CHAPTER 1

1st order Partial Differential Equations

Summary

The basic object of study in this book is the existence solutions to differential equations in geometric interpretations of equations. We first discuss in this chapter the basic facts on the 1st order differential equations.

1. 1st order Partial Differential Equations

We consider the 1st order differential equations defined on a domain $\Omega \subset \mathbb{R}^n$. Let $u(x)$ be the unknown function for $x := (x_1, \ldots, x_n) \in \Omega$. Derivative of u is denoted by $u_j := \frac{\partial u}{\partial x_j}$. By *the 1st order differential equations* on u, we mean $F(x, u, u_1, \ldots, u_n) = 0$ for a function F with $(\frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_n}) \neq 0$. Basic classes of 1st order partial differential equations are as follows.

Definition 1.1. Differential equation $F(x, u, u_1, \ldots, u_n) = 0$ is quasi-linear if F is linear in $u_1, \ldots, u_n$ for some functions $a_j(x, u)$ and $b(x, u)$ as in

$$a_1(x, u)u_1 + \cdots a_n(x, u)u_n = b(x, u)$$

and is almost linear if a_j s are functions of x as in

$$a_1(x), u_1 + \cdots + a_n(x)u_n = b(x, u)$$

and is linear if $b(x, u) = c(x)u + d(x)$ for some functions c and d such that

$$a_1(x)u_1 + \ldots a_n(x)u_n = b(x)u + c(x).$$

1.1. 1st order linear homogeneous partial differential equations. Let $V := a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a nowhere vanishing $\mathcal{C}^1$ vector field on Ω. Then 1st order linear differential equation $a_1(x)u_1 + \ldots a_n(x)u_n = 0$ is

$$V \cdot u = 0$$

and the solution $u(x)$ is constant along integral curves of V.

Definition 1.2. A $\mathcal{C}^1$ function $u(x)$ is a *first integral of* V if $V \cdot u = 0$ i.e. $V \cdot \nabla u = 0$.

Definition 1.3. Functions $u_1, \ldots, u_k$ are said to be functionally dependent if $G(u_1, \ldots, u_k) = 0$ for some nontrivial function G.

Definition 1.4. $u_1, \ldots, u_k$ are functionally independent if they are not functionally dependent on any open subset of Ω.

Generically $V \cdot \nabla u = 0$ has $n - 1$ functionally independent first integrals. Suppose $V = \sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j}$ is defined on $\Omega \subset \mathbb{R}^n$. V is presumably a coordinate vector field since we can always find a local diffeomorphic coordinate change $\varphi : \Omega \to \mathbb{R}^n$ with $\varphi_*(V) = \frac{\partial}{\partial x_1}$. Letting $V = \frac{\partial}{\partial x_1}$ in new coordinates, the other $n - 1$ coordinate functions $x_2, \ldots, x_n$ are the first integrals.

The heuristics to calculate them explicitly is given. Let $V = \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}$ and its integral curve have a infinitesimal line element $(dx_1, \dots, dx_n)$. Along the integral curve

$$\frac{dx_1}{a_1} = \cdots = \frac{dx_n}{a_n}.$$

Equating any two terms above leads to $n-1$ equations, which we assume to be in the form$d$(some function) $= 0$. These functions[1] are the first integrals. Note that these are also calld the *constants of motions* for their total derivative is zero along their motion i.e. the integral curve. See [**Zach**] for details.

EXAMPLE 1.5. Let $V = (1, 0, 0)$ be a vector field in $\mathbb{R}^3$. The first integrals are solutions for $V \cdot \nabla u = 0$ i.e. $\frac{\partial u}{\partial x_1} = 0$. Then $u(x_1, x_2, x_3) = x_2$ or x_3 are two functionally independent first integrals. For any $\mathcal{C}^1$ function F in two variables, $F(x_1, x_2)$ is a first integral. The same solution is obtained by solving

$$\frac{dx_1}{1} = \frac{dx_2}{0} = \frac{dx_3}{0}$$

to get $x_2 =$ constant and $x_3 =$ constant.

EXAMPLE 1.6. Let $V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $\mathbb{R}^2$. Along its integral curves

$$\frac{dx}{-y} = \frac{dy}{x}$$

or $xdx + ydy = 0$. Now $d(x^2 + y^2) = 0$ and $\phi = x^2 + y^2$ is the first integral or the constant of the motion.

EXAMPLE 1.7. Let $V = (x, y, z)$ be a vector field on $(x, y, z) \in \mathbb{R}^3$. They point in the radial directions away from the origin. Its first integrals are $u(x, y, z)$ which solves $xu_x + yu_y + zu_z = 0$. The first equation of

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

gives $\ln x = \ln y + \text{const}$ i.e. $y/x = \text{const}$. Similarly the second equation gives $z/x = \text{const}$. Now y/x and z/x are functionally indenpendent first integrals and the general solution is $u(x, y, z) = F(y/x, z/x)$ for any $\mathcal{C}^1$ function F.

EXERCISE 1.8. Find the first integrals for $V = (y + z, y, x - y)$ in $\mathbb{R}^3$.

1.2. Integral submanifolds for vector fields on domains in $\mathbb{R}^n$. Let $V = \sum_{j=1}^{n} a_i(x)\frac{\partial}{\partial x_j}$ be a vector field defined on $\Omega \subset \mathbb{R}^n$. A k-dimensional submanifold $\mathcal{S} \subset \Omega$ for $k = 1, 2, \dots, n-1$ is an *integral submanifold of* V if V is tangent to $\mathcal{S}$. Integral submanifolds are sometimes called *integral surfaces* and 1 dimensional integral submanifolds are preferrably called *integral curves.* The most basic theorems related are as follows.

THEOREM 1.9.

(1) *If a curve C is transversal, that is, not tangential to V at x_0, then there exists a unique integral surface $\mathcal{S}$ of V containing C.*
(2) *If Γ is k-dimensional submanifold of Ω, transversal to V at $x_0 \in \Gamma$, then on a neighborhood of x_0, there exists the unique integral surface $\mathcal{S}$ of dimension $k+1$ for $k = 1, 2, \dots, n-2$ containing Γ.*

[1] These are explicitly calculated only for special cases.

REMARK 1.10. The curves C and the surfaces $\mathcal{S}$ above are called respectively *initial curves* and *initial surfaces*.

EXAMPLE 1.11. Let $V = (x, y, z)$ be a vector field on $(x, y, z) \in \mathbb{R}^3$ and C a curve defined by $x = 1$,$y = t$ and $z = \cos t$ for real number t. Find an integral surface containing C near $C(0) = (1, 0, 1)$.

SOLUTION. Note that $C'(0) = (0, 1, 0)$ and $V = (1, 0, 1)$ at $C(0) = (1, 0, 1)$ and C and V are transversal at this point. Hence there exists the unique integral surface by the theorem . To get the integral surfaces explicitly, we seek for the first integral u of V since $u =$ const defines integral surfaces. Let (dx, dy, dz) be the infinitesimal line element of an integral curve. Then

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

The first identity yields $\phi_1 := y/x =$ const and the second $\phi_2 := z/x =$ const. Hence the general form of the first integral is $u(x, y, z) := F(\phi_1(x, y, z), \phi_2(x, y, z))$ for any $\mathcal{C}^1$ function F. Now we fix F so that $u(x, y, z) = 0$ contains the initial curve C. Restricted on C,

$$\phi_1 = y/x = t/1 = t, \qquad \phi_2 = z/x = \cos t.$$

Hence $\phi_2 - \cos\phi_1 = 0$ and we fix $F(\phi_1, \phi_2) = \phi_2 - \cos\phi_1$. The integral surface that contains C is

$$\frac{z}{x} - \cos\frac{y}{x} = 0$$

EXERCISE 1.12. $V = (1, 1, z)$ is a vector field on $(x, y, z) \in \mathbb{R}^3$. Find the integral surface containing the curve C: $x = t$, $y = 0$ and $z = \sin t$ for $t \in \mathbb{R}$.

EXERCISE 1.13. Find the integral surface of $V = (y - z, z - x, x - y)$ for the initial curve $C : x = t$, $y = 2t$ and $z = 0$ in the same setting as the previous exercise.

1.3. General Solutions to Quasi-linear 1st order Partial Differential Equations. Keep the notation and let $x := (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$ and $u(x)$ be the unknown function. Consider a quasi-linear 1st order P.D.E

$$a_1(x, u)u_1 + a_2(x, u)u_2 + \cdots + a_n(x, u)u_n = b(x, u) \tag{1.1}$$

To analyze it in geometric viewpoints as before, consider the vector field in $\mathbb{R}^{n+1} = \{(x, u)\}$

$$V = a_1\frac{\partial}{\partial x_1} + \cdots + a_n\frac{\partial}{\partial x_n} + b\frac{\partial}{\partial u}$$

associated with (1.1). Let $\phi(x, u)$ be the first integral such that $\phi(x, u) = 0$ can be solved for $u = \psi(x)$ by the implicit function theorem, for which we require that $\phi_u \neq 0$. Then $u = \psi(x)$ *is a solution* to (1.1).

PROOF. Since $\phi(x, \psi(x)) = 0$ for $x \in \Omega$, differentiate it with respect to x_i to have for $i = 1, \ldots, n$

$$\phi_i + \phi_u \cdot u_i = 0. \tag{1.2}$$

$\phi(x, u)$ is the first integral of V and satisfies

$$a_1\phi_1 + \cdots + a_n\phi_n + b\phi_u = 0 \tag{1.3}$$

Combining (1.2) and (1.3),

$$a_1(-\phi_u u_1) + \cdots + a_n(-\phi_u u_n) + b\phi_u = 0.$$

Cancelling out $\phi_u \neq 0$,

$$a_1 u_1 + \cdots + a_n u_n = b$$

as desired.

EXERCISE 1.14. Let $u(x, y)$ be defined on some open subset in $\mathbb{R}^2$ solving

$$x^2 u_x + y^2 u_y = 2xy.$$

Find the general solution.

EXERCISE 1.15. Find the general solution to

$$x u_x + y u_y = u$$

in the same setting as above.

REMARK 1.16. The approach in this section may be reformulated as follows focusing more on geometric aspect thereof. Let $u(x_1, \ldots, x_n)$ be the unknown $\mathcal{C}^2$ function that solves

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \cdot u_i = b(x, u).$$

let $V := (a_1, \ldots, a_n, b)$ the associated vector field in $\mathbb{R}^{n+1}$ We first find the n dimensional integral submanifold making the best use of the fact that this submanifold is *foliated by integral curves* of V. Assume that our integral submanifold is given as the graph of $u = u(x)$. Denoting the infinitesimal line element of an integral curve of V by $(dx_1, .., dx_n, du)$ we have for some function λ

$$\frac{dx_1}{a_1} = \cdots = \frac{dx_n}{a_n} = \frac{du}{b} = \lambda. \tag{1.4}$$

Since the integral curve is embedded in $u = u(x)$

$$du = u_1 dx_1 + \cdots + u_n dx_n. \tag{1.5}$$

Applying (1.4) upon (1.5),

$$\lambda b = (u_1 a_1 + \cdots + u_n a_n)\lambda.$$

Cancelling out λ,

$$b = u_1 a_1 + \ldots u_n a_n$$

as desired.

1.4. Initial value problem of Quasi-linear 1st order Partial Differential Equations. We restrict our consideration to the case that $u = u(x, y)$ is a unknown function in two variables x and y. Given

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y = b(x, y, u)$$

with some initial data along the curve $(x(t), y(t), u(t))$, let $V = (a_1, a_2, b) \in \mathbb{R}^3$ and find two functionally independent first integrals ϕ_1 and ϕ_2. The General solution is $F(\phi_1, \phi_2)$ for any function F.

We discuss the *geometric configuration* between the initial data and the initial curve to guarantee the unique existence of the solution *or* the submanifold containing the initial curve.

Recall that the vector field V transversal to the curve $C(t) = (x(t), y(t), u(t))$ has the unique integral manifold containing the curve. Note that the vector field V

defined on C is transversal to C locally near $t=0$ if and only if $V(x(0),y(0),u(0))$ is transversal to $C'(0)$.

DEFINITION 1.17. The initial curve $C(t)=(x(t),y(t),u(t))$ is non-characteristic if

$$\det\begin{pmatrix} x'(t) & y'(t) \\ a_1(x(t),y(t),u(t)) & a_2(x(t),y(t),u(t)) \end{pmatrix} \neq 0$$

For the non-characteristic initial curve given, we state without a proof the following basic fact.

THEOREM 1.18. *If the initial curve $C(t)$ is non-characteristic at $t=0$, then there exists the unique solution to the initial value problem.*

REMARK 1.19.

(1) If $\det\begin{pmatrix} x'(0) & y'(0) \\ a_1(x(0),y(0),u(0)) & a_2(x(0),y(0),u(0)) \end{pmatrix} = 0$ and $x'(0)/a_1(0)=y'(0)/a_2(0)\neq u'(0)/b$ then there exists no solution.
(2) If $x'(t)/a_1(t)=y'(t)/a_2(t)=u'(t)/b(t)$ i.e. $C(t)$ is an integral curve of V then there exist infinitely many solutions. Note that we let $a_i(t):=a_i(x(t),y(t),u(t))$ and $b(t):=b(x(t),y(t),u(t))$ here.

Generally let $u(x_1,\dots,x_n)$ defined on $x=(x_1,\dots,x_n)\in\Omega\subset\mathbb{R}^n$ be a function that solves

$$a_1(x,u)u_1+\cdots+a_n(x,u)u_n=b(x,u) \tag{1.6}$$

with initial data along a $n-1$ dimensional submanifold C. We let C be parametrized in $t:=(t_1,\cdots,t_{n-1})$ such that C is given by

$$\begin{cases} x_1=x_1(t) \\ \vdots \\ x_n=x_n(t). \end{cases}$$

Then the initial data is given by $u(t)=u(C(t))$.

DEFINITION 1.20. The initial data $(x(t),u(t))$ is non-characteristic if

$$\det\begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ a_1(x(t),u(t)) & \cdots & a_n(x(t),u(t)) \end{pmatrix} \neq 0$$

along C.

REMARK 1.21. If the equation is almost linear i.e.the coefficients function $a_j=a_j(x)$, we say that an initial surface i.e. a $n-1$ dimensional submanifold $x=x(t)$ is non-characteristic if

$$\det\begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \\ a_1(x(t)) & \cdots & a_n(x(t)) \end{pmatrix} \neq 0$$

Note that it is an initial *data* that is called non-characteristic for *quasi-linear equations* and an initial *hypersurface* for *almost linear equations.*

As for two dimensional case, we have the follwing.

THEOREM 1.22. *A quasi-linear partial differential equation (1.6) is given. If the initial data* $(x(t), u(t))$ *is non-characteristic on a neighborhood of* $t = 0$*, then there exists unique solution* $u = u(x)$ *of the initial value problem on a neighborhood of* $x(0)$*.*

COROLLARY 1.23. *For an almost linear 1st order partial differential equation, let* $\mathcal{S}$ *be an* $n-1$ *dimensional submanifold of* $\Omega \subset \mathbb{R}^n$*. If* $\mathcal{S}$ *is non-characteristic, there exists the unique solution for arbitrary initial data along* $\mathcal{S}$*.*

EXAMPLE 1.24. Find $u(x, y)$ defined on $(x, y) \in \mathbb{R}^2$ that solves

$$(y+u) \cdot u_x + y \cdot u_y = x - y$$

Initial data: $u = 1 + x$ on $y = 1$.

SOLUTION. First find the integral surface of the associated vector field $V = (y+u, y, x-y)$.

$$\frac{dx}{y+u} = \frac{dy}{y} = \frac{du}{x-y} \quad = \quad \frac{d(x+u)}{x+u} = \frac{d(x-y)}{u}$$

the first three are equations for integral curves and the fourth is obtained by combining the first and third ones, the fifth by combining the first and second ones. Equating the second and the fourth terms $\log y = \log(x+u) + \text{constant}$ hence $(x+u)/y =: \phi_1 = \text{constant}$. Equating the third and the fifth $(x-y)d(x-y) = u\, du$ hence $(x-y)^2 - u^2 =: \phi_2 = \text{constant}$. Now the general solution is $F((x+u)/y, (x-y)^2 - u^2) = 0$ for a function F. Along the initial curve,

$$\phi_1 = 2x + 1, \qquad \phi_2 = (x-1)^2 - (x+1)^2 = -4x,$$

hence our solution is $2(\phi_1 - 1) + \phi_2 = 0$ i.e.

$$2\left(\frac{x+u}{y} - 1\right) + (x-y)^2 - u^2 = 0.$$

The initial curve $C : x \to (x, 1, x+1)$ has $C'(x) = (1, 0, 1)$ and $V = (x+2, 1, x-1)$ on C. $\det \begin{pmatrix} 1 & 0 \\ x+1 & 1 \end{pmatrix} = 1 \neq 0$ and the initial data is non-characteristic, which implies the uniqueness of our solution .

EXERCISE 1.25.

(1) $V = (1, 1, z)$ is a vector field defined on $(x, y, z) \in \mathbb{R}^3$.
   (a) Find integral curves.
   (b) Find the integral surface containing the curve $C(t) = (t, 0, \cos t)$ for $-\epsilon < t < \epsilon$.
   (c) Find the solution $z(x, y)$ to the following initial value problem

$$\begin{cases} z_x + z_y = z \\ z(x, 0) = \cos x. \end{cases}$$

(2) Find the solution $z = z(x, y)$ to

$$x(y-z)z_x + y(z-x)z_y = z \cdot (x-y).$$

(3) For $z \cdot z_x + z \cdot z_y = x$ we impose the following initial conditions on the curve $x = t,\ y = t,\ t > 0$. Discuss the existence and uniqueness of the solutions.
   (a) $z = 2t$
   (b) $z = \sin(\pi/2t)$
   (c) Find $f(t)$ such that there are infinitely many solutions for the initial condition $z = f(t)$.

1.5. One dimensional conservation law. Let x denote the position on the real line and t the time. Consider some fluids flowing on the real line. Define $\rho(x,t)$ to be the density of the fluid at the specific position and time and $q(x,t)$ the flux.

$$\boxed{\rho_t + q_x = 0}$$

is called *one dimensional conservation law*, which is quivalent to $\mathrm{Div}(\rho(x,t), q(x,t)) = 0$. It is the mass conservation law for fluids.

Physical motivation. Consider a small compartment $I = [x, x+dx]$, an interval on the real line and the fluid which stay on this compartment at the moment. Total mass of the fluids that stay on I is $\int_x^{x+dx} \rho(x,t)dx$ and the out-flow rate of fluids is the time derivative of this total mass. But the out-flow occurs only at the endpoints x, $x + dx$ and the rate of out-flow is the sum of *flux* at the endpoints $-(q(x+dx,t) - q(x,t))$ taking into accout the sign. We have two expressions for out-flow rate

$$\frac{d}{dt}\int_x^{x+dx} \rho(x,t)dx = -(q(x+dx,t) - q(x,t))$$

,which we divide by dx, pass $dx \to 0$ to get the desired partial differential equation.[2]

[2] This model is used also for traffic control problem.

Bibliography

[Zach] Zachmanoglou

1.5. Initial value problem for 1 dimensional conservation law. Let $\rho(x,t)$ and $q(x,t)$ be density and flux of fluid at $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

$$\boxed{\rho_t + q_x = 0}$$

is called *1 dimensional conservation law.* This is mass conservation law of fluid. We restrict our consideration to ρ and q which are dependent only on a function $u(x,t)$ i.e. $\rho = \rho(u(x,t))$ and $q = q(u(x,t))$.

EXAMPLE 1.1. Let $u(x,t)$ solve

$$\begin{cases} a(u) \cdot u_x + u_t = 0 \\ u(x,0) = f(x). \end{cases}$$

for some function $a(\cdot)$ and $f(\cdot)$. Let $A'(u) = a(u)$ for a function $A(\cdot)$ then (1.1) becomes $(A(u))_x + u_t = 0$, which is a conservation law if u and $A(u)$ are regarded as density and flux respectively. So (1.1) is the initial value problem of 1 dimensional conservation law. To find the solution, consider the associated vector field $(a(u(x,t)), 1, 0)$ on $\{(x,t,u)\}$ and obtain its 1st integrals from the following equation for integral curves.

$$\frac{dx}{a(u)} = \frac{dt}{1} = \frac{du}{0}$$

From the second identity $\phi_1(x,t,u) := u$ is constant along integral curves. The first identity implies that $dx - a(u)dt = 0$ so that $\phi_2(x,t,u) := x - a(u)t$ is constant along integral curves. Note that $a(u)$ is kept constant since u is. Now ϕ_1 and ϕ_2 are functionally independent 1st integrals so that every integral surface is given by $F(\phi_1, \phi_2) = 0$ for a function F. Solve this for u to get the general solution for (1.1). Along initial curve $(x, 0, f(x))$, $\phi_1 = f(x)$ and $\phi_2 = x$. Hence $\phi_1 - f(\phi_2) = 0$. Let $F(\phi_1, \phi_2) := u - f(x - a(u)t) = 0$ and we solve this for $u = u(x,t)$.

To solve it for u requires the implicit function theorem condition

$$F_u = 1 - f'(x - a(u)t) \cdot (-a(u)t) = 1 + f'(x - a(u)t) \cdot a'(u)t \neq 0. \tag{1.1}$$

Now will this initial value problem have the unique solution? We need to check that the initial value $u = f(x)$ at $t = 0$ is noncharacteristic. Actually

$$\det \begin{bmatrix} 1 & 0 \\ a(f(x)) & 1 \end{bmatrix} \neq 0$$

at $t = 0$. Note that this noncharacteristic conditon holds true whatever f is given. In view of (1.1), if $|t|$ is sufficiently small there exists the solution of the form $u = u(x,t)$. But what if the time t elapses further? From $F(\phi_1(x,t,u), \phi_2(x,t,u)) = F(x,t,u) = 0$,

$$u_x = -\frac{F_x}{F_u} = -\frac{-f'}{1 + f'(x - a(u)t)a'(u)t} \tag{1.2}$$

$$u_t = -\frac{F_t}{F_u} = -\frac{-f'a(u)}{1 + f'(x - a(u)t)a'(u)t}. \tag{1.3}$$

In case $f' \neq 0$, if $|t|$ increases on to make denominators of (1.2) and (1.3) approach 0, u_x and u_t blow up to $\pm\infty$, which we call *shock*.

EXAMPLE 1.2. Let $x \in \mathbb{R}$ and $u(x,t)$ be the solution to

$$\begin{cases} u \cdot u_x + u_t = 0 \\ u(x,0) = -x. \end{cases} \tag{1.4}$$

Consider in $\{(x,t,u)\}$

$$\frac{dx}{u} = \frac{dt}{1} = \frac{du}{0}.$$

Solutions are $\phi_1 := u =$ constant from the second identity and $\phi_2 := x - ut =$constant by $dx - udt = 0$ from the first identity. The general solution is $F(u, x - ut) = 0$ for some function F. Along the initial curve $\phi_1 = -x$ and $\phi_2 = x$, hence $\phi_1 + \phi_2 = 0$ i.e. $u + x - ut = 0$. Hence the solution to the initial value problem is $F(x,t,u) = (1-t)u + x = 0$, which is $u = -\frac{x}{1-t}$ for $|t| \approx 0$. $F_u = 1 - t$ indicates that there is a shock at $t = 1$. The level curves of the solution describes the shock in geometric manner. Level curves for $u = 0$, $u = 1$ and $u = 2$ are $x = 0$, $t = x + 1$ and $t = x/2 + 1$ respectively, which intersect one another at $x = 0$ and $t = 1$. This means that the *flows* continues smoothly while $t < 1$ but it runs into the infinite increase or decrease, namely *shock* at $t = 1$.

EXERCISE 1.3. Let $u(x,t)$, $x \in \mathbb{R}$ be the function that solves an initial value problem of 1 dimensional conservation law

$$\begin{cases} u_t + 2uu_x = 0 \\ u(x,0) = 10 - x. \end{cases}$$

(1) Find a local solution near $(x,t) = (0,0)$.
(2) Find level curves in $\{(x,t)\}$ plane, for example $u = 5$, $u = 10$ etc.
(3) When does the shock occur?

CHAPTER 2

Cauchy Kowalesky Theorem

1. Characteristic of linear partial differential oprators

Let $x = (x_1, \dots, x_n) \in \Omega$ an open subset in $\mathbb{R}^n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

For $x \in \mathbb{R}^n$ and $u(x)$ a function in Ω we put $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha u(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u(x)$.

Definition 1.1. For a nonnegative integer k and functions $a_\alpha(x)$ and $f(x)$

$$\sum_{|J| \le k} a_\alpha(x) \cdot \partial^\alpha u(x) = f(x)$$

is called *linear partial differential equation* of order k.

Remark 1.2. Note that the coefficient functions a_α depend only on x not on u.

Our first concern about such equations is the *characteristic* of the linear partial differential operator involved. Roughly speaking, the notion of *characteristic* is the "strength" of a linear partial differential operator

$$L = \sum_{|\alpha| \le k} a_\alpha(x) \cdot \partial^\alpha \tag{1.1}$$

in a certain direction.

Definition 1.3. For (1.1), the *characteristic form* at $x \in \Omega$ is the homogeneous polynominal of degree k defined by

$$\chi_L(x, \xi) := \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha \tag{1.2}$$

for nonzero vector ξ in $\mathbb{R}^n$. This is also called *principal symbol* of (1.1).

Definition 1.4. A vector $\xi \ne 0$ is characteristic for L at x if $\chi_L(x, \xi) = 0$. The set of characteristic vectors, denoted by $\mathsf{char}_x L$, is called *characteristic variety.*

Proposition 1.5. *For (1.1) the following holds.*

(1) $\mathsf{char}_x L$ *is intrinsically defined i.e. independent of co-ordinates.*
(2) L *is said to be elliptic if* $\chi_L(x, \xi) \ne 0$ *for any* $x \in \Omega$ *and nonzero* $\xi \in \mathbb{R}^n$. *Such notion of ellipticity is intrinsic.*

Proof. Suppose that (1.1) is defined on an open set U_p of n dimensional manifold M with $p \in M$. Let $\mathcal{X} : U_p \to \Omega$ and $\mathcal{Y} : U_p \to \Omega'$ be two local co-ordinate charts. Their transition map is a diffeomorphism $y = F(x)$ from $x \in \Omega$ to $y \in \Omega'$

where $F := \mathcal{Y} \circ \mathcal{X}^{-1}$. Jacobian of F is

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

How the vector and co-vector fields components change per co-ordinates changes are as follows. Tangent vectors in two co-ordinate expressions are

$$a_j \frac{\partial}{\partial x_j} = b_i \frac{\partial}{\partial y_i}.$$

Noting that $\frac{\partial}{\partial x_j} = \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i}$ we have

$$a_j \frac{\partial y_i}{\partial x_j} = b_i \qquad \text{or} \qquad \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = J \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

i.e. $\underbrace{\mathbf{b}}_{new} = J \cdot \underbrace{\mathbf{a}}_{old}$. *Tangent vector components transform is Jacobian multiplication.* Cotangent vectors in two co-ordinates expressions are

$$\eta_j dx_j = \xi_i dy_i.$$

Noting $dy_i = \frac{\partial y_i}{\partial x_j} dx_j$,

$$\eta_j = \xi_i \frac{\partial y_i}{\partial x_j} \qquad \text{or} \qquad \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} = J^t \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

i.e. $\eta = J^t \xi$ hence $\underbrace{\xi}_{new} = (J^t)^{-1} \cdot \underbrace{\eta}_{old}$. The diffeomorphic transition map $y = F(x)$ and transformation rule $\frac{\partial}{\partial x_i} = \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$ change (1.1) and (1.2) into

$$L' = \sum_{|\alpha| \le k} a_\alpha(F^{-1}(y))(J^t \partial_y)^\alpha \tag{1.3}$$

$$\chi_{L'}(y, \xi) = \sum_{|\alpha| = k} a_\alpha(F^{-1}(y))(J^t \xi)^\alpha \tag{1.4}$$

respectively. Comparison of (1.2) and (1.4) shows that

$$\xi \in \mathsf{char}_y(L') \quad \Rightarrow \quad (J^t \xi) \in \mathsf{char}_{\underbrace{F^{-1}(y)}_{=x}}(L)$$

and characteristic forms obey tangent vector transformation rules. Hence we define the characteristic variety as a subset of cotangent space to have them intrinsic. It now follows easily that the ellipticity is also intrinsic in view of its definition.

Definition 1.6. A hypersurface $\mathcal{S}$ in Ω is *characteristic* at x for L in (1.1) if normal vector $\nu(x)$ to $\mathcal{S}$ is in $\mathsf{char}_x(L)$. $\mathcal{S}$ is *non-characteristic* if $\mathcal{S}$ is not characteristic at any point.

2. Non-characteristic directions

For the linear partial differential operator $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ and $\xi \in \mathsf{char}_x(L)$, we can choose suitable co-ordinates[1] to have $\xi \in (0,\dots,0,\underbrace{1}_{jth},0,\dots,0)$ after co-ordinate transformation. Then

$$\begin{cases} \xi \in \mathsf{char}_x(L) & \Leftrightarrow \quad \text{the coefficient to } \left(\frac{\partial}{\partial x_j}\right)^k \text{ vanishes at } x, \\ \xi \notin \mathsf{char}_x(L) & \Leftrightarrow^{2} \quad \text{the coefficient to } \left(\frac{\partial}{\partial x_j}\right)^k \text{ is nonzero at } x. \end{cases}$$

For the second case $Lu = f$ can be solved for $\partial_j^k u$ to give

$$\partial_j^k u = G(x, \partial^\alpha u : |\alpha| \leq k, \alpha \neq (0,\dots,\underbrace{1}_{jth},\dots,0)),$$

which shows that the partial differential equation $Lu = f$ has *control* over the solution u in ξ direction.
Note that L is elliptic at x if $\mathsf{char}_x(L)$ is an empty set and L is elliptic on Ω if it is elliptic at every $x \in \Omega$.

EXAMPLE 2.1.

(1) For $u(x,y)$ defined on $\mathbb{R}^2$, consider $\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$. Note that the coefficients are constants, hence independent of x. $\chi_\Delta(x,\xi) = \xi_1^2 + \xi_2^2 \neq 0$ for all$(\xi_1,\xi_2) \neq 0$ which implies that it is elliptic.

(2) Consider the wave operator $L = \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2$. The coefficients are also constant, independent of x. $\chi_L(x,\xi) = \xi_1^2 - \xi_2^2$ vanishes for some nonzero vector ξ and hence L is not elliptic.

(3) For $u(x,t)$ defined on an open set of $\mathbb{R}^2$, consider the heat operator $L = \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2$. $\chi_L(x,\xi) = (\xi_1)^2$ admits a non-characteristic vector $\xi = (0,1)$ and hence L is not elliptic.

[1] Rotations and dilations etc.

1. Solution manifolds in Jet space and its local diffeomorphic transformation

Let M be an open subset of $\mathbb{R}^n$ or more generally N-manifold , $F = (F_1, \ldots, F_l)$ be a system of $\mathcal{C}^\infty$ functions for positive integers $l < N$ and $\mathcal{S}_F$ be a zero set of $F = \{x \in \mathbb{R}^N : F_1(x) = \cdots = F_l(x) = 0\}$. Assume F is of maximal rank on $\mathcal{S}_F$ i.e. $\left(\frac{\partial F_\nu}{\partial x_\mu}\right)$ is of rank l or equivalently $dF_1, \ldots, dF_l$ are linearly independent on $\mathcal{S}_F$. Then $\mathcal{S}_F$ is a $\mathcal{C}^\infty$ manifold.

Proposition 1.1. *A smooth function f is defined on M. f vanishes on $\mathcal{S}_F$ if and only if $f = Q_1F_1 + \cdots + Q_lF_l$ for some $\mathcal{C}^\infty$ functions $Q_1, \ldots, Q_l$, i.e. f belongs to the ideal generated by $F_1, \ldots, F_l$ in the ring of $\mathcal{C}^\infty$ functions.*

Let X be an open subset of $\mathbb{R}^p$ and $U := \{(u^1, \ldots, u^q)\}$ be an open set in $\mathbb{R}^q$. Suppose there is a function f such that $u := f(x)$ for $x \in X$ and $u \in U$. The graph $\Gamma_f = \{(x, f(x)) \in X \times U\}$ is a p-dimensional submanifold of $X \times U$.
Let g be a local diffeomorphism $X \times U \to X \times U$. We let

$$g(\Gamma_f) = \{(\tilde{x}, \tilde{u}) = g(x, u) : (x, u) \in \Gamma_f\} := \Gamma_{\tilde{f}}.$$

Note that we consider only the infinitesimal transform of the identity component so that the graph of the function $u = f(x)$ is transformed by a diffeomorphism g to define graph of another function $\tilde{u} = \tilde{f}(\tilde{x})$. We write

$$g \circ f := \tilde{f}$$

, which we call the *transform* of f by g.

Example 1.2. Let $p = q = 1$, $X = \mathbb{R}$ and $G = SO(2)$[1]. Take the rotation $\Theta \in G$ as our diffeomorphic transformation. Then $\Theta(x, u) = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta) = (\tilde{x}, \tilde{u})$. Consider the graphs $u = ax + b = f(x)$. Substituting $u = -\tilde{x}\sin\theta + \tilde{u}\cos\theta$ and $x = \tilde{x} + \tilde{u}\sin\theta$ for $u = ax + b$, we have the graph $\tilde{u} = \frac{a\cos\theta + \sin\theta}{\cos\theta - a\sin\theta}\tilde{x} + b := \tilde{f}(\tilde{x})$.

Definition 1.3. For $x \in (x^1, \ldots, x^p) \in X$ and $u \in (u^1, \ldots, u^q) \in U$, the n-th jet space of $X \times U$ is

$$X \times U^{(n)} := \{(x, u^{(n)})\}$$

, which is endowed with Euclidean structure and smooth topology.

Definition 1.4. Given a system of partial differential equations of order n

$$\Delta_\nu(x, u^{(n)} = 0, \qquad \nu = 1, 2, \ldots, l$$

, where $\Delta = (\Delta_1, \ldots, \Delta_l)$, the system of $\mathcal{C}^\infty$ functions defined on $X \times U^{(n)}$, We define

$$\mathcal{S}_\Delta := \text{ zero set of } \Delta \text{ i.e. } \{\Delta = 0\}.$$

Remark 1.5. We only consider the case for which $\mathcal{S}_\Delta$ is smooth manifold i.e. $d\Delta_1, \ldots, d\Delta_l$ is of maximal rank.

[1] $O(2)$ has two components with the signature of the determinant ± 1. $SO(2)$ is the identity component of the two.

Hence we have the following equivalent notions.

$$u = f(x) \text{ is a solution of } \Delta = 0 \tag{1.1}$$

$$\iff \quad \Delta_\nu(x, f^{(n)}(x)) = 0, \quad \nu = 1, 2, \ldots, l \tag{1.2}$$

$$\iff \quad (x, f^{(n)}(x)) \in \mathcal{S}_\Delta. \tag{1.3}$$

2. Prolongation of vector fields and infinitesimal symmetries

2.1. Prolongation of local diffeomorphisms.

DEFINITION 2.1. Let M be an open subset of $X \times U$ and g a local diffeomorphism $M \to M$. Then $\mathsf{pr}^n g : M^{(n)} \to M^{(n)}$, the n-th prolongation of g on $M^{(n)} = \{(x, u^{(n)}) : (x, u) \in M\}$ is defined as follows. For all $(x_0, u_0^{(0)}) \in M^{(n)}$, take any function $u = f(x)$ such that $(x_0, f^{(n)}(x_0)) = (x_0, u_0^{(n)})$ and let $\tilde{u}(\tilde{x}) = (g \circ f)(\tilde{x})$. Then

$$\mathsf{pr}^n g(x_0, u_0^{(n)}) := (\tilde{x_0}, \tilde{u_0}^{(n)}(\tilde{x_0}))$$

, where $(\tilde{x_0}, \tilde{u_0}) = g(x_0, u_0)$. This is well-defined i.e. independent of choice of f.

REMARK 2.2. In (x, u) space, 1-jet of $u = f(x)$ may be considered as slopes of some line elements in its graph. Transform image of this graph by a local diffromorphism g is put $\tilde{u} = \tilde{f}(\tilde{x})$ in new coordinates. Calculate the slopes of this new graph. The process of assigning new slopes to old slopes when the graph is being transformed by g is 1st prolongation of g in naive sense.

2.2. Prolongation of group actions.

Let G be a local group of transformation acting on M. Then $\mathsf{pr}^n G := \{\mathsf{pr}^n g : g \in G\}$ acts on $M^{(n)}$.

EXAMPLE 2.3. Example1.2 continued. Suppose that $\mathsf{pr}^1\Theta : X \times U^{(1)} \to X \times U^{(1)}$ sends $(x_0, u_0, u_0') \to (\tilde{x}, \tilde{u}, \tilde{u}')$. Then $\mathsf{pr}^1\theta(x_0, u_0, u_0') = (x_0 \cos\theta - u_0 \sin\theta, x_0 \sin\theta + u\cos\theta, \frac{u_0'\cos\theta + \sin\theta}{\cos\theta - u_0'\sin\theta})$. Dropping 0 subscripts we have generally

$$\mathsf{pr}^1\theta(x, u, u') = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{u_x\cos\theta + \sin\theta}{\cos\theta - u_x\sin\theta})$$

2.3. Prolongation of Vector fields.

DEFINITION 2.4. Let M be an open subset of $X \times U$. Let V be an vector field on M and $\varphi_\varepsilon := \exp(\varepsilon V)$ is 1 parameter group of local diffeomorphisms, which are *flows*. Then the prolongation of vector field V, $\mathsf{pr}^n V$ is a vector field on $M^{(n)}$ defined by

$$\mathsf{pr}^n V(x, u^{(n)}) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathsf{pr}^n(\exp\varepsilon V)(x, u^{(n)}).$$

EXAMPLE 2.5. Let $p = q = 1$, $X = \mathbb{R}$ and $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$. Then $\exp(\varepsilon V)$ is a rotation by angle ε which is calculated as follows. Noting $V = (-u, x)$,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x(\varepsilon) \\ u(\varepsilon) \end{pmatrix} = e^{\varepsilon A} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$e^{\varepsilon A} = I + \varepsilon A + \frac{\varepsilon^2}{2} A^2 + \cdots = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}.$$

Its action on jets is given by

$$\begin{aligned} \mathsf{pr}^1 V(x,u,u_x) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathsf{pr}^1 \exp(\varepsilon V)(x,u,u_x) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (x\cos\varepsilon - u\sin\varepsilon, x\sin\varepsilon + u\cos\varepsilon, \frac{u_x\cos\varepsilon + \sin\varepsilon}{\cos\varepsilon - u_x\sin\varepsilon}) \\ &= (-u, x, 1+u_x^2). \end{aligned}$$

Hence $\mathsf{pr}^1 V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}$.

EXAMPLE 2.6. Given $u(x,y)$ and Laplace equation $u_{xx} + u_{yy} = 0$. 2nd jet space is $\{(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})\} \subset X \times U^{(2)} \subset \mathbb{R}^8$. Let the equation be $\Delta(x,u^{(2)}) := u_{xx} + u_{yy} = 0$ then $\mathcal{S}_\Delta = \{\Delta = 0\}$ is a hypersurface since Δ is of maximal rank on its zero set with the Jacobian $(0,\ldots,0,\underbrace{1}_{6th},0,1)$.

2.4. Symmetry groups of partial differential equations.

DEFINITION 2.7. Let G be a local group of transformations acting on $X \times U$ and $\Delta = 0$ with $\Delta = (\Delta_1, \ldots, \Delta_l)$ be a system of partial differential equations of order n. G is a symmetry group of $\Delta = 0$ if $\mathsf{pr}^n g$ sends $\mathcal{S}_\Delta$ into $\mathcal{S}_\Delta$ for every $g \in G$ or equivalently,

$$\mathsf{pr}^n V(\Delta_\nu) = 0 \text{ on } \mathcal{S}_\Delta$$

for every $\nu = 1, 2, \ldots, l$ and every infinitesimal generator V of G.

DEFINITION 2.8. By a differential function of order k we mean a $\mathcal{C}^\infty$ function $P(x,u^{(n)})$ defined on an open subset of $X \times U^{(n)}$. By the *total derivative* of P we mean

$$D_i P = D_{x_i} := \frac{\partial P}{\partial x_i} + \sum_{\substack{\alpha=1,\ldots,q \\ |J| \le n}} \frac{\partial P}{\partial u_J^\alpha} u_{J,i}^\alpha.$$

The total derivative of P is a differential function of order $n+1$.

EXAMPLE 2.9. Let $u(x,y)$ be defined on $\mathbb{R}^2$ then a tota derivative

$$D_x(xu + u_x + u_y^2) = u + xU_x + u_{xx} + 2u_y u_{xy}$$

Lecture 08

1. Review

Prolongation of diffeomorphisms. A local diffromorphism $g : X \times U \to X \times U$ induces $\mathsf{pr}^n g : X \times U^{(n)} \to X \times U^{(n)}$, which is defined as follows. Take any function $u = f(x)$ such that $u^{(n)}(x) = f^{(n)}(x)$. Then transform this by g [1] to get new function $\tilde{u} = \tilde{f}(\tilde{x})$ for which we let $\tilde{f} =: g \circ f$. Now we define the image of $(x, u^{(n)})$ under $\mathsf{pr}^n g$ is $(\tilde{x}, \tilde{f}^{(n)}(\tilde{x}))$.
For V a vector field on $X \times U$, $\mathsf{pr}^n V$ is a vector field on $X \times U^{(n)}$ defined as follows. Let $g_\varepsilon := \exp(\varepsilon V)$ which is a local diffeomorphism of $X \times U$. Then $\mathsf{pr}^n g_\varepsilon(x, u^{(n)})$ is a curve in $X \times U^{(n)}$ parametrized by ε. We identify $\mathsf{pr}^n V$ with $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathsf{pr}^n g_\varepsilon(x, u^{(n)})$.

Symmetries of differential equations. Given a system of partial differential equations $\Delta(x, u^{(n)}) = 0$ with $\Delta = (\Delta_1, \ldots, \Delta_l)$ and $\mathcal{S}_\Delta$ the zero set of Δ, a symmetry g refers to a diffeomorphism $X \times U$ to itself such that $\mathsf{pr}^n g : \mathcal{S}_\Delta \to \mathcal{S}_\Delta$.

Remark 1.1. Note this definition is different from saying $\mathsf{pr}^n g$ sends a solution to another solution. For a certain class of differential equations has the solution at every point of $\mathcal{S}_\Delta$. But some differential equation e.g. over-determined system of differential equation has the solution only at some points of $\mathcal{S}_\Delta$.

If g is such a diffeomorphism and if $u = f(x)$ is a solution of $\Delta = 0$ then $\tilde{u} = (g \circ f)(\tilde{x})$ is also a solution of $\Delta = 0$. In fact, let f be a solution then $(x, f^{(n)}(x)) \in \mathcal{S}_\Delta$ for all $x \in X$. Now $(\tilde{x}, \tilde{f}^{(n)}(\tilde{x})) = \mathsf{pr}^n g(x, f^{(n)}(x)) \in \mathcal{S}_\Delta$. Hence $\tilde{u} = \tilde{f}(\tilde{x})$ is a solution. This justifies why we call g a *symmetry* of $\Delta = 0$.

2. Infinitesimal symmetries

Infinitesimal symmetries. We look for vector fields V on $X \times U$ such that $\mathsf{pr}^n V$ is tangent to $\mathcal{S}_\Delta \subset X \times U^{(n)}$. Such V is called an *infinitesimal symmetry* of $\Delta = 0$.

Remark 2.1. The space of infinitesimal symmetries is essentially equivalent to the local symmetric group of identity component. If one is finite dimensional so is the other and vice versa. But the infinitesimal symmetries are preferred since it admits concrete calculation many times.

Note $(\mathsf{pr}^n V)\Delta = 0$ on $\Delta = 0$ if and only if $(\mathsf{pr}^n V)\Delta = \sum_{i=1}^q Q_i \cdot \Delta_i$ for some differential function $Q_i(x, u^{(n)})$.

Properties of pr^n*: prolongation of vector fields.* Let $\mathcal{X}(\cdot)$ denote the space of smooth vector fields defined locally on the given manifold. pr^n is a map $\mathcal{X}(X \times U) \to \mathcal{X}(X \times U^{(n)})$ such that

(1) $\mathsf{pr}^n(aV + bW) = a\mathsf{pr}^n V + b\mathsf{pr}^n W$
(2) $\mathsf{pr}^n[V, W] = [\mathsf{pr}^n V, \mathsf{pr}^n W]$

for constants a, b and local smooth vector fields V, W. Hence pr^n is a Lie algebra homomorphism.

Example 2.2. Let $p = q = 1$ and $u(x)$ solve $\Delta(x, u, u_x) = (u - x)u_x + u + x = 0$. Show $SO(2)$ is a symmetry group.

Solution. Symmetric group itself is difficult to find by calculation whereas infinitesimal generators thereof are *more calculable*. The reason is that they are solved

[1] We consider the case g is close enough to identity mapping.

from the linearized differential equations. Once the generator V found, $\exp(\varepsilon V)$ gives the symmetric group element. In the viewpoint above, we are well enough to show $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$, the generator of $SO(2)$ is an infinitesimal symmetry of $\Delta = 0$. Recalling $\mathsf{pr}^1 V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}$, we have

$$\begin{aligned}(\mathsf{pr}^1 V)\Delta &= -u(-u_x+1) + x(u_x+1) + (1+u_x^2)(u-x)\\ &= u_x\cdot(u + x + uu_x - xu_x)\\ &= u_x \cdot \Delta\end{aligned}$$

which is 0 on $\Delta = 0$. ||

This is how this example was found. An ordinary differential equation

$$\frac{dr}{d\theta} = r$$

in polar co-ordinates (r,θ) has the solution $r(\theta) = r(0)\exp(\theta)$ whose graph spirals out. This obviously solves the same ordinary differential equation after *rotation* and so $SO(2)$ is a symmetric group of this ordinary differential equation. We convert this into $(x,u) \in \mathbb{R}^2$ co-ordinates by $x = r\cos\theta$ and $u = r\sin\theta$ to get $(u-x)u_x + u + x = 0$.

3. Prolongation formula for vector fields

THEOREM 3.1. *Let*

$$V = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field on $X \times U$. *Then*

$$\mathsf{pr}^n V = V + \sum_{\alpha=1}^{q}\sum_{|J|\le n} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha},$$

where

$$\phi_\alpha^{J,k} = D_k(\phi_\alpha^J) - \sum_{i=1}^{p}(D_k\xi^i)u_{Ji}^\alpha.$$

The multi index $J = (j_1 \dots j_m)$, $j_i = 1\dots p$ *is understood as unordered,* $|J\,| := m$ *and* $Jk := (j_1 \dots j_m\ k)$, $j_i, k = 1\dots p$.

EXAMPLE 3.2. We verify this formula by the previous example. For $p = q = 1$, let $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$ and $\mathsf{pr}^1 V = V + \phi^x\frac{\partial}{\partial u_x}$. According to the formula above, $\phi^x = D_x\phi - ((D_x\xi)u_x = 1 - (-u_x)u_x = 1 + u_x^2$ the same result as before.

3.1. Derivation of formula for special vector fields.

(1). Let $V = \sum_{i=1}^{p} \xi^i(x)\frac{\partial}{\partial x^i}$ with $q = 1$ and $\exp(\varepsilon V) := g_\varepsilon$. Let $g_\varepsilon \cdot (x,u) = (\tilde{x},\tilde{u}) = (\Xi_\varepsilon(x),u)$ then $\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Xi^i = \xi^i(x)$. Put $(\mathsf{pr}^1 g_\varepsilon)(x,u^{(1)}) = (\tilde{x},\tilde{u}^{(1)}) = (\Xi_\varepsilon(x),u,\tilde{u}_j)$ where $\tilde{u}_j$ is to find. Let $u = f(x)$ be any function that fits $(x,u^{(1)})$ and let $\tilde{f}_\varepsilon = g_\varepsilon \cdot f$ be the transformed function which is given by $\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) =$

$f(\Xi_\varepsilon^{-1}(\tilde{x})) = f(\ \Xi_{-\varepsilon}(\tilde{x})$. Then $\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial \tilde{x^j}}(\tilde{x}) = \frac{\partial f}{\partial x^k}(\Xi_{-\varepsilon}(\tilde{x}))\cdot \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x}_j}(\tilde{x}) = \sum_{k=1}^p \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x^j}}(\tilde{x})\cdot u_k$ and so $\mathsf{pr}^1 g_\varepsilon(x, u^{(1)}) = (\ \Xi_\varepsilon(x), u, \sum_{k=1}^p \frac{\partial\ \Xi^k_{-\varepsilon}}{\partial \tilde{x^j}}(\tilde{x})\cdot u_k)$.

$$\begin{aligned}
\mathsf{pr}^1 V &= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + 0\cdot\frac{\partial}{\partial u} + \sum_{k=1}^p \frac{\partial}{\partial \tilde{x^j}}\left(\frac{\partial}{\partial \varepsilon}\Xi^k_{-\varepsilon}(\tilde{x})\right)\cdot u_k \\
&= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{k=1}^p \left[-\frac{\partial \xi^k}{\partial \tilde{x^j}}(\tilde{x}) + \sum_{i=1}^p \frac{\partial^2\ \Xi^k_{-\varepsilon}}{\partial \tilde{x}^j \partial \tilde{x}^i}\xi^i\right]\cdot u_k \Bigg|_{\varepsilon=0} \\
&= \sum_{i=1}^p \xi^i \frac{\partial}{\partial x^i} + \sum_{k=1}^p \left(-\frac{\partial \xi^k}{\partial x^j}(x)\right)\cdot u_k.
\end{aligned}$$

Therefore

$$\mathsf{pr}^1 V = V + \sum_{j=1}^p \left(-\sum_{k=1}^p \frac{\partial \xi^k}{\partial x^j}(x)\cdot u_k\right)\cdot\frac{\partial}{\partial u^j}$$

where we put $\phi^j = -\sum_{k=1}^p \frac{\partial \xi^k}{\partial x^j}(x)\cdot u_k$.

(2). Let $V = \phi(x,u)\frac{\partial}{\partial u}$ with $q = 1$ and $\exp(\varepsilon V) =: g_\varepsilon$. Set $g_\varepsilon(x,u) = (x, \Phi_\varepsilon(x,u))$. Then $\frac{\partial \Phi}{\partial \varepsilon}|_{\varepsilon=0} = \phi(x,u)$. For any function $u = f(x)$ let $\tilde{f}_\varepsilon := g_\varepsilon \cdot f$ then $\tilde{u} = \tilde{f}(\tilde{x}) = \Phi_\varepsilon(x, f(x))$ and $\tilde{u}_j = \frac{\partial \tilde{u}}{\partial \tilde{x^j}} = \frac{\partial \tilde{u}}{\partial x^j} = \frac{\partial \Phi_\varepsilon}{\partial x^j} + \frac{\partial \Phi_\varepsilon}{\partial u}\frac{\partial f}{\partial x^j}$. So, $g_\varepsilon(x, u, u^j, 1\le j\le p) = (x, \Phi_\varepsilon(x,u), \frac{\partial \Phi_\varepsilon}{\partial x^j} + \frac{\partial \Phi_\varepsilon}{\partial u}u_j, 1\le j\le p)$, which we differentiate in ε and evaluate at $\varepsilon = 0$ to get

$$\begin{aligned}
\mathsf{pr}^1 V &= \left(0, \phi(x,u), \frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u}\cdot u_j\right) \\
&= \phi\frac{\partial}{\partial u} + \sum_{j=1}^p \left(\frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u}\cdot u_j\right)\frac{\partial}{\partial u_j} \\
&= \phi\frac{\partial}{\partial u} + \sum_{j=1}^p \phi^j \frac{\partial}{\partial u_j}
\end{aligned}$$

where $\phi^j := \frac{\partial \phi}{\partial x^j} + \frac{\partial \phi}{\partial u_j}$.

1. Prolongation formula for general vector fields

Prolongation formula. Let $x \in (x^1, \dots, x^p)$ and $u = (u^1, \dots, u^q)$ and a vector field $V = \xi^i \frac{\partial}{\partial x^i} + \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$.[1] Then we have the following prolongation formula.

THEOREM 1.1. *For the vector field above,*

$$\mathsf{pr}^{(n)} V = V + \sum_{\alpha=1}^{q} \sum_{|J| \le n} \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

where

$$\phi_\alpha^J = D_J(\phi_\alpha - \xi^i \cdot u_i^\alpha) + \xi^i \cdot u_{J,i}^\alpha \tag{1.1}$$

Furthermore for all J,

$$\phi_\alpha^{J,k} = D_k(\phi_\alpha^j) - (D_k \xi^i) u_{J,i}^\alpha. \tag{1.2}$$

REMARK 1.2. Recall that $\mathsf{pr}^{(n)} V$ is a vector field on $X \times U^{(n)}$ and defined as follows. Let $\exp(\varepsilon V) =: g_\varepsilon$ and $\mathsf{pr}^{(n)} g_\varepsilon(x, u^{(n)}) = (\tilde{x}(\varepsilon), \tilde{u}(\varepsilon))$, which is a curve parametrized by ε in $X \times U^{(n)}$. Now $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \mathsf{pr}^{(n)} g_\varepsilon(x, u^{(n)}) = \mathsf{pr}^{(n)} V(x, u^{(n)})$.

PROOF. Assume $n = 1$. Let $(\tilde{x}, \tilde{u}) = g_\varepsilon(x,u) := (\Xi_\varepsilon(x,u), \Phi_\varepsilon(x,u)$. Then $\frac{\partial\, \Xi}{\partial \varepsilon}\big|_{\varepsilon=0} = \xi(x,u)$, $\frac{\partial\, \Phi}{\partial \varepsilon}\big|_{\varepsilon=0} = \phi(x,u)$. Given $(x, u^{(1)}) \in X \times U^{(1)}$, let $f(x)$ be any function that fits this point i.e. $f^{(1)}(x) = u^{(1)}$. Then

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) := (g_\varepsilon \cdot f)(\tilde{x}) = [\Phi_\varepsilon \circ (\mathbf{1} \times f)](x) = [\Phi_\varepsilon \circ (\mathbf{1} \times f)] \circ [\, \Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x}).$$

To get $\frac{\partial \tilde{u}^\alpha}{\partial \tilde{x}^k}$, we compute the Jacobian using the chain rule to have

$$[J\tilde{f}_\varepsilon](\tilde{x}) = J[\Phi_\varepsilon \circ (\mathbf{1} \times f)](x) \cdot J[\Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x})$$

whose (α, k) entry is $\tilde{u}_k^\alpha(\varepsilon)$. Differentiate in ε and evaluate the above at $\varepsilon = 0$ to find ϕ_α^k. Especially the right hand side becomes

$$J[\phi \circ (1 \times f)](x) \cdot I + Jf(x) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} J[\, \Xi_\varepsilon \circ (\mathbf{1} \times f)]^{-1}(\tilde{x})$$

which is $J[\phi \circ (\mathbf{1} \times f)](x) \cdot I - Jf(x) \cdot I \cdot J[\xi \circ (\mathbf{1} \times f)](x) \cdot I$ whose (α, k) entry is

$$\frac{\partial \phi^\alpha}{\partial x^k}(x, f(x)) - \frac{\partial f^\alpha}{\partial x^i} \cdot \frac{\partial \xi^i}{\partial x^k}(x, f(x)) = D_k \phi^\alpha - u_i^\alpha (D_k \xi^i).$$

That is,

$$\phi_\alpha^k := D_k \phi^\alpha - D_k \xi^i \cdot u_i^\alpha = D_k(\phi^\alpha - \xi^i u_i^\alpha) + \xi^i u_{ik}^\alpha$$

which is (1.2) for $n = 1$. We use induction. First note that $(n+1)$st jet $X \times U^{(n+1)}$ can be viewed as a subspace of $(X \times U^{(n)})^{(1)}$ as follows.

EXAMPLE 1.3. $p = 2$, $q = 1$. Then

$$\begin{aligned} X \times U^{(1)} &= \{(x, y, u, u_x, u_y)\} \\ X \times U^{(2)} &= \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\} \\ (X \times U^{(1)})^{(1)} &= \{(x, y, u, v, w, u_x, u_y, v_x, v_y, w_x, w_y)\} \end{aligned}$$

[1]We follow *Einstein* summation convention over *repeated indices.*

with $v = u_x$ and $w = u_y$. Regard $X \times U^{(2)}$ as a subset of $(X \times U^{(1)})^{(1)}$ defined by $u_x = v$, $u_y = w$ and $v_y = w_x$ etc.

We proceed on the induction on $|J|$. Note that $\mathsf{pr}^{(n-1)}V$ is a vector field on $X \times U^{(n-1)}$. For J with $|J| = n-1$

$$\phi_\alpha^{J,k} = D_k \phi_\alpha^J - D_k \xi^i \cdot u_{Ji}^\alpha$$

by the 1st prolongation formula, which still holds true for J with $|J| = n$. By induction hypothesis, this is in turn equal to

$$\begin{aligned} & D_k[D_j(\phi_\alpha - \xi^i u_i^\alpha) + \xi^i u_{Ji}^\alpha] - D_k \xi^i \cdot u_{Ji}^\alpha \\ = \quad & D_k D_J(\phi_\alpha - \xi^i u_i^\alpha) + D_k \xi^i \cdot u_{Ji}^\alpha + \xi^i u_{Jik}^\alpha - D_k \xi^i \cdot u_{Ji}^\alpha \end{aligned}$$

fulfilling (1.1).

Example 1.4. Let $V = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}$ be infinitesimal rotation on $\{(x,u)\} = \mathbb{R}^2$. then

$$\begin{aligned} \mathsf{pr}^{(1)}V &= V + (1+u_x^2)\tfrac{\partial}{\partial x}, \\ \mathsf{pr}^{(2)}V &= \mathsf{pr}^{(1)}V + (3u_x \cdot u_{xx})\tfrac{\partial}{\partial u_{xx}} \end{aligned}$$

since $D_x(1+u_x^2) - \underbrace{D_x(-u)}_{=D_x\xi} \cdot u_{xx} = 2u_x u_{xx} + u_x u_{xx}$. Our differential equation is $u_{xx} = 0$ whose solutions are group of straight lines. Rotation of straight lines gives also straight lines hence we know our V is an infinitesimal symmetry. To prove it, it is enough to show $\mathsf{pr}^{(2)}V u_{xx} = 0$ on $u_{xx} = 0$. Actually

$$\begin{aligned} \mathsf{pr}^{(2)}V u_{xx} &= \left(-u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_{xx}} + (3u_x u_{xx})\frac{\partial}{\partial u_{xx}}\right) u_{xx} \\ &= 3u_x u_{xx} \end{aligned}$$

which is 0 on $u_{xx} = 0$.

Example 1.5. *Differential Invariant.* Given the graph of the function $u = f(x)$, $x \in \mathbb{R}$, its curvature $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$ is rotation invariant. We remark here that κ is a function defined on $X \times U^{(2)}$, it is not V but $\mathsf{pr}^{(2)}V$ that is supposed to act on κ. Now $\mathsf{pr}^{(2)}V\left(\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right) = \frac{3u_x u_{xx}(1+u_x^2)^{3/2} - u_{xx}(1+u_x^2)^{3/2}(1+u_x^2)^{1/2}u_x}{(1+u_x^2)^3} = 0$. Hence κ is a differential invariant of second order under rotation group.

Theorem 1.6. *Let $\Delta = (\Delta_1, \ldots, \Delta_l)$ be a system of differential equations defined on an open subset M of $\times U$. Then the set $\mathfrak{g}$ of all infinitesimal symmetries forms a Lie algebra. If $\mathfrak{g}$ is finite dimensional, the connected component of the symmetric group of $\Delta = 0$ is a local Lie group of transformations acting on M.* [2]

Proof. Let V, W be infinitesimal symmetries of $\Delta = 0$. Suppose that $\Delta = 0$ is of order n. In view of $\mathsf{pr}^{(n)}[V,W] = [\mathsf{pr}^{(n)}V, \mathsf{pr}^{(n)}W]$, we have $\mathsf{pr}^{(n)}[V,W]\Delta = [\mathsf{pr}^{(n)}V, \mathsf{pr}^{(n)}W]\Delta = \mathsf{pr}^{(n)}V(\mathsf{pr}^{(n)}W\Delta) - \mathsf{pr}^{(n)}W(\mathsf{pr}^{(n)}V\Delta)$ which vanishes on $\mathcal{S}_\Delta$ since $\mathsf{pr}^{(n)}V$, $\mathsf{pr}^{(n)}W$ are tangent to $\mathcal{S}_\Delta$ and $\mathsf{pr}^{(n)}V\Delta$ and $\mathsf{pr}^{(n)}W\Delta$ are 0 on $\mathcal{S}_\Delta$. Therefore $[V,W]$ is an infinitesimal symmetry.

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[2] We only consider the Lie group of finite dimension.

2. Characteristic of Symmetries

For $V = \xi^i \frac{\partial}{\partial x^i} + \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$, let $Q_\alpha(x, u^{(1)}) := \phi_\alpha - \xi^i u_i^\alpha$, $\alpha = 1, \ldots, q$. The q tuple $Q(x, u^{(1)}) = (Q_1, \ldots, Q_q)$ is called *characteristic* of the vector field V. Then $\phi_\alpha^J = D_J Q_\alpha + \xi^i u_{Ji}^\alpha$ in (1.1) and

$$\begin{aligned}
\mathsf{pr}^{(n)}V &= \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{|J|\le n} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha} \\
&= \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{|J|\le n} (D_J Q_\alpha + \xi^i u_{Ji}^\alpha)\frac{\partial}{\partial u_J^\alpha} \\
&= \sum_{\alpha}\sum_{J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^{p} \xi^i \left(\frac{\partial}{\partial x^i} + u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}\right)
\end{aligned}$$

Here we define $V_Q := \sum_{\alpha=1}^q Q_\alpha(x,u^{(1)})\frac{\partial}{\partial u^\alpha}$ and $\mathsf{pr}^{(n)}V_Q := \sum_{\alpha=1}^q \sum_{|J|\le n} D_J Q_\alpha \frac{\partial}{\partial u^\alpha}$. Noting $D_i = \frac{\partial}{\partial x^i} + u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}$, we have

$$\mathsf{pr}^{(n)}V = \mathsf{pr}^{(n)}V_Q + \sum_{i=1}^{p} \xi^i D_i.$$

EXERCISE 2.1. Complete a symmetric group for your choice differential equation.

3. Symmetric group of Heat equation

Let $p = n+1$, $q = 1$ and $u(x^1, \ldots, x^p, t)$ be a $\mathcal{C}^2$ function defined on $\mathbb{R}^{n+1}$ that solves

$$u_t = k\Delta u$$

where $\Delta = \left(\frac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x^p}\right)^2$. This is called the heat conduction equation. For convenience sake, we assume $k = 1$ hereafter.

Physical motivation. Many physical laws are conservation laws and so is the heat equation. Let $\Omega \subset \mathbb{R}^n$ and $u(x,t)$ denote the temperature at $x \in \Omega$ and t. Then the vector $-\nabla_x u$ stands for the heat flux at (x,t). Total heat in Ω is $\int_\Omega u(x,t)dV(x)$, whose time derivative is the rate of heat increase in Ω. This should be caused by total heat flux into Ω across $\partial\Omega$. Hence

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(x,t)dV(x) &= \int_{\partial\Omega} (-\nabla u)\cdot(-\vec{n})d\sigma \\
&= \int_\Omega \mathrm{div}\nabla u dV \\
&= \int_\Omega \Delta u dV.
\end{aligned}$$

Since Ω was arbitrary, we deduce $u_t = \Delta u$.

3.1. Symmetric group of 1 dimensional heat equation. Let $u(x,y)$ be defined on $(x,t) \in \mathbb{R}^2$ that solves $u_t = u_{xx}$. We look for infinitesimal symmetry in the form $V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}$. Let $\Delta(x,t,u,u_t,u_x,u_{xx},u_{xt},u_{tt}) := u_{xx} - u_t$. Then

$$(\mathsf{pr}^{(2)}V)\Delta = 0 \text{ on } \Delta = 0$$

is the equation to give the symmetry. Let $\mathsf{pr}^{(2)}V = V + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}$.

1 Exterior Differential System

Let M^n be a smooth $\mathcal{C}^n$-manifold and $\phi_1, ..., \phi_s$ be differential forms on M^n. The exterior differential system is a system of equations

$$\phi_i = 0, \quad i = 0, ..., s.$$

The goal is to find a submanifold of M^n on which $\phi_i = 0$. Especially, if all ϕ_i's are 1-forms, it is called a Pfaffian system.

Definition 1.1. An integral manifold is an immersion $f : N \to M$ such that

$$f^*\phi_i = 0, \quad i = 0, ..., s.$$

If $f : N \to M$ is an integral manifold, then $f^*(d\phi_i) = d(f^*\phi_i) = 0$ and $f^*(\psi \wedge \phi_i) = f^*\psi \wedge f^*\phi_i = 0$ for any form ψ on M. Thus we are really working with the differential ideal generated by $\{\phi_1, \ldots, \phi_s\}$.

Example 1.2 (Pfaff problem). In $\mathbb{R}^n$, let

$$x = (x^1, \ldots, x^n)$$

be the coordinates of $\mathbb{R}^n$ and

$$\omega = a_1(x)dx^1 + \cdots + a_n(x)dx^n.$$

Clearly, the equation $\omega = 0$ has a solution since we have an integral curve of a vector field which is orthogonal to $(a_1, \cdots, a_n)$ by the existence theorem of ordinary differential equations. We want to find a k $(< n)$ dimensional integral manifold $f : \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^k$. Put

$$f = (f^1, \ldots, f^n)$$

and let

$$y = (y^1, \ldots, y^k)$$

be the coordinates of $\mathbb{R}^k$. Then

$$\begin{aligned} f^*\omega &= a_1(f(y))df^1 + \cdots + a_n(f(y))df^n \\ &= \sum_{\lambda=1}^{k}\left(a_1(f(y))\frac{\partial f^1}{\partial y^\lambda} + \cdots + a_n(f(y))\frac{\partial f^n}{\partial y^\lambda}\right)dy^\lambda \\ &= 0. \end{aligned}$$

Therefore

$$a_1(f(y))\frac{\partial f^1}{\partial y^\lambda} + \cdots + a_n(f(y))\frac{\partial f^n}{\partial y^\lambda} = 0, \quad \lambda = 1, \ldots, k.$$

This is an underdetermined system of PDE with n unknowns and k equations. The Pfaff problem is finding an integral manifold of maximal dimension.

We use the following notations :

(i) $\Omega^0(M) = C^\infty(M)$: the 0-forms,

(ii) $\Omega^p(M)$: the set of smooth p-forms on M for $p = 1, \ldots, n$,

(iii) $\Omega^*(M) = \bigoplus_{p=0}^{n} \Omega^p(M)$: graded module over $C^\infty(M)$.

$\{\Omega^*(M), \wedge, d\}$ is called **exterior algebra of differential forms**.

Definition 1.3. Exterior differntial system(EDS) is a pair $(M, \mathfrak{I})$ where M is a smooth manifold and $\mathfrak{I} \subset \Omega^*(M)$ is an ideal in the graded ring $\Omega^*(M)$ of differential forms on M that is closed under exterior differentiation, that is, $d\phi \in \mathfrak{I}$ for any $\phi \in \mathfrak{I}$.

Definition 1.4. A subalgebra $\mathfrak{I} \subset \Omega^*(M)$ is called an **(algebraic) ideal** if the following are satisfied.

(i) If $\phi \in \mathfrak{I}$, then $\psi \wedge \phi \in \mathfrak{I}$ for any $\psi \in \Omega^*(M)$.

(ii) If $\phi \in \mathfrak{I}$, each homogeneous component of ϕ is in $\mathfrak{I}$.

Definition 1.5. A subalgebra $\mathfrak{I} \subset \Omega^*(M)$ is called a **differential ideal** if

(i) $\mathfrak{I}$ is an algebraic ideal,

(ii) $d\mathfrak{I} \subset \mathfrak{I}$, that is, if $\phi \in \mathfrak{I}$, then $d\phi \in \mathfrak{I}$.

Thus definition 1.3 implies that an EDS is a pair $(M, \mathfrak{I})$, where M is a smooth manifold and $\mathfrak{I} \subset \Omega^*(M)$ is a differential ideal.

Let $\mathfrak{I} \subset \Omega^*(M)$ be an algebraic ideal. Then $\mathfrak{I} = \bigoplus_{q=0}^{n} \mathfrak{I}^q$, where $\mathfrak{I}^q = \mathfrak{I} \bigcap \Omega^q(M)$. Hence $\mathfrak{I}$ itself is a graded algebra.

In most cases, generators are given: $\phi_1, \ldots, \phi_s \in \Omega^*(M)$. The algebraic ideal $< \phi_1, \ldots, \phi_s >_{alg}$ generated by $\{\phi_1, \ldots, \phi_s\}$ is the set of differential forms $\phi = \gamma^1 \wedge \phi^1 + \cdots + \gamma^s \wedge \phi^s, \quad \gamma^j \in \Omega^*(M)$ and the differential ideal $< \phi_1, \ldots, \phi_s >$ generated by $\{\phi_1, \ldots, \phi_s\}$ is the set of differential forms $\phi = \gamma^1 \wedge \phi^1 + \cdots + \gamma^s \wedge \phi^s + \beta^1 \wedge d\phi^1 + \cdots + \beta^s \wedge d\phi^s, \quad \gamma^j, \beta^k \in \Omega^*(M)$, that is, the algebraic ideal generated by ϕ's and $d\phi$'s.

The fundamental problem in EDS is to study integral manifolds of differential ideals.

Let Ω be a decomposable p-form,

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p, \qquad \omega^j : \text{1-form}$$

and $\mathfrak{I}$ a differential ideal. Then the pair $(\mathfrak{I}, \Omega)$ is called an **EDS with independence condition** Ω. The integral manifold of $(\mathfrak{I}, \Omega)$ is an integral manifold of $\mathfrak{I}$ such that $f^*\Omega \neq 0$. We use this system when we wish to keep some variables independent.

Remark. Every PDE(ODE) system can be written as an EDS with independence condition. A couple of examples are shown below. The independent variables of PDE make the independence condition of the EDS.

Example 1.6. Consider the PDE of order 2

$$y'' = F(x, y, y').$$

Then we obtain $dy = y'dx$ and $dy' = y''dx = F(x, y, y')dx$. Thus, on $M = \{(x, y, y')\} = \mathbb{R}^3$, the above PDE gives an EDS of 1-forms

$$\begin{cases} dy - y'dx, \\ dy' - Fdx, \end{cases}$$

with independence condition $dx \neq 0$.

Example 1.7. Consider the PDE of order 2

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Let M be the second jet space $J^2(\mathbb{R}^2, \mathbb{R})$ such that

$$M = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\} = \mathbb{R}^8 = \{(x, y, u, p, q, r, s, t)\}.$$

We obtain the EDS made by

0-form $\quad F(x, y, u, p, q, r, s, t) = 0,$

$$\text{1-forms} \begin{cases} du &= u_x dx + u_y dy &= pdx + qdy &\rightarrow du - pdx - qdy &= 0, \\ du_x &= u_{xx} dx + u_{xy} dy &= rdx + sdy &\rightarrow dp - rdx - sdy &= 0, \\ du_y &= u_{xy} dx + u_{yy} dy &= sdx + tdy &\rightarrow dq - sdx - tdy &= 0 \end{cases}$$

with independence condition $dx \wedge dy \neq 0$.

Let $\alpha^1, \ldots, \alpha^{n-r}$ be the given 1-forms on M^n which are independent and $\mathfrak{I}$ the ideal generated by $\alpha^1, \ldots, \alpha^{n-r}$. $\mathfrak{I}$ is said to be **closed** if it satisfies the following condition:

$$\left. \begin{aligned} & d\mathfrak{I} \subset \mathfrak{I} \\ \Leftrightarrow \quad & d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^{n-r} \\ \Leftrightarrow \quad & d\alpha^i = \phi_1 \wedge \alpha^i + \cdots + \phi_{n-r} \wedge \alpha^{n-r} \end{aligned} \right\} \tag{1}$$

A Pfaffian system $\alpha^i = 0$, $i = 1, \ldots, n - r$ is called completely integrable if the condition (1) holds.

Theorem 1.8 (Frobenius, [1]). *Let $\mathfrak{I}$ be a differential ideal generated by 1-forms $\alpha^1, \ldots, \alpha^{n-r}$ so that the condition (1) is satisfied. Then, in a sufficiently small neighborhood, there exists a coordinate system $y^1, \ldots, y^n$ such that $\mathfrak{I}$ is generated by $dy^{r+1}, \ldots, dy^n$.*

Example 1.9. In $\mathbb{R}^3$, let $\omega = Rdx + Sdy + Tdz$. Then $d\omega \equiv 0 \mod \omega$ if and only if there exists a function μ such that $\mu\omega$ is exact.

2 Jet Bundle(Jet Space)

Let N and M be manifolds of dimensions k and n, respectively. For each $r = 0, 1, 2, \ldots$, the r-th **jet space(jet bundle)** is roughly the set of all partial derivatives up to order r of maps $f : N \to M$.

Definition 2.1. The maps f, $g : N \to M$ are said to have the **same r-th jet** at p if partial derivatives of f and g up to order r are equal. Then the relation is an equivalence relation and the equivalence class with the representative $f : N \to M$ is denoted by $j^r_p(f)$. Let $J^r_{p,q}$ denote the set of all r-jets of mappings from N into M with source p and target q. Then define the set

$$J^r(N, M) = \bigcup_{p\in N, q\in M} J^r_{p,q}(N, M).$$

$J^r_{p,q}$ is the doubly fibred manifold with the natural projections α and β, where $\alpha : J^r(N, M) \to N$ and $\beta : J^r(N, M) \to M$ defined by $\alpha(j^r_p(f)) = p$ and $\beta(j^r_p(f)) = f(p)$.

Let (U, x) and (V, z) be the coordinate charts of N and M, respectively. Then $\alpha^{-1}(U) \cap \beta^{-1}(V)$ is a coordinate neighborhood of $J^r(N, M)$. We may define a coordinate system by

$$h(j^r_p(f)) = \big(x^i(p), z^j(f(p)), D^\alpha_x(z \circ f)(p)\big),$$
$$1 \le i \le k, 1 \le j \le n, 1 \le |\alpha| \le r.$$

The chain rule guarantees that a differentiable change of local coordinates in N and M will induce a differentiable change of local coordinates in $J^r(N, M)$.

The r-th graph $j^r(f) : N \to J^r(N, M)$ of a map f is defined by $j^r(f)(p) = j^r_p(f)$. On $J^r(N, M)$, we write the natural coordinates as

$$x^i(p), z^\alpha(f(p)), p^\alpha_i, p^\alpha_{i_1,i_2}, \ldots, p^\alpha_{i_1,\cdots,i_r}, \quad 1 \le i, i_1, \ldots, i_r \le k,\ 1 \le \alpha \le n.$$

Then the Pfaffian system

$$\begin{cases} dz^\alpha - p_i^\alpha dx^i, \\ p_{i_1}^\alpha - p_{i_1,i_2}^\alpha dx^{i_2}, \\ \quad \vdots \\ p_{i_1,\dots,i_{r-1}}^\alpha - p_{i_1,\dots,i_{r-1},i_r}^\alpha dx^{i_r} \end{cases}$$

is called the **contact system** $\Omega^r(N, M)$.

References

[1] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems* (1991), Springer-Verlag, New York, Berlin, Heidelberg.

[2] R. Bryant, P. Griffiths and D. Yang, *Characteristics and existence of isometric embeddings*, Duke Math. J. **50** (1983), 893-994.

1 Pfaffian System and Involutivity

Let M be a manifold. Pfaffian system (I, J) with independence condition J satisfies the following conditions on M:

$$\theta^\alpha = 0, \quad \alpha = 1, \ldots, s,$$

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^p \neq 0$$

where $I = \text{span}\{\theta^1, \ldots, \theta^s\}$ and $J = \text{span}\{\omega^1, \ldots, \omega^p\}$.

Definition 1.1. For each $q = 1, \ldots, p$, a **q-dimensional integral manifold** is a manifold N of dimension q such that $\theta^\alpha|_N = 0,\ \alpha = 1, \ldots, s$, which implies that $d\theta|_N = 0$, so that $\phi|_N = 0$, for any ϕ in $\mathfrak{I}$, where $\mathfrak{I}$ is the differential ideal generated by θ^α's.

Definition 1.2. An **integral element** (x, E) of dimension q is a q-dimensional subspace E of T_xM such that $\phi|_E = 0$ for any $\phi \in \mathfrak{I}$. $V_q(\mathfrak{I})$ is defined to be the set of all q-dimensional integral elements for $q = 1, \ldots, p-1$. $V(I, J)$ is defined to be the set of p-dimensional integral elements (x, E) such that $\Omega|_E \neq 0$.

Let (x, E) be a q-dimensional integral element of $\mathfrak{I}$. Let $e_1, \ldots, e_q$ be a basis of E. Its **polar equations** are linear equations for the subspace of all $v \in T_xM$ such that $< \phi(x), e_1 \wedge \cdots \wedge e_q \wedge v >= 0$ for any $(q+1)$-form ϕ in $\mathfrak{I}$.

Definition 1.3. An integral element (x, E) is **K-regular** if (x, E) is a smooth point of $V_q(\mathfrak{I})$ and the rank of the polar equations is constant near (x, E). A **regular flag** of E is a sequence of K-regular elements such that $(0) = E^0 \subset E^1 \subset \cdots \subset E^{p-1} \subset E^p = E$.

Definition 1.4. A Pfaffian system (I, J) is **involutive** if a general integral element $(x, E) \in V(I, J)$ admits a regular flag.

Theorem 1.5 (Cartan-Kähler, [?]). *Let M be a C^ω manifold and $\mathfrak{I}$ be a C^ω differential ideal. Suppose that $(x, E) \in V(I, J)$ has a regular flag. Then there exists a C^ω integral manifold N of dimension p passing through x with $T_x N = E$. In other words, if C^ω Pfaffian system is involutive, it has C^ω solutions.*

If the Pfaffian system (I, J) is involutive, it does not imply any additional equations and the map $V(I, J) \to M$ is surjective or it satisfies the Frobenius condition $d\theta^\alpha = 0 \mod \theta$. Even if the map $V(I, J) \to M$ is surjective, still it may happen that the Pfaffian system implies additional equations.

Definition 1.6. Let (I, J) be a Pfaffian system with independence condition on M such that the map $V(I, J) \to M$ is surjective. Let

$$M^{(1)} = V(I, J) \subset G_p M.$$

Define the **first prolongation** $(I^{(1)}, J^{(1)})$ to be the restriction to $M^{(1)}$ of the canonical Pfaffian system of $G_p M$. Define $(I^{(q)}, J^{(q)})$ to be the first prolongation of $(I^{(q-1)}, J^{(q-1)})$ inductively for $q = 1, 2, \ldots$.

Cartan and Kuranishi showed it can happen that $(I^{(1)}, J^{(1)})$ is involutive while (I, J) is not.

Theorem 1.7 (Cartan-Kuranishi, [?]). *Given a Pfaffian system (I, J), there exists q_0 such that for $q \geq q_0$, $(I^{(q)}, J^{(q)})$ are involutive (under a mild regularity assumption).*

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1 Local solvability of generic over-determined PDE systems

Let $u = (u^1, \cdots, u^q)$ be a system of real-valued functions of independent variables $x = (x^1, \cdots, x^p)$. Consider a system of partial differential equations of order m

$$\Delta_\lambda(x, u^{(m)}) = 0, \quad \lambda = 1, \cdots, \ell, \tag{1.1}$$

where $u^{(m)}$ denotes all the partial derivatives of u of order up to m. We assume (1.1) is over-determined, that is, $\ell > q$. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \cdots, j_r)$ with $1 \leq j_s \leq p$. The order of a multi-index J is denoted by $|J|$. By u^α_J we denote the $|J|$-th order partial derivative of u^α with respect to $x^{j_1}, \cdots, x^{j_{|J|}}$. For a smooth function $\Delta(x, u^{(m)})$, the total derivative of Δ with respect to x^i is the function in the arguments $(x, u^{(m+1)})$ defined by the chain rule:

$$D_i\Delta = \frac{\partial \Delta}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{|J| \leq m} \frac{\partial \Delta}{\partial u^\alpha_J} u^\alpha_{Ji},$$

where Ji dentes the multi-index $(j_1, \cdots, j_{|J|}, i)$. Compatibility conditions are those equations obtained from (1.1) by differentiation and algebraic operations, that is, the ideal generated by Δ and the total derivatives of Δ. By (1.1)'s being generic we shall mean that the compatibility conditions determine all the partial derivatives of u of a sufficiently high order, say $k (k \geq m)$, as functions of derivatives of lower order, namely,

$$u^\alpha_K = H^\alpha_K(x, u^{(k-1)}), \tag{1.2}$$

for all multi-index K with $|K| = k$, and for all $\alpha = 1, \cdots, q$. This is the case where there exists a system of compatibility conditions that satisfies the non-degeneracy hypothesis of the implicit function theorem so that the system is solvable for all u_K's with $|K| = k$ in terms of lower order derivatives. If this is the case (1.2) is called a complete system of order k and (1.1) is said to admit prolongation to a complete system (1.2). Now we consider the ring of smooth functions in the arguments $(x, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \cdots)$. For each non-negative integer r let $\Delta^{(r)}$ be the algebraic ideal generated by Δ and the total derivatives of Δ up to order r, where $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ as in (1.1). Suppose that the complete system (1.2) is obtained from $\Delta^{(r)}$. Let $J^{k-1}(X, U)$ be the space of $(k-1)$th jets $(x, u^{(k-1)})$. Let $\mathcal{S} \subset J^{k-1}(X, U)$ be the common zero set of those functions in the arguments $(x, u^{(k-1)})$ that are elements of $\Delta^{(r)}$. We assume $\mathcal{S}$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge x^p \neq 0$. Then there exist disjoint sets of indices

$$A := \{(a, I)\} \quad \text{and } B := \{(b, J)\},$$

where $a, b \in \{1, \cdots, q\}$, I and J are multi-indices of order $\leq k-1$ so that $\mathcal{S}$ is the graph

$$u^b_J = \Phi^b_J(x, u^a_I :), \quad \text{for all } (b, J) \in B. \tag{1.3}$$

We take $(x, u^a_I : (a, I) \in A)$, as local coordinates of $\mathcal{S}$. Observe that if $u = u(x)$ is a solution of (1.1) then its $(k-1)$th jet graph $(x, u^{(k-1)}(x))$ is contained in $\mathcal{S}$ and for each $(a, I) \in A$, we have

$$du^a_I(x) = \begin{cases} \sum_{i=1}^p u^a_{Ii}(x) dx^i & \text{for } |I| \leq k-2, \\ \sum_{i=1}^p H^a_{Ii}(x, u^{(k-1)}) dx^i & \text{for } |I| = k-1, \end{cases}$$

where H's are as in (1.2).
Substituting (1.3) for all u^b_J with $(b, J) \in B$ we obtain

$$du^a_I(x) = \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L(x)) dx^i,$$

where all the indices (α, L) are in A. Thus on $\mathcal{S}$ we define independent 1-forms

$$\theta^a_I := du^a_{Ii} - \sum_{i=1}^p \Psi^a_{Ii}(x, u^\alpha_L : (\alpha, L) \in A) dx^i, \tag{1.4}$$

for all $(a, I) \in A$. Then the smooth solutions of (1.1) are in one-to-one correspondence with the smooth integral manifolds of the Pfaffian system (1.4). Let $\theta = \{\theta^a_I : (a, I) \in A\}$. We set

$$d\theta^a_I \equiv \sum_{i<j} T^a_{Iij}(x, u^\alpha_J : (\alpha, J) \in A) dx^i \wedge dx^j, \qquad \text{mod } \theta.$$

If all of $T := \{T^a_{Iij} : (a, I) \in A\}$ are identically zero then $\mathcal{S}$ is foliated by integral manifolds by the Frobenius theorem. Otherwise, we pull back the Pfaffian system (1.4) to a subset $\mathcal{S}' \subset \mathcal{S}$, where $\mathcal{S}'$ is the common zero set of $\{\Delta, T\}$ and their compatibility conditions that are functions in $(x, u^{(k-1)})$. We assume that $\mathcal{S}'$ is a smooth manifold on which $dx^1 \wedge \cdots \wedge dx^p \neq 0$. Making use of the following theorem we further check the integrability of (1.4) on $\mathcal{S}'$.

Theorem 1.1 *Let M be a smooth manifold of dimension n. Let $\theta := (\theta^1, \cdots, \theta^s)$ be a set of independent 1-forms on M and $\mathcal{D} :=< \theta >^{\perp}$ be the $(n-s)$ dimensional plane field annihilated by θ. Suppose that N is a submanifold of M of dimension $n-r$ defined by $T_1 = \cdots = T_r = 0$, where T_i are smooth real-valued functions of M such that $dT_1 \wedge \cdots \wedge dT_r \neq 0$ on N. Then the following are equivalent :*

(i) *$\mathcal{D}$ is tangent to N.*

(ii) $dT_j \equiv 0 \mod \theta^1, \ldots, \theta^s$ *on N, $1 \leq j \leq r$,*

(iii) $i^*\theta^1, \cdots, i^*\theta^s$ *have rank $s-r$, where $i : N \hookrightarrow M$ is the inclusion.*

Proof. (i) $\Leftrightarrow$ (ii)

Let $P \in N$. Then

$\mathcal{D}_P$ is tangent to N.
$\Leftrightarrow < \theta^1, \cdots, \theta^s >^{\perp} \subset < dT_1, \cdots, dT_r >^{\perp}$ at P.
$\Leftrightarrow dT_i \in < \theta^1, \cdots, \theta^s >$ at P for each $i = 1, \cdots, r$.
$\Leftrightarrow dT_i \equiv 0, \mod \theta$.

(ii) $\Leftrightarrow$ (iii)

Assuming (ii), we have $dT_j = \sum_{k=1}^{s} a_{jk}\theta^k$ on N, $j = 1, \ldots, r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, the matrix (a_{jk}) has rank r. Then we have

$$0 = i^*dT_j = \sum_{k=1}^{s} a_{jk} i^*\theta^k, \ 1 \leq j \leq r.$$

This means that $i^*\theta^1, \cdots, i^*\theta^s$ have rank no greater than $k-r$. Since N is of dimension $n-r$, $i^*\theta^1, \cdots, i^*\theta^s$ have rank no less than $k-r$. Therefore we obtain (iii).
Conversely, (iii) implies that (after suitable index changes) $i^*\theta^{s-r+l} = \sum_{j=1}^{s-r} b_{lj} i^*\theta^j$, $1 \leq l \leq r$ for some functions on N. Then $i^*(\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j) = 0$, $1 \leq l \leq r$. Since $dT_1, \ldots, dT_r$ are linearly independent on N, we have $\theta^{s-r+l} - \sum_{j=1}^{s-r} b_{lj}\theta^j = \sum_j c_{lj} dT_j$, $1 \leq l \leq r$ on N, where c_{ij} are functions on N and the matrix (c_{ij}) is nonsingular. Therefore we can solve dT_j in terms of θ^k's. □

Thus $\mathcal{S}'$ is foliated by integral manifolds if $dT^a_{Iij} \equiv 0 \mod \theta$ on $\mathcal{S}'$. Otherwise, we repeat the same argument, eventually to reach a submanifold $\tilde{\mathcal{S}}$ of dimension $\leq p$. If $\tilde{\mathcal{S}}$ is of dimension p then the Frobenius integrability is simply

$$i^*\theta = 0,$$

where $i : \tilde{\mathcal{S}} \to \mathcal{S}$ is the inclusion map. If the dimension of $\tilde{\mathcal{S}}$ is less than p, no solution exists.

As for the prolongation to a complete system we refer to [**Han**].
Our standard reference on exterior differential system are [**Br**] and [**BCGGG**].

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2 Existence of infinitesimal isometries on Riemannian manifolds of dimension 2

Let (M, g) be a smooth Riemannian manifold of dimension n. A smooth vector field ξ on M is an infinitesimal isometry (or a Killing field) if and only if ξ satisfies

$$L_\xi g = 0, \tag{2.1}$$

where L is the Lie derivative. In terms of local coordinates $x = (x^1, \cdots, x^n)$ (2.1) becomes

$$\xi_i^\lambda g_{\lambda j} + \xi_j^\lambda g_{\lambda i} - \xi^\lambda g_{ij,\lambda} = 0, \quad i, j = 1, \cdots, n, \tag{2.2}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda}$ (summation convention for $\lambda = 1, \cdots, n$). Since (2.2) is symmetric in (i, j) the number of equations in (2.2) is $\frac{n(n+1)}{2}$ whereas the number of unknowns is n so that (2.2) is overdetermined if $n \geq 2$. In this section we shall present a coordinate-free computation of prolongation of (2.1) with $n = 2$ to a complete system of order 2 and discuss the existence of solutions. Let $\{e_1, e_2\}$ be an orthonormal frame over a 2-dimensional Riemannian manifold M and let ω^1, ω^2 be the dual coframe. Then

$$g = \omega^1 \circ \omega^1 + \omega^2 \circ \omega^2,$$

where $\phi \circ \eta := \frac{1}{2}(\phi \otimes \eta + \eta \otimes \phi)$ is the symmetric product of 1-forms. Recall also that there exist a uniquely determined 1-form ω_2^1 (Levi-Civita connection) and a function K (Gaussian curvature) satisfying

$$\left.\begin{array}{l} d\omega^1 = -\omega_2^1 \wedge \omega^2 \\ d\omega^2 = \ \ \omega_2^1 \wedge \omega^1 \end{array}\right\} \tag{2.3}$$

and

$$d\omega_2^1 = K\omega^1 \wedge \omega^2. \tag{2.4}$$

Furthermore, Lie derivatives of ω^i, $i = 1, 2$ with respect to a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ are

$$\begin{array}{rcl} L_\xi \omega^1 & = & d(\xi \lrcorner \omega^1) + \xi \lrcorner d\omega^1 \\ & = & d\xi^1 - \omega_2^1(\xi)\omega^2 + \xi^2 \omega_2^1 \ \text{ by (2.3)} \end{array} \tag{2.5}$$

and similarly

$$\begin{array}{rcl} L_\xi \omega^2 & = & d(\xi \lrcorner \omega^2) + \xi \lrcorner d\omega^2 \\ & = & d\xi^2 + \omega_2^1(\xi)\omega^1 - \xi^1 \omega_2^1. \end{array} \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}\frac{1}{2}L_\xi g &= (L_\xi\omega^1)\circ\omega^1 + (L_\xi\omega^2)\circ\omega^2\\ &= (d\xi^1+\xi^2\omega_2^1)\circ\omega^1 + (d\xi^2-\xi^1\omega_2^1)\circ\omega^2.\end{aligned}$$

By setting

$$\begin{cases} d\xi^1 &= -\xi^2\omega_2^1+\xi_1^1\omega^1+\xi_2^1\omega^2,\\ d\xi^2 &= \xi^1\omega_2^1+\xi_1^2\omega^1+\xi_2^2\omega^2\end{cases} \tag{2.7}$$

and substituting in the above we have

$$\frac{1}{2}L_\xi g = \xi_1^1\omega^1\circ\omega^1 + (\xi_2^1+\xi_1^2)\omega^1\circ\omega^2 + \xi_2^2\omega^2\otimes\omega^2.$$

By (2.1), ξ is an infinitesimal isometry if and only if

$$\xi_1^1=\xi_2^2=0,\ \ \xi_2^1+\xi_1^2=0. \tag{2.8}$$

Substituting (2.8) in (2.7) we see that a vector field $\xi=\xi^1e_1+\xi^2e_2$ is an infinitesimal isometry if and only if

$$\begin{cases} d\xi^1 &= -\xi^2\omega_2^1+\xi_2^1\omega^2,\\ d\xi^2 &= \xi^1\omega_2^1+\xi_1^2\omega^1,\end{cases} \tag{2.9}$$

which is a coordinate-free version of (2.2) with $n=2$ expressed as an exterior differential system. Prolongation of (2.9) to a complete system is differentiating (2.9) and expressing $(d\xi^1, d\xi^2, d\xi_2^1)$ in terms of (ξ^1,ξ^2,ξ_2^1) :
We apply d to (2.9) and substitute (2.9), (2.3) and (2.4) for $d\xi^i$, $d\omega^i$ and $d\omega_2^1$, respectively, to obtain

$$\begin{aligned}(d\xi_2^1-K\xi^2\omega^1)\wedge\omega^2=0,\\ (d\xi_2^1+K\xi^1\omega^2)\wedge\omega^1=0.\end{aligned}$$

Hence we have

$$d\xi_2^1 = K(\xi^2\omega^1-\xi^1\omega^2). \tag{2.10}$$

The system (2.9) and (2.10) is a prolongation of (2.1) to a complete system. Now consider the Euclidean space $\mathbb{R}^3$ of variables (ξ^1,ξ^2,ξ_2^1). Then the submanifold of the first jet space of ξ defined by (2.8) may be identified with

$\mathcal{S} := M \times \mathbb{R}^3$.
On $M \times \mathbb{R}^3$ consider the Pfaffian system $\theta = (\theta^1, \theta^2, \theta^3)$ given by

$$\begin{array}{rcl} \theta^1 &=& d\xi^1 + \xi^2\omega^1_2 - \xi^1_2\omega^2, \\ \theta^2 &=& d\xi^2 - \xi^1\omega^1_2 + \xi^1_2\omega^1, \\ \theta^3 &=& d\xi^1_2 - K\xi^2\omega^1 + K\xi^1\omega^2. \end{array} \tag{2.11}$$

We check the Frobenius integrability conditions for (2.11): By (2.3) and (2.4) we have

$$d\theta^1, d\theta^2 \equiv 0 \mod \theta$$

and

$$d\theta^3 \equiv (K_1\xi^1 + K_2\xi^2)\omega^1 \wedge \omega^2 \qquad \mod \theta$$

where $K_i = dK(e_i),\ \ i = 1, 2$ so that $dK = K_1\omega^1 + K_2\omega^2$.
Thus (2.11) is integrable if and only if $T := K_1\xi^1 + K_2\xi^2$ is identically zero on $M \times \mathbb{R}^3$, which is equivalent to $K_1 = K_2 = 0$ i.e. K is constant. In this case, there exist 3 parameter family of solutions by the Frobenius theorem. Otherwise, assuming $dT \neq 0$ on $T = 0$, we consider a submanifold $\mathcal{S}'$ of dimension 4 defined by $T = 0$.
Differentiating $dK = K_1\omega^1 + K_2\omega^2$, we see by (2.3) that

$$\begin{array}{rcl} 0 &=& d^2K \\ &=& (dK_1 + K_2\omega^1_2)\omega^1 + (dK_2 - K_1\omega^1_2)\omega^2. \end{array} \tag{2.12}$$

Thus we put

$$dK_1 = -K_2\omega^1_2 + K_{11}\omega^1 + K_{12}\omega^2, \tag{2.13}$$
$$dK_2 = K_1\omega^1_2 + K_{21}\omega^1 + K_{22}\omega^2. \tag{2.14}$$

By substituting (2.13), (2.14) in (2.12) we have $K_{12} = K_{21}$.

On $\mathcal{S}'$, we have by (2.11), (2.13) and (2.14)

$$\begin{array}{rcl} dT &=& \xi^1 dK_1 + K_1 d\xi^1 + \xi^2 dK_2 + K_2 d\xi^2 \\ &\equiv& (K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi^1_2)\omega^1 + (K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi^1_2)\omega^2 \qquad \mod \theta. \end{array}$$

We set

$$\left\{ \begin{array}{rcl} T_1 &=& K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi^1_2, \\ T_2 &=& K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi^1_2. \end{array} \right. \tag{2.15}$$

If $T_1, T_2 \equiv 0$ on $\mathcal{S}'$, $i^*\theta^1, i^*\theta^3, i^*\theta^3$ have rank 2 by Theorem 1.1. Then $\mathcal{S}'$ is foliated by two dimensional integral manifolds and therefore there are 2 parameter family of solutions. But this implies that $K_1 = K_2 = 0$ which is impossible.

Let $A = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \end{pmatrix}$.

If $\det A = 0$, A has rank 2 and $\mathcal{S}'' = \{T = T_1 = T_2 = 0\}$ is a 3-dimensional submanifold of $\mathcal{S}$. If we have $dT_1, dT_2 \equiv 0 \mod \theta^1, \theta^2, \theta^3$ on $\mathcal{S}''$, Theorem 1.1 and the Frobenius theorem imply that $\mathcal{S}''$ is foliated by two dimensional integral manifolds and therefore there exists 1 parameter family of solutions. To calculate dT_1, dT_2 we differentiate (2.13). Then we have

$$\begin{aligned} 0 &= d^2K_1 \\ &= (dK_{11} + 2K_{12}\omega_2^1 + K_2K\omega^2)\omega^1 + (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^2. \end{aligned} \tag{2.16}$$

Thus we put

$$dK_{11} = -2K_{12}\omega_2^1 + K_{111}\omega^1 + K_{112}\omega^2, \tag{2.17}$$
$$dK_{12} = (K_{11} - K_{22})\omega_2^1 + K_{121}\omega^1 + K_{122}\omega^2. \tag{2.18}$$

By substituting (2.17), (2.18) in (2.16) we have $K_{112} = K_{121} - K_2K$.

Differentiating (2.14), we have

$$\begin{aligned} 0 &= d^2K_2 \\ &= (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^1 + (dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1)\omega^2. \end{aligned} \tag{2.19}$$

By substituting (2.17), (2.18) in (2.19) we have

$$(dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1 - K_{122}\omega^1)\omega^2 = 0.$$

Thus we put

$$dK_{22} = 2K_{12}\omega_2^1 + (K_{122} - K_1K)\omega^1 + K_{222}\omega^2. \tag{2.20}$$

On $\mathcal{S}''$, we have by (2.11), (2.17), (2.18) and (2.20)

$$\begin{aligned} dT_1 \equiv\ & (K_{111}\xi^1 + (K_{121} - K_2K)\xi^2 - 2K_{12}\xi^1_2)\omega^1 \\ & + (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^2 \quad \mod \theta \end{aligned}$$

and

$$\begin{aligned} dT_2 \equiv\ & (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi^1_2)\omega^1 \\ & + ((K_{122} - K_1K)\xi^1 + K_{222}\xi^2 + 2K_{12}\xi^1_2)\omega^2 \quad \mod \theta. \end{aligned}$$

We summarize the discussions of this section in the following

Theorem 2.1 *Let M be a Riemannian manifold of dimension 2.*

Let $\mathbf{K} = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \\ K_{111} & K_{121} - K_2K & -2K_{12} \\ K_{121} & K_{122} & K_{11} - K_{22} \\ K_{122} - K_1K & K_{222} & 2K_{12} \end{pmatrix}$.

(i) If $\mathbf{K}$ has rank 0, there exist 3 parameter family of infinitesimal isometries,

(ii) If $\mathbf{K}$ has rank 2 and $(K_1, K_2) \neq 0$, there exist 1 parameter family of infinitesimal isometries,

(iii) If $\mathbf{K}$ has rank 3, there exists only trivial infinitesimal isometry.

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3 Existence of CR automorphisms on three dimensional CR manifolds of non-degenerate Levi form

Let M be a smooth manifold of dimension $2n+1$.

Definition 3.1 *$(H(M), J)$ is a CR-structure (of hypersurface type) if H is a 2n-dimensional sub-bundle of TM and $J : H \to H$ satisfies $J^2 = -id$.*

The map J extends to a complex linear map of $\mathbb{C} \otimes H$ to itself and we obtain the decomposition $\mathbb{C} \otimes H = H_{1,0} \oplus H_{0,1}$ with $H_{1,0}$ as its i eigenspace and $H_{0,1}$ as its $-i$ eigenspace. A CR-structure $(H(M), J)$ is called integrable if $[Z, \tilde{Z}] \in H_{1,0}$ for any local sections Z and $\tilde{Z}$ of $H_{1,0}$. A manifold with an integrable CR-structure is called a CR manifold.

Now let M be a CR-manifold and $\{Z_1, \ldots, Z_n\}$ be a basis for $H_{1,0}$, near some point x. Let U be a smooth real vector field that is transversal to H. Then $\{Z_1, \ldots, Z_n, \overline{Z_1}, \ldots, \overline{Z_n}\}$ is a basis for $\mathbb{C} \otimes H$ near x. For each $i, j = 1, \ldots, n$, let

$$[Z_i, \overline{Z_j}] = \sqrt{-1} g_{ij} U \qquad \mod \{Z_1, \ldots, Z_n, \overline{Z_1}, \ldots, \overline{Z_n}\}.$$

Then the matrix (g_{ij}) is hermitian, which we call the Levi form. M is called a nondegenerate CR-manifold if the matrix (g_{ij}) is nonsingular.

A smooth map f of M into another CR manifold $\tilde{M}$ is called a CR mapping if

(i) df maps $H(M)$ to $H(\tilde{M})$

(ii) $df \circ J = J \circ df$.

We reformulate the definition of CR structures in terms of forms as follows. Given $(H(M), J)$, we choose a real nonzero form $\theta \in H^\perp$ and then find $\theta^1, \cdots, \theta^n$ so that $\theta, \theta^1, \cdots, \theta^n$ span $H_{0,1}^\perp$ linearly. Thus we have $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0$. Integrability can be expressed as $d\theta, d\theta^i \equiv 0$

mod $\theta, \theta^1, \dots, \theta^n$.
Conversely, given forms $\theta, \theta^1, \cdots, \theta^n$ on M where

$$\begin{aligned} &\theta \text{ is real,} \\ &\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \overline{\theta^1} \wedge \cdots \wedge \overline{\theta^n} \neq 0, \end{aligned}$$

we can define the CR structure on M by setting

$$\begin{aligned} &H = \theta^\perp, \\ &H_{0,1} = \{\theta, \theta^1, \dots, \theta^n\}^\perp. \end{aligned}$$

Tanaka-Chern-Moser theory [**T1**], [**CM**] asserts that there exists a complete system of local invariants for non-degenerate CR structures. In particular, when $n = 1$ we have the following.

Theorem 3.2 (p140 of [Ja], [BS]) *Let M be a nondegenerate CR manifold of dimension 3. Then there exists an eight-dimensional bundle Y over M and there is a completely determined set of 1-forms ω, ω^1, ϕ^1_1, ϕ^1, ψ on Y, of which ω, ψ are real and which satisfy the following :*

$$\begin{array}{lcl} d\omega & = & i\omega^1\overline{\omega^1} + \omega(\phi^1_1 + \overline{\phi^1_1}), \\ d\omega^1 & = & \omega^1\phi^1_1 + \omega\phi^1, \\ d\phi^1 & = & \frac{1}{2}\omega^1\psi + \overline{\phi^1_1}\phi^1 + Q\overline{\omega^1}\omega, \\ d\phi^1_1 & = & i\overline{\omega^1}\phi^1 + 2i\omega^1\overline{\phi^1} + \frac{1}{2}\omega\psi, \\ d\psi & = & 2i\phi^1\overline{\phi^1} + (\phi^1_1 + \overline{\phi^1_1})\psi + (R\omega^1 + \overline{R\omega^1})\omega. \end{array} \tag{3.1}$$

Futhermore, if M is another nondegenerate CR manifold with correspong notions $\tilde{Y}$, $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\phi}^1_1$, $\tilde{\phi}^1$, $\tilde{\psi}$, then there exists a CR diffeomorphism $f : M \to \tilde{M}$ if only if there exists a diffeomorphism $F : Y \to \tilde{Y}$ such that

(*i*) *the diagram commutes :*

$$\begin{array}{ccc} Y & \xrightarrow{F} & \tilde{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & \tilde{M} \end{array}$$

(*ii*) *Pull back of the forms, $\tilde{\omega}$, $\tilde{\omega}^1$, $\tilde{\phi}^1_1$, $\tilde{\phi}^1$, $\tilde{\psi}$ by F are exactly those forms, ω, ω^1, ϕ^1_1, ϕ^1, ψ, respectively.*

We choose 1-forms ω and ω^1 that define the CR structure of M^3 and that satisfy $d\omega = i\omega^1\overline{\omega^1} + \omega\phi$. This frame $\{\omega, \omega^1, \phi\}$ determines a section $\sigma : M \to Y$ and we have

$$
\begin{array}{rcl}
d\omega & = & i\omega^1\overline{\omega^1} + \omega\phi \quad \phi = \phi_1^1 + \overline{\phi_1^1}, \\
d\omega^1 & = & \omega^1\phi_1^1 + \omega\phi^1, \\
d\phi^1 & = & \frac{1}{2}\omega^1\psi + \overline{\phi_1^1}\phi^1 + q\overline{\omega^1}\omega, \\
d\phi_1^1 & = & i\overline{\omega^1}\phi^1 + 2i\omega^1\overline{\phi^1} + \frac{1}{2}\omega\psi, \\
d\psi & = & 2i\phi^1\overline{\phi^1} + (\phi_1^1 + \overline{\phi_1^1})\psi + (r\omega^1 + \overline{r}\overline{\omega^1})\omega,
\end{array}
\tag{3.2}
$$

where $q = \sigma^* Q$, $r = \sigma^* R$.

Differentiating $d\phi^1$, $d\psi$ in (3.2), we have

$$
\begin{array}{rcl}
0 & = & d^2\phi^1 \\
& = & (dq - q\phi_1^1 - 3q\overline{\phi_1^1} - \dfrac{1}{2}\overline{r}\omega^1)\overline{\omega^1}\omega,
\end{array}
$$

$$
\begin{array}{rcl}
0 & = & d^2\psi \\
& = & (dr - 3r\phi_1^1 - 2r\overline{\phi_1^1} + 2i\overline{q}\phi^1)\omega^1\omega + (d\overline{r} - 3\overline{r}\overline{\phi_1^1} - 2\overline{r}\phi_1^1 + 2ir\overline{\phi^1})\overline{\omega^1}\omega.
\end{array}
$$

Thus we put

$$
dq = q_0\omega + \frac{1}{2}\overline{r}\omega^1 + \overline{q_1}\overline{\omega^1} + q\phi_1^1 + 3q\overline{\phi_1^1}, \tag{3.3}
$$

$$
dr = r_0\omega + r_1\omega^1 + \tilde{r}_1\overline{\omega^1} + 3r\phi_1^1 + 2r\overline{\phi_1^1} + 2i\overline{q}\phi^1 \tag{3.4}
$$

for some q_0, q_1, r_0, r_1 and $\tilde{r}_1$ with $\tilde{r}_1$ real.

Remark.

1. The function $q(x)$ on M is not an invariant but a relative invariant of M : A different choice of ω and ω^1 gives a different function $\tilde{q}(x)$, but $q(x)$ and $\tilde{q}(x)$ are either both zero or both nonzero.

2. From (3.3), we know that $q \equiv 0$ implies $r \equiv 0$. In this case, M^3 is CR equivalent to the real hyperquadric Q^3.

Let M and $\tilde{M}$ be real hypersurfaces in $\mathbb{C}^n$.
A pseudo-conformal mapping of M into $\tilde{M}$ is a smooth mapping that can be extended to a biholomorphism of a neighborhood of M into a neighborhood of $\tilde{M}$. A pseudo-conformal mapping is obviously a CR diffeomorphism. If a hypersurface is connected and non-degenerate at a point, then the group of all pseudo-conformal automorphisms $Aut(M)$ is a Lie group of transformations with

$$\dim Aut(M) \leq n^2 + 2n,$$

and the equality holds if and only if M is the real hyperquadric(See [**T1**] and [**Ya**]).

Definition 3.3 *A smooth real vector field X on M is an infinitesimal CR-automorphism if $L_X V \in H$ and $L_X JV = J(L_X V)$ for any section V of H.*

Proposition 3.4 *Let X be a smooth vector field on a CR manifold (M, H, J). Then the followings are equivalent :*

(i) X *is an infinitesimal CR-automorphism,*

(ii) $L_X \bar{Z} \in H_{0,1}$ *for any section* $\bar{Z}$ *of* $H_{0,1}$*,*

(iii) $L_X \omega \in H_{0,1}^{\perp}$ *for any section* ω *of* $H_{0,1}^{\perp}$*.*

Proof.

- $(i) \Rightarrow (ii)$ $J(L_X \bar{Z}) = L_X J\bar{Z} = -iL_X \bar{Z}, \forall \bar{Z} \in H_{0,1}$.
- $(ii) \Rightarrow (i)$ Note that $J(V + iJV) = -i(V + iJV)$ for any section V of H i.e. $V + iJV \in H_{0,1}$. Then (ii) implies $J(L_X(V + iJV)) = -iL_X(V + iJV)$ and we get $L_X JV = J(L_X V)$ by comparing the real part of both sides.
- $(ii) \Leftrightarrow (iii)$ is easily checked. □

Let X be an infinitesimal CR-automorphism on M^3 with $\omega(X) = \eta$ and

$\omega^1(X) = \xi$.
From the property of Lie derivative and (3.2), we have

$$\begin{aligned} L_X\omega &= d(X\lrcorner\omega) + X\lrcorner d\omega \\ &= d\eta + X\lrcorner(i\omega^1\overline{\omega^1} + \omega\phi) \\ &= d\eta + i\xi\overline{\omega^1} - i\bar{\xi}\omega^1 + \eta\phi - \phi(X)\omega, \end{aligned}$$

$$\begin{aligned} L_X\omega^1 &= d(X\lrcorner\omega^1) + X\lrcorner d\omega^1 \\ &= d\xi + X\lrcorner(\omega^1\phi_1^1 + \omega\phi^1) \\ &= d\xi + \xi\phi_1^1 - \phi_1^1(X)\omega^1 + \eta\phi^1 - \phi^1(X)\omega. \end{aligned}$$

By Proposition 3.4 we have

$$d\eta = a\omega + i\bar{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \tag{3.5}$$
$$d\xi = b\omega + c\omega^1 - \xi\phi_1^1 - \eta\phi^1 \tag{3.6}$$

for some functions a, b and c.

The exterior differentiations of (3.5) and (3.6) give respectively

$$\begin{aligned} 0 &= d^2\eta \\ &= (da - i\bar{b}\omega^1 + ib\overline{\omega^1} + i\xi\overline{\phi^1} - i\bar{\xi}\phi^1 + \eta\psi)\omega + i(a - c - \bar{c})\omega^1\overline{\omega^1}, \\ 0 &= d^2\xi \\ &= (db - \eta q\overline{\omega^1} - b\overline{\phi_1^1} + \bar{c}\phi^1 + \frac{1}{2}\xi\psi)\omega + (dc - ib\overline{\omega^1} + i\bar{\xi}\phi^1 + 2i\xi\overline{\phi^1} + \frac{1}{2}\eta\psi)\omega^1. \end{aligned}$$

Thus we have

$$a = c + \bar{c}, \tag{3.7}$$
$$da = f\omega + i\bar{b}\omega^1 - ib\overline{\omega^1} + i\bar{\xi}\phi^1 - i\xi\overline{\phi^1} - \eta\psi, \tag{3.8}$$
$$db = g\omega + h\omega^1 + \eta q\overline{\omega^1} + b\overline{\phi_1^1} - \bar{c}\phi^1 - \frac{1}{2}\xi\psi, \tag{3.9}$$
$$dc = h\omega + l\omega^1 + ib\overline{\omega^1} - i\bar{\xi}\phi^1 - 2i\xi\overline{\phi^1} - \frac{1}{2}\eta\psi \tag{3.10}$$

for some functions f, g, h and l.

From (3.7), (3.8) and (3.10), we get $l = 2i\bar{b}$ and $f = h + \overline{h}$.

Differentiating (3.10) we have

$$\begin{aligned} 0 &= d^2c \\ &= \{dh - 2i(\overline{g} + \xi\overline{q})\omega^1 - i(g + \overline{\xi}q)\overline{\omega^1} - \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) - h\phi - i\overline{b}\phi^1 \\ &\quad +ib\overline{\phi^1} + \frac{1}{2}a\psi\}\omega + 2i(h - \overline{h})\omega^1\overline{\omega^1}. \end{aligned}$$

This gives $h = \overline{h}$, hence $g + \overline{\xi}q = 0$, and

$$dh = k\omega + \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) + h\phi + i\overline{b}\phi^1 - ib\overline{\phi^1} - \frac{1}{2}a\psi$$

for some function k.

Differentiating (3.9) we have

$$\begin{aligned} 0 &= d^2b \\ &= (3\overline{c}q + cq + \eta q_0 + \frac{1}{2}\xi\overline{r} + \overline{\xi}\overline{q_1})\omega\overline{\omega^1} + (k + \frac{1}{2}\xi r + \frac{1}{2}\overline{\xi}\overline{r})\omega\omega^1, \end{aligned}$$

which implies that $k = -\frac{1}{2}\xi r - \frac{1}{2}\overline{\xi}\overline{r}$.

Thus we obtain a complete system of order 3 for η and ξ, which can be expressed as

$$\left\{ \begin{array}{lcl} d\eta &=& a\omega + i\overline{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \qquad a = c + \overline{c},\ \phi = \phi_1^1 + \overline{\phi_1^1} \\ d\xi &=& b\omega + c\omega^1 - \xi\phi_1^1 - \eta\phi^1 \\ db &=& -\overline{\xi}q\omega + h\omega^1 + \eta q\overline{\omega^1} + b\overline{\phi_1^1} - \overline{c}\phi^1 - \frac{1}{2}\xi\psi \\ dc &=& h\omega + 2i\overline{b}\omega^1 + ib\overline{\omega^1} - i\overline{\xi}\phi^1 - 2i\xi\overline{\phi^1} - \frac{1}{2}\eta\psi \\ dh &=& -\frac{1}{2}(\xi r + \overline{\xi}\overline{r})\omega + \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) + h\phi + i\overline{b}\phi^1 - ib\overline{\phi^1} - \frac{1}{2}a\psi. \end{array} \right. \tag{3.11}$$

Now define 1-forms on the 11-dimensional manifold $\mathcal{S} := M \times \mathbb{R} \times \mathbb{C}^3 \times \mathbb{R} = \{(x, \eta, \xi, b, c, h) | x \in M\}$ by

$$\begin{array}{lcl} \theta^1 &=& d\eta - a\omega - i\overline{\xi}\omega^1 + i\xi\overline{\omega^1} + \eta\phi \\ \theta^2 &=& d\xi - b\omega - c\omega^1 + \xi\phi_1^1 + \eta\phi^1 \\ \theta^3 &=& db + \overline{\xi}q\omega - h\omega^1 - \eta q\overline{\omega^1} - b\overline{\phi_1^1} + \overline{c}\phi^1 + \frac{1}{2}\xi\psi \\ \theta^4 &=& dc - h\omega - 2i\overline{b}\omega^1 - ib\overline{\omega^1} + i\overline{\xi}\phi^1 + 2i\xi\overline{\phi^1} + \frac{1}{2}\eta\psi \\ \theta^5 &=& dh + \frac{1}{2}(\xi r + \overline{\xi}\overline{r})\omega - \frac{1}{2}\eta(r\omega^1 + \overline{r}\overline{\omega^1}) - h\phi - i\overline{b}\phi^1 + ib\overline{\phi^1} + \frac{1}{2}a\psi. \end{array} \tag{3.12}$$

We want to apply the Frobenius theorem to $\theta^1,\ Re\,\theta^i,\ Im\,\theta^i,\ \theta^5,\quad i = 2,\ 3,\ 4$

From the previous calculation we have

$$\begin{aligned} d\theta^1 &= 0 \\ d\theta^2 &= 0 \\ d\theta^3 &= -(3\bar{c}q + cq + \eta q_0 + \tfrac{1}{2}\xi\bar{r} + \bar{\xi}\overline{q_1})\omega\overline{\omega^1} \\ d\theta^4 &= 0 \end{aligned}$$

mod $\quad \theta^1,\ \theta^i,\ \overline{\theta^i},\ \theta^5,\quad i = 2,\ 3,\ 4.$

We have

$$\begin{aligned} d\theta^5 &\equiv -\frac{1}{2}(\eta r_0 + \xi r_1 + \bar{\xi}\tilde{r}_1 + (3c + 2\bar{c})r + 2ib\bar{q})\omega\omega^1 \\ &\quad -\frac{1}{2}(\eta\overline{r_0} + \bar{\xi}\overline{r_1} + \xi\tilde{r}_1 + (3\bar{c} + 2c)\bar{r} - 2i\bar{b}q)\omega\overline{\omega^1} \end{aligned}$$

mod $\quad \theta^1,\ \theta^i,\ \overline{\theta^i},\ \theta^5,\quad i = 2,\ 3,\ 4.$

Thus we put

$$\begin{aligned} T_1 &= 3\bar{c}q + cq + \eta q_0 + \frac{1}{2}\xi\bar{r} + \bar{\xi}\overline{q_1}, \\ T_2 &= \eta r_0 + \xi r_1 + \bar{\xi}\tilde{r}_1 + (3c + 2\bar{c})r + 2ib\bar{q}. \end{aligned}$$

If $q \equiv 0$, we know that $r \equiv 0$ from the remark following Theorem 3.2 and hence $T_1 \equiv T_2 \equiv 0$. By the Frobenius theorem, there is a foliation of $M \times \mathbb{R} \times \mathbb{C}^3 \times \mathbb{R}$ by three dimensional integral manifolds, which gives the eight-parameter family of solutions. If $q \neq 0$, we solve $T_1 = T_2 = 0$ to get $b = b(x, \eta, \xi)$, $c = c(x, \eta, \xi)$ and (3.11) is reduced to a complete system of order 1 :

$$\left\{ \begin{aligned} d\eta &= (c(x,\eta,\xi) + \overline{c(x,\eta,\xi)})\omega + i\bar{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \\ d\xi &= b(x,\eta,\xi)\omega + c(x,\eta,\xi)\omega^1 - \xi\phi_1^1 - \eta\phi^1. \end{aligned} \right.$$

We may keep analyzing this pfaffian system on $\mathcal{S}' := M \times \mathbb{R} \times \mathbb{C}$ by checking the Frobenius condition. If the system is integrable on $\mathcal{S}'$ in the sense of Frobenius there exits 3 parameter family of solutions. We summarize the above discussions of this section in the following

Theorem 3.5 *Let M be a nondegenerate CR manifold of dimension 3. Let q be the relative CR invariant as in (3.2). Then*

(i) *If $q \equiv 0$, M^3 is CR equivalent to the real hyperquadric Q^3 and there exist eight-parameter family of infinitesimal CR automorphisms.*

(ii) *If $q \neq 0$, we obtain a complete system of order 1 for infinitesimal CR automorphisms :*

$$\begin{cases} d\eta &= (c(x,\eta,\xi) + \overline{c(x,\eta,\xi)})\omega + i\bar{\xi}\omega^1 - i\xi\overline{\omega^1} - \eta\phi, \\ d\xi &= b(x,\eta,\xi)\omega + c(x,\eta,\xi)\omega^1 - \xi\phi_1^1 - \eta\phi^1, \end{cases}$$

where b and c are functions on $M \times \mathbb{R} \times \mathbb{C}$ given by $T_1 = T_2 = 0$. In particular, there are infinitesimal CR automorphisms on M of at most three parameters.

As for the dimension of CR automorphism group of M we refer [**Ja**]. The second part of Theorem 3.5 can also be proved by using Theorem 2 and its corollary of [**Ja**].

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1st order PDE

$u(x,y)$: unknown

$$\begin{cases} F(x,y,u,u_x,u_y) = 0 \\ u(\gamma(t)) = \phi(t) : \text{prescribed initial data} \end{cases}$$

If the initial data is **non-characteristic**, then there is the unique solution of the differential equation.

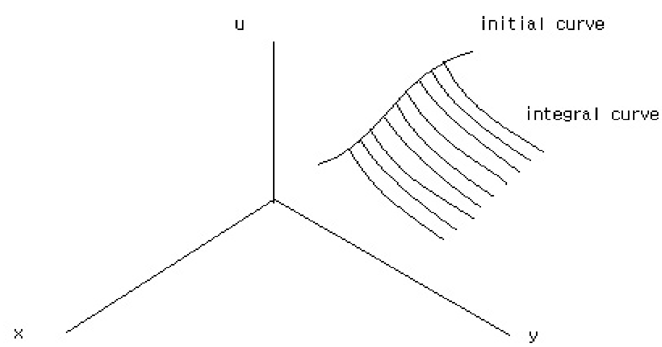
0.1 Quasi-linear case

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ u(x(t),y(t)) = \phi(t) \end{cases}$$

Consider a vector field in $\mathbb{R}^3 = \{(x,y,u)\}$

$$\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial u} : \text{characteristic vector field}$$

and a space curve $(x(t), y(t), \phi(t))$.



Considering integral curves of ξ starting from the initial curve, we get a surface $u = u(x,y)$. Then $u = u(x,y)$ satisfies the initial condition trivially and $a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = (a(x,y,u), b(x,y,u), c(x,y,u)) \cdot (u_x, u_y, -1) = 0$ because ξ is tangent to the surface $u = u(x,y)$.

0.2 General case

We use a similar argument to the quasi-linear case, in $\{(x, y, u, p, q)\}$ space
$\begin{cases} F(x, y, u, p, q) = 0 \\ u(x(t), y(t)) = \phi(t) \end{cases}$

F gives 1-parameter family $(p(\lambda), q(\lambda))$ of tangent spaces of possible solutions $u = u(x, y)$ at each point (x_0, y_0, u_0) which defines Monge cone with vertex (x_0, y_0, u_0) .

If $u = u(x, y)$ is a solution, then at each point $(x_0, y_0, u(x_0, y_0))$, the tangent space is tangent to the Monge cone. So there is the unique generating line of the cone that is tangent to the solution surface.

In fact, given 1-jet $(x_0, y_0, u_0, p_0, q_0)$ in $F = 0$

$$\begin{cases} F(x, y, u, p(\lambda), q(\lambda)) &= 0 \\ du - p(\lambda)dx - q(\lambda)dy &= 0 \\ p'(\lambda)dx + q'(\lambda)dy &= 0 \\ F_p p'(\lambda) + F_q q'(\lambda) &= 0 \end{cases}$$

Hence, along generating line

$$dx : dy : du = F_p : F_q : pF_p + qF_q \qquad \text{characteristic direction}$$

Thus, at (x_0, y_0, u_0) we have 1-parameter family of characteristic directions.
In quasi-linear case, only one characteristic direction($F_p : F_q : pF_p + qF_q = a : b : c$)

A curve $(x(s), y(s), u(s))$ is called a focal curve(Monge curve) if

$$(*) \begin{cases} \frac{dx}{ds} &= F_p \\ \frac{dy}{ds} &= F_q \\ \frac{du}{ds} &= pF_p + qF_q \end{cases}$$

A curve $(x(s), y(s), u(s), p(s), q(s))$ in 1st jet space $\{(x, y, u, p, q)\}$ that satisfies (*) and $F(x(s), y(s), u(s), p(s), q(s)) = 0$ is called a focal strip.

For a focal curve to be embedded into an solution surface, we need two additional conditions.

By differentiating $F(x, y, u, p, q) = 0$ with respect to x and y, we get

$$\begin{cases} F_x + F_u p + F_p p_x + F_q q_x &= 0 \\ F_y + F_u q + F_p p_y + F_q q_y &= 0 \end{cases}$$

Since $\begin{cases} F_p p_x + F_q q_x &= \frac{dx}{ds} p_x + \frac{dx}{ds} p_y &= \frac{dp}{ds} \\ F_p p_y + F_q q_y &= \frac{dx}{ds} q_x + \frac{dy}{ds} p_y &= \frac{dq}{ds} \end{cases}$

Hence,

$$\begin{aligned} \frac{dp}{ds} &= -(F_x + F_u p) \\ \frac{dq}{ds} &= -(F_y + F_u p) \end{aligned}$$

Thus, we get the equation for a focal curve embedded in the solution surface.

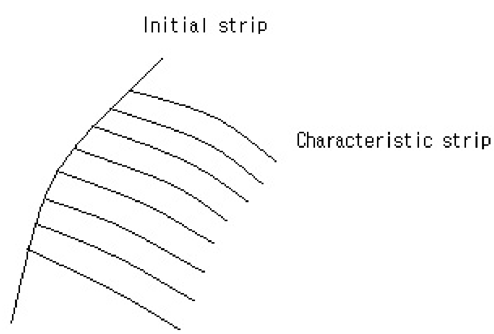
$$\begin{cases} \frac{dx}{ds} &= F_p \\ \frac{dy}{ds} &= F_q \\ \frac{du}{ds} &= pF_p + qF_q \\ \frac{dp}{ds} &= -(F_x + F_u p) \\ \frac{dq}{ds} &= -(F_y + F_u p) \end{cases}$$

Such curve is called a characteristic strip.

A curve $(x(t), y(t), u(t), p(t), q(t))$ in 1st jet space is called a initial strip if

$$\begin{cases} du - pdx - qdy &= 0 \\ F(x(t), y(t), u(t), p(t), q(t)) &= 0 \end{cases}$$

In $J^1(\mathbb{R}^2, \mathbb{R}) = \{(x, y, u, p, q)\}$



we get 2-dimensional surface $(x(t,s), y(t,s), u(t,s), p(t,s), q(t,s))$
So, $u = u(x, y)$ is obtained.

1 Method of moving frame

Let M^n be a manifold with geometric structure. The **method of moving frame** devised by E. Cartan consists of

(i) finding a natural frame $\theta = (\theta^1, \cdots, \theta^n)$ over M,

(ii) expressing $d\theta$ in terms of θ,

(iii) finding a complete system of invariants.

Example 1.1 (Frenet frame for a curve in $\mathbb{R}^3$).
Let $\beta(s)$ be a curve in $\mathbb{R}^3$ parameterized by arc length s. Let $T = \beta'(s)$ be a unit tangent vector. Then $T'(s) = k(s)N$ where N is a unit normal vector to β. Assume $k > 0$. Let $B = T \times N$ be the binormal vector. Then $B' = -\tau N$. We have the Frenet formula :

$$\begin{cases} T' &= kN, \\ N' &= -kT + \tau B, \\ B' &= -\tau N. \end{cases}$$

A pair (T, N, B) is a moving frame along the curve β. We expressed $(T, N, B)'$ in terms of (T, N, B) and obtained a complete system of invariants $\{k, \tau\}$. Suppose there are two curves $\alpha(s)$ and $\beta(s)$ in $\mathbb{R}^3$ which are parameterized by arc length s. If $k_\alpha = k_\beta$ and $\tau_\alpha = \tau_\beta$, then α and β are congruent.

Let G be a Lie group and $\mathfrak{g}$ the associated Lie algebra.

Definition 1.2. A **Maurer-Cartan form** is a $\mathfrak{g}$-valued 1-form ω which satisfies the Maurer-Cartan equation :

$$d\omega = -\frac{1}{2}[\omega, \omega],$$

that is, for any tangent vectors X and Y to G, $d\omega(X, Y) = -[\omega(X), \omega(Y)]$.

In the cases that G is a Lie subgroup of $Gl(n,\mathbb{R})$, let $g : G \hookrightarrow Gl(n,\mathbb{R})$ be the inclusion map. Then $\eta := g^{-1}dg$ is a Maurer-Cartan form. It has the following properties:

(i) η is a $\mathfrak{g}$-valued 1-form.
For $g_0 \in G$ and $v \in T_{g_0}G$,

$$\begin{aligned}\eta(v) &= g_0^{-1}dg(v)\\ &= g_0^{-1}\frac{d}{dt}\Big|_{t=0} g(\alpha(t)) \text{ where } \alpha(0)=g_0,\ \alpha'(0)=v\\ &= \frac{d}{dt}\Big|_{t=0} g_0^{-1}g(\alpha(t)) \in T_eG = \mathfrak{g}.\end{aligned}$$

(ii) η is left invariant.
For any $a \in G$,

$$\begin{aligned}L_a^*\eta &= (ag)^{-1}d(ag)\\ &= g^{-1}a^{-1}adg\\ &= g^{-1}dg\\ &= \eta.\end{aligned}$$

(iii) $d\eta = -\frac{1}{2}[\eta,\eta]$.
Since $g^{-1}g = I$, $(dg^{-1})g + g^{-1}dg = 0$. Thus $d(g^{-1}) = -g^{-1}(dg)g^{-1}$.
Now,

$$\begin{aligned}d\eta &= d(g^{-1})\wedge dg + g^{-1}ddg\\ &= -g^{-1}dgg^{-1}dg\\ &= -\eta\wedge\eta.\end{aligned}$$

In $\mathbb{R}^n$, a frame is an ordered set of vectors

$$F = (x, e_1, \ldots, e_n),$$

where $x \in \mathbb{R}^n$ and e_i's are orthonormal tangent vectors at x. Such frames form the group $E(n)$ of Euclidean motions. $E(n)$ is the set of all the matrices of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_1 & & & \\ \vdots & e_1 & \cdots & e_n \\ x_n & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \quad \text{where } \ {}^tAA = I.$$

$E(n)$ is a group since

$$\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x + Ay & AB \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -{}^tAx & {}^tA \end{bmatrix}.$$

Now compute the Maurer-Cartan form of $E(n)$.
Let $g : E(n) \hookrightarrow Gl(n+1, \mathbb{R})$ be the inclusion. Then we have

$$\begin{aligned} \eta &= g^{-1}dg \\ &= \begin{bmatrix} 1 & 0 \\ -{}^tAx & {}^tA \end{bmatrix} \begin{bmatrix} 0 & 0 \\ dx & dA \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ {}^tAdx & {}^tAdA \end{bmatrix}. \end{aligned}$$

From the bundle point of view, $\pi : E(n) \longrightarrow \mathbb{R}^n$ given by $(x, e_1, \cdots, e_n) \mapsto x$ gives a principal fibration :

$$E(n) \xrightarrow{\pi} E(n)/O(n) \approx \mathbb{R}^n$$

with structure group $O(n)$. Let $\sigma(x) = (x, e_1, \cdots, e_n)$ be an orthonormal frame field. Pull back η by σ. Then we obtain

$$\sigma^*\eta = \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \theta_1 & & & \\ \vdots & & \omega^i_j & \\ \theta_n & & & \end{bmatrix}.$$

In order to find θ and ω, we let $A = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}$. Then ${}^tAdx = \theta$ and ${}^tAdA = \omega$. Observe that the Maurer-Cartan equation $d\eta + \eta \wedge \eta = 0$ implies

$$\begin{aligned} 0 &= \sigma^*(d\eta + \eta \wedge \eta) \\ &= d(\sigma^*\eta) + \sigma^*\eta \wedge \sigma^*\eta \\ &= \begin{bmatrix} 0 & 0 \\ d\theta & d\omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ d\theta + \omega \wedge \theta & d\omega + \omega \wedge \omega \end{bmatrix}. \end{aligned}$$

Hence $d\theta + \omega \wedge \theta = 0$ and $d\omega + \omega \wedge \omega = 0$.

In general, let G be a Lie group of $Gl(n, \mathbb{R})$ and H a closed subgroup of G. G may be regarded as the set of frames of G/H. Then the Maurer-Cartan forms appear in the structure equations of a moving frame. The Maurer-Cartan equations give a complete set of relations for the structure equations of a moving frame. The question of describing the position of a submanifold $M \subset G/H$ may be thought of attaching to M a natural frame or equivalently, a cross section of $G \to G/H$ over M. The Maurer-Cartan form for G when restricted to the natural frame becomes a complete set of invariants for M in G/H.

2 Local geometry of a submanifold $M^m \subset \mathbb{R}^N$

Choose a natural frame(adapted frame) of M, that is, an orthonormal frame $e_1, \ldots, e_m, e_{m+1}, \ldots, e_N$ where $e_1, \ldots, e_m$ are tangent to M. Let $\sigma : M \to E(N)$ be the map $x \mapsto (x, e_1, \ldots, e_N)$.

Definition 2.1. Pull back by σ of the Maurer-Cartan form η of $E(N)$ is called the **complete system of (local) invariants** of M with respect to the group of euclidean motions.

Let M, M' be submanifolds of $\mathbb{R}^N$ of dimension m. Let σ, σ' be adapted frames and $(\theta, \omega), (\theta', \omega')$ the complete systems of local invariants of M and M', respectively. Then M and M' are congruent if and only if there exists $\varphi : M \to M'$ such that $\varphi^*(\theta', \omega') = (\theta, \omega)$.

The previous example of Frenet frame is a special case :

$$k = \omega_2^1(T), \quad \tau = \omega_3^2(T).$$

Now we consider the existence and uniqueness of maps into Lie groups. Let G be a Lie group with the Lie algebra $\mathfrak{g} = T_eG$ and let η be a $\mathfrak{g}$-valued 1-form on G such that

(i) $\eta_e : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the identity map,

(ii) $L_a^*\eta = \eta$ for all $a \in G$.

Such an η exists and is unique. If $G \hookrightarrow Gl(n, \mathbb{R})$, then $\eta = g^{-1}dg$ is such a Maurer-Cartan form.

The following theorem is due to E. Cartan.

Theorem 2.2. *Let N be a connected and simply connected manifold and let γ be a smooth $\mathfrak{g}$-valued 1-form on N such that $d\gamma = -\frac{1}{2}[\gamma, \gamma]$. Then, there is a smooth map $g : N \to G$ which is unique up to composition with a constant left multiplication so that $g^*\eta = \gamma$.*

Proof. We assume that G and $\mathfrak{g}$ are matrix groups. Let $M = N \times G$ and consider a 1-form $\theta = \eta - \gamma$. Then we have

$$\begin{aligned} d\theta &= d\eta - d\gamma \\ &= -\eta \wedge \eta + \gamma \wedge \gamma \\ &= -(\theta + \gamma) \wedge (\theta + \gamma) + \gamma \wedge \gamma \\ &= -\theta \wedge (\theta + \gamma) - \gamma \wedge \theta. \end{aligned}$$

We write $\theta = \theta^1 x_1 + \cdots + \theta^s x_s$ where $\{x_1, \ldots, x_s\}$ is a basis of $\mathfrak{g}$ and $\theta^1, \ldots, \theta^s$ are 1-forms on M. Then the algebraic ideal $I = < \theta^1, \ldots, \theta^s >$

satisfies $dI \subset I$. Moreover, $\theta^1, \ldots, \theta^s$ are linearly independent because they restrict to each fibre $\{n\} \times G$ to be linearly independent. By Frobenius theorem, M is foliated by maximal connected integral manifolds of I, each of which projects onto N to be a covering map. Observe that the foliation is invariant under $Id \times L_a : N \times G \to N \times G$ since η is left invariant. Since N is connected and simply connected, each integral leaf projects diffeomorphically onto N and hence the graph of a map $g : N \to G$. $\square$

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1 Isometric Embedding

Let M^n be an n-dimensional Riemannian manifold with metric locally given by

$$ds^2 = g_{ij}(x)dx^i dx^j,$$

where $x = (x^1, \dots, x^n)$ are local coordinates on M.

Isometric embedding means a one-to-one C^∞-mapping

$$u : M^n \to \mathbb{R}^N$$

such that

$$< du, du >= ds^2$$

or in local coordinates

$$\sum_{\lambda=1}^{N} \frac{\partial u^\lambda}{\partial x^i} \frac{\partial u^\lambda}{\partial x^j} = g_{ij}, \quad i, j = 1, \dots, n. \tag{1}$$

So a local isometric embedding problem is reduced to a PDE system. There are three different cases to deal with according to the number of equations and the number of unknowns. The number of equations of (1) is $\frac{n(n+1)}{2}$ and the number of unknowns is $N = n + d$. The system (1) is

(i) underdetermined if $\frac{n(n+1)}{2} < N$,

(ii) determined if $\frac{n(n+1)}{2} = N$,

(iii) overdetemined if $\frac{n(n+1)}{2} > N$.

In the determined case, there exists an analytic embedding by the following theorem.

Theorem 1.1 (Cartan-Janet,[3]). *If $N = \frac{1}{2}n(n+1)$ and $g_{ij} \in C^\omega$, then there exists a C^ω-solution $u = (u^1, \dots, u^{\frac{1}{2}n(n+1)})$.*

Some of the results on the underdetermined case were obtained by J. Nash[4].

Theorem 1.2. *Any Riemannian n-manifold with C^k positive metric, where $3 \leq k \leq \infty$, has a C^k isometric embedding in $(\frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n)$-space, in fact in any small portion of this space.*

In overdetermined case, we consider the case of codimension one.

2 Isometric Embedding of Codimension One

Isometric embedding of codimension one is an isometric embedding

$$u : M^n \to \mathbb{R}^{n+1}. \tag{2}$$

This is determined if $n = 2$ and overdetermined if $n > 2$. The question of finding a necessary and sufficient condition for the existence of local isometric embedding (2) is reduced to the problem of solving the Gauss and Codazzi equations.

Let $(e_1, \ldots, e_{n+1})$ be an adapted orthonormal frame and $\theta = (\theta^1, \ldots, \theta^n)^t$ be a dual frame of $(e_1, \ldots, e_n)$. For any 1-forms η and ψ, the **symmetric product** is defined by

$$\eta \circ \psi = \frac{1}{2}(\eta \otimes \psi + \psi \otimes \eta).$$

Let $I = \sum_{i=1}^{n} \theta^i \circ \theta^j = \sum_{i=1}^{n} (\theta^i)^2$ be the first fundamental form of M.

Let X be a tangent vector field on M and $Y = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x^i}$ a vector field on M which is not necessarily tangent to M. Define

$$\overline{\nabla}_X Y = \sum_{i=1}^{n+1} (X a_i) \frac{\partial}{\partial x_i}.$$

Proposition 2.1. *If X and Y are tangent vector fields to M, then $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y]$. So $[X, Y]$ is also a tangent vector field to M.*

Proof. If X and Y are tangent vector fields, then $X = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$. Thus

$$\begin{aligned}
[X, Y] &= XY - YX \\
&= \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\Big(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big) - \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\Big) \\
&= \sum_{i,j} b_j \frac{\partial}{\partial x_j}\Big(a_i\Big)\frac{\partial}{\partial x_i} + \sum_{i,j} a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
&\quad - \sum_{i,j} a_i \frac{\partial}{\partial x_i}\Big(b_j\Big)\frac{\partial}{\partial x_j} - \sum_{i,j} a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
&= \overline{\nabla}_X Y - \overline{\nabla}_Y X.
\end{aligned}$$

Since $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$ are tangent to M, so is $[X, Y]$. □

Definition 2.2. For tangent vector fields X, Y and normal vector field N, the second fundamental form Π of M is defined by

$$\Pi(X, Y) = - < \overline{\nabla}_X N, Y > .$$

Proposition 2.3. *For tangent vector fields X, Y and normal vector field N, the second fundamental form has the properties :*

$$\Pi(X, Y) =< \overline{\nabla}_X Y, N >,$$

$$\Pi(X, Y) = \Pi(Y, X).$$

Proof. Since N is a normal vector field, $< Y, N >= 0$. Thus

$$\begin{aligned}
X < Y, N > &= < \overline{\nabla}_X Y, N > + < Y, \overline{\nabla}_X N > \\
&= 0.
\end{aligned}$$

By definition 2.2, we have

$$\begin{aligned}
\Pi(X, Y) &= - < \overline{\nabla}_X N, Y > \\
&= < Y, \overline{\nabla}_X N > \\
&= < \overline{\nabla}_X Y, N > .
\end{aligned}$$

Using the previous proposition, we get

$$\begin{aligned}\Pi(X,Y)-\Pi(Y,X) &= <\overline{\nabla}_X Y, N> - <\overline{\nabla}_Y X, N> \\ &= <\overline{\nabla}_X Y - \overline{\nabla}_Y X, N> \\ &= <[X,Y], N> \\ &= 0.\end{aligned}$$

□

Since Π is a symmetric 2-form on M, we write

$$\Pi = \sum_{i,j=1}^{n} h_{ij}\theta^i \otimes \theta^j,$$

where $h_{ij} = h_{ji}$. Then $h_{ij} = \Pi(e_i, e_j)$. Since (h_{ij}) is symmetric, its eigenvalues are real. Let $k_1, \ldots, k_n$ be eigenvalues. We call them the principal curvatures.

Theorem 2.4. *Let* $({\omega_i}^j) = A^{-1}dA$, *where* $A = (e_1, \ldots, e_{n+1})$. *On* M,

$$\omega_i^{n+1} = \sum_{\lambda=1}^{n} h_{i\lambda}\theta^\lambda.$$

Proof. We know that $\overline{\nabla}_X e_i = \sum_{j=1}^{n+1} \omega_i^j(X)e_j$. Since ω is generated by $\theta^1, \ldots, \theta^n$, it is enough to show that $\omega_i^{n+1} = h_{i\lambda}$. Since (h_{ij}) is symmetric, $h_{i\lambda} = h_{\lambda i}$ and since $(e_1, \ldots, e_n, e_{n+1})$ is the adapted orthonormal frame, e_{n+1} is a normal vector. Therefore, we have

$$\begin{aligned} h_{i\lambda} &= \Pi(e_\lambda, e_i) \\ &= -<\overline{\nabla}_\lambda e_{n+1}, e_i> \\ &= -<\sum_{j=1}^{n+1} \omega_{n+1}^j(e_\lambda)e_j, e_i> \\ &= -\omega_{n+1}^i(e_\lambda) \\ &= \omega_i^{n+1}(e_\lambda).\end{aligned}$$

The last equality follows from the fact that ω is skew-symmetric as shown below. Since $< e_i, e_j >= \delta_{ij}$, we have

$$\begin{aligned} 0 &= d < e_i, e_j > \\ &= < de_i, e_j > + < e_i, de_j > \\ &= < \sum_{\lambda=1}^{n+1} \omega_i^\lambda e_\lambda, e_j > + < e_i, \sum_{\lambda=1}^{n+1} \omega_j^\lambda e_\lambda > \\ &= \omega_i^j + \omega_j^i. \end{aligned}$$

□

From now on we consider the case of $n = 3$. In order to obtain the structure equations, consider $E(4) \hookrightarrow Gl(5, \mathbb{R})$ with Maurer-Cartan form $\gamma = g^{-1}dg$ of $E(4)$. $E(4)$ is the set of all matrices $\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}$ with ${}^tAA = I$. Let $\sigma : M \longrightarrow E(4)$ be an adpated frame $\sigma(x) = (e_1, e_2, e_3, e_4)_x$. Then it follows that

$$\begin{aligned} \sigma^*(\gamma) &= \begin{bmatrix} 0 & 0 \\ {}^tAdX & {}^tAdA \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \theta^1 & 0 & -\omega_1^2 & -\omega_1^3 & -\eta_1 \\ \theta^2 & \omega_1^2 & 0 & -\omega_2^3 & -\eta_2 \\ \theta^3 & \omega_1^3 & \omega_2^3 & 0 & -\eta_3 \\ 0 & \eta_1 & \eta_2 & \eta_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix}, \end{aligned}$$

where $\eta_i = {\omega_i}^4$, $A = (e_1, \ldots, e_4)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ and

$$\omega = \begin{bmatrix} 0 & -{\omega_1}^2 & -{\omega_1}^3 \\ {\omega_1}^2 & 0 & -{\omega_2}^3 \\ {\omega_1}^3 & {\omega_2}^3 & 0 \end{bmatrix}.$$

Maurer-Cartan equation $d\gamma = -\gamma \wedge \gamma$ implies that

$$d(\sigma^*\gamma) = (-\sigma^*\gamma) \wedge (\sigma^*\gamma).$$

Thus

$$d\begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -{}^t\eta \\ 0 & \eta & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ d\theta & d\omega & -{}^t(d\eta) \\ 0 & d\eta & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 \\ \omega \wedge \theta & \omega \wedge \omega - {}^t\eta \wedge \eta & -\omega \wedge {}^t\eta \\ \eta \wedge \theta & \eta \wedge \omega & -\eta \wedge {}^t\eta \end{bmatrix}.$$

By Theorem 2.4, we have $\eta_i = \omega_i^4 = \sum_{\lambda=1}^3 h_{i\lambda}\theta^\lambda$, that is, $\eta = {}^t\theta H$.

Thus we obtain

$$\begin{aligned} d\theta &= -(\omega \wedge \theta), \\ d\omega &= -\omega \wedge \omega + {}^t\eta \wedge \eta \quad \text{(Gauss equation)}, \\ d\eta &= -\eta \wedge \omega \quad \text{(Codazzi equation)}, \\ \eta \wedge \theta &= 0, \\ \eta \wedge {}^t\eta &= 0, \\ \eta &= {}^t\theta H. \end{aligned}$$

It is enough to show that there exists the second fundamental form $\Pi = (h_{ij}) = H$ by the following theorem

Theorem 2.5 (Bonnet, [5]). *Suppose that two hypersurfaces M and $\widetilde{M} \subset \mathbb{R}^{n+1}$ have the same first and second fundamental forms. Then they are congruent.*

Let us summarize the process of solving the case of $n = 3$ as follows:

(i) Start with metric $g = I$.

(ii) Find orthonormal frame θ such that $I = \sum_{i=1}^{3}(\theta)^2$.

(iii) Find Levi-Civita connection for (ω_i^j) $i, j = 1, 2, 3$ such that $d\theta = -\omega \wedge \theta$ and ${}^t\omega = -\omega$. Then compute curvature $\Phi = d\omega + \omega \wedge \omega = {}^t\eta \wedge \eta$.

(iv) Solve the algebraic equation $\Phi = H\theta\, {}^t\theta H$ for H. Compute ${}^t\eta \wedge \eta = H\theta \wedge {}^t\theta H$. Let $\Phi = (\Phi_i^j)$. Compare both sides of $\Phi = H\theta \wedge {}^t\theta H$. Both sides are skew-symmetric. Then we obtain the following three equations.

$$\begin{aligned}(h_{22}h_{33} - h_{23}^2)\theta^2 \wedge \theta^3 \quad &+ \quad (h_{23}h_{13} - h_{12}h_{33})\theta^3 \wedge \theta^1 \\ &+ \quad (h_{12}h_{23} - h_{22}h_{13})\theta^1 \wedge \theta^2 = \Phi_2^3,\end{aligned}$$

$$\begin{aligned}(h_{13}h_{23} - h_{12}h_{33})\theta^2 \wedge \theta^3 \quad &+ \quad (h_{11}h_{33} - h_{13}^2)\theta^3 \wedge \theta^1 \\ &+ \quad (h_{12}h_{13} - h_{11}h_{23})\theta^1 \wedge \theta^2 = -\Phi_1^3,\end{aligned}$$

$$\begin{aligned}(h_{12}h_{23} - h_{13}h_{22})\theta^2 \wedge \theta^3 \quad &+ \quad (h_{13}h_{12} - h_{11}h_{23})\theta^3 \wedge \theta^1 \\ &+ \quad (h_{11}h_{22} - h_{12}^2)\theta^1 \wedge \theta^2 = \Phi_1^2.\end{aligned}$$

In matrix form, these equations are

$$\operatorname{adj}(H) \begin{bmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{bmatrix} = \begin{bmatrix} \Phi_2^3 \\ -\Phi_1^3 \\ \Phi_1^2 \end{bmatrix}.$$

To compute $\operatorname{adj}(H) = K$, evaluate on (e_k, e_l). Let $\Phi_{ikl}^j = \Phi_i^j(e_k, e_l)$. Then

$$K = \begin{bmatrix} \Phi_{223}^3 & \Phi_{231}^3 & \Phi_{212}^3 \\ -\Phi_{123}^3 & -\Phi_{131}^3 & -\Phi_{112}^3 \\ \Phi_{123}^2 & \Phi_{131}^2 & \Phi_{112}^2 \end{bmatrix}.$$

Since $K = \mathrm{adj}(H) = (\det H)H^{-1}$,

$$\begin{aligned} H &= \frac{1}{\det H} K^{-1}, \\ \det K &= (\det H)^3 (\det H)^{-1} \\ &= (\det H)^2. \end{aligned}$$

Thus $\det H = \pm\sqrt{\det K}$. If $\det K > 0$, Gauss equation is solvable and the solution is unique up to sign and if $\det K < 0$, there is no solution.

(v) Check whether H satisfies Codazzi equation $d({}^t\theta H) = -({}^t\theta H) \wedge \omega$. If it holds, then H is a solution.

Here is a more general result of the codimension one case under some restrictions for M^n for $n \geq 5$. This result was shown by J. Vilms[6].

Let V be an n-dimensional real vector space with inner product. Let $\Lambda^2 V$ denote the $\binom{n}{2}$- dimensional space of bivectors of V. A linear map $L : V \to V$ defines a linear map $L \wedge L : \Lambda^2 V \to \Lambda^2 V$ by $(L \wedge L)(x \wedge y) = Lx \wedge Ly$. When V is taken to be the tangent space at a point of M^n, the curvature tensor R at that point can be thought of as a symmetric linear map $R : \Lambda^2 V \to \Lambda^2 V$. Letting L denote the second fundamental form operator and denoting the covariant derivative by ∇, we can express the Gauss and Codazzi equations as $R = L \wedge R$ and ∇L is symmetric. On the above setting, the problem of locally isometrically embedding into $\mathbb{R}^{n+1}$ a C^3 Riemannian manifold M^n with curvature of rank ≥ 6 is reduced to the following algebraic question: *Given a symmetric linear map $R : \Lambda^2 V \to \Lambda^2 V$, find necessary and sufficient condition in order that there exists a symmetric linear map $L : V \to V$ satisfying $R = L \wedge L$.*

Theorem 2.6 (J. Vilms[6]). *Let M^n, with $n \geq 5$, be a Riemannian manifold with nonsingular curvature tensor R. Then M^n admits local isometric imbedding into $\mathbb{R}^{n+1}$ if and only if*

(1) $R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) + R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) = 0$, *for all* $x_i \in V$, *and*

(2) $R^{ij}_{kl} R^{kp}_{iq} R^{lq}_{jp} + \frac{1}{4} R^{ij}_{kl} R^{kl}_{pq} R^{pq}_{ij} > 0.$

Moreover, if $n \equiv 3 \mod 4$, *then* (1) *can be replaced by* $\det R > 0$.

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1 Method of moving frame for intrinsic equivalence problem

Let M^n be a smooth manifold and G be a Lie subgroup.

Definition 1 *A G-structure on M is a reduction of the bundle of frame to a sub-bundle with the structure group G.*

Example 1

- *$O(n)$-structure is a Riemannian structure.*
- *$Gl(m, \mathbb{C})$-structure is an almost complex structure.*
- *$Gl^+(n, \mathbb{R})$-structure is an orientation.*

Let M and $\tilde{M}$ be manifolds of dimension n with G-structure. The equivalence problem is deciding whether there exists a structure-preserving mapping $f : M \mapsto \tilde{M}$. Locally, this is a question of existence of solutions for an overdetermined system of partial differential equations of 1st order.

1.1 Cartan's method for the equivalence problem of G-structure.

Let $\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \vdots \\ \theta^n \end{bmatrix}$ and $\tilde{\theta} = \begin{bmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \vdots \\ \tilde{\theta}^n \end{bmatrix}$ be the fixed frames over M and $\tilde{M}$ respectively, adapted to the G-structure.

Question : when does there exist $f : M \mapsto \tilde{M}$ satisfying $f^*\tilde{\theta}^j = \Sigma_{i=1}^n a_i^j(x)\theta^i$ where $[a_i^j]$ is a G-valued function on M.