Élie Cartan and Geometric Duality

by

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0. Introduction

It is a great honor for me to be asked to give a lecture about the work of Élie Cartan at the institute that was founded in his name. When I was asked to do this, I was immediately beset by doubts as to whether I could actually say anything of value. Nowadays, the work of Cartan is so widely and deservedly praised and studied, the originality and central importance of his insights so universally acknowledged, that there would seem to be little point in my adding my own small testimonial, however deeply felt, to his pile of honors.

Mathematicians better qualified have written in-depth surveys of Cartan's work and comments on its significance for modern geometry. In addition to the masterful obituaries by Chern and Chevalley [CC] and by J. H. C. Whitehead [Wh], written shortly after his death, there has been a recent book [AR] and a beautiful recent survey containing much valuable information about his life and work by P. Libermann [Li]. Moreover, we have the survey by Élie Cartan himself [Ca13], in which he outlines the great themes of his work in a style that I cannot hope to match.

Thus, my plan in this lecture is not to try to describe the whole tapestry of Cartan's work but, instead, to follow one tiny thread, explain its significance for Cartan as best I can and then expose some of the ramifications of this work in modern geometry. The thread that I will follow is the geometry of the rank 2 simple Lie groups, particularly the geometry of G_2.

1. Lie groups

In 1893, Cartan published his first papers, including the famous "Über die einfachen Transformationsgruppen" [Ca1], in which he announces, in particular, that he has found examples of Lie groups corresponding to each of the 'exceptional' root systems found by Killing. One of the things that I find remarkable about this work is the way that Cartan found interpretations of the exceptional groups as transformation groups.

I want to describe this in a little detail because it will serve as the basis of the rest of my talk. Of course, this material is extremely well-known nowadays and there are many excellent expositions. Helgason's masterful treatment [He] is also an indispensable guide to the literature. I apologize in advance to the experts in the audience, for whom this will be an unnecessary review.

I want to begin by recalling some of the origins of the theory of Lie groups. For Sophus Lie, a *transformation group* in n-space was essentially a set of (local) transformations of $\mathbb{R}^n$ (or $\mathbb{C}^n$) that is closed under composition and inverse.

Examples in $\mathbb{R}^n$*:*

1. Translations: $f_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$,
2. Linear: $f_A(\mathbf{x}) = A\mathbf{x}$,
3. Affine: $f_{A,\mathbf{b}}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$,
4. Projective:

$$f_M(\mathbf{x}) = \frac{A\mathbf{x} + \mathbf{b}}{\mathbf{c}\cdot\mathbf{x} + d}, \qquad M = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{c} & d \end{pmatrix} \in \mathrm{SL}(n{+}1, \mathbb{R})$$

(not defined when $\mathbf{c}\cdot\mathbf{x} + d = \mathbf{0}$.)

A *Lie transformation group* depends smoothly on a finite number of parameters, as in the above cases. We will also assume it to be generated by elements near the identity.

If $f_t : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth curve in the group with $f_0(\mathbf{x}) = \mathbf{x}$, then

$$f_t(\mathbf{x}) = \mathbf{x} + t\,X(\mathbf{x}) + O(t^2).$$

Lie considered X as a *infinitesimal generator* vector field (summation understood):

$$X = X^i \frac{\partial}{\partial x^i}.$$

For example, when $n = 1$, and one defines a curve of projective transformations of $\mathbb{R}^1$ by taking

$$M(t) = \begin{pmatrix} 1 + ta & tb \\ tc & 1 + td \end{pmatrix}$$

then

$$\begin{aligned} f_{M(t)}(x) &= \frac{(1{+}ta)x + tb}{tcx + (1{+}td)} \\ &= x + t\,(b + (a{-}d)\,x - c\,x^2) + O(t^2), \end{aligned}$$

so the infinitesimal generators of the linear fractional transformation group are of the form

$$X = (a + bx + cx^2)\frac{\partial}{\partial x}\,.$$

They form a three dimensional linear space. This explains the relationship of the linear system

$$\begin{pmatrix} \dot u(t) \\ \dot v(t) \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

to the (nonlinear) Riccati equation for $x = u/v$,

$$\dot x(t) = b(t) + \big(a(t){-}d(t)\big)\,x(t) - c(t)\,x(t)^2.$$

Lie showed how this classical 'linearization' of the Riccati equation generalized to an arbitrary Lie transformation group and made it the basis of his method of integrating ordinary and partial differential equations.

As part of his general program, Lie showed that the flow of an infinitesimal generator X generates a 1-dimensional subgroup of the given Lie transformation group, that the vector fields whose flows lie in the group form a finite dimensional vector space of vector fields $\mathfrak{g}$ and that every element of the group that is sufficiently close to the identity is part of a flow generated by an element of $\mathfrak{g}$.

The group multiplication in a Lie group need not be abelian, of course, and Lie found that this was reflected in $\mathfrak{g}$ as follows: If $f_t, g_t : \mathbb{R}^n \to \mathbb{R}^n$ are smooth curves in the group, with $f_0(\mathbf{x}) = g_0(\mathbf{x}) = \mathbf{x}$ and

$$\begin{aligned} f_t(\mathbf{x}) &= \mathbf{x} + t\,X(\mathbf{x}) + O(t^2) \\ g_t(\mathbf{x}) &= \mathbf{x} + t\,Y(\mathbf{x}) + O(t^2) \end{aligned}$$

then

$$g_{-t} \circ f_{-t} \circ g_t \circ f_t(\mathbf{x}) = \mathbf{x} + \tfrac{1}{2}t^2\,Z(\mathbf{x}) + O(t^3)$$

where $Z = [X, Y]$ is (now) called the *Lie bracket* of X and Y. It is another infinitesimal generator of the group and can be calculated directly in terms of X and Y as

$$Z = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}\,.$$

Lie proved that a finite dimensional linear subspace $\mathfrak{g}$ of vector fields on $\mathbb{R}^n$ that is closed under Lie bracket is the space of infinitesimal generators of a (local) Lie group.

In addition to the evident skewsymmetry $[X, Y] = -[Y, X]$, the Lie bracket satisfies the *Jacobi identity*:

$$\bigl[[X, Y], Z\bigr] + \bigl[[Y, Z], X\bigr] + \bigl[[Z, X], Y\bigr] = 0.$$

An algebra $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with these two properties is now known as a Lie algebra. Lie showed that two transformation groups are locally isomorphic if their Lie algebras are isomorphic (as algebras).

What I mean by 'local isomorphism' is not that the two transformation groups are conjugate by a change of variables, but that the two groups are isomorphic as abstract groups. In the Riccati example above, the same group $\mathrm{SL}(2, \mathbb{R})$ acts both as linear transformations of $\mathbb{R}^2$ and as linear fractional transformations of $\mathbb{R}^1$ (or, more globally, as projective transformations of $\mathbb{P}^1 = \mathbb{R}^1 \cup \{\infty\}$).

A more substantial example, and one that I will return to at some length below, is the example of $\mathrm{SL}(3, \mathbb{R})$ acting in two distinct ways on $\mathbb{P}_2$ (= lines in 3-space). Namely, $A \in \mathrm{SL}(3, \mathbb{R})$ can act either as

$$A \cdot [\mathbf{x}] = [A\mathbf{x}] \qquad \text{or as} \qquad A \cdot [\mathbf{y}] = [A^*\mathbf{y}] = [(A^{-1})^T\mathbf{y}]$$

(where $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^3$, $[\mathbf{x}]$ denotes the line spanned by $\mathbf{x}$, and A^* denotes the *contragredient* of A, i.e., the inverse of the transpose of A). Both actions of $\mathrm{SL}(3, \mathbb{R})$ are transitive on $\mathbb{P}_2$, but they are not intertwined by any diffeomorphism of $\mathbb{P}_2$.

However, these two representations are related via *projective duality*. The product action of $\mathrm{SL}(3,\mathbb{R})$ on $\mathbb{P}_2 \times \mathbb{P}_2$ defined by

$$A \cdot \big([\mathbf{x}],[\mathbf{y}]\big) = \big([A\mathbf{x}],[A^{*}\mathbf{y}]\big)$$

preserves the locus $\mathbb{I} \subset \mathbb{P}_2 \times \mathbb{P}_2$ defined by $\mathbf{y}^T\,\mathbf{x} = 0$ and acts transitively on its complement. The set $\mathbb{I}$ is the classical *incidence correspondence* that mediates projective duality:

$$\begin{array}{ccccc} & & \mathbb{I} & & \\ & \lambda\swarrow & & \searrow\pi & \\ \mathbb{P}_2 & & & & {\mathbb{P}_2}^{*} \end{array} .$$

The notion of duality is important in projective geometry, having profound consequences for its development. Felix Klein's *Erlanger* lectures had promoted the idea that the geometry of transformation groups was the proper setting for generalizations of this relationship.

Lie's results naturally led to the following two questions: "What are all the possible Lie algebras?", an abstract algebra question, and "How can they appear as Lie algebras of vector fields?", a more geometric question.

A Lie algebra $\mathfrak{g} \neq 0$ is said to be *simple* if it has no proper ideals. It was quickly realized that the algebraic classification problem would depend on a classification of the simple Lie algebras. It was to this problem that Killing and Cartan turned.

Let $\mathfrak{g}$ be a Lie algebra defined over $\mathbb{C}$. For $x \in \mathfrak{g}$, let $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ be defined by $\mathrm{ad}(x)(y) = [x,y]$. What Killing and Cartan showed is that, when $\mathfrak{g}$ is simple and of dimension $d > 0$, there is a maximal abelian subalgebra (nowadays called a 'Cartan subalgebra') $\mathfrak{t} \subset \mathfrak{g}$ of dimension $r > 0$ and a set $\Lambda \subset \mathfrak{t}^*$ (the root system) consisting of $(d{-}r)$ non-zero, distinct covectors so that

$$\det\big(\lambda\, I_d - \mathrm{ad}(x)\big) = \lambda^r \prod_{\alpha\in\Lambda} \big(\lambda - \alpha(x)\big)$$

for all $x \in \mathfrak{t}$ and so that

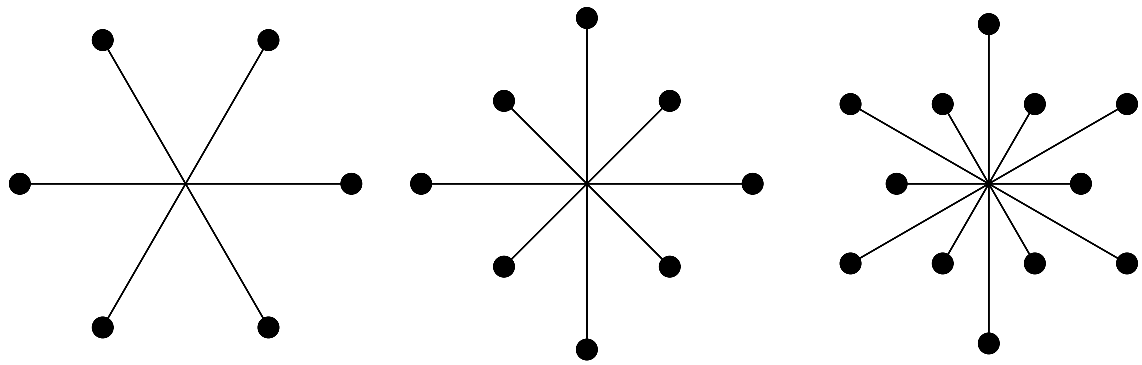
$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Lambda} \mathbb{C}\cdot X_\alpha$$

where $X_\alpha \in \mathfrak{g}$ satisfies $[x, X_\alpha] = \alpha(x)\, X_\alpha$ for all $x \in \mathfrak{t}$. The Jacobi identity then implies that $[X_\alpha, X_\beta] = c_{\alpha\beta}\, X_{\alpha+\beta}$ for some constants $c_{\alpha\beta}$ when $\alpha{+}\beta \neq 0$, while $[X_\alpha, X_{-\alpha}]$ lies in $\mathfrak{t}$. The line $\mathbb{C}\cdot X_\alpha$ is known as the *root space* belonging to $\alpha \in \Lambda$.

The root system $\Lambda \subset \mathfrak{t}^*$ turns out to have remarkable properties. It spans a real subspace $\mathfrak{r} \subset \mathfrak{t}^*$ of dimension r and $c\,\alpha$ lies in Λ for $c \in \mathbb{C}$ and $\alpha \in \Lambda$ if and only if $c = \pm 1$. In fact, for every $\alpha \in \Lambda$, there is a linear map $r_\alpha : \mathfrak{r} \to \mathfrak{r}$ that preserves Λ, satisfies $(r_\alpha)^2 = 1$ and whose (-1)-eigenspace is $\mathbb{R}\cdot\alpha$. The simplicity of $\mathfrak{g}$ turns out to imply that $\Lambda \subset \mathfrak{r}$ is *irreducible*, i.e., it does not lie in the union of two proper subspaces $\mathfrak{r}', \mathfrak{r}'' \subset \mathfrak{r}$.

Killing (with a few later corrections by Cartan) classified the finite subsets $\Lambda \subset \mathfrak{r} \simeq \mathbb{R}^r$ with these properties. He found that there were four families, nowadays called the 'classical root systems' and labeled A_r $(r \geq 1)$, B_r $(r \geq 2)$, C_r $(r \geq 3)$, and D_r $(r \geq 4)$, and five 'exceptional root systems', nowadays labeled as G_2, F_4, E_6, E_7, and E_8 (the subscript indicates the rank r).

Thus, for example, there are three distinct, irreducible root systems of rank 2 (drawn here so that the involutions r_α are reflections):



A_2 B_2 G_2

The four classical root systems corresponded to the classically known (complex) Lie groups: A_r to $\mathrm{SL}(r{+}1,\mathbb{C})$, B_r to $\mathrm{SO}(2r{+}1,\mathbb{C})$, C_r to $\mathrm{Sp}(r,\mathbb{C})$, and D_r to $\mathrm{SO}(2r,\mathbb{C})$. However, in 1888, Killing did not have examples of Lie groups for the five exceptional root systems. While Killing was able to argue that a root system came from at most one Lie algebra, he did not give a convincing argument that there actually existed a Lie algebra corresponding to each of the exceptional root systems.

In 1893, Cartan and Engel, independently, found examples of 14-dimensional simple Lie algebras of vector fields in $\mathbb{R}^5$ whose root system was of type G_2. Contrary to what one might expect, these examples apparently had nothing to do with the octonions or Cayley numbers (more about this below). Instead, their examples were described in terms of differential equations.

What they claimed (although neither gave a proof at the time) was that the Lie algebra of vector fields on $\mathbb{C}^5$ whose flows preserve the 2-plane field $E \subset T\mathbb{C}^5$ defined by the equations

$$dx_2 - x_4\,dx_1 = 0\,, \qquad dx_3 - x_2\,dx_1 = 0\,, \qquad dx_5 - x_4\,dx_2 = 0\,.$$

is such a Lie algebra. Cartan further claimed that there is no simple Lie algebra of vector fields on $\mathbb{C}^4$ whose root system is of type G_2.

In addition, Cartan and Engel found a second simple Lie algebra of vector fields on $\mathbb{C}^5$ of dimension 14 and whose root system is of type G_2. The flows of the vector fields in this algebra preserve a contact 4-plane field but no 2-plane field. I will return to these examples below.

It is interesting that, in his thesis [Ca2], even when representing the exceptional groups as matrix groups, Cartan did not give a purely algebraic model of G_2, as he did for E_6 (the stabilizer of a certain cubic form on $\mathbb{C}^{27}$), E_7 (the stabilizer of a certain quartic form on $\mathbb{C}^{56}$), and F_4 (the stabilizer of a certain quadratic form on $\mathbb{C}^{26}$ and a certain 15-dimensional subvariety $X_{15} \subset \mathbb{P}_{25}$). Instead Cartan characterizes G_2 as the subgroup of $\mathrm{SO}(7,\mathbb{C})$ that preserves a certain set of differential equations for curves in $\mathbb{C}^7$.

Nowadays, we are accustomed to thinking of G_2 as being connected with the algebra of octonions $\mathbb{O}$. The algebra of octonions was discovered independently by Graves (1844) and Cayley (1845) and was reasonably well known by 1890. However, there does not appear to be any indication that Cartan (or anyone else, for that matter) linked the octonions with the exceptional groups until much later. To my knowledge, the first mention of this relationship that appears in print is in Cartan's 1908 encyclopedia article [Ca3]. On the penultimate page of that article, Cartan remarks (without reference or indication of proof) that the group of automorphisms of the "octaves of Graves and Cayley" is a 14-dimensional simple group. (This is also, to my knowledge, the first place where *any* discussion of the automorphism group of the octonions appears in print, almost 50 years after their discovery.) As far as I can tell, it is not until six years after this that Cartan explicitly states [Ca6] that the compact form of G_2 is the group of automorphisms of the octonions. The relation of the octonions with the other exceptional simple groups did not come to light until much later.

In the next three sections, I am going to examine these three different rank 2 groups and explain certain analogies between them that, I believe, helped lead Cartan to some of his most profound discoveries.

For simplicity, I will take the ground field to be $\mathbb{C}$ instead of $\mathbb{R}$, and work in the holomorphic category. Nearly everything that I say will have an obvious analog over the reals (with 'smooth' replacing 'holomorphic'), but, as was customary one hundred years ago, I will not always make explicit remarks about real versus complex when discussing algebraic objects.

I will discuss the three groups in order of their increasing complexity, but it should be noted that this is historically backwards. In fact, Cartan thoroughly analyzed the G_2 case first [Ca4].

2. Classical Duality: A_2

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ of the group $G = \mathrm{SL}(3,\mathbb{C})$ has its root system of type A_2.

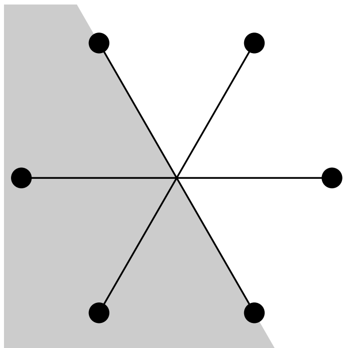
Now, G acts transitively on $\mathbb{P}_2$, the space of lines $L \subset \mathbb{C}^3$, and it also acts transitively on ${\mathbb{P}_2}^*$, the space of 2-planes $E \subset \mathbb{C}^3$. As I already mentioned, the product action on $\mathbb{P}_2 \times {\mathbb{P}_2}^*$ acts transitively on the 3-dimensional *incidence correspondence*

$$\mathbb{I} = \bigl\{\, [L,E] \in \mathbb{P}_2 \times {\mathbb{P}_2}^* \;\bigl|\; L \subset E \bigr\}.$$

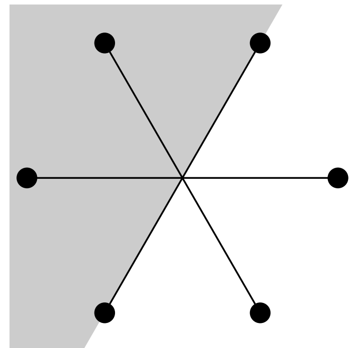
Let $[\bar{L},\bar{E}] \in \mathbb{I}$ be fixed. Let $P_1 \subset G$ denote the stabilizer of $\bar{L}$ and let $P_2 \subset G$ denote the stabilizer of $\bar{E}$. Then the spaces and subgroups introduced so far fit into a double fibration of the following kind:

$$\begin{array}{ccc}
 & \mathbb{I} & \\
 & \| & \\
 & G/(P_1\cap P_2) & \\
{}^{\lambda}\swarrow & & \searrow^{\pi} \\
\mathbb{P}_2 = G/P_1 & & G/P_2 = {\mathbb{P}_2}^*
\end{array}$$

Let $\mathfrak{p}_i \subset \mathfrak{g}$ denote the corresponding Lie algebras. The intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a Cartan subalgebra $\mathfrak{t}$ such that $\mathfrak{p}_i$ is equal to the sum of $\mathfrak{t}$ and the root spaces corresponding to the roots that lie in a closed half-plane, as illustrated below:



$\mathfrak{p}_1$



$\mathfrak{p}_2$

Via the incidence correspondence $\mathbb{I}$, the points of ${\mathbb{P}_2}^*$ define the projective lines in $\mathbb{P}_2$ and, conversely, the points of $\mathbb{P}_2$ define the projective lines in ${\mathbb{P}_2}^*$. This is classical duality.

Correspondence of curves. Although the two spaces are not isomorphic as homogeneous spaces of $G = \mathrm{SL}(3, \mathbb{C})$, there is a correspondence between the curves in the two spaces. A smooth curve $C \subset \mathbb{P}_2$ defines a dual curve $C^* \subset {\mathbb{P}_2}^*$ by assigning to each point $p \in C$ its projective tangent line $p^* \in {\mathbb{P}_2}^*$. When C is free of flexes, the assignment $p \mapsto p^*$ is an immersion of C into ${\mathbb{P}_2}^*$ and this is a true duality in the sense that the corresponding dual mapping of C^* into $\mathbb{P}^2$ simply recovers the curve C.

When the curve C is algebraic and nonlinear (i.e., no component of C is a projective line in $\mathbb{P}_2$), the dual curve C^* is also an (isomorphic) algebraic curve and the singularities, flexes, double tangents and so forth of C correspond to the singularities, flexes, double tangents, and so forth of C^*. The classical Plücker relations describe these relations and they are an important tool in the study of algebraic plane curves.

Duality of differential equations in the plane. The picture described so far is very classical and familiar to every algebraic geometer. However, in [Ca12], Cartan considered generalizations of this duality that played an important role in his development of the theory of projective geometry and projective connections. Cartan wanted to generalize the idea of projective plane geometry as the geometry of a 2-parameter family of lines in the plane by considering, instead, an essentially arbitrary 2-parameter family of curves on a surface.

More precisely, Cartan considered what we would now call a double fibration

$$\begin{tikzcd} & I \arrow[dl, "\lambda"'] \arrow[dr, "\pi"] & \\ \Lambda & & S \end{tikzcd}$$

where Λ and S are surfaces and $I \subset S \times \Lambda$ is a 3-manifold with the nondegeneracy properties that, first, each of the projections $\pi : I \to S$ and $\lambda : I \to \Lambda$ is a submersion and, second, for each $(x, \xi) \in I$, the two curves $\lambda^{-1}(\xi)$ and $\pi^{-1}(x)$ meet transversely at (x, ξ), their tangents spanning a 2-plane $E_{(x,\xi)} \subset T_{(x,\xi)} I$ that defines a contact structure on I.

Cartan showed [Ca9] that I carries a canonical $(P_1 \cap P_2)$-structure $\tau : B \to I$ with what is now called a Cartan connection θ, a 1-form on B with values in $\mathfrak{sl}(3,\mathbb{C})$ so that, at every point $b \in B$, the map $\theta_b : T_b B \to \mathfrak{sl}(3,\mathbb{C})$ is an isomorphism. This form θ has the property that $\theta^{-1}(\mathfrak{p}_1) = \ker(\pi \circ \tau)'$ while $\theta^{-1}(\mathfrak{p}_2) = \ker(\lambda \circ \tau)'$ and that its curvature takes values in the nilpotent part of $\mathfrak{p}_1 \cap \mathfrak{p}_2$. The curvature of θ vanishes if and only if the double fibration is locally equivalent to that of the double fibration of the projective plane.

There are two 'real forms' of this picture that show up in Cartan's work.

The first is the 'split' real form got by simply redoing everything over $\mathbb{R}$ instead of over $\mathbb{C}$. This gives path geometry on surfaces as we now understand it and the duality is the classical duality of second order differential equations. Namely, one thinks of S as having local coordinates (x,y), Λ as having local coordinates (a,b) and $I \subset S \times \Lambda$ as being defined by an equation $f(x,y;a,b) = 0$. Assuming that the 1-form $\rho = f_x\,dx + f_y\,dy = -f_a\,da - f_b\,db$ is a contact form on I, this equation can be regarded as defining a 2-parameter family of paths on P.

Eliminating a and b from the equations

$$f = f_x + f_y\,y' = f_{xx} + 2f_{xy}y' + f_{yy}(y')^2 + f_y y'' = 0$$

leads to a second order equation $y'' = F(x,y,y')$ and conversely, a second order equation of this form defines a path geometry on S, where the paths are the graphs of the solutions of this equation. Doing the analogous operation after switching the variables leads to the *dual equation* $b'' = G(a,b,b')$ for curve on Λ.

In this case, Cartan showed that, after replacing $\mathrm{SL}(3,\mathbb{C})$ by $\mathrm{SL}(3,\mathbb{R})$ and the two parabolic subgroups P_i by their real counterparts, the geometry of this double fibration was captured by a $\mathfrak{sl}(3,\mathbb{R})$-valued Cartan connection θ on a bundle over I. He showed, moreover, that this bundle with connection could be regarded as a P_1 bundle with connection over S if and only if the paths are the geodesics of what is now known as a projective connection on S.

The second 'real form' is now known as the geometry of a CR-hypersurface $I^3 \subset \mathbb{C}^2$. One can understand this as a different real form of the above path geometry by simply thinking of (x,y) as complex coordinates on $S \simeq \mathbb{C}^2$ and setting $(a,b) = (\bar{x},\bar{y})$. The real hypersurface $I \subset \mathbb{C}^2$ can now be regarded as the locus of points satisfying $f(x,y;\bar{x},\bar{y}) = 0$. In [Ca10,11], Cartan shows that, after replacing $\mathrm{SL}(3,\mathbb{C})$ by its real form $\mathrm{SU}(2,1)$, one can define a bundle over I endowed with a Cartan connection θ with values in $\mathfrak{su}(2,1)$ whose invariants capture the geometry of the given real hypersurface in $\mathbb{C}^2$. In fact, understanding Cartan's papers on hypersurfaces in $\mathbb{C}^2$ is made much simpler by keeping this model in mind since it clearly guided his solution of the equivalence problem.

3. The Lie correspondence: B_2

The Lie algebra $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$ of the group $G = \mathrm{Sp}(2,\mathbb{C}) \subset \mathrm{SL}(4,\mathbb{C})$ of linear transformations that preserve a nondegenerate skewsymmetric pairing $\Omega : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$, has its root system of type B_2.

Now, G acts transitively on $\mathbb{P}_3$, the space of lines $L \subset \mathbb{C}^4$, and it also acts transitively on $\mathbb{Q}_3$, the space of 2-planes $E \subset \mathbb{C}^4$ to which Ω restricts to become zero. In fact, one can understand this as follows: G acts on $\Lambda^2(\mathbb{C}^4) \simeq \mathbb{C}^6$ and preserves the 5-dimensional subspace $\ker \Omega \subset \Lambda^2(\mathbb{C}^4)$. Since the symplectic form Ω defines a volume form on $\mathbb{C}^4$, i.e., an isomorphism $\Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$, there is a natural quadratic form on $\Lambda^2(\mathbb{C}^4)$ given by sending $e \in \Lambda^2(\mathbb{C}^4)$ to $e{\wedge}e \in \Lambda^4(\mathbb{C}^4) \simeq \mathbb{C}$. This quadratic form is nondegenerate, so the representation of G on $\ker \Omega \simeq \mathbb{C}^5$ is as a subgroup of $\mathrm{SO}(5, \mathbb{C})$. This defines a double cover $\mathrm{Sp}(2, \mathbb{C}) \to \mathrm{SO}(5, \mathbb{C})$, yielding a local isomorphism of these two groups that was first noticed by Lie. The non-zero null elements of the quadratic form on $\ker \Omega$ are evidently the same as the decomposable 2-vectors in $\ker \Omega$, i.e., projectively the same as the 2-planes $E \subset \mathbb{C}^4$ to which Ω restricts to become zero, as desired. This shows, by the way, that $\mathbb{Q}_3$ is, indeed, as the notation suggests, a 3-quadric. Moreover, $G = \mathrm{Sp}(2, \mathbb{C})$ acts on $\mathbb{Q}_3$ as the full $\mathrm{SO}(5, \mathbb{C})$. In particular, this action is transitive.

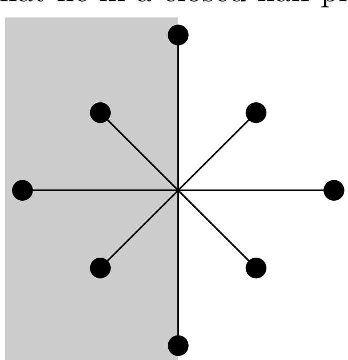
Now, the product action on $\mathbb{P}_3 \times \mathbb{Q}_3$ acts transitively on the 4-dimensional *incidence correspondence*

$$\mathbb{I} = \left\{ [L, E] \in \mathbb{P}_3 \times \mathbb{Q}_3 \,\middle|\, L \subset E \right\}.$$

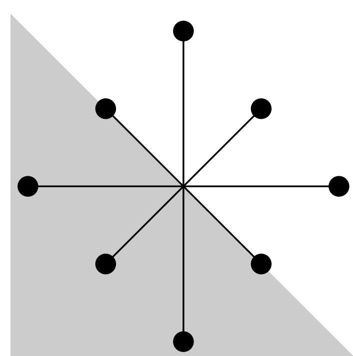
Let $[\bar{L}, \bar{E}] \in \mathbb{I}$ be fixed. Let $P_1 \subset G$ denote the stabilizer of $\bar{L}$ and let $P_2 \subset G$ denote the stabilizer of $\bar{E}$. Then the spaces and subgroups introduced so far fit into a double fibration of the following kind:

$$\begin{tikzcd}
 & \mathbb{I} \arrow[d, equal] & \\
 & G/(P_1{\cap}P_2) \arrow[dl, "\lambda"'] \arrow[dr, "\pi"] & \\
\mathbb{P}_3 = G/P_1 & & G/P_2 = \mathbb{Q}_3
\end{tikzcd}$$

Let $\mathfrak{p}_i \subset \mathfrak{g}$ denote the corresponding Lie algebras. The intersection $\mathfrak{p}_1{\cap}\mathfrak{p}_2$ contains a Cartan subalgebra $\mathfrak{t}$ such that $\mathfrak{p}_i$ is equal to the sum of $\mathfrak{t}$ and the root spaces corresponding to the roots that lie in a closed half-plane, as illustrated below:



$\mathfrak{p}_1$



$\mathfrak{p}_2$

Via the incidence correspondence $\mathbb{I}$, the points of $\mathbb{Q}_3$ define a 3-parameter family of projective lines in $\mathbb{P}_3$, the lines tangent to a contact 2-plane field on $\mathbb{P}_3$. Conversely, the points of $\mathbb{P}_3$ are represented as the lines in $\mathbb{Q}_3 \subset \mathbb{P}_4$ that are null with respect to the induced (holomorphic) conformal structure.

Correspondence of curves. Just as in projective duality in the plane, there is a correspondence between certain curves in the two spaces. A smooth contact curve $C \subset \mathbb{P}_3$ (i.e., a curve tangent to the $\mathrm{Sp}(2,\mathbb{C})$-invariant contact field on $\mathbb{P}_3$) defines a dual curve $C^* \subset \mathbb{Q}_3$ by assigning to each point $p \in C$ its projective tangent line $p^* \in \mathbb{Q}_3$. When C is free of flexes, the assignment $p \mapsto p^*$ is an immersion of C into $\mathbb{Q}_3$ as a null curve.

Conversely, a smooth null curve $D \subset \mathbb{Q}_3$, defines a dual curve $D^* \subset \mathbb{P}_3$ by assigning to each point $p \in D$ its (null) projective tangent line $p^* \in \mathbb{P}_3$. As long as D does not osculate too closely to the straight null lines in $\mathbb{Q}_3$, the assignment $p \mapsto p^*$ is an immersion of D into $\mathbb{P}_3$. A straightforward computation shows that D^* is a contact curve and, moreover, that $C^{**} = C$ and $D^{**} = D$ as long as these curves are not straight lines.

This correspondence between null curves in $\mathbb{Q}_3$ and contact curves in $\mathbb{P}_3$ was discovered by Lie. It takes nonlinear algebraic curves to nonlinear algebraic curves and there are generalizations of the Plücker relations that relate the singularities of one curve to the 'dual' singularities of the other. For one particular application of this to the study of Willmore surfaces, see [Br2].

Conformal geometry in dimension 3. One of two possible generalizations of this picture is to replace $\mathbb{Q}_3$ by an arbitrary complex 3-manifold S endowed with a holomorphic conformal structure $[g]$. Assuming certain reasonable global hypotheses that I won't spell out here, the space Λ of (holomorphic) null geodesics of $[g]$ is a complex 3-manifold (The global hypotheses needed are just to ensure that the space Λ is Hausdorff.) The incidence correspondence $I \subset S \times \Lambda$ consists of the set of pairs (p,ξ) where $p \in S$ lies on the null geodesic $\xi \in \Lambda$. Again, the projections of I onto the two factors defines a double fibration

$$\begin{array}{ccc} & I & \\ {\scriptstyle\lambda}\swarrow & & \searrow{\scriptstyle\pi} \\ \Lambda & & S \end{array}$$

and the space Λ carries a natural contact structure, where the contact 2-plane field $E \subset T\Lambda$ is defined at $\xi \in \Lambda$, by letting E_ξ be the tangent plane at ξ to the surface $\Sigma_\xi \subset \Lambda$ consisting of the set of null geodesics in S that meet ξ.

Each point $p \in S$ defines a contact curve D_p in Λ by letting D_p consist of the curve of null geodesics that pass through p. Thus, the double fibration defines, in addition to a contact structure on Λ, a 3-parameter family of contact curves.

Moreover, just as in the 'flat case', there is a duality between certain curves in the two spaces. Each $[g]$-null curve C in S gives rise to a contact curve C^* in Λ and conversely. The correspondence is just given by sending each $p \in C$ to the null geodesic p^* that is tangent to C at p. For a contact curve $D \subset \Lambda$, one gets a dual null curve D^* by sending $\xi \in D$ to the point $\xi^* \in S$ for which D_{ξ^*} is tangent to D at ξ.

Cartan [Ca8] developed a theory of conformal connections that, in dimension 3, directly generalizes the geometry of the fibration $\mathrm{Sp}(2,\mathbb{C})/P_2 = \mathbb{Q}_3$. In fact, what Cartan shows is that in the above situation, there is a canonical principal right P_2-bundle $B \to S$ endowed with a $\mathfrak{sp}(2,\mathbb{C})$-valued 1-form θ whose invariants capture the conformal geometry of $\bigl(S,[g]\bigr)$ completely. This bundle B is, in a natural way, a $(P_1 \cap P_2)$-bundle over I, though it does not, in general, have the structure of a P_1-bundle over Λ in a natural way.

Of course, there are two real forms of this picture. One is the case of a conformal structure on a real 3-manifold and the other is the case of a conformal Lorentzian structure on a real 3-manifold. It is the second one that has the clearest analog of the complex picture since it is in this case that there exist null geodesics.

Third order differential equations. The other possible generalization of the flat picture is to generalize the P_1-bundle $\mathrm{Sp}(2,\mathbb{C}) \to \mathbb{P}_3$. In this case, one starts with a contact 3-manifold Λ endowed with a three-dimensional family S of contact curves. The necessary nondegeneracy hypothesis one imposes is that the incidence correspondence $I \subset \Lambda \times S$ should be a smooth 4-dimensional submanifold that submerses onto each of the factors and that the tangents at any $\xi \in \Lambda$ of the set of curves in S that pass through ξ should fill out an open set in the set of lines in the contact plane at ξ.

One source of this geometry is to consider a third order differential equation $y''' = F(x, y, y', y'')$ in the plane. On the space Λ of first order contact elements in the plane, with coordinates (x, y, y'), each line that lies in a contact plane is tangent to a unique lifted solution of the above equation. Thus, the space S of solutions to the given third order equation defines a structure of the desired kind on Λ. To study this geometry is to study the geometry of third order equations in the plane up to contact transformations. Cartan [Ca14] studied the geometry of these equations up to point transformations in the plane as a generalization of the geometry of second order equations discussed in the last section. Chern [Ch] studied the more general equivalence of such equations up to contact transformations and showed that one could, in fact, associate a bundle of dimension 10 with connection over Λ to this geometry that generalized the bundle $\mathrm{Sp}(2,\mathbb{R})/(P_1 \cap P_2) \to \mathbb{I}$ in the 'flat' case (which turns out to correspond to the equation $y''' = 0$). He showed, that if a certain invariant, originally defined by Wünschmann, vanishes then this bundle has the structure of a P_2-bundle over S, in fact the P_2-bundle associated to a conformal structure on S.

Thus, again, the notion of a 'generalized space' in Cartan's sense (roughly speaking, a 'deformation' of a homogeneous bundle with connection) turns out to be reflected in the geometry of so system of differential equations. Cartan points out in [Ca12] that these situations provide an important way to generalize the notion of 'integrability' of an ordinary differential equation. I think that this set of ideas is still very profound and needs to be better understood.

4. Cartan's correspondence: G_2

Cartan showed that there is a 14-dimensional Lie group $G \subset \mathrm{SO}(7,\mathbb{C})$ whose Lie algebra $\mathfrak{g}$ has its root system of type G_2. It is interesting that, in his thesis, Cartan does not describe this group as the subgroup preserving some algebraic structure on $\mathbb{C}^7$. In fact, his original description is in terms of the inner product and a set of differential equations for curves in $\mathbb{C}^7$. What he says is that G is the subgroup of $\mathrm{GL}(7,\mathbb{C})$ that preserves the quadratic form

$$J = z^2 + x_1\,y_1 + x_2\,y_2 + x_3\,y_3$$

and the system of 7 Pfaffian equations (where (i, j, k) is any even permutation of $(1, 2, 3)$)

$$\begin{aligned} z\,dx_i - x_i\,dz + y_j\,dy_k - y_k\,dy_j &= 0, \\ z\,dy_i - y_i\,dz + x_j\,dx_k - x_k\,dx_j &= 0, \\ x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2 + x_3\,dy_3 - y_3\,dx_3 &= 0. \end{aligned}$$

Since G preserves a quadratic form on $\mathbb{C}^7$, it cannot act transitively on $\mathbb{P}_6$, the space of lines in $\mathbb{C}^7$. However, Cartan showed that G does act transitively on $\mathbb{Q}_5 \subset \mathbb{P}_6$, the space of J-null lines in $\mathbb{C}^7$.

Now G does not act transitively on the space of J-null 2-planes in $\mathbb{C}^7$ (a 7-dimensional homogeneous space of $\mathrm{SO}(7,\mathbb{C})$), but it does act transitively on the 5-dimensional space $\mathbb{N}_5$ consisting of the J-null 2-planes on which the following 2-forms vanish (again, (i,j,k) is any even permutation of $(1,2,3)$):

$$\begin{aligned} dz \wedge dx_i + dy_j \wedge dy_k &= 0, \\ dz \wedge dy_i + dx_j \wedge dx_k &= 0, \\ dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 &= 0. \end{aligned}$$

Cartan does not seem to have been aware that G can be defined more simply as the stabilizer of the 3-form

$$\phi = dz \wedge \bigl(dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3\bigr) + dy^1 \wedge dy^2 \wedge dy^3 - dx^1 \wedge dx^2 \wedge dx^3,$$

even as late as 1914, when he finally identified the two real forms of type G_2 as the stabilizers of certain algebraic structures on $\mathbb{R}^7$. In fact, this form has a very simple interpretation in terms of the octonions $\mathbb{O}$, an eight dimensional algebra with unit $\mathbf{1}$ and nondegenerate inner product $\langle,\rangle$ that satisfies $\langle \mathbf{xy}, \mathbf{xy}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle \langle \mathbf{y}, \mathbf{y}\rangle$. (Over $\mathbb{C}$ there is a unique such algebra while, over $\mathbb{R}$, there are two, one for which $\langle,\rangle$ is positive definite and one for which it is of type $(4,4)$.) Letting $\mathrm{Im}\mathbb{O} = \{\mathbf{x} \in \mathbb{O} \mid \langle \mathbf{x}, \mathbf{1}\rangle = 0\}$, the 3-form ϕ can be defined as $\phi(\mathbf{x},\mathbf{y},\mathbf{z}) = \langle \mathbf{xy}, \mathbf{z}\rangle$. Then, in each of the three possible cases (one complex and two real), the stabilizer of ϕ is a Lie group of dimension 14 whose root system is of type G_2.

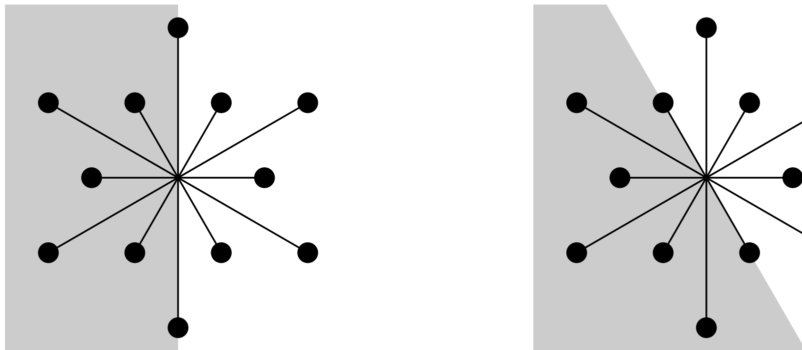
The product action of G on $\mathbb{Q}_5 \times \mathbb{N}_5$ acts transitively on the 6-dimensional *incidence correspondence*

$$\mathbb{I} = \bigl\{\, [L,E] \in \mathbb{Q}_5 \times \mathbb{N}_5 \bigm| L \subset E \bigr\}.$$

Let $[\bar{L},\bar{E}] \in \mathbb{I}$ be fixed. Let $P_1 \subset G$ denote the stabilizer of $\bar{L}$ and let $P_2 \subset G$ denote the stabilizer of $\bar{E}$. Then the spaces and subgroups introduced so far fit into a double fibration of the following kind:

$$\begin{array}{ccccc} & & \mathbb{I} & & \\ & & \| & & \\ & & G/(P_1 \cap P_2) & & \\ & {}^{\lambda}\swarrow & & \searrow^{\pi} & \\ \mathbb{Q}_5 = G/P_1 & & & & G/P_2 = \mathbb{N}_5 \end{array}$$

Let $\mathfrak{p}_i \subset \mathfrak{g}$ denote the corresponding Lie algebras. The intersection $\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a Cartan subalgebra $\mathfrak{t}$ such that $\mathfrak{p}_i$ is equal to the sum of $\mathfrak{t}$ and the root spaces corresponding to the roots that lie in a closed half-plane, as illustrated below:



$\mathfrak{p}_1$ $\mathfrak{p}_2$

Via the incidence correspondence $\mathbb{I}$, the points of $\mathbb{Q}_5$ define lines in $\mathbb{N}_5$ and, conversely, the points of $\mathbb{N}_5$ define lines in $\mathbb{Q}_5$. These are not arbitrary lines, but, instead satisfy certain conditions that I now want to describe in terms of the Lie algebras.

As we now know, the infinitesimal geometry of the homogeneous space G/P_i is reflected in the representation of $\mathfrak{p}_i$ on $\mathfrak{g}/\mathfrak{p}_i$ induced by the adjoint representation.

In the case of $\mathbb{Q}_5 = G/P_1$, this representation preserves a filtration

$$0 \subset \mathfrak{n}_1^{(-1)} \subset \mathfrak{n}_1^{(-2)} \subset \mathfrak{n}_1^{(-3)} = \mathfrak{g}/\mathfrak{p}_1 \simeq \mathbb{C}^5$$

with associated graded spaces $\mathfrak{n}_1^{(-1)} \simeq \mathbb{C}^2$, $\mathfrak{n}_1^{(-2)}/\mathfrak{n}_1^{(-1)} \simeq \mathbb{C}$, and $\mathfrak{n}_1^{(-3)}/\mathfrak{n}_1^{(-2)} \simeq \mathbb{C}^2$. The induced representation of $\mathfrak{p}_1$ on each graded piece has image the full general linear representation. In particular, it is irreducible. It follows that G acts on $\mathbb{Q}_5 = G/P_1$ preserving a 2-plane field $E \subset T\mathbb{Q}_5$. The plane field E is null with respect to the conformal structure on $\mathbb{Q}_5$. Via the correspondence $\mathbb{I}$, each point of $\mathbb{N}_5$ represents a line in $\mathbb{Q}_5$ that is tangent to this 2-plane field. Now, the 2-plane field E is as non-integrable as possible. The system $[E, E]$ spanned by the Lie brackets of vector fields tangent to E is a 3-plane field and the system $\big[[E, E], E\big]$ generated by the triple brackets of vector fields in E is equal to the whole of $T\mathbb{Q}_5$. As we will see, this 'generic' behavior is what makes it a model for the 'generic' 2-plane field in dimension 5.

This 2-plane field is exactly the 2-plane field discovered by Cartan and Engel (more precisely, there is an affine open set $\mathbb{C}^5 \subset \mathbb{Q}_5$ such that the restriction of the 2-plane field E to $\mathbb{C}^5$ is the one identified by Cartan and Engel). Consequently, G is the symmetry group of this 2-plane field. Moreover, this plane field has turned up in other ways. For example, I used it in [Br1] to give a description of the null-torsion pseudo-holomorphic curves in the 6-sphere, thereby managing to prove that every compact Riemann surface appears as such a pseudo-holomorphic curve.

In the case of $\mathbb{Q}_5 = G/P_2$, this representation preserves a filtration

$$0 \subset \mathfrak{n}_2^{(-1)} \subset \mathfrak{n}_2^{(-2)} = \mathfrak{g}/\mathfrak{p}_2 \simeq \mathbb{C}^5$$

with associated graded spaces $\mathfrak{n}_2^{(-1)} \simeq \mathbb{C}^4$ and $\mathfrak{n}_2^{(-2)}/\mathfrak{n}_2^{(-1)} \simeq \mathbb{C}$. The induced representation of $\mathfrak{p}_2$ on $\mathfrak{n}_1^{(-1)}$ is isomorphic to the representation of $\mathfrak{gl}(2, \mathbb{C})$ on the third symmetric power of $\mathbb{C}^2$. This is a conformally symplectic representation, so it preserves a

2-dimensional cone $F \subset \mathfrak{n}_2^{(-1)}$ of degree 3 that corresponds to the cone of perfect cubes in $S^3(\mathbb{C}^2)$. Thus, G acts on $\mathbb{Q}_5 = G/P_2$ preserving a field $F \subset T\mathbb{N}_5$ of 2-dimensional cubic cones. These cones lie in the contact plane field on $\mathbb{N}_5$ defined by the 4-plane field corresponding to $\mathfrak{n}_2^{(-1)}$. Via the correspondence $\mathbb{I}$, each point of $\mathbb{Q}_5$ represents a line in $\mathbb{N}_5$ that is tangent to this 2-cone field.

Correspondence of curves. Now the analog of the Lie-Klein correspondence can be explained. Every curve $C \subset \mathbb{Q}_5$ that is tangent to the plane field E defines a curve C^* in $\mathbb{N}_5$ by the rule $p \mapsto p^*$ where $p^* \in \mathbb{N}_5$ is the projective tangent line to C at p. The curve C^* is tangent to the cone field F. Conversely, every curve $D \subset \mathbb{N}_5$ that is tangent to the cone field F defines a curve D^* in $\mathbb{Q}_5$ by the rule $p \mapsto p^*$ where $p^* \in \mathbb{Q}_5$ is the projective tangent line to D at p. The curve D^* is tangent to the plane field E. Moreover, assuming that the curves C and D are not linear, these transforms are dual, i.e., $C^{**} = C$ and $D^{**} = D$.

2-plane fields in dimension 5. Cartan used the model geometry of the double fibration to study the geometry of generic 2-plane fields on 5-manifolds [Ca4]. The genericity condition can be described as follows: Let Q be a 5-manifold and let $E \subset TQ$ be a 2-plane field. Let us say that E is of *Cartan type* if the brackets of local sections of E span a 3-plane field at each point of Q (a generic condition) and the triple brackets of local sections of E span the entire tangent space at each point of Q (also a generic condition).

Cartan showed how one can canonically associate to each 2-plane field $E \subset TQ$ of Cartan type a principal P_1-bundle B endowed with a $\mathfrak{g}$-valued coframing 1-form θ (in fact, what is now known as a Cartan connection) that captures the local geometry of the plane field E completely in the sense that any equivalence $(Q, E) \simeq (Q', E')$ of Cartan type 2-plane fields on 5-manifolds is covered by a unique P_1-bundle isomorphism $B \simeq B'$ that identifies θ with θ'. Using this construction, for example, Cartan was able to show that the fundamental curvature tensor (the analog, in this situation, of the Riemann curvature tensor for metrics) of a Cartan-type 2-plane field E is a section of $S^4(E^*)$ and that it vanishes if and only if the 2-plane field is locally equivalent to the 'flat' example described above on $\mathbb{Q}_5$.

His argument is a true *tour-de-force*, still striking in its originality today. It was, by far, the most elaborate application of his method of equivalence to be fully worked out in his lifetime and, I believe, served as the model for many of his later applications of the method to conformal and projective geometries.

I do not have time today to discuss all of the ways this example was applied or the effects that it had on the development and application of the general theory. For example, Hilbert and Cartan [Ca7] dealt with exactly this case in their studies of certain foundational problems in the calculus of variations, and Cartan had used it to illustrate his notion of 'absolute equivalence' of differential equations and its relation with the integration problem for such systems. In fact, Cartan's motivation for studying this problem in the first place was to understand the geometry of the characteristics of a certain type of parabolic equation in the plane.

Instead, I just want to point out one particular application that emphasizes the analogy of it with the first two cases A_2 and B_2. One might wonder what the analog of the 'second base' of the double fibration should be in this case. This went unidentified for a long time. However, when Lucas Hsu and I were trying to understand the geometry of the abnormal curves in the calculus of variations, the example of 2-plane fields in five (or more) dimensions was very important. What we found [BH] was that, regarding a Cartan-type 2-plane field E on a 5-manifold Q as a holonomic mechanical system (for example, one surface in space rolling over another without slipping or twisting), the abnormal curves have the property that there is exactly one such curve passing through each point of Q in each direction tangent to E. In other words, the space of abnormal curves for such a 2-plane field is a 5-manifold N. In the 'flat' case discussed above, this is exactly the manifold $\mathbb{N}_5$, but even in the general case, N inherits the structure of a contact 5-manifold. The geometry of the resulting double fibration has not at all been explored, but it might be the key to understanding such fundamental problems as the conditions for global rigidity for such abnormal curves.

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