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On New Generalizations of the Burgers and Korteweg-de Vries Equations

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Abstract

We describe new classes of nonlinear Galilean–invariant equations of Burgers and Korteweg–de Vries type and study symmetry properties of these equations.

1 Introduction

Equations which are writen below, simple wave, Burgers, Korteweg–de Vries, Korteweg-de Vries-Burgers and Kuramoto-Sivashinski equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{5}$$

are widely used for the mathematical modelling of various physical and hydrodynamic processes [4, 7, 9].

These equations possess very important properties:

1. All these equations have the same nonlinearity $u\dfrac{\partial u}{\partial x}$, and the operator $\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x} \equiv \dfrac{d}{dt}$ is "the material derivative".

2. All these equations are compatible with the Galilean relativity principle.

This talk is based on the results obtained in collaboration with Prof. W. Fushchych [2] and dedicated to his memory.

The last assertion means that equations (1)–(5) are invariant with respect to the Galilean transformations

$$t \to t' = t, \qquad x \to x' = x + vt, \qquad u \to u' = u + v, \tag{6}$$

where v is the group parameter (velocity of an inertial system with respect to another inertial system).

It is evident that equations (1)–(5) are invariant also with respect to the transformations group

$$t \to t' = t + a, \qquad x \to x' = x + b, \qquad u \to u' = u, \tag{7}$$

where a and b are group parameters.

In terms of the Lie algebra, invariance of equations (1)–(5) with respect to transformations (6)–(7) means that the Galilean algebra, which will be designated as $AG(1,1) = \langle P_0, P_1, G \rangle$ [3] with basis elements

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad G = t\partial_x + \partial_u. \tag{8}$$

is an invariance algebra of the given equations.

Let us recall some well-known facts on symmetry properties of equations (1)–(5). The equation of a simple wave (1) has general solutions of the form $u = f(x - ut)$ [9] and admits an infinite invariance algebra.

Equation (2) admits a five-dimensional invariance algebra [3], besides, let us note that this equation can be reduced to the heat equation by means of the Cole-Hopf transformation [9].

The Korteweg-de Vries equation (3) admits a four-dimensional invariance algebra [5], besides equation (2) is the classical example of an integrable equation [1].

Unfortunately, the symmetry of equations (4) and (5) is rather poor (the maximal invariance algebra is a three-dimensional algebra (8)), though, the presence of members which contain u_{xxx}, u_{xxxx} in these equations, is very important from the physical point of view.

Thus, linearity of equations (3)–(5) with respect to u_{xxx}, u_{xxxx} is bad from the point of view of symmetry, linearity of these equations causes the essential narrowing of symmetry the compared to the Burgers equation (2). The question arises how we can "correct" equations (3)–(5) so as at least to preserve the symmetry of the Burgers equation or to obtain some new generalization of the Galilean algebra (8).

To solve the problem, let us consider a natural generalization of all adduced equations, namely, the equation of the form

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right), \tag{9}$$

and, as particular case, the equation

$$u_{(0)} + uu_{(1)} = F\left(u_{(n)}\right). \tag{10}$$

Here and further, we use the following designations: $u = u(t,x)$; $u_{(0)} = \dfrac{\partial u}{\partial t}$; $u_{(n)} = \dfrac{\partial^n u}{\partial x^n}$; $F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right)$, $F\left(u_{(n)}\right)$ are arbitrary smooth functions.

Evidently, equations (9)–(10) will be invariant with respect to transformation (6)–(7), so they are compatible with the Galilean relativity princeple, and thus equations (9), (10) with an arbitrary function F will be invariant with respect to the Galilean algebra (8).

To have a hope to construct at least partial solutions of equations (9), (10), we need to specify (to fix) the function F. One of approaches to this problem is based on description of equations (9), (10) which admit wider invariance algebras than the Galilean algebra $AG(1,1)$ [3]. Wide symmetries of nonlinear equations, as is well known [3, 5, 6], enable to describe ansatzes reducing partial differential equations to ordinary differential equations which can often be solved exactly or approximately, or for which qualitative properties of solutions, asymptotic properties, etc. can be studied.

The principal aim of our work is as follows: to give a description of equations (9), (10) which have wider symmetry properties than the algebra $AG(1,1)$ or to describe nonlinear smooth functions F for which these equations are invariant with respect to Lie algebras which are extensions of the Galilean algebra $AG(1,1)$; using the symmetry of equations, to construct ansatzes and to reduce partial differential equations to ordinary differential equations.

The paper is organized as follows. In Section 2, we present all principal theorems and corollaries on symmetry classification of equations (9), (10) which admit wider symmetry than the Galilean algebra $AG(1,1)$. We do not give proofs of theorems, because they are extremely cumbersome, though simple from the point of view of ideas. In Section 3, we adduce finite group transformations, construct anzatzes and some classes of exact solutions.

2 Symmetry classification

Let us first formulate the statements on the Lie symmetry of certain equations of the type (10). Consider the following equations:

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}\right), \tag{11}$$

$$u_{(0)} + uu_{(1)} = F\left(u_{(3)}\right), \tag{12}$$

$$u_{(0)} + uu_{(1)} = F\left(u_{(4)}\right). \tag{13}$$

Theorem 1 *The maximal invariance algebras of equation (11) depending on* $F\left(u_{(2)}\right)$ *are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(2)}\right)$ *is arbitrary;*

2. $\langle P_0, P_1, G, Y_1\rangle$ *if* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^k$, $k = \mathrm{const}$*;* $k \neq 0$*;* $k \neq 1$*;* $k \neq \frac{1}{3}$*;*

3. $\langle P_0, P_1, G, Y_2\rangle$ *if* $F\left(u_{(2)}\right) = \ln u_{(2)}$*;*

4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(2)}\right) = \lambda u_{(2)}$*;*

5. $\langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$ *if* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^{1/3}$.

Here, $\lambda = \mathrm{const}$*,* $\lambda \neq 0$*, and basis elements of the Lie algebras have the following repre-*

sentation:

$$Y_1 = (k+1)t\partial_t + (2-k)x\partial_x + (1-2k)u\partial_u, \quad Y_2 = t\partial_t + \left(2x - \frac{3}{2}t^2\right)\partial_x + (u-3t)\partial_u,$$
$$D = 2t\partial_t + x\partial_x - u\partial_u, \quad \Pi = t^2\partial_t + tx\partial_x + (x-tu)\,\partial_u, \quad R_1 = 4t\partial_t + 5x\partial_x + u\partial_u,$$
$$R_2 = u\partial_x, \quad R_3 = (2tu - x)\,\partial_x + u\partial_u, \quad R_4 = (tu-x)\,(t\partial_x + \partial_u)\,.$$

Theorem 1 makes the result obtained in [8] more precise. The Burgers equation (2) as a particular case of (11) is includes in Case 4 of Theorem 1.

Note that the following equation has the widest symmetry in the class of equations (11) (7–dimensional algebra):

$$u_{(0)} + uu_{(1)} = \lambda\left(u_{(2)}\right)^{1/3}. \tag{14}$$

Theorem 2 *The maximal invariance algebras of equation (7) depending on* $F\left(u_{(3)}\right)$ *are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(3)}\right)$ *is arbitrary;*

2. $\langle P_0, P_1, G, Y_3\rangle$ *if* $F\left(u_{(3)}\right) = \lambda\left(u_{(3)}\right)^k$, $k = \text{const}$*;* $k \neq 0$*;* $k \neq \frac{3}{4}$*;*

3. $\langle P_0, P_1, G, Y_4\rangle$ *if* $F\left(u_{(3)}\right) = \ln u_{(3)}$*;*

4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(3)}\right) = \lambda\left(u_{(3)}\right)^{3/4}$.

Here, $\lambda = \text{const}$, $\lambda \neq 0$,

$$Y_3 = (2k+1)t\partial_t + (2-k)x\partial_x + (1-3k)u\partial_u, \quad Y_4 = t\partial_t + \left(2x - \frac{5}{2}t^2\right)\partial_x + (u-5t)\partial_u.$$

Case 2 of Theorem 2 for $k = 1$ includes the Korteweg–de Vries equation (3) as a particular case of (12).

Theorem 3 *The maximal invariance algebras of equation (8) depending on* $F\left(u_{(4)}\right)$ *are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(4)}\right)$ *is arbitrary;*

2. $\langle P_0, P_1, G, Y_5\rangle$ *if* $F\left(u_{(4)}\right) = \lambda\left(u_{(4)}\right)^k$, $k = \text{const}$*;* $k \neq 0$*;* $k \neq \frac{3}{5}$*;*

3. $\langle P_0, P_1, G, Y_6\rangle$ *if* $F\left(u_{(4)}\right) = \ln u_{(4)}$*;*

4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(4)}\right) = \lambda\left(u_{(4)}\right)^{3/5}$.

Here, $\lambda = \text{const}$, $\lambda \neq 0$,

$$Y_5 = (3k+1)t\partial_t + (2-k)x\partial_x + (1-4k)u\partial_u, \quad Y_6 = t\partial_t + \left(2x - \frac{7}{2}t^2\right)\partial_x + (u-7t)\partial_u.$$

Theorems 1–3 give the exhaustive symmetry classification of equations (11)–(13).

On the basis of Theorems 1–3, let us formulate some generalizations concerning the symmetry of equations (10), namely, investigate symmetry properties of equation (10) with fixed functions $F(u_{(n)})$.

Theorem 4 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \ln u_{(n)} \tag{15}$$

is the four–dimensional algebra $\langle P_0, P_1, G, A_1 \rangle$, where

$$A_1 = t\partial_t + \left(2x - \frac{2n-1}{2}t^2\right)\partial_x + \Big(u - (2n-1)t\Big)\partial_u.$$

Theorem 5 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(n)}\Big)^k \tag{16}$$

is the four–dimensional algebra $\langle P_0, P_1, G, A_2 \rangle$, where

$$A_2 = ((n-1)k+1)\,t\partial_t + (2-k)x\partial_x + (1-nk)u\partial_u,$$

k, λ are real constants, $k \neq 0$, $k \neq \dfrac{3}{n+1}$, $\lambda \neq 0$; for $n = 2$, there is the additional condition: $k \neq \dfrac{1}{3}$ (see Case 5 of Theorem 1).

Theorem 6 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(n)}\Big)^{3/(n+1)}, \quad \lambda = \text{const}, \lambda \neq 0 \tag{17}$$

is the five–dimensional algebra

$$\langle P_0, P_1, G, D, \Pi \rangle. \tag{18}$$

Remark. If $n = 1$ in (17), then we get the equation

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(1)}\Big)^{3/2}. \tag{19}$$

Theorem 7 *The maximal invariance algebra of equation (19) is the four–dimensional algebra $\langle P_0, P_1, G, D \rangle$.*

Remark. It is interesting that (18) defines an invariance algebra for equation (17) for any natural $n \geq 2$. With $n = 2$, (17) is the classical Burgers equation (2). Let us note that operators (18) determine a representation of the generalized Galilean algebra $AG_2(1,1)$ [3].

Now let us investigate the invariance of equation (9) with respect to representation (18) or point out from the class of equations (9) those which are invariant with respect of the invariance algebra of the classical Burgers equation. The following statement is true:

Theorem 8 *Equation (9) is invariant under the generalized Galilean algebra $AG_2(1,1)$ (18) iff it has the form*

$$u_{(0)} + uu_{(1)} = u_{(2)}\Phi\Big(\omega_3, \omega_4, \ldots, \omega_n\Big), \tag{20}$$

where Φ is an arbitrary smooth function,

$$\omega_k = \frac{1}{u_{(2)}}\Big(u_{(k)}\Big)^{3/(k+1)}, \quad u_{(k)} = \frac{\partial^k u}{\partial x^k}, \quad k = 3, \ldots, n.$$

The class of equations (20) contains the Burgers equation (2) (for $\Phi = \text{const}$) and equation (17). Equation (20) includes as a particular case the following equation which can be interpreted as a generalization of the Burgers equation and used for description of wave processes:

$$u_{(0)} + uu_{(1)} = \lambda_2 u_{(2)} + \lambda_3 \left(u_{(3)}\right)^{3/4} + \cdots + \lambda_n \left(u_{(n)}\right)^{3/(n+1)}, \tag{21}$$

$\lambda_2, \lambda_3, \ldots, \lambda_n$ are an arbitrary constant.

Let us note that the maximal invariance algebra of equation (21) is a generalized Galilean algebra (18).

Below we describe all second–order equations invariant under the generalized Galilean algebra (18). The following assertions are true:

Theorem 9 *A second–order equation is invariant under the generalized Galilean algebra $AG_2(1,1)$ iff it has the form*

$$\Phi\left(\frac{\left(u_{00}u_{11} - (u_{01})^2 + 4u_0u_1u_{11} + 2uu_{11}(u_1)^2 - 2u_{01}(u_1)^2 - (u_1)^4\right)^3}{(u_{11})^8};\right.$$
$$\left.\frac{u_0 + uu_1}{u_{11}}; \frac{\left(u_{01} + uu_{11} + (u_1)^2\right)^3}{(u_{11})^4}\right) = 0, \tag{22}$$

where Φ is an arbitrary function.

3 Finite group transformations, ansatzes, solutions

Operators of the algebra $L = \langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$ which define the invariance algebra equation (14), satisfy the following group relations:

	P_0	P_1	G	R_1	R_2	R_3	R_4
P_0	0	0	P_1	$4P_0$	0	$2R_2$	R_3
P_1	0	0	0	$5P_1$	0	$-P_1$	$-G$
G	$-P_1$	0	0	G	P_1	G	0
R_1	$-4P_0$	$-5P_1$	$-G$	0	$-4R_2$	0	$4R_4$
R_2	0	0	$-P_1$	$4R_2$	0	$-2R_2$	$-R_3$
R_3	$-2R_2$	P_1	$-G$	0	$2R_2$	0	$-2R_4$
R_4	$-R_3$	$-G$	0	$-4R_4$	R_3	$2R_4$	0

Let us note that it is possible to specify three subalgebras of the algebra L, which are Galilean algebras: $\langle P_0, P_1, G\rangle$, $\langle P_1, G, -R_4\rangle$, $\langle -R_2, P_1, G\rangle$.

The finite transformations corresponding to the operators R_1, R_2, R_3, R_4 are the following:

$$\begin{aligned}
R_1:\quad & t \to \tilde{t} = t\exp(4\theta), & R_2:\quad & t \to \tilde{t} = t\\
& x \to \tilde{x} = x\exp(5\theta), & & x \to \tilde{x} = x + \theta u,\\
& u \to \tilde{u} = u\exp(\theta), & & u \to \tilde{u} = u,\\
R_3:\quad & t \to \tilde{t} = t, & R_4:\quad & t \to \tilde{t} = t,\\
& x \to \tilde{x} = x\exp(-\theta) + tu\exp(\theta), & & x \to \tilde{x} = x + \theta t\,(ut - x),\\
& u \to \tilde{u} = u\exp(\theta), & & u \to \tilde{u} = u + \theta(ut - x),
\end{aligned}$$

where θ is the group parameter.

Let us represent the exact solution of (14) (below, we point out the operator, the ansatz, the reduced equation, and the solution obtained by means of reduction and integration of the reduced equation)

the operator: $R_3 = (2tu - x)\,\partial_x + u\partial_u$,

the ansatz: $xu - tu^2 = \varphi(t)$,

the reduced equation: $\varphi' = \lambda(2\varphi)^{1/3}$,

the solution:

$$xu - tu^2 = \frac{1}{2}\left(\frac{4}{3}\lambda t + C\right)^{3/2}. \qquad (23)$$

Relation (23) determines the set of exact solutions of equation (14) in implicit form.

The following Table contains the commutation relations for operators (18):

	P_0	P_1	G	D	Π
P_0	0	0	P_1	$2P_0$	D
P_1	0	0	0	P_1	G
G	$-P_1$	0	0	$-G$	0
D	$-2P_0$	$-P_1$	G	0	2Π
Π	$-D$	$-G$	0	-2Π	0

The finite group transformations corresponding to the operators D, Π in representation (18) are the following:

$$D: \quad t \to \tilde{t} = t\exp(2\theta), \qquad \Pi: \quad t \to \tilde{t} = \frac{t}{1-\theta t},$$
$$x \to \tilde{x} = x\exp(\theta), \qquad x \to \tilde{x} = \frac{x}{1-\theta t},$$
$$u \to \tilde{u} = u\exp(-\theta), \qquad u \to \tilde{u} = u + (x - ut)\,\theta,$$

where θ is the group parameter.

The ansatz

$$u = t^{-1}\varphi(\omega) + \frac{x}{t}, \qquad \omega = 2xt^{-1}$$

constructed by the operator Π reduces equation (17) to the following ordinary differential equation

$$\varphi\varphi' = \lambda_1 2^{(2n-1)/(n+1)}\left(\varphi^{(n)}\right)^{3/(n+1)}. \qquad (24)$$

A partial solution (24) has the form

$$\varphi = -2\left(\lambda_1{}^{(n+1)}(n!)^3\right)^{1/(2n-1)}\omega^{-1},$$

and whence we get the following exact solution of equation (17)

$$u = -2\left(\lambda_1{}^{(n+1)}(n!)^3\right)^{1/(2n-1)}\frac{1}{2x} + \frac{x}{t}.$$

In general case, it is necessary to use nonequvilent one-dimensional subalgebras to obtain solutions. In Table, nonequivalent one–dimensional subalgebras for algebra (18) and corresponding ansatzes are adduced. (Classification of one–dimensional subalgebras is carried out according to the scheme adduced in [5].)

	Ansatz
P_1	$u = \varphi(t)$
G	$u = \varphi(t) + xt^{-1}$
$P_0 + \alpha G,\ \alpha \in \mathbf{R}$	$u = \varphi\left(x - \frac{\alpha}{2}t^2\right) + \alpha t$
D	$u = t^{-1/2}\varphi\left(xt^{-1/2}\right)$
$P_0 + \Pi$	$u = (t^2+1)^{-1/2}\varphi\left(\frac{x}{(t^2+1)^{1/2}}\right) + \frac{tx}{t^2+1}$

The ansatzes constructed can be used for symmetry reduction and for construction of solutions for equations (17), (20), (21).

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