Notes on point-set topology

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Contents

Chapter 1

Metric spaces

1.1 Introduction

1.1.1. In this chapter we will introduce the notion of a *metric space*. Metric spaces are examples of *topological spaces* that are the true objects of these notes, and we will not develop any theory exclusively for metric spaces. (The only exception is the appendix on completion of metric spaces.) This chapter merely gives a good number of examples and some techniques that apply to produce even more examples.

1.1.2. Consider two points x, y in a set X. We want a measure of the 'distance' between x and y. Intuitively, this distance should measure the shortest path from x to y without actually specifying any path.

In some cases there is a rather natural definition of distance: If $X = \mathbb{R}$ we may take the distance to be $|x - y|$. In other cases it may not be so obvious what to understand by the distance between x and y.

Example 1.1.3. Let X be the surface of the earth. What is the distance between the Eiffel Tower in Paris and the Statue of Liberty in New York? If you travel by plane, you'll probably realize that the shortest way is by flying just above sea level following the great circle that contains the two relevant points. But if you were able to dig your way through the interior of the earth, you could probably find a shorter path.

1.1.4. Note that in some sense the first definition of distance on the surface of the earth X is the most intrinsic in the sense that our shortest paths never leave the surface X (if we fly low enough), while the second definition obviously depends on the fact that X is a subset of a larger ambient space, namely the earth it self.

1.1.5. What kind of properties do we want the distance to have? It is natural to require the distance between x and y to be non-negative. Moreover, it should be 0 exactly when $x = y$. We also want it to be symmetric in the sense that the distance from x to y is the same as the distance from y to x.

What more? If we think in terms of 'shortest paths', then we would expect that if we pick a third point z, then the total distance of following the shortest path from x to y and then the shortest path from y to z should be at least as big as the shortest distance from x to z.

If we forget about 'paths' (you better do this from now on, or you may get into troubles with some of the examples) this amounts to the following definition.

Definition 1.1.6. Let X be a set. A *metric* (or *distance function*) on X is a function $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ which satisfies

1. For any $x, y \in X$: $d(x, y) = 0$ if and only if $x = y$ (faithfulness)
2. For any $x, y \in X$: $d(x, y) = d(y, x)$ (symmetry)
3. For any $x, y, z \in X$: $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

The pair (X, d) is called a *metric space.*

Example 1.1.7. The most important example of a metric is the Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ on $\mathbb{R}^n$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are points in $\mathbb{R}^n$, let

$$d(x, y) = |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

This is clearly faithful and symmetric. To prove the triangle inequality, recall that we have an inner product $< -, - >: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ on $\mathbb{R}^n$ such that $|x|^2 =< x, x >$. Schwartz' inequality says that $| < x, y > | \leq |x||y|$ for all x, y. Thus, for $x, y, z \in \mathbb{R}^n$ we have

$$\begin{aligned} |x - z|^2 &=< x - z, x - z > \\ &=< (x - y) + (y - z), (x - y) + (y - z) > \\ &=< x - y, x - y > + < y - z, y - z > +2 < x - y, y - z > \\ &\leq |x - y|^2 + |y - z|^2 + 2|x - y||y - z| = (|x - y| + |y - z|)^2. \end{aligned}$$

Taking square roots we get the triangle inequality.

Exercise 1.1.8. Let X be a set, and define $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ by the assignment

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Show that d defines a metric.

Exercise 1.1.9. Let (X, d) be a metric space, and let $a \in \mathbb{R}_{>0}$. Define a function $d_a : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ by the assignment

$$d_a(x, y) = \begin{cases} d(x, y), & \text{if } d(x, y) < a \\ a, & \text{if } d(x, y) \geq a. \end{cases}$$

Show that d_a defines a metric.

Recall our Example 1.1.3 from above where X was the surface of the earth. In mathematical terms this becomes

Example 1.1.10. Let $X = S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ be the 2-sphere. We define a metric $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ by $d(x, y) = \cos^{-1}(< x, y >)$ for $x, y \in X$, where $\cos : [0, \pi] \longrightarrow [-1, 1]$ is the restriction of the cosine function. This way cos is monotone decreasing and $\cos^{-1}$ is well-defined.

Geometrically, $d(x, y)$ may be described as follows. If $x = y$, put $d(x, y) = 0$; if $x = -y$, put $d(x, y) = \pi$; otherwise, there exists a unique great circle (geodesic) that passes through both x and y, and $d(x, y)$ is the length of the shortest arc connecting x and y on this great circle.

Let us prove that d is a metric. Only the triangle inequality causes problems. For convenience, let $a = d(x, y)$ and $b = d(y, z)$ and $c = d(x, z)$. We must show that $c \leq a + b$.

Since $a \leq \pi$, then $\sin(a) = \sqrt{1 - \cos^2(a)} = \sqrt{1 - < x, y >^2}$. Let $\bar{x} = x - < x, y > y$ denote the projection of x onto the orthogonal complement in $\mathbb{R}^3$ to the vector y. Then a direct calculation shows that $|\bar{x}|^2 = < \bar{x}, \bar{x} > = 1 - < x, y >^2$, so $\sin(a) = |\bar{x}|$. Similarly, if $\bar{z} = z - < z, y > y$, then $\sin(b) = |\bar{z}|$. Moreover, $< x, z > - < \bar{x}, \bar{z} > = < x, y > < y, z > = \cos(a)\cos(b)$. Using the addition formula for cosine and Schwartz' inequality we get

$$\begin{aligned} \cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ &= < x, z > - (< \bar{x}, \bar{z} > + |\bar{x}||\bar{z}|) \\ &\leq < x, z > = \cos(c). \end{aligned}$$

If $a + b \leq \pi$, apply the monotone decreasing function $\cos^{-1}$ to this inequality to get $c \leq a + b$. Otherwise, $c \leq \pi \leq a + b$.

1.2 Normed vector spaces

An important class of examples of metric spaces are the *normed vector spaces.*

Definition 1.2.1. Let V be a vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$. A *norm* on V is a map $N : V \longrightarrow \mathbb{R}_{\geq 0}$ that satisfies

1. For all $x \in V$: $N(x) = 0$ if and only if $x = 0$ (faithfulness)

2. For all $x \in V$ and $a \in k$: $N(ax) = |a|N(x)$. (homogeneity)

3. For all $x, y \in V$: $N(x + y) \leq N(x) + N(y)$ (subadditivity)

The pair (V, N) is called a *normed vector space*.

1.2.2. Let (V, N) be a normed vector space. Then we may define a metric $d : V \times V \longrightarrow \mathbb{R}_{\geq 0}$ on V by letting $d(x, y) = N(x - y)$ for all $x, y \in V$. Then d is faithful, since N is faithful:

$$d(x, y) = 0 \Leftrightarrow N(x - y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y.$$

Also, d is symmetric since $d(x, y) = N(x-y) = N((-1)(y-x)) = |-1|N(y-x) = d(y, x)$, and the triangle inequality is a consequence of subadditivity of N in the following way

$$d(x, z) = N(x-z) = N((x-y)+(y-z)) \leq N(x-y)+N(y-z) = d(x, y)+d(y, z).$$

Example 1.2.3. It is obvious that the Euclidean metric on $\mathbb{R}^n$ is the one induced from the norm $N(x) = |x|$ on $\mathbb{R}^n$.

Exercise 1.2.4. Let k be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $X = [0, 1]$ be the closed unit interval. By $C(X)$ we denote the set of k-valued continuous functions, $f : X \longrightarrow k$. Show that $C(X)$ is a k-vector space. For $f \in C(X)$ we define $\|f\|_\infty = \sup_{x \in X} |f(x)|$. Show that the map $\|-\|_\infty : C(X) \longrightarrow \mathbb{R}_{\geq 0}$ defines a norm on $C(X)$. This norm is called the *supremum norm* or the *uniform norm*.

Example 1.2.5. With $C(X)$ as in Exercise 1.2.4, we may actually define quite a lot of other norms on the vector space $C(X)$. For $p \geq 1$ a real number, the L^p-norm $\|-\|_p : C(X) \longrightarrow \mathbb{R}_{\geq 0}$ is defined by the assignment

$$\|f\|_p = \left(\int_X |f(x)|^p dx \right)^{1/p}.$$

It is an easy exercise to prove that $\|-\|_p$ is faithful and homogeneous. Subadditivity is known as *Minkowski's inequality* and is a little harder to prove (see e.g. [Rudin], Theorem 3.5).

For $k = \mathbb{C}$, let $L^p(X)$ denote the completion of $C(X)$ in the L^p-norm (see Theorem 1.6.15 in the Appendix on completion). Then $L^p(X)$ is a Banach space. For $p = 2$, the L^2-norm is actually an inner-product norm, and hence $L^2(X)$ becomes a Hilbert space. Hilbert and Banach spaces play an important rôle in analysis.

1.3 Subspace metrics

1.3.1. Let (Y, d_Y) be a metric space, and let $X \subset Y$ be a subset of Y. Then we may restrict d_Y to a metric $d_X : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ on X: For $x, y \in X$, let $d_X(x, y) = d_Y(x, y)$. It is an easy exercise to convince oneself that d_X satisfies 1., 2. and 3. of Definition 1.1.6 given that d_Y does. We call d_X the *subspace metric* or *induced metric* on X.

Exercise 1.3.2. As before, $X = S^2$ is naturally a subset of $Y = \mathbb{R}^3$. Equip Y with the Euclidean metric d_Y of Example 1.1.7. Then the induced metric d_X is different from the metric d of Example 1.1.10. Prove that

$$\frac{2}{\pi} \leq \frac{d_X(x, y)}{d(x, y)} \leq 1$$

for all $x, y \in X$ for which $x \neq y$. Maybe you want to reread the geographical example 1.1.3.

Exercise 1.3.3. Let n and p be natural numbers. For which pairs (n, p) is it possible to find a subset $X \subset \mathbb{R}^n$ with p elements, such that the metric d_X on X induced from the Euclidean metric on $\mathbb{R}^n$ coincides with the metric d of Exercise 1.1.8?

1.4 Open subsets and continuous maps

We are now going to consider maps between metric spaces. Often in mathematics we are interested in identifying those maps which preserve the structures on the spaces it maps between. In our case, the relevant structures would be the metrics. So, if $f : X \longrightarrow Y$ is a map of metric spaces (X, d_X) and (Y, d_Y), we might require that $d_X(x, y) = d_Y(f(x), f(y))$. A map satisfying this requirement is called an *isometry.*

But isometries are rare, and very often we are only interested in a weaker property like 'if x and y are close to each other, then $f(x)$ and $f(y)$ are also close to each other'. Mathematically, this may be formulated as follows.

Definition 1.4.1. Let $x \in X$. Then f is *continuous* at x if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, z) < \delta \Rightarrow d_Y(f(x), f(z)) < \epsilon \qquad \text{for all } z \in X.$$

If f is continuous at all $x \in X$, then we say that f is *continuous.*

This definition should be familiar to the reader if X and Y are subsets of $\mathbb{R}^n$.

1.4.2. Let (X, d) be a metric space. For $x \in X$ a point in X and $r > 0$, the *open ball around* x *of radius* r is the set $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$.

We define a subset $U \subset X$ to be an *open subset of* X, if for each point $x \in U$ there exists an $r_x > 0$ such that $B(x, r_x) \subset U$.

1.4.3. With this definition at hand we see that a map $f : X \longrightarrow Y$ as before is continuous at $x \in X$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$.

We will give one statement about continuous maps and otherwise refer to the treatment in the next chapter.

Theorem 1.4.4. *f is continuous if and only if $f^{-1}(V)$ is an open subset of X for any open subset V of Y.*

Proof. Suppose f is continuous. Let V be an open subset of Y and $x \in f^{-1}(V)$ a point. Then by openness of V there exists an $\epsilon > 0$ with $B_{d_Y}(f(x), \epsilon) \subset V$. Now choose $\delta > 0$ such that $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$. Then $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon)) \subset f^{-1}(V)$. This holds for all $x \in f^{-1}(V)$ and shows that $f^{-1}(V)$ is open.

For the converse, suppose that $f^{-1}(V)$ is an open subset of X for each open subset V of Y. Let $x \in X$ be a point. We will show that f is continuous at x. Given $\epsilon > 0$, let $V = B_{d_Y}(f(x), \epsilon)$. Then V is an open subset in Y by Exercise 1.4.7 below, so $f^{-1}(V)$ is an open subset of X by assumption. Since $x \in f^{-1}(V)$, this implies that there exist $\delta > 0$ such that $B_{d_X}(x, \delta) \subset f^{-1}(V)$. Or, put differently, $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$. This shows that f is continuous at x. □

1.4.5. This theorem suggests that we should focus more on the open subsets of a metric space rather than on the metric itself. In particular, we may ask when two different metrics on a set give rise to the same open subsets.

We will say that two metrics d and d' on a set X are *equivalent*, if for all $x \in X$ and $\epsilon > 0$ there exist $\delta, \delta' > 0$ such that

$$B_d(x, \delta) \subset B_{d'}(x, \epsilon) \quad \text{and} \quad B_{d'}(x, \delta') \subset B_d(x, \epsilon).$$

Proposition 1.4.6. *The following are equivalent for a set X with two metrics d and d'.*

1. *d and d' are equivalent.*

2. *Any subset $U \subset X$ which is open with respect to one of the metrics is also open with respect to the other.*

Proof. 1. $\Rightarrow$ 2.: Suppose d and d' are equivalent and U is open with respect to one of the metrics, say d. Fix $x \in U$. By openness, $B_d(x, \epsilon) \subset U$ for some $\epsilon > 0$. By equivalence of d and d' there exist $\delta' > 0$ such that $B_{d'}(x, \delta') \subset B_d(x, \epsilon) \subset U$. This shows that U is also open with respect to d'.

2. $\Rightarrow$ 1.: Assume 2., and let $x \in X$ and $\epsilon > 0$ be given. Then $U = B_d(x, \epsilon)$ is open with respect to d (1.4.7), hence also with respect to d' by assumption. This implies that there exists $\delta' > 0$ with $B_{d'}(x, \delta') \subset U = B_d(x, \epsilon)$. By interchanging d and d' we see that there also exists $\delta > 0$ with $B_d(x, \delta) \subset B_{d'}(x, \epsilon)$. Together, this shows that d and d' are equivalent. □

Exercise 1.4.7. Let (X, d) be a metric space. Show that $B_d(x, r)$ is open for all $x \in X$ and $r > 0$.

Exercise 1.4.8. For a metric space (X, d), prove the following assertions:

1. Let I be any set, and assume that for each $i \in I$ we have an open subset U_i of X. Then $\bigcup_{i \in I} U_i$ is an open subset of X.

2. Let I be a *finite* set, and assume that for each $i \in I$ we have an open subset U_i of X. Then $\bigcap_{i \in I} U_i$ is an open subset of X.

3. $\emptyset$ and X are open subsets of X.

Exercise 1.4.9. Consider again the metric space (X, d) from Exercise 1.1.8. Describe the open subsets of X.

Exercise 1.4.10. Let (X, d) be a metric space, and let d_a be the metric from Exercise 1.1.9 for some $a > 0$. Show that d and d_a are equivalent.

Exercise 1.4.11. In Exercise 1.3.2 we considered two different metrics on S^2. Use the result of that exercise to show that they are equivalent.

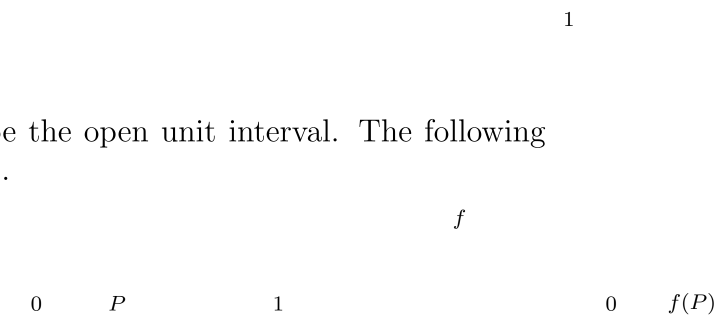
Exercise 1.4.12. Let V be a vector space over k with $k = \mathbb{R}$ or $k = \mathbb{C}$. Assume that N_1, N_2 are two norms on V. We say that N_1 and N_2 are *equivalent* if there exist positive real numbers $A, B > 0$ such that $A \cdot N_1(x) \leq N_2(x) \leq B \cdot N_1(x)$ for all $x \in V$.

1. Show that this is an equivalence relation on the set of norms on V.

2. Assume N_1 and N_2 are equivalent norms. Show that the associated metrics are equivalent.

3. Assume N_1 and N_2 define equivalent metrics, show that N_1 and N_2 are equivalent norms.

Exercise 1.4.13. Let $X = (0, 1)$ be the open unit interval. The following picture indicates a map $f : X \longrightarrow \mathbb{R}^2$.



1. Find/guess/outline an expression for f. (The curved part is circular.)

Consider the function $d' : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ defined by $d'(x, y) = |f(x) - f(y)|$ for $x, y \in X$.

2. Argue why d' is a metric on $(0, 1)$.
3. Let $U \subset X$ be an open subset of X with respect to d', and assume that $P \in U$. Show that U contains the interval $(1 - \epsilon, 1)$ for some $\epsilon > 0$.
4. Show that any subset of X which is open with respect to d' is also open with respect to the usual metric, d, on $X = (0, 1)$. Describe the subsets that are open with respect to d but not with respect to d'.

1.5 Metrics on products

1.5.1. In this section we will discuss the possibility of defining a metric on a space which arises as the product of other metric spaces. If we are dealing with only a finite product of metric spaces, then there are various possible definitions of a 'product metric', all of which are equivalent in the sense of 1.4.5. But for an infinite product there is no satisfactory general construction. In the next chapter we will see that despite of this, we may still give a precise definition of the 'open subsets' of an arbitrary product space.

1.5.2. Let I be a set, and assume that for each $i \in I$ we have a metric space (X_i, d_i). The product space

$$X = \prod_{i \in I} X_i$$

is the space that consists of all I-tuples $(x_j)_{j \in I}$ with $x_j \in X_j$. If $I = \{1, \ldots, n\}$ we also write $X = X_1 \times \cdots \times X_n$.

Definition 1.5.3. Assume $I = \{1, \ldots, n\}$. We may define a metric $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ on X as follows. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are elements of X, let

$$d(x, y) = \sup_{i=1,\ldots,n} d_i(x_i, y_i).$$

It is an easy exercise to check that d satisfies the requirements of Definition 1.1.6. We will call d the *product metric* on X.

1.5.4. This is just one possible choice of a metric on the product space. Another choice might be $d'(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) + \cdots + d_n(x_n, y_n)$, which is also easily seen to define a metric on X.

It follows directly from the definitions that d' and the product metric d satisfy $d(x, y) \leq d'(x, y) \leq nd(x, y)$ for $x, y \in X$. From this it is easy to check that d and d' are equivalent. (cf. 1.4.5).

Exercise 1.5.5. The Euclidean metric on $\mathbb{R}^n$ is *not* the same as the product metric when we consider $\mathbb{R}^n$ the n-fold product of $\mathbb{R}$. Show that the two metrics are equivalent.

Exercise 1.5.6. (You may consider this a hard one...) Given metric spaces $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ and a positive real number $p > 0$, define a function $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ ($X = X_1 \times X_2 \times \cdots \times X_n$) by the assignment

$$d^{(p)}(x, y) = \big(d_1(x_1, y_1)^p + d_2(x_2, y_2)^p + \cdots + d_n(x_n, y_n)^p\big)^{1/p},$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are elements of X.

1. Show that $d^{(p)}$ is faithful and homogeneous for all $p > 0$
2. Show that $d^{(p)}$ is a metric for $p \geq 1$ by establishing subadditivity. (Hint: Use a discrete version of the Minkowski inequality [Rudin], Theorem. 3.5.)
3. Show that $d^{(p)}$ is equivalent to the product metric for all $p \geq 1$.
4. Show by example that $d^{(p)}$ need not be subadditive, hence not a metric, if $p < 1$.

1.5.7. Consider again a product $X = \prod_{i \in I} X_i$, where for each $i \in I$, X_i is a metric space with metric d_i. Suppose now that I is infinite.

If we try to mimic the defininition of the product metric from the finite situation above, we put $d(x, y) = \sup_{i \in I} d_i(x_i, y_i)$, where $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ are two points in X. But this is only well-defined if there is a common upper bound on the metrics d_i. By replacing each d_i by an equivalent metric, we may actually achieve this (cf. 1.4.10). But then the next problem appears: As Exercise 1.5.9 below shows, the equivalence class of d is not uniquely determined by the equivalence classes of the d_i.

1.5.8. Another (and even more dubious) approach to obtaining a metric on X would be by addition: $d(x, y) = \sum_{i\in I} d_i(x_i, y_i)$. This will not make sense in general, but if $I = \mathbb{N}$ and the bounds on the d_i are decaying rapidly, then there is a chance. In Example 3.3.8 we will see that this may be fruitful in certain cases.

Exercise 1.5.9. For all $i \in \mathbb{N}$, let $X_i = [0, 1]$. We consider two metrics on X_i: $d_i(x, y) = |x - y|$ and $d'_i(x, y) = |x - y|/i$. Let $X = \prod_{i\in\mathbb{N}} X_i$, and let d and d' be the metrics on X defined as in 1.5.7 with respect to the d_i and the d'_i respectively.

1. Show that d_i and d'_i are equivalent for all $i \in \mathbb{N}$.
2. Let $x = (0)_{i\in I} \in X$ and let $\delta' > 0$ be any positive real number. Show that $B_{d'}(x, \delta') \nsubseteq B_d(x, 1)$. Conclude that d and d' are not equivalent.

1.6 Appendix: Completion

1.6.1. The Cauchy sequences in $\mathbb{R}$, which are usually discussed in introductory calculus courses, can also be defined for general metric spaces. (We will have more to say about sequences in 2.5.)

Given a metric space (X, d) you can consider

Definition 1.6.2. A *Cauchy sequence* in X is a sequence $\{x_n\}$ of points in X such that for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for $m, n \geq N$.

A sequence $\{x_n\}$ *converges* towards $x \in X$ if and only if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for $n > N$.

Remark 1.6.3. A Cauchy sequence converges to at most one point. To see this, suppose that $\{x_n\}$ converges to both x and x'. If we put $\epsilon = d(x, x')/2$ and assume $x \neq x'$ then $\epsilon > 0$. By convergence there exists an N such that $d(x_n, x) < \epsilon$ and $d(x_n, x') < \epsilon$ for $n > N$. From the triangle inequality we get $d(x, x') \leq d(x, x_{N+1}) + d(x_{N+1}, x') < 2\epsilon = d(x, x')$, which is impossible. This contradicts the assumption $x \neq x'$ and proves our claim.

1.6.4. It is an easy exercise to show that a *convergent* sequence in a metric space is a Cauchy sequence. Conversely, a Cauchy sequence in the metric space of real numbers always converges towards some x. But this is not so in general. For instance, let $X = \mathbb{Q}$ be the subset of rational numbers in $\mathbb{R}$. Let $\{x_n\}$ be a Cauchy sequence of rational numbers that converges towards an irrational number. As a Cauchy sequence in X, this sequence does not converge to any point in X.

Definition 1.6.5. A metric space (X, d) is *complete* if every Cauchy sequence in X converges.

Exercise 1.6.6. Let $A \subset X$ where X is a complete metric space. Prove that A is closed in X if and only if each Cauchy sequence in A converges towards an element of A (so A is complete).

Definition 1.6.7. Let (X, d) and $(\hat{X}, \hat{d})$ be metric spaces. A distance preserving map $i : X \to \hat{X}$ is called a *completion* if

- X is dense in $\hat{X}$.
- $\hat{X}$ is complete.

That X is dense in $\hat{X}$ means that for any point $x \in \hat{X}$ there exists a sequence in X converging to x.

Exercise 1.6.8. Prove that the inclusion $\mathbb{Q} \subset \mathbb{R}$ (with the usual metrics) is a completion.

1.6.9. If a metric space is not complete, we can construct a completion of it. The points of this completion are defined in a very formal and abstract way, which 'patches the holes' in the space we are considering.

The philosophy of this is the following: A point x in the completion $\hat{X}$ will be represented by a Cauchy sequence $\{x_n\}$ in X, which converges towards the point x. Since X is dense in $\hat{X}$, it is possible to represent every point of $\hat{X}$ in this way. But of course, different Cauchy sequences in X might converge towards the same point in $\hat{X}$. In that case, we call those two Cauchy sequences 'equivalent'.

To turn this into a program for constructing $\hat{X}$ from X, we first need to formulate the equivalence relation on Cauchy sequences without any reference to the space $\hat{X}$, since we have not constructed that space yet.

Definition 1.6.10. Two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are *equivalent* if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

1.6.11. We write $\{x_n\} \sim \{y_n\}$ if the two sequences are equivalent. The reader may check that this defines an equivalence relation.

Let $\mathcal{C}(X)$ be the set of Cauchy sequences in X.

Given a point $x \in X$ we consider the constant Cauchy sequence $i(x) = \{x_n\}$, which satisfies that $x_n = x$ for all n.

Definition 1.6.12. The set $\hat{X} = \mathcal{C}(X)/_\sim$ is the set of equivalence classes of Cauchy sequences in X. The distance $\hat{d}$ is given by $\hat{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, x_n)$.

Remark 1.6.13. The distance does not depend on the choice of Cauchy sequence in an equivalence class. If $\{y_n\} \sim \{z_n\}$, by the triangle inequality we have that $d(x_n, y_n) - d(y_n, z_n) \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$ Taking limits, we get $\lim_{n\to\infty} d(x_n, y_n) - \lim_{n\to\infty} d(y_n, z_n) \leq \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)$ which evaluates to $d(\{x_n\}, \{y_n\}) - 0 \leq d(\{x_n\}, \{z_n\}) \leq d(\{x_n\}, \{y_n\}) + 0$. This shows that the distance d is welldefined on the quotient $\mathcal{C}(X)/_\sim$.

Theorem 1.6.14. *$(\hat{X}, \hat{d})$ is a metric space. The map i from X to the subspace of constant Cauchy sequences $x_i = x$ preserves the distance, that is $\hat{d}(i(x), i(y)) = d(x, y)$*

Proof. We check the conditions.

- $\hat{d}(\{x_n\}, \{y_n\}) = 0$ implies that $\lim_{n\to\infty}(d(x_n, y_n)) = 0$. By definition, this implies that $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy sequences. so $\{x_n\} = \{y_n\} \in \mathcal{C}(X)/_\sim$.

- Symmetry and the triangle inequality for $\hat{d}$ follow directly from the symmetry and triangle inequality for d.

- Let $i(x) = \{x_n\}$ be the constant Cauchy sequence $x_n = x$, and $i(y) = \{y_n\}$ the constant Cauchy sequence $y_n = y$. Then,

$$\hat{d}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y)$$

□

We now consider (X, d) as a subspace of $(\hat{X}, \hat{d})$ by the inclusion i of the constant Cauchy sequences.

Theorem 1.6.15. *The inclusion $i : (X, d) \subset (\hat{X}, \hat{d})$ is a completion.*

Proof. We already know that i is distance preserving. There are two things to check.

First, $i(X)$ is dense in $\hat{X}$. If $\{x_n\} \in \mathcal{C}(X)$, then the sequence $\{x_n\}$ is a Cauchy sequence in X. Since i is distance preserving, $\{i(x_n)\}$ is a Cauchy sequence in $\hat{X}$. We claim that this Cauchy sequence converges towards $\{x_n\} \in \mathcal{C}(X)$. To see this, we just observe that

$$\lim_{n\to\infty} \hat{d}(i(x_n), \{x_m\}) = \lim_{n\to\infty} \lim_{m\to\infty} d(x_n, x_m) = 0$$

since $\{x_n\}$ is Cauchy.

Secondly, to see that $\hat{X}$ is complete, we have to show that a Cauchy sequence $\{z_n\} \in \hat{X}$ converges. This is a Cauchy sequence of Cauchy sequences. Written down in detail, z_n is the equivalence class of some $\{(x_n)_m\}$ (where n is fixed).

As before, the Cauchy sequence $i((x_n)_m)$ converges in $\hat{X}$ towards z_n.

So for each z_n we can choose a particular element $x_n = (x_n)_{m_n}$. We think of x_n as an approximation to z_n in the sense that by choosing m_n big, we can make $i(x_n)$ arbitrary close to z_n. We choose these approximations closer and closer with increasing n, so that $\lim_{n\to\infty}(\hat{d}(z_n, i(x_n))) = 0$.

I claim that the sequence x_n is a Cauchy sequence in X. For $\epsilon > 0$, we find N so that if $n > N$, then $\hat{d}(z_n, i(x_n)) < \frac{\epsilon}{3}$, and simultaneously, if $m_1 > N$ and $m_2 > N$, then $\hat{d}(z_{m_1}, z_{m_2}) < \frac{\epsilon}{3}$

For $m_1 > N$ and $m_2 > N$ we have that

$$\begin{aligned} d(x_{m_1}, x_{m_2}) &= \hat{d}(i(x_{m_1}), i(x_{m_2})) \\ &\le \hat{d}(i(x_{m_1}), z_{m_1}) + \hat{d}(z_{m_1}, z_{m_2}) + \hat{d}(z_{m_2}, i(x_{m_2})) \\ &< \epsilon. \end{aligned} \tag{1.1}$$

We write $x \in \hat{X}$ for the element represented by $\{x_n\}$. Finally, the inequality

$$\hat{d}(z_n, x) \le \hat{d}(z_n, i(x_n)) + \hat{d}(i(x_n), x)$$

shows that the Cauchy sequence $\{z_n\}$ in $\hat{X}$ converges towards $z = \{x_n\}$. □

1.6.16. The completion of a space depends on the metric in a sensitive way. If you change the metric to an equivalent metric, the completion of (X, d) will change, and not necessarily to an equivalent metric space.

Exercise 1.6.17. Compute the completion of the following metric spaces. Find the completions. Show that the number of points in $\hat{X} \setminus X$ is different in the three cases.

- $X = \mathbb{R}$ with its usual metric
- $X = (0, 1) \subset \mathbb{R}$
- $X = (0, \infty) \subset \mathbb{R}$

When you have read the following chapters, you will realize that these completions are pairwise non-homeomorphic despite the fact that $\mathbb{R}$, $(0, 1)$ and $(0, \infty)$ are homeomorphic (cf. 2.2.11).

Remark 1.6.18. Without scruples we have been using the real numbers throughout this course (already in the definition of a metric). If you look back in your first year books, you will probably find a list of properties of the real numbers but no formal construction. That is, you might not have seen a proof of the existence of $\mathbb{R}$. One way of defining $\mathbb{R}$ is by completion of $\mathbb{Q}$.

Chapter 2

Topological spaces

2.1 Introduction

2.1.1. In the first chapter saw a lot of examples of metric spaces. In particular, we saw that very often one may equip a set X with many different metrics (cf. Example 1.1.9), but that these different metrics may define the same *open* subsets of X (cf. the examples and exercises of Section 1.4).

For many purposes it is sufficient to know just the family open subsets of X, regardless of how these are defined (e.g. which metric has been used). For example, continuity of maps is entirely a question of manipulations with open subsets. In its most basic form, *topology* could be called the theory of *open subsets*. But we will will also use the word topology in a different way:

Definition 2.1.2. Let X be a set. A *topology* on X is family τ of subsets of X that satisfies the following axioms:

1. Let I be any set, and assume that for each $i \in I$ we have an element $U_i \in \tau$. Then $\bigcup_{i \in I} U_i \in \tau$.

2. Let I be a *finite* set, and assume that for each $i \in I$ we have an element $U_i \in \tau$. Then $\bigcap_{i \in I} U_i \in \tau$

The pair (X, τ) is called a *topological space*. An element of τ is called an *open* subset of X. If x is a point in X, then an open subset of $U \subset X$ containing x is also called an *open neighbourhood* of x.

2.1.3. We should emphasize that for any topological space (X, τ) the following condition is automatically satisfied

$$3. \quad \emptyset \in \tau \quad \text{and} \quad X \in \tau.$$

This follows from applying the two axioms for a topology to the case $I = \emptyset$: By convention, an empty union of subsets is void, while an empty intersection of subsets is equal to X.

In some books you will find 3. included as an extra axiom in the definition of a topology.

2.1.4. In the following you will often meet sentences beginning with "Let X be a topological space...". By this we mean that X is a set that comes equipped with a topology. Whenever we speak about open subsets in X, open neighbourhoods of points in X etc., it is with reference to this unnamed topology.

Example 2.1.5. On any set X we may define at least two different topologies. The *trivial* topology is the smallest possible, namely $\tau_{\text{trivial}} = \{\emptyset, X\}$. The *discrete* topology is the biggest possible, namely $\tau_{\text{discrete}} = 2^X$, which by definition is the family of all subsets of X. It is an easy exercise to show that these two definitions really define topologies.

2.1.6. In the above example, $\tau_{\text{trivial}} \subset \tau_{\text{discrete}}$. In any such situation where a space has two topologies, τ_{coarse} and τ_{fine} say, with $\tau_{\text{coarse}} \subset \tau_{\text{fine}}$, then we say that the topology τ_{fine} is *finer* than the topology τ_{coarse} (or that τ_{coarse} is *coarser* than τ_{fine}).

Exercise 2.1.7. Describe all topologies on a set with two elements. Do the same for a set with three elements. (If you feel like doing it for a set with four or more elements, go ahead!)

Example 2.1.8. For a metric space (X, d), let

$$\tau = \{U \subset X \mid U \text{ is an open subset of } X\},$$

where open subsets are defined as in Section 1.4.2. Then Exercise 1.4.8 shows that τ is a topology on X. We call τ the topology *induced* from the metric d. We will also say that τ is a *metric topology*.

Note, as Exercise 2.1.10 will show: There exist non-metric topologies.

2.1.9. The above construction applies in particular to $X = \mathbb{R}^n$ with the Euclidean metric. The induced *Euclidean* topology consists of all subsets of $\mathbb{R}^n$ which are open in the traditional sense.

Exercise 2.1.10. Let X be a set.

1. Show that the topology defined by the metric in Exercise 1.1.8 defines the discrete topology on X. (If you have done Exercise 1.4.9 correctly, this should not be too hard.)

2. Show that if X contains more than one point, then there is no metric on X whose induced topology coincides with the trivial topology.

Example 2.1.11. In Exercise 1.4.13 we displayed two different metrics, d and d', on $X = (0, 1)$. Let τ and τ' be the topologies induced from d and d' respectively. We then saw that τ is finer than τ', since we have a (proper) inclusion $\tau' \subset \tau$.

2.1.12. Let X be a topological space and $Y \subset X$ a subset. A point $x \in Y$ is called an *interior* point of Y, if there exists an open neighbourhood U_x of x with $U_x \subset Y$. The set of interior points of Y is denoted Y°.

We claim that Y° is an open subset of X. More precisely, Y° is the biggest open subset of X contained in Y in the sense that if $U \subset Y$ is an open subset of X, then $U \subset Y^\circ$.

To see this, first observe that if $U \subset Y$ is an open subset of X, then each $x \in U$ is an interior point of Y (just take $U_x = U$), so $U \subset Y^\circ$. On the other hand, if $x \in Y^\circ$, then x is an interior point of Y, so there exists an open neighbourhood U_x contained in Y. By what we just showed, $U_x \subset Y^\circ$, so Y° is a neighbourhood of each of its points. This proves that Y° is open, and indeed the biggest open subset of X contained in Y.

The set Y° is called *the interior* of Y. We say that Y is a *neighbourhood* of $x \in Y$ if x is an interior point of Y. This extends our previous definition of an *open* neighbourhood, since Y is open in X if and only if $Y = Y^\circ$.

2.1.13. Assume that $X \setminus Y$ is open, i.e. $X \setminus Y \in \tau$. Then Y is called a *closed* subset of X. Exercise 2.1.15 below shows that any intersection of closed sets and any finite union of closed sets is again closed. This allows us to define, for any subset $Y \subset X$, the *closure* of Y in X to be the set $\overline{Y} = \bigcap C$, where the intersection is taken over all closed subsets C of X containing Y. Then $\overline{Y}$ is closed and contained in any other closed set containing Y. From the definition we see that Y is a closed subset of X if and only if $\overline{Y} = Y$.

The notions of 'interior' and 'closure' are each others counterparts due to the equality $X \setminus \overline{Y} = (X \setminus Y)^\circ$.

We say that Y is *dense* in X, if $X = \overline{Y}$.

The difference $\partial Y = \overline{Y} \setminus Y^\circ$ is called the *boundary* of Y. A point in ∂Y is called a *boundary point* for Y and may or may not be a member of Y.

Remark 2.1.14. Note that a subset $Y \subset X$ of a topological space X is open if and only if $Y = Y^\circ$. This may be phrased as follows:

Y is open if and only if Y is a neighbourhood of x for all $x \in Y$.

This simple criterion is often convenient when we want to check if a given subset $Y \subset X$ is open or not.

Exercise 2.1.15. Let (X, τ) be a topological space. Recall that a subset $C \subset X$ of X is closed if $X \setminus C \in \tau$. Let τ^{cl} denote the family of closed sets, and show that τ^{cl} satisfies the following axioms.

1. Let I be any set, and assume that for each $i \in I$ we have an element $C_i \in \tau^{\mathrm{cl}}$. Then $\bigcap_{i \in I} C_i \in \tau^{\mathrm{cl}}$. (In particular, $X \in \tau^{\mathrm{cl}}$).

2. Let I be a *finite* set, and assume that for each $i \in I$ we have an element $U_i \in \tau^{\mathrm{cl}}$. Then $\bigcup_{i \in I} U_i \in \tau^{\mathrm{cl}}$. (In particular, $\emptyset \in \tau^{\mathrm{cl}}$).

Conversely, show that given a familiy τ^{cl} of subsets of X which satisfies 1. and 2., then $\tau = \{X \setminus C \mid C \in \tau^{\mathrm{cl}}\}$ defines a topology on X.

Exercise 2.1.16. (This is an easy one...) Let Y be a subset of a topological space (X, τ). Show

1. $\partial Y \subset Y$ if and only if Y is closed in X.

2. $\partial Y \cap Y = \emptyset$ if and only if Y is open in X.

2.2 Continuous maps

Definition 2.2.1. Suppose (X, τ_X) and (Y, τ_Y) are topological spaces. A map $f : X \longrightarrow Y$ is *continuous* at a point $x \in X$ if $f^{-1}(V) \subset X$ is a neighbourhood of x for any neighbourhood $V \subset Y$ of $y = f(x)$.

The map f is *continuous* if it is continuous at all points $x \in X$.

Exercise 2.2.2. When X and Y are metric spaces, this definition of continuity is equivalent to the definition from 1.4.1. Prove this!

Proposition 2.2.3. *The following conditions are equivalent for a map $f : X \longrightarrow Y$ of topological spaces and a point $x \in X$.*

1. *f is continuous at x.*

2. *For any subset $A \subset X$ with $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.*

Proof. Note that $x \in \overline{A}$ is equivalent to $x \notin X \setminus \overline{A} = (X \setminus A)^\circ$, which says that $X \setminus A$ is *not* a neighbourhood of x. Similarly, $f(x) \in \overline{f(A)}$ is equivalent to saying that $Y \setminus f(A)$ is not a neighbourhood of $f(x)$. Hence, 2. is equivalent to

2'. For any subset $A \subset X$: If $Y \setminus f(A)$ is a neighbourhood of $f(x)$, then $X \setminus A$ is a neighbourhood of x.

1. $\Rightarrow$ 2'.: Let $A \subset X$ be given, such that $Y \setminus f(A)$ is a neighbourhood of $f(x)$, and assume that f is continuous at x. Then $f^{-1}(Y \setminus f(A)) \subset X \setminus A$ is a neighbourhood of x, as wanted.

2'. $\Rightarrow$ 1.: Given a neighbourhood V of $f(x)$, let $A = X \setminus f^{-1}(V)$. Then $Y \setminus f(A) \supset V$ is a neighbourhood of $f(x)$, so by applying 2'. we see that $X \setminus A = f^{-1}(V)$ is a neighbourhood of x. Since this holds for any neighbourhood V of $f(x)$, then f is continuous at x, as desired. □

Theorem 2.2.4. *The following conditions are equivalent for a map $f : X \longrightarrow Y$ of topological spaces.*

1. *f is continuous*

2. *$f^{-1}(V) \subset X$ is open whenever $V \subset Y$ is open.*

3. *$f^{-1}(C) \subset X$ is closed whenever $C \subset Y$ is closed.*

4. *$f(\overline{A}) \subset \overline{f(A)}$ for any subset $A \subset X$.*

Proof. 1. $\Rightarrow$ 2.: Let $V \subset Y$ be open and f continuous. Put $U = f^{-1}(V)$. We must show that U is open in X. But for each $x \in U$, V is a neighbourhood of $f(x)$, so U is a neighbourhood of x by continuity of f at x. By Remark 2.1.14, this implies that U is open.

2. $\Rightarrow$ 3.: If $C \subset Y$ is closed, then $V = Y \setminus C$ is open. If we assume 2., then $X \setminus f^{-1}(C) = f^{-1}(V)$ is open, so $f^{-1}(C)$ is closed. This shows 3.

3. $\Rightarrow$ 4.: Given any subset $A \subset X$, then $C = \overline{f(A)}$ is a closed subset of Y. If we assume 3., then $f^{-1}(C)$ is a closed subset of X containing A, hence $\overline{A} \subset f^{-1}(C)$. Apply f to this inclusion to get $f(\overline{A}) \subset f(f^{-1}(C)) \subset C = \overline{f(A)}$, as desired.

4. $\Rightarrow$ 1.: Assuming 4., we must show that f is continuous at any point $x \in X$. Let $A \subset X$ be any subset with $x \in \overline{A}$. Then by 4., $f(x) \in f(\overline{A}) \subset \overline{f(A)}$. But according to Proposition 2.2.3 this implies that f is continuous at x, as desired. □

Remark 2.2.5. From the above theorem we see that if a set X has two topologies, τ_1 and τ_2, then τ_1 is finer than τ_2 if and only if the identity map $(X, \tau_1) \xrightarrow{id} (X, \tau_2)$ is continuous.

Proposition 2.2.6. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps between topological spaces. Suppose f is continuous at a point $x \in X$, and g is continuous at $f(x)$. Then $g \circ f : X \longrightarrow Z$ is continuous at x.*

In particular, if f and g are continuous, then $g \circ f$ is continuous.

Proof. This is quite simple: Let $W \subset Z$ be a neighbourhood of $(g \circ f)(x) = g(f(x))$. Then $V = g^{-1}(W)$ is a neighbourhood of $f(x)$ by continuity of g at $f(x)$, and so $(g \circ f)^{-1}(W) = f^{-1}(V)$ is a neighbourhood of x by continuity of f at x. This proves that $g \circ f$ is continuous at x. $\square$

Exercise 2.2.7. Equip $\mathbb{R}$ with the Euclidean topology and consider the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by the expression

$$f(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x \in \mathbb{R}$ if and only if $x \notin \{0, 1\}$. Suppose we replaced all occurrences of 'neighbourhood' in Definition 2.2.1 with 'open neighbourhood'. At how many points would f then be continuous?

2.2.8. Suppose $f : X \longrightarrow Y$ is a map from a set X to a topological space (Y, τ_Y). Then f induces a topology on X,

$$\tau(f) = \left\{ f^{-1}(V) \mid V \in \tau_Y \right\}.$$

It is straightforward to check that τ_X satisfies the axioms of a topology, and that f is continuous as a map from $(X, \tau(f))$ to (Y, τ_Y). We call $\tau(f)$ the *least fine topology* that makes f continuous.

If X comes equipped with a topology τ_X, then f is continuous as a map between (X, τ_X) and (Y, τ_Y) if and only if $\tau(f) \subset \tau_X$.

2.2.9. Suppose (Y, τ_Y) is a topological space and $X \subset Y$ is a subset. Then we may equip X with the least fine topology, $\tau(\iota)$, that makes the inclusion $\iota : X \hookrightarrow Y$ continuous. This is called the *subspace topology* on X.

By definition,

$$\tau(\iota) = \{V \cap X \mid V \text{ is open in } Y\}.$$

Note the curious fact that even if X is neither open nor closed as a subset of Y, then X is always both open and closed in its subspace topology by definition.

Exercise 2.2.10. Let (Y, d_Y) be a metric space, and and let τ_Y be the metric topology. Let X be a subset of Y. Then we have an induced metric d_X on X (cf. 1.3.1). Show that the metric topology on X coincides with the subspace topology.

2.2.11. Let $f : X \longrightarrow Y$ be a map between two topological spaces. We call f *open*, if $f(U)$ is open in Y for any open subset U of X. Similarly, f is *closed* if $f(C)$ is closed in Y for any closed subset C of X.

If f is bijective, continuous and open, then f is called a *homeomorphism*. This is equivalent to saying that both f and its inverse $f^{-1} : Y \longrightarrow X$ are continuous. Whenever there exists a homeomorphism between two topological spaces X and Y, we say that X and Y are *homeomorphic*.

A continuous injective map f is called an *embedding*, if the induced map $f : X \longrightarrow f(X)$ is open, when $f(X) \subset Y$ is equipped with the subspace topology. This is equivalent to demanding that $f : X \longrightarrow f(X)$ is a homeomorphism, or, as we will also say, that f is a *homeomorphism on its image*. In Exercise 2.2.13 and Proposition 3.2.10 we will see examples of this phenomena.

A homeomorphism may also be called an isomorphism of topological spaces: it preserves all the relevant structure.

Exercise 2.2.12. For $a < b$ real numbers, find a homeomorphism $\mathbb{R} \longrightarrow (a, b)$. Argue why there cannot be a homeomorphism $(a, b) \longrightarrow [a, b]$. (Compare with Exercise 2.7.2.)

Exercise 2.2.13. Let $f : (0, 1) \longrightarrow \mathbb{R}^2$ be the map from 1.4.13. Let $Y = f(0, 1)$ be the image of f. Argue why f is *not* an embedding (cf. 2.2.11). How about the map $g : (0, 1) \longrightarrow \mathbb{R}^2$, $g(x) = (x, 0)$?

2.3 Bases

2.3.1. You will probably be familiar with the notion of a basis for a finite dimensional vector space: It is a minimal collection of vectors that spans the vector space. If you know a basis, you can always recover the vector space.

A basis for a topology is in a quite similar fashion a family of open subsets that 'span' the topology. If you know a basis, then you know the topology. There is no good notion of minimality for a basis here, so there is no such requirement, but it is often convenient to have a basis with as few elements as possible.

Definition 2.3.2. Let (X, τ) be a topological space.

1. A subfamily $\sigma \subset \tau$ is called a *basis* for the topology τ, if any element $U \in \tau$ can be expressed as a union of elements in σ, i.e. there is a subfamily $\rho \subset \sigma$ such that $U = \bigcup_{V \in \rho} V$.

2. Let $x \in X$ be a point of X. A family σ_x of neighbourhoods of x in X is called a *neighbourhood basis* for x, if for any neighbourhood U of x there exists an element $V \in \sigma_x$ such that $V \subset U$.

Proposition 2.3.3. *Given a family σ of subsets of X, for each $x \in X$ let $\sigma_x = \{U \in \sigma \mid x \in U\}$. Equip X with a topology τ. Then σ is a basis for τ if and only if σ_x is a neighbourhood basis for x for all $x \in X$.*

Proof. Suppose σ is a basis for τ, and x is a point in X. Given a neighbourhood U of x we may write U° as a union of elements in σ. At least one of these elements, say V, will contain x, hence $V \in \sigma_x$ and $V \subset U$. This shows that σ_x is a neighbourhood basis for x.

Conversely, assume σ_x is a neighbourhood basis for x for all $x \in X$, and let $U \in \tau$. Then for each $x \in U$ we may find $U_x \in \sigma_x \subset \sigma$ with $x \in U_x \subset U$, and $U = \bigcup_{x \in U} U_x$ is a union of elements in σ. This shows that σ is a basis for τ. □

Exercise 2.3.4. Let a set X be equipped with two topologies, τ and τ', and let σ be a basis for τ. Show that τ is coarser than τ' (i.e. $\tau \subset \tau'$) if for any $U \in \sigma$ and any $x \in X$, there exists an open neighbourhood V of x in the topology τ' such that $V \subset U$.

Example 2.3.5. Let X be a set equipped with the discrete topology. Then a basis for the topology is given by $\sigma = \{\{x\} \mid x \in X\}$.

Moreover, assume σ' is another basis, and let $x \in X$. Then since $\{x\}$ is an open neighbourhood of x, there exists an element $U \in \sigma'$ such that $x \in U \subset \{x\}$. That is, $\{x\} \in \sigma'$. This shows that σ is contained in any basis for the discrete topology.

Example 2.3.6. Let (X, d) be a metric space and τ the induced topology. By definition (cf. 1.4.2), $U \subset X$ is open if for each $x \in U$ there exists a ball $B(x, r_x)$ of some radius $r_x > 0$ which is contained in U. Thus, $U = \bigcup_{x \in U} B(x, r_x)$. According to our definitions, this shows that

$$\sigma = \{B(x, r) \mid x \in X \text{ and } r \in \mathbb{R}_{>0}\}$$

is a basis for τ, and $\sigma_x = \{B(x, r) \mid r \in \mathbb{R}_{>0}\}$ is a neighbourhood basis for $x \in X$.

2.3.7. Let X be a set, and assume that for each element i in some index set I we have a topology τ_i on X. Then it is easy to check that the intersection of all these topologies, $\tau = \bigcap_{i \in I} \tau_i$, is again a topology on X.

Now suppose we have a family σ of subsets on X. Then there is a *least fine* topology containing σ, namely

$$\tau(\sigma) = \bigcap_{\tau \supset \sigma} \tau,$$

where the intersection is taken over all topologies τ on X containing σ. This intersection is non-trivial since any family σ is contained in the discrete topology.

In general, σ is not a basis for $\tau(\sigma)$ (see Exercise 2.3.8). The problem is that any finite intersection of elements in σ will necessarily belong to $\tau(\sigma)$ but may not be expressible as a union elements of σ. Instead, consider the family σ' of all finite intersections of elements in σ. We claim that σ' is a basis for $\tau(\sigma)$.

To see this, let τ' be the familiy consisting of arbitrary unions of elements of σ'. It is straightforward to check that τ' is a topology, and that σ' is a basis for τ'. From the axioms it follows that any topology containing σ must also contain τ'. We conclude that $\tau' = \tau(\sigma)$, and σ' is indeed a basis for $\tau(\sigma)$.

Whenever (X, τ) is a topological space and $\sigma \subset \tau$ is a subfamily with $\tau = \tau(\sigma)$, then σ is called a *subbasis* for the topology τ. Clearly, a basis for τ is also a subbasis.

Exercise 2.3.8. Show that a subbasis σ for a topology on a set X is a basis for the topology if and only if any finite intersection of elements in σ can be expressed as a union of elements in σ.

2.4 The axioms of countability

Definition 2.4.1. Let (X, τ) be a topological space.

1. (X, τ) is *separable* if there exists an at most countable dense subset of X.

2. (X, τ) satisfies the *first axiom of countability* if every point in X has an at most countable neighbourhood basis.

3. (X, τ) satisfies the *second axiom of countability* if τ has an at most countable basis

Proposition 2.4.2. *Let (X, τ) satisfy the second axiom of countability. Then (X, τ) is separable and satisfies the first axiom of countability.*

Proof. It is clear that (X, τ) satisfies the first axiom of countability, and we only need to show separability.

Let $\sigma \subset \tau$ be an at most countable basis. For each $U \in \sigma$, pick a point $y_U \in U$, and define $Y = \{y_U \mid U \in \sigma\}$. Clearly, Y is at most countable. We must show that $\overline{Y} = X$. Put $U = X \setminus \overline{Y}$. Then U is open, and if U is not empty, then it is a non-trivial union of elements of σ. But each element of σ contains a point in Y, hence so does U. This is impossible, so $U = \emptyset$, as desired. $\square$

Exercise 2.4.3. Let $\tau = \{\mathbb{C} \setminus F \mid F \subset \mathbb{C} \text{ is finite or empty}\} \cup \{\emptyset\}$.

1. Show that τ defines a topology on $\mathbb{C}$. (This is called the Zariski topology.)

2. Show that τ is separable but does not satisfy the first axiom of countability.

Lemma 2.4.4. *Let (X, d) be a metric space. Then the induced topology satisfies the first axiom of countability.*

Proof. Let $x \in X$. As we saw in Example 2.3.6, $\sigma_x = \{B(x, r) \mid r \in \mathbb{R}_{>0}\}$ is a neighbourhood basis for x. But then $\sigma'_x = \{B(x, r) \mid r \in \mathbb{Q}_{>0}\}$ is a countable neighbourhood basis for x. In fact, it will suffice to show that for any $r \in \mathbb{R}_{>0}$, the ball $B(x, r)$ contains a ball $B(x, r')$ from σ'_x, and this is not hard: Given r, choose any rational r' with $0 < r' < r$. $\square$

Remark 2.4.5. In general, metric spaces are not separable (and hence, do not satisfy the second axiom of countability). As an example, take $X = \mathbb{R}$ with the discrete topology which is metric by 2.1.10. Since any subset of X is closed, then the only dense subset is X itself, and X is not countable.

Lemma 2.4.6. *The Euclidean topology on $\mathbb{R}^n$ satisfies the second axiom of countability. In particular, $\mathbb{R}^n$ is separable.*

Proof. By induction in n, one easily proves that $\mathbb{Q}^n$ is a countable subset of $\mathbb{R}^n$. Let $\mathbb{Q}_{>0}$ be the positive rational numbers and put

$$\sigma = \{B(q, r) \mid q \in \mathbb{Q}^n \text{ and } r \in \mathbb{Q}_{>0}\}$$

Then σ is in bijective correspondance with the countable set $\mathbb{Q}^n \times \mathbb{Q}_{>0}$ and is thus countable. We must show that σ is a basis for the topology.

Let $U \subset \mathbb{R}^n$ be open, and let $x \in U$. Then we may choose $r_x \in \mathbb{R}_{>0}$ such that $B(x, r_x) \subset U$. Now pick $q_x \in \mathbb{Q}^n$ such that $q_x \in B(x, r_x/2)$, and $r'_x \in \mathbb{Q}_{>0}$ such that $|q_x - x| < r'_x < r_x/2$. Then by the triangle inequality, $x \in B(q_x, r'_x) \subset B(x, r) \subset U$. Since $B(q_x, r'_x) \in \sigma$ and $U = \bigcup_{x \in U} B(q_x, r'_x)$, this shows that σ is indeed a basis. $\square$

2.5 Sequences

Let X be a set. A sequence in X is simply a map $x. : \mathbb{N} \longrightarrow X$. We write it as $\{x_i\}_{i \in \mathbb{N}}$ where $x_i \in X$ is the image of $i \in \mathbb{N}$.

A subsequence of a sequence $\mathbb{N} \xrightarrow{x.} X$ is a sequence arising from the following construction: Take a strictly increasing function $c : \mathbb{N} \longrightarrow \mathbb{N}$. Then the composite $\mathbb{N} \xrightarrow{c} \mathbb{N} \xrightarrow{x.} X$ is a subsequence. If $c(j) = i_j$, $j \in \mathbb{N}$, then the subsequence is denoted $\left\{x_{i_j}\right\}_{j \in \mathbb{N}}$.

Definition 2.5.1. Let X be a set, $A \subset X$ a subset of X, and $\{x_i\}_{i \in \mathbb{N}}$ a sequence in X.

1. We say that $\{x_i\}_{i \in \mathbb{N}}$ is *frequently* in A, if there is a subsequence contained in A. (Equivalently, there are infinitely many $i \in \mathbb{N}$ such that $x_i \in A$.)

2. We say that $\{x_i\}_{i \in \mathbb{N}}$ is *eventually* (read as: sooner or later) in A, if any subsequence is frequently in X. (Equivalently, there exists $N \in \mathbb{N}$ such that $x_i \in A$ for $i \geq N$.)

Definition 2.5.2. Let (X, τ) be a topological space, $x \in X$ a point, and $\{x_i\}_{i \in \mathbb{N}}$ a sequence in X.

1. We say that $\{x_i\}_{i \in \mathbb{N}}$ *converges* to x, if $\{x_i\}_{i \in \mathbb{N}}$ is eventually in U for all open neighbourhoods U of x. In this case we also say that x is a *limit point* for $\{x_i\}_{i \in \mathbb{N}}$ and write $x = \lim_{i \to \infty} x_i$.

2. We say that $\{x_i\}_{i \in \mathbb{N}}$ *accumulates* at x, or that x is an accumulation point for $\{x_i\}_{i \in \mathbb{N}}$, if $\{x_i\}_{i \in \mathbb{N}}$ is frequently in U for each open neighbourhood U of x.

Remark 2.5.3. Warning! A sequence $\{x_i\}_{i \in \mathbb{N}}$ has in general no limit points, and if it has, it need not be unique (cf. Exercise 2.5.5). When we write $x = \lim_{i \to \infty} x_i$, then x is one of possibly many limit points for $\{x_i\}_{i \in \mathbb{N}}$.

Exercise 2.5.4. Assume that (X, τ) satisfies the first axiom of countability, and that $\{x_i\}_{i \in \mathbb{N}}$ is a sequence in X. Show that a point x is an accumulation point for $\{x_i\}_{i \in \mathbb{N}}$ if and only if there exists a subsequence of $\{x_i\}_{i \in \mathbb{N}}$ converging to x.

Exercise 2.5.5. Prove that if X is a set equipped with the trivial topology, then any sequence $\{x_i\}_{i \in \mathbb{N}}$ in X converges to any $x \in X$.

2.5.6. Given a topological space (X, τ) and a subset $A \subset X$, we define the *sequential closure* of A by

$$\overline{A}^s = \{x \in X \mid \text{there exists a sequence in } A \text{ converging to } x\}.$$

If $x \in A$, then the sequence which is constantly equal to x converges to x, so we always have $A \subset \overline{A}^s$.

Proposition 2.5.7. *We have an inclusion $\overline{A}^s \subset \overline{A}$. Conversely, if $x \in \overline{A}$ and x has an at most countable neighbourhood basis in X, then $x \in \overline{A}^s$.*

Corollary 2.5.8. *If X is a metric space or more generally, if X satisfies the first axiom of countability, then $\overline{A}^s = \overline{A}$ for any subset $A \subset X$.*

Proof. (Of 2.5.7) Let $x \in \overline{A}^s$. Then there exists a sequence $\{y_i\}_{i \in \mathbb{N}}$ of points in A converging to x, so for any open neighbourhood U of x in X, the sequence is eventually in U. But if x does not belong to $\overline{A}$, then $U = X \setminus \overline{A}$ is an open neighbourhood of x in X which does not contain any points from the sequence. This gives a contradiction, and we see that $\overline{A}^s \subset \overline{A}$.

Now suppose $x \in \overline{A}$, and that x has an at most countable neighbourhood basis, $\sigma_x \subset \tau$. By Lemma 2.5.9 below we may assume that there exists a surjective descending map $c : \mathbb{N} \longrightarrow \sigma_x$. For any $i \in \mathbb{N}$, c(i) is an open neighbourhood of x, so it must intersect A non-trivially. Choose $x_i \in A \cap c(i)$. This defines a sequence $\{x_i\}_{i \in \mathbb{N}}$ in A, which we claim converges to x. In fact, if $i \in \mathbb{N}$, then for $j \geq i$ we have $x_j \in c(j) \subset c(i)$, so $\{x_i\}_{i \in \mathbb{N}}$ is eventually in $c(i)$. Since the $c(i)$ run through a neighbourhood basis for x, this implies that $x = \lim_{i \to \infty} x_i$, as claimed. We conclude that $x \in \overline{A}^s$. □

Lemma 2.5.9. *Suppose $x \in X$ has an at most countable neighbourhood basis. Then there exist a neighbourhood basis σ_x of x and a surjective map $c : \mathbb{N} \longrightarrow \sigma_x$ which is* descending, *i.e. $i \leq j \Rightarrow c(i) \supset c(j)$.*

Proof. Suppose σ'_x is an at most countable neighbourhood basis for x. Then there exists a surjective map $c' : \mathbb{N} \longrightarrow \sigma'_x$. For $i \in \mathbb{N}$, put $c(i) = c'(1) \cap c'(2) \cap \cdots \cap c'(i)$. Then it is easy to check that $\sigma_x = \{c(i) \mid i \in \mathbb{N}\}$ is a neighbourhood basis for x, and the map $c : \mathbb{N} \longrightarrow \sigma_x$, $i \mapsto c(i)$ is clearly surjective. □

Proposition 2.5.10. *Suppose $f : X \longrightarrow Y$ is a map between topological spaces, and let $x \in X$ be a point. Then 1. implies 2.*

1. *f is continuous at x.*
2. *$f(x) = \lim_{i \to \infty} f(x_i)$ for any sequence $\{x_i\}_{i \in \mathbb{N}}$ in X with $x = \lim_{i \to \infty} x_i$.*

If moreover x has an at most countable neighbourhood basis, then 1. and 2. are equivalent.

Proof. 1. $\Rightarrow$ 2.: Suppose f is continuous at x and $\{x_i\}_{i\in\mathbb{N}}$ is a sequence in X with $x = \lim_{i\to\infty} x_i$. Consider a neighbourhood $V \subset Y$ of $f(x)$. By continuity, $f^{-1}(V) \subset X$ is a neighbourhood of x, so eventually x_i will be in $f^{-1}(V)$. But then eventually, $f(x_i)$ will be in V. This shows that $f(x) = \lim_{i\to\infty} f(x_i)$, as claimed.

2. $\Rightarrow$ 1.: Assume that x has a countable neighbourhood basis, and let $A \subset X$ be any subset with $x \in \overline{A}$. By Proposition 2.5.7, this implies that $x \in \overline{A}^s$. Now, condition 2. immediately gives that $f(x) \in \overline{f(A)}^s \subset \overline{f(A)}$, and by Proposition 2.2.3 we conclude that f is continuous at x. □

Corollary 2.5.11. *If $f : X \longrightarrow Y$ is a map between topological spaces and X satisfies the first axiom of countability, then f is continuous if and only if $f(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} f(x_i)$ for any convergent sequence $\{x_i\}_{i\in\mathbb{N}}$.*

In the above Corollary, we are abusing our notation: By $f(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} f(x_i)$ we mean that if $\{x_i\}_{i\in\mathbb{N}}$ converges to a point $x \in X$, then $\{f(x_i)\}_{i\in\mathbb{N}}$ converges to $f(x)$. The abuse lies in the fact that a convergent sequence $\{x_i\}_{i\in\mathbb{N}}$ potentially has more than one limit point, so $\lim_{i\to\infty} x_i$ is not a well-defined point.

2.6 Product topologies

Let I be a set, and assume that for each $i \in I$ we have a topological space (X_i, τ_i). Let $X = \prod_{i\in I} X_i$ be the product space. For each $i \in I$ there is a natural projection map $pr_i : X \longrightarrow X_i$, $pr_i((x_j)_{j\in I}) = x_i$.

Definition 2.6.1. The *product topology* on X is the least fine topology that makes all the pr_i, $i \in I$, continuous.

2.6.2. Let us give a more precise description of the product topolgy, τ. For each $i \in I$, $pr_i : X \longrightarrow X_i$ should be continuous. This amounts to saying that τ is the least fine topology that contains the family

$$\sigma = \left\{pr_i^{-1}(U_i) \subset X \mid i \in I, U_i \in \tau_i\right\},$$

so $\tau = \tau(\sigma)$, and σ is a subbasis for τ. Note that $pr_i^{-1}(U_i) = \prod_{j\in I} U'_j$, where $U'_i = U_i$ and $U'_j = X_j$ for $j \neq i$. If we intersect $pr_i^{-1}(U_i)$ with another set of the same type, $pr_{i'}^{-1}(U_{i'})$ say, with $i \neq i'$, then we get

$$pr_i^{-1}(U_i) \cap pr_{i'}^{-1}(U_{i'}) = \prod_{j\in I} U'_j$$

with $U'_i = U_i$, $U'_{i'} = U_{i'}$ and $U'_j = X_j$ for $j \notin \{i, i'\}$. From the construction in 2.3.7 it is now clear that a basis for the product topology is given by

$$\sigma' = \left\{ \prod_{i \in I} U_i \subset X \mid U_i \in \tau_i \text{ for all } i \in I, \text{ and } U_i = X_i \text{ for all but finitely many } i \in I \right\}.$$

Proposition 2.6.3. *Let Z and X_i, $i \in I$, be topological spaces, and let $f : Z \longrightarrow \prod_{i \in I} X_i$ be a map. Then f is continuous if and only if the "coordinate function" $f_i = pr_i \circ f : Z \to X_i$ is continuous for all $i \in I$.*

Proof. Assume f is continuous. Since pr_i is continuous by definition of the product topology, then the composite $f_i = pr_i \circ f$ is indeed continuous.

Conversely, assume that all the f_i are continuous. Then for an open subset $U_i \subset X_i$, $f_i^{-1}(U_i) = f^{-1}(pr_i^{-1}(U_i))$ is open in Z. But the subbasis σ from 2.6.2 for the product topology consists of elements of the form $pr_i^{-1}(U_i)$, so indeed $f^{-1}(U)$ will be open in Z for any element U of σ. But this means that $f^{-1}(U)$ will be open in Z for *any* open U in $\prod_{i \in I} X_i$, and hence f is continuous. □

A refinement of the above proof will show that f is continuous at a point $z \in Z$ if and only if all of the f_i are continuous at z.

Exercise 2.6.4. Prove once and for all the following subtle claim (that one may so easily use without ever realizing that there is something to prove). We will use this claim several times in the sequel.

For each i in some index set I_0, let X_i be a topological space. Let $I_1 \subset I$ be a subset, and put $I_2 = I \setminus I_1$. For $j = 0, 1, 2$, define $X^{(j)} = \prod_{i \in I_j} X_i$ and equip it with the product topology. Then there is an obvious bijection $X^{(1)} \times X^{(2)} \longrightarrow X^{(0)}$. Show that this is an identification of topological spaces (i.e. a homeomorphism).

Exercise 2.6.5. Another subtle fact that may be harder to appreciate than to prove:

Let X and Y be topological spaces, and let $A \subset X$ and $B \subset Y$ be subspaces equipped with the subspace topologies. Then $A \times B$ may be equipped with the product topology τ_P. But we may also consider $A \times B$ a subset of $X \times Y$ and give it the subspace topology τ_S. Show that $\tau_S = \tau_P$.

Example 2.6.6. If Y and Z are sets, then the product $X = \prod_{y \in Y} Z$ may be identified with the set $\mathrm{Map}(Y, Z)$ of all maps $Y \longrightarrow Z$. Indeed, $(z_y)_{y \in Y} \in X$ defines the map $y \mapsto z_y$.

If Z is a topological space, then $\mathrm{Map}(Y, Z) = X$ may be given the product topology. Given a finite set of points $\{y_1, \ldots, y_n\} \subset Y$ and a family $\{U_1, \ldots, U_n\}$ of open subsets of Z,

$$\{g \in \mathrm{Map}(Y, Z) \mid g(y_i) \in U_i \text{ for } i = 1, \ldots, n\}$$

defines an open subset of $\mathrm{Map}(Y, Z)$. The family of open subsets of this form corresponds to the basis for X given in 2.6.2.

An application: If Y is a normed vector space over $\mathbb{C}$, let Y^* denote the set of continuous linear maps $Y \longrightarrow \mathbb{C}$. Then Y^* is a subset of $\mathrm{Map}(Y, \mathbb{C})$. The subspace topology on Y^* is called the W^*-topology ("weak-star-topology") on Y^* and is frequently considered in the theory of Banach spaces.

Proposition 2.6.7. *Let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be a finite collection of metric spaces. Equip X_i with the metric topology τ_i for all i. Let $X = \prod_{i=1}^n X_i = X_1 \times X_2 \times \cdots \times X_n$ be the product, and let d be the product metric on X (cf. 1.5.3). Then the metric topology on X coincides with the product topology.*

Proof. Observe that for any $x = (x_1, \ldots, x_n) \in X$ and any $r > 0$,

$$B_d(x, r) = B_{d_1}(x_i, r) \times \cdots \times B_{d_n}(x_n, r)$$

by definition of d. From this we see that $B_d(x, r)$ is open in the product topology, so any d-open subset is product-open.

Conversely, assume $U \subset X$ is open in the product topology and $x = (x_1, \ldots, x_n) \in U$ is a point. Then there exists for all $i = 1, \ldots, n$ an open neighbourhood $U_i \subset X_i$ of x_i such that $U_1 \times \cdots \times U_n \subset U$. We may take U_i to be $B_{d_i}(x_i, r_i)$ for some $r_i > 0$. Let $r = \inf_i r_i$. Then $B_d(x, r) \subset \prod_i U_i \subset U$, and U is open in the metric topology. $\square$

2.7 Appendix: Countability

By definition, a set S is *countable*, if there is a bijection $c : \mathbb{N} \longrightarrow S$. The term countable comes from the fact that c allows us to count the (infinitely many!) elements of S: $c(1)$ is the first element of S, $c(2)$ the second, $c(3)$ the third, ... Each element in S gets a number.

We will say that a set S is *at most countable* if S is either finite (i.e. has only finitely many elements) or countable. Note that according to our definition, a finite set is *not* countable. An *infinite* set which is not countable is said to be *uncountable*.

Let us list some properties of (at most) countable sets.

1. If S is countable and $S \longrightarrow T$ is an isomorphism, then T is countable.

2. Any subset of a countable set is at most countable.

3. A set S is at most countable if and only if there exists a surjective map $\mathbb{N} \longrightarrow S$.

4. If S and T are countable, then $S \times T$ is countable.

5. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

6. An at most countable union of at most countable sets is at most countable.

Proof. The proof of 1. and 2. are easy exercises. For 3. "if", suppose we have a surjective map $\phi : \mathbb{N} \longrightarrow S$. Define

$$T = \big\{ i \in \mathbb{N} \mid i \text{ is the least element in } \phi^{-1}(\phi(i)) \big\}.$$

Then the restriction of ϕ to T defines a bijection $\phi|S : T \longrightarrow S$, and by by 1. and 2 we conclude that T is at most countable. 3. "only if" is an easy exercise.

For the proof of 4. it will suffice to show that $\mathbb{N} \times \mathbb{N}$ is countable. For this, consider the set

$$C = \{(n, i) \mid n \in \mathbb{N} \text{ and } i = 0, \ldots, n-1\} = \bigcup_{n=1}^{\infty} C_n$$

with $C_n = \{n\} \times \{0, 1, \ldots, n-1\}$. We may define a counting of C by first taking the element of C_1, then the two elements of C_2, then the three elements of C_3 etc. This way we will sooner or later get to any element of C. Thus, C is countable and we are done by checking that the expression $(n, i) \mapsto (n - i, i + 1)$ defines a bijection $C \longrightarrow \mathbb{N} \times \mathbb{N}$.

The proof of 5. is an exercise, using 1. – 4. □

Note that $\mathbb{R}$ is *uncountable* – this is the famous diagonalization argument of Cantor.

Exercise 2.7.1. Let $a < b$ be real numbers. Given that $\mathbb{R}$ is uncountable, show that the open interval (a, b) is uncountable.

Exercise 2.7.2. For $a < b$ real numbers, find bijections $\mathbb{R} \longrightarrow (a, b)$ and $(a, b) \longrightarrow [a, b]$.

Chapter 3

Compact spaces

3.1 The Hausdorff separation axiom

3.1.1. Before starting the discussion of compactness, we introduce the Hausdorff axiom.

The first axiom for a metric is the requirement that the distance between two distinct points x, y is always strictly positive. Thus, we can always find an open set which contains x but not y – or even better, we can find *disjoint* neighbourhoods of x and y. (Recall that two subsets are disjoint if their intersection is empty.)

On the other hand, if we consider a set (with at least two elements) equipped with the trivial topology, then the only non-empty open subset contains all points in our set, so there is no chance of 'separating' points with the topology. This, for instance, has the unfortunate and somewhat counter-intuitive consequence that a sequence may have more than one limit point.

For many purposes, the least measure of decency for a topological space is that is satisfies the Hausdorff axiom. Sequences in Hausdorff spaces have at most one limit point, and this property in fact characterizes Hausdorff spaces among topological spaces satisfying the first axiom of countability, as we will soon see.

Just to mention it, the Zariski-topology on an algebraic scheme is one of the few examples of a very important non-Hausdorff topology. It is a remarkable fact that even if most of what we have to say in this chapter does not apply to such spaces, a lot of effort in algebraic geometry has been put into establishing constructions that on a geometric level amount to much the same.

Definition 3.1.2. A topological space X is a *Hausdorff space*, if for any two distinct points x, y in X there exist an open neighbourhood U of x and an open neighbourhood V of y such that $U \cap V = \emptyset$.

3.1.3. As we argued above, metric spaces are Hausdorff spaces. A set equipped with the discrete topology is also a Hausdorff space. It is straightforward to see that if Y is a Hausdorff space and $X \subset Y$ is equipped with the subspace topology, then X is likewise a Hausdorff space.

3.1.4. An important property of a Hausdorff space X is that $\{x\}$ is a closed subset for any point $x \in X$. In fact, let $W = X \setminus \{x\}$. Then for any $y \in W$ we may by the Hausdorff axiom choose an open neighbourhood V of y which does not contain x, so $y \in V \subset W$. This shows that W is a neighbourhood of each of its elements, so W is open, and hence $\{x\}$ is closed.

This is a special case of 3.2.9.

Exercise 3.1.5. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x, x) \in X \times X \mid x \in X\}$ is a closed subset of $X \times X$.

Proposition 3.1.6. *Let X be a topological space. If X is Hausdorff, then every seqence in X converges to at most one point.*

Conversely, if X satisfies the first axiom of countability and any sequence converges to at most one point, then X is a Hausdorff space.

Proof. Suppose X is Hausdorff and $\{x_i\}_{i \in \mathbb{N}}$ is a sequence converging to $x \in X$. If $y \in X \setminus \{x\}$ then we may choose open disjoint neighbourhoods U and V of x and y respectively. By convergence, $\{x_i\}_{i \in \mathbb{N}}$ is eventually in U, so only finitely many x_i are in V. This means that $\{x_i\}_{i \in \mathbb{N}}$ does not converge to y, and x must be the only point to which $\{x_i\}_{i \in \mathbb{N}}$ converges.

For the converse, assume that X satisfies the first axiom of countability. If X is not Hausdorff, then we may find two distinct points x and y for which any two open neighbourhoods will intersect non-trivially. From Lemma 2.5.9 we see that there exists a neighbourhood basis σ_x for x and a surjective descending map $c_x : \mathbb{N} \longrightarrow \sigma_x$. Let $c_y : \mathbb{N} \longrightarrow \sigma_y$ be a similar thing for y. Then for each $i \in \mathbb{N}$, the intersection $c_x(i) \cap c_y(i)$ has non-empty interior, and we may pick a point x_i in there. This defines a sequence $\{x_i\}_{i \in \mathbb{N}}$ in X. Given a neighbourhood U of x, $c(i) \subset U$ for i big enough, say $i \geq N$. But then $x_i \in c_x(i) \subset U$ for $i \geq N$, so the sequence is eventually in U. This holds for any neighbourhood U of x, so $\{x_i\}_{i \in \mathbb{N}}$ converges to x. By symmetry, $\{x_i\}_{i \in \mathbb{N}}$ will also converge to y.

This shows that if every sequence converges to at most one point, then X is Hausdorff. □

For future use we prove the following proposition.

Proposition 3.1.7. *If $X = \prod_{i \in I} X_i$ is a product of Hausdorff spaces, then X is a Hausdorff space in the product topology.*

Proof. This is most elegantly done using Exercise 3.1.5: All the maps $pr_i \times pr_i : X \times X \longrightarrow X_i \times X_i$ are continuous, and $\Delta(X_i)$ is closed in $X_i \times X_i$, so

$$\Delta(X) = \bigcap_{i \in I} (pr_i \times pr_i)^{-1}(\Delta(X_i))$$

is closed. This implies that X is Hausdorff. $\square$

3.2 Compactness

3.2.1. The aim of this chapter is to define and get some intuition for the notion of compactness for a topological space. Many readers will know the Borel–Heine characterization of compactness for subsets of $\mathbb{R}^n$: $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.

The abstract definition of compactness for a general topological space K does not refer to an embedding of K in a larger space, but this definition – however convenient to work with – does not in itself reveal much of the geometric intuition behind the concept.

What we will try to advocate here is the idea that whenever a compact space K is embedded in some larger (Hausdorff) space X, then it is indeed a closed and 'bounded' subset. (And conversely, any closed and 'bounded' subset of X is compact).

To give proper meaning to the word 'bounded', recall that a sequence $\{x_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^n$ is said to *tend to infinity*, if for any $r > 0$, the sequence is eventually outside the closed ball $\overline{B(0, r)}$. (That is, for any $r > 0$ exists $N \in \mathbb{N}$ such that $|x_i| > r$ for $i \geq N$.) Indeed, a sequence that tends to infinity is eventually outside any bounded subset. If you think of ∞ as an extra point, then this suggests that $\mathbb{R}^n \setminus K$ should be a punctured neighbourhood of ∞ for any closed and bounded (=compact) subset. This is precisely the idea behind the one-point-compactification of a Hausdorff space X: Let $X_\infty = X \cup \{\infty\}$ where ∞ is a new abstract point which is not already in X, and declare $X_\infty \setminus K$ to be an open neighbourhood of ∞ for all compact $K \subset X$. Then the compact subsets of X are precisely the closed subsets which are 'bounded' in the sense that they lie outside a neighbourhood of ∞.

The construction can also be modified to non-Hausdorff spaces.

3.2.2. Let us give some definitions.

Let (X, τ) be a topological space. An *open cover* of X is a familiy (possibly infinite) $\{U_i\}_{i \in I}$ of open subsets of X, such that $X = \bigcup_{i \in I} U_i$. This is a *finite* cover, if I is a finite set.

A *subcover* of an open cover $\{U_i\}_{i\in I}$ is an open cover of X of the form $\{U_i\}_{i\in J}$ for some $J \subset I$.

Definition 3.2.3. A topological space X is *quasi-compact* if any open cover of X has a finite subcover. A quasi-compact Hausdorff space is called *compact.*

Proposition 3.2.4. *Let (X,τ) be a (quasi-)compact topological space, and let $C \subset X$ be a* closed *subset. Then C is (quasi-)compact in the subspace topology.*

Proof. If X is Hausdorff, then C is also Hausdorff by 3.1.3, so it is enough to consider the case when X is quasi-compact and show that then C is also quasi-compact.

Let $\{V_i\}_{i\in I}$ be an open cover of C in the subspace topology. By definition of this topology there exists for all $i \in I$ an open subset U_i of X such that $V_i = U_i \cap C$. Since $X \setminus C$ is open, $\{U_i\}_{i\in I} \cup \{X \setminus C\}$ is an open cover of X. By quasi-compactness of X we may find a finite subset $\{i_1, i_2, \dots, i_k\} \subset I$ such that

$$\{X \setminus C, U_{i_1}, U_{i_2}, \dots, U_{i_k}\}$$

is an open cover of X. Intersect this cover with C to conclude that

$$\{V_{i_1}, V_{i_2}, \dots, V_{i_k}\}$$

is a finite subcover of $\{V_i\}_{i\in I}$. This shows that any open cover of C has a finite subcover, so C is quasi-compact. ☐

Exercise 3.2.5. Show that $\mathbb{C}$ with the (Zariski-)topology introduced in Exercise 2.4.3 is quasi-compact but not compact.

Exercise 3.2.6. Suppose $K_1, \dots, K_n \subset X$ are quasi-compact subsets of a topological space X.

1. Show that $\bigcup_{i=1}^n K_i$ is quasi-compact.
2. Show that $\bigcap_{i=1}^n K_i$ is quasi-compact if each K_i is a closed subset of X.
3. Below we give an example showing that $\bigcap_{i=1}^n K_i$ need not be quasi-compact if the K_i are not closed. Try to cook up your own example before looking at 3.2.7.

Exercise 3.2.7. This is supposed to be amusing! We will produce an example of a topology on $\mathbb{R}$ for which there exist two non-closed quasi-compact subsets $K_1, K_2 \subset \mathbb{R}$ whose intersection is not quasi-compact, cf. Exercise 3.2.6.

For A a subset of $\mathbb{R}$, let $-A$ be the set $\{-a \mid a \in A\}$.

1. Show that

$$\tau = \{U \subset \mathbb{R} \mid U \text{ is open in the Euclidean topology and } U = -U\}$$

   defines a new topology on $\mathbb{R}$.

2. Show that $K_1 = [-1, 1)$ and $K_2 = (-1, 1]$ are quasi-compact and not closed with respect to τ.

3. Show that $K_1 \cap K_2 = (-1, 1)$ is not quasi-compact with respect to τ.

Exercise 3.2.8. Let X be a set equipped with the discrete topology. Show that X is compact if and only if X is finite.

When X is Hausdorff, 3.2.4 has a converse (cf. 3.2.9): Any quasi-compact subset is closed. Without the Hausdorff assumption this is no longer true. For instance, let X be any set with at least two elements equpped with the trivial topology; then any subset of X is quasi-compact (there are only two open subsets of X so any open cover is finite), but only $\emptyset$ and X are closed.

Proposition 3.2.9. *Let X be a Hausdorff space, and let $K \subset X$ be a compact subset. Then K is a closed subset of X.*

Moreover, for each $x \in X \setminus K$ there exist open subsets $U, V \subset X$ with $x \in U$, $K \subset V$ and $U \cap V = \emptyset$.

Proof. We first prove the second part of the proposition.

Fix $x \in X \setminus K$. By the Hausdorff property we may to each $y \in K$ find open neighbourhoods in X of x and y which do not intersect. Let us call them U_y and V_y, respectively. By the choice of V_y, $K = \bigcup_{y \in K} \{y\} \subset \bigcup_{y \in K} V_y$, so $\{V_y \cap K\}_{y \in K}$ is an open cover of the compact space K. Take a subcover $\{V_y \cap K\}_{y \in F}$ with $F \subset K$ a finite subset, and define $U = \bigcap_{y \in F} U_y$ and $V = \bigcup_{y \in F} V_y$. Then U and V are open and disjoint with $x \in U$ and $K \subset V$, as required.

We may use this to see that K is closed in X. If $x \subset X \setminus K$, then by what we just proved there exists a neighbourhood U_x of x whose intersection with K is empty. That is, $x \in U_x \subset X \setminus K$. This expresses that x is an interior point of $X \setminus K$. Since this holds for any point x in $X \setminus K$, then $X \setminus K$ is open, and K must be closed. $\square$

Proposition 3.2.10. *Suppose X and Y are topological spaces with X quasi-compact. Let $f : X \longrightarrow Y$ be a continuous map. Then $f(X)$ is quasi-compact.*

If moreover f is injective and Y is Hausdorff, then f is an embedding with closed image (cf. 2.2.11).

Proof. To show that $f(X)$ is quasi-compact, let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. Then by continuity, $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is quasi-compact, this cover has a finite subcover, $\{f^{-1}(U_i)\}_{i\in F}$, for some finite $F \subset I$. But then $\{U_i\}_{i\in F}$ is a finite cover of $f(X)$. This shows that $f(X)$ is quasi-compact.

Now suppose that Y is Hausdorff. Then $f(X)$ is compact and hence closed in Y by 3.2.9.

If in addition we assume that f is injective, then for f to be an embedding we need to show that $f : X \longrightarrow f(X)$ is an open map. For this we consider an open subset $U \subset X$ and aim to prove that $f(U)$ is open in the subspace topology on $f(X)$. The complement $C = X \setminus U$ is a closed subset of X, hence C is quasi-compact. By the first part of our proposition, $f(C)$ is quasi-compact and hence compact since Y is Hausdorff. Since $f(X)$ inherits the Hausdorff-property from Y, then the compact subset $f(C) \subset f(X)$ is closed in $f(X)$ by 3.2.9, and we conclude that the complement $f(X) \setminus f(C) = f(X \setminus C) = f(U)$ is open in $f(X)$. This is what we had to prove.

Note that since X is homeomorphic to $f(X)$, then X is necessarily a Hausdorff space. □

Corollary 3.2.11. *Let X be a quasi-compact space, Y a Hausdorff space, and $f : X \longrightarrow Y$ a bijective continuous map. Then f is a homeomorphism.*

Proof. From 3.2.10 we know that f is an embedding. This means that $f : X \longrightarrow f(X) = Y$ is a homeomorphism. □

Exercise 3.2.12. Show that if $K_i \subset X$, $i \in I$, is a family of compact subsets of a Hausdorff space X with $\bigcap_{i\in I} K_i = \emptyset$, then there is a finite subset $F \subset I$ such that $\bigcap_{i\in F} K_i = \emptyset$.

Theorem 3.2.13. *Let (X, τ) be a topological space. Then 1. implies 2.:*

1. *X is quasi-compact.*
2. *Any sequence in X has an accumulation point. (Equivalently, any sequence has a convergent subsequence.)*

If the topology τ satisfies the second axiom of countability, then 1. and 2. are equivalent.

Proof. 1. $\Rightarrow$ 2.: Let X be quasi-compact, and assume that $\{x_i\}_{i\in\mathbb{N}}$ is a sequence in X with no accumulation points. Then for each $x \in X$ there will be an open neighbourhood U_x of X for which $I_x = \{i \in \mathbb{N} \mid x_i \in U_x\}$ is finite. The open cover $\{U_x\}_{x\in X}$ of X has a finite subcover $\{U_x\}_{x\in F}$, $F \subset X$ finite, since X is quasi-compact. Each x_i is contained in some U_x, $x \in F$, so

$\mathbb{N} = \bigcup_{x \in F} I_x$. But this is clearly impossible since F and all the I_x are finite. This contradicts our assumtion that $\{x_i\}_{i \in \mathbb{N}}$ has no accumulation point and proves the claim.

2. $\Rightarrow$ 1.: For this we have to assume τ has a countable basis. We must prove that any open cover of a space satisfying 2. has a finite subcover. We do this in two steps. In the first step we prove that a countable open cover has a finite subcover. The second step proves that also an uncountable cover has a finite subcover by reducing to the countable case. This is where the assumption that τ has a countable basis is used.

Step 1: Let $\{U_i\}_{i \in I}$ be an open cover of X. We first treat the case where I is countable, and we may of course assume that $I = \mathbb{N}$. By $V_i = \bigcup_{j=1}^{i} U_j$ we define an increasing sequence of open subsets of X, $V_1 \subset V_2 \subset V_3 \subset \dots$. If $V_i = X$ for some $i \in \mathbb{N}$, then $\{U_1, \dots, U_i\}$ is a finite cover of X and we are done. Otherwise, $X \setminus V_i$ is non-empty for all i and we may choose a point x_i in there. Let x be an accumulation point for the sequence $\{x_i\}_{i \in \mathbb{N}}$ thus defined. Since the V_i cover X, there exists an $N \in \mathbb{N}$ so that $x \in V_N$. By the definition of an accumulation point, infinitely many of the x_i are contained in V_N. This contradicts the definition of the x_i, and we conclude that $\{U_i\}_{i \in I}$ has a finite subcover.

Step 2: Now suppose that I is infinite and uncountable. By assumption we may find a countable basis $\sigma \subset \tau$ for τ. Define

$$\rho = \{V \in \sigma \mid V \subset U_i \text{ for some } i \in I\}.$$

Then ρ is a countable cover of X since any U_i is a union of elements in σ. By step 1, ρ has a finite subcover, say $\{V_1, \dots, V_k\} \subset \rho$. Since each V_j is contained in some U_{i_j}, then $\left\{U_{i_1}, \dots, U_{i_j}\right\}$ is a finite subcover of $\{U_i\}_{i \in I}$. $\square$

3.2.14. A Hausdorff space X is said to be *normal*, if for any two disjoint closed subsets $C, D \subset X$ there exist disjoint open subsets $U, V \subset X$ with $C \subset U$ and $D \subset V$. Many reasonable spaces (metric spaces, for instance) have this useful property.

Lemma 3.2.15. *Any compact space is normal.*

Proof. Let X be a compact space and C, D disjoint closed subsets. Then C and D are compact as well by 3.2.4.

Apply the last statement in 3.2.9 to the compact subset D and a point $x \in C$ to obtain open disjoint subsets $U_x, V_x \subset X$ with $x \in U_x$ and $D \subset V_x$. If we do this for all $x \in C$ we obtain an open cover $\{U_x \cap C\}_{x \in C}$ of C. Since C is compact, we may find a subcover $\{U_x \cap C\}_{x \in F}$ for some finite $F \subset C$. Put $U = \bigcup_{x \in F} U_x$ and $V = \bigcap_{x \in F} V_x$. Then U is an open set in X containing

C, and V is an open set (since F is finite) containing D. It is easy to see that $U \cap V = \emptyset$ by construction.

The existence of U and V with the given properties is precisely what is required for X to be normal. □

In Exercise 3.4.12 we will show that $\mathbb{R}^n$ and many other non-compact spaces are normal.

3.3 Products of compact spaces

In 2.6 we introduced a topology on the product of a family of topological spaces. This product topology has the remarkable property of being compact if all the spaces that enter the product are compact.

Theorem 3.3.1. *Let $X = \prod_{i \in I} X_i$ be an arbitrary product of (quasi-)compact spaces X_i, $i \in I$. Then X is (quasi-)compact in the product topology.*

3.3.2. This theorem is known as Tychonoff's theorem. Since any product of Hausdorff spaces is again Hausdorff by 3.1.7, then it suffices to consider only the quasi-compact situation. We will not prove 3.3.1 in full generality but restrict ourselves to two useful special cases.

Claim 3.3.3. *Theorem 3.3.1 holds if the index set I is finite.*

3.3.4. We will prove Claim 3.3.3 by combining the two lemmas 3.3.5 and 3.3.6 below. In total, this may not give the shortest possible proof of the claim, but it should make the structure of the proof more transparent. In addition, both lemmas will be useful for us later on.

Proof. (Of Claim 3.3.3.) We proceed by induction in the number n of elements in I.

For $n = 2$ we have to show that the product $X_1 \times X_2$ of two quasi-compact spaces X_1 and X_2 is quasi-compact. But this is the conclusion of Lemma 3.3.5 when applied to the projection map $pr_1 : X_1 \times X_2 \longrightarrow X_1$; it follows directly from 3.3.6 that pr_1 satisfies the assumptions.

Suppose $n > 2$. By Exercise 2.6.4 we may consider X as the topological product of $X' = X_1 \times \cdots \times X_{n-1}$ with X_n. But X' is quasi-compact by induction, and hence so is X by the case "$n = 2$". □

Lemma 3.3.5. *Let X, Y be topological spaces and $f : X \longrightarrow Y$ a continuous closed map. Assume that the fiber $f^{-1}(y)$ of f over y is quasi-compact for all $y \in Y$, and that Y is quasi-compact as well. Then X is quasi-compact.*

Proof. Consider an open cover $\{U_j\}_{j\in J}$ of X. We must show that it has a finite subcover. Fix a point $y \in Y$ for a moment. The fiber $f^{-1}(y)$ is quasi-compact by assumption, so the open cover $\{U_j \cap f^{-1}(y)\}_{j\in J}$ has a finite subcover $\{U_j \cap f^{-1}(y)\}_{j\in F}$ for some finite subset $F \subset J$. Put $U'_y = \bigcup_{j\in F} U_j$. Then U'_y is an open subset of X containing $f^{-1}(y)$. Since f is a closed map then $f(X \setminus U'_y)$ is a closed subset of Y which does not contain y. Put $V_y = Y \setminus f(X \setminus U'_y)$. Then V_y is a neighbourhood of y, and $f^{-1}(V_y) \subset U_y$ by construction.

If we do this for all $y \in Y$, we get an open cover $\{V_y\}_{y\in Y}$ of Y. Since Y is quasi-compact, this cover has a finite subcover, $\{V_{y_1}, \ldots, V_{y_k}\}$ say. Hence,

$$X = f^{-1}(Y) = f^{-1}(V_{y_1}) \cup \cdots \cup f^{-1}(V_{y_k}) \subset U'_{y_1} \cup \cdots \cup U'_{y_k}.$$

Since each U'_{y_i} is a finite union of U_j's, this shows that $\{U_j\}_{j\in J}$ indeed has a finite subcover. □

Lemma 3.3.6. *Let X and Z be topological spaces, and let $pr_X : X \times Z \longrightarrow X$ be the projection on the first factor. Assume Z is quasi-compact. Then pr_X is continuous and closed with quasi-compact fibers.*

Proof. We already know from 2.6.3 that pr_X is continuous, and for each $x \in X$ the fiber $pr_X^{-1}(x) = \{x\} \times Z$ is homeomorphic to Z, so pr_X has quasi-compact fibers. We must show that pr_X is closed.

For this we consider a closed subset $C \subset X \times Z$ and show that $f(C)$ is closed in X. More precisely, we prove that $X \setminus pr_X(C)$ is a neighbourhood of each of its elements.

Fix $x \in X \setminus pr_X(C)$. For all $z \in Z$ the set $(X \times Z) \setminus C$ is an open neighbourhood of (x, z) in the product topology so it has a subset of the form $U_z \times V_z \subset (X \times Z) \setminus C$ with U_z and V_z open neighbourhoods of x in X and z in Z, respectively. Since Z is quasi-compact, the cover $\{V_z\}_{z\in Z}$ has a finite subcover, $\{V_{z_1}, \ldots, V_{z_k}\}$ say. Put $U'_x = U_{z_1} \cap \cdots \cap U_{z_k}$. Then U'_x is open in X, and $U'_x \times Z \subset (X \times Z) \setminus C$ by construction. This implies that $x \in U'_x \subset X \setminus pr_X(C)$ and proves that $X \setminus pr_X(C)$ is a neighbourhood of x. This is what we had to show. □

Claim 3.3.7. *Theorem 3.3.1 holds if $I = \mathbb{N}$ and each X_i satisfies the second axiom of countability. In this case, also the product space X satisfies the second axiom of countability.*

Proof. Let σ_i be an at most countable basis for the topology on X_i, $i \in \mathbb{N}$. Then

$$\sigma = \left\{ \prod_{i\in\mathbb{N}} U_i \mid \text{for some } N \in \mathbb{N},\ U_i \in \sigma_i \text{ for } i \leq N \text{ and } U_i = X_i \text{ for } i > N \right\}$$

is a basis for the product topology on X. It is an (easy?) exercise to show that σ is countable, so X satisfies the second axiom of countability. By Theorem 3.2.13 we just have to check that any sequence in X has an accumulation point.

Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in X, where $x_j = (x_{j,i})_{i\in\mathbb{N}}$ with $x_{j,i} \in X_i$.

Claim: Let $N \in \mathbb{N}$, and suppose $(x_{0,1}, x_{0,2}, \dots, x_{0,N})$ is an accumulation point for the projection of the sequence $\{x_j\}_{j\in\mathbb{N}}$ on $X_1 \times X_2 \times \cdots \times X_N$. Then there exists $x_{0,N+1} \in X_{N+1}$ such that $(x_{0,1}, x_{0,2}, \dots, x_{0,N+1})$ is an accumulation point for the projection of $\{x_j\}_{j\in\mathbb{N}}$ on $X_1 \times X_2 \times \cdots \times X_{N+1}$.

To prove the claim we may take a subsequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ of $\{x_j\}_{j\in\mathbb{N}}$ whose projection on $X_1 \times X_2 \times \cdots \times X_N$ converges to $(x_{0,1}, x_{0,2}, \dots, x_{0,N})$. By quasi-compactness of X_{N+1} the sequence $\{pr_{N+1}(x_{j_k})\}_{k\in\mathbb{N}}$ has an accumulation point $x_{0,N+1}$ in X_{N+1}. This clearly shows that $(x_{0,1}, x_{0,2}, \dots, x_{0,N+1})$ is an accumulation point for the projection of $\{x_{j_k}\}_{k\in\mathbb{N}}$ – hence also of $\{x_j\}_{j\in\mathbb{N}}$ – on $X_1 \times X_2 \times \cdots \times X_{N+1}$, as predicted by the claim.

The claim allows us to construct inductively an element $x_0 = (x_{0,i})_{i\in\mathbb{N}}$ of X whose projection onto any finite product of the X_i's is an accumulation point for the projection of the sequence $\{x_j\}_{j\in\mathbb{N}}$. We want to show that x_0 is an accumulation point for $\{x_j\}_{j\in\mathbb{N}}$ in X. For this, let U be an open neighbourhood of x_0 in X. We must show that x_j is frequently in U. By shrinking U, we may assume that U is of the form $\prod_{i\in\mathbb{N}} U_i$, where U_i is an open neighbourhood of $x_{0,i}$ in X_i for all i, and there exists an $N \in \mathbb{N}$ such that $U_i = X_i$ for $i > N$. But since the projection of $\{x_j\}_{j\in\mathbb{N}}$ on $X_1 \times \cdots \times X_N$ accumulates to $(x_{0,1}, \dots, x_{0,N})$, then this projection is frequently in $U_1 \times \cdots \times U_N$, whose pre-image in X is U. This implies that $\{x_j\}_{j\in\mathbb{N}}$ is frequently in U, as desired. □

Example 3.3.8. Claim 3.3.7 applies to show that the *Tychonoff cube* $T = \prod_{i\in\mathbb{N}}[0,1]$ is compact and satisfies the second axiom of countability. Indeed, we just have to observe that the compact unit interval $[0,1]$ satisfies the second axiom of countability since it is a subset of $\mathbb{R}$ (cf. 2.4.6).

Exercise 3.3.9. This exercise will show that the product topology on the Tychonoff cube T from Example 3.3.8 is a metric topology. For $x, y \in T$, define $d(x,y) = \sum_{i=1}^{\infty} 2^{-i}|pr_i(x) - pr_i(y)|$.

1. Show that d defines a metric on T.

2. Show that this metric defines the product topology on T.

As an application of the previous theorems we get the well-known theorem of Borel–Heine.

Theorem 3.3.10. *A subset K of $\mathbb{R}^n$ is compact (in the subspace topology) if and only if it is a closed and bounded subset.*

Proof. By 2.4.6, $\mathbb{R}^n$ and hence any subset of $\mathbb{R}^n$ has a countable basis for the topology, so the two conditions 1. and 2. of Theorem 3.2.13 are equivalent for any subspace of $\mathbb{R}^n$.

Let us first show that a closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact. For this, consider an arbitrary sequence $\{x_i\}_{i\in\mathbb{N}}$ in $[a, b]$ and let

$$x = \limsup_{i\to\infty} x_i = \lim_{i\to\infty} (\sup_{j\geq i} x_j).$$

Then x is an accumulation point for $\{x_i\}_{i\in\mathbb{N}}$ which is contained in $[a, b]$ (Exercise!). So $[a, b]$ is indeed compact.

Now, if K is closed and bounded in $\mathbb{R}^n$, then K is a closed subset of the product $K' = [-r, r]^n$ for r big enough. By 3.3.3 and what we just proved, K' is compact, and hence K is compact by 3.2.4.

Conversely, assume $K \subset \mathbb{R}^n$ is compact. Then K is a closed subset of $\mathbb{R}^n$ by 3.2.9. If K were not bounded then we could choose a sequence $x_i \in K$ with $|x_i| \geq i$ for all i. This has no accumulation point, and we get a contradiction. $\square$

3.4 The one-point compactification

The construction of the one-point-compactification applies to any topological space (X, τ) and goes as follows. Let $X_\infty = X \sqcup \{\infty\}$, where $\infty \in X_\infty$ denotes an abstract point not in X. We define a topology, τ_∞, on X_∞ by demanding that a subset $U \subset X_\infty$ is open if and only if the following conditions are satisfied

1. If $\infty \notin U$, then U is open in X.

2. If $\infty \in U$, then $X_\infty \setminus U$ is a closed and quasi-compact subset of X.

Note that if X is Hausdorff, then 2. is equivalent to saying that $X_\infty \setminus U$ is compact.

Before we check that 1. and 2. really define a topology on X_∞, observe that $X \cap U$ is open in X for any open subset $U \subset X_\infty$ of X_∞, and that any open subset of X is also open in X_∞. This expresses the fact that the subspace topology on $X \subset X_\infty$ coincides with the original topology on X, so the inclusion $X \hookrightarrow X_\infty$ is a homeomorphism on its image.

We check that τ_∞ satisfies the axioms for a topology.

1. Assume $\{U_i\}_{i\in I}$ is a collection of open subsets of X_∞. We check that $U = \bigcup_{i\in I} U_i$ is open.

   If $\infty \notin U$, then $\infty \notin U_i$ for all i and hence U_i is open in X. Therefore U is open in X and hence also in X_∞.

   If $\infty \in U$, then $\infty \in U_{i_0}$ for some $i_0 \in I$, and $K = X_\infty \setminus U_{i_0}$ is a closed and quasi-compact subset of X. But then

   $$X_\infty \setminus U = K \cap \bigcap_{i\in I\setminus\{i_0\}} (X_\infty \setminus U_i) = K \cap \bigcap_{i\in I\setminus\{i_0\}} (X \setminus (X \cap U_i))$$

   is a closed subset of the closed and quasi-compact subset K, hence $X_\infty \setminus U$ is closed and quasi-compact. This is what is required for U to be open in X_∞.

2. Assume $\{U_i\}_{i=1}^k$ is a finite collection of elements in τ_∞. We check that $U = \bigcap_{i=1}^k U_i$ is open.

   If $\infty \in U$, then $\infty \in U_i$ for all i and so $X \setminus U = \bigcup_{i=1}^k (X \setminus U_i)$ is a finite union of closed and quasi-compact subset of X. This is again closed and quasi-compact by 3.2.6, so U is open.

   If $\infty \notin U$ then $\infty \notin U_i$ for some i and then $U = X \cap U = \bigcap_{i=1}^k (X \cap U_i)$ is open in X since it is a finite intersection of open subsets of X. Again, U is open.

3. The reader should check that $\emptyset$ and X_∞ are open.

Exercise 3.4.1. Consider the map $f : \mathbb{R}^2 \longrightarrow S^2$ given by the expression $f(u, v) = (4u, 4v, u^2 + v^2 - 4)/(u^2 + v^2 + 4)$. This is the inverse of the so called *stereographic projection* (see e.g. [do Carmo] p. 67 for a nice picture).

1. Show that f is injective, and that the image of f is $S^2 \setminus \{(0, 0, 1)\}$.

2. Show that if $X = \mathbb{R}^2$, then the extension of f to a map $X_\infty \longrightarrow S^2$ defined by mapping ∞ to $(0, 0, 1)$ is a homeomorphism.

Exercise 3.4.2. Let $\mathbb{N}_\infty$ be the one-point-compactification of $\mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology. Show that a sequence $\{x_i\}_{i\in\mathbb{N}}$ in a topological space X is convergent if and only if the map $\mathbb{N} \longrightarrow X$ defined by $i \mapsto x_i$ extends to a continuous map $\mathbb{N}_\infty \longrightarrow X$.

Proposition 3.4.3. *The one-point compactification X_∞ of X is quasi-compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of X_∞. Then there exists an $i_0 \in I$ such that $\infty \in U_{i_0}$. Let $K = X_\infty \setminus U_{i_0}$. Then K is a closed and quasi-compact subset of X, and $\{U_i \cap K\}_{i\in I\setminus\{i_0\}}$ is an open cover of K. By quasi-compactness, this has a finite subcover, $\{U_{i_1} \cap K, \ldots, U_{i_k} \cap K\}$ say. But then $\{U_{i_0}, U_{i_1}, \ldots, U_{i_k}\}$ is a finite subcover of the original cover of X_∞. This proves that X_∞ is quasi-compact. □

Remark 3.4.4. Let X be a Hausdorff space. We may call a subset $A \subset X$ of X *bounded*, if ∞ is *not* contained in the closure of A inside X_∞. This is the same as saying that there exists an open neighbourhoood U of ∞ which does not intersect A. The complement of U is by definition a closed and quasi-compact subset of X, and since X is Hausdorff, this is compact. Thus, $A \subset X$ is bounded if and only if its closure inside X is compact.

In this way the compact subsets of X are exactly the closed and bounded subsets.

This sentence is not a theorem! All we have been doing is to design our notion of boundedness so conveniently that the statement of the sentence is true. In particular, for $X = \mathbb{R}^n$ this is *not* a new proof of the Borel–Heine theorem. The Borel–Heine theorem guarantees us that the new definition of boundedness in $\mathbb{R}^n$ agrees with the old one.

Nevertheless, the above sentence is a good way to get some intuition for compactness, and it fits well into the discussion of proper maps in the next section.

To complete the discussion of the one-point-compactification and for later use we introduce a few more concepts.

Definition 3.4.5. A topological space is *locally compact*, if it Hausdorff and every point has a compact neighbourhood.

Exercise 3.4.6. Show that any point in a locally compact space has a neighbourhood basis consisting of compact subsets. (Equivalently, any neighbourhood of a point contains a compact neighbourhood of the same point).

Proposition 3.4.7. *X_∞ is Hausdorff (and hence compact) if and only if X is locally compact.*

Proof. Assume X_∞ is Hausdorff. Then X as a subspace of X_∞ is also Hausdorff. Moreover, if $x \in X$ then we may choose open neighbourhoods U and V of x and ∞ respectively, such that $U \cap V = \emptyset$. But then, by definition of the topology on X_∞, $X_\infty \setminus V$ is a compact subset of X which contains U. Thus, there exists in X a compact neighbourhood of x.

Conversely, assume X is locally compact, and let $x, y \in X_\infty$ be two distinct points. Since a locally compact space is implicitly Hausdorff it is straightforward to find open disjoint neighbourhoods of x and y if both of them lie in X.

So, suppose $y = \infty$ and pick a compact neighbourhood K of x in X. By definition of the topology on X_∞, $X_\infty \setminus K$ is an open neighbourhood of ∞ which is disjoint from the open neighbourhood K° of x. This shows that X_∞ is indeed Hausdorff. $\square$

3.4.8. A Hausdorff space X is called *σ-compact* if it is a countable union of compact subsets.

Exercise 3.4.9. Prove the following assertions.

1. $\mathbb{R}^n$ is both σ-compact and locally compact.
2. $\mathbb{Q}$ is σ-compact but *not* locally compact in the subspace topology from $\mathbb{R}$.
3. Challenge: $\mathbb{R} \setminus \mathbb{Q}$ is neither σ-compact nor locally compact.

Exercise 3.4.10. Let X be a locally compact space. Show that X is σ-compact if and only if there exists an increasing sequence of compact subsets

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset K_i \subset K_{i+1} \subset \ldots$$

such that $\bigcup_{i=1}^\infty K_i = X$ and $K_i \subset K_{i+1}^\circ \subset K_{i+1}$ for all i.

Exercise 3.4.11. Show that any locally compact topological space which satisfies the second axiom of countability is σ-compact.

Exercise 3.4.12. Show that a space which is both locally compact and σ-compact is also normal (cf. 3.2.14). Use Exercise 3.4.9 to conclude that $\mathbb{R}^n$ is normal.

(This exercise is harder than the previous ones. To get some inspiration, it may actually be a good idea to consider $\mathbb{R}^n$ first.)

3.5 Properness

3.5.1. Proper maps play an important rôle in many geometric situations. The general idea is that a proper map $X \longrightarrow Y$ is one which maps a point 'near infinity' of X to a point 'near infinity' of Y. There are various almost-equivalent definitions of properness in the literature – we have adopted one where this geometric idea may be given a precise meaning in terms of the one-point-compactifications of X and Y.

We shall make no attempt of doing things in the greatest possible generality. For instance, we will always assume the spaces we map between to be Hausdorff.

Definition 3.5.2. A continuous map $f : X \longrightarrow Y$ between Hausdorff spaces is *proper* if $f^{-1}(K)$ compact in X for any compact subspace K of Y.

Example 3.5.3. If X and Z are Hausdorff spaces with Z compact, then the projection $pr_X : X \times Z \longrightarrow X$ onto the first factor is proper.

Exercise 3.5.4. Show that if X is compact and Y is a Hausdorff space, then any continuous map $f : X \longrightarrow Y$ is proper.

3.5.5. Given a continuous map $f : X \longrightarrow Y$ of Hausdorff spaces, we may define a map from the one-point compactification $X_\infty = X \sqcup \{\infty_X\}$ of X to the one-point compactification $Y_\infty = Y \sqcup \{\infty_Y\}$ of Y as follows.

$$f_\infty(x) = \begin{cases} f(x), & \text{if } x \in X \\ \infty_Y, & \text{if } x = \infty_X. \end{cases}$$

Proposition 3.5.6. *f is proper if and only if f_∞ is continuous.*

Proof. Assume f is proper, and let U be open in Y_∞. If $\infty \notin U$, then U is open in Y, and $f_\infty^{-1}(U) = f^{-1}(U)$ is open in X. If $\infty \in U$, then $Y_\infty \setminus U$ is compact in Y, hence $f_\infty^{-1}(Y_\infty \setminus U) = f^{-1}(Y_\infty \setminus U)$ is compact in X. The complement $f_\infty^{-1}(U)$ is then open in X_∞. This shows that f_∞ is continuous.

Conversely, assume f_∞ is continuous, and let $K \subset Y$ be compact. Since Y is Hausdorff, K is a closed subset of Y, and thus $K' = f^{-1}(K) = f_\infty^{-1}(K)$ is a closed subset of X_∞. Since X_∞ is quasi-compact, then K' is quasi-compact, and since X is Hausdorff then K' is compact. □

Remark 3.5.7. In continuation of Remark 3.4.4 we see that Proposition 3.5.6 gives a very concrete meaning to the slogan:

Proper maps are the ones that send infinity to infinity.

Think about this and make your own sense of it!

3.5.8. We will now present a couple of alternative characterizations of properness. A very attractive approach is to define the subset $Z(f) \subset Y$ of 'improper points'. Under certain extra conditions, f is proper if and only if $Z(f) = \emptyset$ (i.e. there are no 'improper points'). This way, $Z(f)$ gives a measure of how far f is from being proper, and it also helps locating the non-proper behaviour.

Definition 3.5.9. Let $f : X \longrightarrow Y$ be a continuous map between Hausdorff spaces X and Y.

1. A point $y \in Y$ is called an *improper point* for f, if there exists a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with no accumulation points, such that $y = \lim_{i\to\infty} f(x_i)$.

2. The *improper point set* for f is the following subset of Y,

$$Z(f) = \{y \in Y : y \text{ is an improper point for } f\}.$$

Exercise 3.5.10. Compute $Z(f)$ for each of the following maps $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}^2$: $x \mapsto 0$, $x \mapsto 1/x$, $x \mapsto \log|x|$, $x \mapsto e^{-x^2}$. Which of them are proper?

Compute $Z(id)$ for the identity map $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Exercise 3.5.11. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a locally compact space X. Show that the sequence tends to ∞ (in X_∞) if and only if it has no accumulation points in X. Conclude that in the situation of Definition 3.5.9, a point $y \in Y$ is improper for f if and only if there exists a sequence in X tending to ∞, whose image sequence tends to y.

Exercise 3.5.12. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in a Hausdorff space X converging to some $x \in X$. Show that $K = \{x_i \mid i \in \mathbb{N}\} \cup \{x\} \subset X$ is quasi-compact.

Lemma 3.5.13. *Let $f : X \longrightarrow Y$ be a continuous map of Hausdorff spaces. Assume that X satisfies the second axiom of countability and Y satisfies the first axiom of countability. Then $f^{-1}(K)$ is compact for any compact $K \subset Y \setminus Z(f)$.*

Proof. Let $K \subset Y \setminus Z(f)$ be compact and let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in $f^{-1}(K)$. Since K is compact, then the sequence $\{f(x_i)\}_{i\in\mathbb{N}}$ in K has a subsequence $\{f(x_{i_j})\}_{j\in\mathbb{N}}$ which converges to some $y \in Y$. By assumption $y \notin Z(f)$, so $\{x_{i_j}\}_{j\in\mathbb{N}}$ must have an accumulation point in $x \in X$. In particular, x is an accumulation point for $\{x_i\}_{i\in\mathbb{N}}$, and x is contained in $f^{-1}(K)$ since this is a closed subset of X. Now apply Theorem 3.2.13: Since X (and hence $f^{-1}(K)$) satisfies the second axiom of countability, and any sequence in $f^{-1}(K)$ has an accumulation point, then $f^{-1}(K)$ is compact. □

Proposition 3.5.14. *Let $f : X \longrightarrow Y$ be a continuous map of Hausdorff spaces.*

1. *If f is proper, then $Z(f) = \emptyset$.*

2. *If $Z(f) = \emptyset$ and X satisfies the second axiom of countability and Y satisfies the first axiom of countability, then f is proper.*

Proof. 1.: Let f be proper. Assume $Z(f) \neq \emptyset$ and pick $y \in Z(f)$. Then there is a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X with no accumulation point, such that $y = \lim_{i\to\infty} f(x_i)$. Let $K = \{f(x_i) \mid i \in \mathbb{N}\} \cup \{x\} \subset Y$. Then K is compact (Y is

Hausdorff by assumption) by Exercise 3.5.12, so by properness of f, $f^{-1}(K)$ is a compact set containing $\{x_i\}_{i\in\mathbb{N}}$. By 3.2.13, $\{x_i\}_{i\in\mathbb{N}}$ has an accumulation point in $f^{-1}(K)$ and hence in X. This contradicts the assumption $Z(f) \neq \emptyset$, so $Z(f)$ must be empty.

2. is immediate from Lemma 3.5.13. □

The following theorem collects our findings up to now and adds yet another criterion for properness.

Theorem 3.5.15. *Consider the following four conditions on a continuous map $f : X \longrightarrow Y$ between Hausdorff spaces X and Y.*

1. *f is closed and all fibers are compact.*
2. *f is proper.*
3. *f_∞ is continuous (cf. 3.5.5).*
4. *$Z(f) = \emptyset$.*

Then 1. $\Rightarrow$ 2. $\Leftrightarrow$ 3. $\Rightarrow$ 4.
If Y is locally compact, then 1. is equivalent to 2. and 3.
If X satisfies the second axiom of countability and Y satisfies the first axiom of countability, then 4. is equivalent to 2. and 3.

Proof. We only have to show that 1. implies 2., and that 2. implies 1. provided Y is locally compact. The other assertions have already been proved above.

1. $\Rightarrow$ 2.: Let $K \subset Y$ be compact. We must show that $f^{-1}(K)$ is compact as well. Consider the map $f' : f^{-1}(K) \longrightarrow K$ given by restriction of f. Clearly f' has compact fibers since f has. We claim that f' is a closed map. To see this, let C be a closed subset (in the subspace topology) of $f^{-1}(K)$. Since $f^{-1}(K)$ is closed in X, then C is also closed as a subset of X, and $f'(C) = f(C)$ is closed since f is a closed map. This closes the discussion: f' is indeed a closed map.

We are now in a position where we may apply Lemma 3.3.5 to the map f', and the conclusion is that $f^{-1}(K)$ is compact, as desired.

2. $\Rightarrow$ 1.: Assume that f is proper and Y is locally compact. For any $y \in Y$, $\{y\}$ is compact so the fiber $f^{-1}(y)$ is compact by properness of f. It remains to be proved that f is closed. So, let $C \subset X$ be a closed subset, and let y be a point in $Y \setminus f(C)$. We must show that $Y \setminus f(C)$ is a neighbourhood of y.

Let $K \subset Y$ be a compact neighbourhood of y. (Recall that this means that K is compact and y is an interior point of K.) Then $f^{-1}(K)$ is compact by properness of f and hence closed since X is Hausdorff. The intersection $C' = C \cap f^{-1}(K)$ is then compact. By continuity of f, $f(C')$ is compact, hence closed, and does not contain y. We conclude that $K \setminus f(C')$ is a neighbourhood of y. Since $K \setminus f(C') \subset Y \setminus f(C)$, then also $Y \setminus f(C)$ is a neighbourhood of y, as desired. $\square$

Chapter 4

Quotient topology and gluing.

4.1 The quotient topology

4.1.1. One of the nice things about topological spaces, is that it is easy to glue them together to make new spaces.

For instance, a sheet of cloth is a nice topological space. You can take this sheet, cut it a little, sew things together, and you might obtain a shirt. Which is a completely different topological space.

Let us take a more general topological space X. We assume that we are given an equivalence relation $\sim$ on X, and we want to give the set of equivalence classes $X/\sim$ a topology.

There is a quotient map of sets $\pi : X \to X/_\sim$. We also write $\tilde{x}$ for $\pi(x)$.

Definition 4.1.2. Let $U \subset X/_\sim$ be open if and only if $\pi^{-1}(U) \subset X$ is open.

Lemma 4.1.3. *With this definition, the open sets in $X/_\sim$ form a topology. The map $X \to X/_\sim$ is continuous, and if Y is a topological space, a map*

$$X/_\sim \to Y$$

is continuous if and only if the composite

$$X \to X/_\sim \to Y$$

is continuous

Proof. To see that we have defined a topology, we have to check the axioms.

- $\pi^{-1}(\emptyset) = \emptyset$, so $\emptyset \subset X/_\sim$ is open.
- $\pi^{-1}(X/_\sim) = X$, so $X/_\sim \subset X/_\sim$ is open.

- If $U, V \subset X/_\sim$ is open, then $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$ is open. So $U \cap V \in X/_\sim$ is open. Hence any finite intersection of open sets is open.
- If $U_i \subset X/_\sim$ is open, $i \in I$, then $\pi^{-1}(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} \pi^{-1}(U_i)$ is open, so that $\bigcup_{i\in I} U_i \in X/_\sim$ is open.

That the map π is continuous means exactly that if $U \subset X/_\sim$ is open, then $\pi^{-1}(U) \subset X$ is open. But this is a direct consequence of the definition of the topology on $X/_\sim$.

Now, let $f : X/_\sim \to Y$ be any map. By the definition of continuity, it is continuous if and only if for each open set $V \subset Y$, the set $f^{-1}(V) \subset X/_\sim$ is open. But this is so, if and only if $\pi^{-1}(f^{-1}(V)) \subset X$ is open. However, $\pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V)$, so the condition that f is continuous can also reformulated as: f is continuous if and only if for each open set $U \subset Y$, the set $(f \circ \pi)^{-1}(U)$ is open. This is exactly the same as saying that the composite $f \circ \pi$ is continuous. □

4.1.4. One interesting point is that it is much easier to glue topological spaces than to glue metric spaces. If X has a metric, there is no gluing construction of a metric on $X/_\sim$. The price we pay for this greater flexibility of topological spaces is that in some situations, the topology on $X/_\sim$ can be rather mysterious!

Exercise 4.1.5. Let $X = [0, 1]$, the closed interval in $\mathbb{R}$ with its usual topology. Let $\sim$ be the equivalence relation for which $a \sim b$ if and only if $a - b$ is a rational number. Show that the only open sets in $X/_\sim$ are the empty set and the whole space. That is, $X/_\sim$ has the trivial topology. Conclude that $X/_\sim$ is not even a Hausdorff space.

4.1.6. The lesson from this is that sometimes a quotient space of a reasonable space by an equivalence relation is not a Hausdorff space. This is a *bad thing.* What it is usually telling you is that one should not consider those quotient spaces anyhow.

But from a more enthusiastic point of view, this phenomenon also shows that the definition of topological space is pretty clever. The way it is defined, an equivalence relation on a topological space always defines a topology on the quotient. If we had insisted on building the Hausdorff property into the definition, we would have had to check some properties to show that the quotient is even defined!

Often we want to show that a space we have obtained from a gluing construction agrees with a space we already know.

Exercise 4.1.7. Let $X = [0, 1]$ as in 4.1.5. But this time, we pick the equivalence relation generated by requirering that $0 \sim 1$. So if $a \sim b$, either $a = b$ or $\{a, b\} = \{0, 1\}$. There is a map $f : X/_\sim \to S^1 \subset \mathbb{R}^2$ given by

$$f(x) = (\cos(2\pi x), \sin(2\pi x)).$$

Show that this map induces a homeomorphism $X/_\sim \to S^1$.

4.1.8. The previous exercise is an instance of a more general procedure. If $f : X \to Y$ is a map, it induces an equivalence relation on X by $a \sim b$ if and only if $f(a) = f(b)$. The map $f : X \to Y$ factors as a map of sets as $X \xrightarrow{\pi} X/_\sim \xrightarrow{\tilde{f}} Y$.

Remark 4.1.9. Recall that a map f is said to be injective if $f(a) = f(b)$ only can happen in case $a = b$. Note that $\tilde{f}$ is injective by its construction: If $\tilde{a}, \tilde{b} \in X/_\sim$, and $\tilde{f}(\tilde{a}) = \tilde{f}(\tilde{b})$, then $f(a) = f(b)$ so $a \sim b$; this means that $\tilde{a} = \tilde{b}$.

Lemma 4.1.3 informs us that the map $\tilde{f}$ is continuous if and only if f is continuous.

Definition 4.1.10. If $\tilde{f}$ is a homeomorphism, we say that f is a quotient map.

4.1.11. If f is a quotient map, the map $\tilde{f}$ is bijective, so in particular it is surjective. But this is only possible if the original map f is surjective.

Recall that $f : X \to Y$ is said to be surjective if every $y \in Y$ can be written as $y = f(x)$ for some $x \in X$.

Proposition 4.1.12. *Assume that X is quasi-compact, and Y is a Hausdorff space. Then any continuous surjective map $f : X \to Y$ is a quotient map.*

Proof. The space $X/_\sim$ is quasi-compact by 3.2.10 since $\pi : X \to X/_\sim$ is a surjective continuous map. The induced map $\tilde{f} : X/_\sim \to Y$ is thus a map from a quasi-compact space to a Hausdorff space. It is also a bijection, so by refering to 3.2.10 once more we see that it is a homeomorphism. □

Remark 4.1.13. The compactness of X is essential for this! Below, in 4.4.2, we give an embarrassingly simple counterexample to 4.1.12, if the compactness assumption is foolishly dropped.

There is a converse to 4.1.3.

Lemma 4.1.14. *Let $f : X \to Y$ be a surjective map. The following conditions are equivalent.*

- *f is a quotient map.*
- *A map $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Z$ is continuous.*

Proof. One direction is easy. If f is a quotient map, it is continuous. So the first condition implies the second (using 4.1.3). In the other direction, suppose that $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Y$ is. The map $\tilde{f}$ is a surjection by assumption.

It is also injective by 4.1.9, so it is bijective, and we can define the map $(\tilde{f})^{-1} : Y \to X/_\sim$ as a map of sets.

To show that $\tilde{f}$ is a homeomorphism, we have to show that $(\tilde{f})^{-1}$ is continuous.

But by the assumption on f, this map is continuous if and only if the composite

$$X \xrightarrow{f} Y \xrightarrow{(\tilde{f})^{-1}} X/_\sim$$

is continuous. We have to compute $(\tilde{f})^{-1}(f(x))$. The recipe for computing $(\tilde{f})^{-1}(y)$ is: "Find a $z \in X$ so that $y = f(z)$. Then $(\tilde{f})^{-1}(y) = \pi(z)$". We have to do this for $y = f(x)$. But by definition of $\tilde{f}$, we can chose $z = x$, which shows that

$$(\tilde{f})^{-1}(f(x)) = \pi(z) = \pi(x).$$

So, $(\tilde{f})^{-1} \circ f = \pi$, which is a continuous map. It follows from the assumption on f that $(\tilde{f})^{-1}$ is continuous, so that $\tilde{f}$ is a homeomorphism. □

Corollary 4.1.15. *If $q_2 : X_2 \to X_1$ and $q_1 : X_1 \to X_0$ are quotient maps, so is $q_1 \circ q_2$.*

Proof. A map $f : X_0 \to Y$ is continuous if and only if $f \circ q_1$ is, and this is continuous if and only if $f \circ q_1 \circ q_2$ is. □

Exercise 4.1.16. Prove the last corollary directly from the definition of quotient topology.

4.2 Gluing surfaces out of charts.

4.2.1. The following example might feel especially relevant for those who have been/will be exposed to an introduction to surfaces, like the one in [do Carmo].

Example 4.2.2. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Also, $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. We like this disc so much that we make six copies of it, labeling them $D_1, D_2, D_3, D_4, D_5, D_6$. Let

$$X = \coprod_{1 \leq i \leq 6} D_i.$$

That is, X is as a topological space the disjoint union of six discs. Now, consider the following six maps, $f_i : D_i \longrightarrow S^2$,

$$\begin{aligned}
f_1(x, y, z) &= (x, y, \sqrt{1 - x^2 - y^2}),\\
f_2(x, y, z) &= (x, y, -\sqrt{1 - x^2 - y^2}),\\
f_3(x, y, z) &= (x, \sqrt{1 - x^2 - z^2}, z),\\
f_4(x, y, z) &= (x, -\sqrt{1 - x^2 - z^2}, z),\\
f_5(x, y, z) &= (\sqrt{1 - y^2 - z^2}, y, z),\\
f_6(x, y, z) &= (-\sqrt{1 - y^2 - z^2}, y, z).
\end{aligned}$$

Exercise 4.2.3. Show that these maps combine to define a continuous, surjective map $f : X \to S^2$. Show that f is a quotient map. Conclude that a map $g : S^2 \to Y$ is continuous if and only if all the composites $g \circ f_i : D^2 \to Y$ are continuous.

The last example can be vastly extended.

Exercise 4.2.4. Let X be a topological Hausdorff space and I an index set, such that for each $\phi \in I$ there exists a topological space U_ϕ, and a continuous map $f_\phi : U_\phi \to X$. Assume that each $f_\phi : U_\phi \to f_\phi(U_\phi) \subset X$ is a homeomorphism to an open subset of X. Finally, assume that $X = \bigcup_\phi f_\phi(U_\phi)$. Show that these maps induce a continuous, surjective map

$$F : \coprod_\phi U_\phi \to X$$

which is a quotient map.

4.2.5. **??** Consider the special case where all the spaces U_ϕ are open subsets of the plane : $U_\phi \subset \mathbb{R}^2$. Then, a space X satisfying 4.2.4 is called a surface. The maps $f_\phi : U_\phi \to X$ are called charts on X. And the point of it is that the topology of X is completely described by those charts.

Examples of surfaces are the subspaces of $\mathbb{R}^3$, which are called surfaces in [do Carmo]. For the purely topological properties of surfaces, the above definition is more general than do Carmo's. In particular, the ambient Euclidean space $\mathbb{R}^3$ has completely evaporated from the picture!

One problem with the definition is that there are certain large, strange, but not very interesting spaces, which locally are homeomorphic to open subsets of a plane. (For an example of this, see for instance [Bredon] I.17.5) To exclude these from being honored by the name of surface, one usually includes the additional condition that X should satisfy the second axiom of countability.

Even if the study of surfaces usually involves properties (like curvature) that are not merely topological, this definition captures at least some of their properties.

Remark 4.2.6. If we more generally glue together open sets in $\mathbb{R}^n$ instead of open sets in $\mathbb{R}^2$ we obtain what is called an n-dimensional (topological) manifold.

Exercise 4.2.7. Find all homeomorphism classes of compact, 1-dimensional manifolds. (This exercise is probably too difficult to do formally! But at least try to guess the answer.)

Example 4.2.8. Let

$$I^2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1; 0 \leq y \leq 1\}.$$

We introduce an equivalence relation by requirering that $(x, 0) \sim (x, 1)$ and $(0, x) \sim (1, x)$. The quotient space is a torus $T = I^2/_\sim$. Se figure below.

Exercise 4.2.9. Let $a > r > 0$. Show that the map $f : T \to \mathbb{R}^3$ given by

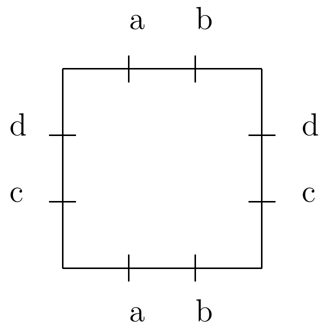
$$f(x, y) = ((a + r\cos(2\pi x))\cos(2\pi y), (a + r\cos(2\pi x))\sin(2\pi y), r\sin(2\pi x))$$

defines a homeomorphism $f : T \to f(T)$. Sketch the image $f(T)$.

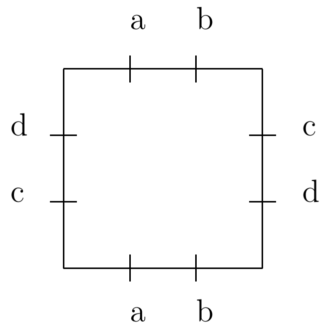
Exercise 4.2.10. Prove that the torus of 4.2.8 is a surface in the sense of **??**!

Exercise 4.2.11. We can define a different equivalence relation on the space I^2 from 4.2.8 as the equivalence relation generated by the requirement $(x, 0) \sim (x, 1)$ and $(0, x) \sim (1, 1 - x)$. The quotient space K is called a "Klein bottle". Prove that K is a surface in the sense of 4.2.4. Se figure below.

Remark 4.2.12. The Klein bottle is *not* homeomorphic to a subset of $\mathbb{R}^3$, so it is not a surface in the sense of [do Carmo].



Torus



Klein bottle

4.2.13. The posh way of defining new geometrical spaces from old is to glue them together in this fashion from old ones, using equivalences.

A particularly simple and useful case is the following:

Example 4.2.14. Let X and Y be topological spaces, $A \subset X$ be a subspace, and $f : A \to Y$ a continuous map. Then we form the space obtained from X by *attaching* Y along f as the quotient space

$$X \bigcup_{f:A\to Y} Y = \left(X \coprod Y\right) /_{\sim}$$

where the equivalence relation is generated by the requirement that $x \sim y$ if $x \in A \subset X$ and $f(x) = y$.

4.3 The importance of being Hausdorff.

It is nice to have a way of telling that a certain quotient space is Hausdorff.

Here is a necessary criterion:

Lemma 4.3.1. *Let X be compact, $\sim$ an equivalence relation on X, and assume that $X/_{\sim}$ is a compact space. If $A \subset X$ is a closed set, then the set $\pi^{-1}\pi(A)$ of all points equivalent to a point in A is a closed set.*

Proof. Since A is a closed subset of a compact space, it is itself a compact space. The image $\pi(A)$ of A in $X/_{\sim}$ is quasi-compact by 3.2.10, and since $X/_{\sim}$ is Hausdorff this implies that $\pi(A)$ is a closed subset by 3.2.9. But by the definition of the quotient topology, the closedness of this set is exactly the statement of the lemma. □

Remark 4.3.2. As you can see by taking complements, the condition in the lemma is exactly equivalent to the following condition. If $U \subset X$ is open, and

$$S(U) = \{x \in X \mid \text{ If } x \sim y \text{ then } y \in U\} = X \setminus \pi^{-1}\pi(X \setminus U)$$

is the union of all equivalence classes that are entirely contained in U, then S(U) is an open set. It also follows that $\pi(S(U)) \subset \pi(U)$ is an open set in $X/_\sim$, since $\pi^{-1}\pi(S(U)) = S(U)$.

Actually, lemma 4.3.1 has a converse.

Theorem 4.3.3. *Let X be a compact topological space, Y a set. Let $\pi : X \to Y$ be a surjective map of sets. We give Y the quotient topology. Suppose that for every closed set $A \in X$, the set $\pi^{-1}\pi(A)$ of all points equivalent to points in A is a closed set. Then, Y is a Hausdorff space.*

Proof. We use the reformulation in 4.3.2 of the assumption of the theorem into an assumption about open sets.

Let $a, b \in Y$. By assumption, the sets $\pi^{-1}(a)$ and $\pi^{-1}(b)$ are closed in X. Since X is compact, by 3.2.15 it is a normal space. That means that we can find disjoint open sets U, V in X, so that $\pi^{-1}(a) \subset U$ and $\pi^{-1}(b) \subset V$.

But following the notation in 4.3.2, $a \in \pi(S(U))$ and $b \in \pi(S(V))$. And these sets are disjoint open sets. So we have checked the Hausdorff property. $\square$

Exercise 4.3.4. Use 4.3.3 to show that the torus defined in 4.2.8 is a Hausdorff space.

Exercise 4.3.5. Show that if $A \subset X$ is compact subset of a Hausdorff space, and $f : A \to Y$ is a continuous map to a Hausdorff space, then the space we get by attaching Y to X by means of f (as in 4.2.14) is a Hausdorff space.

Example 4.3.6. Let X be the union of the closed intervals $[0, 1]$ and $[2, 3]$. Let $\sim$ be the equivalence relation generated by requirering that $x \sim x + 2$ unless $x = \frac{1}{2}$.

- Show that every point of $X/_\sim$ has an open neighbourhood homeomorphic to an open interval.
- Show that X is not a Hausdorff space.

4.4 Compatibility of quotient topology with products.

4.4.1. It is sometimes not so easy to combine quotient constructions with other constructions.

For instance, if $Y \subset X$, and $\sim$ is an equivalence relation, we can give $Y/_\sim$ a topology in two different ways.

We can either give $Y/_\sim$ the subspace topology from $X/_\sim$. Let us denote $\tilde{Y} = Y/_\sim$ with this topology by $\tilde{Y}_{\text{ss}}$. We could also give it the quotient topology from the equivalence relation $\sim$ restricted to Y. Let us denote $\tilde{Y}$ with this topology by $\tilde{Y}_{\text{qt}}$.

Since $Y \to X \to X/_\sim$ is continuous, the map $\tilde{Y}_{\text{qt}} \to X/_\sim$ is continuous. So by the definition of the subspace topology, the "identity" map $\tilde{Y}_{\text{qt}} \to \tilde{Y}_{\text{ss}}$ is continuous. But it is not necessarily a homeomorphism! That is, the two topologies might not agree.

Exercise 4.4.2. Recall the definitions of 4.1.7. Let Y be the half-open interval $Y = \{x \mid 0 \leq x < 1\}$. Then $Y \subset X = [0, 1]$ is an open subset, and it has the subspace topology. The map f restricted to Y is bijective. Is it also a quotient map? Realize that this gives an example of a situation where the map $\tilde{Y}_{\text{qt}} \to \tilde{Y}_{\text{ss}}$ is *not* a homeomorphism. (Cf. Remark 4.1.13.)

4.4.3. The situation with respect to products is also not too satisfactory: If X, Y and Z are topological spaces and $\pi : X \longrightarrow Y$ is a quotient map, then $\pi \times id_Z : X \times Z \longrightarrow Y \times Z$ is not necessarily a quotient map. We will give an example of this in 4.4.7 below, but let us first show some results in a positive direction. Once you have understood them, you will probably realize that one has to be rather inventive to cook up a counter-example.

One comparatively easy result, which we will use later, is the following:

Theorem 4.4.4. *Let $\pi : X \to Y$ be an open quotient map, and Z a topological space. Then, the product map*

$$\rho = \pi \times \text{id}_Z : X \times Z \to Y \times Z$$

is also an open quotient map.

Proof. We first check that the map is open. But this is really obvious, since it is enough to show that if $U_X \subset X$ and $U_Z \subset Z$ are open sets, then $\rho(U_X \times U_Z)$ is an open set. But $\rho(U_X \times U_Z) = \pi(U_X) \times U_Z$ is open.

To prove that ρ is a quotient map, we have to prove that a subset $A \subset Y \times Z$ is open if and only if $\rho^{-1}(A) \subset X \times Z$ is open. Since ρ is continuous, if A is open, so is $\rho^{-1}(A)$, and we only have to prove that if $\rho^{-1}(A)$ is open, so is A.

Let $(y, z) \in A \subset Y \times Z$. Since π is a quotient map, it is surjective, and we can pick an $x \in X$ so that $\pi(x) = y$. Assume that $\rho^{-1}(A)$ is open. We claim that we can find an open neighborhood of (y, z) which is contained in A. There are open subsets $U_X \subset X$ and $U_Z \subset Z$ so that $(x, z) \in U_X \times U_Z \subset \pi^{-1}(A)$. Since π is an open map, $\pi(U_X) \times Z$ is an open set in $Y \times Z$. And $(y, z) = (\pi(x), y) \in \pi(U_X) \times Z \subset A$. □

A more difficult result is the following lemma, which is useful in applications of the theory to geometrical situations.

Lemma 4.4.5. *Let $\pi : X \to Y$ be a quotient map, and let Z be a locally compact space. Then the product map*

$$\pi \times \mathrm{id}_Z : X \times Z \to Y \times Z$$

is also quotient map.

Proof. The criterion for showing this is that a map $g : Y \times Z \to W$ is continuous if and only if the composite

$$f : X \times Z \xrightarrow{\pi \times \mathrm{id}_Z} Y \times Z \xrightarrow{g} W$$

is continuous.

If g is continuous, f is continuous as the composition of continuous functions.

Now assume that f is continuous. We need to show that g is continuous. We do this by showing that it is continuous at an arbitrary $(y_0, z_0) \in Y \times Z$.

Let $V \subset W$ be an open subset, such that $g(y_0, z_0) \in V$. We must show that $g^{-1}(V)$ is a neighbourhood of (y_0, z_0).

Chose $x_0 \in X$ so that $y_0 = \pi(x_0)$. By the assumption that f is continuous, there are open sets $U_X \subset X$ and $U_Z \subset Z$ so that $(x_0, z_0) \subset U_X \times U_Z$, and $f(U_X \times U_Z) \subset V$.

Since we are assuming that Z is locally compact, then by Exercise 3.4.6 z_0 has a compact neighbourhood K_Z contained in U_Z, i.e. $z_0 \in K_Z^0 \subset K_Z \subset U_Z$.

Let

$$A = \{y \in Y \mid g(y \times K_Z) \subset V\}.$$

Then $(y_0, z_0) \in A \times K_Z^0 \subset A \times K_Z \subset g^{-1}(V)$, so if we can show that A is open in Y, then $g^{-1}(V)$ will be a neighbourhood of (y_0, z_0) as desired. The rest of the proof is devoted to checking this.

First reduction. Since $\pi : X \to Y$ is a quotient map, $A \subset Y$ is open if and only if the subset $\pi^{-1}(A) \subset X$ is open. We compute this set, in the hope that it will turn out to be so:

$$\pi^{-1}(A) = \{x \in X \mid g(\pi(x) \times K_Z) \subset V\} = \{x \in X \mid x \times K_Z \subset f^{-1}(V)\}.$$

It is not yet visible to the naked eye whether this set is open or not. But we know that it is open if and only if its complement is a closed set.

Second reduction. The complement $X \setminus \pi^{-1}(A)$ consists of points $x \in X$ with the property that there exists a $z \in K_Z$ such that $f(x, z) =\notin V$. That is,

$$\begin{aligned} X \setminus \pi^{-1}(A) &= \{x \in X \mid \exists z \in K_Z \text{ so that } (x, z) \notin f^{-1}(V)\} \\ &= pr_X(X \times K_Z \setminus f^{-1}(V)), \end{aligned}$$

where $pr_X : X \times K_Z \longrightarrow X$ denotes the projection on the first factor. Since K_Z is compact, this is a closed map by 3.3.6. And since $X \times K_Z \setminus f^{-1}(V)$ is a closed subset of $X \times K_Z$, this implies that $X \setminus \pi^{-1}(A)$ is closed in X. This is what we needed to show.

□

4.4.6. The two previous results show that we have been rather unfortunate if the product of a quotient map $\pi : X \longrightarrow Y$ with an identity map $id_Z : Z \longrightarrow Z$ turns out not to be a quotient map: in such a situation the quotient map π cannot be open, and Z is not locally compact. But, as the following exercise shows, there do exist examples of this phenomena.

Exercise 4.4.7. Let $\mathbb{Q}_{>0}$ denote the positive rational numbers. Let $\sim_1$ be the equivalence relation on the numbers generated by the requirement that $x \sim_1 y$ if both x and y are integers. Let $\mathbb{Q}_{>0}/_{\sim_1}$ be the quotient space, with quotient map $\pi_1 : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}/_{\sim_1}$. In the same way, consider the equivalence relation $\sim_2$ on $\mathbb{Q}_{>0} \times \mathbb{Q}$, generated by requirering that $(x, y) \sim_2 (u, y)$ if both x and u are integers. Denote the corresponding quotient map by $\pi_2 : \mathbb{Q}_{>0} \times \mathbb{Q} \to (\mathbb{Q}_{>0} \times \mathbb{Q})/_{\sim_2}$.

1. Show that there is unique map (of sets)

$$f : (\mathbb{Q}_{>0} \times \mathbb{Q})/_{\sim_2} \to \mathbb{Q}_{>0}/_{\sim_1} \times \mathbb{Q}$$

   with the property that $f(\pi_2(x, y)) = (\pi_1(x), y)$. Show that f defines a continuous, bijective map.

2. For each natural number $n \geq 1$, we define

$$U_n = \{(x,y) \in \mathbb{Q}_{>0} \times \mathbb{Q} \big| \; |x-n| < \min\{|y - \frac{\sqrt{2}}{n}|, \frac{1}{2}\} \; \},$$

and we put

$$U = \bigcup_{n \geq 1} U_n.$$

Show that $\pi_2(U)$ is open. (Hint: Use that $\sqrt{2}$ is irrational to prove that $\pi_2^{-1}\pi_2(U) = U$.)

3. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Assume that $\frac{\sqrt{2}}{n} < \epsilon$. Prove that there are no $a < n < b$ so that

$$\{(x,y) \in \mathbb{Q}_{>0} \times \mathbb{Q} \big| a < x < b \text{ and } -\epsilon < y < \epsilon\} \subset U_n.$$

4. Show that $f(\pi_2(U))$ is not a neighborhood of the element $(\pi_1(1), 0)$. In particular, the set is not open.

5. Conclude that f is not a homeomorphism.

Chapter 5

Topological groups

Sometimes, a topological space has a group structure. If the algebraic operations defining the group structure are continuous, we say that the group is a topological group.

5.1 Definition. The group $GL_n(\mathbb{R})$.

Definition 5.1.1. A Hausdorff space G is a *topological group* if there are continuous maps

$$\mu : G \times G \to G$$
$$\chi : G \to G$$

and an element $e \in G$, so that μ defines an associative multiplication with e as unit element, and so that $\mu(x, \chi(x)) = e$. As usual, we write $gh := \mu(g, h)$ and $g^{-1} := \chi(g)$.

Just like the guy who had been speaking in prose all his life without knowing it, you already know many example of topological groups.

Exercise 5.1.2. Addition makes $\mathbb{R}$ into a topological group with unit 0.

Exercise 5.1.3. Multiplication makes $\mathbb{R} \setminus \{0\}$ into a topological group with unit 1.

Exercise 5.1.4. Show that a topological Hausdorff space G with a group multiplication is a topological group if and only if the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is continuous.

More fancy examples are matrix groups.

Example 5.1.5. Recall that $GL_n(\mathbb{R})$ is the set of all invertible $n \times n$-matrices with entries in $\mathbb{R}$. Matrix multiplication makes $GL_n(\mathbb{R})$ into a topological group. There are things to check! For instance, we have to say in which way we define the topology on $GL_n(\mathbb{R})$. To specify this topology, we first put a topology and a metric on the space of all matrices, invertible or not.

We can identify the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices with an Euclidean space of dimension n^2, simply by sending a matrix to the vector consisting of a list of its entries.

Let us temporarily call this isomorphism of vector spaces $L_n : M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$. This makes $M_n(\mathbb{R})$ into a vector space with inner product

$$< A, B >_{M_n(\mathbb{R})} = < L_n(A), L_n(B) >_{\mathbb{R}^{n^2}} .$$

Using the inner product we make $M_n(\mathbb{R})$ into a normed vector space.

Exercise 5.1.6. The following fact has nothing to do with topology, but it is cute: Let A^T be the transpose of A. Show that

$$< A, B >_{M_n(\mathbb{R})} = \mathrm{Tr}(A^T B).$$

5.1.7. We give $GL_n(\mathbb{R})$ the induced metric, and thus the induced topology from its inclusion in the normed space of all matrices:

$$GL_n(\mathbb{R}) \overset{i}{\subset} M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

The composed map $g = f \circ i : GL_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ has n^2 components, g_{ij}. Each of these components gives a particular entry in the matrix we feed it with. That is, it is given as $A \mapsto g_{ij}(A) = A_{ij}$ for fixed i and j. In particular, by its very definition, the topology on $GL_n(\mathbb{R})$ has the property that the map $g_{ij} : GL_n(\mathbb{R}) \to \mathbb{R}$ taking a matrix to its (i, j)'th entry is a continuous map.

5.1.8. Armed with this remark, we go to check that multiplication is also continuous. That is, we have to see that the composite map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \xrightarrow{\mu} GL_n(\mathbb{R}) \to \mathbb{R}^{n^2}$$

is continuous. Fortunately, multiplication extends to matrix multiplication, and we have a commutative diagram of maps:

$$\begin{array}{ccc} GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) & \xrightarrow{\mu} & GL_n(\mathbb{R}) \\ {\scriptstyle (i,i)}\downarrow & & {\scriptstyle i}\downarrow \\ M_n(\mathbb{R}) \times M_n(\mathbb{R}) & \xrightarrow{\mu_M} & M_n(\mathbb{R}). \end{array}$$

So it is enough to show that matrix multiplication μ_M is a continuous map. But this is true, since we know that multiplication is given by formulas entirely consisting of polynomials in the matrix entries. For instance, in dimension 2 we have the well known formula

$$\mu((a_{11}, a_{12}, a_{21}, a_{22}), (b_{11}, b_{12}, b_{21}, b_{22})) = (a_{11}b_{11} + a_{12}b_{21}, \dots).$$

This proves that the multiplication is continuous. However, this is not enough. We also need to prove that the inverse is continuous.

5.1.9. In the same way as before, it suffices to prove that the composite

$$GL_n(\mathbb{R}) \xrightarrow{\chi} GL_n(\mathbb{R}) \xrightarrow{i} M_n(\mathbb{R})$$

is continuous. Unfortunately, this time the map does not extend to the space of all matrices. (Why?)

5.1.10. We do remember that there is a formula for computing the inverse of a matrix. The entries in the inverse matrix g^{-1} are given by polynomials in the entries in g, divided by the determinant $\det(g)$ of g. Those polynomials on the entries are continuous functions of g, and so is the determinant. But since the determinant of g cannot be 0, the inverse of the determinant is also a continuous function:

$$GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R} \setminus \{0\} \xrightarrow{1/x} \mathbb{R}.$$

This makes each entry of $\chi(g)$ into a continuous function of g, and proves that matrix inversion χ is continuous.

There are many other topological groups!

Exercise 5.1.11. Prove that if you give the orthogonal matrices $O_n \subset GL_n(\mathbb{R})$ the subset topology, then O_n with the matrix multiplication is a topological group. (Recall that an $n \times n$-matrix A is in O_n if $A^T A = I$.)

5.2 Quotient groups.

5.2.1. If we are dealing with groups, one thing we often want to do is to form quotient groups. And we want to be able to say that if $H \subset G$ is a normal subgroup, then G/H is a topological group.

At least it is clear what the topology should be : G/H is defined as the set of equivalence classes of elements in G under the equivalence relation $\sim$, where $g \sim gh$ for $h \in H$. So we give G/H the quotient topology, as in 4.1.2.

Remark 5.2.2. The quotient map $\pi : G \longrightarrow G/H$ has a special property: It is an open map. That is, if $U \subset G$ is open, so is $\pi(U) \subset G/H$. This is true because $\pi^{-1}\pi(U) = \bigcup_{h \subset H} Uh$ is a union of open sets. (Here we use that the map $\mu(-, h)$ is a homeomorphism for any h.)

Exercise 5.2.3. Let $H \subset G$ be an open subgroup. Prove that H is also closed.

Lemma 5.2.4. *G/H is a Hausdorff space exactly if H is a closed subgroup.*

Proof. If G/H is Hausdorff, the set consisting only of the point $\pi(e)$ is closed. So, by definition of the quotient topology, $H = \pi^{-1}(e) \subset G$ is closed.

In the other direction, suppose that H is closed. Then we consider the continuous map

$$f : G \times G \to G$$
$$f(g, h) = g^{-1}h.$$

Since H is closed, the inverse image of its complement $f^{-1}(G \setminus H)$ is open. Suppose that $g_1, g_2 \in G$ define two distinct points $\pi(g_1)$ and $\pi(g_2)$ in G/H. This amounts to saying that $g_1^{-1}g_2 = f(g_1, g_2) \notin H$. Or equivalently, (g_1, g_2) belongs to the open subset $f^{-1}(G \setminus H)$ of $G \times G$.

By the definition of the product topology we can find open sets U_1 and U_2 in G such that $g_1 \in U_1$, $g_2 \in U_2$ and $f(U_1 \times U_2) \cap H = \emptyset$. But the last condition immediately translates into $\pi(U_1) \cap \pi(U_2) = \emptyset$. The $\pi(U_i)$ are open sets because of 5.2.2, so we can use them as the open sets asked for by the Hausdorff property. □

5.2.5. Let us now consider the case where $H \subset G$ is a *normal* subgroup. Then G/H inherits a group structure from G. We definitely want this quotient group to be a Hausdorff space, so we now only consider *closed* normal subgroups $H \subset G$, even though this is not strictly necessary.

5.2.6. To show that G/H is a topological group, we need to show that the multiplication and the inverse are continuous maps.

It is easy to show that the inverse is continuous since $G \to G/H$ is a quotient map. So to check that $\chi_{G/H} : G/H \to G/H$ is continuous, we have to check that the composite $\pi \circ \chi_{G/H} : G \to G/H$ is continuous. But this map is also the composition $\chi_G \circ \pi$ which is continuous, being the composition of continuous maps.

The key to the continuity of the group composition is the following

Lemma 5.2.7. *Let G be a topological group, and H a closed, normal subgroup. The map*

$$\pi \times \pi : G \times G \to G/H \times G/H$$

is a quotient map.

Proof. It follows from 5.2.2 that the map

$$\pi : G \to G/H$$

is an open quotient map. By 4.4.4,

$$\mathrm{id}_G \times \pi : G \times G \to G \times G/H$$

is also an open a quotient map. Apply the same argument once more to obtain that

$$\pi \times \mathrm{id}_{G/H} : G \times G/H \to G/H \times G/H$$

is also an open quotient map. And the composition of quotient maps is a quotient map by 4.1.15. □

Theorem 5.2.8. *Let G be a compact topological group, and H a closed normal subgroup. Then G/H is a topological group.*

Proof. We have to check that product $\mu_{G/H} : G/H \times G/H \longrightarrow G/H$ is continuous. But because of the previous lemma, we only have to show that

$$\pi \times \pi : G \times G \to G/H \times G/H \stackrel{\mu_{G/H}}{\to} G/H$$

is continuous. This follows from the following commutative diagram of maps.

$$\begin{array}{ccc} G \times G & \stackrel{\mu}{\longrightarrow} & G \\ {\scriptstyle \pi\times\pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ G/H \times G/H & \stackrel{\mu_{G/H}}{\longrightarrow} & G/H. \end{array}$$

□

Chapter 6

Connectedness and countable product of finite sets.

6.1 Connectedness

6.1.1. Connectedness is a simple property, which tells you whether a space X consists of one single piece.

Definition 6.1.2. A topological space X is *connected* if it is *not* a union of two non-empty, open, disjoint subsets.

Since the complement of an open set is closed, this is equivalent to saying that the only subsets of X which are both open and closed are X and $\emptyset$.

Example 6.1.3. A set with the discrete topology is not connected - unless it is empty or consists of only one point.

The fundamental example of a connected space is the following:

Lemma 6.1.4. *The unit interval* $[0, 1]$ *is connected.*

Proof. Assume that $[0, 1] = U \cup V$, where U and V are disjoint open sets. Then they are also closed sets. Without restriction on the assumption, $0 \in U$. Let $s = \inf\{x \in V\}$. This is some number, $0 \leq s \leq 1$. By its definition, in every neighborhood of s, there is some element of V (This element could be s itself). Since V is closed, $s \in V$. In particular, $s > 0$.

But every number less than s is not in V, since s is the infimum. So these points have to be in U. Also, since $s > 0$, we have an entire half open inteval $[0, s) \subset U$. Since U is closed, $s \in U$.

This contradicts that $U \cap V = \emptyset$. □

Exercise 6.1.5. Prove that an open interval $(a,b) \subset \mathbb{R}$ is connected.

Exercise 6.1.6. Prove that the unit interval with one point removed $[0,1] \setminus \{\frac{1}{2}\}$ is not connected.

Exercise 6.1.7. Show that if $a \in [0,1] \times [0,1] \subset \mathbb{R}^2$, the space $[0,1] \times [0,1] \setminus \{a\}$ is still connected. Conclude from this and 6.1.6 that $[0,1]$ and $[0,1] \times [0,1]$ are not homeomorphic (as if anyone ever suspected they would be...).

Definition 6.1.8. A map $f : X \to Y$ from a topological space X to a set Y is locally constant if every $x \in X$ has an open neighborhood U in X, so that f is constant on U.

This is equivalent to demanding that f is continuous when Y is equipped with the discrete topology.

One important use of connectedness is the following principle:

Remark 6.1.9 *(The principle of connectedness).* Let $f : X \to Y$ be a map from a *connected* topological space X to a set Y. If f is locally constant, then f is constant.

In fact, for each $y \in Y$, the set $f^{-1}(y)$ is open. So is $f^{-1}(Y \setminus \{y\}) = \bigcup_{z \neq y} f^{-1}(z)$. Since these two sets are open, disjoint and with union X, one of them has to be empty. That is, if $f(x) = a$ for one single $x \in X$, then $f(x) = a$ for all $x \in X$. In other words, f is constant on the space X.

Exercise 6.1.10. Let U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a differentiable function. Assume that for every partial derivative, $\frac{\partial f}{\partial x_i} = 0$.

- Prove that f is constant if U is connected.
- Give an example of this situation where f is not constant on U.

There is a stronger property, which is often of interest

Definition 6.1.11. A topological space X is path connected if for every pair of points $a, b \in X$ there is a map $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$.

A continuous map $f : [0,1] \to X$ is called a path from $f(0)$ to $f(1)$ in X.

Remark 6.1.12. If there is a path f from a to b and a path g from b to c, the map $h : [0,1] \to X$

$$h(x) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path from a to c. We have to check that h is continuous. h is obviously continuous at all points except possibly at $\frac{1}{2}$. We have to prove that for an open U such that $f(\frac{1}{2}) \in U$, the set $h^{-1}(U)$ contains an open interval around $\frac{1}{2}$.

To do this, we argue that by the continuity of f and g it contains half open intervals $(f(\frac{1}{2}) - \epsilon, f(\frac{1}{2})]$ and $[f(\frac{1}{2}), f(\frac{1}{2}) + \epsilon)$. That is, since h is left and right continuous at $\frac{1}{2}$, it is continuous.

Exercise 6.1.13. For $a, b \in X$ we define that $a \sim b$ if there is a path from a to b. Prove that this defines an equivalence relation.

Lemma 6.1.14. *A path connected space is connected.*

Proof. Assume that X is path connected. Assume that $X = U \cup V$, where U and V are two disjoint, non empty, open subsets. Pick $a \in U$ and $b \in V$. There is a path f from a to b as in 6.1.11. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open, disjoint subsets of $[0, 1]$. This contradicts lemma 6.1.4. □

In general, the two notions are not equivalent. But if you have some assumptions on the local structure of X they might be. Here is an example:

Lemma 6.1.15. *An open subset $U \subset \mathbb{R}^n$ is connected if and only if it is path connected.*

Proof. In view of lemma 6.1.14 we only have to show that if U is connected, then it is also path connected.

Let $a, b \in U$ be two points in U. We must show that there exists a path in U from a to b. For this, we use the principle of connectedness.

Let Y be the set $\{0, 1\}$. Let $f : U \to Y$ be the function defined by

$$f(x) = \begin{cases} 0 & \text{if there is a path in } U \text{ from } a \text{ to } x \\ 1 & \text{if there is no path in } U \text{ from } a \text{ to } x. \end{cases}$$

If x and y are in the same open ball, they can be connected by a linear path $f(t) = tx + (1 - t)y$. So, using 6.1.12, we see that either both x and y or neither of them can be connected to a by a continuous path.

This translates exactly to that f is locally constant. Using remark 6.1.9, we see that f is constant on U. This allow us to conclude that $f(b) = f(a) = 0$, so there is indeed a path in U from a to b. □

Exercise 6.1.16. Let X be a connected topological space, and $f : X \to Y$ be a continuous surjective map. Show that Y is connected.

Exercise 6.1.17. Let X, Y be connected spaces. Show that $X \times Y$ is connected.

6.2 The Cantor set

6.2.1. If a Hausdorff topological space has only finitely many points, then it is a discrete space. Surprisingly, if you form the product of infinitely (but countably) many of these discrete spaces, we get a topological space with an interesting topology.

Definition 6.2.2.

$$C_p \cong \prod_{i=1}^{\infty} X_i.$$

Each X_i is a set with p elements.

Here is another manifestation of the infinite product C_2.

Example 6.2.3. The Cantor set is a closed subset of the closed unit interval $[0, 1]$. It can be defined in various ways. Here is a simple way of doing it:

$$C = \left\{ x \in [0,1] \mid x = \sum_{i=1}^{i=\infty} \frac{a_i}{3^i}, \text{where each } a_i \text{ either equals 0 or 2.} \right\}$$

Exercise 6.2.4. A more complicated way of defining the Cantor set is the following. Let $[0, 1]$ be the closed interval. We remove infinitely many open intervals from it as follows.

The first interval we remove is the open middle third of the original interval, that is $(\frac{1}{3}, \frac{2}{3})$. Then two closed intervals $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$ remain. From each of these, we remove the open middle third. What is left is the union of four closed intervals, and we continue in the same way, removing open middle intervals forever.

Show that the closed set that remains after we have finished doing this interval removing infinitely many times is the Cantor set.

Exercise 6.2.5. Show that there are points in C which are not end points of intervals in $[0, 1] \setminus C$.

Lemma 6.2.6. *A point $x \in C$ determines the coefficients a_i uniquely. This defines for all i a map $a_i : C \longrightarrow \{0, 2\}$. This map is continuous.*

Proof. Assume that $x, y \in C$. Let us write

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \qquad y = \sum_{i=1}^{\infty} \frac{b_i}{3^i}.$$

Claim:

$$\text{If } |x-y| < 3^{-n}, \text{ then } a_i = b_i \text{ for } i \leq n. \tag{6.1}$$

We compute the distance in terms of the a_i and b_i:

$$|x-y| = |\sum_{i=1}^{\infty} \frac{a_i - b_i}{3^i}|$$

Let us assume that x and y give a counterexample to the statement. That is, $|x-y| < 3^{-n}$ but $a_m \neq b_m$ for some $m \leq n$. Without restriction, we can assume that $a_m = 2$ and $b_m = 0$. We can also as well assume m is minimal with this property. Stated in formulas: $a_i = b_i$ for $i < m$. To arrive at a contradiction, we compute the distance from x to y using the previous formula.

$$|x-y| = |\frac{2}{3^m} + \sum_{i=m+1}^{\infty} \frac{a_i - b_i}{3^i}| \geq \frac{2}{3^m} - (\sum_{i=m+1}^{\infty} \frac{2}{3^i}) = \frac{1}{3^m} \geq \frac{1}{3^n}$$

which is in contradiction to the assumption. This proves our claim.

It immediately follows from the claim that $x \in C$ determines a_i.

To show that a_n defines a continuous map, we resort to an ϵ and δ argument. Given $\epsilon > 0$, choose $\delta = 3^{-n}$ (Yes, that *is* independent of ϵ). The inequality 6.1 shows that if $|x-y| < \delta$, then $a_n(x) - a_n(y) = 0 < \epsilon$. □

6.2.7. We conclude that we have a continuous map

$$A = \prod_{i=0}^{\infty} a_i : C \to \prod_{i=1}^{\infty} \{0, 2\} = C_2.$$

This map is surjective, since a given sequence $\{a_i\}$ is equal to $A(x)$, where $x = \sum_{i=1}^{i=\infty} \frac{a_i}{3^i}$. It is injective by 6.2.6.

Theorem 6.2.8. *The map A is a homeomorphism*

Proof. Since C is a closed bounded set in $\mathbb{R}$, it is compact. Because A is a continuous bijective map from a compact space to a Hausdorff space, it is a homeomorphism. □

Corollary 6.2.9. *The topology of the space C_2 is induced from a metric.*

Exercise 6.2.10. Show that the Cantor set does not have the discrete topology.

Exercise 6.2.11. Show that given two distinct points $a, b \in C$, we can always find two disjoint open sets U, V so that $C = U \cup V$, and $a \in U$, $b \in V$. Conclude that any continuous map from a connected space into C is constant.

6.3 Space filling curves

6.3.1. The homeomorphism of 6.2.8 can be used in a surprising way to construct a so called space filling curve. We will construct a surjective, continuous map from the interval to the square. The existence of such maps should be viewed with scepticism. They hint that the concept of continuity might be more liberal than what our intuition assumes.

We will need the following remarks

Lemma 6.3.2. *Let $A \subset \mathbb{R}^1$ be a closed subset. The complement $U = \mathbb{R}^1 \setminus A$ is the union $U = \cup_i (a_i, b_i)$ of disjoint, open intervals where each a_i is either $-\infty$ or in A, each b_i is either ∞ or in A.*

Proof. It is enough to show that any $x \in U$ is contained in some such interval $x \in (a, b) \subset U$, where $a \in A \cup \{-\infty\}$ and $b \in A \cup \{\infty\}$, since any two intervals have to be disjoint. Let a be the infimum of the set $\{y \mid (y, x) \subset U\}$ and b the supremum of the set $\{z \mid (x, z) \subset U\}$. Then $(a, b) \subset U$, and $a \notin U$, $b \notin U$. The lemma follows. □

Lemma 6.3.3. *Let X be a topological space, and let $f_n : X \to \mathbb{R}$ be a sequence of continuous maps. Suppose that there exists a convergent series $\sum_{n=0}^{\infty} a_n$ of non-negative numbers such that $|f_n(x)| \leq a_n$ for all $x \in X$. Then*

$$\sum_{n=0}^{\infty} f_n(x)$$

converges to a continuous function $F(x)$.

Proof. For each $x \in X$ the sum converges by the comparison criterion for infinite sums. So the function $F(x)$ is defined, and the only issue is whether it is continuous. To check continuity at x, given an ϵ, we first find N so that $\sum_{n=N+1}^{\infty} a_n < \frac{\epsilon}{3}$. Let $F_N(x) = \sum_{n=0}^{N} f_n(x)$. By the choice of N, $|F(y) - F_N(y)| < \frac{\epsilon}{3}$ for all $y \in X$. Moreover, F_N is continuous, so there is a neighbourhood $U \subset X$ of x such that $|F_N(y) - F_N(x)| < \frac{\epsilon}{3}$ for all $y \in U$. But then, for $y \in U$, we have the estimate

$$\begin{aligned} |F(x) - F(y)| &\leq |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that F is continuous at x for any $x \in X$. □

Theorem 6.3.4. *There exists a continuous, surjective map*

$$f : [0, 1] \to [0, 1] \times [0, 1].$$

Proof. We first show that there is a continuous, surjective map $g : C \to [0,1]$. The map can be written explicitly as

$$g(x) = g\left(\sum_{i=1}^{\infty} \frac{a_i(x)}{3^i}\right) = \sum_{i=1}^{\infty} \frac{a_i(x)}{2(2^i)}.$$

This map is clearly surjective, since every number has a binary 'decimal' expansion.

Here is a formal proof of the last statement: Let $0 \leq x \leq 1$. For each $n \in \mathbb{N}$, let x_n be the biggest number of the form $x_n = \frac{y_n}{2^n}$ such that y_n is an integer, and $x_n \leq x$. There are uniquely determined choices of numbers $a_i(n)$, each one either equalling 0 or 2, so that you can write $x_n = \sum_{i=1}^{n} \frac{a_i(n)}{2(2^i)}$.

The numbers x_n form a nondecreasing sequence, converging towards x. Moreover, $a_i(n) = a_j(n)$ for $i, j \leq n$. Put $a_i = a_i(n)$ for $n \geq i$. Then $x = \sum_{i=1}^{\infty} \frac{a_i}{2(2^i)}$.

The map g is continuous by 6.3.3.

There is also a homeomorphism $\Delta : C \to C \times C$. This homeomorphism is easiest to establish if we replace C by C_2, which after all is homeomorphic to C by 6.2.8. We define a map $C_2 \to C_2 \times C_2$ by

$$(a_1, a_2, \dots) \mapsto ((a_1, a_3, a_5, \dots), (a_2, a_4, a_6, \dots))$$

according to the principle of the zipper. This is clearly bijective, and by applying 2.6.3 one may check that both the map itself and its inverse are continous. This way we obtain a homeomorphism $C_2 \to C_2 \times C_2$ and hence a homeomorphism $\Delta : C \to C \times C$.

By composition we obtain a surjective, continuous map

$$f_C : C \overset{\Delta}{\to} C \times C \overset{g \times g}{\to} [0,1] \times [0,1].$$

Finally, we claim that there is an extension of this map from C to the interval $[0,1]$. That is, there is a map $f : [0,1] \to [0,1] \times [0,1]$ so that the restriction of f to the Cantor set agrees with f_C. Note that it does not matter what this extension is, as long as it is continuous and agrees with f on the subset $C \subset [0,1]$.

In general, it is often a difficult or even impossible task to extend a continuous functions defined on a subset of a space to a continuous function defined on the entire space. But in this very special case we can do the extension easily.

The point is that since C is a closed subset of an interval, the complement $[0,1] \setminus C$ is a union of disjoint open intervals (6.3.2). So to extend the map $f_C : C \to [0,1] \times [0,1]$ we have to extend it on each such interval $[p, q]$, when it is already defined on the endpoints p and q. We can do this linearly, since the square we are mapping to is a convex set. For $0 \leq \lambda \leq 1$ we define:

$$f(\lambda p + (1-\lambda)q) := \lambda f(p) + (1-\lambda)f(q) \subset [0,1] \times [0,1]$$

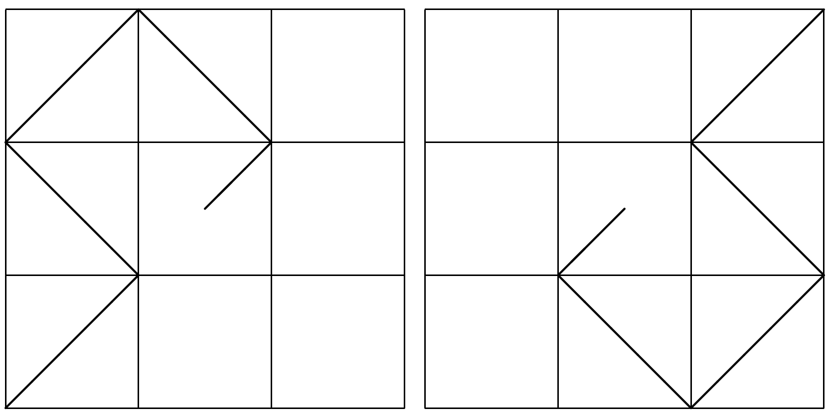
This gives an extension of $f_C : C \to [0,1] \times [0,1]$ to a continuous surjective map $f : [0,1] \to [0,1] \times [0,1]$. This map is continuous at every point of the open intervals (since it is linear in a neighborhood). By the same argument, f is left continuous at any point $q \in C$ which is a right endpoint of one of those intervals. (Left continuity at q of course means that for any open neighbourhood V of $f(q)$ there exists a neighbourhood U of q such that $f(y) \subset V$ for all $y \in U$ with $y \leq q$.) To prove that f is left continuous everywhere on $[0,1]$, it remains to check the left continuity of f at points $x \in C$ which are not right endpoints of intervals. At such a point x, any nontrivial interval U around x contains points in C smaller than x. Given an open interval $V \in [0,1]$ so that $f(x) \in V$, we can find an open interval $U \subset [0,1]$ such that $U \cap C \subset f_C^{-1}(V)$ by continuity of f_C. Pick $y \in C$ so that $y < x$ and $y \in U$. Then, $(y,x] \cap U \subset f^{-1}(V)$ since V is convex. This shows that f is left continuous at x.

We prove right continuity in the same way, and the continuity of f follows.

□

Exercise 6.3.5. There are other ways to construct such space filling curves. Try to complete the following outline to an inductive construction:

- Let $f_1 : [0,1] \to [0,1]^2$ be the diagonal map $f_1(x) = (x,x)$.
- Let $f_2 : [0,1] \to [0,1]^2$ be a map that is linear on every interval $[\frac{a}{9}, \frac{a+1}{9}]$ for an integer a. The image of the first and second half of the interval is specified by the following drawings:



$f([0,\frac{1}{2}])$ $f([\frac{1}{2},1])$

The curve passes diagonally through every of the nine small squares exactly once.

- Let $f_3(x) = f_2(x)$ whenever $x = \frac{a}{9}$ for some integer a. On each interval $[\frac{a}{9}, \frac{a+1}{9}]$, we change f_2, in the same way as we changed f_1 to f_2. That is, where f_1 ran through the diagonal of each $\frac{1}{3} \times \frac{1}{3}$ square once, f_3 takes a longer road, passing through each $\frac{1}{9} \times \frac{1}{9}$ square exactly once. The map f_3 will be linear on every interval $[\frac{a}{81}, \frac{a+1}{81}]$ for integers a.

- Continue this procedure. This is the hard part if you want to do it in details! Show that f_i converges uniformly towards a function f.

- Prove that f is continuous and surjective.

Exercise 6.3.6. Show that there exists a continuous, surjective map $f : \mathbb{R} \to \mathbb{R}^2$.

Exercise 6.3.7. Show that there exists a continuous, surjective map $[0, 1] \to [0, 1]^n$.

6.3.8. The existence of space filling curves leads to the question whether there are homeomorphisms $[0, 1] \to [0, 1]^2$, or perhaps homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ for $n \neq m$.

Such homeomorphisms intuitively seem very implausible, but after you have accepted that space filling curves exist, you realise that without a proof one cannot be sure that there are no such things.

Actually, if one could modify the map we just constructed so that it were bijective instead of merely surjective, the usual trick about bijective maps from compact spaces would show that the map was a homeomorphism.

We have already decided - in 6.1.7 - that there are no homeomorphisms between a closed interval and a square. It is much harder to decide the more general question, whether there are homeorphisms $[0, 1]^n \to [0, 1]^m$.

But it is a famous result of algebraic topology, the *"invariance of domain"*, that there is no homeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^m$ unless $m = n$.

Exercise 6.3.9. Assuming the "invariance of domain", show that there is no homeomorphism between $[0, 1]^m$ and $[0, 1]^n$ for $m \neq n$.

6.4 The p-adic integers

6.4.1. A very different example of a space homeomorphic to the product spaces C_n is from number theory.

We fix a prime p. The p-adic valuation $v_p(n)$ of an integer n is defined as $v_p(n) = r$, where p^r is the highest power of p that divides n.

We define a metric on the set of integers in the following way:

$$d(m, n) = p^{-v_p(m-n)}.$$

(By definition, $v_p(0) = +\infty$ and $p^{-\infty} = 0$.)

Exercise 6.4.2. Prove that this is a metric. Prove that the sets $a + p^r\mathbb{Z}$ for all a and r form a basis for the topology on $\mathbb{Z}$.

6.4.3. The p-adic integers $\mathbb{Z}_{(p)}$ are the completion of the integers with respect to this metric. That is, an element $x \in \mathbb{Z}_{(p)}$ is an equivalence class of Cauchy sequences of integers.

6.4.4. You can write a number in a unique way in the number system based on the prime p. That is, for $x \in \mathbb{Z}$, there is a unique sequence of numbers $a_i(x)$ so that $0 \leq a_i(x) \leq p - 1$ and

$$x = \sum_{i=0}^{\infty} a_i(x)p^i.$$

The remainder we get from dividing x by p^r is the truncated sum

$$x = \sum_{i=0}^{r-1} a_i(x)p^i.$$

6.4.5. Recall that if x and y give the same remainder after division by n, we say that $x \equiv y \mod (n)$. We see that $x \equiv y \mod (p^r)$ is equivalent to $a_i(x - y) = 0$ for $0 \leq i < r$. It is also equivalent to $v_p(x - y) \geq r$, which again is equivalent to $d(x, y) \leq p^{-r}$.

This gives a simple reformulation of what it means to be a Cauchy sequence in $(\mathbb{Z}, d)$.

Lemma 6.4.6. *A sequence of integers $\{x_i\}$ is a Cauchy sequence for the p-adic metric, exactly if for each natural number r there is a number N_r so that if i, j are greater than N_r, then $x_i \equiv x_j \mod (p^r)$.*

Proof. If $\{x_i\}$ is a Cauchy sequence, we can find an N_r so that $d(x_i, x_j) \leq p^{-r}$ for i, j greater than N_r. But this says that $x_i - x_j$ is divisible by p^r, so that $x_i \equiv x_j \mod (p^r)$. Conversely, if the condition is true, then $d(x_i, x_j) \leq p^{-r}$ for i, j greater than N_r. So $\{x_i\}$ is indeed a Cauchy sequence. □

Exercise 6.4.7. Show that a_n is a well defined continuous map from $\mathbb{Z}_{(p)}$ to the discrete set X_p of integers in the interval $[0, p - 1]$.

6.4.8. By the previous exercise, we get a continuous map

$$A = \prod_{i=0}^{\infty} a_i : \mathbb{Z}_{(p)} \to \prod_{i=0}^{\infty} X_p.$$

Exercise 6.4.9. Show that the map A is bijective.

6.4.10. It follows that A is a homeomorphism, so $\mathbb{Z}_{(p)}$ is homeomorphic to the space C_p of Definition 6.2.2.

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