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Special conformal groups of a Riemannian manifold and Lie point symmetries of the nonlinear Poisson equation

Yuri Bozhkov [a,*], Igor Leite Freire [a,b]

[a] *Instituto de Matemática, Estatística e Computação Científica - IMECC, Universidade Estadual de Campinas - UNICAMP, C.P. 6065, 13083-970 Campinas, SP, Brazil*
[b] *Centro de Matemática, Computação e Cognição, Universidade Federal do ABC - UFABC, Rua Catequese, 242, Jardim, 09090-400 Santo André, SP, Brazil*

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A B S T R A C T

We obtain a complete group classification of the Lie point symmetries of nonlinear Poisson equations on generic (pseudo) Riemannian manifolds M. Using this result we study their Noether symmetries and establish the respective conservation laws. It is shown that the projection of the Lie point symmetries on M are special subgroups of the conformal group of M. In particular, if the scalar curvature of M vanishes, the projection on M of the Lie point symmetry group of the Poisson equation with critical nonlinearity is the conformal group of the manifold. We illustrate our results by applying them to the Thurston geometries.

1. Introduction

The study of differential equations on manifolds is the corner stone of the Geometric Analysis. For this purpose various methods have been applied: fixed point theorems, continuity method, maximum principles, a priori estimates, Schauder theory, etc. However it seems that it is little known how the symmetries of the considered equation (or system) and the geometry of the manifold are related. (By a ‘symmetry’ we understand a Lie point symmetry [9,10,27,35–37].)

* Corresponding author.
E-mail addresses: bozhkov@ime.unicamp.br (Y. Bozhkov), igor.freire@ufabc.edu.br (I.L. Freire).

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Let M^n be a (pseudo) Riemannian manifold of dimension $n \geqslant 3$ endowed with a (pseudo) Riemannian metric $g = (g_{ij})$ given in local coordinates $\{x^1, \dots, x^n\}$. In this paper we shall study the Lie point symmetries of the Poisson equation on M^n:

$$\Delta_g u + f(u) = 0, \tag{1}$$

where

$$\Delta_g u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right) = g^{ij} \nabla_i \nabla_j u = \nabla^j \nabla_j u = \nabla_i \nabla^i u$$

is the Laplace–Beltrami operator, f is a smooth function, (g^{ij}) is the inverse matrix of (g_{ij}), ∇_i is the covariant derivative corresponding to the Levi-Civita connection and we have used the Einstein summation convention, that is, summation from 1 to n over repeated indices is understood.

Eq. (1) can be equivalently written as

$$H \equiv g^{ij} u_{ij} - \Gamma^i u_i + f(u) = 0, \tag{2}$$

where $\Gamma^i := g^{pq} \Gamma^i_{pq}$, Γ^i_{pq} being the Christoffel symbols, $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$.

We observe that Eq. (1) includes elliptic and hyperbolic equations, depending on whether (M^n, g) is a Riemannian or a pseudo Riemannian manifold. Such equations appear in various geometric and mathematical physics contexts which we shall not going into here. We merely mention the Poisson equations in $\mathbb{R}^n$, in particular those involving critical exponents, taking in (1) $f(u) = u^{\frac{n+2}{n-2}}$ and the Euclidean metric; the Klein–Gordon equation, taking in (1) the metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and $f(u) = u$; the semilinear wave equations in $\mathbb{R}^{1+n}$, with $f''(u) \neq 0$ and the metric $ds^2 = -dt^2 + \delta_{ij} dx^i dx^j$ in (1); and the Klein–Gordon equation on the $\mathbb{S}^2$ sphere. These particular equations have been studied in [27,35,6,7,24,43,5,22,25,39]. The interested reader may also consult the book [27] of Ibragimov, where various aspects of symmetry analysis of differential equations on manifolds are presented. We would also like to mention the paper by Ratto and Rigoli [38] in which these authors obtain gradient bounds and Liouville type theorems for the Poisson equation (1) on complete Riemannian manifolds.

The purpose of this paper is three-fold. First we shall obtain a complete group classification for the semilinear Poisson equations (1) by applying the S. Lie symmetry theory. Then we shall study the Noether symmetries of (1). (Noether symmetry = variational or divergence symmetry.) The latter will be used to establish the corresponding conservation laws via the Noether's theorem.

Since we suppose that the reader is familiar with the basic notions and methods of contemporary group analysis [9,10,27,35–37], we shall not present preliminaries concerning Lie point symmetries of differential equations. For a geometric viewpoint of Lie point symmetries, see [34,31]. We would just like to recall that, following Olver [35, p. 182], to perform a group classification on a differential equation involving a generic function f consists of finding the Lie point symmetries of the given equation with arbitrary f, and, then, to determine all possible particular forms of f for which the symmetry group can be enlarged. Usually there exists a geometrical or physical motivation for considering such specific cases.

Our first result is the following group classification theorem.

Theorem 1. 1.) *The Lie point symmetry group of the Poisson equation* (1) *with an arbitrary* $f(u)$ *coincides with the isometry group of* M^n. *In this case the infinitesimal generator is given by*

$$X = \xi = \xi^i(x) \frac{\partial}{\partial x^i} \tag{3}$$

where

$$\mathcal{L}_\xi g_{ij} = 0$$

and $\mathcal{L}_\xi$ *is the Lie derivative with respect to the vector filed* ξ.

For some special choices of the function $f(u)$ *it can be extended in the cases listed below.*

2.) *If* $f(u) = 0$ *then the symmetries have the form*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \left[\left(\frac{2-n}{4}\mu(x) + c\right)u + b(x)\right]\frac{\partial}{\partial u}, \tag{4}$$

where c is an arbitrary constant,

$$\Delta_g b = 0, \tag{5}$$

$$\Delta_g \mu = 0 \tag{6}$$

and $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ *is a conformal Killing vector field such that*

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}. \tag{7}$$

3.) *In the case* $f(u) = k = const. \neq 0$ *the symmetries are generated by*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \left[\frac{n-2}{n+2}\left(\frac{1}{k}(\Delta_g b) - c\right)u + b(x)\right]\frac{\partial}{\partial u}, \tag{8}$$

where c is an arbitrary constant,

$$\Delta_g^2 b = 0,$$

and $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ *is a conformal Killing vector field such that*

$$\mathcal{L}_\xi g_{ij} = \frac{4}{(n+2)}\left(c - \frac{1}{k}\Delta_g b\right)g_{ij}. \tag{9}$$

4.) *If the function* f *is a linear function*: $f(u) = u$, *then the symmetry generator is given by* (4) *with* $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ *satisfying* (7),

$$\Delta_g b + b = 0, \tag{10}$$

$$\frac{2-n}{4}\Delta_g \mu + \mu = 0. \tag{11}$$

5.) *For exponential nonlinearity* $f(u) = e^u$ *we have the generator*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} - \mu\frac{\partial}{\partial u}, \tag{12}$$

where μ *is a constant and* $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ *is a homothety such that*

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}, \qquad \mu = const. \tag{13}$$

6.) *For power nonlinearity $f(u) = u^p$, $p \neq 0$, $p \neq 1$, the Lie point symmetry group is generated by*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{\mu}{1-p}u\frac{\partial}{\partial u}, \tag{14}$$

where μ is a constant and the vector field $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ is a homothety such that

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}, \qquad \mu = const. \tag{15}$$

6.1) *If $p = \frac{n+2}{n-2}$, $n \neq 6$, the infinitesimal generator of the Lie point symmetries has the form*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{2-n}{4}\mu u\frac{\partial}{\partial u}, \tag{16}$$

where μ is a harmonic function on M^n:

$$\Delta_g \mu = 0 \tag{17}$$

and

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}. \tag{18}$$

6.2) *If $p = 2$ and $n = 6$, the symmetry group is determined by*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \left(-\mu(x)u + \frac{1}{2}\Delta_g \mu\right)\frac{\partial}{\partial u}, \tag{19}$$

where μ is a biharmonic function on M^n:

$$\Delta_g^2 \mu = 0 \tag{20}$$

and $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ is a conformal Killing vector field such that

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}. \tag{21}$$

It is clear that the projection of the Lie point symmetries listed in Theorem 1 on the space of independent variables are, in fact, special conformal Killing vector fields generating some subgroups of the conformal group of (M^n, g), which we shall call special conformal groups generated by symmetries.

Theorem 1 applied to the situations studied in [43,5,23,39] immediately gives the results on group classification, obtained in these works, for semilinear Poisson and wave equations. In this way the projection on the space of independent variables of the symmetry group of the critical Klein–Gordon equation in $\mathbb{R}^{1+n}$

$$u_{tt} - \Delta u = u^{(n+3)/(n-1)}$$

is the conformal group of $\mathbb{R}^{1+n}$.

In [5] the wave equation on the sphere $\mathbb{S}^2$

$$u_{tt} = u_{xx} + (\cot x)u_x + \frac{1}{\sin^2 x}u_{yy}$$

has been studied using the standard Lie approach. In [22] one of us, using the results of the present paper, generalized [5] for the semilinear wave equation on the sphere $\mathbb{S}^2$:

$$u_{tt} = u_{xx} + (\cot x)u_x + \frac{1}{\sin^2 x}u_{yy} + f(u).$$

Corollary 1. *Let (M^n, g) be a compact manifold without boundary. Then the Lie point symmetry group of* (1) *with an arbitrary $f(u)$ coincides with the isometry group Isom(M^n, g).*

If $f(u) = 0$ the symmetry group is generated by

$$X = \xi + (cu + b)\frac{\partial}{\partial u},$$

where ξ is a Killing vector field (that is $\mathcal{L}_\xi g_{ij} = 0$), c and b—arbitrary constants.

If $f(u) = u$, then the symmetry generator has the form

$$X = \xi + \big(cu + b(x)\big)\frac{\partial}{\partial u},$$

where ξ is a Killing vector field, $\int_M b(x)\,dV = 0$, $\Delta_g b + b = 0$ and c is an arbitrary constant.

By the results in [48] one can easily obtain the following estimates on the dimension of the symmetry Lie algebras:

Corollary 2. *Let $\mathfrak{S}$ be a Lie algebra generated by the symmetries of the nonlinear Poisson equation on (M^n, g). Then, the dimension of $\mathfrak{S}$ with an arbitrary $f(u)$ does not exceed $\frac{n(n+1)}{2}$ and the equality holds if and only if the sectional curvature of (M^n, g) is constant.*

For some special choices of the function $f(u)$ the dimension of $\mathfrak{S}$ can be enlarged.

1. *If $f(u) = k$, $k = const.$ or $f(u) = u$, then $\dim(\mathfrak{S}) = \infty$ and all finite dimensional subalgebras possess dimension less than $\frac{(n+1)(n+2)}{2} + 1$ and the equality holds if and only if (M^n, g) is a flat manifold.*
2. *If $f(u) = e^u$ or $f(u) = u^p$, with $p(p-1)(p - \frac{n+2}{n-2}) \neq 0$, then $\dim(\mathfrak{S}) \leqslant \frac{n(n+1)}{2} + 1$ and the equality holds if and only if (M^n, g) is Euclidean. In particular, if $\mathfrak{p}$ and $\mathfrak{e}$ denote the symmetry Lie algebras corresponding to the cases of power and exponential nonlinearity, then $\mathfrak{p} \approx \mathfrak{e}$, for any manifold (M^n, g).*
3. *If $f(u) = u^{\frac{n+2}{n-2}}$, then $\dim(\mathfrak{S}) \leqslant \frac{(n+1)(n+2)}{2}$ and the equality holds if and only if (M^n, g) is a flat manifold.*

In order not to loose the generality we have not made specific assumptions on the manifold M^n except $n \geqslant 3$. (The case $n = 2$ will be treated elsewhere. Some partial results can be found in [16].) Rather we provide a scheme which can be followed and specialized for any concrete manifold, for which one should extract further information using its geometrical properties.

Another related point to be emphasized concerns the integrability conditions for $\mathcal{L}_\xi g_{ij} = \mu g_{ij}$ with $\mu = 0$ (isometry), $\mu = const.$ (homothety) or $\mu = \mu(x)$ corresponding to a general conformal transformation ('conformal motion'). These conditions are in terms of the Riemannian curvature tensor and have been thoroughly studied in 1950s–1960s. See for instance [48,40] and the references therein. Although we shall not explicitly state the integrability conditions corresponding to the cases of Theorem 1, we shall suppose that such conditions hold. (Otherwise the symmetry determining equations might define the set {0}. The latter, of course, may occur for certain manifolds. This simply means that there are no nontrivial Lie point symmetries of the Poisson equation (1) on such manifolds.) Moreover, the number of the integrability conditions is undetermined for a generic manifold. This would create another considerable difficulty in treating the group classification problem in such a general setting. For this reason we have presented in Theorem 1 just the relations determining the symmetry groups as special conformal groups, without entering in differential-geometrical details regarding M^n like, e.g., positivity, negativity, vanishing or boundedness of its scalar, sectional or Ricci curvatures as well as of the respective consequences. For a variety of such results see [49] and the references therein.

Our next purpose in this paper is to find out which of the above symmetries are variational or divergence symmetries.

Theorem 2. 1.) *For an arbitrary $f(u)$ any symmetry of* (1) *is a variational symmetry, that is, the isometry group of (M^n, g) and the variational symmetry group of* (1) *coincide.*

2.) *In the exponential case $f(u) = e^u$ the only variational symmetries are the isometries of (M^n, g).*

3.) *In the power case $f(u) = u^p$, $p \neq 0$, $p \neq 1$, the symmetry* (14) *is variational if and only if*

$$p = \frac{n+2}{n-2}, \tag{22}$$

that is, $p+1$ equals to the critical Sobolev exponent.

3.1) *If $p = \frac{n+2}{n-2}$, $n \neq 6$, then the symmetry generated by* (16) *is a divergence symmetry.*
3.2) *If $p = 2$ and $n = 6$, then the symmetry generated by* (19) *is a divergence symmetry.*

4.) *In the linear cases we have*:

4.1) *If $f(u) = 0$, the symmetry* (4) *is a Noether symmetry if and only if $c = 0$.*
4.2) *If $f(u) = k$, $k \neq 0$, the symmetry* (8) *is a Noether symmetry if and only if $b = c = 0$.*
4.3) *If $f(u) = u$, the symmetry* (4) *is a Noether symmetry if and only if $c = 0$.*

From case 4.2) we conclude that the Noether symmetry group for the nonhomogeneous case $f(u) = k$ coincides with the isometry group of (M^n, g). Also from case 4.3), if $c = 0$ in (4), then, except the term $b\frac{\partial}{\partial u}$ in (4), the Noether symmetry group of the homogeneous case coincides with the symmetry group of the critical case (16).

We would also like to observe that in the critical cases 3.1) and 3.2) of Theorem 2 all Lie point symmetries are Noether symmetries. The general property stating that a Lie point symmetry of an equation (or system) is a Noether symmetry if and only if the equation parameters assume critical values has been established and discussed in [12]. Recall that, as it is well known, the so-called critical exponent is found as the critical power for embedding theorems of Sobolev type. It is also related to some numbers dividing the existence and nonexistence cases for the solutions of differential equations, in particular semilinear differential equations with power nonlinearities involving the Laplace operator. The above mentioned property traces a connection between these two notions: the Noether symmetries and the 'criticality' of the equation. It relates two important theorems, namely, the Sobolev theorem and the Noether theorem. Theorem 2 shows that this property holds also in the context of Riemannian manifolds. In fact, this is another motivation to write the present paper. In this regard, we can see that the widest symmetry group admitted by the Poisson equation may be the full conformal group of M^n. Namely:

Corollary 3. *If the scalar curvature of (M^n, g), $n \geqslant 3$, vanishes, then the widest symmetry group admitted by the Poisson equation* (1) *is achieved for the critical equation*

$$\Delta_g u + u^{(n+2)/(n-2)} = 0.$$

In this case it coincides with the conformal group of (M^n, g).

The latter equation and its invariant properties have been studied in [27].

We remark that if the manifold is flat, then the symmetry group of the nonlinear cases is maximal in the critical case.

Now we shall state the conservation laws corresponding to the found Noether symmetries. Before doing this it is worth mentioning that there are powerful modern methods to obtain conservation laws due to George Bluman et al. [2–4,1,28–30,33,47]. We believe that these methods can be very useful in the study of various differential geometric problems. However, we have chosen here the classical approach since we have at our disposal explicit formulae for the potentials ensured by the Noether's theorem (see Sections 8 and 9), whose determination is usually the major difficulty. Thus it is immediate, simple and natural to apply the classical approach.

Corollary 4. *The conservation laws corresponding to the Noether symmetries of Eq.* (1) *are classified as follows*:

1. *If $f(u)$ is an arbitrary function, then the conservation law is $D_iA^i=0$, where*

$$A^k=\sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k-g^{kj}\xi^i\right)u_iu_j-\sqrt{g}\xi^kF(u), \tag{23}$$

and $X=\xi^i\frac{\partial}{\partial x^i}$ is a Killing vector field on (M^n,g).

2. *If $f(u)=0$, then the conservation law is $D_iA^i=0$, where*

$$A^k=\sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k-g^{kj}\xi^i\right)u_iu_j+\frac{2-n}{4}\sqrt{g}g^{kj}\left(\mu uu_j-\frac{1}{2}\mu_ju^2\right)+\sqrt{g}g^{jk}(bu_j-b_ju), \tag{24}$$

and ξ, b μ satisfy (5), (6), (7).

3. *If $f(u)=k$, $k=const.$, then the conservation law is given by Eq.* (24) *with $b=0$.*
4. *If $f(u)=u$, then the conservation law is $D_iA^i=0$, where*

$$A^k=\sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k-g^{kj}\xi^i\right)u_iu_j+\frac{2-n}{4}\sqrt{g}g^{kj}\left(\mu uu_j-\frac{1}{2}\mu_ju^2\right)$$
$$+\sqrt{g}g^{jk}(bu_j-b_ju)-\frac{1}{2}\xi^k\sqrt{g}u^2 \tag{25}$$

and ξ, b μ satisfy (7), (10), (11).

5. *If $n\neq 6$ and $f(u)=u^{\frac{n+2}{n-2}}$, then the conservation law is $D_iA^i=0$, where*

$$A^k=\sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k-g^{kj}\xi^i\right)u_iu_j+\frac{2-n}{2n}\sqrt{g}\xi^ku^{\frac{2n}{n-2}}$$
$$+\frac{2-n}{4}\sqrt{g}g^{kj}\left(\mu uu_j-\frac{1}{2}\mu_ju^2\right), \tag{26}$$

and ξ, μ satisfy (17), (18).

6. *If $n=6$ and $f(u)=u^2$, then the conservation law is $D_iA^i=0$, where*

$$A^k=\sqrt{g}\left(\frac{1}{2}g^{ij}\xi^k-g^{kj}\xi^i\right)u_iu_j-\frac{1}{3}\sqrt{g}\xi^ku^3$$
$$+\sqrt{g}g^{kj}\left[\frac{1}{2}\left(\Delta_g\mu u_j+\mu_ju^2\right)-\left(\mu uu_j+(\Delta_g\mu)_ju\right)\right] \tag{27}$$

and ξ, μ satisfy (20), (21).

For some applications of symmetries and conservation laws see [10,13,15,27,22].

We observe that in [26,27] Ibragimov has established the fact that the projection of the Lie symmetry group on the space of independent variables is a sub-group of the conformal group of the considered manifold. For this purpose he has obtained the symmetry determining Eqs. (50)–(52) (see below and [27]). However, a complete group classification, up to the authors' knowledge, as carried out in Theorem 1 without restrictions on the curvature, is new. Moreover, the study of Noether's symmetries (Theorem 2) and corresponding to them conservation laws is also original. In this way our results complement Ibragimov's ones. We hope that they give useful insights. In fact, the results on conformally invariant equations established in [27] are our main motivation to write the present work.

This paper is organized as follows. Section 2 includes the geometric preliminaries and introduces notations and conventions used in this paper. Further, the determining equations for the symmetry coefficients are obtained in Section 3. The connections between isometry groups and symmetry groups are established in Section 4. The group classification for the linear cases is obtained in Section 5 and for the nonlinear cases in Sections 6 and 7. The proof of Corollary 1 concerning the Lie point symmetries in the case of compact manifolds without boundary is presented in Section 8. The Noether symmetries are found in Sections 9 and 10. In order to illustrate the main results, some examples are presented in Section 11 in which we perform the group classification and establish the Noether symmetries and their respective conservation laws for the nonlinear Poisson equations in the Thurston geometries, namely: $\mathbb{R}^3$, the three-dimensional hyperbolic space $\mathbb{H}^3$, the sphere $\mathbb{S}^3$, the three-dimensional solvable group Sol, the product spaces $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, the universal covering of $SL_2(\mathbb{R})$ and the three-dimensional Heisenberg group.

2. Preliminaries

In this section we introduce notation. We also state some results which will be used later.

The Riemann tensor of (M^n,g) is given by

$$R^i{}_{jks}=\Gamma^i_{jk,s}-\Gamma^i_{js,k}+\Gamma^i_{ls}\Gamma^l_{jk}-\Gamma^i_{lk}\Gamma^l_{js}. \tag{28}$$

The Ricci tensor

$$R^i{}_s=g^{jk}R^i{}_{jks} \tag{29}$$

and its trace $R:=R^i{}_i$ is the scalar curvature of M.

For any contravariant vector field $T=(T^i)$ the following commutation relation holds

$$(\nabla_k\nabla_l-\nabla_l\nabla_k)T^i=-R^i{}_{skl}T^s. \tag{30}$$

See [21].

We observe that the Riemann and Ricci tensors used in this paper coincide with those in Yano's book [48] and in Dubrovin, Fomenko and Novikov's book [21]; they are negatives of the respective tensors used by Ibragimov in [27].

We shall need some auxiliary results which are presented in a sequence of lemmas.

Lemma 1. *If ξ is a conformal Killing vector field satisfying*

$$\nabla^i\xi^j+\nabla^j\xi^i=\mu g^{ij}, \tag{31}$$

then the covariant divergence

$$div(\xi)=\nabla_j\xi^j=\frac{n}{2}\mu. \tag{32}$$

Proof. Just take the trace in (31). □

Lemma 2. *If ξ is a conformal Killing vector field* (*see Eq.* (31)) *then*

$$\Delta_g \xi^i + R^i{}_j \xi^j = \frac{2-n}{2} g^{ij} \mu_j. \tag{33}$$

Proof. Applying the covariant derivative operator ∇_j to equality (31) and summing up we obtain

$$\nabla_j \nabla^i \xi^j + \Delta_g \xi^i = g^{ij} \mu_j. \tag{34}$$

On the other hand, from (30) it follows that

$$\nabla_j \nabla^i \xi^j - \nabla^i \nabla_j \xi^j = R^i{}_s \xi^s$$

and hence

$$\nabla_j \nabla^i \xi^j = \nabla^i \, div(\xi) + R^i{}_s \xi^s. \tag{35}$$

Then (32), (34) and (35) imply the relation (33). □

Lemma 3. *(See Yano [48].) If ξ is a conformal vector field such that*

$$\mathcal{L}_\xi g_{ij} = \mu g_{ij}, \tag{36}$$

then

$$\Delta_g \mu = -\frac{1}{n-1} (\mathcal{L}_\xi R + \mu R). \tag{37}$$

The next two lemmas concern the form of the semilinear Poisson equation (1) and its variational structure.

Lemma 4. *The Poisson equation* (1) *can be written in the equivalent form* (2).

Proof. This can be seen by performing explicitly the partial differentiations in the Laplace–Beltrami operator and using the formula

$$\left(\sqrt{g} g^{ik}\right)_{,k} = -g^{pq} \Gamma^i_{pq} \sqrt{g}. \qquad \square \tag{38}$$

Lemma 5. *The Poisson equation* (1) *has a variational structure and it is* (*formally*) *the Euler–Lagrange equation of a functional $\int_M L\, dx$, where the Lagrangian*

$$L = \frac{\sqrt{g}}{2} g^{ij} u_i u_j - F(u)\sqrt{g}, \tag{39}$$

and $F'(u) = f(u)$.

Proof. First we apply to L the Euler operator

$$E = \frac{\partial}{\partial u} - D_k \frac{\partial}{\partial u_k},$$

where

$$D_k = \frac{\partial}{\partial x^k} + u_k \frac{\partial}{\partial u} + u_{ks} \frac{\partial}{\partial u_s} + \cdots$$

is the total derivative operator. Then, after simplifying, Eq. (1) is obtained. $\square$

3. The determining equations

In this section we shall obtain the set of linear partial differential equations determining the Lie point symmetries of the Poisson equation (1).

To begin with, let

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

be a partial differential operator on $M^n \times \mathbb{R}$ which is infinitesimal generator of such a symmetry. Let $X^{(1)}$ be the first-order prolongation of X. (See [9,10,27,35,37] for the corresponding definitions.) First we shall simplify the form of X.

Proposition 1. *If $n \geqslant 2$, then the infinitesimals of the symmetry X take the form*

$$\begin{cases} \xi^i = \xi^i(x), \\ \eta = a(x)u + b(x), \end{cases} \tag{40}$$

where $a = a(x)$ and $b = b(x)$ are smooth functions.

Proof. This proposition follows from two theorems of Bluman [8,10]. $\square$

The following intermediate result, up to notation, is the same as in Ibragimov's book [27, pp. 114–115].

Proposition 2. *(See Ibragimov [27].) The infinitesimals of the Lie point symmetries of Eq.* (1) *satisfy the relations*:

$$\xi^k g^{ij}{}_{,k} - g^{ik}\xi^j{}_{,k} - g^{jk}\xi^i{}_{,k} + ag^{ij} = \lambda g^{ij}, \tag{41}$$

$$2g^{ij}a_j - g^{jk}\xi^i{}_{,jk} + \Gamma^j\xi^i{}_{,j} - \Gamma^i{}_{,j}\xi^j - a\Gamma^i = -\lambda\Gamma^i, \tag{42}$$

$$(\Delta_g a)u + (au+b)f'(u) + \Delta_g b = \lambda f(u), \tag{43}$$

where $\lambda = \lambda(x)$ and $\Gamma^i := g^{pq}\Gamma^i_{pq}$.

Proof. This proposition follows from Proposition 1 and the definition of Lie point symmetry. $\square$

Proposition 3. *Let $\xi = \xi^i \frac{\partial}{\partial x^i}$. Then the determining equations can be written in the following equivalent form:*

$$\nabla_i \xi_j + \nabla_j \xi_i = \mathcal{L}_\xi g_{ij} = \mu g_{ij}, \tag{44}$$

$$2g^{ij} a_j - \left(\Delta_g \xi^i + R^i{}_j \xi^j\right) = 0, \tag{45}$$

$$auf'(u) + bf'(u) - \lambda f(u) + (\Delta_g a)u + \Delta_g b = 0, \tag{46}$$

where $\mu := a - \lambda$ and $R^i{}_j$ is the Ricci tensor of g.

Corollary 5. *The relation* (45) *is equivalent to*

$$a_i = \frac{2-n}{4}\mu_i \tag{47}$$

which, itself, is equivalent to

$$\lambda_i = \frac{n+2}{n-2} a_i. \tag{48}$$

Thus, the determining equations are (44), (47) *and*

$$auf'(u) + bf'(u) + (\mu - a) f(u) + \frac{2-n}{4}(\Delta_g \mu)u + \Delta_g b = 0. \tag{49}$$

Corollary 6. *The determining equations are:*

$$\nabla_i \xi_j + \nabla_j \xi_i = \mathcal{L}_\xi g_{ij} = \mu g_{ij}, \tag{50}$$

$$a_i = \frac{2-n}{4}\mu_i, \tag{51}$$

$$auf'(u) + bf'(u) + (\mu - a) f(u) + \frac{n-2}{4(n-1)}\left[\xi^i R_{,i} + \mu R\right]u + \Delta_g b = 0. \tag{52}$$

The statement of Corollary 6 is explicitly announced in [27, pp. 115–116], without a detailed proof. For the sake of clearness and completeness we have decided to present here the corresponding proof, dividing the procedure in three steps: Proposition 3, Corollary 5 and Corollary 6.

Proof of Proposition 3. The equivalence between (43) and (46) is obvious. The equivalence between (41) and (44) is clear from the definitions of Lie derivative and conformal Killing vectors. Indeed, from (41):

$$\xi^k g^{ij}{}_{,k} - g^{is} \xi^j{}_{,s} - g^{js} \xi^i{}_{,s} = -\mu g^{ij}.$$

The formula

$$g^{ij}{}_{,k} = -\Gamma^i_{ks} g^{sj} - \Gamma^j_{ks} g^{is},$$

substituted above, implies

$$\nabla^i \xi^j + \nabla^j \xi^i = \mu g^{ij}.$$

Hence and by the definition of Lie derivative, it follows that (44) holds. And vice versa: (44) implies (41).

Further, we shall prove the equivalence of (42) and (45). From (42) we have

$$2g^{ij}a_j - \left(g^{jk}\xi^i{}_{,jk} - \Gamma^j\xi^i{}_{,j} + \Gamma^i{}_{,j}\xi^j + \mu\Gamma^i\right) = 0. \tag{53}$$

We denote

$$A^i := g^{jk}\xi^i{}_{,jk}, \qquad B^i := \Gamma^j\xi^i{}_{,j}, \qquad C^i := \Gamma^i{}_{,j}\xi^j.$$

Then (53) reads

$$2g^{ij}a_j - \left(A^i - B^i + C^i + \mu\Gamma^i\right) = 0.$$

We aim to express A^i, B^i and C^i in the terms of covariant derivatives. From

$$\nabla_j\xi^i = \xi^i{}_{,j} + \Gamma^i_{js}\xi^s \tag{54}$$

and

$$\nabla_j\Gamma^i = \Gamma^i{}_{,j} + \Gamma^i_{js}\Gamma^s \tag{55}$$

we obtain

$$B^i = \Gamma^j\left(\nabla_j\xi^i - \Gamma^i_{js}\xi^s\right) = \Gamma^j\nabla_j\xi^i - \Gamma^j\Gamma^i_{js}\xi^s \tag{56}$$

and

$$C^i = \left(\nabla_j\Gamma^i - \Gamma^i_{js}\Gamma^s\right)\xi^j = \left(\nabla_j\Gamma^i\right)\xi^j - \Gamma^i_{js}\Gamma^s\xi^j. \tag{57}$$

Further:

$$\begin{aligned}
\nabla_k\xi^i{}_{,j} &= \xi^i{}_{,jk} + \Gamma^i_{kl}\xi^l{}_{,j} - \Gamma^l_{kj}\xi^i{}_{,l},\\
\xi^i{}_{,jk} &= \nabla_k\left(\nabla_j\xi^i - \Gamma^i_{js}\xi^s\right) - \Gamma^i_{kl}\left(\nabla_j\xi^l - \Gamma^l_{js}\xi^s\right) + \Gamma^l_{kj}\left(\nabla_l\xi^i - \Gamma^i_{ls}\xi^s\right)\\
&= \nabla_k\nabla_j\xi^i - \left(\nabla_k\Gamma^i_{js}\right)\xi^s - \Gamma^i_{js}\nabla_k\xi^s - \Gamma^i_{kl}\nabla_j\xi^l\\
&\quad + \Gamma^i_{kl}\Gamma^l_{js}\xi^s + \Gamma^l_{kj}\nabla_l\xi^i - \Gamma^i_{ls}\xi^s\Gamma^l_{kj}.
\end{aligned} \tag{58}$$

From (53), (56), (57) and (58) it follows that

$$\begin{aligned}
&2g^{ij}a_j - \Delta_g\xi^i + g^{jk}\left(\nabla_k\Gamma^i_{js}\right)\xi^s + g^{jk}\Gamma^i_{js}\nabla_k\xi^s + \Gamma^i_{kl}g^{jk}\nabla_j\xi^l - g^{jk}\Gamma^i_{kl}\Gamma^l_{js}\xi^s\\
&\quad - \Gamma^l\nabla_l\xi^i + \Gamma^l\Gamma^i_{ls}\xi^s - \left(\nabla_j\Gamma^i\right)\xi^j + \Gamma^s\Gamma^i_{js}\xi^j + \Gamma^j\nabla_j\xi^i - \Gamma^j\Gamma^i_{js}\xi^s - \mu\Gamma^i = 0.
\end{aligned} \tag{59}$$

On the other hand, from (44)

$$\nabla^k\xi^l = -\nabla^l\xi^k + \mu g^{lk}$$

and hence

$$-\Gamma^i_{kl} g^{jk}\nabla_j \xi^l = \Gamma^i_{kl}\nabla^l \xi^k - \mu g^{lk}\Gamma^i_{lk}. \tag{60}$$

Substituting (60) into (59) we obtain after renaming the indices and canceling some terms:

$$2g^{ij}a_j - \left(\Delta_g \xi^i + w^i_s \xi^s\right) = 0, \tag{61}$$

where

$$w^i_s = -g^{jk}\left(\nabla_k \Gamma^i_{js}\right) + g^{jk}\Gamma^i_{kl}\Gamma^l_{js} + \nabla_s \Gamma^i - \Gamma^l \Gamma^i_{sl}. \tag{62}$$

Then we express the covariant derivatives in (62) in the terms of usual partial derivatives. In this way we get that

$$w^i_s = g^{jk}\left(\Gamma^i_{jk,s} - \Gamma^i_{js,k} + \Gamma^i_{ls}\Gamma^l_{jk} - \Gamma^i_{lk}\Gamma^l_{js}\right) = g^{jk}R^i{}_{jks} = R^i{}_s, \tag{63}$$

where $R^i{}_{jks}$ and $R^i{}_s$ are the components of the Riemann and Ricci tensors respectively. Thus (61), (62) and (63) imply (45). □

Proof of Corollary 5. From (33) and (45) we obtain

$$2g^{ij}a_j - \frac{2-n}{2}g^{ij}\mu_j = 0,$$

that is,

$$a_i = \frac{2-n}{4}\mu_i.$$

The rest of the proof is straightforward. □

Proof of Corollary 6. The conclusion follows from Corollary 4 and Lemma 3 (see (37)). □

4. The isometry group and the Lie point symmetry group for arbitrary $f(u)$

In this section we prove the first part of Theorem 1.

Let X be a symmetry of (1). Then X has necessarily the form given in Proposition 1. From (46), equating to zero the terms involving u, we obtain

$$a = b = \lambda = \mu = 0.$$

Hence, $\eta = 0$ and $X = \xi^i \frac{\partial}{\partial x^i}$. From (44) with $\mu = 0$, it follows that X is an isometry.

Let $X = \xi^i \frac{\partial}{\partial x^i}$ be an infinitesimal isometry of M^n. Therefore $\mathcal{L}_X g_{ij} = 0$. Hence Eq. (44) holds with $\mu = 0$. By the form of X, $\eta = 0$ and thus $a = b = 0$. Hence the relation (49) is satisfied. Obviously (47) is satisfied since $a = \mu = 0$. Therefore $X = \xi^i \frac{\partial}{\partial x^i}$ is a Lie point symmetry.

In this way we have proved that

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of (1) with arbitrary $f(u)$ if and only if $\eta = 0$ and $X = \xi^i(x)\frac{\partial}{\partial x^i} = \xi$ is an infinitesimal isometry of M^n. In other words, the isometry group of (M^n, g) is the invariance group of its Laplace–Beltrami operator!

In this case, by (37) we have $\mathcal{L}_\xi R = 0$, an integrability condition for $\mathcal{L}_\xi g_{ij} = 0$, which we suppose holds true. See [48] for details concerning the integrability conditions for $\mathcal{L}_\xi g_{ij} = 0$ or, more generally, $\mathcal{L}_\xi g_{ij} = \mu g_{ij}$.

5. The Lie point symmetries in the linear cases

In this section we prove parts 2.), 3.) and 4.) of Theorem 1.

Let $f(u) = 0$. Then from (49) we conclude that $\Delta_g b = 0$ and $\Delta_g \mu = 0$. Integrating (47) we obtain $a = \frac{2-n}{4}\mu + c$, where c is an arbitrary constant. This completes the proof of the second part of Theorem 1.

Further, let $f(u) = k = const. \neq 0$. From (49) we have

$$(\mu - a)k + \frac{2-n}{4}(\Delta_g \mu)u + \Delta_g b = 0.$$

Hence, equating to zero the coefficient of u and the free term, we obtain

$$\Delta_g \mu = 0 \tag{64}$$

and

$$(\mu - a)k + \Delta_g b = 0. \tag{65}$$

Again from (47) we get $a = \frac{2-n}{4}\mu + c$, where c is an arbitrary constant. Substituting this expression for a in (65) we find

$$\mu = \frac{4}{(n+2)}\left(c - \frac{1}{k}\Delta_g b\right).$$

It remains to put the latter into (64) to conclude that $\Delta_g^2 b = 0$.

Let now $f(u) = u$. From (49) we have

$$au + b + (\mu - a)u + \frac{2-n}{4}(\Delta_g \mu)u + \Delta_g b = 0.$$

Hence, equating to zero the coefficient of u and the free term, we obtain $\Delta_g b + b = 0$ and $\frac{2-n}{4}\Delta_g \mu + \mu = 0$. This completes the proof of Theorem 1 in the linear cases.

6. The Lie point symmetries in the case of exponential nonlinearity

Let $f(u) = e^u$. From (49) we obtain

$$aue^u + (b + \mu - a)e^u + \frac{2-n}{4}(\Delta_g \mu)u + \Delta_g b = 0.$$

Hence, equating to zero the coefficients of ue^u, e^u, u and the free term, we obtain $a = 0$, $b = -\mu$ and $\Delta_g b = \Delta_g \mu = 0$. From (47)

$$\frac{2-n}{4}\mu_i = 0$$

since $a = 0$. Thus $\mu_i = 0$ because $n \geqslant 3$. Hence $\mu = const$. This completes the proof.

We observe that this case, in fact, is a Liouville–Gelfand problem on Riemannian manifolds.

7. The Lie point symmetries in the case of power nonlinearity

In this section we shall prove parts 6.), 6.1) and 6.2) of Theorem 1.

Let $f(u) = u^p$. The cases $p = 0$ and $p = 1$ have already been considered and we may suppose that $p \neq 0$ and $p \neq 1$.

From (49) we obtain

$$\big[(p-1)a + \mu\big]u^p + pbu^{p-1} + \frac{2-n}{4}(\Delta_g \mu)u + \Delta_g b = 0.$$

Let $p \neq 2$. Then, equating to zero the coefficients of u^p, u^{p-1}, u and the free term, implies that

$$a = \frac{1}{1-p}\mu, \tag{66}$$

$pb = 0$ and $\Delta_g \mu = 0$. Since $p \neq 0$ it follows that $b = 0$. From (47) and (66) it follows that

$$\left[\frac{1}{1-p} + \frac{n-2}{4}\right]\mu_i = 0. \tag{67}$$

(i) If $p \neq \frac{n+2}{n-2}$, then (67) implies $\mu_i = 0$ for all i and thus $\mu = const.$ and ξ is a homothety.

(ii) Let $p = \frac{n+2}{n-2}$ and $n \neq 6$. (Otherwise $p = 2$.) From (66) it is clear that the symmetry has the form announced in 6.1) of Theorem 1.

Let now $p = 2$. From (49) we have

$$(a+\mu)u^2 + \left(2b + \frac{2-n}{4}(\Delta_g \mu)\right)u + \Delta_g b = 0. \tag{68}$$

Hence

$$a = -\mu, \tag{69}$$

$$2b + \frac{2-n}{4}(\Delta_g \mu) = 0 \tag{70}$$

and

$$\Delta_g b = 0. \tag{71}$$

From (47) and (69) it follows that

$$\frac{6-n}{4}\mu_i = 0. \tag{72}$$

Again we have to consider two cases.

I.) If $n \neq 6$ then from (72) we obtain $\mu_i = 0$ for all i and thus $\mu = const$. Hence and from (70) it follows that $b = 0$. We see that this case is included in case 6.) of Theorem 1.

II.) Let $n = 6$. Now we cannot conclude from (72) that μ is a constant. From (70) we express b as a function of μ:

$$b = \frac{n-2}{8}\Delta_g \mu = \frac{1}{2}\Delta_g \mu.$$

Substituting this expression for b into (71) we obtain $\Delta_g^2 \mu = 0$. This completes the proof of Theorem 1.

Proof of Corollary 3. If $R = 0$ then by Lemma 3, (37) any conformal transformation (36) satisfies $\Delta_g \mu = 0$ and hence the conclusion follows immediately. □

8. The Lie point symmetries in the case of compact manifolds without boundary

In this section we prove Corollary 1.

1.) Let $f(u)$ be an arbitrary function. By Theorem 1 the symmetry group coincides with the isometry group of (M^n, g).

2.) If $f(u) = 0$ by Theorem 1 the symmetry is determined by (4), (5), (6), (7). Since μ and b are harmonic, by the E. Hopf's maximum principle it follows that $\mu = const.$ and $b = const.$ But $\mu = 2\,div(\xi)/n$ (see Lemma 1). Hence

$$\mu\, Vol\bigl(M^n\bigr) = \int\limits_{M^n} \mu\, dV = \frac{2}{n}\int\limits_{M^n} div(\xi)\, dV = 0$$

by the Green's theorem. Thus $\mu = 0$.

3.) Here we observe that the Poisson equation with $f(u) = k$ on compact manifolds without boundary makes sense only if $k = 0$. This follows directly from the Green's theorem.

The proof of Corollary 1 in the rest of the cases in Theorem 1 is similar to the above presented. For this reason we shall not present further details merely pointing out that the constancy of the conformal factor μ, the harmonicity of μ or biharmonicity of μ imply that $\mu = 0$ by the maximum principle and the Green's theorem.

9. The Noether symmetries in nonlinear cases

In this and the next sections we present the proof of Theorem 2 divided in several propositions and lemmas.

In order to apply the infinitesimal criterion for invariance [35, p. 257], we need the following

Proposition 4. *Let*

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \bigl[a(x)u + b(x)\bigr]\frac{\partial}{\partial u}, \tag{73}$$

where a, b and ξ^i are smooth functions, be a partial differential operator on $M^n \times \mathbb{R}$. Then

$$\begin{aligned} X^{(1)}L + LD_i\xi^i = {} & \frac{1}{2}\bigl[g^{ks}\,div(\xi) + 2ag^{ks} - \nabla^k\xi^s - \nabla^s\xi^k\bigr]\sqrt{g}u_k u_s \\ & - \sqrt{g}\,div(\xi)F(u) - \sqrt{g}auf(u) - \sqrt{g}bf(u) \\ & + (a_i u + b_i)\sqrt{g}g^{is}u_s, \end{aligned} \tag{74}$$

where $X^{(1)}$ is the first-order prolongation of X and the function L is given in (39).

Proof. By a straightforward calculation we obtain the first-order prolongation

$$X^{(1)} = X + \big(a_i u + b_i + \big(a\delta_i^j - \xi^j{}_{,i}\big)u_j\big)\frac{\partial}{\partial u_i}$$

where ',' means partial derivative: $\xi^j{}_{,i} = \partial\xi^j/\partial x^i$ and δ_i^j is the Kronecker symbol. We apply $X^{(1)}$ to L given by (39) and then change some of the indices in the obtained expression. In this way we get that

$$\begin{aligned} X^{(1)}L + L\frac{\partial\xi^i}{\partial x^i} = {} & \left[\frac{1}{2}\xi^i\big(g^{ks}\sqrt{g}\big)_{,i} + \big(a\delta_i^k - \xi^k{}_{,i}\big)g^{is}\sqrt{g} + \frac{1}{2}\xi^i{}_{,i}g^{ks}\sqrt{g}\right]u_k u_s \\ & - auf(u)\sqrt{g} - bf(u)\sqrt{g} + (a_i u + b_i)u_s g^{is}\sqrt{g} \\ & - F(u)\xi^i(\sqrt{g})_i - F(u)\xi^i{}_{,i}\sqrt{g}. \end{aligned} \tag{75}$$

Further we shall make use of the formulae

$$\big(g^{ks}\sqrt{g}\big)_{,i} = -g^{sl}\Gamma^k_{li} - g^{kl}\Gamma^s_{li}, \qquad (\sqrt{g})_{,i} = \Gamma^k_{ik}\sqrt{g}, \tag{76}$$

where Γ's are the Christoffel symbols. Then by the definition of the covariant derivative operators ∇^i, corresponding to the Levi-Civita connection ∇, and the second formula in (76), the last two terms in (75) can be written as

$$-\xi^i(\sqrt{g})_{,i}F(u) - \xi^i{}_{,i}\sqrt{g}F(u) = -div(\xi)F(u)\sqrt{g}. \tag{77}$$

(We recall that $div(\xi) = \nabla_i\xi^i$ is the covariant divergence of ξ.)

Now we denote by A the expression in the right-hand side of the first line of (75) containing $u_k u_s$. Using (76) we obtain that

$$\begin{aligned} A &= \left\{\frac{1}{2}\xi^i\sqrt{g}\big[-g^{sl}\Gamma^k_{li} - g^{kl}\Gamma^s_{li} + g^{ks}\Gamma^l_{il}\big] + ag^{ks}\sqrt{g}\right. \\ &\quad \left. - \frac{1}{2}g^{is}\sqrt{g}\xi^k{}_{,i} - \frac{1}{2}g^{ki}\sqrt{g}\xi^s{}_{,i} + \frac{1}{2}g^{ks}\sqrt{g}\xi^i{}_{,i}\right\}u_k u_s \\ &= \left\{-\frac{1}{2}\big(g^{is}\xi^k{}_{,i} + g^{sl}\Gamma^k_{li}\xi^i\big) - \frac{1}{2}\big(g^{ki}\xi^s{}_{,i} + g^{kl}\Gamma^s_{li}\xi^i\big)\right. \\ &\quad \left. + \frac{1}{2}g^{ks}\big(\xi^i{}_{,i} + \Gamma^l_{il}\xi^i\big) + ag^{ks}\right\}\sqrt{g}u_k u_s. \end{aligned} \tag{78}$$

From the definition of the covariant derivative we have that

$$\begin{aligned} \nabla^s\xi^k &= g^{is}\xi^k{}_{,i} + g^{sl}\Gamma^k_{li}\xi^i, \\ \nabla^k\xi^s &= g^{ki}\xi^s{}_{,i} + g^{kl}\Gamma^s_{li}\xi^i, \\ \nabla_i\xi^i &= \xi^i{}_{,i} + \Gamma^l_{il}\xi^i. \end{aligned}$$

We substitute these formulae into (78). Thus

$$A = \frac{1}{2}\big[-\nabla^s\xi^k - \nabla^k\xi^s + g^{ks}\,div(\xi) + 2ag^{ks}\big]\sqrt{g}u_k u_s. \tag{79}$$

From (75), (77) and (79) we obtain (74). □

Now we shall prove the first part of Theorem 2, namely

Lemma 6. *For an arbitrary $f(u)$ any symmetry of* (1) *is a variational symmetry, that is, the isometry group of M^n and the variational symmetry group of* (1) *coincide.*

Proof. We have already seen in Section 4 that in this case $a = b = \mu = div(\xi) = 0$. Substituting this data into (74) we obtain $X^{(1)}L + LD_i\xi^i = 0$. Thus $X = \xi^i(x)\frac{\partial}{\partial x^i}$ is a variational symmetry. And vice-versa: from (74) it follows that any variational symmetry of (1) with arbitrary $f(u)$ is an isometry. $\square$

Lemma 7. *In the exponential case $f(u) = e^u$ the only variational symmetries are the isometries of M.*

Proof. Substituting $a = 0$, $b = -\mu = const.$, $div(\xi) = n\mu/2$ and $L = g^{ks}u_k u_s/2 - e^u$ into (74) we obtain

$$X^{(1)}L + LD_i\xi^i = (n-2)\mu L/2. \tag{80}$$

Hence it is clear that X is never a divergence symmetry. Again from (80), the symmetry X is variational if and only if $n = 2$ or $\mu = 0$. Since $n \geqslant 3$ it follows that the only variational symmetries in the exponential case are the isometries ($\mu = 0$). $\square$

Lemma 8. *In the power case $f(u) = u^p$, $p \neq 0$, $p \neq 1$, the symmetry* (14) *is variational if and only if*

$$p = \frac{n+2}{n-2},$$

that is, $p + 1$ equals to the critical Sobolev exponent.

Proof. We put $a = \mu/(1-p)$, $\mu = const.$, $b = 0$, $div(\xi) = n\mu/2$ and $L = g^{ks}u_k u_s/2 - u^{p+1}/(p+1)$ into (74). We obtain

$$X^{(1)}L + LD_i\xi^i = \left(-1 + \frac{n}{2} + \frac{2}{1-p}\right)\mu\sqrt{g}g^{ks}u_k u_s - \left(\frac{n}{2}\frac{1}{p+1} + \frac{1}{1-p}\right)\mu\sqrt{g}u^{p+1}. \tag{81}$$

Hence X is a variational symmetry if and only if in (81) the coefficients of the terms containing u and its derivatives vanish, which holds if and only if $p = (n+2)/(n-2)$. $\square$

Lemma 9. *Let X be the Lie point symmetry* (16)*. Then*

$$X^{(1)}L + LD_i\xi^i = \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i u u_j. \tag{82}$$

Proof. Substituting $a = (2-n)\mu/4$, $b = 0$, $div(\xi) = n\mu/2$ and $L = g^{ks}u_k u_s/2 - (n-2)u^{2n/(n-2)}/(2n)$ into (74) we obtain (82). $\square$

Lemma 10. *If $\Delta_g\mu = 0$, the following equality holds*:

$$\frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i = D_i\varphi^i, \tag{83}$$

where

$$\varphi^i = \varphi^i(x,u) = \frac{2-n}{8}\sqrt{g}g^{ij}\mu_j u^2.$$

Proof. We calculate the total divergence of $\varphi = (\varphi^i)$:

$$\begin{aligned}
D_i\varphi^i &= \left(\frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ik}\frac{\partial}{\partial u_k} + \cdots\right)\left(\frac{2-n}{8}\sqrt{g}g^{ij}\mu_j u^2\right)\\
&= \frac{2-n}{8}\big(\sqrt{g}g^{ij}\mu_j\big)_i u^2 + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i\\
&= \frac{2-n}{8}\big[\nabla_i\big(\sqrt{g}g^{ij}\mu_j\big) - \Gamma^i_{ki}\sqrt{g}g^{kj}\mu_j\big]u^2 + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i\\
&= \frac{2-n}{8}\big[\nabla_i(\sqrt{g})g^{ij}\mu_j + \sqrt{g}\nabla_i\big(g^{ij}\mu_j\big) - \Gamma^i_{ki}\sqrt{g}g^{kj}\mu_j\big]u^2 + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i\\
&= \frac{2-n}{8}\big[\big(\nabla_i(\sqrt{g}) - \Gamma^i_{ki}\sqrt{g}\big)g^{ij}\mu_j + \sqrt{g}\Delta_g\mu\big]u^2 + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i\\
&= \frac{2-n}{4}\sqrt{g}g^{ij}\mu_j u u_i
\end{aligned}$$

since the metric is parallel with respect to the Levi-Civita connection, $\Delta_g\mu = 0$ and $\Gamma^i_{ki}\sqrt{g} = (\ln\sqrt{g})_i\sqrt{g} = (\sqrt{g})_i = \nabla_i(\sqrt{g})$. $\square$

Then from (82) and (83) it follows that

$$X^{(1)}L + LD_i\xi^i = D_i\varphi^i,$$

that is, X is a divergence symmetry. In this way we have proved the part 3.1) of Theorem 2.

Lemma 11. *Let X be the Lie point symmetry* (19)*. Then*

$$X^{(1)}L + LD_i\xi^i = -\frac{1}{2}\sqrt{g}\Delta_g\mu u^2 - \sqrt{g}g^{ij}\mu_i u u_j + \frac{1}{2}\sqrt{g}g^{ij}(\Delta_g\mu)_i u_j. \tag{84}$$

Proof. Substituting $a = -\mu$, $b = \Delta_g\mu/2$, $div(\xi) = 3\mu$ and $L = g^{ks}u_k u_s/2 - u^3/3$ into (74) we obtain (84). $\square$

Lemma 12. *If $\Delta_g^2\mu = 0$, the following equality holds*:

$$-\frac{1}{2}\sqrt{g}\Delta_g\mu u^2 - \sqrt{g}g^{ij}\mu_i u u_j + \frac{1}{2}\sqrt{g}g^{ij}(\Delta_g\mu)_i u_j = D_i\phi^i, \tag{85}$$

where

$$\phi^i = \phi^i(x,u) = -\frac{1}{2}\sqrt{g}g^{ij}\mu_j u^2 + \sqrt{g}g^{ij}(\Delta_g\mu)_j u.$$

Proof. We calculate the total divergence of $\phi = (\phi^i)$:

$$\begin{aligned}
D_i\phi^i &= \left(\frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ik}\frac{\partial}{\partial u_k} + \cdots\right)\left(-\frac{1}{2}\sqrt{g}g^{ij}\mu_j u^2 + \sqrt{g}g^{ij}(\Delta_g\mu)_j u\right)\\
&= -\frac{1}{2}\big(\sqrt{g}g^{ij}\mu_j\big)_i u^2 - \sqrt{g}g^{ij}\mu_j u u_i + \big(\sqrt{g}g^{ij}(\Delta_g\mu)_j\big)_i u + \sqrt{g}g^{ij}(\Delta_g\mu)_j u_i
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\big[\nabla_i\big(\sqrt{g}g^{ij}\mu_j\big) - \Gamma^k_{ik}\sqrt{g}g^{ij}\mu_j\big]u^2 - \sqrt{g}g^{ij}\mu_j uu_i \\
&\quad + \nabla_i\big(\sqrt{g}g^{ij}(\Delta_g\mu)_j\big)u - \Gamma^k_{ik}\sqrt{g}g^{ij}(\Delta_g\mu)_j u + \sqrt{g}g^{ij}(\Delta_g\mu)_j u_i \\
&= -\frac{1}{2}\big[\nabla_i(\sqrt{g}) - \Gamma^k_{ik}\sqrt{g}\big]g^{ij}\mu_j u^2 - \frac{1}{2}\sqrt{g}\Delta_g\mu u^2 - \sqrt{g}g^{ij}\mu_j uu_i \\
&\quad + \big[\nabla_i(\sqrt{g}) - \Gamma^k_{ik}\sqrt{g}\big]g^{ij}(\Delta_g\mu)_j u + \sqrt{g}\Delta_g^2\mu u + \sqrt{g}g^{ij}(\Delta_g\mu)_i u_j \\
&= -\frac{1}{2}\sqrt{g}\Delta_g\mu u^2 - \sqrt{g}g^{ij}\mu_i uu_j + \frac{1}{2}\sqrt{g}g^{ij}(\Delta_g\mu)_i u_j
\end{aligned}
$$

since the metric is parallel with respect to the Levi-Civita connection, $\Delta_g^2\mu = 0$ and $\Gamma^i_{ki}\sqrt{g} = (\ln\sqrt{g})_i\sqrt{g} = (\sqrt{g})_i = \nabla_i(\sqrt{g})$. □

Then from (84) and (85) it follows that

$$X^{(1)}L + LD_i\xi^i = D_i\phi^i,$$

that is, X is a divergence symmetry. In this way we have proved the part 3.2) of Theorem 2.

10. The Noether symmetries in linear cases

Here we shall prove the part 4.) of Theorem 2.

Lemma 13. *Let X be the symmetry* (4)*. Then*

$$X^{(1)}L + LD_i\xi^i = 2cL + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i uu_j + \sqrt{g}g^{ij}b_i u_j. \tag{86}$$

Proof. Substituting $a = (\frac{2-n}{4}\mu(x) + c)$, $div(\xi) = \frac{n}{2}\mu$, $F(u) = f(u) = 0$ and $L = \sqrt{g}g^{ks}u_k u_s/2$ into (74) we obtain (86). □

Lemma 14. *If $\Delta_g\mu = \Delta_g b = 0$, the following equality holds*:

$$\frac{2-n}{4}\sqrt{g}g^{ij}\mu_j uu_i + \sqrt{g}g^{ij}b_i u_j = D_i\phi^i, \tag{87}$$

where

$$\phi^i = \phi^i(x,u) = \frac{2-n}{8}\sqrt{g}g^{ij}\mu_j u^2 + \sqrt{g}g^{ij}b_j u. \tag{88}$$

Proof. We have $\phi^i = \varphi^i + \psi^i$, where $\varphi^i = \frac{2-n}{8}\sqrt{g}g^{ij}\mu_j u^2$ and $\psi^i = \sqrt{g}g^{ij}b_j u$. Since the divergence of φ^i was already calculated in Lemma 10 we have that

$$
\begin{aligned}
D_i\phi^i = D_i\varphi^i + D_i\psi^i &= \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i uu_j + \left(\frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ik}\frac{\partial}{\partial u_k} + \cdots\right)\big(\sqrt{g}g^{ij}b_j u\big) \\
&= \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i uu_j + \big(\sqrt{g}g^{ij}b_j\big)_i u + \sqrt{g}g^{ij}b_j u_i \\
&= \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i uu_j + \sqrt{g}\Delta_g bu + \sqrt{g}g^{ij}b_i u_j
\end{aligned}
$$

$$= \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i u u_j + \sqrt{g}g^{ij}b_i u_j$$

since $\Delta_g b = 0$. □

Lemma 15. *Let X be the symmetry* (4)*. Then X is a Noether symmetry if and only if $c = 0$.*

Proof. From (86) and (87) it follows that

$$X^{(1)}L + LD_i\xi^i = 2cL + D_i\phi^i. \tag{89}$$

Then the conclusion of Lemma 15 follows from (89). This proves part 4.1) of Theorem 2. □

We observe that the symmetry (4) can be written as

$$X = \xi + \frac{2-n}{4}\mu(x)\frac{\partial}{\partial u} + b(x)\frac{\partial}{\partial u} + cu\frac{\partial}{\partial u},$$

where $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ is a conformal Killing vector field satisfying (7), $c = const.$, $\Delta_g b = 0$ and $\Delta_g \mu = 0$. The potentials φ^i, ψ^i are the potentials of the symmetries $\xi + \frac{2-n}{4}\mu(x)\frac{\partial}{\partial u}$ and $b(x)\frac{\partial}{\partial u}$, respectively. The symmetry $u\frac{\partial}{\partial u}$ corresponds to a non-Noetherian symmetry and it reflects the linearity of the equation.

Lemma 16. *Let X be the symmetry* (8)*. Then*

$$X^{(1)}L + LD_i\xi^i = 2cL + \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i u u_j + \sqrt{g}g^{ij}b_i u_j + \sqrt{g}\Delta_g b u - \sqrt{g}bk. \tag{90}$$

Proof. Substituting

$$a = \left(\frac{2-n}{4}\mu(x) + c\right), \qquad div(\xi) = \frac{n}{2}\mu, \qquad F(u) = ku, \qquad f(u) = k,$$

$$L = \sqrt{g}g^{ks}u_k u_s/2 - ku\sqrt{g}$$

into (74) and using $\Delta_g b = ck - \frac{n+2}{4}\mu k$ (see (9)) we obtain (90). □

Observe that by (9): $\mu = \frac{4}{(n+2)}(c - \frac{1}{k}\Delta_g b)$. Hence $\Delta_g \mu = 0$ since $\Delta_g^2 b = 0$.

Lemma 17. *Let ϕ^i be the potential* (88) *and $\Delta_g \mu = 0$. Then*

$$D_i\phi^i = \frac{2-n}{4}\sqrt{g}g^{ij}\mu_i u u_j + \sqrt{g}g^{ij}b_i u_j + \sqrt{g}\Delta_g b u. \tag{91}$$

The proof of this lemma is similar to that of Lemma 10 and Lemma 14.

Lemma 18. *Let X be the symmetry* (8)*. Then X is a Noether symmetry if and only if $c = b = 0$.*

Proof. Substituting $\Delta_g b = -\frac{n+2}{4}k\mu + kc$ and (91) into (90), we obtain

$$X^{(1)}L + LD_i\xi^i = 2cL + D_i\phi^i - \sqrt{g}bk \tag{92}$$

which implies Lemma 18. This proves part 4.2) of Theorem 2. □

Now let X be the symmetry (4) of (1) with $f(u) = u$. We shall only sketch the proof of 4.3) of Theorem 2.

Substituting $a = (2-n)\mu/4 + c$, $div(\xi) = n/2$, $f(u) = u$, $F(u) = u^2/2$, $L = \sqrt{g}g^{ij}u_iu_j/2 - \sqrt{g}u^2/2$ into (74), we obtain

$$X^{(1)}L + LD_i\xi^i = 2cL + D_i\phi^i - \sqrt{g}u^2\left(\frac{2-n}{2}\Delta_g\mu + \mu\right)\Big/2 - \sqrt{g}u(\Delta_g b + b), \tag{93}$$

where ϕ^i is given in (88). Then from (10), (11) and (93) we get

$$X^{(1)}L + LD_i\xi^i = 2cL + D_i\phi^i.$$

Hence X is a Noether symmetry if and only if $c = 0$.

Thus, we have concluded the proof of Theorem 2.

11. Examples: Poisson equations on Thurston geometries

In this section we apply our results to Poisson equations on the Thurston geometries. The presentation is very schematic in order not to increase the volume of the paper.

All examples presented here involve elliptic forms of Eq. (1). Examples of symmetry analysis involving some particular hyperbolic cases of (1) can be found in [5,22,24].

Some of the results presented in this section were verified using the SYM package [19,20].

To the authors' knowledge, the results in Sections 11.3, 11.4, 11.5, 11.6, 11.7, 11.8 and 11.9 are original.

11.1. Thurston geometries

A manifold M^n is said to be homogeneous if, for every $x, y \in M$, there exists an isometry of M^n such that it leaves x in y. Let X be the universal covering of M^n and G its isometry group.

A geometry consists of a pair (X, G) as above, where X is a connected manifold and G is a group that acts effectively and transitively on X, and where all stabilizers G_x are compact. This is also equivalent to the data of a connected Lie group G and a compact Lie subgroup H of G, if we associate to this data the homogeneous space $X = G/H$ endowed with the natural left action of G.

Two geometries (X, G) and (X', G') are identified if there is a diffeomorphism from X to X' which sends the action of G to the action of G'. (X, G) is said to be maximal if there is no larger geometry (X', G') with $G \subseteq G'$ and $G \neq G'$. For more details, see [11,41].

There are exactly 8 three-dimensional maximal geometries (X, G), the so-called Thurston geometries.

Thurston [45] has classified the three-dimensional, simply-connected, homogeneous manifolds as follows (see also [44,46,11,41]):

- the Euclidean space $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$, with canonical metric

$$ds^2 = dx^2 + dy^2 + dz^2;$$

- the hyperbolic space $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, with metric

$$ds^2 = \left(dx^2 + dy^2 + dz^2\right)/z^2;$$

- the sphere $\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 x_i^2 = 1\}$, with induced metric from $\mathbb{R}^4$;
- the solvable group Sol, which can be defined as the Lie group $(\mathbb{R}^3, *)$ where

$$(x, y, z) * \left(x', y', z'\right) = \left(x + x', y + e^{-x} y', z + e^x z'\right),$$

  with left-invariant metric $ds^2 = dx^2 + e^{2x} dy^2 + e^{-2x} dz^2$;
- the space $\mathbb{S}^2 \times \mathbb{R}$, with product metric;
- the space $\mathbb{H}^2 \times \mathbb{R}$, with product metric. Here $\mathbb{H}^2$ is the two-dimensional hyperbolic space;
- the universal covering of $SL_2(\mathbb{R})$, or $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, with metric

$$ds^2 = \left(dx + \frac{dy}{z}\right)^2 + \frac{(dy^2 + dz^2)}{z^2}$$

  and
- the Heisenberg group H^1, whose group structure is given by

$$\phi\left((x, y, t), \left(x', y', t'\right)\right) = \left(x + x', y + y', t + t' + 2\left(yx' - xy'\right)\right)$$

  and the left-invariant metric is $ds^2 = dx^2 + dy^2 + (dz + 2y\,dx - 2x\,dy)^2$.

Three of them are isotropic geometries: if the curvature is positive, then the isotropic geometry is the 3-sphere $\mathbb{S}^3$. If the curvature is negative, the isotropic geometry is the hyperbolic space $\mathbb{H}^3$. If the curvature vanishes, the isotropic geometry is the Euclidean space $\mathbb{R}^3$.

Four of the Thurston geometries, the product spaces $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and two Lie groups, $\widetilde{SL_2}(\mathbb{R})$ and H^1, are known as the four Seifert type geometries.

Finally, we have the Sol group, which possesses this name because the group G of the pair (Sol, G) is solvable and it is the only one of the Thurston geometries with this property.

For more details about the Thurston geometries, see [44–46,11,41].

11.2. The Euclidean space

The application presented in this subsection is well known. It corresponds to the group classification, Noether symmetries and conservation laws of nonlinear Poisson equations in $\mathbb{R}^3$. The group classification of these equations can be found in [43] as a particular case. The Noether symmetries and conservation laws are established in [13] in a more general context.

Here we shall consider the three-dimensional vector space $\mathbb{R}^3$ with the Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

and the Poisson equation

$$u_{xx} + u_{yy} + u_{zz} + f(u) = 0. \tag{94}$$

11.2.1. The group classification

1. For any arbitrary function $f(u)$, the symmetry group of (94) coincides with the isometry group of $\mathbb{R}^3$. It is well known (see e.g. [27,21]) that the latter is generated by translations and rotations given by

$$R_1 = \frac{\partial}{\partial x}, \qquad R_2 = \frac{\partial}{\partial y}, \qquad R_3 = \frac{\partial}{\partial z},$$
$$R_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad R_5 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad R_6 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}. \tag{95}$$

   Hence (95) determine the symmetry group of (94) for arbitrary function $f(u)$.
2. If $f(u) = 0$ then the additional to (95) symmetries are

$$R_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{u}{2}\frac{\partial}{\partial u},$$
$$R_8 = xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + \frac{(z^2 - x^2 - y^2)}{2}\frac{\partial}{\partial z} - zu\frac{\partial}{\partial u},$$
$$R_9 = xy\frac{\partial}{\partial x} + \frac{y^2 - x^2 - z^2}{2}\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} - \frac{yu}{2}\frac{\partial}{\partial u},$$
$$R_{10} = \frac{x^2 - y^2 - z^2}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} - \frac{xu}{2}\frac{\partial}{\partial u}, \tag{96}$$
$$R_{11} = u\frac{\partial}{\partial u}, \qquad R_\infty = b(x, y, z)\frac{\partial}{\partial u}, \tag{97}$$

   where $\Delta b = 0$.
3. The case $f(u) = k = const. \neq 0$ reduces to the homogeneous case under the change $u \to u - kx^2/2$.
4. If the function f is a linear function, $f(u) = u$, then the additional symmetry generators are given by (97), with $\Delta b + b = 0$.
5. For exponential nonlinearity $f(u) = e^u$, the additional generator is

$$R_{13} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}.$$

6. For power nonlinearity $f(u) = u^p$, $p \neq 0$, $p \neq 1$, and $p \neq 5$, the additional generator is

$$R_{14} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{2}{1-p}u\frac{\partial}{\partial u}.$$

7. If $p = 5$, then the additional infinitesimal generators of the Lie point symmetries are given in (96).

We observe that the critical Sobolev exponent $\frac{n+2}{n-2}$ in this case is exactly 5 and, if $p(p-1)(p-5) \neq 0$, then the symmetry group of (94), with $f(u) = u^p$, is isomorphic to the symmetry group of (94) with $f(u) = e^u$ and the special conformal group generated by the symmetries is the group of homothetic motions in $\mathbb{R}^3$.

11.2.2. The Noether symmetries

1. The isometry group is a variational symmetry group of the nonlinear Poisson equation in $\mathbb{R}^3$. In particular, it is the Noether symmetry group of the cases $f(u) = e^u$ and $f(u) = u^p$, with $p \neq 0, 1, 5$.
2. The conformal group of $\mathbb{R}^3$ and the symmetry R_∞ generate a Noether symmetry group for $\Delta u = 0$.
3. The isometry group and the symmetry R_∞ generate a Noether symmetry group for $\Delta u + u = 0$.
4. The full conformal group of $\mathbb{R}^3$ is a Noether symmetry group for $\Delta u + u^5 = 0$.

The corresponding conservation laws can be obtained as a particular case of those, more general, established in [13] and for this reason they are not stated explicitly here.

11.3. The hyperbolic space

We consider the Klein's model of the hyperbolic space $\mathbb{H}^3$ represented by the set of $(x, y, z) \in \mathbb{R}^3$, with $z > 0$, and endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

This metric has constant negative scalar curvature $R = -6$ and its sectional curvature is equal to -1 (see [18, p. 160]). Thus one immediately concludes from [48, p. 57], that the isometry group of $\mathbb{H}^3$ possesses a six-dimensional Lie algebra.

It is easy to check that the following vector fields

$$\begin{gathered} H_1 = \frac{\partial}{\partial x}, \qquad H_2 = \frac{\partial}{\partial y}, \qquad H_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \\ H_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\ H_5 = \frac{x^2 - y^2 - z^2}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, \\ H_6 = xy\frac{\partial}{\partial x} + \frac{-x^2 + y^2 - z^2}{2}\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} \end{gathered} \tag{98}$$

are Killing fields on $(\mathbb{H}^3, g)$ and, for maximality, they form a basis of generators of $Isom(\mathbb{H}^3, g)$.

The nonlinear Poisson equation on $(\mathbb{H}^3, g)$ is given by

$$z^2 \Delta u - z u_z + f(u) = 0, \tag{99}$$

where Δ is the Laplace operator on $\mathbb{R}^3$ and $u_z = \frac{\partial u}{\partial z}$.

11.3.1. The group classification

1. For any function $f(u)$, the symmetry group coincides with $Isom(\mathbb{H}^3, g)$.
2. If $f(u) = 0$ then the additional symmetries are

$$H_\infty = b\frac{\partial}{\partial u}, \tag{100}$$

with $\Delta_g b = 0$, and

$$H_7 = u\frac{\partial}{\partial u}. \tag{101}$$

3. The case $f(u) = k = const. \neq 0$ is equivalent to case $f(u) = 0$ under the change $v = u + (k/2)\ln z$.
4. If the function f is a linear function, $f(u) = u$, then the additional symmetry generator is given by (100), with $\Delta b + b = 0$, and (101).

11.3.2. The Noether symmetries

1. The isometry group of $(\mathbb{H}^3, g)$ is a variational symmetry group.
2. Symmetries (100), with $\Delta_g b = 0$ or $\Delta_g b + b = 0$, are the Noether symmetries to the cases $f(u) = 0$ or $f(u) = u$, respectively.

11.3.3. The conservation laws

Here we present the conservation laws corresponding to the Noether symmetries of Eqs. (99) with arbitrary $f(u)$.

1. For the symmetry H_1, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$A_1 = \frac{y^2+z^2-x^2}{4z}\left(u_x^2 - u_y^2 - u_z^2\right) - \frac{1}{z}(xyu_xu_y + xzu_xu_z) + \frac{y^2+z^2-x^2}{4z^2}F(u),$$
$$A_2 = \frac{xy}{2z}\left(u_x^2 - u_y^2 + u_z^2\right) + \frac{y^2+z^2-x^2}{2z}u_xu_y - xzu_yu_z - \frac{xy}{2z^2}F(u),$$
$$A_3 = \frac{x}{2}\left(u_x^2 + u_y^2 - u_z^2\right)\frac{y^2+z^2-x^2}{2z}u_xu_z - xyu_yu_z - \frac{x}{2z}F(u).$$

2. For the symmetry H_2, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$B_1 = \frac{xy}{2z}\left(u_y^2 + u_z^2 - u_x^2\right) + \frac{x^2-y^2+z^2}{2z}u_xu_y - yu_xu_z - \frac{xy}{2z^2}F(u),$$
$$B_2 = \frac{x^2+y^2+z^2}{2z}\left(u_y^2 - u_x^2 - u_z^2\right) - \frac{xy}{z}u_xu_y - yu_yu_z - \frac{1}{4z^2}F(u),$$
$$B_3 = \frac{y}{2}\left(u_x^2 + u_y^2 - u_z^2\right) - \frac{xy}{z}u_xu_y + \frac{x^2-y^2+z^2}{2z}u_yu_z - \frac{y}{2z}F(u).$$

3. For the symmetry H_3, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$C_1 = \frac{x}{2z}\left(u_y^2 + u_z^2 - u_x^2\right) - \frac{y}{z}u_xu_y - u_xu_z - \frac{x}{2z^2}F(u),$$
$$C_2 = \frac{y}{2z}\left(u_x^2 - u_y^2 + u_z^2\right) - \frac{x}{z}u_xu_y - u_yu_z - \frac{y}{2z^2}F(u),$$
$$C_3 = \frac{1}{2}\left(u_x^2 + u_y^2 - u_z^2\right) - \frac{x}{z}u_xu_z - \frac{y}{z}u_yu_z - \frac{1}{2z}F(u).$$

4. For the symmetry H_4, the conservation law is $Div(D) = 0$, where $D = (D_1, D_2, D_3)$ and

$$D_1 = \frac{y}{2z}\left(u_y^2 + u_z^2 - u_x^2\right) + \frac{x}{z}u_xu_y - \frac{y}{2z^2}F(u),$$
$$D_2 = \frac{x}{2z}\left(u_y^2 - u_x^2 - u_z^2\right) - \frac{y}{z}u_xu_y + \frac{1}{2z^2}F(u),$$
$$D_3 = \frac{x}{z}u_yu_z - \frac{y}{z}u_xu_z.$$

5. For the symmetry H_5, the conservation law is $Div(E)=0$, where $E=(E_1,E_2,E_3)$ and

$$E_1=\frac{1}{2z}\left(u_y^2+u_z^2-u_x^2\right)-\frac{1}{2z^2}F(u),$$
$$E_2=-\frac{1}{z}u_xu_y,$$
$$E_3=-\frac{1}{z}u_xu_z.$$

6. For the symmetry H_6, the conservation law is $Div(F)=0$, where $F=(F_1,F_2,F_3)$ and

$$F_1=-\frac{1}{z}u_xu_y,$$
$$F_2=\frac{1}{2z}\left(u_x^2-u_y^2+u_z^2\right)-\frac{1}{2z^2}F(u),$$
$$F_3=-\frac{1}{z}u_yu_z.$$

7. For the symmetry H_∞, with $\Delta_g b=0$, the conservation law is $Div(G)=0$, where $G=(G_1,G_2,G_3)$ and

$$G_1=\frac{bu_x-b_xu}{z},$$
$$G_2=\frac{bu_y-b_yu}{z},$$
$$G_3=\frac{bu_z-b_zu}{z}. \tag{102}$$

8. For the symmetry H_∞, with $\Delta_g b+b=0$, the conservation law is $Div(G)=0$, where G is given in (102).

11.4. The sphere

Let us now consider the 3-sphere $\mathbb{S}^3$. Its metric is given by the restriction to $\mathbb{S}^3$ of the canonical metric of $\mathbb{R}^4$. Or, more specifically

$$ds^2=\frac{4}{(1+x^2+y^2+z^2)^2}\left(dx^2+dy^2+dz^2\right). \tag{103}$$

This metric determines the following Poisson equation on $\mathbb{S}^3$

$$\Delta u-\frac{2}{1+x^2+y^2+z^2}(xu_x+yu_y+zu_z)+f(u)=0, \tag{104}$$

where Δ denotes the Laplacian in $\mathbb{R}^3$.

The $Isom(\mathbb{S}^3, g)$ is generated by the following vector fields:

$$
\begin{aligned}
S_1 &= \left(1 + x^2 - y^2 - z^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z},\\
S_2 &= 2xy\frac{\partial}{\partial x} + \left(1 - x^2 + y^2 - z^2\right)\frac{\partial}{\partial y} + 2zy\frac{\partial}{\partial z},\\
S_3 &= 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + \left(1 - x^2 - y^2 + z^2\right)\frac{\partial}{\partial z},\\
S_4 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
S_5 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\\
S_6 &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.
\end{aligned}
\tag{105}
$$

The scalar curvature of $(\mathbb{S}^3, g)$ is $R = 6$.

11.4.1. Group classification

1. **Arbitrary $\boldsymbol{f(u)}$:** It is immediate that the vector fields (105) are symmetries. (See Theorem 1 and Corollary 1.)
2. **Linear case:** In addition to $Isom(\mathbb{S}^3, g)$, we have the symmetries

$$S_7 = u\frac{\partial}{\partial u} \tag{106}$$

and

$$S_\infty = b\frac{\partial}{\partial u}, \tag{107}$$

where $\Delta_g b + b = 0$, $\int_{M^n} b\,dV = 0$.
3. **Homogeneous case:** In this case, the symmetries are given by (105), (106) and

$$S_8 = \frac{\partial}{\partial u}.$$

11.4.2. The Noether symmetries

1. For arbitrary $f(u)$, the isometry group of $(\mathbb{S}^3, g)$ is a variational symmetry group.
2. If $f(u) = 0$, in addition to the variational symmetries $Isom(\mathbb{S}^3, g)$, we have the divergence symmetry $\frac{\partial}{\partial u}$.
3. If $f(u) = u$, the additional divergence symmetry is (107), with $\Delta_g b + b = 0$.

11.4.3. The conservation laws

1. For the symmetry S_1, with arbitrary $f(u)$, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$
\begin{aligned}
A_1 = {} & \frac{(1 + x^2 - y^2 - z^2)(u_y^2 + u_z^2 - u_x^2) - 4xyu_xu_y + 4xzu_xu_z}{(1 + x^2 + y^2 + z^2)^2}\\
& - 4\frac{1 + x^2 - y^2 - z^2}{(1 + x^2 + y^2 + z^2)^3}F(u),
\end{aligned}
$$

$$A_2 = \frac{2(xyu_x^2 - xyu_y^2 + xzu_z^2) - (1 + x^2 - y^2 - z^2)u_x u_y}{(1 + x^2 + y^2 + z^2)^2} - \frac{4xy}{(1 + x^2 + y^2 + z^2)^3} F(u),$$

$$A_3 = \frac{2((xzu_x^2 + xzu_y^2 - xzu_z^2) - (1 + x^2 - y^2 - z^2)u_x u_y)}{(1 + x^2 + y^2 + z^2)^2} - \frac{8xy}{(1 + x^2 + y^2 + z^2)^3} F(u).$$

2. For the symmetry S_2, with arbitrary $f(u)$, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$B_1 = \frac{2(xy(u_y^2 + u_z^2 - u_x^2) - 2yzu_x u_z - (1 - x^2 - y^2 + z^2)u_x u_y)}{(1 + x^2 + y^2 + z^2)^2} - \frac{8xy}{(1 + x^2 + y^2 + z^2)^3} F(u),$$

$$B_2 = \frac{(1 - x^2 + y^2 - z^2)(u_x^2 - u_y^2 + u_z^2) - 4xyu_x u_y - 4yzu_y u_z}{(1 + x^2 + y^2 + z^2)^2} + \frac{4(1 - x^2 + y^2 - z^2)}{(1 + x^2 + y^2 + z^2)^3} F(u),$$

$$B_3 = \frac{2(yz(u_x^2 + u_y^2 - u_z^2) - 2xyu_x u_z - (1 - x^2 + y^2 - z^2)u_x u_z)}{(1 + x^2 + y^2 + z^2)^2} - \frac{8yz}{(1 + x^2 + y^2 + z^2)^3} F(u).$$

3. For the symmetry S_3, with arbitrary $f(u)$, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$C_1 = \frac{2(xz(u_y^2 + u_z^2 - u_x^2) - 2yzu_x u_y - (1 - x^2 - y^2 + z^2)u_x u_z)}{(1 + x^2 + y^2 + z^2)^2} - \frac{8xz}{(1 + x^2 + y^2 + z^2)^3} F(u),$$

$$C_2 = \frac{2(yz(u_x^2 - u_y^2 + u_z^2) - 2xzu_x u_y - (1 - x^2 - y^2 + z^2)u_y u_z)}{(1 + x^2 + y^2 + z^2)^2} - \frac{8yz}{(1 + x^2 + y^2 + z^2)^3} F(u),$$

$$C_3 = \frac{(1 - x^2 - y^2 + z^2)(u_x^2 + u_y^2 - u_z^2) - 4xzu_x u_z - 4yzu_y u_z}{(1 + x^2 + y^2 + z^2)^2} - \frac{4(1 - x^2 - y^2 + z^2)}{(1 + x^2 + y^2 + z^2)^3} F(u).$$

4. For the symmetry S_4, with arbitrary $f(u)$, the conservation law is $Div(D) = 0$, where $D = (D_1, D_2, D_3)$ and

$$D_1 = \frac{2y(u_y^2 + u_z^2 - u_x^2) - 4xu_xu_y}{(1+x^2+y^2+z^2)^2} - \frac{8y}{(1+x^2+y^2+z^2)^3}F(u),$$

$$D_2 = \frac{-2x(u_x^2 - u_y^2 + u_z^2) - 4yu_xu_y}{(1+x^2+y^2+z^2)^2} + \frac{8x}{(1+x^2+y^2+z^2)^3}F(u),$$

$$D_3 = \frac{4xu_yu_z - 4yu_xu_z}{(1+x^2+y^2+z^2)^2}.$$

5. For the symmetry S_5, with arbitrary $f(u)$, the conservation law is $Div(E) = 0$, where $E = (E_1, E_2, E_3)$ and

$$E_1 = \frac{2z(u_y^2 + u_z^2 - u_x^2) + 4xu_xu_z}{(1+x^2+y^2+z^2)^2} - \frac{8z}{(1+x^2+y^2+z^2)^3}F(u),$$

$$E_2 = \frac{4xu_yu_z - 4zu_xu_y}{(1+x^2+y^2+z^2)^2},$$

$$E_3 = \frac{2x(u_x^2 + u_y^2 - u_z^2) + 4zu_xu_z - 4yu_xu_z}{(1+x^2+y^2+z^2)^2} - \frac{8x}{(1+x^2+y^2+z^2)^3}F(u).$$

6. For the symmetry S_6, with arbitrary $f(u)$, the conservation law is $Div(F) = 0$, where $F = (F_1, F_2, F_3)$ and

$$F_1 = \frac{4yu_xu_z - 4zu_xu_y}{(1+x^2+y^2+z^2)^2},$$

$$F_2 = \frac{2z(u_x^2 + u_y^2 - u_z^2) + 4yu_yu_z}{(1+x^2+y^2+z^2)^2} - \frac{8z}{(1+x^2+y^2+z^2)^3}F(u),$$

$$F_3 = \frac{2y(u_z^2 - u_x^2 - u_y^2) - 4zu_yu_z}{(1+x^2+y^2+z^2)^2} - \frac{8y}{(1+x^2+y^2+z^2)^3}F(u).$$

7. For the symmetry S_∞, with $\Delta_g b + b = 0$ or $\Delta_g b = 0$, the conservation law is $Div(G) = 0$, where $G = (G_1, G_2, G_3)$ and

$$G_1 = \frac{bu_x - b_xu}{1+x^2+y^2+z^2},$$

$$G_2 = \frac{bu_y - b_yu}{1+x^2+y^2+z^2},$$

$$G_3 = \frac{bu_z - b_zu}{1+x^2+y^2+z^2}.$$

8. For the symmetry $S_8 = \frac{\partial}{\partial u}$, the conservation law is $Div(J) = 0$, where $J = (J_1, J_2, J_3)$ and

$$J_1 = \frac{u_x}{1+x^2+y^2+z^2},$$

$$J_2 = \frac{u_y}{1+x^2+y^2+z^2},$$
$$J_3 = \frac{u_z}{1+x^2+y^2+z^2}.$$

11.5. The Sol group

The solvable group Sol topologically is the real vector space $\mathbb{R}^3$. Its Lie group structure is determined by the product

$$(x, y, z) * (x', y', z') = (x + x', y + e^{-x} y', z + e^x z'),$$

where $(x, y, t), (x', y', t') \in \mathbb{R}^3$. See [17].

The left-invariant metric on Sol is

$$ds^2 = dx^2 + e^{2x} dy^2 + e^{-2x} dz^2 \tag{108}$$

and it determines the semilinear Poisson equation

$$u_{xx} + e^{-2x} u_{yy} + e^{2x} u_{zz} + f(u) = 0. \tag{109}$$

The sectional curvature of (Sol, g) is nonconstant. See [11]. Its scalar curvature $R = -2$.

The dimension of $Isom(Sol, g)$ is 3 (see [11,42]) and a basis of Killing vector fields on Sol is given by

$$So_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad So_2 = \frac{\partial}{\partial y}, \qquad So_3 = \frac{\partial}{\partial z}. \tag{110}$$

11.5.1. The group classification

1. **Arbitrary case:** The Lie point symmetry group of (109) is $Isom(Sol, g)$ generated by (110).
2. **Linear case:** In addition to the isometry group, we have the symmetries

$$So_4 = u\frac{\partial}{\partial u},$$
$$So_\infty = b(x)\frac{\partial}{\partial u}, \tag{111}$$

   where b is a function such that $\Delta_g b + b = 0$.
3. **Homogeneous case:** We have the same symmetries as in the linear case, but the function b in (111) satisfies $\Delta_g b = 0$.

11.5.2. The Noether symmetries

1. The isometry group $Isom(Sol, g)$ is a variational symmetry group of (109) for any function $f(u)$.
2. If $f(u) = 0$ in (109), the additional divergence symmetry is (111), with $\Delta_g b = 0$.
3. In the remaining linear case, the Noether symmetries are $Isom(Sol)$ and (111), where $\Delta_g b + b = 0$.

11.5.3. The conservation laws

1. For the symmetry So_1, with arbitrary $f(u)$, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$A_1 = \frac{1}{2}\left(e^{-2x}u_y^2 + e^{2x}u_z^2 - u_x^2\right) + yu_xu_y - zu_xu_z - F(u),$$
$$A_2 = -\frac{1}{2}\left(yu_x^2 + ye^{2x}u_z^2 - ye^{-2x}u_y^2\right) - e^{-2x}u_xu_y - e^{-2x}zu_yu_z + yF(u),$$
$$A_3 = \frac{z}{2}\left(u_x^2 + e^{-2x}u_y^2 - e^{2x}u_z^2\right) - e^{2x}u_xu_z + e^{2x}yu_yu_z + yF(u).$$

2. For the symmetry So_2, with arbitrary $f(u)$, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$B_1 = -u_xu_y,$$
$$B_2 = \frac{1}{2}\left(u_x^2 - e^{-2x}u_y^2 + e^{2x}u_z^2\right) - F(u),$$
$$B_3 = -e^{2x}u_yu_z.$$

3. For the symmetry So_3, with arbitrary $f(u)$, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$C_1 = -u_xu_z,$$
$$C_2 = -e^{-2x}u_yu_z,$$
$$C_3 = \frac{1}{2}\left(u_x^2 + e^{-2x}u_y^2 - e^{2x}u_z^2\right) - F(u).$$

4. For the symmetry So_∞, with $\Delta_g b = 0$, the conservation law is $Div(S) = 0$, where $S = (S_1, S_2, S_3)$ and

$$S_1 = bu_x - b_xu,$$
$$S_2 = e^{-2x}bu_y - e^{-2x}b_yu,$$
$$S_3 = e^{2x}bu_z - e^{2x}b_zu. \tag{112}$$

5. For the symmetry So_∞, with $\Delta_g b + b = 0$, the conservation law is $Div(S) = 0$, where S is given in (112).

11.6. The product space $\mathbb{S}^2 \times \mathbb{R}$

Let us now consider the set $\mathbb{S}^2 \times \mathbb{R}$ endowed with the product metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} + dz^2. \tag{113}$$

The product manifold $(\mathbb{S}^2 \times \mathbb{R}, g)$ is a manifold with constant scalar curvature $R = 2$. The isometry group $Isom(\mathbb{S}^2 \times \mathbb{R}, g)$ has the following generators:

$$S_1' = \left(1 + x^2 - y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},$$

$$S_2' = 2xy\frac{\partial}{\partial x} + \left(1 - x^2 + y^2\right)\frac{\partial}{\partial y},$$

$$S_3' = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$S_4' = \frac{\partial}{\partial z}. \tag{114}$$

The nonlinear Poisson equation on $(\mathbb{S}^2 \times \mathbb{R}, g)$ is given by

$$\left(1 + x^2 + y^2\right)^2(u_{xx} + u_{yy}) + u_{zz} + f(u) = 0. \tag{115}$$

11.6.1. Group classification

1. **Arbitrary case:** For any function $f(u)$, the symmetry group coincides with $Isom(\mathbb{S}^2 \times \mathbb{R})$.
2. **Homogeneous case:** If $f(u) = 0$, then the additional symmetries are

$$S_\infty' = b\frac{\partial}{\partial u}, \tag{116}$$

   where $\Delta_g b = 0$ and

$$S_5' = u\frac{\partial}{\partial u}. \tag{117}$$

3. **Constant case:** The case $f(u) = k$ is reduced to the earlier under the change $u \mapsto u - kz^2/2$.
4. **Linear case:** The isometry group and the symmetry S_∞', with $\Delta_g b = 0$ in (116), generate a basis to the symmetry group generators.

11.6.2. The Noether symmetries

1. The isometry group of $(\mathbb{S}^2 \times \mathbb{R}, g)$ is a variational symmetry group of Eq. (115).
2. Symmetries (116), with $\Delta_g b = 0$ or $\Delta_g b + b = 0$, are the Noether symmetries in the cases $f(u) = 0$ or $f(u) = u$, respectively.

11.6.3. The conservation laws

1. For the symmetry S_1', with arbitrary $f(u)$, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$\begin{aligned}
A_1 = {} & \frac{(1 + x^2 - y^2)}{2}\left(u_y^2 - u_x^2\right) - 2xyu_xu_y + \frac{1 + x^2 - y^2}{2(1 + x^2 + y^2)^2}u_z^2 \\
& - \frac{1 + x^2 - y^2}{(1 + x^2 + y^2)^2}F(u), \\
A_2 = {} & xyu_x^2 - xyu_y^2 - \left(1 + x^2 - y^2\right)u_xu_y + \frac{xy}{(1 + x^2 + y^2)^2}u_z^2 \\
& - \frac{2xy}{(1 + x^2 + y^2)^2}F(u), \\
A_3 = {} & -\frac{1 + x^2 + y^2}{(1 + x^2 + y^2)^2}u_xu_z - \frac{2xy}{(1 + x^2 + y^2)^2}u_yu_z.
\end{aligned}$$

2. For the symmetry S_2', with arbitrary $f(u)$, the conservation law is $Div(B)=0$, where $B=(B_1,B_2,B_3)$ and

$$B_1 = xy\left(u_y^2-u_x^2\right)+\frac{xy}{(1+x^2+y^2)^2}u_z^2-\left(1-x^2+y^2\right)u_xu_y-\frac{2xy}{(1+x^2+y^2)^2}F(u),$$
$$B_2=\frac{1-x^2+y^2}{2}\left(u_x^2-u_y^2\right)+\frac{1-x^2+y^2}{2(1+x^2+y^2)}u_z^2-2xyu_xu_y$$
$$-\frac{(1-x^2+y^2)}{(1+x^2+y^2)^2}F(u),$$
$$B_3=\frac{2xy}{1+x^2+y^2}u_xu_z+\frac{1-x^2+y^2}{(1+x^2+y^2)^2}u_yu_z.$$

3. For the symmetry S_3', with arbitrary $f(u)$, the conservation law is $Div(C)=0$, where $C=(C_1,C_2,C_3)$ and

$$C_1=\frac{y}{2}\left(u_y^2-u_x^2\right)+\frac{y}{2(1+x^2+y^2)^2}u_z^2+xu_xu_z-\frac{y}{(1+x^2+y^2)^2}F(u),$$
$$C_2=\frac{x}{2}\left(u_y^2-u_x^2\right)-\frac{x}{(1+x^2+y^2)^2}u_z^2-yu_xu_y+\frac{x}{(1+x^2+y^2)^2}F(u),$$
$$C_3=\frac{-y}{(1+x^2+y^2)^2}u_xu_y+\frac{-x}{(1+x^2+y^2)^2}u_yu_z.$$

4. For the symmetry S_4', with arbitrary $f(u)$, the conservation law is $Div(D)=0$, where $D=(D_1,D_2,D_3)$ and

$$D_1=-u_xu_z,$$
$$D_2=-u_yu_z,$$
$$D_3=\frac{1}{2}\left(u_x^2+u_y^2\right)-\frac{u_z^2}{2(1+x^2+y^2)^2}-\frac{F(u)}{(1+x^2+y^2)^2}.$$

5. For the symmetry S_∞', with $\Delta_g b=0$ or $\Delta_g b+b=0$, the conservation law is $Div(E)=0$, where $E=(E_1,E_2,E_3)$ and

$$E_1=bu_x-b_xu,$$
$$E_2=bu_y-b_yu,$$
$$E_3=\frac{bu_z-b_zu}{(1+x^2+y^2)^2}. \qquad (118)$$

11.7. The product space $\mathbb{H}^2\times\mathbb{R}$

Let $\mathbb{H}^2=\{(x,y)\in\mathbb{R}^2\colon\ y>0\}$ endowed with metric

$$\frac{dx^2+dy^2}{y^2}$$

be the hyperbolic plane (Klein's model) and consider the set $(H_2 \times \mathbb{R}, g)$ endowed with the product metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + dz^2. \tag{119}$$

The Lie algebra of the infinitesimal isometries of $(\mathbb{H}^2 \times \mathbb{R}, g)$, $Isom(\mathbb{H}^2 \times \mathbb{R}, g)$, is given by (see [32])

$$X_1 = \frac{x^2 - y^2}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial z}. \tag{120}$$

The scalar curvature of $(\mathbb{H}^2 \times \mathbb{R}, g)$ is $R = -2$ and the nonlinear Poisson equation is given by

$$y^2(u_{xx} + u_{yy}) + u_{zz} + f(u) = 0. \tag{121}$$

11.7.1. Group classification

1. **Arbitrary case:** For any function $f(u)$, the symmetry group coincides with $Isom(\mathbb{H}^2 \times \mathbb{R}, g)$.
2. **Homogeneous case:** If $f(u) = 0$, then the additional symmetries are

$$X_\infty = b\frac{\partial}{\partial u}, \tag{122}$$

where $\Delta_g b = 0$ and

$$X_5 = u\frac{\partial}{\partial u}. \tag{123}$$

3. **Constant case:** The case $f(u) = k$ is reduced to the earlier under the change $u \mapsto u - kz^2/2$.
4. **Linear case:** The isometry group and the symmetry X_∞, with $\Delta_g b = 0$ in (122), generate a basis to the symmetry group generators.

11.7.2. The Noether symmetries

1. The isometry group of $Isom(\mathbb{H}^2 \times \mathbb{R}, g)$ is a variational symmetry group of Eq. (121).
2. Symmetries (122), with $\Delta_g b = 0$ or $\Delta_g b + b = 0$, are the Noether symmetries in the cases $f(u) = 0$ or $f(u) = u$, respectively.

11.7.3. The conservation laws

1. For the symmetry X_1, with arbitrary $f(u)$, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$A_1 = \frac{x^2 - y^2}{4}\left(u_y^2 - u_x^2\right) + \frac{x^2 - y^2}{4y^2}u_z^2 - xyu_xu_y - \frac{x^2 - y^2}{2y^2}F(u),$$

$$A_2 = \frac{xy}{2}\left(u_x^2 - u_y^2\right) + \frac{x}{2y}u_z^2 - \frac{x^2 - y^2}{2}u_xu_y - \frac{x}{y}F(u),$$

$$A_3 = -\frac{x^2 - y^2}{2y^2}u_xu_z - \frac{x}{y}u_yu_z.$$

2. For the symmetry X_2, with arbitrary $f(u)$, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$B_1 = \frac{u_y^2 - u_x^2}{2} + \frac{u_z^2}{2y^2} - \frac{F(u)}{y^2},$$
$$B_2 = -u_x u_y,$$
$$B_3 = -u_x u_z.$$

3. For the symmetry X_3, with arbitrary $f(u)$, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$C_1 = \frac{x}{2}\left(u_y^2 - u_x^2\right) + \frac{x}{2y^2}u_z^2 - yu_x u_y - \frac{x}{y^2}F(u),$$
$$C_2 = \frac{y}{2}\left(u_x^2 - u_y^2\right) + \frac{u_z^2}{2y} - xu_x u_y + \frac{F(u)}{y},$$
$$C_3 = -\frac{x}{y^2}u_x u_z - \frac{u_y u_z}{y}.$$

4. For the symmetry X_4, with arbitrary $f(u)$, the conservation law is $Div(D) = 0$, where $D = (D_1, D_2, D_3)$ and

$$D_1 = -u_x u_z,$$
$$D_2 = -u_y u_z,$$
$$D_3 = \frac{u_x^2 + u_y^2}{2} - \frac{u_z^2}{2y^2} - \frac{F(u)}{y^2}.$$

5. For the symmetry X_∞, with $\Delta_g b = 0$ or $\Delta_g b + b = 0$, the conservation law is $Div(E) = 0$, where $E = (E_1, E_2, E_3)$ and

$$E_1 = bu_x - b_x u,$$
$$E_2 = bu_y - b_y u,$$
$$E_3 = \frac{bu_z - b_z u}{y^2}. \tag{124}$$

11.8. The universal covering of $SL_2(\mathbb{R})$

The universal covering of the Lie group of 2×2 matrices with determinant equal to 1, denoted by $\widetilde{SL_2}(\mathbb{R})$, topologically is $\mathbb{R}^3_+ := \{(x, y, z) \in \mathbb{R}^3\colon\ z > 0\}$, endowed with the Riemannian metric

$$ds^2 = \left(dx + \frac{dy}{z}\right)^2 + \frac{dy^2 + dz^2}{z^2}. \tag{125}$$

$(\widetilde{SL_2}(\mathbb{R}), g)$ possesses scalar curvature $R = -5/2$.

The nonlinear Poisson equation induced by metric (125) is given by

$$2u_{xx} - 2zu_{xy} + z^2(u_{yy} + u_{zz}) + f(u) = 0. \tag{126}$$

We have not found references giving the explicit form of $Isom(\widetilde{SL}_2, g)$. (At least in the information sources available to us.) However it is known that its dimension is 4. See [41]. Then we shall proceed in a way opposite to that in Sections 11.2–11.7.

Since the scalar curvature of $(\widetilde{SL}_2, g)$ is constant ($R = -5/2$), from the proof of Theorem 1 we conclude that the symmetry group of (126), with $f(u) \neq \lambda u$, $\lambda = const.$, is reduced to the symmetry group of the arbitrary case. Then, using the package SYM [19,20] by Stelios Dimas et al., we obtain that the symmetries of Eq. (126) are determined by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad X_4 = z\frac{\partial}{\partial x} + \frac{y^2 - z^2}{2}\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}. \tag{127}$$

To see that (127) is the isometry group of $(\widetilde{SL}_2, g)$ we have two alternatives. The first one is to check whether the fields (127) are Killing vector fields. Indeed, a simple substitution of (127) into the Killing equations confirms this claim. Then, from [41], $\dim(Isom(\widetilde{SL}_2, g)) = 4$. Thus, (127) generate a basis of the generators of the isometry group of $(\widetilde{SL}_2, g)$.

The second one is as follows. From Theorem 1 is proved that the isometry group of $(\widetilde{SL}_2, g)$ and the symmetry group of (126) are the same. Since (127) generate a basis of the symmetry group of (126), we conclude that (127) is a basis of the isometry group of $(\widetilde{SL}_2, g)$.

This procedure suggests that the existing programs for symbolic calculation of symmetries of differential equations may be used to calculate the isometry group of the considered manifold.

11.8.1. Group classification

1. **Arbitrary case:** For any function $f(u)$, the symmetry group coincides with $Isom(\widetilde{SL}_2(\mathbb{R}), g)$.
2. **Homogeneous case:** If $f(u) = 0$, then the additional symmetries are

$$X_\infty = b\frac{\partial}{\partial u}, \tag{128}$$

where $\Delta_g b = 0$ and

$$X' = u\frac{\partial}{\partial u}. \tag{129}$$

3. **Constant case:** The case $f(u) = k$ is reduced to the preceding case by the change $u \mapsto u - kx^2/2$.
4. **Linear case:** The isometry group and the symmetry X_∞, with $\Delta_g b + b = 0$ in (128), generate a basis of the symmetry algebra.

11.8.2. The Noether symmetries

1. The isometry group $Isom(\widetilde{SL}_2(\mathbb{R}), g)$ is a variational symmetry group of Eq. (127).
2. Symmetries (128), with $\Delta_g b = 0$ or $\Delta_g b + b = 0$, are the Noether symmetries to the cases $f(u) = 0$ or $f(u) = u$, respectively.

11.8.3. The conservation laws

1. For the symmetry X_1, with arbitrary $f(u)$, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$A_1 = -\frac{u_x^2}{z^2} + \frac{u_y^2}{2} + \frac{u_y^2}{2} - \frac{F(u)}{z^2},$$
$$A_2 = \frac{u_x^2}{z} - u_x u_y,$$
$$A_3 = -u_x u_z.$$

2. For the symmetry X_2, with arbitrary $f(u)$, the conservation law is $Div(B)=0$, where $B=(B_1,B_2,B_3)$ and

$$B_1=-\frac{2}{z^2}u_xu_y+\frac{u_y^2}{z},$$
$$B_2=\frac{u_x^2}{z^2}-\frac{u_y^2}{2}+\frac{u_z^2}{2}-\frac{F(u)}{z^2},$$
$$B_3=-u_yu_z.$$

3. For the symmetry X_3, with arbitrary $f(u)$, the conservation law is $Div(C)=0$, where $C=(C_1,C_2,C_3)$ and

$$C_1=-\frac{2y}{z}u_xu_y-\frac{2}{z}u_xu_z+\frac{y}{z}u_y^2+u_yu_z,$$
$$C_2=\frac{y}{z^2}u_x^2+u_xu_y-\frac{y}{2}u_y^2-zu_yu_z+\frac{y}{2}u_z^2-\frac{y}{z^2}F(u),$$
$$C_3=\frac{u_x^2}{z}-u_xu_y+\frac{z}{2}u_y^2-yu_yu_z-\frac{z}{2}u_z^2-\frac{F(u)}{z}.$$

4. For the symmetry X_4, with arbitrary $f(u)$, the conservation law is $Div(D)=0$, where $D=(D_1,D_2,D_3)$ and

$$D_1=-\frac{u_x^2}{z}+\frac{z^2-y^2}{z^2}u_xu_y-\frac{2y}{z}u_xu_z+\frac{y^2}{2z}u_y^2+yu_yu_z+\frac{z}{2}u_z^2-\frac{F(u)}{z},$$
$$D_2=\frac{y^2+z^2}{2z^2}u_x^2-zu_xu_y+yu_xu_z-yzu_yu_z+\frac{y^2-z^2}{4}u_z^2+\frac{z^2-y^2}{2z^2}F(u),$$
$$D_3=\frac{y}{z}u_x^2-yu_xu_y-zu_xu_z+\frac{yz}{2}u_y^2+\frac{z^2-y^2}{2}u_yu_z-\frac{yz}{2}u_z^2-\frac{y}{z}F(u).$$

5. For the symmetry X_∞, with $\Delta_g b=$ or $\Delta_g b+b=0$, the conservation law is $Div(E)=0$, where $E=(E_1,E_2,E_3)$ and

$$E_1=\frac{2}{z^2}(bu_x-b_xu)-\frac{1}{z}(bu_y-b_yu),$$
$$E_2=-\frac{1}{z}(bu_x-b_xu)+(bu_y-b_yu),$$
$$E_3=bu_z-b_zu.$$

11.9. The Heisenberg group

The three-dimensional nilpotent Lie group, also called Heisenberg group H^1, topologically is the real three-dimensional vector space $\mathbb{R}^3$ endowed with the group structure determined by the composition law $\phi:\mathbb{R}^3\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ defined by

$$\phi\big((x,y,t),\big(x',y',t'\big)\big):=\big(x+x',y+y',t+t'+2\big(x'y-xy'\big)\big).$$

This composition law determines the left invariant vector fields

$$T=\frac{\partial}{\partial t},\qquad X=\frac{\partial}{\partial x}+2y\frac{\partial}{\partial t},\qquad Y=\frac{\partial}{\partial y}-2x\frac{\partial}{\partial t} \tag{130}$$

and the left invariant metric on H^1

$$ds^2=dx^2+dy^2+(dz+2y\,dx-2x\,dy)^2. \tag{131}$$

The scalar curvature of (H^1,g) is $R=-8$ and the operators (130) satisfy the following commutation relations:

$$[X,Y]=-4T,\qquad [X,T]=[Y,T]=0.$$

These formulae represent in an abstract form the commutation relations for the quantum-mechanical position and momentum operators. This justifies the name Heisenberg group.

It is well known that the metric (131) determines the following generators of the isometry group of H^1, denoted by $Isom(H^1,g)$:

$$T=\frac{\partial}{\partial t},\qquad \tilde{X}=\frac{\partial}{\partial x}-2y\frac{\partial}{\partial t},\qquad \tilde{Y}=\frac{\partial}{\partial y}+2x\frac{\partial}{\partial t},\qquad R=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}. \tag{132}$$

Note that T corresponds to translations in t, R—to rotations in the (x,y) plane and $\tilde{X},\tilde{Y}$ determine the right multiplication.

For more details, see [14,15,32].

The nonlinear Poisson equation on H^1 is given by

$$u_{xx}+u_{yy}+\big[4\big(x^2+y^2\big)+1\big]u_{tt}+4yu_{xt}-4xu_{yt}+f(u)=0. \tag{133}$$

11.9.1. Group classification

1. **Arbitrary case:** For any function $f(u)$, the symmetry group coincides with $Isom(H^1,g)$.
2. **Homogeneous case:** If $f(u)=0$, then the additional symmetries are

$$H_\infty=b\frac{\partial}{\partial u}, \tag{134}$$

   where $\Delta_g b=0$ and

$$H_1=u\frac{\partial}{\partial u}. \tag{135}$$

3. **Constant case:** The case $f(u)=k$ is reduced to the earlier under the change $u\mapsto u-kx^2$.
4. **Homogeneous case:** The isometry group and the symmetry H_∞, with $\Delta_g b=0$ in (134), generate a basis of the symmetry group generators.

11.9.2. The Noether symmetries

1. The isometry group of (H^1,g) is a variational symmetry group of Eq. (133).
2. Symmetries (134), with $\Delta_g b=0$ or $\Delta_g b+b=0$, are the Noether symmetries to the cases $f(u)=0$ or $f(u)=u$, respectively.

11.9.3. The conservation laws

1. For the symmetry T, with arbitrary $f(u)$, the conservation law is $Div(A)=0$, where $A=(A_1,A_2,A_3)$ and

$$A_1=-u_xu_t-2yu_t^2,$$
$$A_2=-u_yu_t+2xu_t^2,$$
$$A_3=\frac{u_x^2+u_y^2}{2}-\frac{4(x^2+y^2)+1}{2}u_t^2-F(u).$$

2. For the symmetry $\tilde{X}$, with arbitrary $f(u)$, the conservation law is $Div(B)=0$, where $B=(B_1,B_2,B_3)$ and

$$B_1=\frac{u_y^2-u_x^2}{2}+2yu_xu_t-2xu_yu_t+\frac{4(x^2+3y^2)+1}{2}u_t^2-F(u),$$
$$B_2=-u_xu_y-2yu_yu_t+2xu_xu_t-4xyu_t^2,$$
$$B_3=-3yu_x^2-yu_y^2+2xu_xu_y+y\big[4\big(x^2+y^2\big)+1\big]u_t^2$$
$$-\big[4\big(x^2+y^2\big)+1\big]u_xu_t+2yF(u).$$

3. For the symmetry $\tilde{Y}$, with arbitrary $f(u)$, the conservation law is $Div(C)=0$, where $C=(C_1,C_2,C_3)$ and

$$C_1=-u_xu_y-2xu_xu_t-2yu_yu_t-4xyu_t^2,$$
$$C_2=\frac{u_x^2-u_y^2}{2}+2yu_xu_t-2xu_yu_t+\frac{4(3x^2+y^2)+1}{2}u_t^2-F(u),$$
$$C_3=-xu_x^2+3xu_y^2-x\big[4\big(x^2+y^2\big)+1\big]u_t^2-2yu_xu_y$$
$$-\big[4\big(x^2+y^2\big)+1\big]u_yu_t-2xF(u).$$

4. For the symmetry R, with arbitrary $f(u)$, the conservation law is $Div(D)=0$, where $D=(D_1,D_2,D_3)$ and

$$D_1=-\frac{y}{2}\big(u_y^2-u_x^2\big)+\frac{y}{2}\big[4\big(x^2+y^2\big)+1\big]u_t^2+xu_xu_y-yF(u),$$
$$D_2=-\frac{x}{2}\big(u_y^2+u_x^2\big)-\frac{x}{2}\big[4\big(x^2+y^2\big)+1\big]u_t^2-yu_xu_y+xF(u),$$
$$D_3=-2y^2u_x^2-2x^2u_y^2+4xyu_xu_y-y\big[4\big(x^2+y^2\big)+1\big]u_xu_t$$
$$+x\big[4\big(x^2+y^2\big)+1\big]u_yu_t.$$

5. For the symmetry H_∞, with $\Delta_g b=0$ or $\Delta_g b+b=0$, the conservation law is $Div(E)=0$, where $E=(E_1,E_2,E_3)$ and

$$E_1=b(u_x+2yu_t)-u(b_x+2yu_t),$$
$$E_2=b(u_y-2xu_t)-u(b_y-2xu_t),$$

$$E_3 = b\{2yu_x - 2xu_y + [4(x^2+y^2)+1]u_t\} \\ - u\{2yb_x - 2xb_y + [4(x^2+y^2)+1]b_t\}. \tag{136}$$

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