Divergence Symmetries of Critical Kohn-Laplace Equations on Heisenberg groups

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Abstract

We show that any Lie point symmetry of semilinear Kohn-Laplace equations on the Heisenberg group H^1 with power nonlinearity is a divergence symmetry if and only if the corresponding exponent assumes critical value.

1 Introduction

Recently a great renewed and increasing interest in the Heisenberg group has been manifested from both analysts and geometers. In this regard, in the last few decades partial differential equations on the Heisenberg group were studied by various authors using different methods. There is a big variety of results among which we shall mention a few. In [6] Garofalo and Lanconelli obtained existence, regularity and nonexistence results for semilinear PDEs involving Kohn-Laplace operators. General nonexistence results for solutions of differential inequalities on the Heisenberg group were established by Pokhozhaev and Veron in [10]. The work [7] by Lanconelli is a survey on a series of results concerning critical semilinear equations on the Heisenberg group. For further details we direct the interested reader to these works and the references therein as well as to the existing internet instruments for search of mathematical information.

In this paper we apply the S. Lie symmetry theory of differential equations ([8, 9]) to the study of a model Kohn-Laplace equation on the Heisenberg group. Namely, we study the

variational and divergence symmetries of the following differential equation on the Heisenberg group H^1:

$$\Delta_{H^1} u + u^p = 0, \tag{1}$$

where $\Delta_{H^1} = X^2 + Y^2$ is the Kohn-Laplace operator,

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$$

and

$$Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are the generators of *left* multiplication in H^1. In more details, the equation (1) for $u = u(x, y, t) : \mathbb{R}^3 \to \mathbb{R}$ reads

$$u_{xx} + u_{yy} + 4(x^2 + y^2)u_{tt} + 4yu_{xt} - 4xu_{yt} + u^p = 0. \tag{2}$$

In a previous work [2] we obtained a complete group classification of semilinear Kohn-Laplace equations on H^1. In the case of nonlinearity of power type, the result states that the symmetry group of (1) for $p \neq 0$, $p \neq 1$ consists of translations in t, rotations in the $x - y$ plane, right multiplications in the Heisenberg group H^1 and a dilation, generated respectively by

$$T = \frac{\partial}{\partial t}, \quad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t} \tag{3}$$

and

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}. \tag{4}$$

Moreover, if $p = 3$ the symmetry group can be expanded by the following generators:

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}, \tag{5}$$

$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u}, \tag{6}$$

$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}. \tag{7}$$

The purpose of this paper is to find out which of the above symmetries are variational or divergence symmetries.

We denote by G the five-parameter Lie group of point transformations generated by $T, R, \tilde{X}, \tilde{Y}$ and Z. Then our first result can be formulated as follows.

Theorem 1. *The Lie point symmetry group G of the Kohn-Laplace equation (1) is a variational symmetry group if and only if $p = 3$.*

We recall that the homogeneous dimension of the Heisenberg group H^n is given by $Q = 2n + 2$ and that the critical Sobolev exponent is $(Q+2)/(Q-2)$. Hence the Theorem 1 means that the dilation Z is a variational symmetry if and only if p equals to the critical exponent. Actually, the latter property holds for H^n, $n > 1$, and we shall come back to this point later.

Further we show that in the critical case the additional symmetries V_1, V_2, V_3 are divergence symmetries. For this purpose we find explicitly the vector-valued 'potentials' which determine V_1, V_2 and V_3 as divergence symmetries.

The main result in this paper is the following

Theorem 2. *Any Lie point symmetry of the Kohn-Laplace equation*

$$\Delta_{H^1} u + u^3 = 0 \tag{8}$$

is a divergence symmetry.

As it is well-known, the divergence symmetries determine conservation laws via the Noether Theorem [8]. Thus the next steps in this research would be to establish the conservation laws corresponding to the already studied variational and divergence symmetries as well as to study the invariant solutions of the Kohn-Laplace equations. These problems will be treated elsewhere [3, 4].

We observe that, by Theorem 2, all Lie point symmetries of the Kohn-Laplace equation (8) are divergence symmetries. This confirms the validity of the general property, established and discussed in [1], stating that the Lie point symmetries of critical quasilinear differential equations with power nonlinearities are divergence symmetries. The fact that this property should be valid for differential equations on Heisenberg groups was conjectured by Enzo Mitidieri in June 2003 [5].

This paper is organized as follows. In sections 2 and 3 we prove theorems 1 and 2 respectively. In section 3 we discuss the generalization of the obtained results to the Heisenberg group H^n, $n > 1$.

2   The variational symmetries

To begin with, we note that the Kohn-Laplace equation

$$\Delta_{H^1} u + f(u) = 0 \tag{9}$$

is the Euler-Lagrange equation for the functional

$$\int L(x, y, t, u, u_x, u_y, u_t) dx dy dt,$$

where the integration is performed over $\mathbb{R}^3$, the function of Lagrange is given by

$$L = \frac{1}{2}(Xu)^2 + \frac{1}{2}(Yu)^2 - \int_0^u f(s) ds,$$

or, equivalently,

$$L = \frac{1}{2} u_x^2 + \frac{1}{2} u_y^2 + 2(x^2 + y^2) u_t^2 + 2y u_x u_t - 2x u_y u_t - \int_0^u f(s) ds, \tag{10}$$

3

and the function u is assumed to satisfy appropriate vanishing conditions as $d = (t^2 + (x^2 + y^2))^{1/4} \to \infty$.

Proof of Theorem 1. By the general theory of symmetries of differential equations [8] it is enough to show that the generators of G determine variational symmetries. Indeed, by the infinitesimal criterion for invariance [8], p. 257, G is a variational symmetry group if and only if

$$W^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 0 \tag{11}$$

for all $(x, y, t, u, u_x, u_y, u_t)$ and every infinitesimal generator

$$W = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}.$$

(Recall that $W^{(1)}$ is the first order extension of W, see [8].)

Aiming to verify (11) for $T, R, \tilde{X}, \tilde{Y}$ and Z, we first calculate the corresponding first order extensions using the formulae for the extended infinitesimals [8]:

$$T^{(1)} = T, \tag{12}$$

$$R^{(1)} = R + u_y\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial u_y}, \tag{13}$$

$$\tilde{X}^{(1)} = \tilde{X} + 2u_t\frac{\partial}{\partial u_y}, \tag{14}$$

$$\tilde{Y}^{(1)} = \tilde{Y} - 2u_t\frac{\partial}{\partial u_x}, \tag{15}$$

and

$$Z^{(1)} = Z + \frac{1+p}{1-p}u_x\frac{\partial}{\partial u_x} + \frac{1+p}{1-p}u_y\frac{\partial}{\partial u_y} + \frac{2p}{1-p}u_t\frac{\partial}{\partial u_t}. \tag{16}$$

Then from (3), (10), (12), (13), (14), (15) we obtain easily that $T, R, \tilde{X}, \tilde{Y}$ satisfy (11). Hence they determine variational symmetries for arbitrary $f(u)$.

Further, let $\xi = x, \phi = y, \tau = 2t, \eta = 2u/(1 - p)$ be the infinitesimals of the dilation Z. Then the left-hand side of (11) with $W = Z$ reads

$$Z^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u} \right.$$

$$\left. + \frac{1+p}{1-p}u_x\frac{\partial}{\partial u_x} + \frac{1+p}{1-p}u_y\frac{\partial}{\partial u_y} + \frac{2p}{1-p}u_t\frac{\partial}{\partial u_t} \right] L + 4L$$

where

$$L = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{1}{p+1}u^{p+1}.$$

After a differentiation and simplifying we obtain

$$Z^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) =$$

$$\frac{3-p}{1-p}(u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t) + \frac{2(3-p)}{p^2 - 1}u^{p+1}.$$

Hence Z is a variational symmetry if and only if $p = 3$, which completes the proof of Theorem 1.

3 The divergence symmetries

In this section we prove Theorem 2.

Recall that a point transformation with infinitesimal generator

$$W = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}$$

is a divergence symmetry for $\int L$ if and only if there exists a vector function $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of u and its derivatives up to some finite order such that

$$W^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = Div\,(\varphi). \tag{17}$$

Since the variational symmetries are divergence symmetries with $\varphi = 0$, by Theorem 1 it is enough to prove that V_1, V_2, V_3 are divergence symmetries. For this purpose we shall find the corresponding 'potentials' φ.

For the symmetry V_1 we have

$$\xi = xt - x^2y - y^3, \quad \phi = yt + x^3 + xy^2,$$

$$\tau = t^2 - (x^2 + y^2)^2, \quad \eta = -tu.$$

We calculate the first order extension of V_1:

$$V_1^{(1)} = V_1 + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_y^{(1)}\frac{\partial}{\partial u_y} + \eta_t^{(1)}\frac{\partial}{\partial u_t},$$

where the extended infinitesimals are given by

$$\eta_x^{(1)} = 2(xy - t)u_x - (3x^2 + y^2)u_y + 4x(x^2 + y^2)u_t,$$

$$\eta_y^{(1)} = (x^2 + 3y^2)u_x - 2(t + xy)u_y + 4y(x^2 + y^2)u_t,$$

$$\eta_t^{(1)} = -u - xu_x - yu_y - 3tu_t.$$

Then, after some tedious work, we obtain

$$V_1^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 2xuu_y - 2yuu_x - 4(x^2 + y^2)u_t. \tag{18}$$

Thus V_1 is not a variational symmetry. Let

$$A_1 = -yu^2, \quad A_2 = xu^2, \quad A_3 = -2(x^2 + y^2)u^2. \tag{19}$$

Then from (18) and (19)

$$V_1^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = Div(A),$$

where $A = (A_1, A_2, A_3)$. Hence V_1 is a divergence symmetry.

Analogously, for the symmetries V_2 and V_3, after another tedious work, we obtain

$$V_2^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = 2uu_y - 4xuu_t \tag{20}$$

and

$$V_3^{(1)}L + L(D_x\xi + D_y\phi + D_t\tau) = -2uu_x - 4yuu_t. \tag{21}$$

Now we denote

$$B = (0, u^2, -2xu^2) \tag{22}$$

and

$$C = (-u^2, 0, -2yu^2). \tag{23}$$

Then by (20)-(23) we see that V_2 and V_3 satisfy (17) with B and C, respectively, in the place of φ. Thus V_2 and V_3 are divergence symmetries.

4   On the generalization to H^n, $n > 1$

In this section we comment briefly on the generalization of our approach to the Heisenberg group H^n, $n > 1$.

First we observe that the Kohn-Laplace equation

$$\Delta_{H^n}u + f(u) = 0, \tag{24}$$

or equivalently

$$u_{x_ix_i} + u_{y_iy_i} + 4(x_i^2 + y_i^2)u_{tt} + 4y_iu_{x_it} - 4x_iu_{y_it} + f(u) = 0$$

is the Euler-Lagrange equation of the functional

$$J[u] = \int L,$$

with

$$L = \frac{1}{2}(X_iu)^2 + \frac{1}{2}(Y_iu)^2 - \int_0^u f(s)ds$$

$$= \frac{1}{2}u_{x_i}^2 + \frac{1}{2}u_{y_i}^2 + 2(x_i^2 + y_i^2)u_t^2 + 2y_iu_{x_i}u_t - 2x_iu_{y_i}u_t - \int_0^u f(s)ds,$$

where $X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}$, $Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}$ and a summation over $i = 1, 2, ...n$ is assumed.

Using the definition of Lie point symmetry of a differential equation, one can show that the scaling transformation

$$\begin{cases} x_j^* &= \lambda x_j, \\ y_j^* &= \lambda y_j, \\ t^* &= \lambda^2 t, \\ u^* &= \lambda^{\frac{2}{1-p}}u \end{cases}$$

is admitted by the equation

$$\Delta_{H^n}u + u^p = 0. \tag{25}$$

Then performing this change of variables in the functional J it is easy to see that the dilation

$$Z = x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}$$

6

is a variational symmetry if and only if

$$p = \frac{n+2}{n} = \frac{Q+2}{Q-2}.$$

Thus the equation (25) admits a variational symmetry group containing Z if and only if p assumes the critical value.

In conclusion, we note that the complete group classification of the Kohn-Laplace equations (24) is available only for $n = 1$ ([2]). We conjecture that in the critical case the symmetry group of (25) can be expanded and, in addition, that all Lie point symmetries in this case are divergence symmetries.

Acknowledgements

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