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An Exterior Differential System for a Generalized Korteweg-de Vries Equation and its Associated Integrability

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Abstract The method of Cartan is reviewed by applying it to the classical Korteweg-de Vries equation. The method is then applied to a new generalized Korteweg-de Vries equation for which a prolongation is obtained. As a consequence, a Bäcklund transformation for the equation is derived as well as the associated potential equation.

Keywords Generalized KdV equation · Exterior differential system · Cartan prolongation · Solitons

Mathematics Subject Classification (2000) 35A30 · 32A25 · 35C05

1 Introduction

The concept of integrability of certain kinds of partial differential equations or systems of such equations has generated a great deal of attention over the last several years. There are many implications and related applications of the idea of integrability [1]. For example, a Lax pair could be determined in theory, as well as an infinite number of conservation laws and perhaps more importantly a Bäcklund transformation might be written down [1, 2]. A Bäcklund transformation has more important practical consequences, since it can be used to determine solutions to an associated equation, usually referred to as the potential equation, from solutions of a given equation. Of course, determining any of these in practice in a particular case is not a given. However, there are approaches which often yield results. It has been shown by Wahlquist and Estabrook [3] that by applying Cartan calculus of differential forms as well as prolongation techniques, it is possible to ascertain integrability of a given equation and use the prolongation results to determine a Lax pair for the given nonlinear problem. Nonlinear fields then occur as coefficients in the linear Lax equations, which for the classical Korteweg-de Vries (KdV) case is related to the stationary Schrödinger equation.

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Of course, the existence of an infinite number of conservation laws and the notion of integrability has been connected to the existence of soliton solutions associated with these equations. The study of solitons recently has been of interest due to their appearance in many physical applications, and so the investigation of a new equation is to be of interest. The equation which has been studied by Wahlquist and Estabrook is the classical KdV equation, which has been known to describe the long time evolution of finite amplitude waves. The equation has many applications which extend beyond the original applications to solitary surface water waves, and further applications have been mentioned in [4]. Many studies would seem to indicate that nonlinear dispersion can act to compactify solitary waves and generate solitons with a finite wavelength. Such modes have been determined in the form of solutions to a particular generalized KdV equation, which will be the focus of study here [5, 6]. The symmetry group has been determined for this equation, and several classes of solutions have been obtained, which include both soliton solutions and solutions which have compact support [7].

It is the intention here to review some of the mathematical preliminaries by applying Cartan's method to the classical KdV equation as first done by Wahlquist and Estabrook. It will be shown next how the analysis can be extended to the case of the generalized KdV equation, essentially in the form given in [7]. An exterior differential system which reproduces the equation on the transverse manifold is presented, so integrability can be established. A Cartan prolongation can then be obtained corresponding to this exterior differential system and from this, a Bäcklund transformation and its associated potential equation can be given as well at the end. These equations have been of great interest recently due to their applications to areas such as coupled autonomous oscillators and soliton theory, where in the former instance these have been a subject of interest since the discovery of their synchronization by Huygens [8, 9].

2 Exterior Differential Systems and Cartan Prolongations

Consider the space $M = \mathbb{R}^n(x, t, u, p, q, \dots)$ in which there is defined a closed exterior differential system

$$\alpha_1 = 0, \dots, \alpha_l = 0, \tag{2.1}$$

and let I be the ideal generated by the set $\{\alpha_i\}_{i=1}^l$ in (2.1) given as

$$I = \left\{ \omega = \sum_{i=1}^{l} \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda(M) \right\}. \tag{2.2}$$

If ideal (2.2) is closed, we have $dI \subset I$ and so (2.1) is integrable by a theorem. It is important to stress that system (2.1) is chosen such that the solutions $u = u(x, t)$ of an equation

$$u_t = F(x, t, u, u_x, u_{xx}, \dots), \tag{2.3}$$

correspond with the two-dimensional integral manifolds of (2.1). These are the integral manifolds given by sections S of the projection

$$\pi : M \to \mathbb{R}^2, \quad \pi(x, t, u, p, q, \dots) = (x, t). \tag{2.4}$$

These sections S are given by a mapping

$$S : \mathbb{R}^2 \to M, \quad S(x, t) = (x, t, u(x, t), p(x, t), q(x, t), \dots). \tag{2.5}$$

Introduce the fiber bundle $(\tilde{M}, \tilde{p}, M)$ over M with $M \subset \tilde{M}$ and $\tilde{p}$ a projection of $\tilde{M}$ onto M, so points of $\tilde{M}$ are denoted by $\tilde{m}$, those in M by m hence $\tilde{p}(\tilde{m}) = m$. A Cartan-Ehresmann connection in the fiber bundle $(\tilde{M}, \tilde{p}, M)$ is a system of one forms $\tilde{\omega}^i$, $i = 1, \dots, k$ in $T^*(\tilde{M})$ with the property that the mapping $\tilde{p}_*$ from the vector space

$$H_{\tilde{m}} = \{\tilde{X} \in T_{\tilde{m}} \mid \tilde{\omega}^i(\tilde{X}) = 0, i = 1, 2, \dots, k\},$$

to the tangent space T_m is a bijection. We consider in $\tilde{M}$ the exterior differential system

$$\begin{aligned} \tilde{\alpha}_i &= \tilde{p}^* \alpha_i = 0, \quad & i &= 1, \dots, l, \\ \tilde{\omega}^j &= 0, & j &= 1, \dots, k, \end{aligned} \tag{2.6}$$

with $\tilde{\omega}^j$ a Cartan-Ehresmann connection in $(\tilde{M}, \tilde{p}, M)$.

The system (2.6) is called a Cartan prolongation of (2.1) if (2.6) is closed and whenever S is a transversal solution of (2.1), there should also exist a transversal local solution $\tilde{S}$ of (2.6) with $\tilde{p}(\tilde{S}) = S$. There is then a theorem which states that (2.6) is a Cartan prolongation of (2.1) if and only if (2.6) is closed. A necessary and sufficient condition for the existence of this is given by

$$d\tilde{\omega}^i = \tilde{\beta}^i_j \wedge \tilde{\omega}^j, \quad \operatorname{mod} p^*(I) \tag{2.7}$$

where I is the ideal generated by $\{\alpha_i\}_{i=1}^l$, and the summation convention holds in (2.7). Consider a trivial bundle of the form $\tilde{M} = M \times \mathbb{R}^k$ with $y = (y^1, \dots, y^k) \in \mathbb{R}^k$ and define connections

$$\tilde{\omega}^i = dy^i - \eta^i, \quad i = 1, \dots, k, \tag{2.8}$$

with $\eta^i = A^i\, dx + B^i\, dt$, where A^i and B^i are C^∞ functions on $\tilde{M}$. Substituting into the prolongation condition (2.7), it reads

$$-d\eta^i = \tilde{\beta}^i_j \wedge \left(dy^j - \eta^j\right), \quad \operatorname{mod} \tilde{p}^*(I).$$

From this, it follows that $\tilde{\beta}^i_j$ may be chosen such that they do not depend on $dy^m - \eta^m$, $m = 1, \dots, k$, $m \neq j$. Moreover, it is possible to show that the prolongation condition boils down to the following condition

$$d\eta + \frac{1}{2}[\eta, \eta] = 0, \quad \operatorname{mod} \tilde{p}^*(I). \tag{2.9}$$

These results can be summarized in the following theorem, which will be made use of in what follows: A necessary and sufficient condition for the connection forms (2.8) to be a Cartan-prolongation is the vanishing of its curvature form.

3 Cartan Prolongation of the Korteweg-de Vries Equation

Begin by defining the following exterior system over the space $M = \mathbb{R}^5(x, t, u, p, q)$ as done by Wahlquist and Estabrook [3]

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt = 0, \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt = 0, \\ \alpha_3 &= du \wedge dx - dq \wedge dt + 6up\, dx \wedge dt = 0. \end{aligned} \tag{3.1}$$

Differentiating the forms in (3.1), we calculate that

$$
\begin{aligned}
d\alpha_1 &= -dp\wedge dx\wedge dt = dx\wedge\alpha_2,\\
d\alpha_2 &= -dq\wedge dx\wedge dt = -dx\wedge\alpha_3,\\
d\alpha_3 &= 6p\,du\wedge dx\wedge dt + 6u\,dp\wedge dx\wedge dt = -6dx\wedge(p\alpha_1 + u\alpha_2).
\end{aligned}
\tag{3.2}
$$

Therefore the ideal $I = \{\omega \mid \omega = \sum_{i=1}^{3}\sigma_i\wedge\alpha_i : \sigma_i\in\Lambda(M)\}$ is closed $dI\subset I$, and system (3.1) is integrable. For sections of the projection $\pi: M\to\mathbb{R}^2$, it follows that

$$
\begin{aligned}
0 &= \alpha_1|_S = S^*\alpha_1 = (u_x - p)\,dx\wedge dt,\\
0 &= \alpha_2|_S = S^*\alpha_2 = (p_x - q)\,dx\wedge dt,\\
0 &= \alpha_3|_S = S^*\alpha_3 = (-u_t - q_x + 6up)\,dx\wedge dt.
\end{aligned}
\tag{3.3}
$$

The conditions that result from (3.3) imply that $u_x = p$, $p_x = q$ and $-u_t - q_x + 6pu = 0$. Thus the transversal integral manifolds of (3.1) give solutions of the following Korteweg-de Vries equation

$$
u_t - 6uu_x + u_{xxx} = 0. \tag{3.4}
$$

To construct a Cartan prolongation [10], the procedure outlined in the first section is followed. Consider the trivial fiber bundle $\tilde{M} = M\times\mathbb{R}^k$ with coordinates $y = (y^1,\dots,y^k)\in\mathbb{R}^k$ and the connection forms $\tilde{\omega}^i$ are specified in (2.8) with

$$
\eta^i = A^i\,dx + B^i\,dt,\quad A^i = A^i(x,t,u,p,q,y),\quad B^i = B^i(x,t,u,p,q,y). \tag{3.5}
$$

The prolongation condition (2.9) has the form

$$
\begin{aligned}
d\eta + \frac{1}{2}[\eta,\eta] &= \lambda_1(du\wedge dt - p\,dx\wedge dt) + \lambda_2(dp\wedge dt - q\,dx\wedge dt)\\
&\quad + \lambda_3(du\wedge dx - dq\wedge dt + 6pu\,dx\wedge dt).
\end{aligned}
$$

Using η as given in (2.5) on the left and comparing the coefficients of the differential forms on both sides of the prolongation condition (2.9), the following set of conditions holds

$$
\begin{aligned}
&-A_t + B_x + [A,B] = -\lambda_1 p - \lambda_2 q + 6\lambda_3 pu,\\
&A_u = \lambda_3,\qquad A_p = 0,\qquad A_q = 0,\\
&B_u = \lambda_1,\qquad B_p = \lambda_2,\qquad B_q = -\lambda_3.
\end{aligned}
\tag{3.6}
$$

If we were to assume that A and B do not depend explicitly on (x,t), then we can put $A_x = A_t = 0$, $B_x = B_t = 0$, and so system (3.6) reduces to

$$
\begin{aligned}
&[A,B] = -pB_u - qB_p + 6puB_q,\\
&A_p = 0,\qquad A_q = 0,\qquad A_u = -B_q
\end{aligned}
\tag{3.7}
$$

where $A = A(u,y)$ and $B = B(u,p,q,y)$. To obtain an elementary prolongation, simplifications on the vector fields A and B can be imposed. Assuming that A is independent of u does not lead to a useful result, since it implies that B is independent of q which, when combined with the commutator equation, implies that $A = A(y)$ and $B = B(y)$. However, it can be assumed that in terms of vector fields X_1 and X_2 we have

$$
A = A(u,y) = X_1 + uX_2,\quad X_i = X_i(y),\ i = 1,2. \tag{3.8}
$$

Substituting the expression for A in (3.8) into (3.7), it follows that B is determined up to a function C as

$$B = -qX_2 + C(u, p, y). \tag{3.9}$$

To obtain an equation for C, substitute A and B into the first equation of (3.7) to give

$$pC_u + qC_p - 6puX_2 = q[X_1, X_2] - [X_1, C] - u[X_2, C]. \tag{3.10}$$

Now X_1 and X_2 define a new vector field $X_3 = X_3(y)$ by means of the commutator $X_3 = [X_1, X_2]$, so (3.10) can be written as,

$$pC_u + qC_p = -[X_1, C] - u[X_2, C] + 6puX_2 + qX_3. \tag{3.11}$$

Since C is independent of q, the coefficients of q on both sides of (3.11) must match, which gives

$$C_p = X_3.$$

This simple equation can be integrated to give

$$C(u, p, y) = pX_3 + D(u, y). \tag{3.12}$$

Substituting C in (3.12) into (3.11) and equating the coefficients of p on both sides gives the following pair of equations for D,

$$\begin{aligned} &[X_1, D] + u[X_2, D] = 0, \\ &D_u = 6uX_2 - [X_1, X_3] - u[X_2, X_3]. \end{aligned} \tag{3.13}$$

Here we will only look for a prolongation with an uncomplicated structure. In particular suppose D is taken to have the form

$$D = \xi\big(uX_1 + u^2X_2\big). \tag{3.14}$$

Then the first equation in (3.13) is satisfied. Differentiating D and substituting it into the second equation of (3.13) gives

$$2\xi uX_2 + \xi X_1 = 6uX_2 - u[X_2, X_3] - [X_1, X_3]. \tag{3.15}$$

This can be solved if the X_i satisfy a particular algebra, in particular, if the X_i satisfy the commutation relations

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = 2X_2, \qquad [X_3, X_1] = 2X_1. \tag{3.16}$$

Substituting the relations (3.16) into (3.15), it simplifies into,

$$2\xi uX_2 + \xi X_1 = 4uX_2 + 2X_1.$$

This is satisfied provided that we take $\xi = 2$ so that $D = 2(uX_1 + u^2X_2)$ in this case. The commutation relations in (3.16) are isomorphic to $sl(2, \mathbb{R})$. This is clearer if we take as a basis for $sl(2, \mathbb{R})$ the matrices

$$\hat{X}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{X}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{X}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.17}$$

It can be verified that these matrices satisfy (3.16), and the isomorphism between them is given by $X_i \leftrightarrow \tilde{X}_i$, $i = 1, 2, 3$. The prolongation condition is given in terms of η specified in (3.5) where A, B, C and D are given by

$$\begin{aligned} A &= X_1 + uX_2, \\ B &= 2uX_1 + \big(2u^2 - q\big)X_2 + pX_3, \\ C &= pX_3 + 2\big(uX_1 + u^2X_2\big), \\ D &= 2\big(uX_1 + u^2X_2\big) \end{aligned} \tag{3.18}$$

where the X_i satisfy (3.16). On account of the isomorphism, any realization of the $sl(2, \mathbb{R})$ algebra as vector fields on $\mathbb{R}^k$ will give a Cartan-prolongation of the exterior differential system.

4 A Cartan Prolongation of the Generalized KdV Equation

Consider the more general exterior differential system on the space M which is given by

$$\begin{aligned} \alpha_1 &= nu^{n-1}\, du \wedge dt - p\, dx \wedge dt = 0, \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt = 0, \\ \alpha_3 &= du \wedge dx - dq \wedge dt - \gamma p u^s\, dx \wedge dt = 0 \end{aligned} \tag{4.1}$$

where γ is a constant. From system (4.1) we calculate that

$$\begin{aligned} d\alpha_1 &= -dp \wedge dx \wedge dt = dx \wedge \alpha_2, \\ d\alpha_2 &= -dq \wedge dx \wedge dt = -dx \wedge \alpha_3, \\ d\alpha_3 &= -\gamma s p u^{s-1} du \wedge dx \wedge dt - \gamma u^s\, dp \wedge dx \wedge dt \\ &= dx \wedge \left(\gamma \frac{s}{n} p u^{s-n} \alpha_1 + \gamma p u^s \alpha_2\right). \end{aligned} \tag{4.2}$$

Therefore the ideal $I = \{\omega \mid \omega = \sum_{i=1}^{3} \sigma_i \wedge \alpha_i : \sigma_i \in \Lambda(M)\}$ is closed, $dI \subset I$, and the system $\{\alpha_i\}$ in (4.1) is integrable.

On the transversal integral manifold,

$$\begin{aligned} 0 &= \alpha_1|_S = S^*\alpha_1 = \big((u^n)_x - p\big)\, dx \wedge dt, \\ 0 &= \alpha_2|_S = S^*\alpha_2 = (p_x - q)\, dx \wedge dt, \\ 0 &= \alpha_3|_S = S^*\alpha_3 = \big(u_t\, dt \wedge dx - q_x\, dx \wedge dt - \gamma p u^s\, dx \wedge dt\big). \end{aligned} \tag{4.3}$$

The transversal integral manifolds imply the equations

$$p = (u^n)_x, \qquad q = p_x = (u^n)_{xx}, \qquad u_t + q_x + \gamma p u^s = 0.$$

Let $n + s \neq 0$, then substituting p and q, it can be seen that u must satisfy the following generalized KdV equation

$$u_t + (u^n)_{xxx} + \gamma \frac{n}{n+s} (u^{s+n})_x = 0. \tag{4.4}$$

This can be put in more familiar form by setting $m = s + n$ and defining $\beta = \gamma n/(n+s)$, so (4.4) takes the form

$$u_t + (u^n)_{xxx} + \beta(u^m)_x = 0. \tag{4.5}$$

The prolongation condition (2.9) making use of (4.1) now leads to the relations

$$\begin{aligned}
&A_t\, dt \wedge dx + A_u\, du \wedge dx + A_p\, dp \wedge dx + A_q\, dq \wedge dx \\
&\quad + B_x\, dx \wedge dt + B_u\, du \wedge dt + B_p\, dp \wedge dt + B_q\, dq \wedge dt + [A, B]\, dx \wedge dt \\
&= \lambda_1 (n u^{n-1}\, du \wedge dt - p\, dx \wedge dt) + \lambda_2 (dp \wedge dt - q\, dx \wedge dt) \\
&\quad + \lambda_3 (du \wedge dx - dq \wedge dt - \gamma p u^s\, dx \wedge dt).
\end{aligned}$$

Comparison of both sides of this equation yields the following set of conditions,

$$\begin{aligned}
&-A_t + B_x + [A, B] = -\lambda_1 p - \lambda_2 q - \gamma \lambda_3 p u^s, \\
&A_u = \lambda_3, \qquad A_p = 0, \qquad A_q = 0, \\
&B_u = n\lambda_1 u^{n-1}, \qquad B_p = \lambda_2, \qquad B_q = -\lambda_3.
\end{aligned} \tag{4.6}$$

As in the KdV case, assume that A and B do not explicitly depend on (x, t) so that $A_x = A_t = 0$ and $B_x = B_t = 0$. The system in (4.6) can be put in the form

$$\begin{aligned}
&[A, B] = -\frac{1}{n} u^{-n+1} p B_u - q B_p + \gamma p u^s B_q, \\
&A_p = 0, \qquad A_q = 0, \qquad A_u = -B_q.
\end{aligned} \tag{4.7}$$

These conditions imply that $A = A(u, y)$ and $B = B(u, p, q, y)$. To obtain a class of prolongations for this system, let us take the following form for the vector field A,

$$A = A(u, y) = X_1 + u X_2, \quad X_i = X_i(y),\ i = 1, 2. \tag{4.8}$$

Using $A_u = X_2$ and (4.7), A in (4.8) is sufficient to determine B in the form

$$B = -q X_2 + C(u, p, y). \tag{4.9}$$

Therefore the first equation in (4.7) takes the form

$$[X_1 + u X_2, -q X_2 + C] = -\frac{p}{n} u^{-n+1} C_u - q C_p - \gamma p u^s X_2.$$

Simplifying this, it follows that

$$\frac{p}{n} C_u + q u^{n-1} C_p = -\gamma p u^{s+n-1} X_2 + q u^{n-1} [X_1, X_2] - u^{n-1} [X_1, C] - u^n [X_2, C]. \tag{4.10}$$

Define the vector field $X_3 = [X_1, X_2]$, then whenever C is independent of q, we obtain from (4.10) that

$$C(u, p, y) = p X_3 + D(u, y). \tag{4.11}$$

Substituting C in (4.11) into (4.10), we have

$$\begin{aligned}
\frac{p}{n} D_u &= p\big\{-\gamma u^{s+n-1} X_2 - u^{n-1} [X_1, X_3] - u^n [X_2, X_3]\big\} \\
&\quad - u^{n-1} \big\{[X_1, D] - u [X_2, D]\big\}.
\end{aligned} \tag{4.12}$$

Since D does not depend on p, the last term in (4.12) must vanish and we have two conditions on D

$$\begin{aligned}&[X_1, D] - u[X_2, D] = 0,\\ &\frac{1}{n} D_u = -\gamma u^{s+m-1} X_2 - u^{n-1}[X_1, X_3] - u^n [X_2, X_3].\end{aligned} \tag{4.13}$$

Integrating the second equation for D in (4.13) with respect to u

$$D(u, y) = -\gamma \frac{n}{m} u^m X_2 - u^n [X_1, X_3] - \frac{n}{n+1} u^{n+1} [X_2, X_3] + X_4 \tag{4.14}$$

where $m = s + n$. Substituting D from (4.14) into the first equation with commutator in (4.13), it can be simplified to the following form

$$\begin{aligned}&-\gamma \frac{n}{m} u^m [X_1, X_2] + u^n \big[X_1, [X_1, X_3]\big] + [X_1, X_4]\\ &\quad - u^{n+1} \left\{ \frac{n}{n+1} \big[X_1, [X_2, X_3]\big] + \big[X_2, [X_1, X_3]\big] \right\}\\ &\quad + \frac{n}{n+1} u^{n+2} \big[X_2, [X_2, X_3]\big] - u[X_2, X_4] = 0.\end{aligned} \tag{4.15}$$

If it is required that m and n not be equal to one, then it follows from (4.15) that the following brackets must vanish

$$[X_1, X_4] = 0, \qquad [X_2, X_4] = 0.$$

To satisfy these brackets, one way in which this can be done is to take $X_4 = \mu X_2$ and $X_4 = \kappa X_1$, from which it follows that $X_1 = \lambda X_2$, where μ, κ and λ are real constants. Moreover, substituting these results into the definition of X_3, it follows that $X_3 = 0$. Using all of these results in (4.15), it follows that the remaining terms in (4.15) vanish, hence (4.15) is satisfied identically and we have one solution. To summarize these results for the vector fields, we have

$$X_1 = \lambda X_2, \qquad X_2 = X, \qquad X_3 = 0, \qquad X_4 = \mu X_2. \tag{4.16}$$

Since there is only one independent vector field left, we have set $X = X_2$ in (4.16). In this case, the prolongation structure reduces to the following set of vector fields

$$\begin{aligned}&A = (\lambda + u) X,\\ &B = -qX + C,\\ &C = D = -\gamma \frac{n}{m} u^m X + \mu X = \left(-\gamma \frac{n}{m} u^m + \mu\right) X,\\ &X = X(y), \quad \lambda, \mu \in \mathbb{R}.\end{aligned} \tag{4.17}$$

Given the results for A and B in (4.17), the connection form $\tilde{\omega}$ in (2.8) is given by

$$\tilde{\omega} = dy - \left\{ (\lambda + u)\, dx + \left(-q - \gamma \frac{n}{m} u^m + \mu\right) dt \right\} X(y). \tag{4.18}$$

The connection $\tilde{\omega}$ can always be chosen on $\mathbb{R}$ with coordinate y and $X = \partial/\partial y$, thus

$$\tilde{\omega} = dy - (\lambda + u)\, dx - \left(-q - \gamma \frac{n}{m} u^m + \mu\right) dt, \quad \lambda, \mu \in \mathbb{R},$$

and solutions of the system (4.1) determine transversal sections of the fibre bundle such that, substituting $q = (u^n)_{xx}$, the sections are defined by

$$\begin{aligned} y_x &= \lambda + u, \\ y_t &= -(u^n)_{xx} - \gamma \frac{n}{m} u^m + \mu. \end{aligned} \tag{4.19}$$

Since (4.19) implies that $u = y_x - \lambda$, we can eliminate u to obtain an equation for $y = y(x, t)$,

$$y_t + \big((y_x - \lambda)^n\big)_{xx} + \gamma \frac{n}{m} (y_x - \lambda)^m - \mu = 0.$$

It follows then that for $\lambda = \mu = 0$, a potential equation in terms of y results

$$y_t + \big((y_x)^n\big)_{xx} + \gamma \frac{n}{m} (y_x)^m = 0. \tag{4.20}$$

Although the prolongation or solution for the vector fields (4.17) is not extremely complicated, in effect a Bäcklund transformation has been determined in the form of the equations presented in (4.19). This set of equations transforms the original equation into the form of its potential equation. Given a solution u of (4.4) then integrating (4.19) gives a corresponding solution y to (4.20).

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