hamiltonian and Conformally Hamiltonian Form

Another important invariant encountered in nonholonomic systems is a Poisson structure that can be used to represent the equations in the Hamiltonian form.

Recall that the most general form of a Hamiltonian system looks like (for details, see [29])

$$\dot{\boldsymbol{x}} = \mathbf{J}(\boldsymbol{x})\nabla H, \tag{3.5}$$

where $H(\boldsymbol{x})$ is a Hamiltonian function and $\mathbf{J}(\boldsymbol{x}) = \|J_{ij}(\boldsymbol{x})\|$ is a Poisson structure (also referred to as Poisson tensor), which is a skew-symmetric tensor field whose component satisfy the Jacobi identity

$$\frac{\partial J_{ij}(\boldsymbol{x})}{\partial x_k} + \frac{\partial J_{jk}(\boldsymbol{x})}{\partial x_i} + \frac{\partial J_{ki}(\boldsymbol{x})}{\partial x_j} = 0. \tag{3.6}$$

A Poisson structure enables definition of the Poisson bracket of two functions f, g by the formula

$$\{f(\boldsymbol{x}), g(\boldsymbol{x})\} = \sum_{i,j} J_{ij}(\boldsymbol{x}) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.$$

Obviously, the Hamiltonian system (3.5) *has the Hamiltonian function* $H(\boldsymbol{x})$ *as an integral of motion and preserves measure by Liouville's theorem.*

Hamiltonian systems are a class of dynamical systems, which is most extensively studied from several points of view: theory of integrability, stability analysis, topological analysis, etc. In this connection, the study of possibility (or impossibility) to represent equations from nonholonomic mechanics in the Hamiltonian form seems to be a very important issue.

Remark 8. According to Lie's definition, a tensor field $\mathbf{T}(\boldsymbol{x})$ is *a tensor invariant* of the system (3.1), if its Lie derivative is zero, that is,

$$\mathcal{L}_{\boldsymbol{v}}(\mathbf{T}(\boldsymbol{x})) = 0.$$

It is easy to show that $\mathbf{J}(\boldsymbol{x})$ is a tensor invariant of (3.5) due to Jacobi's identity. In connection with this it should be noted that writing a particular system of differential equations in the form of (3.5) but with a skew-symmetric $\mathbf{J}(\boldsymbol{x})$ that does not obey Jacobi's identity is useless because in this case the form (3.5) implies no conservation laws.

There is one special class of systems that are practically indistinguishable from Hamiltonian systems, these are so-called *conformally Hamiltonian systems*:

$$\dot{\boldsymbol{x}} = \mathcal{N}(\boldsymbol{x})\mathbf{J}(\boldsymbol{x})\nabla H. \tag{3.7}$$

Here $\mathbf{J}(\boldsymbol{x})$ is Poisson tensor and $\mathcal{N}(\boldsymbol{x})$ is a non-constant function. Following Chaplygin, we call the function $\mathcal{N}(\boldsymbol{x})$ *a reducing multiplier*.

Conformally Hamiltonian systems also preserve measure and have the Hamiltonian as a constant of motion.

If the sign $\mathcal{N}(\boldsymbol{x})$ is fixed, then, upon the change of time $\mathcal{N}(\boldsymbol{x})\,dt = d\tau$, the system (3.7) is reduced to a Hamiltonian form. If the function takes zero value in some points, then the change of time results in a Hamiltonian system with, on occasion, intriguing topological features [30].

Most nonholonomic systems can be represented in the conformally Hamiltonian form (3.7) (see below).

3.4. Canonical Form of Conformally Hamiltonian Systems

Consider some properties of (conformally) Hamiltonian systems that will be used below.

Most nonholonomic systems possess a degenerate Poisson structure, that is, $\operatorname{rank}\mathbf{J} < \dim\mathcal{M}$. Thus, here we encounter another important "personage" — a Casimir function $\Phi_\alpha(\boldsymbol{x})$, which is defined on the whole manifold $\mathcal{M}$ and commutes with any function on $\mathcal{M}$:

$$\forall f(\boldsymbol{x}) : \{\Phi_\alpha(\boldsymbol{x}), f(\boldsymbol{x})\} \equiv 0, \quad \alpha = 1\dots m.$$

It is obvious that

- a Casimir function is an integral of the system (3.7);
- $\nabla\Phi_\alpha(\boldsymbol{x})$ lies in the kernel of the Poisson structure $\mathbf{J}(\boldsymbol{x})$.

Locally, the structure of a manifold endowed with a Poisson structure is described by *the generalized Darboux theorem* [31, 32].

Theorem 2. *For any Poisson structure* $\mathbf{J}$ *on* $\mathcal{M}$ *whose rank is constant in a neighborhood of a point* $\boldsymbol{x}_0 \in \mathcal{M}$*, there exist local coordinates* q_1, $p_1, \dots, q_n$, p_n, $z_1, \dots, z_k$ *in a neighborhood of this point such that*

$$\{z_\alpha, q_i\} = \{z_\alpha, p_i\} = 0, \quad \alpha = 1\dots k,$$
$$\{q_i, p_i\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0, \quad i, j = 1\dots n.$$

Therefore, $z_\alpha(\boldsymbol{x})$ are (locally defined) Casimir functions, while the manifold $\mathcal{M}$ itself (at least in a neighborhood of the point $\boldsymbol{x}_0$) is foliated into submanifolds

$$\tilde{\mathcal{M}}_c = \{\boldsymbol{x} \in \mathcal{M} | z_1(\boldsymbol{x}) = c_1, \dots, z_k(\boldsymbol{x}) = c_k\},$$

which are called symplectic leaves. The restriction of the Poisson structure $\mathbf{J}|_{\tilde{\mathcal{M}}_c}$ is non-degenerate.

Remark 9. Obviously, the number of local Casimir functions $z_\alpha(\boldsymbol{x})$ (in the vicinity of point $\boldsymbol{x}_0$) is equal to the dimension of the kernel of the matrix $\mathbf{J}(\boldsymbol{x}_0)$.

Consider two possible situations.

1) The number of globally defined Casimir functions $\Phi_\alpha(\boldsymbol{x})$ equals the number of locally defined Casimir functions (i.e. $\dim(\operatorname{Ker}\mathbf{J}) = m$). This case is of common occurrence in applications and described in [29, 31–33]. In this case, a symplectic leaf $\mathcal{M}_c \subset \mathcal{M}$ is a globally defined symplectic manifold, that is,

$$\mathcal{M}_c = \{\boldsymbol{x} \in \mathcal{M} | \Phi_1(\boldsymbol{x}) = c_1, \dots, \Phi_m(\boldsymbol{x}) = c_m\}.$$

Moreover, by Darboux's theorem, the restriction of the system (3.7) onto $\mathcal{M}_c$ can be locally represented in the (conformally) canonical form

$$\dot{\boldsymbol{q}} = N_c(\boldsymbol{q}, \boldsymbol{p})\frac{\partial H_c(\boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = -N_c(\boldsymbol{q}, \boldsymbol{p})\frac{\partial H_c(\boldsymbol{q}, \boldsymbol{p})}{\partial \boldsymbol{q}},$$

where $N_c = N|_{\mathcal{M}_c}$ and $H_c = H|_{\mathcal{M}_c}$ are the restriction of the reducing multiplier and the Hamiltonian onto the symplectic leaf.

In this case, to prove system's integrability (after the change of time) one should use the Liouville–Arnold theorem. For the system to be integrable the existence of $n = \frac{1}{2}\dim\mathcal{M}_c$ additional first integrals is necessary.

2) The number of globally defined Casimir functions $\Phi_\alpha(\boldsymbol{x})$ is less than the number of locally defined Casimir functions (i.e. $\dim(\operatorname{Ker}\mathbf{J}) = m$). This case is not discussed in the literature (even in the comprehensive survey [34]). Here we consider an example of Poisson structure, given by a solvable five-dimensional Lie algebra, which occurs in Suslov's problem. The problem will be considered in greater detail in our subsequent papers. The Hamiltonian form of the equations that we employ in this paper is similar to that obtained in [35]. Thus, the Hamiltonian form of the equations is as follows:

$$\dot{\gamma}_i = \{\gamma_i, H\}, \quad \dot{p}_j = \{p_j, H\}, \quad i = 1, 2, 3; j = 1, 2,$$
$$H = \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2)$$

where the Poisson structure $\mathbf{J}$ is given by the relations

$$\{\gamma_1, \gamma_2\} = p_1, \quad \{\gamma_2, \gamma_3\} = dp_2, \quad \{p_1, \gamma_3\} = -\gamma_1, \quad \{p_2, \gamma_3\} = -d\gamma_2 \tag{3.8}$$

(the unwritten brackets are zero).

The rank of the bracket (3.8) is two, and though the symplectic leaves are two-dimensional, in the case of irrational d the Poisson structure has only two independent globally defined Casimir functions

$$\Phi_1 = p_1^2 + \gamma_1^2, \quad \Phi_2 = p_2^2 + \gamma_2^2.$$

Therefore, in this case none of the symplectic leaves of the bracket (3.8) can be represented as a level surface of a set of analytical functions.

Remark 10. The third "Casimir function" can be written in complex form as

$$\Phi_3 = \frac{(p_1 + i\gamma_1)^d}{p_2 + i\gamma_2}.$$

4. INTEGRABILITY AND REDUCTION

Traditionally, a system is called *integrable* if it is possible to express the solutions in terms of quadratures, that is, analytical formulas for solutions must consist of only "algebraic" operations, inverse functions, formulas for a change of local coordinates, and "quadratures" (which are integrals of known functions). In nonholonomic mechanics there are three "mechanisms" for explicit integration of systems in terms of quadratures:

1) *integrability by Lie's theorem* (some simplified versions of this theorem were known already to Euler, Lagrange and Jacobi) there are enough of first integrals and symmetry fields to reduce the system to a single first-order differential equation;

2) *integrability by the Euler–Jacobi theorem* is based on the reduction of a dynamical system to a measure-preserving phase flow on a two-dimensional manifold;

3) *representation of a system in the conformally Hamiltonian form* (3.7), after a change of time, integrability is proved with the help of Liouville's theorem.

Let us consider these three approaches in greater detail.

4.1. Integration Based on the Use of Symmetry Fields and First Integrals

The following result due to S. Lie is well known (for a modern exposition see [36])).

Theorem 3 (Lie). *Suppose that a system of equations $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ has k first integrals and $n-k-1$ symmetry fields, which are tangent to the common level surfaces of the integrals and form a solvable Lie algebra, then this system can be integrated in terms of quadratures.*

In this case, on a common level surface one can choose coordinates so that the given vector field takes the form

$$\dot{y}_1 = \widetilde{v}_1(y_1), \quad \dot{y}_2 = \widetilde{v}_2(y_1), \quad \dot{y}_3 = \widetilde{v}_3(y_1, y_2), \ldots, \dot{y}_{n-k} = \widetilde{v}_{n-k}(y_1, \ldots, y_{n-k-1}).$$

It is therefore concluded that *Lie-integrable systems preserve measure*.

For the proof it is sufficient to find the divergence of the vector field in coordinates $F_1, \ldots, F_k, y_1, \ldots, y_{n-k}$.

There is one special subclass of such systems, whose dynamics is practically identical to the dynamics of integrable Hamiltonian systems. On an invariant surface there exists a "special" coordinate such that all the other coordinates can be expressed as quadratures of this coordinate. In rigid body dynamics an example of such systems is the Lagrange top, for which the evolution of the precession angle $\psi(t)$ and of the intrinsic rotation angle $\varphi(t)$ can be expressed as independent quadratures of the nutation angle $\theta(t)$ [37], that is,

$$\dot{\psi} = \frac{p_\psi - p_\varphi \cos\theta}{\sin^2\theta}, \quad \dot{\varphi} = (I_3^{-1} - I_1^{-1})p_\varphi + \frac{p_\psi - p_\varphi \cos\theta}{\sin^2\theta},$$

Here p_ψ, p_φ are constants of motion and I_1, I_3 are the top's moments of inertia. Hereinafter we will say that such systems are *Lagrange–Lie integrable*.

A well-known example of such systems is the problem of motion of an axisymmetric body on a plane; in the context of the classical nonholonomic model (with spinning allowed) this system was independently integrated by S.A. Chaplygin [38] and M. Gallop [39] (see also [1, 40, 41, 83]). The motion of an axisymmetric body on a plane (without spinning) has been discussed in Section 5.

There we present formal results concerned with reduction to the Hamiltonian form after a change of time for the case when a common level surface of the first integrals is a torus. (A similar result is valid for the averaging procedure in one-frequency systems [12].)

Theorem 4 (on reducibility). *Suppose that an integrable system on $\mathcal{M}^n$ has k first integrals $\boldsymbol{F} = (F_1, \ldots, F_k)$, and their common level surface is a $(n-k)$-dimensional torus. Suppose that in a neighborhood of this torus there are angular coordinates $(\theta_1, \ldots, \theta_{n-k})$ mod 2π such that the vector field $\boldsymbol{v}(\boldsymbol{x})$ takes the form*

$$\dot{\theta}_1 = \omega_1(\boldsymbol{F}), \quad \dot{\theta}_2 = \zeta_2(\theta_1, \boldsymbol{F}), \ldots, \dot{\theta}_{n-k} = \zeta_{n-k}(\theta_1, \boldsymbol{F}). \tag{4.1}$$

Then there exists a differentiable (analytic) change of variables that reduces the system (4.1) *to the form*

$$\dot{\theta}_1 = \omega_1(\boldsymbol{F}), \quad \dot{\theta}_2 = \omega_2(\boldsymbol{F}), \ldots, \dot{\theta}_{n-k} = \omega_{n-k}(\boldsymbol{F}). \tag{4.2}$$

Prior to the proof of the theorem let us formulate

Corollary 1. *If an integrable system can be written in the form* (4.1)*, then this system is Hamiltonian* (*and even multi-Hamiltonian* [29]).

Proof. Expand each function ζ_i, $i = 2, \ldots, n-k$ in a convergent Fourier system

$$\zeta_i(\theta_1, \boldsymbol{F}) = \sum_{m\in\mathbb{Z}} \zeta_i^m(\boldsymbol{F}) e^{im\theta_1}.$$

A change of variables that reduces the system to the form (4.2) is given by the convergent series of the form

$$\widetilde{\theta}_i = \theta_i - \frac{1}{\omega_1(\boldsymbol{F})} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\zeta_i^{(m)}(\boldsymbol{F})}{im} e^{im\theta_1}.$$

The frequencies are defined as

$$\omega_i(\boldsymbol{F}) = \zeta_i(\boldsymbol{F}) = \frac{1}{2\pi}\int\limits_0^{2\pi} \zeta_i(\theta_1, \boldsymbol{F})d\theta_1, \quad i = 2, \ldots, n-k.$$

$\square$

Remark 11. In a more general case a vector field on a torus looks like

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \zeta_2(\theta_1), \quad \dot{\theta}_3 = \zeta_3(\theta_1, \theta_2), \ldots, \dot{\theta}_{n-k} = \zeta_{n-k}(\theta_1, \ldots, \theta_{n-k-1});$$

such a representation can be impossible if $n - k > 2$, whereas the solution itself can be expressed in quadratures. Nevertheless, the system can be non-Hamiltonian.

Remark 12. Although the reducibility theorem guaranties that near a torus the system can be reduced to the Hamiltonian form, in the whole phase space such a reduction may turn out to be impossible. In some cases there are topological obstructions [42, 43].

4.2. Euler–Jacobi Integrability

Another method for integration of dynamical systems is given by the following theorem.

Theorem 5 (Euler–Jacobi). *Consider a system of equations $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$ that preserves measure and has $n-2$ first integrals $\boldsymbol{F} = F_1, \ldots, F_{n-2}$. Moreover, suppose that on the invariant set $\mathcal{M}_c = \{\boldsymbol{x} \in \mathcal{M},\ F_i(\boldsymbol{x}) = c_i = \mathrm{const}\}$ the functions F_i, $i = 1, \ldots, n-2$ are functionally independent. Then*

1) *those solutions of the system that lie on $\mathcal{M}_c$ can be expressed in terms of quadratures; if $\mathcal{L}_c$ is a connected and compact component of the level surface $\mathcal{M}_c$ and $\boldsymbol{v}(\boldsymbol{x}) \neq 0$ on $\mathcal{L}_c$,*

2) *then $\mathcal{L}_c$ is a smooth manifold diffeomorphic to a two-dimensional torus;*

3) *there exists a neighborhood of $\mathcal{L}_c$ one can choose angular coordinates $(\theta_1, \theta_2) \bmod 2\pi$ so that the equations of motion take the form*

$$\dot{\theta}_1 = \frac{\lambda_1(\boldsymbol{F})}{\Phi(\theta_1, \theta_2, \boldsymbol{F})}, \quad \dot{\theta}_2 = \frac{\lambda_2(\boldsymbol{F})}{\Phi(\theta_1, \theta_2, \boldsymbol{F})}. \tag{4.3}$$

This statement extends the statement from [28].

The non-trivial part of this theorem is the assertion that a system can be reduced to the form (4.3) in a neighborhood of an invariant torus. The proof of this result resides on the famous Kolmogorov's theorem on reducibility of a measure-preserving flow on a torus to the standard form

$$\dot{\theta}_1 = \frac{\lambda_1}{\Phi(\theta_1, \theta_2)}, \quad \dot{\theta}_2 = \frac{\lambda_2}{\Phi(\theta_1, \theta_2)}, \quad \lambda_{1,2} = \mathrm{const}.$$

The proof can be found, for example, in the book [44, Lecture 11]; for some preliminary statements necessary for the proof (the existence of Zigel's curve) see [45]. One can show that this proof can be easily extended to the case where the flow on a torus depend on parameters (this dependence is assumed to be sufficiently smooth for all algebraic manipulations in the proof to be valid). Thus taking the values of the first integrals as these parameters yields the equations of the form (4.3).

Using Theorem 5, one concludes that

Corollary 2. *At least in the vicinity of invariant tori, all Euler–Jacobi integrable systems are conformally Hamiltonian.*

At the same time, unlike the previous case (Lagrange-Liouville integrable systems), *a Euler–Jacobi integrable system cannot generally (without resort to changes of time) be reduced to the Hamiltonian form.*

Indeed, consider an invariant torus for which the rotation number $\frac{\lambda_1(\boldsymbol{F})}{\lambda_2(\boldsymbol{F})}$ from equations (4.3) is rational; such a torus is called resonant and is foliated into periodic trajectories. In the general case, since $\Phi(\theta_1, \theta_2, \boldsymbol{F})$ depends on θ_i the periods of motion along these trajectories are different [46]. This results in occurrence of mixing on resonant tori of nonholonomic systems [47] thereby preventing them from being Hamiltonian.

For Hamiltonian systems a simple corollary of Liouville's theorem is valid.

Proposition 4. *Consider an integrable Hamiltonian system that has two-dimensional invariant tori. The periods of the solutions that belong to one and the same resonant torus are equal.*

The proof is based on the existence of angle-action variables in the vicinity of non-singular tori of integrable systems [12, 48].

4.3. Generalized Chaplygin Systems

Consider a class of frequently-encountered systems which turn out to be conformally Hamiltonian if preserve measure. Moreover, the reduction to the conformally Hamiltonian form (3.7) can be performed explicitly.

Consider a mechanical system with two degrees of freedom governed by the equations of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \dot{q}_2 S, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = -\dot{q}_1 S, \qquad S = a_1(\boldsymbol{q})\dot{q}_1 + a_2(\boldsymbol{q})\dot{q}_2 + b(\boldsymbol{q}), \tag{4.4}$$

where L is a function of generalized coordinates $\boldsymbol{q} = (q_1, q_2)$ and velocities $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2)$; this function will be referred to as the Lagrangian of this system.

If the function S is specially chosen, one gets an ordinary Chaplygin system and $b(\boldsymbol{q}) = 0$) [49]. We will call (4.4) *a generalized Chaplygin system* (not to be confused with [50, 51], where a different generalization of Chaplygin systems is suggested!).

S.A. Chaplygin proved [49] the governing equations for a so-called Chaplygin sleigh can be reduced to the form (4.4) with $b(\boldsymbol{q}) = 0$. The equations can be integrated using the method of reducing multiplier (which is discussed below) and solution of a Hamilton–Jacobi equation. In [52] it is shown Veselova's problem (see for example [16, 27]) is a Chaplygin system. Recall that in this problem a rigid body rotates about a fixed point and obeys a non-integrable constraint that the projection of the angular velocity onto a fixed in space axis is zero. We will show in this paper that the problem of rolling motion of a dynamically non-symmetric, balanced Chaplygin ball [15] can also be reduced to the form (4.4) but with $b(\boldsymbol{q}) \neq 0$. Some multidimensional generalizations of the Chaplygin system are considered in [53].

It was shown by Chaplygin that for $b(\boldsymbol{q}) = 0$ [49] the equations preserve their form under the change of time

$$\mathcal{N}(\boldsymbol{q})\, dt = d\tau,$$

if $\mathcal{N}$ does not depend on velocities. Let us show that this also holds true for equations (4.4).

Denote differentiation with respect to the new time as $q_i' = \dfrac{dq_i}{d\tau}$; then

$$\dot{q}_i = \mathcal{N} q_i', \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{\mathcal{N}}\frac{\partial \overline{L}}{\partial q_i'}, \quad \frac{\partial L}{\partial q_i} = \frac{\partial \overline{L}}{\partial q_i} - \frac{1}{\mathcal{N}}\frac{\partial \mathcal{N}}{\partial q_i}\sum_{k=1}^{2} q_k' \frac{\partial \overline{L}}{\partial q_k'},$$

where $\overline{L}(q, q') = L(q, \mathcal{N} q')$.

Substituting this into (4.4) gives

$$\frac{d}{d\tau}\left(\frac{\partial\overline{L}}{\partial q_1'}\right)-\frac{\partial\overline{L}}{\partial q_1}=q_2'\overline{S},\quad \frac{d}{d\tau}\left(\frac{\partial\overline{L}}{\partial q_2'}\right)-\frac{\partial\overline{L}}{\partial q_2}=-q_1'\overline{S},$$
$$\overline{S}=\mathcal{N}S+\frac{1}{\mathcal{N}}\left(\frac{\partial\mathcal{N}}{\partial q_2}\frac{\partial\overline{L}}{\partial q_1'}-\frac{\partial\mathcal{N}}{\partial q_1}\frac{\partial\overline{L}}{\partial q_2'}\right). \tag{4.5}$$

For an ordinary Chaplygin system it is established [49] that if there exists an invariant measure whose density is a function of coordinates only, one can choose $\mathcal{N}(\boldsymbol{q})$ so that $\overline{S}=0$; therefore, in the new time τ the system can written in the canonical Hamiltonian form. This result can be extended to generalized Chaplygin systems of the form (4.4) if it is additionally assumed that the Lagrangian is a quadratic form (not necessarily homogeneous) in velocities $\dot{q}_i$.

Theorem 6. *Let* $\det\left\|\dfrac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}\right\|\neq 0$ *and assume that the system* (4.4) *preserves measure whose density depends only on coordinates; then there exists a change of time* $\mathcal{N}(\boldsymbol{q})\,dt=d\tau$ *such that*

1) *the function* $\overline{S}$ *given by* (4.5) *depends only on coordinates*: $\overline{S}=\overline{S}(\boldsymbol{q})$*;*

2) *with the new time the equations of motion can be written in the Hamiltonian form*

$$\frac{dq_i}{d\tau}=\{q_i,\overline{H}\},\quad \frac{dp_i}{d\tau}=\{p_i,\overline{H}\},$$

where

$$p_i=\frac{\partial\overline{L}}{\partial q_i'},\quad \overline{H}=\sum_{k=1}^{2}p_kq_k'-\overline{L}\Big|_{q_i'\to p_i},$$

and the Poisson bracket is defined as

$$\{q_i,p_j\}=\delta_{ij},\quad \{p_1,p_2\}=\overline{S}(\boldsymbol{q}),\quad \{q_1,q_2\}=0. \tag{4.6}$$

Proof. Perform the Legendre transformation of (4.4):

$$P_i=\frac{\partial L}{\partial\dot{q}_i},\quad H=\sum_i P_i\dot{q}_i-L\Big|_{\dot{q}_i\to P_i},$$

at that

$$\dot{q}_i=\frac{\partial H}{\partial P_i},\quad \dot{P}_1=-\frac{\partial H}{\partial q_1}+\frac{\partial H}{\partial P_2}S,\quad \dot{P}_2=-\frac{\partial H}{\partial q_2}-\frac{\partial H}{\partial P_1}S,$$
$$S=a_1(\boldsymbol{q})\dot{q}_1+a_2(\boldsymbol{q})\dot{q}_2+b(\boldsymbol{q})=A_1(\boldsymbol{q})P_1+A_2(\boldsymbol{q})P_2+B(\boldsymbol{q}). \tag{4.7}$$

Using Liouville's equation for the density of the invariant measure $\rho(\boldsymbol{q})\,dP_1\,dP_2\,dq_1\,dq_2$ of equations (4.7), we get

$$\dot{q}_1\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q_1}-A_2(q)\right)+\dot{q}_2\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q_2}+A_1(q)\right)=0,$$

Since ρ is a function of coordinates only, each bracket must independently vanish:

$$\frac{1}{\rho}\frac{\partial\rho}{\partial q_1}-A_2(\boldsymbol{q})=0,\quad \frac{1}{\rho}\frac{\partial\rho}{\partial q_2}+A_1(\boldsymbol{q})=0.$$

With consideration of $\dfrac{1}{\mathcal{N}}\dfrac{\partial\overline{L}}{\partial q_i'}=P_i$, equation (4.5) for $\overline{S}$ becomes

$$\overline{S}=\left(\mathcal{N}A_1(q)+\frac{\partial\mathcal{N}}{\partial q_2}\right)P_1+\left(\mathcal{N}A_2(q)-\frac{\partial\mathcal{N}}{\partial q_1}\right)P_2+B(q).$$

Thus if we put $\mathcal{N}(\boldsymbol{q})=\rho(\boldsymbol{q})$, then $\overline{S}=B(\boldsymbol{q})$ and the first statement of the theorem is proved.

The proof of the second statement is a straightforward verification of the equations and Jacobi's identity. □

Hence *in the original time t generalized measure-preserving Chaplygin systems are conformally Hamiltonian.*

Hamiltonian systems with bracket (4.6) frequently occur in the description of generalized potential systems, such as the motion of charged particles in a magnetic field, and systems with gyroscopic forces [12, 29, 31, 36]. In the latter case, the closed 2-form $\Omega = \overline{S}(\boldsymbol{q})\, dq_1 \wedge dq_2$ is known as the 2-form of gyroscopic forces. Locally, it can written as a total differential $\Omega = d\omega$, $\omega = W_1(\boldsymbol{q})dq_1 + W_2(\boldsymbol{q})dq_2$, and the equations of motion can be written in the form of the Lagrange-Euler equations

$$\frac{d}{d\tau}\left(\frac{\partial L_W}{\partial q_i'}\right) - \frac{\partial L_W}{\partial q_i} = 0,$$
$$L_W = \overline{L} + W(\boldsymbol{q}, \boldsymbol{q}'), \quad W(\boldsymbol{q}, \boldsymbol{q}') = W_1(\boldsymbol{q})q_1' + W_2(\boldsymbol{q})q_2',$$

whereas using new momenta $\widetilde{p}_i = p_i + W_i(\boldsymbol{q})$, the Poisson bracket is reduced to the canonical form.

If the manifold $\mathcal{M}$, on which the coordinates q_1 and q_2 are defined, is compact, then the form Ω is exact iff

$$\int\limits_{\mathcal{M}} \Omega = 0.$$

Thus if $\int \Omega \neq 0$, meaning that the form 2-form is not exact, then the generalized potential W, the corresponding Lagrangian and Hamiltonian functions have singularities (a so-called monopole) [12, 31]. Such a situation is characterized as impossibility of global reduction of the equations of motion to the Hamiltonian form. A generalization of the Hamilton–Jacobi method to non-holonomic systems is given in [54].

5. HEAVY RUBBER BODY OF REVOLUTION ON A PLANE

As shown above, in this case the evolution of the variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$ is governed by equations (1.18). The equation of the surface of the body is $f(r_1^2 + r_2^2, r_3) = 0$, and two principal moments of inertia are equal $I_1 = I_2$. We will employ the following parameterization of the vector $\boldsymbol{r}(\boldsymbol{\gamma})$:

$$\boldsymbol{r} = (\chi_1(\gamma_3)\gamma_1, \chi_1(\gamma_3)\gamma_2, \chi_2(\gamma_3)).$$

The functions χ_1, χ_2 are determined by the type of the body's surface; these functions are not independent because they are related by the equation $f(\chi_1^2(1-\gamma_3^2), \chi_2) = 0$.

The potential energy of the body (1.16) can be written as

$$U(\gamma_3) = -mg(\chi_1(1-\gamma_3^2) + \chi_2\gamma_3).$$

Besides the geometric integral of motion and the constraint

$$\boldsymbol{\gamma}^2 = 1, \quad (\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0,$$

the system, by Theorem 1, admits the integral of energy

$$\mathcal{E} = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + U(\gamma_3).$$

Moreover, in this case the system (1.18) obviously has a symmetry field that corresponds to the rotation about the symmetry axis

$$\boldsymbol{u}(\boldsymbol{\gamma}, \boldsymbol{\omega}) = \omega_1\frac{\partial}{\partial\omega_2} - \omega_2\frac{\partial}{\partial\omega_1} + \gamma_1\frac{\partial}{\partial\gamma_2} - \gamma_2\frac{\partial}{\partial\gamma_1}.$$

In addition to these quite obvious conservation laws, the system has another first integral

$$K = A(\gamma_3)\omega_3, \quad A(\gamma_3) = \sqrt{I_1\gamma_3^2 + I_3(1-\gamma_3^2) + m(\boldsymbol{r}, \boldsymbol{\gamma})^2}. \tag{5.1}$$

The existence of such a linear in $\boldsymbol{\omega}$ integral is quite expectable [1] (Sect. 2); but the fact that this integral can be expressed in elementary functions of $\boldsymbol{\omega}, \boldsymbol{\gamma}$ for any body's shape is far non-trivial. For example, in the problem of motion of a body of revolution on a plane when spinning is allowed this integral can be expressed in elementary functions only when the surface of the body is a sphere.

By Lie's theorem (Section 4), we have enough of first integrals and symmetry fields to solve the system in terms of quadratures. Parameterizing the vector $\boldsymbol{\gamma}$ by the angle of nutation θ and the angle of intrinsic rotation φ, one gets

$$\boldsymbol{\gamma} = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta).$$

Then on the surface $\mathcal{M}_{h,k} = \{\boldsymbol{\omega}, \boldsymbol{\gamma} | \boldsymbol{\gamma}^2 = 1, (\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0, \mathcal{E} = h, K = k\}$ the following relations can be obtained

$$\begin{gathered}\frac{1}{2}B(\gamma_3)\dot{\theta}^2 + \frac{k^2}{2\sin^2\theta} + U(\gamma_3) = h, \quad \omega_3 = \frac{k}{A(\gamma_3)}, \\ \dot{\varphi} = \frac{\omega_3}{\sin^2\theta}, \quad \omega_1 = \dot{\theta}\cos\varphi - \omega_3 \operatorname{ctg}\theta\sin\varphi, \quad \omega_2 = -\dot{\theta}\sin\varphi - \omega_3\operatorname{ctg}\theta\cos\varphi,\end{gathered} \tag{5.2}$$

where $B(\gamma_3) = I_1 + m(\boldsymbol{r}, \boldsymbol{r})$.

Remark 13. Here we have used the well-know equation for the intrinsic rotation angle [37]:

$$\dot{\varphi} = \omega_3 - \frac{\gamma_3(\omega_1\gamma_1 + \omega_2\gamma_2)}{\gamma_1^2 + \gamma_2^2}.$$

As mentioned above, Lie-integrable systems preserve measure, and in this case the density of the measure $\rho\, d^3\boldsymbol{\omega}\, d^3\boldsymbol{\gamma}$ can be expressed in elementary functions

$$\rho(\gamma_3) = B(\gamma_3)A(\gamma_3).$$

Remark 14. The multipliers that from the density of the measure also enter into the expansion of the function

$$\det\tilde{\mathbf{I}}(\boldsymbol{\gamma}, \tilde{\mathbf{I}}^{-1}\boldsymbol{\gamma}) = B(\gamma_3)A^2(\gamma_3).$$

Let us show that

Proposition 5. *If $k \neq 0$, then the governing equations for a heavy rubber body of revolution can be represented (without change of time) in therefore Hamiltonian form.*

Proof. To prove this, one has to show (according to Theorem 4 on reducibility) that the common level surface of the integrals is a torus (or a disconnected union of two-dimensional tori). In this case it is sufficient to show that on this surface there exists a vector field that does not vanish and has no singularities. In view of the first equation in (5.2), the condition $\dot{\varphi} > 0$ is fulfilled everywhere. □

For several simple-shape bodies we present the quadrature formulas for the angle of nutation θ:

- *disk with displaced center of mass* (*Fig.* 5):

$$\chi_1 = -\frac{R}{\sin\theta}, \quad \chi_2 = -a$$

$$\frac{1}{2}(I_1 + m(R^2 + a^2))\dot{\theta}^2 = h - mg(R\sin\theta + a\cos\theta) - \frac{1}{2}\frac{k^2}{\sin^2\theta};$$

- *ball with displaced center*:

$$\chi_1 = -R, \quad \chi_2 = -R\cos\theta - a$$

$$\frac{1}{2}(I_1 + m(R^2 + a^2) + 2mRa\cos\theta)\dot{\theta}^2 = h - mg(R + a\cos\theta) - \frac{1}{2}\frac{k^2}{\sin^2\theta}.$$

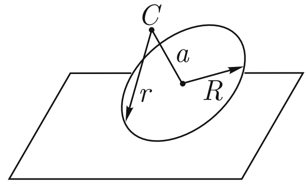

Fig. 5. Disk on a plane.

6. THE VESELOVA SYSTEM

6.1. Equations of Motion and Conservation Laws

As was said above (see Section 1), the Veselova system describes the motion of a rigid body with a fixed point subject to nonholonomic constraint of the form $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, where $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ — are the body's angular velocity vector and the unit vector of the space-fixed axis in the frame of reference fixed to the body. Thus, for the Veselova constraint, the projection of the angular velocity onto a space-fixed axis is zero. This constraint is reciprocal to the Suslov constraint for which the projection of the angular velocity onto a body-fixed axis is zero.

In the moving axes fixed to the body, the equations of motion can be written as follows [16]:

$$\begin{gathered}\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \lambda_0 \boldsymbol{\gamma} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \lambda_0 = \frac{\left(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \mathbf{I}^{-1}\boldsymbol{\gamma}\right)}{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})},\end{gathered} \tag{6.1}$$

where $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the tensor of inertia, $U(\boldsymbol{\gamma})$ is the potential energy.

In the general case the equations (6.1) admit the integral of energy and the geometric integral

$$H = \frac{1}{2}(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad \boldsymbol{\gamma}^2 = 1, \tag{6.2}$$

as well as the invariant measure $\rho_\omega \, d^3\boldsymbol{\omega} \, d^3\boldsymbol{\gamma}$ with density

$$\rho_\omega = \sqrt{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})}. \tag{6.3}$$

When $U = 0$, there is exists additional integral [16]

$$F = (\mathbf{I}\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) - (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma})^2 = |\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\gamma}|^2, \tag{6.4}$$

and hence, the system is integrable according to the theorem 5 (Euler–Jacobi).

Remark 15. The integral (6.4) is generalized when the Bruns $U = (\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})$ potential is added [16, 27]. Some other integrable potentials are given in [52, 55].

Remark 16. The system (6.1) was rediscovered in paper [56] almost ten years after [16, 27]. In [57], an explicit integration was performed using sphero-conical coordinates.

Remark 17. The Veselova system and the nonholonomic systems (considered below) describing the rolling motion of bodies belong to the class of so-called LR- and $L + R$-systems on Lie groups [27, 52]. Several results on the existence of invariant measure for such systems are known. We do not consider here these general results, especially useful for multidimensional generalizations. Note also that the general methods of reduction of nonholonomic systems were examined in many papers, see for example [58].

Remark 18. Generalization of the Veselova constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = d \neq 0$ was considered in [59]. Using Chaplygin's method of integration for a dynamically asymmetric ball with non-zero constant of areas [15], the author presented an explicit integration of the equations.

6.2. Conformally Hamiltonian Representation

It was shown in [52] that the Veselova system is the Chaplygin system (4.4) with $b(q) = 0$ and, therefore, upon the time substitution $\mathcal{N}\,dt = d\tau$, it can be written in the Hamiltonian form, where the reducing multiplier is $\mathcal{N} = \rho_\omega^{-1}$. Let us show this explicitly using the local coordinates (namely, the Euler angles θ, φ, ψ) and then apply the obtained canonical Poisson structure of the cotangent bundle of the sphere T^*S^2 to construct an algebraic Poisson bracket of redundant variables $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$. With such an algebraization of the Poisson structure, one can naturally establish an isomorphism with the Neumann system describing the dynamics of a point on a sphere in a quadratic potential. Originally, this isomorphism was straightforwardly established in [16, 27]. Later, we will see that this analogy can be directly extended to the Chaplygin ball and the general Clebsch system (which includes the Neumann system as a particular case).

In terms of the Euler angles, the body's angular velocity $\boldsymbol{\omega}$ and the unit vector $\boldsymbol{\gamma}$ are given by

$$\boldsymbol{\omega} = (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi, \dot\psi\cos\theta + \dot\varphi), \quad \boldsymbol{\gamma} = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta). \tag{6.5}$$

The equation of the constraint is

$$f = (\boldsymbol{\omega}, \boldsymbol{\gamma}) = \dot\psi + \cos\theta\dot\varphi = 0, \tag{6.6}$$

Eliminating the undetermined Lagrange multiplier from the equations of motion we can write the equation for θ, φ as a Chaplygin system:

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial U}{\partial\theta} = \dot\varphi S, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\varphi}\right) - \frac{\partial T}{\partial\varphi} + \frac{\partial U}{\partial\varphi} = -\dot\theta S,\\ S = \frac{\partial T_0}{\partial\dot\psi}\bigg|_{\dot\psi=-\cos\theta\dot\varphi} = \sin^2\theta\left(\dot\theta(I_2 - I_1)\sin\varphi\cos\varphi - \dot\varphi(I_1\cos^2\varphi + I_2\sin^2\varphi + I_3)\right),\end{gathered} \tag{6.7}$$

where U is the potential energy of the body in an external field, $T_0 = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega})$ is the kinetic energy without the constraint, while T is the kinetic energy from which $\dot\psi$ is eliminated using the constraint

$$\begin{aligned}T = T_0\Big|_{\dot\psi=-\cos\theta\dot\varphi} = \frac{1}{2}I_1(\dot\theta\cos\varphi - \dot\varphi\sin\varphi\sin\theta\cos\theta)^2&\\ + \frac{1}{2}I_2(\dot\theta\sin\varphi + \dot\varphi\cos\varphi\sin\theta\cos\theta)^2 + \frac{1}{2}I_3\dot\varphi^2\sin^4\theta.&\end{aligned} \tag{6.8}$$

Remark 19. The representation (6.7) is obtained upon differentiation under the condition (6.6):

$$\frac{\partial T}{\partial\dot\theta} = \frac{\partial T_0}{\partial\dot\theta}, \quad \frac{\partial T}{\partial\dot\varphi} = \frac{\partial T_0}{\partial\dot\varphi} - \cos\theta\frac{\partial T_0}{\partial\dot\psi}, \quad \frac{\partial T}{\partial\theta} = \frac{\partial T_0}{\partial\theta} + \dot\varphi\sin\theta\frac{\partial T_0}{\partial\dot\psi}, \quad \frac{\partial T}{d\varphi} = \frac{\partial T_0}{\partial\varphi}.$$

Theorem 7 ([52]). *After the time substitution $\mathcal{N}\,dt = d\tau$, $\mathcal{N} = (\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})^{-1/2}$, the equations of motion of the Veselova system take the form of the Euler–Lagrange equations:*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\theta'}\right) - \frac{\partial L}{\partial\theta} = 0, \quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial\varphi'}\right) - \frac{\partial L}{\partial\varphi} = 0, \tag{6.9}$$

where $L = T - U\Big|_{\dot\theta=\mathcal{N}\theta',\,\dot\varphi=\mathcal{N}\varphi'}$ is the Lagrangian function; after time substitution it can be written in the form:

$$L = \frac{1}{2}\frac{(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}, \mathbf{I}(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}))}{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})} - U(\boldsymbol{\gamma}).$$

Proof. The proof is based on a simple computation test: after the time substitution, the right-hand side $\overline{S}$ of (6.9), calculated by virtue of (4.5), should vanish. □

The canonical Hamiltonian form of the equations of motion (6.9) can be obtained using the Legendre transformation

$$
\begin{gathered}
p_\theta = \frac{\partial L}{\partial \theta'}, \quad p_\varphi = \frac{\partial L}{\partial \varphi'}, \quad H = p_\theta \theta' + p_\varphi \varphi' - L, \\
\frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta}, \quad \frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial \theta}, \quad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi}.
\end{gathered} \tag{6.10}
$$

6.3. Algebraic Representation and Isomorphism with the Neumann System

Using the canonical variables of (6.10) and the time substitution $(\rho_\omega \sqrt{\det \mathbf{I}})^{-1} dt = d\tau$, one can write the equations of motion of the Veselova system in the Hamiltonian form on the (co)algebra of the Poisson brackets $e(3)$:

$$
\begin{gathered}
\boldsymbol{M} = \rho_\omega \mathbf{I}^{1/2} \boldsymbol{\omega}, \quad \boldsymbol{\Gamma} = \rho_\omega^{-1} \mathbf{I}^{-1/2} \boldsymbol{\gamma}, \\
\frac{d\boldsymbol{M}}{d\tau} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\Gamma} \times \frac{\partial H}{\partial \boldsymbol{\Gamma}}, \quad \frac{d\boldsymbol{\Gamma}}{d\tau} = \boldsymbol{\Gamma} \times \frac{\partial H}{\partial \boldsymbol{M}},
\end{gathered} \tag{6.11}
$$

$$
H = \frac{1}{2} (\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma}) (\boldsymbol{M}, \boldsymbol{M}) + \widetilde{U}(\boldsymbol{\Gamma}), \tag{6.12}
$$

where $\widetilde{U}(\boldsymbol{\Gamma}) = U\left(\rho_\omega \mathbf{I}^{1/2} \boldsymbol{\Gamma}\right)$, and, respectively,

$$
\begin{gathered}
\boldsymbol{\gamma}^2 = \boldsymbol{\Gamma}^2 = 1, \quad (\boldsymbol{\omega}, \boldsymbol{\gamma}) = (\boldsymbol{M}, \boldsymbol{\Gamma}) = 0, \quad \rho_\omega = (\boldsymbol{\gamma}, \mathbf{I}^{-1} \boldsymbol{\gamma})^{1/2} = (\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma})^{-1/2}, \\
\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \Gamma_j\} = \varepsilon_{ijk} \Gamma_k, \quad \{\Gamma_i, \Gamma_j\} = 0.
\end{gathered}
$$

Thus, we have a Hamiltonian system with Poisson brackets corresponding to the algebra $e(3)$, and the four-dimensional symplectic leaf of the structure corresponding to the real motion is given by $\boldsymbol{\gamma}^2 = 1$, $(\boldsymbol{M}, \boldsymbol{\Gamma}) = 0$ (for the classical Euler–Poisson equations a similar situation takes place if the constant of areas [37] is zero). Note also that the system (6.11), (6.12) determines a certain integrable potential system on a two-dimensional sphere and defines thereby a certain geodesic flow.

The inverse transformation is

$$
\boldsymbol{\omega} = (\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma})^{1/2} \mathbf{I}^{-1/2} \boldsymbol{M}, \quad \boldsymbol{\gamma} = (\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma})^{-1/2} \mathbf{I}^{1/2} \boldsymbol{\Gamma}.
$$

Therefore, a search for integrable potentials for the Veselova system is now reduced to the well-studied problem of search for integrable cases in a Hamiltonian system on $e(3)$ with Hamiltonian (6.12). So, if $U = 0$, then the additional integral (6.4) can be written as

$$
F = (\mathbf{I} \boldsymbol{M}, \boldsymbol{M})(\mathbf{I} \boldsymbol{\Gamma}, \boldsymbol{\Gamma}) - (\mathbf{I} \boldsymbol{M}, \boldsymbol{\Gamma})^2.
$$

(Note that $\{H, F\} = 0$ only on the level $(\boldsymbol{M}, \boldsymbol{\Gamma}) = 0$.)

It was noted in [16, 27] that for $U = 0$ the system (6.1) is equivalent to the Neumann problem. As we can see, such equivalence is not a result of the natural reduction of the Veselova system to the Hamiltonian form (6.11), (6.12) on $e(3)$. It turns out that the isomorphism with the Neumann system is caused by existence of a transformation that does not conserve the Poisson brackets but reduces the vector field to the required form on the level surface $H = \mathrm{const}$.

Indeed, let us consider a Hamiltonian system on $e(3)$ (in the case $(\boldsymbol{M}, \boldsymbol{\Gamma}) = 0$) defined by the Hamiltonian

$$
H = \alpha \frac{1}{2} \boldsymbol{M}^2 (\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma}) + \beta \frac{1}{2} \left((\boldsymbol{M}, \mathbf{I} \boldsymbol{M})(\boldsymbol{\Gamma}, \mathbf{I} \boldsymbol{\Gamma}) - (\boldsymbol{M}, \mathbf{I} \boldsymbol{\Gamma})^2 \right). \tag{6.13}
$$

It is clear that both terms are the first integrals of the system. The following holds true:

Proposition 6. *On a fixed level* $\dfrac{\boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})}{\det \mathbf{I}} = c$ *and* $(\boldsymbol{M}, \boldsymbol{\Gamma}) = 0$, *the vector field generated by the Hamiltonian* (6.13) *is isomorphic to the vector field of the Kirchhoff equations in the Clebsch case with zero value of constant of areas* $(\boldsymbol{L}, \boldsymbol{s}) = 0$; *this field can be written in the form*

$$\begin{aligned} \dot{\boldsymbol{s}} &= k(\alpha \boldsymbol{s} \times \boldsymbol{L} + \beta \boldsymbol{s} \times \mathbf{I}\boldsymbol{L}), \\ \dot{\boldsymbol{L}} &= k\left(\alpha c \boldsymbol{s} \times \mathbf{I}\boldsymbol{s} + \beta(\boldsymbol{L} \times \mathbf{I}\boldsymbol{L} - c(\det \mathbf{I})\boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})\right), \end{aligned} \qquad k = -\sqrt{\det \mathbf{I}}. \tag{6.14}$$

Proof. Let us change the variables

$$\boldsymbol{L} = \mathbf{I}^{-1/2}\boldsymbol{M}, \quad \boldsymbol{s} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2}\mathbf{I}^{1/2}\boldsymbol{\Gamma},$$

so that the relations $\boldsymbol{s}^2 = \boldsymbol{\Gamma}^2 = 1$, $(\boldsymbol{M}, \boldsymbol{\Gamma}) = (\boldsymbol{s}, \boldsymbol{L}) = 0$ hold. By virtue of linearity we consider the two cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ separately. In the first case the equations of motion in terms of the new variables read

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\sqrt{\det \mathbf{I}}\left(\boldsymbol{s} \times \boldsymbol{L} + (\boldsymbol{s}, \boldsymbol{L})\frac{(\boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})}\right), \\ \dot{\boldsymbol{L}} &= -\sqrt{\det \mathbf{I}}\frac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{\det \mathbf{I}(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})}\boldsymbol{s} \times \mathbf{I}\boldsymbol{s}. \end{aligned}$$

Hence, taking into account $(\boldsymbol{s}, \boldsymbol{L}) = 0$, $\dfrac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})} = \boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) = c \det \mathbf{I}$, we get the required result.

The case $\alpha = 0$, $\beta = 1$ can be considered analogously. □

If we consider c as a constant parameter, then the vector field (6.14) is generated on $e(3)$ by the following Hamiltonian

$$H = k\alpha\left(\frac{1}{2}\boldsymbol{M}^2 + \frac{c}{2}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})\right) + k\beta\left(\frac{1}{2}(\boldsymbol{M}, \mathbf{I}\boldsymbol{M}) - \frac{c}{2}\det \mathbf{I}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})\right).$$

If $\alpha = 1$, $\beta = 0$, we obtain the Hamiltonian of the Neumann case, while at $\alpha = 0$, $\beta = 1$ this is the Hamiltonian of the Bruns problem. With arbitrary α, β, this Hamiltonian corresponds to the general Clebsch case in the Kirchhoff equations [12, 37]. Using the representation (6.11), (6.12), we easily obtain the following theorem for the Veselova system:

Theorem 8 ([16, 27]). *After time substitution, the vector field of the Veselova problem* (*with* $U = 0$) *on the fixed level of the integral of energy* $H = h = \mathrm{const}$ *becomes isomorphic to the vector field of the Neumann problem.*

7. THE CHAPLYGIN BALL

7.1. Equations of Motion and Conservation Laws

Equations of motion of this system were derived earlier(1.20), he we will represent then in the form

$$\begin{gathered} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathbf{I}_Q\boldsymbol{\omega}, \quad D = mR^2, \end{gathered} \tag{7.1}$$

where $\boldsymbol{\omega}$ is the ball's angular velocity, $\boldsymbol{\gamma}$ is the vertical unit vector in the moving frame of reference, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the ball's tensor of inertia with respect to its center, m and R are the ball's mass and radius, and $U = U(\boldsymbol{\gamma})$ is the potential of the external axisymmetric field. The vector $\boldsymbol{M}$ is the ball's angular momentum with respect to the point of contact. We present the tensor $\mathbf{I}_Q$ in the form

$$\mathbf{I}_Q = \mathbf{J} - D\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}, \quad \mathbf{J} = \mathbf{I} + D\mathbf{E}.$$

Equations (6.1) (with an arbitrary potential) admit the integral of energy, the geometric integral and the integral of areas:

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const.} \tag{7.2}$$

They also admit the invariant measure indicated by Chaplygin [15], $\rho_\mu \, d^3\boldsymbol{M} \, d^3\boldsymbol{\gamma}$, with density

$$\rho_\mu = (\det \mathbf{I}_Q)^{-1/2} = \left[\det \mathbf{J}\left(1 - D\left(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}\right)\right)\right]^{-1/2}. \tag{7.3}$$

If there is no external field ($U = 0$), the system (6.1) has an additional integral

$$F = (\boldsymbol{M}, \boldsymbol{M}), \tag{7.4}$$

hence, it is integrable according to the Euler–Jacobi theorem [15]. In [15], the solution of (7.1) was given in terms of hyperelliptic functions. A qualitative analysis of the dynamical behaviour of a Chaplygin ball was performed in [60] (stability of partial solutions was also studied in [61]).

Remark 20. The integral (7.4) can be generalized to the cases of the Bruns field $U(\boldsymbol{\gamma}) = \frac{k}{2}(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})$ [28] and gyrostat [62]. Other integrable potentials (with the zero constant of areas $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) can be found using the representation of the system on algebra $e(3)$, which is given below.

Remark 21. Although Chaplygin developed the general method of reducing multiplier [49], he did not apply it to the system (7.1). The paper [28] indicates possible obstructions to the application of this method to the system (7.1). On the other hand, already in [15] the following question was formulated: if the system (7.1) can be represented in Hamiltonian form? The problem of Hamiltonization of the Chaplygin ball was formulated more strictly by Kozlov [36] and Duistermaat [63]. In [46], the authors showed numerically that without a time substitution, the equations of motion of the Chaplygin ball are not Hamiltonian because the periods of motion for the orbits lying on the two-dimensional invariant resonance tori are not the same. However, after an appropriate time substitution the system (7.1) becomes Hamiltonian and Poisson brackets are found explicitly [64] (see below).

Unfortunately, the authors of the review [65] did not succeed in an explicit verifying our result. This result was confirmed in [66]. Here, we will prove the result of [64] using another method and then reveal an interesting *isomorphism between the Chaplygin ball and the Clebsch case* in the Kirchhoff equations. Another isomorphism has been indicated in [67].

7.2. Conformally Hamiltonian Representation

To describe the ball's rotation let us add to the system (7.1) the equations for the remaining direction cosines:

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}.$$

Such a system admits two additional integrals linear in velocities

$$(\boldsymbol{M}, \boldsymbol{\alpha}) = \text{const}, \quad (\boldsymbol{M}, \boldsymbol{\beta}) = \text{const}.$$

Under such an extension, the integral manifolds remain two-dimensional. Hence, the Chaplygin problem with $U = 0$ is degenerate or, as it is sometimes called, superintegrable. From this viewpoint, the Chaplygin ball is a nonholonomic analogue of the Euler–Poinsot top, a well-known noncommutative integrable system, and the phase space of this three-degree-of-freedom Hamiltonian system is foliated into two-dimensional tori (not three-dimensional according to the Liouville theorem).

It was shown in [64] that, for an arbitrary potential, after the time substitution and change of variables

$$\rho_\mu \, dt = d\tau, \quad \boldsymbol{L} = \rho_\mu \boldsymbol{M}, \tag{7.5}$$

the equations of motion (6.1) take in the Hamiltonian form:

$$\frac{dM_k}{d\tau} = \{H, M_k\}, \quad \frac{d\gamma_k}{d\tau} = \{H, \gamma_k\}$$

with the nonlinear Poisson bracket

$$\{L_i, L_j\} = \varepsilon_{ijk}\left(L_k - D(\boldsymbol{L},\boldsymbol{\gamma})\rho_\mu^2 J_i J_j \gamma_k\right), \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{7.6}$$

and the Hamiltonian is the energy (7.2), which can be written in the form:

$$H = \frac{\det \mathbf{J}}{2}\left(\left(1 - D\left(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}\right)\right)\left(\boldsymbol{L}, \mathbf{J}^{-1}\boldsymbol{L}\right) + D\left(\mathbf{J}^{-1}\boldsymbol{L}, \boldsymbol{\gamma}\right)^2\right) + U(\boldsymbol{\gamma}). \tag{7.7}$$

7.3. The Chaplygin Ball as a Generalized Chaplygin System

Now we show that the system (7.1) describing the motion of a Chaplygin ball is a generalized Chaplygin system (4.4), and the bracket (7.6) can be obtained using the method of reducing multiplier (see theorem 6).

As in the Veselova problem, we use now the local coordinates: the Euler angles θ, φ, ψ and the Cartesian coordinates of the ball's center x, y. In the moving frame of reference aligned with the ball's principal axes, the angular velocity vector and the normal to the plane are given by (6.5).

The equations of the constraints (corresponding to the no slip condition at the point of contact) can be written in the form

$$f_x = \dot{x} - R\dot{\theta}\sin\psi + R\dot{\varphi}\sin\theta\cos\psi = 0, \quad f_y = \dot{y} + R\dot{\theta}\cos\psi + R\dot{\varphi}\sin\theta\sin\psi = 0. \tag{7.8}$$

The equations of motion with Lagrange multipliers are follows:

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{x}}\right) = \lambda_x, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{y}}\right) = \lambda_y, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\psi}}\right) = 0, \\
\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\theta}}\right) - \frac{\partial T_0}{\partial \theta} = \lambda_x\frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y\frac{\partial f_y}{\partial \dot{\theta}}, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\varphi}}\right) - \frac{\partial T_0}{\partial \varphi} = \lambda_x\frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y\frac{\partial f_y}{\partial \dot{\varphi}},
\end{gathered} \tag{7.9}$$

where T_0 is the ball's kinetic energy without taking into account the constraints (7.8) (obviously, this energy does not depend on x, y, and ψ):

$$T_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}).$$

Eliminating the undetermined multipliers λ_x and λ_y with the help of the first two equations in (7.9) and the constraints (7.8) we get

$$\begin{aligned}
\lambda_x\frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y\frac{\partial f_y}{\partial \dot{\theta}} &= -mR^2(\ddot{\theta} + \dot{\psi}\dot{\varphi}\sin\theta), \\
\lambda_x\frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y\frac{\partial f_y}{\partial \dot{\varphi}} &= -mR^2(\ddot{\varphi}\sin\theta + \dot{\theta}\dot{\varphi}\cos\theta - \dot{\theta}\dot{\psi})\sin\theta.
\end{aligned}$$

Hence, the equations of motion for the angles θ and φ do not depend on ψ, but only on $\dot{\psi}$. Therefore, ψ is a cyclic variable and can be eliminated using the Routh reduction procedure; after that the equations of motion for θ and φ can be written as

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{R}}{\partial \theta} = -\dot{\varphi}S, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{R}}{\partial \varphi} = \dot{\theta}S, \\
S = mR^2\sin\theta\left(\dot{\varphi}\cos\theta + \dot{\psi}\right).
\end{gathered} \tag{7.10}$$

Here, $\mathcal{R}$ is the Routh function:

$$\mathcal{R}(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = T_0 - \dot{\psi}\frac{\partial T_0}{\partial \dot{\psi}},$$

from which $\dot{x}$ and $\dot{y}$ should be eliminated using the equations of constraints, while $\dot{\psi}$ is eliminated using the equation for the cyclic integral,

$$\begin{gathered}\frac{\partial T_0}{\partial \dot{\psi}} = (I_1 - I_2)\dot{\theta}\sin\theta\sin\varphi\cos\varphi + I_3\dot{\varphi}\cos\theta \\ + \big((I_1\sin^2\varphi + I_2\cos^2\varphi)\sin^2\theta + I_3\cos^2\theta\big)\,\dot{\psi} = c = \text{const.}\end{gathered} \tag{7.11}$$

So, we get the equations of motion in the form of a generalized Chaplygin system (4.4), and, since the system has an invariant measure, it is possible to write equations (7.10) in the Hamiltonian form with the bracket (4.6).

Perform the time substitution of the form

$$N(\theta,\varphi)dt = d\tau, \tag{7.12}$$

where $\mathcal{N} = \rho_\mu$ is the density of the invariant measure (7.3).

According to (4.5), the equations of motion, in terms of the new time, are

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial\overline{\mathcal{R}}}{\partial\theta'}\right) - \frac{\partial\overline{\mathcal{R}}}{\partial\theta} = -\varphi'\overline{S}, \quad \frac{d}{d\tau}\left(\frac{\partial\overline{\mathcal{R}}}{\partial\varphi'}\right) - \frac{\partial\overline{\mathcal{R}}}{\partial\varphi} = \theta'\overline{S}, \\ \overline{S} = c(I_3 + mR^2)mR^2\mathcal{N}^3\sin\theta(I_1\cos^2\varphi + I_2\sin^2\varphi + mR^2),\end{gathered} \tag{7.13}$$

where $\theta' = \dfrac{d\theta}{d\tau} = \mathcal{N}^{-1}\dot{\theta}$, $\varphi' = \dfrac{d\varphi}{d\tau} = \mathcal{N}^{-1}\dot{\varphi}$, $\overline{\mathcal{R}} = \mathcal{R}(\theta,\dot{\theta},\varphi,\dot{\varphi})\big|_{\dot{\theta}=\mathcal{N}\theta',\,\dot{\varphi}=\mathcal{N}\varphi'}$.

By applying the Legendre transformation to the system (7.13) we arrive at

Theorem 9. *Upon the time substitution $\rho_\mu dt = d\tau$ the equations of motion* (7.10) *for a Chaplygin ball can be written in the Hamiltonian form*:

$$\frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial\theta} - \overline{S}\frac{\partial H}{\partial p_\varphi}, \quad \frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \quad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial\varphi} + \overline{S}\frac{\partial H}{\partial p_\theta},$$

with the Poisson bracket of the form

$$\{\theta, p_\theta\} = \{\varphi, p_\varphi\} = 1, \quad \{p_\varphi, p_\theta\} = \overline{S}(\theta,\varphi), \quad \{\theta,\varphi\} = 0, \tag{7.14}$$

where

$$\begin{gathered}p_\theta = \frac{\partial\overline{\mathcal{R}}}{\partial\theta'}, \quad p_\varphi = \frac{\partial\overline{\mathcal{R}}}{\partial\varphi'}, \\ H = \theta' p_\theta + \varphi' p_\varphi - \overline{\mathcal{R}} \\ = \frac{1}{2}p_\theta^2\left(I_3\widetilde{I}_{12} - D(\boldsymbol{\gamma},\mathbf{I}\boldsymbol{\gamma})\right) + \frac{1}{2}\frac{p_\varphi^2}{\sin^2\theta}\left(I_1I_2\sin^2\theta + I_3\widetilde{I}_{12}\cos^2\theta - D(\boldsymbol{\gamma},\mathbf{I}\boldsymbol{\gamma})\right) \\ + \frac{p_\theta p_\varphi}{\sin\theta}I_3(I_1 - I_2)\cos\theta\sin\theta\sin\varphi\cos\varphi - \frac{\mathcal{N}cp_\theta}{\sin\theta}(I_1 - I_2)(I_3 + D\sin^2\theta)\sin\varphi\cos\varphi \\ - \frac{\mathcal{N}cp_\varphi}{\sin^2\theta}I_3(\widetilde{I}_{21} + D) + \frac{\mathcal{N}^2c^2}{\sin^2\theta}(I_3 + D\sin^2\theta)(\widetilde{I}_{21} + D), \\ \widetilde{I}_{12} = I_1\sin^2\varphi + I_2\cos^2\varphi, \quad \widetilde{I}_{21} = I_1\cos^2\varphi + I_2\sin^2\varphi.\end{gathered}$$

Expressing the variables $\boldsymbol{L} = \rho_\mu\boldsymbol{M}$ (7.5) in terms of the local variables θ, φ, p_θ, and p_φ, we find

$$L_1 = p_\theta\cos\varphi - p_\varphi\frac{\cos\theta\sin\varphi}{\sin\theta} + c\mathcal{N}\frac{\sin\varphi}{\sin\theta}, \quad L_2 = -p_\theta\sin\varphi - p_\varphi\frac{\cos\theta\cos\varphi}{\sin\theta} + c\mathcal{N}\frac{\cos\varphi}{\sin\theta}, \quad L_3 = p_\varphi.$$

Using such a transformation one can straightforwardly obtain (7.6).

7.4. The Chaplygin Ball with a Gyrostat

If we add a rotor having a constant angular velocity to a ball rolling over the plane then the equations of motion (7.1) take the form [62]

$$\dot{\boldsymbol{M}} = (\boldsymbol{M}+\boldsymbol{k})\times\boldsymbol{\omega}+\boldsymbol{\gamma}\times\frac{\partial U}{\partial\boldsymbol{\gamma}},\quad \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma}\times\boldsymbol{\omega},\quad \boldsymbol{M}=\mathbf{I}_Q\boldsymbol{\omega},$$

where $\boldsymbol{k}$ is the constant gyrostatic momentum vector. Similarly, for (7.10) we have to choose

$$T_0=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(\boldsymbol{\omega},\mathbf{I}\boldsymbol{\omega})+(\boldsymbol{\omega},\boldsymbol{k}).$$

This system possesses the same invariant measure (7.3) and, obviously, we can also apply the reducing multiplier method to the system (Theorem 6); moreover, for the corresponding function $\overline{S}$ we have

$$\overline{S}=N^3D\sin\theta\left(cJ_3(J_1\cos^2\varphi+J_2\sin^2\varphi)+\det\mathbf{J}(\boldsymbol{\gamma},\mathbf{J}^{-1}\boldsymbol{k}\right).$$

Using (7.14) with the new function $\overline{S}$, we find the following commutation relations (similar to (7.6)) for the components of the vectors $\boldsymbol{L}=\rho_\mu(\boldsymbol{M}+\boldsymbol{k})$, $\boldsymbol{\gamma}$:

$$\{L_i,L_j\}=\varepsilon_{ijk}\left(L_k-D\det\mathbf{J}\left(\rho_\mu^2(\boldsymbol{L},\boldsymbol{\gamma})-(\boldsymbol{\gamma},\mathbf{J}^{-1}\boldsymbol{k})\right)\gamma_k\right),\quad \{L_i,\gamma_j\}=\varepsilon_{ijk}\gamma_k,\quad \{\gamma_i,\gamma_j\}=0.$$

7.5. Two Isomorphisms with the Clebsch Case

1. Consider the integrable case $U=0$ at the zero constant of areas $(\boldsymbol{M},\boldsymbol{\gamma})=(\boldsymbol{L},\boldsymbol{\gamma})=0$ in more detail, because in this case bracket (7.6) corresponds to the algebra $e(3)$. Write the Hamiltonian (7.2) and the additional integral (7.4) in the variables $\boldsymbol{L}$ and $\boldsymbol{\gamma}$ in the form (omitting some inessential factors)

$$H=\frac{1}{2}\boldsymbol{L}^2(\boldsymbol{\gamma},\mathbf{B}\boldsymbol{\gamma})-\frac{1}{2}\left[(\boldsymbol{L},\mathbf{B}\boldsymbol{L})(\boldsymbol{\gamma},\mathbf{B}\boldsymbol{\gamma})-(\boldsymbol{\gamma},\mathbf{B}\boldsymbol{L})^2\right],$$
$$F=\boldsymbol{L}^2(\boldsymbol{\gamma},\mathbf{B}\boldsymbol{\gamma}),\quad \mathbf{B}=\mathbf{E}-D\mathbf{J}^{-1}=\mathbf{I}\mathbf{J}^{-1}.$$

Using Proposition 6, we obtain

Theorem 10. *After the change of time* (7.5) *and the change of variables*

$$\boldsymbol{s}=(\boldsymbol{\gamma},\mathbf{B}\boldsymbol{\gamma})^{-1/2}\mathbf{B}^{-1/2}\boldsymbol{\gamma},\quad \widetilde{\boldsymbol{L}}=\mathbf{B}^{-1/2}\boldsymbol{L}$$

the vector field (7.1) *for* $U=0$ *(on a fixed level* $(\boldsymbol{M},\boldsymbol{M})=\mathrm{const}$ *and* $(\boldsymbol{M},\boldsymbol{\gamma})=0$*) reduces to the vector field of the Clebsch case for the zero constant of areas.*

2. Yu.N. Fedorov indicated a somewhat different isomorphism of equations (7.1) for $U=0$ with the integrable Clebsch case. Indeed, we can easily derive the following equation of the invariant measure:

$$\dot{\rho}_\mu=-D((\boldsymbol{\gamma}\times\boldsymbol{\omega}),\mathbf{J}^{-1}\boldsymbol{\gamma})/\rho_\mu,\qquad \mathbf{J}=\mathbf{I}+D\mathbf{E},$$

which gives the system in $\boldsymbol{Q}=\mathbf{I}\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$:

$$\frac{d}{dt}(\rho_\mu\boldsymbol{Q})=\mathbf{I}\mathbf{J}^{-1}(\boldsymbol{Q}\times\rho_\mu I^{-1}\boldsymbol{Q})+((\boldsymbol{Q}\times\mathbf{I}^{-1}\boldsymbol{Q}),D\mathbf{J}^{-1}\boldsymbol{\gamma})\mathbf{I}\mathbf{J}^{-1}\boldsymbol{\gamma}/\rho_\mu-D\boldsymbol{Q}((\boldsymbol{\gamma}\times\mathbf{I}^{-1}\boldsymbol{Q}),\mathbf{J}^{-1}\boldsymbol{\gamma})/\rho_\mu$$
$$\frac{d\boldsymbol{\gamma}}{dt}=\boldsymbol{\gamma}\times\mathbf{I}^{-1}\boldsymbol{Q}.$$

With the new time τ and new momentum $\boldsymbol{K}$

$$dt=\rho_\mu d\tau,\qquad \boldsymbol{K}=\rho_\mu\boldsymbol{Q}=\rho_\mu\mathbf{I}\boldsymbol{\omega}$$

and using some formulas from the vector algebra on the level $(\boldsymbol{Q}, \boldsymbol{\gamma}) = (\boldsymbol{K}, \boldsymbol{\gamma}) = 0$, we get

$$\frac{d\boldsymbol{K}}{d\tau} = \boldsymbol{K} \times \mathbf{I}^{-1}\boldsymbol{K} - \frac{D}{\det \mathbf{J}} \frac{(\mathbf{J}\boldsymbol{K}, \mathbf{I}^{-1}\boldsymbol{K})}{(\mathbf{I}\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma})} \boldsymbol{\gamma} \times \mathbf{I}\boldsymbol{\gamma},$$
$$\frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma} \times \mathbf{I}^{-1}\boldsymbol{K}.$$

This system has the integral

$$\mathcal{L} = \frac{(\mathbf{J}\boldsymbol{K}, \mathbf{I}^{-1}\boldsymbol{K})}{(\mathbf{I}^{-1}\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})}.$$

Thus, on the level $\mathcal{L} = \beta = \text{const}$ we get a system

$$\frac{d\boldsymbol{K}}{d\tau} = \boldsymbol{K} \times \mathbf{I}^{-1}\boldsymbol{K} - \beta' \boldsymbol{\gamma} \times \mathbf{I}\boldsymbol{\gamma} \qquad \beta' = \frac{D}{\det \mathbf{J}} \beta.$$
$$\frac{d\boldsymbol{\gamma}}{d\tau} = \boldsymbol{\gamma} \times \mathbf{J}^{-1}\boldsymbol{K}$$

The system corresponds to the integrable Clebsch case for the Kirchhoff equations. This isomorphism differs from the one indicated above. It would be interesting to reveal the relation between these results.

8. THE CHAPLYGIN BALL ON A SPHERE

8.1. Equations of Motion and Cases of Integrability

The equations that govern the motion without slipping of a spherical body over a spherical surface can be written as [68]

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega}, \quad k = \frac{a}{a+b}, \tag{8.1}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body, $\boldsymbol{n}$ is the normal at the point of contact, a is the radius of the body, and b is the radius of the surface (Fig. 6). Hereinafter all vectors and tensors are given in the moving coordinate frame whose axes are aligned with the principal axes of inertia of the body. At the point of contact, the angular momentum $\boldsymbol{M}$ linearly depends on the angular velocity $\boldsymbol{\omega}$:

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + d\boldsymbol{n}(\boldsymbol{n} \times \boldsymbol{\omega}), \quad d = mb^2,$$

Here m is the mass of the body and $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the central inertia tensor. The value of the coefficient k is clear from Fig.6.

This is the problem on a spherical suspension. A strategy for investigation of gyroscopes with spherical rolling thrust bearing on a spherical foundation was suggested by Contensou [69] in connection with the problem of stabilization of the Fleuriais gyroscope.

The case $k = 1$ corresponds to $a \to \infty$, i.e. a ball is rolling over a plane (a Chaplygin ball); this system is known to be integrable and studied in many works [1, 15, 28, 60].

For an arbitrary k the system (8.1) has three integrals of motion

$$F_0 = (\boldsymbol{n}, \boldsymbol{n}) = 1, \quad H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}), \quad F_1 = (\boldsymbol{M}, \boldsymbol{M}) \tag{8.2}$$

and preserves measure $\rho\, d\boldsymbol{\omega}\, d\boldsymbol{n}$ whose density reads [70]

$$\rho^2 = (\boldsymbol{n}, \boldsymbol{n}) - d(\boldsymbol{n}, (\mathbf{I} + d)^{-1}\boldsymbol{n}).$$

There is another remarkable case of integrability for (8.1) with $k = -1$ (A.V. Borisov, Yu.N. Fedorov [68]): a spherical body (Fig. 6c) rolls on a ball which is fixed in space. The radii of the body and the ball are connected by the equation $\frac{b}{a} = \frac{1}{2}$. An additional linear integral (which is an analogue of the integral of areas) reads

$$F_2 = (\mathbf{A}\boldsymbol{M}, \boldsymbol{n}), \tag{8.3}$$

where $\mathbf{A} = \operatorname{diag}(\frac{1}{2}(-I_1 + I_2 + I_3), \frac{1}{2}(I_1 - I_2 + I_3), \frac{1}{2}(I_1 + I_2 - I_3))$.

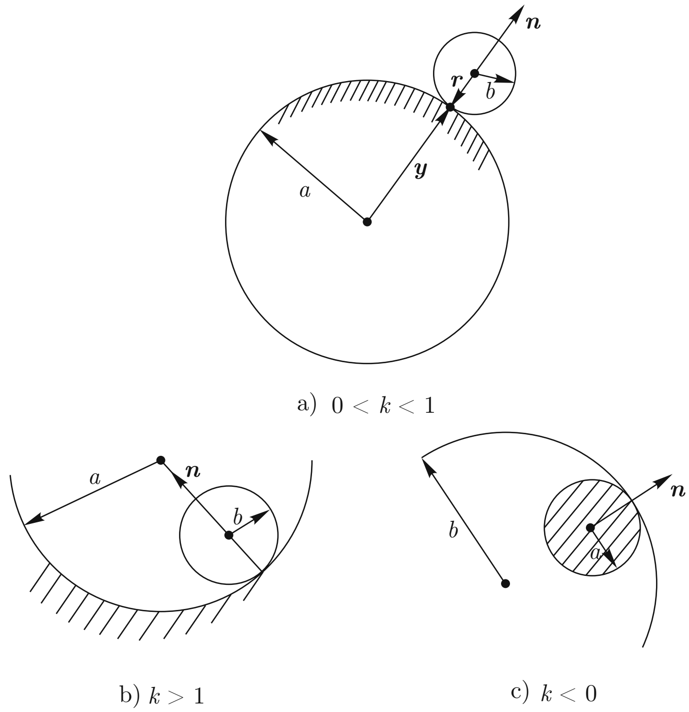

Fig. 6. Rolling of a spherical body over a fixed surface. The surface is shown by hatching.

Note that an analogous system (for $k = -1$) with the additional constraint $(\boldsymbol{\omega}, \boldsymbol{n}) = 0$, which prevents spinning, is also integrable (for its explicit integration see [71]).

8.2. Reduction to the Chaplygin-System Form and the Conformally Hamiltonian Form for $k = -1$

Here we show how to obtain the solutions to (8.1) in terms of quadratures for the case of $k = -1$ and $F_2 = 0$ [72]. This problem is identical to the integration of the equations of motion in Chaplygin's problem (rolling of a sphere on a plane) with zero integral of areas [15] (see also [60, 73]). We do not consider the case $F_2 \neq 0$ (at least for the time being).

First, using $F_2 = 0$, we solve for the angular velocity from (8.1):

$$\boldsymbol{\omega} = \frac{\mathbf{B}\boldsymbol{n} \times \dot{\boldsymbol{n}}}{(\boldsymbol{n}, \mathbf{B}\boldsymbol{n})}, \quad \mathbf{B} = (\mathbf{I} + d - d\boldsymbol{n} \otimes \boldsymbol{n})\mathbf{A}. \tag{8.4}$$

Next, on the sphere $\boldsymbol{n}^2 = 1$ we introduce sphero-conical coordinates u, v by the formulas

$$n_i^2 = \frac{(J_i - u)(J_i - v)}{(J_i - J_j)(J_i - J_k)}, \quad i \neq j \neq k \neq i, \quad J_i = I_i + d. \tag{8.5}$$

Using (8.5), we represent the equations of motion (8.1) in the Chaplygin-system form (see Section 4)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial u} = \dot{u}\Phi, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial T}{\partial v} = -\dot{v}\Phi \tag{8.6}$$

where $T = \frac{1}{2}(b_{uu}\dot{u}^2 + b_{uv}\dot{u}\dot{v} + b_{vv}\dot{v}^2)$ is the integral of energy H (8.2) expressed in the sphero-conical coordinates on the surface $F_2 = 0$ and $\Phi = (a_u\dot{u} + a_v\dot{v})$ is a linear and homogeneous in velocities function. We do not give here the expressions of T, Φ because they are too bulk.

According to Chaplygin's method of reducing multiplier, after the change of time $\mathcal{N}(u, v)dt = d\tau$, the system (8.6) can be represented in the Lagrangian form

$$\frac{d}{d\tau}\frac{\partial T}{\partial u'} - \frac{\partial T}{\partial u} = 0, \quad \frac{d}{d\tau}\frac{\partial T}{\partial v'} - \frac{\partial T}{\partial v} = 0, \quad u' = \frac{du}{d\tau}, \quad v' = \frac{dv}{d\tau}.$$

The reducing multiplier $\mathcal{N}$ is equal to the density of the invariant measure $\mathcal{N}\,du\,dv\,dP_u\,dP_v$ (where $P_u = \frac{\partial T}{\partial \dot u}$, $P_v = \frac{\partial T}{\partial \dot v}$):

$$\mathcal{N} = \frac{2uv + (u+v)(2d+\alpha_1) + \alpha_2 - d\alpha_1}{\sqrt{\det(\mathbf{I} + d - d\boldsymbol{n}\otimes\boldsymbol{n})}}(4\alpha_3 + 2\alpha_1\alpha_2 - \alpha_1^3 - d\alpha_1^2$$
$$+(\alpha_1^2 - 2\alpha_2 + 4d\alpha_1)(u+v) - 4d(u+v)^2)^{-1}.$$

Here $\alpha_1 = \sum J_i$, $\alpha_2 = \sum J_i^2, \alpha_3 = J_1J_2J_3$.

Thus after a change of time we get a Hamiltonian system on the two-dimensional sphere S^2; this system can be represented as equations on a special (zero) orbit of the co-algebra $e(3)$. Finally we get

$$\begin{gathered}
H = \frac{\delta \det \mathbf{J}}{8(\boldsymbol{\gamma},\,\mathbf{B}\boldsymbol{\gamma})^2}\sum_{i=1}^{3} c_i m_i^2, \quad F_2 = \frac{\rho^2}{4(\boldsymbol{\gamma},\,\mathbf{B}\boldsymbol{\gamma})^2}(\delta^2\boldsymbol{m}^2 - 4\sum_{i=1}^{3} d_i m_i^2),\\
\delta = (\boldsymbol{\gamma},\,\mathbf{J}\overline{\mathbf{A}}\boldsymbol{\gamma}) - d(\boldsymbol{\gamma},\,\overline{\mathbf{A}}\boldsymbol{\gamma})^2,\\
c_i = \frac{\rho^2\delta}{J_i} - 4\prod_{k\neq i}(J_i - J_k)\gamma_i^2(\rho^2 - \frac{d\delta}{4J_i\det\mathbf{J}}),\\
d_i = \prod_{k\neq i}(J_i - J_k)\gamma_i^2(\delta(J_i + d) - (\boldsymbol{\gamma},\,\mathbf{J}(\mathbf{J}+d)\overline{\mathbf{A}}^2\boldsymbol{\gamma}) + 2d(\boldsymbol{\gamma},\,\mathbf{J}\overline{\mathbf{A}}\boldsymbol{\gamma})(\boldsymbol{\gamma},\,\overline{\mathbf{A}}\boldsymbol{\gamma})),
\end{gathered}\tag{8.7}$$

where $\overline{\mathbf{A}} = 2\mathbf{A}$. The Poisson brackets are

$$\{m_i,\,m_j\} = \varepsilon_{ijk}m_k, \quad \{m_i,\,\gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i,\,\gamma_j\} = 0,$$

whereas the orbit is determined by the constants of the integrals

$$\boldsymbol{\gamma}^2 = 1, \quad (\boldsymbol{m},\,\boldsymbol{\gamma}) = 0.$$

Remark 22. One can show that for $k = -1$ and $F_2 \neq 0$ the equations of motion of the form (8.6) correspond to the generalized Chaplygin system (4.4) (Section 4). Thus, using Theorem 6, the equations of motion can be represented as Hamiltonian equations with gyroscopic forces. We do not write out these equations here as they are enormously huge (while the general methods for integration (in quadratures) of such systems are not yet sufficiently elaborated).

8.3. Separation of Variables

We will use the following canonical representation of the algebra $e(3)$ (Darboux coordinates):

$$\begin{gathered}
m_1 = p_1(x^2-1) + p_2(y^2-1), \quad m_2 = ip_1(x^2+1) + ip_2(y^2+1),\\
m_3 = 2p_1x + 2p_2y, \quad \gamma_1 = \frac{xy-1}{x-y}, \quad \gamma_2 = i\frac{xy-1}{x-y}, \quad \gamma_3 = \frac{x+y}{x-y}
\end{gathered}\tag{8.8}$$

Using (8.8), we write the pair (H, F_2) in the canonical form [74]

$$\begin{gathered}
H = a(x,\,y)p_1^2 + 2b(x,\,y)p_1p_2 + c(x,\,y)p_2^2,\\
F_2 = A(x,\,y)p_1^2 + 2B(x,\,y)p_1p_2 + C(x,\,y)p_2^2.
\end{gathered}\tag{8.9}$$

It can be shown [74, 75] that to get separable variables it is sufficient to solve the equation

$$(B - bs)^2 = (A - as)(C - cs). \tag{8.10}$$

In the new coordinates the functions H and F take Liouville's form

$$H = \frac{S_1(s_1)}{s_1 - s_2}P_1^2 - \frac{S_2(s_2)}{s_1 - s_2}P_2^2, \quad F = \frac{s_2S_1(s_1)}{s_1 - s_2}P_1^2 - \frac{s_1S_2(s_2)}{s_1 - s_2}P_2^2. \tag{8.11}$$

Let

$$\alpha = (J_2 + J_1 - J_3)(-J_2 + J_2 - J_3)(-J_2 + J_2 + J - 3)$$
$$\beta = J_1^2 + J_2^2 + J_3^2 - 2J_1J_2 - 2J_2J_3 - 2J_3J_1$$
$$\gamma = J_1J_2J_3, \quad \epsilon = J_1 + J_2 + J_3,$$

and the function $S(x)$ now reads

$$S(x) = \frac{2(8x^3 + 8(d-\epsilon)x^2 + (2\epsilon^2\beta - 4d\epsilon)x - 4\gamma - d\beta + \sqrt{\triangle})}{\gamma(2x - \epsilon + 2d)^2}, \tag{8.12}$$

where $\triangle = x^2(\beta^2 + 8\alpha d) + 2x(4\beta\gamma + d\beta^2 - 2d\alpha\epsilon + 4\alpha d^2) + (4\gamma + d\beta)^2$.

If $d = 0$, then the rational function S simplifies to

$$S(x) = 16\frac{(x-J_1)(x-J_2)(x-J_3)}{J_1J_2J_3(2x - J_1 - J_2 - J_3)^2}$$
$$\times\,(x(J_1^2 + J_2^2 + J_3^2 - 2J_1J_2 - 2J_2J_3 - 2J_3J_1) + 4J_1J_2J_3)^2.$$

The case of $d = 0$ corresponds to the classical and thoroughly explored Euler–Poinsot problem (integration of this problem is studied in quite a number of works [37]). Nevertheless, our method for $d = 0$ differs from the classical methods of integration although these approaches are connected with a linear-fractional function.

9. RUBBER BALL ON A SPHERE

9.1. Equations of Motion

The governing equations (8.1) for the rolling motion of a smooth ball (spinning is allowed) over a sphere have been obtained above. Suppose that now an additional constraint is imposed so that there is no spinning:

$$(\boldsymbol{\omega}, \boldsymbol{n}) = 0. \tag{9.1}$$

Here $\boldsymbol{\omega}$ is the angular velocity and $\boldsymbol{n}$ is the normal at the point of contact (Fig. 6). In the ball-fixed coordinate system (the coordinate axes are aligned with the principal inertia axes), the equations of motion can be written as

$$\begin{gathered}\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \lambda\boldsymbol{n} + \boldsymbol{M}_Q, \quad \dot{\boldsymbol{n}} = k\boldsymbol{n} \times \boldsymbol{\omega},\\ \mathbf{J} = \mathbf{I} + mb^2\mathbf{E}, \quad \mathbf{E} = \|\delta_{ij}\|,\end{gathered} \tag{9.2}$$

where

$$\lambda = -\frac{(\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega}, \mathbf{J}^{-1}\boldsymbol{n}) + (\boldsymbol{M}_Q, \mathbf{J}\boldsymbol{n})}{(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})},$$

where $\boldsymbol{M}_Q$ is the moment of the external forces.

Hereafter we assume that no external forces are applied and therefore $\boldsymbol{M}_Q = 0$.

9.2. Conservation Laws and Cases of Integrability

The system (9.2) preserves measure

$$(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{\frac{1}{2k}} d\boldsymbol{\omega}\, d\boldsymbol{n}. \tag{9.3}$$

It should be noted that the measure (9.3) is preserved as well when $\boldsymbol{M}_Q$ depends only on $\boldsymbol{n}$ and is independent of $\boldsymbol{\omega}$. The measure (9.3) was found in [14].

Equations (9.2) possess the integral of energy and the geometric integral:

$$H = \frac{1}{2}(\mathbf{J}\boldsymbol{\omega}, \boldsymbol{\omega}), \quad (\boldsymbol{n}, \boldsymbol{n}) = 1.$$

The constraint (9.1) can be considered as additional partial integral and therefore for the system to be integrable, by the Euler–Jacobi last-multiplier theorem, it is necessary to find one more integral F. It can be found in the following situations.

1) $k = 1$ $(a = \infty)$ — rolling motion of a ball on a horizontal plane, $F = (\mathbf{J}\boldsymbol{\omega} \times \boldsymbol{n}, \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{n})$. As shown above (Section 2), the system that ensue is equivalent to Veselova's system (Section 6) and can be integrated with the help of sphero-conical coordinates.

2) $k = -1$ $(b = -2a)$ — rolling of a dynamically non-symmetric hollow sphere over a fixed ball that touches it from the inside. The additional integral is

$$F = \frac{(\mathbf{J}\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})\boldsymbol{n}^2 + \det \mathbf{J}(\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})(\mathbf{J}^{-1}\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}{(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})}. \tag{9.4}$$

We found this integral using the analogy between the system (9.2) and a certain Hamiltonian system that governs the motion of a particle on a two-dimensional sphere. Let us consider this analogy in greater detail.

9.3. Hamiltonian Structure and Algebraization

On the sphere $|\boldsymbol{n}| = 1$ introduce sphero-conical coordinates (ξ, η). They are roots of the equation

$$f(z) = \frac{n_1^2}{J_1 - z} + \frac{n_2^2}{J_2 - z} + \frac{n_3^2}{J_3 - z} = \frac{(z - \xi)(z - \eta)}{A(z)}, \tag{9.5}$$

where $A(z) = (J_1 - z)(J_2 - z)(J_3 - z)$. One can show that $0 < J_1 < \xi < J_2 < \eta < J_3$. From (9.5) we get

$$n_1^2 = \frac{(J_1 - \xi)(J_1 - \eta)}{(J_1 - J_2)(J_1 - J_3)}, \quad n_2^2 = \frac{(J_2 - \xi)(J_2 - \eta)}{(J_2 - J_1)(J_2 - J_3)}, \quad n_3^2 = \frac{(J_3 - \xi)(J_3 - \eta)}{(J_3 - J_1)(J_3 - J_2)}. \tag{9.6}$$

Using the equation for constraint, one can easily express $\boldsymbol{\omega}$ through $\boldsymbol{n}$ and $\dot{\boldsymbol{n}}$: $\boldsymbol{\omega} = k^{-1}\dot{\boldsymbol{n}} \times \boldsymbol{n}$. From (9.6), we can find the angular velocity $\boldsymbol{\omega}$ as a function of $\dot{\xi}$, $\dot{\eta}$, ξ, η. The kinetic energy $T = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{J}\boldsymbol{\omega})$ can also be expressed through $\dot{\xi}$, $\dot{\eta}$, ξ, η:

$$T = \frac{\xi - \eta}{8k^2}\left(-\frac{\eta\dot{\xi}^2}{A(\xi)} + \frac{\xi\dot{\eta}^2}{A(\eta)}\right). \tag{9.7}$$

One can show that the governing equation can be written in the Chaplygin-system form (Section 4)

$$\begin{gathered} \frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}} - \frac{\partial T}{\partial \xi} = \dot{\eta}S, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) - \frac{\partial T}{\partial \eta} = -\dot{\xi}S, \\ S = \frac{2k - 1}{8k^3}(\xi - \eta)\left(\frac{\dot{\xi}}{A(\xi)} + \frac{\dot{\eta}}{A(\eta)}\right). \end{gathered} \tag{9.8}$$

Instead of the velocities $\dot{\xi}$, $\dot{\eta}$ consider generalized momenta $P_\xi = \frac{\partial T}{\partial \dot{\xi}}$, $P_\eta = \frac{\partial T}{\partial \dot{\eta}}$. In the new variable the invariant measure takes the form $(\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n})^{-1 + \frac{1}{2k}} dP_\xi dP_\eta d\xi d\eta$. Introduce the notation

$$\rho^2 = (\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n}) = \frac{\xi\eta}{\det \mathbf{J}}, \quad \mathcal{N} = \left(\frac{\xi\eta}{\det \mathbf{J}}\right)^{-1 + \frac{1}{2k}}. \tag{9.9}$$

Then we perform the change of time $\mathcal{N}\, dt = d\tau$; the prime will denote differentiation with respect to τ. By Theorem 6 we conclude that after the change of time the equations of motion become canonical

$$\xi' = \frac{\partial T}{\partial p_\xi}, \quad \eta' = \frac{\partial T}{\partial p_\eta}, \quad p_\xi' = -\frac{\partial T}{\partial \xi}, \quad p_\eta' = -\frac{\partial T}{\partial \eta}, \tag{9.10}$$

that is, $\mathcal{N}$ turns out to be a Chaplygin reducing multiplier.

Thus we have shown that *for $k = \pm 1$ the system* (9.2) *is conformally Hamiltonian.*

Using some other techniques, the conformally Hamiltonian form of the system (9.2) was first discovered in [14] (see also [76]). For $k = \frac{1}{2}$ $(a = b)$ the system (9.2), (9.8) is an ordinary Hamiltonian system [14, 76].

The coordinates ξ, η are local. To be able to establish explicitly an isomorphism between (9.2), (9.8) and the problem of motion of a point on a sphere, we use a new algebraic (redundant) set of variables. Let

$$\widetilde{\mathcal{N}} = \frac{\sqrt{\det \mathbf{J}}}{k}\frac{N}{\rho^2} = \frac{\sqrt{\det \mathbf{J}}}{k}(\boldsymbol{n}, J^{-1}\boldsymbol{n})^{\frac{1}{2k}}.$$

Consider three-dimensional vectors

$$\boldsymbol{M} = \widetilde{\mathcal{N}}\mathbf{J}^{1/2}\boldsymbol{\omega}, \quad \boldsymbol{\gamma} = \frac{1}{\rho}\mathbf{J}^{-1/2}\boldsymbol{n}. \tag{9.11}$$

Obviously,

$$(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = \frac{\widetilde{\mathcal{N}}}{\rho}(\boldsymbol{\omega}, \boldsymbol{n}) = 0. \tag{9.12}$$

Using the expressions for $\boldsymbol{\omega}$, $\boldsymbol{n}$ as functions of ξ, η, p_ξ,p_η, we find the Poisson bracket of $\boldsymbol{M}$ and $\boldsymbol{\gamma}$ to be

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0. \tag{9.13}$$

The Hamiltonian reads

$$\begin{aligned} \mathcal{H} = T = \frac{1}{2}\widetilde{\mathcal{N}}^{-2}\boldsymbol{M}^2 &= \frac{1}{2}\frac{k^2}{\det \mathbf{J}}(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{1/k}\boldsymbol{M}^2 \\ &= \frac{2k^2}{\xi - \eta}\left(\frac{\xi\eta}{\det \mathbf{J}}\right)^{(2k-1)/k}\left(\frac{A(\xi)}{\eta}p_\xi^2 - \frac{A(\eta)}{\xi}p_\eta^2\right). \end{aligned} \tag{9.14}$$

The bracket (9.13) is a Lie–Poisson bracket of the (co)algebra $e(3)$. The functions $F_1 = (\boldsymbol{M}, \boldsymbol{\gamma})$, $F_2 = (\boldsymbol{\gamma}, \boldsymbol{\gamma})$ are its Casimir functions.

9.4. Trajectory Isomorphism

The Hamiltonian function (9.14) is the product of two functions, one of which depends on $\boldsymbol{M}$ and the other on $\boldsymbol{\gamma}$: $\mathcal{H} = G(\boldsymbol{\gamma})F(\boldsymbol{M})$. The equations of motion can be written as

$$\dot{\boldsymbol{M}} = G\left(\boldsymbol{M} \times \frac{\partial F}{\partial \boldsymbol{M}} - FG\boldsymbol{\gamma} \times \frac{\partial G^{-1}}{\partial \boldsymbol{\gamma}}\right), \quad \dot{\boldsymbol{\gamma}} = G\boldsymbol{\gamma} \times \frac{\partial F}{\partial \boldsymbol{M}}. \tag{9.15}$$

Perform the change of time $G(\boldsymbol{\gamma})dt = ds$ and fix a level surface of the integral $FG = h$. On this level we get the system

$$\begin{gathered} \frac{d\boldsymbol{M}}{ds} = \boldsymbol{M} \times \frac{\partial \widetilde{\mathcal{H}}}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial \widetilde{\mathcal{H}}}{\partial \boldsymbol{\gamma}}, \quad \frac{d\boldsymbol{\gamma}}{ds} = \boldsymbol{\gamma} \times \frac{\partial \widetilde{\mathcal{H}}}{\partial \boldsymbol{s}}, \\ \widetilde{\mathcal{H}} = F(\boldsymbol{M}) - \frac{h}{G(\boldsymbol{\gamma})}. \end{gathered} \tag{9.16}$$

Thus on a fixed level of the energy integral $\mathcal{H} = h$ the system (9.15) is trajectory-equivalent to the system (9.16) with $\widetilde{\mathcal{H}} = 0$. In celestial mechanics this procedure is known as reguliarization and it traces back to Bolin and Levi-Civita.

For the Hamiltonian (9.14) we have

$$\widetilde{\mathcal{H}} = \frac{1}{2}\boldsymbol{M}^2 - h(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{-1/k}, \quad h = \text{const}. \tag{9.17}$$

It is well known that the Hamiltonian (9.17)describes the motion of material point on a sphere (just under the condition $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) in a potential field with potential $V = h(\boldsymbol{\gamma}, \mathbf{J}\boldsymbol{\gamma})^{-1/k}$. Integrable potentials correspond to $k = 1$ (Braden's system [77]) and $k = -1$ (Neumann's system). In the other cases the Hamiltonian system (9.17) is likely integrable. At least one can prove that an additional integral quadratic in $\boldsymbol{M}$ does not exist. However, integrals of higher degree in momenta still may exist.

9.5. Separation of Variables for $k = -1$

The discussed above isomorphism allows determination of separable variable for the case $k = -1$. As before introduce sphero-conical coordinates as roots of the function

$$g(z) = \sum_{i=1}^{3} \frac{n_1^2}{J_i(J_i - z)} = \frac{(z-u)(z-v)}{(\det \mathbf{J})A(z)},$$

where $A(z) = \prod\limits_i (J_i - z)$ and $0 < J_1 < u < J_2 < v < J_3$. Thus

$$n_1^2 = \rho^2 \frac{J_1(J_1-u)(J_1-v)}{(J_1-J_2)(J_1-J_3)}, \quad n_2^2 = \rho^2 \frac{J_2(J_2-u)(J_2-v)}{(J_2-J_1)(J_2-J_3)}, \quad n_2^2 = \rho^2 \frac{J_3(J_3-u)(J_3-v)}{(J_3-J_1)(J_3-J_2)},$$

where $\rho^2 = (\boldsymbol{n}, \mathbf{J}^{-1}\boldsymbol{n}) = \left(\sum\limits_i J_i - u - v\right)^{-1}$. For the kinetic energy (9.7) we have

$$T = \frac{\det \mathbf{J} \rho^4 (u-v)}{8} \left(\frac{\dot{u}^2}{A(u)} - \frac{\dot{v}^2}{A(v)} \right).$$

From (9.9) we find the reducing multiplier

$$\mathcal{N} = \left(\sum_i J_i - u - v \right)^{-3/2},$$

After the change of time $\mathcal{N}\,dt = d\tau$ we obtain the canonical Hamiltonian equations in u, v, $p_u = \mathcal{N}\frac{\partial T}{\partial \dot{u}}$, $p_v = \mathcal{N}\frac{\partial T}{\partial \dot{v}}$

$$\mathcal{H} = \frac{2}{\det \mathbf{J} \left(\sum\limits_i J_i - u - v\right)(u-v)} \left(A(u)p_u^2 - A(v)p_v^2 \right).$$

Thus the coordinates u and v are separable.

10. DYNAMICALLY NON-SYMMETRIC, UNBALANCED BALL ON A PLANE

Consider in greater detail the motion of a dynamically non-symmetric ball whose center of mass does not generally coincide with its geometric center. As before consider two nonholonomic models of rolling motion:

1) rolling without sliding, dut spinning is allowed (marble ball);

2) both sliding and spinning are prohibited (rubber ball).

Assume that there are no external forces.

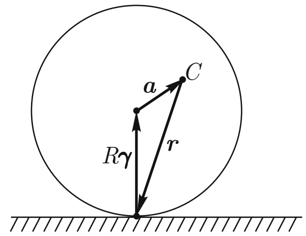

Fig. 7. Ball with a displaced center on a plane.

10.1. Marble Ball

By analogy with problem on the motion of a Chaplygin ball we write the equations of motion of this system as

$$\begin{gathered}\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + m\dot{\boldsymbol{r}} \times (\boldsymbol{\omega} \times \boldsymbol{r}), \quad \dot{\boldsymbol{r}} = (\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{\omega}, \\ \boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}),\end{gathered} \tag{10.1}$$

where $\boldsymbol{a}$ is the vector from the center of mass to the geometric center, $\boldsymbol{r} = -R\boldsymbol{\gamma} - \boldsymbol{a}$ (Fig. 7), and m is the mass of the ball.

The system (10.1) (as any body on an absolutely rough surface) has an energy integral and the geometric integral

$$F_1 = \boldsymbol{\gamma}^2 = 1, \quad H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}). \tag{10.2}$$

Besides these (obvious) integrals, the system (10.1) admits one quadratic integral

$$F_2 = (\boldsymbol{M}, \boldsymbol{M}) - m(\boldsymbol{r}, \boldsymbol{r})(\boldsymbol{M}, \boldsymbol{\omega}). \tag{10.3}$$

This integral is a generalization of the integral $\boldsymbol{M}^2 = \text{const}$ in the problem of motion of a Chaplygin ball (Section 7). We see that to achieve integrability by the Euler–Jacobi theorem and additional integral and invariant measure are needed. Simulations show [1] that in the generic cases the required invariants are missing (Fig. 8).

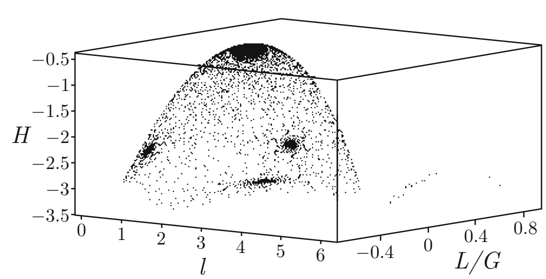

Fig. 8. Three-dimension Poincaré section for the system (10.1) ([1]). One of the trajectories from the problem of motion of an unbalanced ball on a plane. It is seen that all the point lie on the same surface; condensation spots indicate the approach to periodic solutions. The trajectory emanates from the top and approaches the three points at the bottom.

The integral (10.3) can be generalized to the general case of rolling motion of a ball on a sphere.

10.2. Rubber Ball

In the absence of spinning, for an arbitrary body one should put $\boldsymbol{r} = -R\boldsymbol{\gamma} - \boldsymbol{a}$ in the equations of motion thus obtaining

$$\begin{gathered}\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \tilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\omega} - m\boldsymbol{r} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) + \lambda_0\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \lambda_0 = -\frac{(\tilde{\mathbf{I}}^{-1}\boldsymbol{\gamma}, \tilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\omega} - m\boldsymbol{r} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}))}{(\tilde{\mathbf{I}}^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma})}.\end{gathered} \tag{10.4}$$

Besides the integrals (10.2) we have the equation of constraint

$$(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0,$$

moreover, there is an analogue of the integral (10.3), which now takes the form

$$F_2 = |\tilde{\mathbf{I}}\boldsymbol{\omega} \times \boldsymbol{\gamma}|^2 - 2Rm(\boldsymbol{\gamma}, \boldsymbol{a})(\tilde{\mathbf{I}}\boldsymbol{\omega}, \boldsymbol{\omega}). \tag{10.5}$$

For $\boldsymbol{a} = 0$ this integral (as might be expected) transforms into the integral of Veselova's system (Section 6).

Thus only invariant measure is missing for the system (10.4) to be Euler–Jacobi integrable (Section 4).

At the same time the common level surface of the integrals of the system (10.4) (integral manifold) is two-dimensional. At least for small $\boldsymbol{a}$, the integral manifold is generally a two-dimensional torus as the system is obtained by perturbation of the integral manifold (invariant tori) of Veselova's system (Section 6). Therefore this system, due to the fact that the invariant manifold are two-dimensional, has no chaotic trajectories so typical for Hamiltonian systems (that usually fill a three-dimension chaotic layer) and for dissipative systems (in 2D realm there are no strange attractors).

It can be noted that the question of existence (or absence) of invariant measure in this case is reduced to the analysis of the flow on two-dimension invariant tori [45, 78]. If an integral invariant does not exist, only periodic or quasi-periodic trajectories are possible; at the same time, on such a torus asymptotic trajectories and limit cycles may occur. Preliminary numerical experiments indicate the possibility of existence of invariant measure although it is not yet found in the analytical form.

11. DISCUSSION

11.1. The Contensou–Erismann Model of Friction

Having discussed various nonholonomic models for rolling motion of a body in this paper, let us consider in greater detail the mechanism of interaction between a body and a surface at the point of contact. There are several models that describe this interaction. To be able to obtain an adequate model of a real mechanical system one has to resort to various phenomenological theories, especially when dealing with friction. In modern and classical works on dynamics of contacting bodies, the following friction models are encountered: dry Coulomb friction (e.g. [33, 69, 79]), viscous friction (e.g. [80–82]) and more complicated friction models (e.g. in the motion of curling rocks [84, 85]). A comprehensive bibliography on mechanical systems with friction can be found in the book by A. P. Markeev [86].

Another dry-friction model which accounts for both slipping and spinning of the body at the point of contact was suggested by Contensou and Erismann [69, 87]. The Contensou–Erismann model locally utilizes Coulomb friction which is then integrated over the contact area, thus yielding a non-zero net frictional force and a non-zero spinning friction torque. They assume that the contact area is disk and use the classical Hertz contact theory to find the normal contact tension. Using this approach, Contensou studied the motion of a top [69], while Erismann, instead of dynamical problems, concentrated on kinematical problems that arise from the design of integrators [87]. In the modern papers [23, 88–91, 95] the Contensou–Erismann theory is applied to various problems of dynamics. It should be noted that although the Contensou–Erismann model allows an accurate determination of frictional forces and torques, it has a serious drawback in that it uses the static Hertz distribution, whereas to model quite a number of dynamical phenomena it is essential to use dynamical (rather than static) friction. These issues were discussed by Routh from theoretical and experimental viewpoints [79]. Similar considerations may be also found in [33].

Let us consider in greater detail the role of the nonholonomic approach in the theories of rolling motion of rigid bodies and of their connection with various friction models.

11.2. Implementing Nonholonomic Constraints in Rolling Motion

As is well known, there are a variety of ways to implement nonholonomic constraints. One of them is the classical result that goes back to Carathéodory: a nonholonomic constraint can be realized as the limit of infinitely large viscous friction (for a rigorous exposition see [81, 86, 92]). The idea that dissipative forces may be useful in constructing nonholonomic constraints goes back to the classical works of the nineteenth century. In particular, Routh always considered nonholonomic systems through their connection with various friction models. However, there is another source of nonholonomic constraints. As a rule, nonholonomic constraints provide a realistic representation of the final stage of a body's motion. Already to Routh and Painlevé were known the conditions under which a body first slides over a surface and is therefore subject, say, to a Coulomb frictional force; then due to this force, the sliding motion ceases accompanied with the onset of the constraint. At this "constrained" stage (until the constraint is released), the motion of the body can be described by the equations of nonholonomic mechanics.

For example, the classical nonholonomic model in which the velocity of the point of contact is zero but spinning is allowed can be obtained in two ways:

- in the limit of infinitely large viscous friction coefficient;
- as the intermediate stage of motion (the onset of the constraint) of a system in which only the sliding friction at the point of contact is taken into consideration and the frictional torque is neglected (this is the simplest dry-friction model).

Taking the frictional torque into consideration results in a final motion which is adequately described by a new nonholonomic model of rolling-motion — rubber rolling. This model was first considered by Routh (and discussed by Painlevé) who used the Coulomb friction law to find both the force at the point of contact and the torque. For a homogeneous ball, Routh proved that after the onset of the constraint both the velocity at the point of contact and the projection of the angular velocity onto the normal to the surface are zero [79].

Similar final motions of bodies (and in particular of a ball) can be achieved in the Contensou–Erismann model. To do this, it is essential to use the well known result proved in [86] (independently proved in [93, 94] and partly in [95]) that sliding and spinning of a disk or a homogeneous ball terminate simultaneously. Thus onset of the constraint occurs and we arrive at our new nonholonomic model. V.F. Zhuravlev and D.M. Klimov [23, 88, 89] claim that the ensuing equations are not nonholonomic. However, as shown above (Section 2), the constraints imposed on the system are not integrable, and therefore, in the general case, the governing equations must necessarily be from nonholonomic mechanics.

It should be noted that our model can also be considered not just as the final stage of rolling motion of a body over a plane under the action of dry friction and torque. The model allows a simple classical interpretation if we decompose the constraint into two parts: we have the standard non-holonomic constraint (the velocity at the point of contact is zero), and additionally there is Veselova's constraint (the normal component of the angular velocity is zero). In [13] the following model is suggested: on the one hand there are two balls that can roll without slipping each over the surface of the other, and on the other hand the body is supplied with a blade (Chaplygin sleigh) that forces the angular velocity.

11.3. Rolling Models and Dynamical Effects

The papers [23, 88–90, 95] by Zhuravlev and Klimov put forward the proposition that methods of nonholonomic mechanics are absolutely unsuitable for the study of rolling motion of bodies and that the combined-friction model based on the Contensou–Erismann theory should be used instead. In our opinion such an approach is excessively one-sided. In fact, most mechanical systems that involve motion of contacting bodies can be conventionally divided into three following categories.

On the one hand, some dynamical phenomena (e.g. self-excited frictional oscillations, flutter, etc.) cannot be satisfactorily explained without resort to one or another friction model. The book [96] provides plenty of examples of mechanical devices whose operation is essentially based on friction (e.g. the Woodpecker Toy). However, most of these phenomena can be qualitatively (and sometimes even quantitatively) explained in the context of simpler friction models.

Remark 23. In this connection, the loyalty of the above mentioned authors to the Contensou–Erismann model seems incomprehensible. Indeed, as has been mentioned, this model is merely an approximation, which can be further improved to yield more accurate approximations and therefore more complicated governing equations. However, even for the Contensou–Erismann model the analysis in [23, 88–90, 95] is reduced to the numerical determination of just one or two trajectories.

On the other hand, the rolling behavior of some systems (e.g. Celtic stones, wheel pairs, mobile robots, etc.) essentially depends on a body's geometry, the distribution of mass and the absence of absolute sliding, while the preference for any particular friction model produces very little effect on the overall picture of motion. In such cases, one can reasonably use any nonholonomic model to describe the rolling motion. Moreover, the various nonholonomic systems that result are generally more easy to deal with than those incorporating friction. Thus the former can be studied with the help of more advanced tools (Poincaré sections, stability theory, extended KAM-theory, etc.) to obtain a more complete and *qualitative* description of dynamical effects.

Remark 24. It is well known that most problems of rigid-body dynamics can be solved within the framework of Hamiltonian mechanics, where there is no friction. Two of them — the Euler-Poisson equations and the three-body problem — are of special importance and can be used in the study of many practical problems. Another example is provided by the hydrodynamics of perfect fluid and vortex theory, which serve as foundations on which more complicated models that incorporate viscous friction are based.

Besides, there are mechanical systems whose behavior cannot be properly understood without taking into account both friction and the dynamical properties of bodies. A famous example is the Thomson top (also referred to as "tip-top" or Chinese top); for extensive bibliography see [80, 98]. For this toy friction plays a crucial role and is one of several known theories describing its bizarre behavior (see, for example, [39, 97]; modern researches contributed very little in comparison to what is obtained in these classical works). Due to frictional forces, rotation about the axis of symmetry with the center of mass at the lowermost position gradually slows down and finally it becomes unstable; at the same time, rotation with the center of mass at the uppermost position gains stability. Whereas according to [95], this phenomenon of overturn is solely due to the finiteness of the contact area and to the distribution of pressure over it, the phenomenon is essentially independent of the type of contact (i.e. it could be a single point or an area). Generally speaking, any type of friction can serve as an explanation of the overturn: both Coulomb and viscous friction will cause overturn [98]. The choice of friction type is determined by the accuracy required in the calculation of other characteristics of motion. Thus the use of more precise and complicated friction models, such as the Contensou–Erismann model, is justified if some quantitative characteristics (e.g. the overturn time) need to be estimated with higher accuracy. Some friction models in their connection with the analysis of the Thomson top are discussed in [91]. Simulations show that in the case of fast rotations the Contensou–Erismann frictional force is proportional to the sliding velocity and is therefore practically the same as viscous friction, whereas for pure sliding and small rotational velocity one can use the Coulomb friction.

Of course we do not wish to suggest that compound friction models are of no interest. Our objective is to cast doubt on the practice of carrying over one and the same friction model to all rolling-motion problems and the denial of simpler models and nonholonomic approaches. The choice of a particular friction model must be strictly determined by the dynamical effect of interest (e.g. permanent rotations or other regimes), experimental data, and the material property of the contacting bodies.

There may be mentioned besides the two nonholonomic models discussed above, another model with non-integrable constraint: the rolling of a disk over a smooth icy surface. The kinematic constraint is that the velocity at the point of contact is parallel to the horizontal diameter of the disk. Thus, the disk rolls over the surface and does not "bite" the ice, while it is allowed to rotate about its axis of symmetry. This model was suggested in [99], where also the integrability was proved. More complicated integrable problem were found and investigated in [100].

11.4. Loss of Contact Problem

It is well known that during motion a body can lose contact with the surface. Consider Euler's Disk, a famous toy, whose mystery has not been yet uncovered. When spun on a table, its motion at the final stage (before it abruptly comes to rest) is characterized by oscillations and a whirring sound of rapidly increasing frequency (e.g. see [101]). A possible explanation for this may be that the disk alternately loses and gains contact with the surface of the table as it comes to rest.

Such phenomena (bounces invisible to the human eye) were mentioned already in classical texts, see for example Appell [102] (motion of Gervat gyroscopes) and Coriolis [103] (collision of billiard-balls).

Since nonholonomic models are limiting cases of friction models, it should be noted that analysis of the loss of contact phenomenon in the context of nonholonomic mechanics inescapably leads to the familiar Painlevé paradoxes [33] inherent to systems with friction. In particular, "paradoxical behavior" can occur due to the fact that the zero normal reaction might be insufficient condition for a correct characterization of the loss of contact [104, 105]. This phenomenon is very little studied within the realm of nonholonomic mechanics. The example of Euler's Disk clearly shows that loss of contact can lead to intriguing dynamical effects.

12. OPEN PROBLEMS

In conclusion we give an overview of several problems whose solution with the help of the method developed in the paper has not yet met with success. Below we attempt to classify these unsolved problems focusing primarily on the simplest, and at the same time, most natural ones (in our opinion of course).

1. There are nonholonomic systems (other than those considered in this paper) which are known to be reducible to conformally Hamiltonian form with the help of the generalized Chaplygin Theorem (Section 4). Nevertheless, the problem of how to solve the equations analytically and obtain solutions in terms of quadrtatures remains open.

For example, two generalizations of the classical problem on rolling motion of a Chaplygin ball are known: the first one is due to Markeev [62] and the second one — to Kozlov [28] (Markeev added a gyrostat; Kozlov assumed that the ball is subject to a potential Bruns field.) As has been shown in the present paper, both generalizations are conformally Hamiltonian. However, in obtaining solutions to the equations, the following technical difficulty occurs. Upon a change of time, one gets a Hamiltonian system on a sphere which is integrable: the system has a quadratic integral which contains terms linear in velocities. Unfortunately, no general methods for integration of such systems exist. (Kozlov's problem was explicitly solved only in the special case when the integral of areas is zero and therefore these linear summands vanish [106].) Explicit integration of such systems is connected with explicit integration of the Clebsch-type systems with nonzero integral of areas. In spite of the efforts of nineteenth century experts [107] as well as modern researches [108, 109], the problem remains unsolved. The results obtained can hardly be considered complete and their true value is not clear. Indeed, though the equations in the Clebsch case are now recognized as being analytically solved, no attempts are reported of using any of the known analytical formulas in obtaining any qualitative (or quantitative) characteristics of motion.

Similar questions can be formulated regarding the Borisov–Fedorov problem [68] in which a Chaplygin ball rolls over the surface of a sphere. In this work, the equations are integrated in the case of zero linear integral. If this integral is different from zero, then again linear gyroscopic terms arise, thereby seriously complicating analytical integration of the equations. The problem can be generalized by adding a Bruns field; however, the generalized problem is still not solved in terms of quadratures.

2. The following two problems are considerably more complicated. They are only known to be integrable by the Euler–Jacobi Theorem and therefore, theoretically, can be written in conformally Hamiltonian form. Nevertheless, no explicit conformally Hamiltonian representations for these problems are yet known, which makes it difficult to apply any of the numerous available methods for qualitative analysis of Hamiltonian systems along with methods for their analytical solution. The first problem is Fedorov's system (a ball in a ball-type suspension) [110] and the second one is the problem of the motion of a Chaplygin ball along a straight line, considered by Veselov and Veselova [27]. As mentioned, both systems are integrable in the Euler–Jacobi sense and are very close to generalized Chaplygin systems with two degrees of freedom. However, attempts to obtain a Poisson structure or perform reduction of order have not been successful.

Remark 25. Though in this work considerable attention has been paid to explicit analytical integration of nonholonomic systems, this is rather a matter of tradition than an inalienable prerequisite for a qualitative (or quantitative) analysis of concrete systems. In modern dynamics, determination of solutions in analytical form is not of such crucial importance as it was in XIX–XX centuries: at that time, the mainstream idea was to detect integrable systems and to express their solutions in terms of theta-functions. In modern practice, very effective tools for the study of integrable systems are methods of topological and bifurcational analysis (and computer simulations for evaluation of quantitative characteristics) rather than explicit formulas for solutions. Some methods for topological analysis of Hamiltonian systems have been developed in [48, 111]. but they have never been applied to nonholonomic problems. In our opinion the extension of the methods from [48] to nonholonomic systems promises to be productive and does not seem to encounter any fundamental difficulties.

There is one special issue to discuss. For the nonholonomic systems considered in this paper, a conformally Hamiltonian representation has been obtained only for reduced systems (generally, evolution of the precession angle was factored out). An unanswered question is if it is possible to lift the Poisson structure back to the original unreduced system.

3. We should mention some even more complicated open problems connected with multidimensional generalizations of the nonholonomic systems considered in this paper. We mean the multidimensional analogues of Suslov's system, Veselova's system and Chaplygin's problem of the rolling motion of a ball over a plane, which have been considered in a series of publications [112–114]. In these works, invariant measures for the multidimensional Chaplygin and Veselova systems are found and various particular solutions are analyzed. Nevertheless, the property of integrability of these systems is not studied and, moreover, there is no strategy for understanding the mechanisms of their possible integrability. In each case, it seems natural to represent the system in conformally Hamiltonian form and then to use the Liouville theorem. At the same time, even for the four-dimensional analogue of Chaplygin's problem we cannot state with assurance the existence of a Poisson structure and conditions for integrability (even though an invariant measure is already found).

4. Finally, consider one more problem of rolling motion of a rigid body with a fluid-filled cavities over the surface of, say, a plane. This is a classical problem (proposed by Thomson [115]), which is seldom discussed in the modern literature on nonholonomic mechanics (there is nothing but a stability analysis of partial solutions [116]). Various aspects of the hierarchy of dynamics for this system are practically unexplored: in particular, nothing is known about conditions for existence of tensor invariants (first integrals, symmetry fields, measure, and Poisson structure) and cases of integrability.

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