

PROLONGATIONS OF GEOMETRIC OVERDETERMINED SYSTEMS

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ABSTRACT. We show that a wide class of geometrically defined overdetermined semilinear partial differential equations may be explicitly prolonged to obtain closed systems. As a consequence, in the case of linear equations we extract sharp bounds on the dimension of the solution space.

1. INTRODUCTION

For ordinary differential equations, it is clear that the n^{th} order equation

$$\frac{d^n \sigma}{dx^n} = f\left(x, \sigma, \frac{d\sigma}{dx}, \dots, \frac{d^{n-1}\sigma}{dx^{n-1}}\right)$$

is equivalent to the system of first order equations

$$\frac{d\sigma}{dx} = \sigma_1, \dots, \frac{d\sigma_k}{dx} = \sigma_{k+1}, \dots, \frac{d\sigma_{n-1}}{dx} = f(x, \sigma, \sigma_1, \dots, \sigma_{n-1}).$$

This manoeuvre is well-known, for example, in reducing the existence and uniqueness of solutions to ordinary differential equations to the case of first order equations.

For partial differential equations, however, this naïve manoeuvre fails. Even for overdetermined equations, it is necessary to introduce new dependent variables for certain higher derivatives in order to achieve a first order ‘closed system’—one in which all the first partial derivatives of all the dependent variables are determined in terms of the variables themselves. Example 1.1.2 below is typical in this regard—the original equation is first order but the closed system (1.5) implicitly but necessarily involves second derivatives of the original dependent variables σ_a . The introduction of new variables for unknown higher derivatives with the aim of expressing all their derivatives as differential consequences of the original equation is the well-known procedure of ‘prolongation’.

Classically, the prolongations of a semilinear differential operator $D : E \rightarrow F$ between smooth vector bundles E and F on a smooth manifold M are constructed

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from its leading symbol $\sigma(D) : \odot^k \Lambda^1 \otimes E \rightarrow F$ where $\odot^k \Lambda^1$ denotes the bundle of symmetric covariant tensors on M of valence k . At any point of M , denoting by K the kernel of $\sigma(D)$, one considers the vector spaces

$$(1.1) \quad V_i = (\odot^i \Lambda^1 \otimes E) \cap (\odot^{i-k} \Lambda^1 \otimes K) \quad \text{for } i \geq k,$$

declaring the system to be of *finite type* if $V_i = 0$ for i sufficiently large [15]. The solutions of a system of finite type are determined by finitely many jets at a point. Although there is a general criterion that D be of finite type (namely, that its characteristic variety be empty [15, Proposition 1.7.5]) the computation of V_i presents a major obstacle to further progress.

There are two main points to this article. Firstly, for a wide class of geometric overdetermined partial differential equations, we explicitly compute V_i (Lemma 3.1 part (4)). The direct sum $V = \bigoplus_{i=0}^N V_i$ is a vector bundle induced by an irreducible representation of a reductive Lie algebra so N and its rank can be immediately read off. This gives sharp bounds on the jet needed to pin down a solution and, in the linear case, the dimension of the space of solutions. The second point to this article is motivated by geometric considerations. We can deal with all symbols of overdetermined invariant operators for an important class of structures including conformal and quaternionic geometries. Motivated by the machinery of Bernstein-Gelfand-Gelfand sequences [3, 4], we find a uniform procedure to perform the further steps necessary explicitly to rewrite the equation in closed form. For the whole development, representation theory, especially Kostant's algebraic Hodge theory [11] in Lie algebra cohomology, provides the key to our method.

For readers unfamiliar with overdetermined systems, we begin by discussing some examples. The reader should be aware, however, that these examples are far too simple satisfactorily to illustrate the general procedure. In fact, this is inevitable—though our algorithm is explicit, the details in any particular case will generally be fearsome. However, for many purposes, the details are unnecessary. For example, we may deduce without hesitation that, on a Riemannian manifold of dimension $n \geq 3$, the space of solutions of the partial differential equation

$$(1.2) \quad \text{the trace-free part of } \nabla_{(a} \nabla_b \sigma_{c)} = 0$$

is finite-dimensional of dimension at most $n(n+2)(n+4)/3$. This bound is sharp and any solution is determined by its 4-jet at one point.

In (1.2) and throughout, we adopt Penrose's abstract index notation [13]. Thus, indices act as markers to specify the type of a tensor (so σ_a is a 1-form whilst σ^a would be a vector field) and to record symmetries and contractions. Round brackets, as in (1.2), mean that the indices they enclose are symmetrised, square brackets $\phi_{[ab]c}$ take the skew part, and a repeated index ϕ^a_{ab} denotes contraction. On a Riemannian manifold, indices may be raised or lowered with the metric in the usual way. Connections will be denoted ∇_a and on a Riemannian manifold will usually mean the Levi-Civita connection. If ∇_a is a torsion-free connection on the tangent bundle, then its curvature tensor $R_{ab}{}^c{}_d$ is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R_{ab}{}^c{}_d V^d.$$

In particular, $\nabla_b \nabla_a V^b = \nabla_a \nabla_b V^b + R_{ab} V^b$, where $R_{ab} = R_{ca}{}^c{}_b$ is the Ricci tensor.

1.1. Two affine examples. Here we work on a smooth manifold with torsion-free connection ∇_a .

1.1.1. *Example.* Consider the partial differential equation on the function σ :-

$$(1.3) \quad \nabla_a \nabla_b \sigma = 0.$$

If we introduce $\mu_a = \nabla_a \sigma$, then we can rewrite it as a system:-

$$\begin{aligned} \nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= 0. \end{aligned}$$

1.1.2. *Example.* Consider the partial differential equation on the 1-form σ_a :-

$$(1.4) \quad \nabla_{(a} \sigma_{b)} = 0.$$

We can rewrite it as

$$\nabla_a \sigma_b = \mu_{ab} \quad \text{where } \mu_{ab} \text{ is skew.}$$

Naïvely differentiating this equation leads nowhere but notice that, as differential forms, $\mu = d\sigma$ whence $d\mu = 0$. In index notation $\nabla_{[a} \mu_{bc]} = 0$ so

$$\nabla_a \mu_{bc} = \nabla_c \mu_{ba} - \nabla_b \mu_{ca} = \nabla_c \nabla_b \sigma_a - \nabla_b \nabla_c \sigma_a = R_{bc}{}^d{}_a \sigma_d.$$

Therefore, the differential equation (1.4) is equivalent to the system

$$(1.5) \quad \begin{aligned} \nabla_a \sigma_b &= \mu_{ab} \quad \text{where } \mu_{ab} \text{ is skew} \\ \nabla_a \mu_{bc} &= R_{bc}{}^d{}_a \sigma_d. \end{aligned}$$

1.2. Two Riemannian examples. Here we work on n -dimensional Riemannian manifold with metric g_{ab} and Levi-Civita connection ∇_a . We shall suppose that $n \geq 3$.

1.2.1. *Example.* Consider the partial differential equation

$$(1.6) \quad \text{the trace-free part of } \nabla_a \nabla_b \sigma = 0.$$

If we introduce $\mu_a = \nabla_a \sigma$, then we can rewrite it as

$$\nabla_a \mu_b = \rho g_{ab} \quad \text{for some smooth function } \rho.$$

Then

$$\nabla_a \rho = \nabla^b \nabla_a \mu_b = \nabla_a \nabla^b \mu_b + R_a{}^c \mu_c = n \nabla_a \rho + R_a{}^b \mu_b.$$

Therefore, the differential equation (1.6) is equivalent to the system

$$(1.7) \quad \begin{aligned} \nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= \rho g_{ab} \\ \nabla_a \rho &= -\frac{1}{n-1} R_a{}^b \mu_b. \end{aligned}$$

1.2.2. *Example.* Consider the partial differential equation

$$(1.8) \quad \text{the trace-free part of } \nabla_{(a}\sigma_{b)} = 0.$$

Even in this simple case, prolongation is already fairly involved. The details can be omitted on first reading and the main features are described in §1.3 below. We can rewrite (1.8) as

$$(1.9) \quad \nabla_a\sigma_b = \mu_{ab} + \nu g_{ab} \quad \text{where } \mu_{ab} \text{ is skew.}$$

Then $\nabla_{[a}\mu_{bc]} = 0$, so

$$(1.10) \quad \begin{aligned} \nabla_a\mu_{bc} &= \nabla_c\mu_{ba} - \nabla_b\mu_{ca} = \nabla_c(\nabla_b\sigma_a - \nu g_{ba}) - \nabla_b(\nabla_c\sigma_a - \nu g_{ca}) \\ &= R_{bc}{}^d{}_a\sigma_d - g_{ab}\nabla_c\nu + g_{ac}\nabla_b\nu. \end{aligned}$$

Tracing over a and b gives

$$\nabla^b\mu_{bc} = -R_c{}^d\sigma_d - (n-1)\nabla_c\nu.$$

Let us introduce $\rho_c = \frac{1}{n-1}\nabla^b\mu_{bc}$ and rearrange this last equation as

$$(1.11) \quad \nabla_a\nu = -\rho_a - \frac{1}{n-1}R_a{}^b\sigma_b.$$

It may be used to eliminate $\nabla_c\nu$ from (1.10) to obtain

$$(1.12) \quad \nabla_a\mu_{bc} = g_{ab}\rho_c - g_{ac}\rho_b + K_{abc},$$

where

$$(1.13) \quad K_{abc} = R_{bc}{}^d{}_a\sigma_d + \frac{1}{n-1}g_{ab}R_c{}^d\sigma_d - \frac{1}{n-1}g_{ac}R_b{}^d\sigma_d.$$

Notice that K_{abc} is totally trace-free. Now apply ∇_d to (1.12) and skew over a and d to obtain

$$R_{da}{}^e{}_b\mu_{ce} - R_{da}{}^e{}_c\mu_{be} = \nabla_d K_{abc} - \nabla_a K_{dbc} + g_{ab}\nabla_d\rho_c - g_{db}\nabla_a\rho_c - g_{ac}\nabla_d\rho_b + g_{dc}\nabla_a\rho_b.$$

Tracing over a and b gives

$$R_d{}^e\mu_{ce} - R_d{}^{be}{}_c\mu_{be} = -\nabla^b K_{dbc} + (n-2)\nabla_d\rho_c + g_{dc}\nabla^b\rho_b$$

but tracing again, over c and d , gives $0 = 2(n-1)\nabla^b\rho_b$. Therefore,

$$(1.14) \quad \nabla_a\rho_b = \frac{1}{n-2}(R_a{}^c\mu_{bc} - R_a{}^{cd}{}_b\mu_{cd} - \nabla^c K_{abc}).$$

At this point it is clear that the system has closed: it comprises (1.9), (1.11), (1.12), and in (1.14) one has to expand $\nabla^c K_{abc}$ using (1.13) and (1.9).

1.3. Discussion. In each of the examples above, we start with a linear differential operator $D : E \rightarrow F$ between vector bundles and the conclusion is that various auxiliary fields may be introduced so that the equation $D\sigma = 0$ is equivalent to a ‘closed system’ in which all the first partial derivatives of all fields are determined as linear expressions in the fields themselves. It is convenient to regard this system as a vector bundle V with connection $\tilde{\nabla}$. Thus, the conclusion of Example 1.1.2 is that

$$\nabla_{(a}\sigma_{b)} = 0 \quad \text{if and only if} \quad \tilde{\nabla}\Sigma = 0$$

where

$$\Sigma = \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} \quad \text{is a section of the vector bundle} \quad V = \begin{matrix} \Lambda^1 \\ \oplus \\ \Lambda^2 \end{matrix}$$

and $\tilde{\nabla} : V \rightarrow \Lambda^1 \otimes V$ is the connection:–

$$\tilde{\nabla}_a \begin{pmatrix} \sigma_b \\ \mu_{bc} \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a \sigma_d \end{pmatrix}.$$

Our examples, constructing Σ and $\tilde{\nabla}$ from σ and D , follow the well-known method of ‘prolongation’. Our aim in this article, however, is to predict the form of a valid prolongation for a natural and extensive class of examples without having to carry out the prolongation in detail.

The conclusion of Example 1.2.2 is that (1.8) is equivalent to $\tilde{\nabla}\Sigma = 0$ where

$$\Sigma = \begin{pmatrix} \sigma_b \\ \mu_{bc} & \nu \\ \rho_b \end{pmatrix} \quad \text{is a section of the bundle} \quad V = \begin{matrix} \Lambda^1 \\ \Lambda^2 \oplus \Lambda^0 \\ \Lambda^1 \end{matrix}$$

and $\tilde{\nabla} : V \rightarrow \Lambda^1 \otimes V$ is an explicit connection of the form

$$(1.15) \quad \tilde{\nabla} \begin{pmatrix} \sigma \\ \mu & \nu \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla \sigma - \mu - \nu & \nabla \sigma - \mu - \nu \\ \nabla \mu - \rho - R \bowtie \sigma & \nabla \nu - \rho - R \bowtie \sigma \\ \nabla \rho - R \bowtie \mu - R \bowtie \nu - (\nabla R) \bowtie \sigma \end{pmatrix},$$

where each \bowtie indicates an appropriate linear combination of contractions of its ingredients.

Note that Σ is obtained from σ_a by application of a linear second order differential operator, explicitly

$$\sigma_a \longmapsto \begin{pmatrix} \sigma_a \\ \nabla_{[a} \sigma_{b]} & \frac{1}{n} \nabla^a \sigma_a \\ \frac{1}{2(n-1)} (\nabla^b \nabla_b \sigma_a - \nabla^b \nabla_a \sigma_b) \end{pmatrix}.$$

The equation (1.8) is well-known. It says that the vector field σ^a is a conformal Killing field—its flow preserves the metric up to scale. From this geometric interpretation it follows easily that the space of solutions is bounded by $\dim \mathfrak{so}(n+1, 1)$ since $\mathfrak{so}(n+1, 1)$ is the conformal algebra in the flat case. This bound is confirmed by the technique of prolongation:–

$$\text{rank } V = 2 \text{rank } \Lambda^1 + \text{rank } \Lambda^2 + \text{rank } \Lambda^0 = 2n + \frac{n(n-1)}{2} + 1 = \frac{(n+1)(n+2)}{2}.$$

In [14], Semmelmann uses this technique to establish similar bounds on the dimension of spaces of conformal Killing forms. Specifically, he finds an explicit connection (also having the form (1.15)) on the bundle

$$V = \Lambda^{p+1} \oplus \begin{matrix} \Lambda^p \\ \oplus \\ \Lambda^{p-1} \end{matrix} \quad \text{with rank} \begin{pmatrix} n+2 \\ p+1 \end{pmatrix}$$

so that conformal Killing p -forms are equivalent to parallel sections of this bundle. The general procedure, to be explained in this article, includes this case and many more besides.

The corresponding bound for Example 1.1.2 is

$$\text{rank } \Lambda^1 + \text{rank } \Lambda^2 = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

It was pointed out to us by Dan Fox that this is precisely the bound investigated by Eisenhart in [8].

1.4. Semilinear variants. Each of the examples discussed so far persists in a semilinear form. Thus, Example 1.1.1 may be modified as

$$\nabla_a \nabla_b \sigma = f_{ab}(x, \sigma, \nabla_c \sigma)$$

where f_{ab} depends smoothly on its arguments and takes values in symmetric 2-tensors. Evidently, this equation is equivalent to the system

$$\begin{aligned} \nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= f_{ab}(x, \sigma, \mu_a). \end{aligned}$$

Example 1.1.2 may be modified as

$$(1.16) \quad \nabla_{(a} \sigma_{b)} = f_{ab}(x, \sigma_c).$$

The only difficulty in following previous reasoning is that one must be careful as to the meaning of $\nabla_c f_{ab}(x, \sigma_d)$. As it arises, σ_d is a function of x and so f_{ab} may be regarded as a tensor on the manifold and $\nabla_c f_{ab}$ as the usual covariant derivative. On the other hand, we may fix σ_d , regard $f_{ab}(x, \sigma_d)$ as a function of its first argument, and then take its covariant derivative. We shall use the notation $\partial_c f_{ab}$ for the result of this point of view. There is also the partial derivative obtained by fixing x and differentiating with respect to σ_d : let us write $\delta^d = \partial/\partial\sigma_d$. Then, by the chain rule,

$$\nabla_c f_{ab} = \partial_c f_{ab} + (\delta^d f_{ab}) \nabla_c \sigma_d,$$

often referred to as expressing ‘total derivative’ in terms of ‘partial derivative’. The result of following previous reasoning is that (1.16) is equivalent to the system

$$\begin{aligned} \nabla_a \sigma_b &= \mu_{ab} + f_{ab} \\ \nabla_a \mu_{bc} &= R_{bc}{}^d{}_a \sigma_d + 2\partial_{[b} f_{c]a} - 2(\delta^d f_{a[b} \mu_{c]d}) - 2(\delta^d f_{a[b} f_{c]d}), \end{aligned}$$

where μ_{ab} is skew. As a typical nonlinear variant therefore,

$$\nabla_{(a} \sigma_{b)} = \sigma_a \sigma_b + S_{ab}$$

for an arbitrary given symmetric tensor S_{ab} is equivalent to the closed system

$$\begin{aligned} \nabla_a \sigma_b &= \mu_{ab} + \sigma_a \sigma_b + S_{ab} \\ \nabla_a \mu_{bc} &= R_{bc}{}^d{}_a \sigma_d + 2\nabla_{[b} S_{c]a} - 2\sigma_{[b} \mu_{c]a} + 2\sigma_a \mu_{bc} + 2S_{a[b} \sigma_{c]}. \end{aligned}$$

The general semilinear variant of Example 1.2.1 is

$$\nabla_a \nabla_b \sigma - \frac{1}{n} g_{ab} \nabla^c \nabla_c \sigma = f_{ab}(x, \sigma, \nabla_c \sigma),$$

where $f_{ab}(x, \sigma, \sigma_c)$ is symmetric and trace-free. If we write $\delta = \partial/\partial\sigma$, then the chain rule for total derivative in terms of partial derivative is

$$\nabla_c f_{ab} = \partial_c f_{ab} + (\delta f_{ab}) \nabla_c \sigma + (\delta^d f_{ab}) \nabla_c \sigma_d$$

and the closed system generalising (1.7) is

$$\begin{aligned} \nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= \rho g_{ab} + f_{ab} \\ \nabla_a \rho &= -\frac{1}{n-1} R_a{}^b \mu_b + \frac{1}{n-1} (\partial^b f_{ab} + (\delta f_{ab}) \mu^b + (\delta^b f_{ab}) \rho + (\delta^d f_{ab}) f^b{}_d). \end{aligned}$$

A particular semilinear variant of Example 1.2.2 is

$$\text{the trace-free part of } (\nabla_{(a} \sigma_{b)} + \sigma_a \sigma_b + \frac{1}{n-2} R_{ab}) = 0,$$

where R_{ab} is the Ricci tensor. It is the Einstein-Weyl equation and the corresponding closed system is derived in [7] by *ad hoc* methods.

2. FORMULATION OF THE MAIN RESULTS

Firstly, some generalities on differential operators. As detailed in [15], to every smooth vector bundle E on a smooth manifold M there are the canonically associated jet bundles $J^k E$ on M and short exact sequences of homomorphisms of vector bundles

$$0 \rightarrow \odot^k \Lambda^1 \otimes E \rightarrow J^k E \rightarrow J^{k-1} E \rightarrow 0,$$

where $\odot^k \Lambda^1$ denotes the k^{th} symmetric tensor power of Λ^1 . A k^{th} order linear differential operator $D : E \rightarrow F$ between vector bundles E and F is equivalent to a homomorphism of vector bundles $J^k E \rightarrow F$ and the symbol $\sigma(D)$ of D is defined as the composition

$$(2.1) \quad \odot^k \Lambda^1 \otimes E \hookrightarrow J^k E \rightarrow F.$$

A differential operator of the form $D_1 + D_2$ where D_1 is k^{th} order linear and D_2 is $(k-1)^{\text{st}}$ order is called semilinear and its symbol is defined to be $\sigma(D_1)$.

If we now return to the semilinear variants of our affine examples, we see that the form of the equation is independent of the connection. Equation (1.16), for example, says that we have a first order semilinear operator $\Lambda^1 \rightarrow \odot^2 \Lambda^1$ whose symbol

$$\Lambda^1 \otimes \Lambda^1 \longrightarrow \odot^2 \Lambda^1$$

is taking the symmetric part. In particular, a change of torsion-free connection in (1.4) is covered by

$$\nabla_{(a} \sigma_{b)} = \Gamma_{ab}{}^c \sigma_c$$

as a special case of (1.16).

To formulate the semilinear equations on a smooth manifold M to which our prolongation procedure will apply, let us regard the tangent bundle as tautologically associated to the frame bundle under the standard representation of $\text{GL}(n, \mathbb{R})$ on \mathbb{R}^n . Then, an irreducible tensor bundle on M is, by definition, a bundle associated to the frame bundle under an irreducible representation of $\text{GL}(n, \mathbb{R})$. By basic representation theory, any tensor bundle decomposes into a direct sum of irreducible tensor bundles. In fact, for technical reasons, let us fix a volume form on M . This

reduces the structure group of the frame bundle to $\mathrm{SL}(n, \mathbb{R})$ and allows us to use the usual theory of weights to specify an irreducible tensor bundle. Following [2], the irreducible representations are in one-to-one correspondence with attachments of non-negative integers to the nodes of the Dynkin diagram of $\mathfrak{sl}(n, \mathbb{R})$. These numbers represent the coefficients in the expansion of the highest weight of the dual representation (or equivalently the negative of the lowest weight of the given representation) as a linear combination of fundamental weights. Each coefficient is placed over the node representing the simple root that is dual to the fundamental weight. Combining these viewpoints, the tangent bundle is

$$\overset{0}{\bullet} - \overset{0}{\bullet} - \dots - \overset{0}{\bullet} - \overset{1}{\bullet}$$

and so on:-

$$\Lambda^1 = \overset{1}{\bullet} - \overset{0}{\bullet} - \overset{0}{\bullet} - \dots - \overset{0}{\bullet} \quad \Lambda^2 = \overset{0}{\bullet} - \overset{1}{\bullet} - \overset{0}{\bullet} - \dots - \overset{0}{\bullet} \quad \odot^k \Lambda^1 = \overset{k}{\bullet} - \overset{0}{\bullet} - \overset{0}{\bullet} - \dots - \overset{0}{\bullet}.$$

For any irreducible tensor bundle E , the tensor product $\odot^k \Lambda^1 \otimes E$ decomposes into irreducibles amongst which, the one with highest weight obtained by adding the highest weights of $\odot^k \Lambda^1$ and E , occurs with multiplicity one. This is the Cartan product [6] and we shall denote it $\odot^k \Lambda^1 \odot E$. In the notation just established,

$$E = \overset{a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \dots - \overset{d}{\bullet} \quad \Rightarrow \quad \odot^k \Lambda^1 \odot E = \overset{k+a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \dots - \overset{d}{\bullet}.$$

There is a canonical projection $\odot^k \Lambda^1 \otimes E \rightarrow \odot^k \Lambda^1 \odot E$, which we shall also refer to as the Cartan product.

Now we are in a position to state a special case of our main theorem:-

Theorem 2.1. *Suppose M is a smooth manifold equipped with a volume form. Let E be an irreducible tensor bundle on M and $F = \odot^k \Lambda^1 \odot E$. Suppose $D : E \rightarrow F$ is a k^{th} -order semilinear differential operator whose symbol*

$$\sigma(D) : \odot^k \Lambda^1 \otimes E \rightarrow F = \odot^k \Lambda^1 \odot E$$

is given by the Cartan product. Then, there is a vector bundle V and, for every choice of volume-preserving connection ∇ on the tangent bundle, a canonically associated connection $\tilde{\nabla} : V \rightarrow \Lambda^1 \otimes V$ on V so that there is a bijection

$$(2.2) \quad \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + \Phi(\Sigma) = 0\},$$

where $\Phi : V \rightarrow \Lambda^1 \otimes V$ is a fibre-preserving mapping canonically constructed from D . If D is linear, then so is Φ . From left to right (2.2) is implemented by an N^{th} order linear differential operator where N is easily computable from E and k .

We should emphasise that the bundle V is completely determined by E and k . The connection $\tilde{\nabla}$ on V is then determined by a choice of affine connection on M . Finally, the fibre-preserving mapping Φ is determined by D .

In particular, V is defined as follows. Let us embed $\mathrm{SL}(n, \mathbb{R}) \hookrightarrow \mathrm{SL}(n+1, \mathbb{R})$ by

$$\mathrm{SL}(n, \mathbb{R}) \ni A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{SL}(n+1, \mathbb{R}).$$

Corresponding to this embedding, the Dynkin diagram of $\mathfrak{sl}(n + 1, \mathbb{R})$ is obtained from the Dynkin diagram of $\mathfrak{sl}(n, \mathbb{R})$ by adding a node on the left. Let us denote the fundamental weight of $\mathfrak{sl}(n + 1, \mathbb{R})$ corresponding to the additional simple root by ω_0 . Any representation of $SL(n + 1, \mathbb{R})$ restricts to a representation of $SL(n, \mathbb{R})$ and hence gives rise to an associated vector bundle on M . Using these two facts, given $E = \overset{a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \cdots - \overset{d}{\bullet}$ and $k \geq 1$, we define $V := \overset{k-1}{\bullet} - \overset{a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \cdots - \overset{d}{\bullet}$, and it turns out that $N = k - 1 + a + b + c + \cdots + d$. More explicitly, if E is associated to the dual of the irreducible representation of $SL(n, \mathbb{R})$ with highest weight λ , then we consider the irreducible representation of $SL(n + 1, \mathbb{R})$ with highest weight $(k - 1)\omega_0 + \lambda$, restrict its dual to $SL(n, \mathbb{R})$ and let V be the associated vector bundle. When restricted to $SL(n, \mathbb{R})$, an irreducible representation of $SL(n + 1, \mathbb{R})$ splits into a direct sum of irreducible representations of $SL(n, \mathbb{R})$. Correspondingly, we obtain a splitting

$$(2.3) \quad V = \overset{k-1}{\bullet} - \overset{a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \cdots - \overset{d}{\bullet} = \overset{a}{\bullet} - \overset{b}{\bullet} - \overset{c}{\bullet} - \cdots - \overset{d}{\bullet} \oplus \cdots .$$

The representation corresponding to the first summand, which is (isomorphic to) E , can be described as the $SL(n, \mathbb{R})$ -invariant subspace generated by a vector of lowest weight. In particular, there is a canonically defined surjection $\pi : V \rightarrow E$ and it is $\sigma = \pi \circ \Sigma$ that induces the isomorphism (2.2) from right to left.

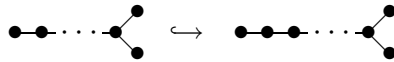
In the situation of Example 1.1.1, E corresponds to the trivial representation and $k = 2$. Thus we obtain $V = \overset{1}{\bullet} - \overset{0}{\bullet} - \overset{0}{\bullet} - \cdots - \overset{0}{\bullet}$. This corresponds to the representation $\mathbb{R}^{(n+1)*}$, which restricted to $SL(n, \mathbb{R})$ splits as $\mathbb{R} \oplus \mathbb{R}^{n*}$. Hence we obtain $V = \mathbb{R} \oplus \Lambda^1$ and $N = 1$.

For Example 1.1.2, we have $E = \Lambda^1$ and $k = 1$, which implies $V = \overset{0}{\bullet} - \overset{1}{\bullet} - \overset{0}{\bullet} - \cdots - \overset{0}{\bullet}$. The corresponding representation $\Lambda^2 \mathbb{R}^{(n+1)*}$ splits as $\mathbb{R}^{n*} \oplus \Lambda^2 \mathbb{R}^{n*}$, so $V = \Lambda^1 \oplus \Lambda^2$ and again $N = 1$.

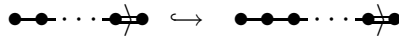
For the Riemannian version of Theorem 2.1 we simply replace the embedding of Lie groups $SL(n, \mathbb{R}) \hookrightarrow SL(n + 1, \mathbb{R})$ by the embedding $SO(n) \hookrightarrow SO(n + 1, 1)$:-

$$SO(n) \ni A \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(n + 1, 1),$$

where $SO(n + 1, 1)$ is realised as preserving the quadratic form $2x_0x_{n+1} + \sum_{i=1}^n x_i^2$. There is a corresponding inclusion of Dynkin diagrams:-



if n is even and



if n is odd. The irreducible tensor bundles on an oriented Riemannian manifold are associated to irreducible representations of $SO(n)$. On an oriented spin manifold, we should use $Spin(n) \hookrightarrow Spin(n + 1, 1)$ instead and there are irreducible spinor bundles too, associated to irreducible spin representations. The Riemannian version

of Theorem 2.1 is obtained by taking $F = \bigodot^k \Lambda^1 \odot E$ where \bigodot denotes trace-free symmetric product. Thus, if n is odd for example, then

$$E = \begin{array}{c} a \quad b \quad \dots \quad c \quad d \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \hline \end{array} \Rightarrow \begin{cases} F = \begin{array}{c} k+a \quad b \quad \dots \quad c \quad d \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \hline \end{array} & V = \begin{array}{c} k-1 \quad a \quad b \quad \dots \quad c \quad d \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \hline \end{array} \\ N = 2(k-1 + a + b + \dots + c) + d \end{cases} .$$

With these replacements, the Riemannian statement is almost identical. The only significant difference is that we may as well use the Levi-Civita connection in the construction of $\tilde{\nabla}$, which thereby becomes canonical.

2.2. Other geometries. Though the affine and Riemannian cases are perhaps the most significant, there is a more general formulation in terms of certain G -structures, which provides a uniform approach and whose proof is no more difficult. It is this approach that we shall adopt for the remainder of this article.

Let G be a Lie group whose Lie algebra \mathfrak{g} is $|1|$ -graded semisimple:–

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

as, for example, discussed in [1, 4, 12]. Let $G_0 \subset G$ be the subgroup consisting of those elements whose adjoint action on \mathfrak{g} preserves the grading. Its Lie algebra is \mathfrak{g}_0 . Let G'_0 be a subgroup of G_0 whose Lie algebra is $[\mathfrak{g}_0, \mathfrak{g}_0]$. It is semisimple and the adjoint action makes \mathfrak{g}_{-1} into a G'_0 -module. We shall suppose that M is a smooth manifold endowed with a first order G'_0 -structure. More specifically, M should have the same dimension as \mathfrak{g}_{-1} and the frame bundle should be reduced under $G'_0 \rightarrow \mathrm{GL}(\mathfrak{g}_{-1})$. If $G = \mathrm{SL}(n+1, \mathbb{R})$, there is a $|1|$ -grading on $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ so that $G'_0 = \mathrm{SL}(n, \mathbb{R})$, included into G as in the discussion after Theorem 2.1. This leads to the standard representation of G'_0 on $\mathfrak{g}_{-1} \cong \mathbb{R}^n$, so the corresponding geometries are n -manifolds endowed with a volume form. For $G = \mathrm{SO}(n+1, 1)$, we may arrange a $|1|$ -grading so that $G'_0 \hookrightarrow G$ becomes the inclusion of $\mathrm{SO}(n)$ described above, and the corresponding geometries are oriented Riemannian n -manifolds.

For M endowed with a G'_0 -structure, as above, we may consider vector bundles on M induced from irreducible representations of G'_0 . If \mathbb{E} is such a representation, we shall write E for the corresponding vector bundle. In particular, the adjoint action of G'_0 on \mathfrak{g}_{-1} is irreducible and induces the tangent bundle. The Killing form on \mathfrak{g} canonically identifies \mathfrak{g}_{-1}^* with \mathfrak{g}_1 as G_0 -modules. Therefore, the G'_0 -module \mathfrak{g}_1 gives rise to the cotangent bundle Λ^1 on M . It is convenient to write $\bigodot^k \Lambda^1 \odot E$ for the vector bundle associated to the Cartan product $\bigodot^k \mathfrak{g}_1 \odot \mathbb{E}$.

A principal G'_0 -connection gives rise to connections on all the associated vector bundles E . Conversely, because the G'_0 -action on \mathfrak{g}_{-1} is infinitesimally effective, a connection on the tangent bundle compatible with the G'_0 -structure, gives rise to a principal connection. Here is the general statement extending Theorem 2.1:–

Theorem 2.3. *Let M be a manifold with G'_0 -structure as above. Suppose E is a vector bundle on M induced from an irreducible representation of G'_0 and fix $k \geq 1$. Then there is a vector bundle V explicitly constructed from E and k and, for every choice of G'_0 -compatible connection ∇ on the tangent bundle, a canonically associated*

connection $\tilde{\nabla} : V \rightarrow \Lambda^1 \otimes V$ on V with the following property. For every k^{th} -order semilinear differential operator $D : E \rightarrow F = \odot^k \Lambda^1 \odot E$ whose symbol

$$\sigma(D) : \odot^k \Lambda^1 \otimes E \rightarrow F = \odot^k \Lambda^1 \odot E$$

is the Cartan product, we have a bijection

$$(2.4) \quad \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + \Phi(\Sigma) = 0\}$$

(implemented by an N^{th} order linear differential operator in one direction and the natural projection in the other), where $\Phi : V \rightarrow \Lambda^1 \otimes V$ is a fibre-preserving mapping canonically constructed from D . If D is linear, then so is Φ .

The proof will occupy §4 but there is a useful and immediate corollary:–

Corollary 2.4. *Any solution of $D\sigma = 0$ is determined by its N -jet. If $D : E \rightarrow F$ is linear, then the dimension of the space of solutions of $D\sigma = 0$ is bounded by $\text{rank } V$.*

Proof. When D is linear Φ is a homomorphism and so $\tilde{\nabla} + \Phi$ is a connection on V . According to (2.4), we seek parallel section of V with respect to this connection. \square

As in the affine and Riemannian cases, the bundle V is induced from an irreducible representation \mathbb{V} of G . Hence, $\text{rank } V = \dim \mathbb{V}$ and N , which is related to the decomposition of \mathbb{V} as a G'_0 -module, can be computed by standard tools from representation theory [9, 10]. For example, the $\text{SO}(n+1, 1)$ -module $\bullet \overset{1}{\bullet} \overset{1}{\bullet} \overset{0}{\bullet} \overset{0}{\bullet} \cdots$ has dimension $n(n+2)(n+4)/3$ and has $N = 4$, the stated bounds for (1.2).

Sharpness of both bounds is observed in the remarks at the end of this article.

3. ALGEBRAIC INGREDIENTS

We shall need some results from Lie algebra cohomology. Specifically, what we need is a special case of Kostant's algebraic Hodge theory [11]. In this section, we state what we need. Proofs may be found in [11]. A more general exposition in a similar context may be found in [3].

The setting is a $|1|$ -graded Lie algebra \mathfrak{g} corresponding to a semisimple Lie group G , as discussed in §2.2. Recall that G'_0 is the semisimple part of G_0 , which is itself a subgroup of G with Lie algebra \mathfrak{g}_0 . Let \mathbb{V} be an irreducible representation of G . We define a complex of G_0 -modules

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial} & \cdots \\ & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & \mathbb{V} & \xrightarrow{\partial} & \text{Hom}(\mathfrak{g}_{-1}, \mathbb{V}) & \xrightarrow{\partial} & \text{Hom}(\Lambda^2 \mathfrak{g}_{-1}, \mathbb{V}) & \xrightarrow{\partial} & \cdots \end{array}$$

where the vertical identifications are by means of the Killing form and

$$\partial : \text{Hom}(\Lambda^p \mathfrak{g}_{-1}, \mathbb{V}) \longrightarrow \text{Hom}(\Lambda^{p+1} \mathfrak{g}_{-1}, \mathbb{V})$$

is defined by

$$\partial\phi(X_0, \dots, X_p) = \sum_{i=0}^p (-1)^i X_i \phi(X_0, \dots, \widehat{X}_i, \dots, X_p).$$

Since \mathfrak{g}_{-1} is Abelian, it is easily verified that $\partial^2 = 0$ and we define the Lie algebra cohomology

$$H^p(\mathfrak{g}_{-1}, \mathbb{V}) = \frac{\ker \partial : \Lambda^p \mathfrak{g}_1 \otimes \mathbb{V} \longrightarrow \Lambda^{p+1} \mathfrak{g}_1 \otimes \mathbb{V}}{\operatorname{im} \partial : \Lambda^{p-1} \mathfrak{g}_1 \otimes \mathbb{V} \longrightarrow \Lambda^p \mathfrak{g}_1 \otimes \mathbb{V}}.$$

Since ∂ is a homomorphism of G_0 -modules, $H^p(\mathfrak{g}_{-1}, \mathbb{V})$ is a G_0 -module. There is also a codifferential

$$(3.2) \quad 0 \longleftarrow \mathbb{V} \xleftarrow{\partial^*} \mathfrak{g}_1 \otimes \mathbb{V} \xleftarrow{\partial^*} \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{V} \xleftarrow{\partial^*} \dots$$

defined by

$$\partial^*(Z_0 \wedge \dots \wedge Z_p \otimes v) = \sum_{i=0}^p (-1)^{i+1} Z_0 \wedge \dots \wedge \widehat{Z}_i \wedge \dots \wedge Z_p \otimes Z_i v.$$

It is also G_0 -equivariant and satisfies $\partial^{*2} = 0$. There is a ‘Hodge decomposition’:-

$$(3.3) \quad \Lambda^p \mathfrak{g}_1 \otimes \mathbb{V} = \operatorname{im}(\partial) \oplus (\ker(\partial) \cap \ker(\partial^*)) \oplus \operatorname{im}(\partial^*)$$

and, in particular, a canonical isomorphism

$$H^p(\mathfrak{g}_{-1}, \mathbb{V}) \cong \ker(\partial) \cap \ker(\partial^*) \quad \text{on } \Lambda^p \mathfrak{g}_1 \otimes \mathbb{V}.$$

The differential ∂ is seen more clearly in the Hodge decomposition

$$\begin{array}{ccc} \Lambda^p \mathfrak{g}_1 \otimes \mathbb{V} & = & \ker(\partial) \oplus \operatorname{im}(\partial^*) \\ \downarrow & & \swarrow \\ \Lambda^{p+1} \mathfrak{g}_1 \otimes \mathbb{V} & = & \operatorname{im}(\partial) \oplus \ker(\partial^*) \end{array}$$

as an isomorphism $\partial : \operatorname{im}(\partial^*) \rightarrow \operatorname{im}(\partial)$. Its inverse is not necessarily ∂^* . Instead, we may define δ^* to be this inverse on $\operatorname{im}(\partial)$ and to annihilate $\ker(\partial^*)$. We obtain a new G_0 -equivariant codifferential defining the same Hodge decomposition as does ∂^* but with the congenial feature that

$$(3.4) \quad \delta^* \partial = \operatorname{id} \text{ on } \operatorname{im}(\delta^*) = \operatorname{im}(\partial^*) \quad \text{and} \quad \partial \delta^* = \operatorname{id} \text{ on } \operatorname{im}(\partial).$$

Now let us be more specific about the representation \mathbb{V} . The description of $|1|$ -gradings is well known: for an appropriate choice of a Cartan subalgebra for the complexification of \mathfrak{g} there is a distinguished simple root α_0 . This has the property that a root space lies in the complexification of \mathfrak{g}_j ($j = -1, 0, 1$) if and only if j is the coefficient of α_0 in the expansion of the given root into simple roots. In particular, the Dynkin diagram of \mathfrak{g}'_0 is obtained by removing in the Dynkin diagram of \mathfrak{g} the node representing α_0 and all edges connected to that node. In the affine and Riemannian cases previously discussed this was the leftmost node. Let ω_0 denote the fundamental weight corresponding to α_0 . Starting with an irreducible representation \mathbb{E} of G'_0 , we may add $(k-1)\omega_0$ to the highest weight of \mathbb{E}^* , and define \mathbb{V} as the dual of the irreducible representation of G with that highest weight.

The subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$ is the nilradical of the parabolic $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, so Kostant’s version of the Bott-Borel-Weil Theorem, see [11], describes the cohomology of \mathfrak{g}_1 with coefficients in an irreducible representation of \mathfrak{g} . It also follows from Kostant’s theory that $H^*(\mathfrak{g}_1, \mathbb{V}^*)$ is dual (as a representation of \mathfrak{g}_0) to $H^*(\mathfrak{g}_{-1}, \mathbb{V})$. Since we use highest weights of dual representations as labels, we can directly apply Kostant’s algorithm.

This describes the highest weights of irreducible components in the cohomology in terms of the actions of the elements of a subset W^p of the Weyl group of \mathfrak{g} .

In particular, $H^0(\mathfrak{g}_1, \mathbb{V}^*)$ is the irreducible representation of \mathfrak{g}'_0 whose highest weight is the restriction of the highest weight of \mathbb{V}^* , whence

$$(3.5) \quad H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{E}.$$

In particular, note that \mathbb{E} has acquired the structure of a G_0 -module.

To deal with the first cohomology, we have to consider elements of the Weyl group which have length one, i.e. are reflections corresponding to simple roots. The only simple reflection which lies in W^p is the one corresponding to α_0 . This means that $H^1(\mathfrak{g}_1, \mathbb{V}^*)$ is an irreducible representation of \mathfrak{g}'_0 , and its highest weight is obtained from the highest weight λ of \mathbb{V}^* by subtracting $(\ell + 1)\alpha_0$, where ℓ is the coefficient of ω_0 in the expansion of λ into a linear combination of fundamental weights. But by definition, $-\alpha_0$ is the highest weight of $\mathfrak{g}_{-1} = \mathfrak{g}'_1$, and we obtain

$$(3.6) \quad H^1(\mathfrak{g}_{-1}, \mathbb{V}) = \odot^k \mathfrak{g}_1 \odot \mathbb{E}.$$

There is a unique element in \mathfrak{g} whose adjoint action is given by multiplication by j on \mathfrak{g}_j for $j = -1, 0, 1$, called the grading element. The representation \mathbb{V} splits into eigenspaces for the action of this element, and it is convenient for our purposes to write this decomposition as

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N, \quad \text{in which } \mathbb{V}_0 = \mathbb{E} \text{ and } \mathfrak{g}_i \mathbb{V}_j \subseteq \mathbb{V}_{i+j}.$$

This is the algebraic source of (2.3) and the number N in Theorems 2.1 and 2.3. The explicit formulae for N in §2 may be obtained by observing that N depends linearly on the coefficients of the fundamental weights in expressing the highest weight and then verifying our formulae for the fundamental representations. By construction, the homomorphisms ∂ and δ^* decrease and increase this grading on \mathbb{V} , respectively. Now (3.5) says that $\mathbb{V}_i \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V}_{i-1}$ is injective $\forall i \geq 1$. The module $\odot^k \mathfrak{g}_1 \odot \mathbb{E}$ appears with multiplicity one in $\mathfrak{g}_1 \otimes \mathbb{V}$. Moreover, since \mathfrak{g}_1 increases the grading, $\odot^k \mathfrak{g}_1 \odot \mathbb{E}$ resides in $\mathfrak{g}_1 \otimes \mathbb{V}_{k-1}$. From (3.6), we conclude that

$$(3.7) \quad \mathbb{V}_i \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V}_{i-1} \xrightarrow{\partial} \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{V}_{i-2} \quad \text{is exact for } 1 \leq i \leq k-1 \text{ and } i > k.$$

Now define $\phi_0 : \mathbb{V}_0 \rightarrow \mathbb{E}$ as the identity and $\phi_i : \mathbb{V}_i \rightarrow \otimes^i \mathfrak{g}_1 \otimes \mathbb{E}$ inductively as the composition:–

$$\mathbb{V}_i \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V}_{i-1} \xrightarrow{\text{id} \otimes \phi_{i-1}} \otimes^i \mathfrak{g}_1 \otimes \mathbb{E}.$$

Also set $\mathbb{K} = \ker : \odot^k \mathfrak{g}_1 \otimes \mathbb{E} \rightarrow \odot^k \mathfrak{g}_1 \odot \mathbb{E}$, the kernel of the Cartan product.

Lemma 3.1. *The homomorphism $\phi_i : \mathbb{V}_i \rightarrow \otimes^i \mathfrak{g}_1 \otimes \mathbb{E}$*

- (1) *is injective for all $i \geq 0$,*
- (2) *has values in $\odot^i \mathfrak{g}_1 \otimes \mathbb{E}$,*
- (3) *is an isomorphism $\mathbb{V}_i \xrightarrow{\cong} \odot^i \mathfrak{g}_1 \otimes \mathbb{E}$, for $0 \leq i \leq k-1$,*
- (4) *is an isomorphism $\mathbb{V}_i \xrightarrow{\cong} (\odot^i \mathfrak{g}_1 \otimes \mathbb{E}) \cap (\odot^{i-k} \mathfrak{g}_1 \otimes \mathbb{K})$, for $i \geq k$.*

Proof. Statements (1)–(3) immediately follow by induction from (3.7). When $i = k$, however, the sequence in (3.7) is no longer exact. Rather, (3.6) implies that $\phi_k : \mathbb{V}_k \hookrightarrow \odot^k \mathfrak{g}_1 \otimes \mathbb{E}$ has $\odot^k \mathfrak{g}_1 \otimes \mathbb{E}$ as cokernel. This yields the isomorphism $\phi_k : \mathbb{V}_k \xrightarrow{\cong} \mathbb{K}$, which is (4) when $i = k$. For $i > k$ the exactness of (3.7) proves (4) by induction. \square

Let us denote by $\phi_i^{-1} : \odot^i \mathfrak{g}_1 \otimes \mathbb{E} \rightarrow \mathbb{V}_i$ the inverse of ϕ_i for $0 \leq i \leq k-1$. Then, by construction and since δ^* inverts ∂ on $\text{im}(\delta^*) = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N$, we have:–

Lemma 3.2. *Although $\delta^* \circ (\text{id} \otimes \phi_{i-1}^{-1})$ is defined on $\mathfrak{g}_1 \otimes \odot^{i-1} \mathfrak{g}_1 \otimes \mathbb{E}$, it coincides with ϕ_i^{-1} on $\odot^i \mathfrak{g}_1 \otimes \mathbb{E}$ for $1 \leq i \leq k-1$.*

We can also be more precise concerning the identification of 1st cohomology in (3.6). From the Hodge decomposition (3.3) and (3.4), the endomorphism π of $\mathfrak{g}_1 \otimes \mathbb{V}$ given by $\pi\varphi = \varphi - \delta^* \partial \varphi - \partial \delta^* \varphi$ is projection onto the unique irreducible G_0 -module isomorphic to $\odot^k \mathfrak{g}_1 \otimes \mathbb{E}$. To fix this isomorphism, we take

$$(3.8) \quad \odot^k \mathfrak{g}_1 \otimes \mathbb{E} \hookrightarrow \mathfrak{g}_1 \otimes \odot^{k-1} \mathfrak{g}_1 \otimes \mathbb{E} \xrightarrow{\text{id} \otimes \phi_{k-1}^{-1}} \mathfrak{g}_1 \otimes \mathbb{V}_{k-1} \xrightarrow{\pi} \ker(\partial) \cap \ker(\delta^*).$$

4. PROOF OF THE MAIN THEOREM

To prove Theorem 2.3, we shall use the algebra of §3 as follows. Recall that M is supposed to have a G'_0 -structure so any representation of G'_0 (and thus any representation of G_0 or G by restriction) induces an associated bundle on M . Of course, E should be the bundle associated to \mathbb{E} and we have already observed that the bundle associated to \mathfrak{g}_1 is the bundle of 1-forms Λ^1 . Now we may transfer the constructions and conclusions of §3 into geometry on M . The G -module \mathbb{V} induces a graded vector bundle

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_N$$

on M . The complex (3.1) induces a complex of vector bundle homomorphisms

$$(4.1) \quad 0 \longrightarrow V \xrightarrow{\partial} \Lambda^1 \otimes V \xrightarrow{\partial} \Lambda^2 \otimes V \xrightarrow{\partial} \cdots$$

and, similarly, (3.2) induces

$$(4.2) \quad 0 \longrightarrow V \xleftarrow{\delta^*} \Lambda^1 \otimes V \xleftarrow{\delta^*} \Lambda^2 \otimes V \xleftarrow{\delta^*} \cdots$$

so that $E = \ker \partial : V \longrightarrow \Lambda^1 \otimes V$ and (3.8) induces

$$(4.3) \quad \odot^k \Lambda^1 \otimes E \cong \frac{\ker \partial : \Lambda^1 \otimes V \longrightarrow \Lambda^2 \otimes V}{\text{im } \partial : V \longrightarrow \Lambda^1 \otimes V} = \ker(\partial) \cap \ker(\delta^*).$$

Lemma 3.1 part (3) yields

$$\phi_j : V_i \xrightarrow{\cong} \odot^j \Lambda^1 \otimes E \quad \text{for } 0 \leq j \leq k-1.$$

Lemma 3.1 part (4) identifies V_i with the classical prolongations (1.1) for $i \geq k$.

A splitting operator. According to the statement of Theorem 2.3 we should choose a connection ∇ on M that is compatible with the G'_0 -structure. From this, we obtain connections on all associated vector bundles, in particular on E and V . Being induced from a principal G'_0 -connection, they respect the grading on V and commute with the homomorphisms in (4.1) and (4.2). We shall denote all of these linear connections by ∇ .

To prove Theorem 2.3 we shall construct $L : E = V_0 \rightarrow V$, an N^{th} order linear differential operator, so that $\sigma \mapsto L\sigma$ induces the isomorphism (2.4). Since the isomorphism in the other direction should simply be given by $\sigma = \Sigma_0$, the component of Σ in $V_0 = E$, the composition $\sigma \mapsto (L\sigma)_0$ should be the identity. For this reason we refer to L as a ‘splitting operator’. Its definition is

$$(4.4) \quad L\sigma = \sum_{i=0}^N (-1)^i (\delta^* \circ \nabla)^i \sigma.$$

Of course, this an N^{th} order linear differential operator. Moreover, since σ is a section of $E = V_0$, we see that $(\delta^* \circ \nabla)^i \sigma$ is a section of V_i and that a section $\Sigma = (\Sigma_0, \Sigma_1, \dots, \Sigma_N)$ of V is of the form $L\sigma$ if and only if

$$(4.5) \quad \Sigma_0 = \sigma \quad \text{and} \quad \Sigma_i = -\delta^* \nabla \Sigma_{i-1} \quad \text{for } 1 \leq i \leq N.$$

Next, we define the connection $\tilde{\nabla}$ on V as $\tilde{\nabla} = \nabla + \partial$. So we simply add the algebraic operator $\partial : V \rightarrow \Lambda^1 \otimes V$ to the component-wise connection ∇ . Of course, this defines a linear connection. Note, however, that whilst for a section Σ_i of V_i , the covariant derivative $\nabla \Sigma_i$ is a 1-form with coefficients in V_i , the algebraic term $\partial \Sigma_i$ is a 1-form with coefficients in V_{i-1} . Otherwise put, for a section $\Sigma = (\Sigma_0, \Sigma_1, \dots, \Sigma_N)$ of V the component of $\tilde{\nabla} \Sigma$ taking values in $\Lambda^1 \otimes V_i$ is $\nabla \Sigma_i + \partial \Sigma_{i+1}$ for $i < N$, while for $i = N$ we simply obtain $\nabla \Sigma_N$.

We should now compute the curvature of $\tilde{\nabla}$. The curvatures of all the connections ∇ are induced by the same 2-form R , which acts on the sections of any associated bundle. On the other hand, for the induced connection on TM we also have the torsion, which we view as a section of $\Lambda^2 \otimes TM$. (In the affine and Riemannian cases we can always choose ∇ to be torsion-free but not with a general G'_0 -structure).

Lemma 4.1. *Let R be the curvature of the connections ∇ and T the torsion of the connection ∇ on TM . Let $\tilde{R} \in \Gamma(\Lambda^2 \otimes \text{End}(V, V))$ be the curvature of $\tilde{\nabla}$. Then for vector fields ξ and η on M and a section $\Sigma = (\Sigma_0, \dots, \Sigma_N)$ of V , the V_i -component of $\tilde{R}(\xi, \eta)\Sigma$ is given by*

$$R(\xi, \eta)\Sigma_i + (\partial \Sigma_{i+1})(T(\xi, \eta)).$$

In particular, $\tilde{\nabla}$ is flat if and only if ∇ has zero curvature and torsion.

Proof. By definition, $\tilde{\nabla}_\xi \tilde{\nabla}_\eta \Sigma = \tilde{\nabla}_\xi (\nabla_\eta \Sigma + (\partial \Sigma)(\eta))$. Writing out the first operator as $\nabla + \partial$, we obtain

$$(4.6) \quad \nabla_\xi \nabla_\eta \Sigma + \nabla_\xi ((\partial \Sigma)(\eta)) + (\partial(\nabla_\eta \Sigma))(\xi) + (\partial(\partial \Sigma)(\eta))(\xi).$$

To obtain $\tilde{R}(\xi, \eta)\Sigma$ we should subtract the same sum with ξ and η exchanged and then subtract

$$(4.7) \quad \tilde{\nabla}_{[\xi, \eta]}\Sigma = \nabla_{[\xi, \eta]}\Sigma + (\partial\Sigma)([\xi, \eta]).$$

On the Lie algebra level $(\partial v)(Y) = Yv$ and thus $\partial((\partial v)(Y))(Z) = Z(Yv)$, which is symmetric in Y and Z since \mathfrak{g}_1 is an Abelian Lie algebra. Hence the last term in (4.6) vanishes after exchange and subtraction. Also, we may write

$$\nabla_\xi((\partial\Sigma)(\eta)) = (\nabla_\xi(\partial\Sigma))(\eta) + (\partial\Sigma)(\nabla_\xi\eta)$$

and, since ∂ is parallel, rewrite the first summand as $(\partial(\nabla_\xi\Sigma))(\eta)$. But this cancels with one of the terms from the other summand of the form (4.6). Altogether, we see that the last three terms in the two summands of the form (4.6) together contribute $(\partial\Sigma)(\nabla_\xi\eta - \nabla_\eta\xi)$. Subtracting the last term in (4.7) we obtain $(\partial\Sigma)(T(\xi, \eta))$ by definition of the torsion. On the other hand, the first terms in the two summands of the form (4.6) add up with the remaining term of (4.7) to $R(\xi, \eta)\Sigma$. Now, the result follows by splitting into components. \square

Having at hand the operators L and $\tilde{\nabla}$, we now define an operator $E = V_0 \rightarrow \Lambda^1 \otimes V$ as the composition $\tilde{\nabla} \circ L$. From (4.3) we know that $F = \odot^k \Lambda^1 \odot E$ sits as the subbundle $\ker(\partial) \cap \ker(\delta^*)$ in $\Lambda^1 \otimes V$, and we can use the algebraic Hodge structure to define a projection onto this subbundle. Indeed, in §3 we arranged that this projection be explicitly given by $\varphi \mapsto \pi\varphi \equiv \varphi - \delta^*\partial\varphi - \partial\delta^*\varphi$. Using (3.8), we now define a differential operator $D^\nabla : E \rightarrow F$ by $D^\nabla \equiv (-1)^{k-1}(\text{id} \otimes \phi_{k-1}) \circ \pi \circ \tilde{\nabla} \circ L$. The main properties of L and D^∇ are collected in:–

Proposition 4.2.

- (1) A section $\Sigma = (\Sigma_0, \dots, \Sigma_N)$ of V lies in the image of L if and only if $\delta^*(\tilde{\nabla}\Sigma) = 0$ and, if this is the case, then $\Sigma = L(\Sigma_0)$.
- (2) Mapping $\sigma \in \Gamma(E)$ to the components of $L\sigma$ in $V_0 \oplus \dots \oplus V_i$ induces a vector bundle homomorphism $J^i V_0 \rightarrow V_0 \oplus \dots \oplus V_i$, which is an isomorphism for $i < k$.
- (3) The differential operator $D^\nabla : E \rightarrow F$ is of order k and its symbol is the Cartan product.

Proof. (1) Since $V \xleftarrow{\delta^*} \Lambda^1 \otimes V$ inverts ∂ on $\text{im}(\partial)$, we may easily compute the components of $\delta^*(\tilde{\nabla}\Sigma)$. We find that $\delta^*(\tilde{\nabla}\Sigma)_0 = 0$ and, for $1 \leq i \leq N$,

$$\delta^*(\tilde{\nabla}\Sigma)_i = \delta^*((\tilde{\nabla}\Sigma)_{i-1}) = \delta^*(\nabla\Sigma_{i-1} + \partial\Sigma_i) = \delta^*(\nabla\Sigma_{i-1}) + \Sigma_i$$

whose vanishing is exactly the criterion (4.5) we already found for $\Sigma = (\Sigma_0, \dots, \Sigma_N)$ to be in the range of L . In (4.5) we also observed that, in this case, $\Sigma = L(\Sigma_0)$.

(2) By construction, mapping σ to the V_i -component of $L\sigma$ is a linear differential operator of order at most i . Thus, we obtain $J^i V_0 \rightarrow V_0 \oplus \dots \oplus V_i$ for all $i = 0, \dots, N$. We can compute the leading terms of $(L\sigma)_i$ quite explicitly as follows. Firstly, $(L\sigma)_0$ is just σ , a section of $V_0 = E$. Next, from its definition (4.4), we have $(L\sigma)_1 = -\delta^*\nabla\sigma$. Assuming that $1 < k$, we see from Lemma 3.2 that δ^* coincides with ϕ_1^{-1} . Therefore,

$(L\sigma)_1 = -\phi_1^{-1}\nabla\sigma$, a section of V_1 . Now $\nabla(L\sigma)_1 = -(\text{id} \otimes \phi_1^{-1})\nabla(\nabla\sigma)$, where $\nabla(\nabla\sigma)$ is a section of $\Lambda^1 \otimes \Lambda^1 \otimes E$. But if we decompose

$$\Lambda^1 \otimes \Lambda^1 \otimes E = (\odot^2 \Lambda^1 \otimes E) \oplus (\Lambda^2 \otimes E),$$

then the component $\nabla \wedge \nabla \sigma$ of $\nabla(\nabla\sigma)$ is a zeroth order operator (made from curvature and torsion). If $2 < k$, then from Lemma 3.2 we conclude that

$$(L\sigma)_2 = -\delta^*\nabla(L\sigma)_1 = \delta^*(\text{id} \otimes \phi_1^{-1})\nabla(\nabla\sigma) = \phi_2^{-1}\nabla \odot \nabla\sigma + \text{lots},$$

where ‘lots’ stands for ‘lower order terms’ (in this case zeroth order). By induction, we claim that

$$(4.8) \quad (L\sigma)_i = (-1)^i \phi_i^{-1} \underbrace{\nabla \odot \nabla \odot \cdots \odot \nabla}_i \sigma + \text{lots}, \quad \text{for } 0 \leq i \leq k-1.$$

For the inductive step, observe that

$$\nabla_a \nabla_b \nabla_c \cdots \nabla_d = \nabla_{(a} \nabla_b \nabla_c \cdots \nabla_d) + \text{lots}$$

as differential operators. Therefore,

$$\begin{aligned} \nabla(L\sigma)_{i-1} &= \nabla((-1)^{i-1} \phi_{i-1}^{-1} \underbrace{\nabla \odot \nabla \odot \cdots \odot \nabla}_{i-1} \sigma + \text{lots}) \\ &= (-1)^{i-1} (\text{id} \otimes \phi_{i-1}^{-1}) (\underbrace{\nabla \odot \nabla \odot \nabla \odot \cdots \odot \nabla}_i \sigma + \text{lots}) \end{aligned}$$

and so, for $i < k$,

$$\begin{aligned} (L\sigma)_i &= -\delta^*\nabla(L\sigma)_{i-1} = (-1)^i \delta^*(\text{id} \otimes \phi_{i-1}^{-1}) (\underbrace{\nabla \odot \nabla \odot \nabla \odot \cdots \odot \nabla}_i \sigma + \text{lots}) \\ &= (-1)^i \phi_i^{-1} \underbrace{\nabla \odot \nabla \odot \nabla \odot \cdots \odot \nabla}_i \sigma + \text{lots}, \end{aligned}$$

the last equality coming from Lemma 3.2. We have shown (4.8) and, clearly, this is sufficient to establish (2).

(3) The projection

$$\Lambda^1 \otimes E \ni \varphi \mapsto \pi\varphi \equiv \varphi - \delta^*\partial\varphi - \partial\delta^*\varphi \in \ker(\partial) \cap \ker(\delta^*)$$

kills $\text{im}(\partial)$ so $D^\nabla\sigma = (-1)^{k-1}(\text{id} \otimes \phi_{k-1})(\pi(\nabla(L\sigma)_{k-1}))$. From (3.8) and (4.8) we see that

$$D^\nabla\sigma = \pi(\underbrace{\nabla(\nabla \odot \nabla \odot \cdots \odot \nabla)}_{k-1} \sigma + \text{lots}),$$

where now $\pi : \Lambda^1 \otimes \odot^{k-1} \Lambda^1 \otimes E \rightarrow \odot^k \Lambda^1 \otimes E = F$ denotes canonical projection onto this irreducible tensor bundle. It is now clear the D^∇ has the Cartan product as its symbol. \square

First step. Now we can perform the first step in rewriting the equation $D\sigma = 0$ on sections of E in terms of sections of V :-

Proposition 4.3. *Let $D : E \rightarrow F$ be a k^{th} order semilinear differential operator as in Theorem 2.3. Then there is a fibre bundle homomorphism $A : V_0 \oplus \cdots \oplus V_{k-1} \rightarrow F$ such that $\sigma \mapsto L\sigma$ induces a set bijection*

$$\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + A(\Sigma) \in \Gamma(\text{im}(\delta^*))\}.$$

If D is linear, then A is linear, i.e. a vector bundle homomorphism.

Proof. From part (3) of Proposition 4.2 we conclude that the operators D and D^∇ have the same symbol. Therefore, we may write $D\sigma = D^\nabla\sigma + \Psi(j^{k-1}\sigma)$ for some bundle map $\Psi : J^{k-1}E \rightarrow F$. By part (2) of Proposition 4.2 there is a unique fibre bundle map $A : V_0 \oplus \cdots \oplus V_{k-1} \rightarrow F$ (which we may extend trivially to V) such that $\Psi(j^{k-1}\sigma) = (-1)^{k-1}A(L\sigma)$ for all $\sigma \in \Gamma(E)$. Of course, if D is linear, then Ψ is a vector bundle homomorphism and hence A is a vector bundle homomorphism too.

Now $\tilde{\nabla}L\sigma$ is a section of $\ker(\delta^*)$ by part (1) of Proposition 4.2 and the same is true for $A(L\sigma)$ since, by construction, A even has values in $F = \ker(\partial) \cap \ker(\delta^*)$. The last observation even shows that $\pi(A(L\sigma)) = A(L\sigma)$ for any σ . Hence, vanishing of $D\sigma = (-1)^{k-1}\pi(\tilde{\nabla}L\sigma + A(L\sigma))$ is equivalent to $\tilde{\nabla}L\sigma + A(L\sigma)$ being a section of the subbundle $\text{im}(\delta^*)$.

Conversely, assume that $\Sigma \in \Gamma(V)$ has the property that $\tilde{\nabla}\Sigma + A(\Sigma)$ is a section of $\text{im}(\delta^*)$. Then, in particular it is a section of $\ker(\delta^*)$ and since $\delta^*(A(\Sigma))$ always vanishes we conclude that $\delta^*(\tilde{\nabla}\Sigma) = 0$. By part (1) of Proposition 4.2 this implies $\Sigma = L(\Sigma_0)$ and, as above, we see that $D(\Sigma_0) = 0$. \square

Second step. The next step in the procedure is to show that, if $\tilde{\nabla}\Sigma + A(\Sigma)$ is a section of $\text{im}(\delta^*)$, then its value can be actually computed. We shall do this in a more general situation than needed for the proof of Theorem 2.3. The motivation for this is that if A is linear, then it can be absorbed into the connection, so dealing with a more general class of connections is helpful. Notice that any smooth section of the bundle $\text{im}(\delta^*) \subset \Lambda^1 \otimes V$ can be written as $\delta^*\psi$ for some smooth $\psi \in \Gamma(\Lambda^2 \otimes V)$.

Proposition 4.4. *Let ∇ be a linear connection on V such that for each $i = 0, \dots, N$ and each smooth section $\Sigma \in \Gamma(V)$ that has values in V_i only, the covariant derivative $\nabla\Sigma$ lies in $\Gamma(\Lambda^1 \otimes (V_i \oplus \cdots \oplus V_N))$ and put $\tilde{\nabla} = \nabla + \partial$. Let $A : V \rightarrow \Lambda^1 \otimes V$ be a fibre bundle map such that for $v = (v_0, v_1, \dots, v_N) \in V$ the component of $A(v)$ in $\Lambda^1 \otimes V_i$ depends only on v_0, \dots, v_i . Then there is a fibre bundle map*

$$B : J^N V = J^N V_0 \oplus \cdots \oplus J^N V_N \rightarrow \Lambda^1 \otimes V$$

such that $\tilde{\nabla}\Sigma + A(\Sigma) \in \Gamma(\text{im}(\delta^))$ is equivalent to $\tilde{\nabla}\Sigma + B(j^N\Sigma) = 0$. Moreover, the component B_i of B with values in $\Lambda^1 \otimes V_i$ factors through*

$$J^i V_0 \oplus J^{i-1} V_1 \oplus \cdots \oplus J^1 V_{i-1} \oplus V_i.$$

If A is linear then B can be chosen to be a vector bundle homomorphism.

Proof. Suppose that $\tilde{\nabla}\Sigma + A(\Sigma) + \delta^*\psi = 0$ for some $\psi \in \Gamma(\Lambda^2 \otimes V)$. Recall that the linear connection $\tilde{\nabla}$ on V extends to an operation $d^{\tilde{\nabla}}$ on V -valued forms called the covariant exterior derivative. For $\alpha \in \Gamma(\Lambda^1 \otimes V)$ the covariant exterior derivative is explicitly given by

$$d^{\tilde{\nabla}}\alpha(\xi, \eta) = \tilde{\nabla}_\xi(\alpha(\eta)) - \tilde{\nabla}_\eta(\alpha(\xi)) - \alpha([\xi, \eta]),$$

for all vector fields ξ and η on M . Clearly, $d^{\tilde{\nabla}}$ is a first order differential operator. Moreover, if $\alpha = \tilde{\nabla}\Sigma$ for some $\Sigma \in \Gamma(V)$, then this definition immediately implies that $d^{\tilde{\nabla}}\tilde{\nabla}\Sigma(\xi, \eta) = \tilde{R}(\xi, \eta)(\Sigma)$.

Now we define B inductively as follows. We put $B_0(\Sigma) \equiv A_0(\Sigma)$. By assumption, this is algebraic (i.e. of order zero) in Σ and depends only on the component Σ_0 . Let $\tilde{R} \bullet \Sigma$ denote the V -valued 2-form $(\xi, \eta) \mapsto \tilde{R}(\xi, \eta)(\Sigma)$. Having defined the components B_j for $j < i$, take the component $(\tilde{R} \bullet \Sigma + d^{\tilde{\nabla}}(B_{i-1}(\Sigma) + \cdots + B_0(\Sigma)))_{i-1}$ in $\Gamma(\Lambda^2 \otimes V_{i-1})$ and define

$$(4.9) \quad B_i(\Sigma) \equiv A_i(\Sigma) - \delta^* \left((\tilde{R} \bullet \Sigma + d^{\tilde{\nabla}}(B_{i-1}(\Sigma) + \cdots + B_0(\Sigma)))_{i-1} + \partial(A_i(\Sigma)) \right).$$

By assumption, A is algebraic in Σ and $A_i(\Sigma)$ depends only on the components $\Sigma_0, \dots, \Sigma_i$. To understand the dependence of $\tilde{R} \bullet \Sigma$, note that by assumption on ∇ , the form $(\tilde{\nabla}\Sigma)_j$ depends only on $\Sigma_0, \dots, \Sigma_{j+1}$. Hence the V_j -component of $\tilde{R}(\xi, \eta)(\Sigma)$ depends at most on $\Sigma_0, \dots, \Sigma_{j+2}$ (since computing curvature needs two derivatives). However, as in the proof of Lemma 4.1, we see that for $\Sigma \in \Gamma(V_{j+2})$ the only contribution of $\tilde{R}(\xi, \eta)(\Sigma)$ in V_j is $\partial((\partial\Sigma)(\eta))(\xi) - \partial((\partial\Sigma)(\xi))(\eta)$ and we have shown that this vanishes. Hence, the term $(\tilde{R} \bullet \Sigma)_{i-1}$ depends only on $\Sigma_0, \dots, \Sigma_i$. Assuming inductively that for $\ell \leq i-1$, the value $B_\ell(\Sigma)(x)$ depends only on $j_x^\ell \Sigma_0, j_x^{\ell-1} \Sigma_1, \dots, \Sigma_\ell(x)$ for each $x \in M$, we immediately conclude from the fact that $d^{\tilde{\nabla}}$ is first order that $B_i(\Sigma)(x)$ depends only on $j_x^i \Sigma_0, j_x^{i-1} \Sigma_1, \dots, \Sigma_i(x)$. Hence our components B_i define a bundle map B whose dependence on jets is exactly as required. Moreover, if A is linear, then obviously B is a vector bundle homomorphism.

Next we show that the equation $\tilde{\nabla}\Sigma + B(\Sigma) = 0$ is equivalent to $\tilde{\nabla}\Sigma + A(\Sigma)$ being a section of $\text{im}(\delta^*)$. On the one hand, we see from the definition (4.9) that $A(\Sigma) - B(\Sigma)$ is a section of $\text{im}(\delta^*)$ for any $\Sigma \in \Gamma(V)$. Thus $\tilde{\nabla}\Sigma + B(\Sigma) = 0$ implies that $\tilde{\nabla}\Sigma + A(\Sigma)$ has values in $\text{im}(\delta^*)$.

Conversely, assume that $\tilde{\nabla}\Sigma + A(\Sigma) + \delta^*\psi = 0$ for some $\psi \in \Gamma(\Lambda^2 \otimes V)$. Then we claim that $A(\Sigma) + \delta^*\psi = B(\Sigma)$. Since δ^* has values in $\Lambda^1 \otimes (V_1 \oplus \cdots \oplus V_N)$ and, by definition, $B_0(\Sigma) = A_0(\Sigma)$, this is true for the component in $\Gamma(\Lambda^1 \otimes V_0)$.

To proceed inductively, we need one more observation concerning $d^{\tilde{\nabla}}$. Suppose that $\alpha \in \Gamma(\Lambda^1 \otimes (V_i \oplus \cdots \oplus V_N))$. Then, from the formula for $d^{\tilde{\nabla}}$, it is manifest that $d^{\tilde{\nabla}}\alpha \in \Gamma(\Lambda^2 \otimes (V_{i-1} \oplus \cdots \oplus V_N))$ and the component $(d^{\tilde{\nabla}}\alpha)_{i-1}$ is easy to compute: expanding $\tilde{\nabla} = \nabla + \partial$ in the above formula, we see that

$$(d^{\tilde{\nabla}}\alpha)_{i-1}(\xi, \eta) = \partial(\alpha_i(\eta))(\xi) - \partial(\alpha_i(\xi))(\eta)$$

and, looking at the definition of ∂ , this means that $(d^{\tilde{\nabla}}\alpha)_{i-1} = \partial(\alpha_i)$.

Now suppose inductively that $(A(\Sigma) + \delta^*\psi)_\ell = B_\ell(\Sigma)$ for $\ell = 0, \dots, i-1$. Denoting by the subscript $\geq i$ the components with values in $\Lambda^1 \otimes (V_i \oplus \dots \oplus V_N)$ we may rewrite the equation $\tilde{\nabla}\Sigma + A(\Sigma) + \delta^*\psi = 0$ as

$$\tilde{\nabla}\Sigma + B_0(\Sigma) + \dots + B_{i-1}(\Sigma) + A_{\geq i}(\Sigma) + (\delta^*\psi)_{\geq i} = 0.$$

Applying $d^{\tilde{\nabla}}$ and looking at the component in $\Lambda^2 \otimes V_{i-1}$ we obtain

$$0 = (\tilde{R} \bullet \Sigma + d^{\tilde{\nabla}}(B_0(\Sigma) + \dots + B_{i-1}(\Sigma)))_{i-1} + \partial(A_i(\Sigma)) + \partial(\delta^*\psi)_i.$$

Applying δ^* , the last term gives $(\delta^*\psi)_i$ and from (4.9) we see $A_i(\Sigma) + (\delta^*\psi)_i = B_i(\Sigma)$, which completes the proof. \square

Third step. The final reduction is now done by solving component by component:–

Proposition 4.5. *Suppose that ∇ is a connection on V satisfying the hypothesis of Proposition 4.4 and*

$$B : J^N V = J^N V_0 \oplus \dots \oplus J^N V_N \rightarrow \Lambda^1 \otimes V$$

*is a fibre bundle map such that the component B_i of B in $T^*M \otimes V_i$ factors through $J^i V_0 \oplus J^{i-1} V_1 \oplus \dots \oplus J^1 V_{i-1} \oplus V_i$.*

Then there is a fibre bundle map $C : V \rightarrow \Lambda^1 \otimes V$ such that $\tilde{\nabla}\Sigma + B(\Sigma) = 0$ is equivalent to $\tilde{\nabla}\Sigma + C(\Sigma) = 0$. If B is a vector bundle homomorphism, then also C can be chosen to be a vector bundle homomorphism.

Proof. Choosing a connection on TM , we may form iterated covariant derivatives of sections of V and by the assumptions on B we may write the components of B (with the obvious meaning of subscripts) as

$$B_i(\Sigma) = B_i(\Sigma_{\leq i}, (\tilde{\nabla}\Sigma)_{\leq i-1}, \dots, (\tilde{\nabla}^i \Sigma)_0).$$

The component in $\Lambda^1 \otimes V_0$ of $\tilde{\nabla}\Sigma + B(\Sigma)$ is given by $(\tilde{\nabla}\Sigma)_0 + B_0(\Sigma_0)$ and we simply put $C_0(\Sigma) \equiv B_0(\Sigma_0)$. The next component has the form $(\tilde{\nabla}\Sigma)_1 + B_1(\Sigma_0, \Sigma_1, (\tilde{\nabla}\Sigma)_0)$. Defining $C_1(\Sigma_0, \Sigma_1) \equiv B_1(\Sigma_0, \Sigma_1, -C_0(\Sigma_0))$, we see that vanishing of $(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq 1}$ is equivalent to vanishing of $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq 1}$, where $C = C_0 + C_1$.

Let us inductively assume that $i > 1$ and we have found a fibre bundle map $C : V \rightarrow \Lambda^1 \otimes (V_0 \oplus \dots \oplus V_{i-1})$ such that vanishing of $(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq i-1}$ is equivalent to vanishing of $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i-1}$ and such that the component $C_j(\Sigma)$ depends only on $\Sigma_0, \dots, \Sigma_j$ for each $j < i$. Let us also assume that we have derived, for any Σ such that $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i-1} = 0$, formulae for $(\tilde{\nabla}^\ell \Sigma)_{\leq i-\ell}$ as algebraic expressions in $\Sigma_{\leq i}$.

So by assumption we have formulae for all the terms going into B_i as algebraic operators in $\Sigma_{\leq i}$ and inserting these formulae, we obtain a bundle map C_i with values in $\Lambda^1 \otimes V_i$, which depends only on $\Sigma_{\leq i}$. By construction, vanishing of $(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq i}$ is equivalent to vanishing of $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i}$. Suppose now that Σ satisfies this equation. By the assumption on $\tilde{\nabla}$, vanishing of $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i}$ implies vanishing of $(\tilde{\nabla}^\ell(\tilde{\nabla}\Sigma + C(\Sigma)))_{\leq i-\ell}$ for each $\ell = 1, \dots, i$. Similarly, $(\tilde{\nabla}^\ell(C(\Sigma)))_{\leq i-\ell}$ depends algebraically on $C(\Sigma)_{\leq i}$, to first order on $C(\Sigma)_{\leq i-1}$ and so on. Hence, expanding this, it can be written as an expression in $\Sigma_{\leq i}, (\tilde{\nabla}\Sigma)_{\leq i-1}, \dots, (\tilde{\nabla}^\ell \Sigma)_{\leq i-\ell}$ and we have

algebraic formulae for all these by inductive hypothesis. Thus we see that vanishing of $(\tilde{\nabla}^\ell(\tilde{\nabla}\Sigma + C(\Sigma)))_{\leq i-\ell}$ gives us an algebraic expression for $(\tilde{\nabla}^{\ell+1}\Sigma)_{\leq i-\ell}$ for each $\ell = 1, \dots, i$, which completes the inductive step. Of course, linearity is never lost in this process, so if one starts with a linear operator B , one will end up with a vector bundle homomorphism C . \square

Since the output of each step of our rewriting procedure is a special case of the input of the next step, this completes the proof of Theorem 2.3.

Remark. As far as the proof of Theorem 2.3 is concerned, the only rôle that ∂^* played was in constructing δ^* as a left inverse to ∂ . Of course, the definition of ∂^* and the resulting algebraic Hodge theory is extremely natural but, in defining ∂ , only the structure of \mathbb{V} as a \mathfrak{g}_{-1} -module is needed. It is also important that ∂ respect the G_0 -action but, as far as δ^* goes, any other G_0 -invariant splittings would work just as well. In practise, there can be considerably simpler *ad hoc* choices.

Remark. The dimension bound of Corollary 2.4 is sharp. The bound is attained by choosing a manifold M endowed with a G'_0 -structure and a compatible connection, such that all the connections ∇ have zero curvature and the connection $\tilde{\nabla}$ on TM also has zero torsion. Such an example is always provided by the constant G'_0 -structure on \mathbb{R}^n (where $n = \dim(\mathfrak{g}_{-1})$) with the standard flat connection. In this case, let us consider the equation $D^\nabla\sigma = 0$. Then our first step of rewriting simply leads to $\tilde{\nabla}\Sigma + \delta^*\psi = 0$. Applying $\delta^*d^{\tilde{\nabla}}$, the first term does not give any contribution, since $\tilde{\nabla}$ has zero curvature by Lemma 4.1. This implies that $\delta^*\psi = 0$. Hence the whole rewriting is already finished and we conclude that the differential splitting $L : E \rightarrow V$ induces a bijection between solutions of $D^\nabla\sigma = 0$ and sections $\Sigma \in \Gamma(V)$ that are parallel for the flat connection $\tilde{\nabla}$. Locally, a flat connection always has the maximal dimension for its space of parallel sections.

Remark. The flat case also shows that the bound N on the order of the jet of σ at a point $p \in M$ needed uniquely to specify a solution of $D^\nabla\sigma = 0$ is sharp. To see this, note that $\tilde{\nabla}\Sigma = 0$ in the flat case is equivalent to $\nabla\Sigma_i = -\partial\Sigma_{i+1}$, for all i . Therefore,

$$\Sigma|_p \in (V_N)_p \Rightarrow \nabla\Sigma|_p \in (V_{N-1} \oplus V_N)_p \Rightarrow \dots \Rightarrow \nabla^{N-1}\Sigma|_p \in (V_1 \oplus \dots \oplus V_N)_p,$$

whence $\nabla^{N-1}\sigma|_p = \nabla^{N-1}\Sigma_0|_p = 0$. But, since $\tilde{\nabla}$ is flat, there is no problem finding a parallel section Σ of V with $\Sigma|_p$ lying in $(V_N)_p$.

Remark. In this flat case, the operator D^∇ is the first in the so-called ‘Bernstein-Gelfand-Gelfand (BGG) resolution’ and one motivation for our study comes from analogues of these first operators on almost Hermitian symmetric manifolds [1] or, more generally, on parabolic geometries [3, 5]. By construction, these analogues are invariant linear differential operators having the same symbol as in the flat case. The G'_0 -geometries studied in this article cover the almost Hermitian symmetric case so Theorem 2.3 covers the first BGG operators on these geometries. This includes the various so-called ‘conformal Killing’ or ‘twistor’ equations in conformal geometry.

Remark. A useful viewpoint on the outcome of Theorem 2.3 is that it restricts the possible jets of σ that might be specified at a point for a solution of $D\sigma = 0$. In the

flat case and the equation $D^\nabla\sigma = 0$, these jets may be freely specified. In general, there are further constraints, which may be obtained by cross-differentiation of the closed system $\tilde{\nabla}\Sigma + \Phi(\Sigma) = 0$.

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