Review of Manifolds, Lie Groups, and Lie Algebras

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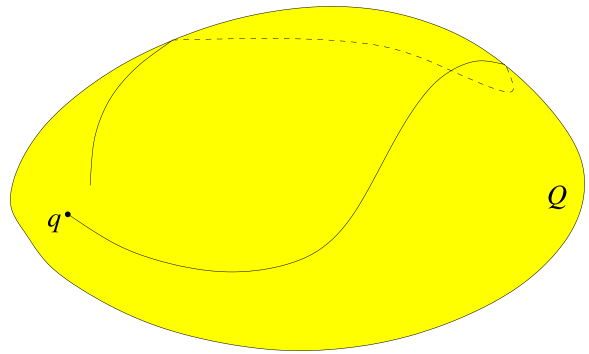
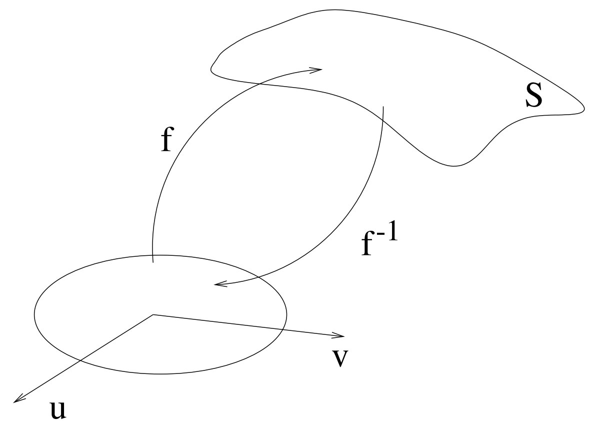
Manifolds

Systems evolve on a manifold, Q.



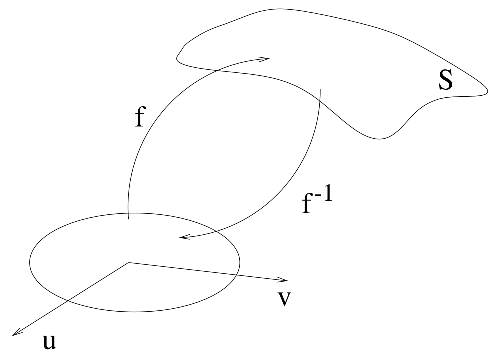
Definition 1 *Let X, Y be subsets of two Euclidean spaces and let $f : X \to Y$ be bijective. If f and f^{-1} are continuous, then f is a* homeomorphism. *If f and f^{-1} are smooth, then f is a* diffeomorphism.



Definition 2 *A k-dimensional* manifold, *M, is locally diffeomorphic to $\mathbb{R}^k$. I.e., for each $x \in M$, there exists a nbhd of x, $V \subset M$, which is diffeomorphic to an open set $U \subset \mathbb{R}^k$.*

Manifolds, continued

Definition 3 *A* coordinatizable surface, *S, is the image of a map $f\colon\ U \rightarrow \mathbb{R}^3$ where*



- *U is an open connected subset of $\mathbb{R}^2$.*
- *The vectors $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are linearly independent for all $(u, v) \in U$.*
- *f is a homeomorphism.*

(f, U) is a *coordinate system* for S with coordinates u, v.

f^{-1} is termed a *local parametrization.*

Example (Unit Sphere)

One coordinate system for the sphere is:

$$U = \{(u, v) \mid -\frac{\pi}{2} < u < \frac{\pi}{2}; -\pi < v < \pi\}$$

$$f(u, v) = \begin{bmatrix} \cos(u)\cos(v) \\ -\cos(u)\sin(v) \\ \sin(u) \end{bmatrix}$$

Note that $\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} = 0$, implying that (f, U) is an orthogonal coordinate system.

Tangent Spaces, Vectors

Definition 4 *The* tangent space *to M at $x \in M$, denoted by T_xM, is the image of $df|_{f^{-1}(x)}$*

Remarks:

1. T_pS, is the closest linear approximation to M at p.
2. Generally, if $p_1 \neq p_2$, then $T_{p_1}S \neq T_{p_2}S$.
3. The *dimension* of a manifold, M, is defined as the dimension of its tangent space: $dim(M) = dim(T_pM)$.
4. The definition of the tangent space is intrinsic.

- A *tangent vector* at $p \in M$ is a vector in T_pM.
- The union of all tangent spaces is the *tangent bundle.*

Example (sphere continued)

Let $p = f(0,0) = [1\ 0\ 0]^T$ (where the x-axis intersects the sphere's surface). Then

$$df_{(0,0)} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $T_p M$ is the plane passing through p and parallel to the y-z plane.

Bundles

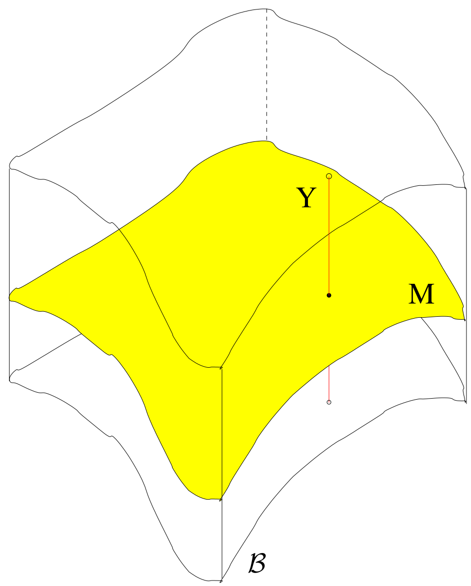
Definition 5 *The manifold $\mathcal{B}$ is a fiber bundle if the following exist:*

1. *a manifold M called the* base space*,*
2. *a projection $\pi : \mathcal{B} \rightarrow M$, and*
3. *a space Y called the* fiber *.*

The set Y_x, defined by

$$Y_x = \pi^{-1}(x)$$

is called the *fiber over the point x of M*. Each Y_x is homeomorphic to Y.

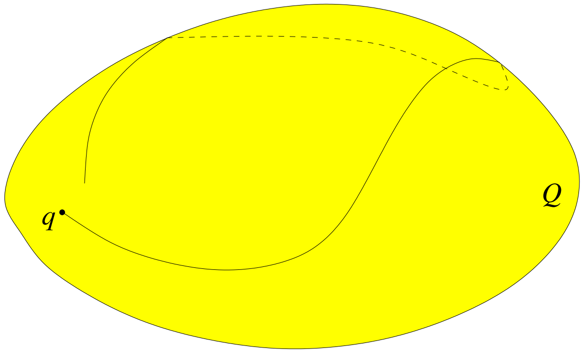


$\mathcal{B}$ is a *vector bundle* if Y is a vector space.

Vector Fields

A vector field is a section defined on the tangent bundle TQ, denoted

$$X : Q \rightarrow TQ$$



For each element $q \in Q$, $X(q) \in T_q Q$.

If the vector field is time dependent, then it is written $X(q, t)$ with shorthand notiation,

$$X_t(\cdot) \equiv X(\cdot, t)$$

The Jacobi-Lie Bracket

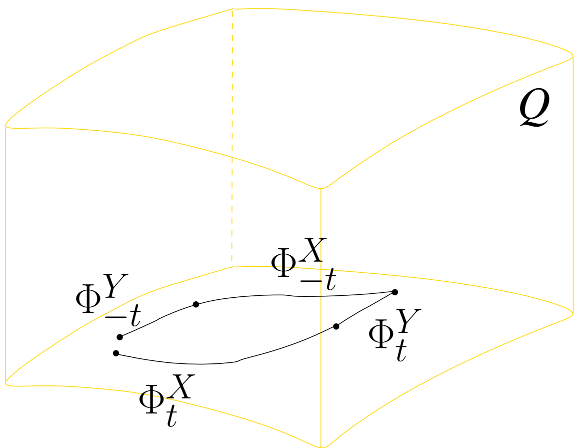
The Jacobi-Lie bracket is defined as,

$$[X, Y] = L_X Y \equiv \lim_{t \to 0} \frac{1}{t} \left(\left(\Phi_t^X \right)^* Y(q) - Y(q) \right)$$

where, Φ_t^X is the flow of the vector field X.

The Jacobi-Lie bracket is used to characterize:

1. Involutivity (closure of bracket).
2. Flows (noncommutativity).



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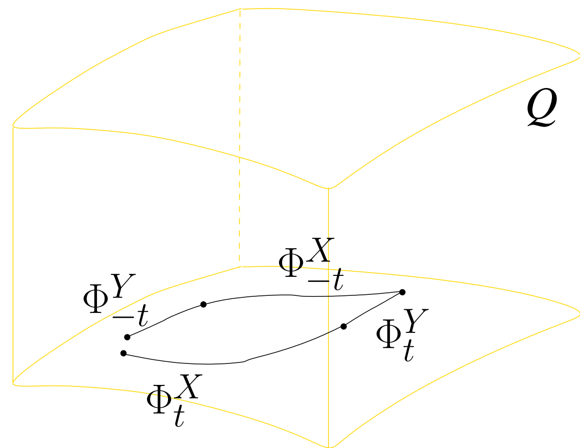
$$[X, Y] \in \Delta, \ \ \forall X, Y \in \Delta$$



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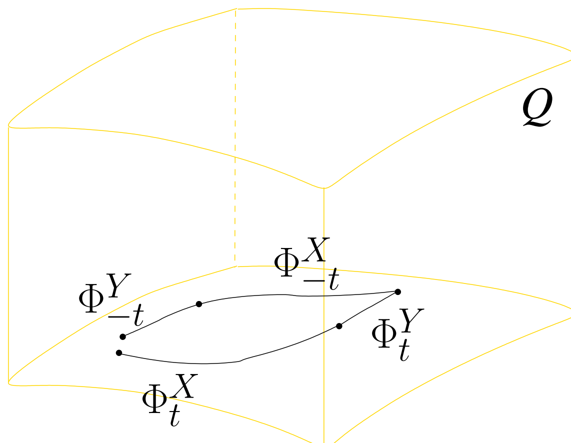
$$[X, Y] \neq 0$$



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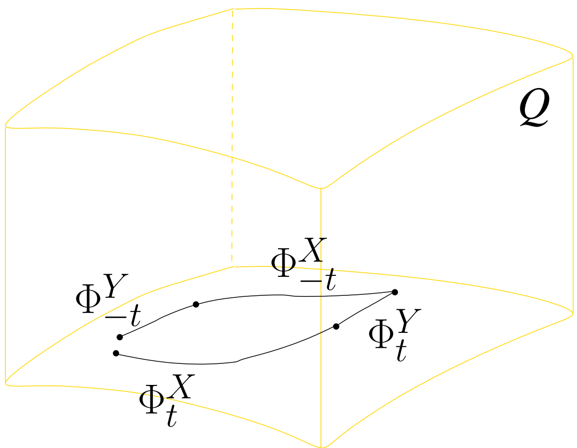


$$[X(z), Y(z)] = \frac{\partial Y}{\partial z} X - \frac{\partial X}{\partial z} Y$$

Lie Groups

Definition of a Lie Group

Definition 5 *A* **group** *is a nonempty set G with a product operation, $*$, such that the following hold:*

1. Associativity Law: $a * (b * c) = (a * b) * c$.
2. Closed Operation: $a * b \in G$ *if* $a, b \in G$
3. Identity: $e * x = x * e = x$.
4. Inverse: $\forall\, x \in G,\ \exists\, y\ :\ x * y = y * x = e$.

Definition 5 *A Lie group is a manifold G whose group structure isconsistent with its manifold structure. I.e., group multiplication,*

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh,$$

is C^∞, as is inversion.

The Classical Matrix Groups

Definition 5 *The set of* $n \times n$ *invertible matrices under matrix multiplication forms group, denoted by* $GL(n)$.

Definition 5 *A subset,* $H \subset G$, *is a* **subgroup** *of* G, *if* H *is itself a group under the operation of* G.

Some of the classical subgroups of $GL(n)$:

1. *SL(n):* $n \times n$ matrices with $det = +1$
2. *O(n):* $n \times n$ orthogonal matrices $(A^T A = I)$
3. *SO(n):* $n \times n$ in both $SL(n)$ and $O(n)$
4. *U(n):* $n \times n$ complex orthogonal matrices
5. *SU(n):* matrices in $U(n)$ with $det = +1$.

Actions of Lie Groups

The product structure can be used to define a *left translation* ,

$$L_g : G \to G, \quad L_g(h) = gh,$$

and similarly a *right translation*,

$$R_g : G \to G, \quad R_g(h) = hg.$$

Note that,

$$L_{g_1} \circ L_{g_2} = L_{g_1\,g_2} \quad \text{and} \quad R_{g_1} \circ R_{g_2} = R_{g_2\,g_1}.$$

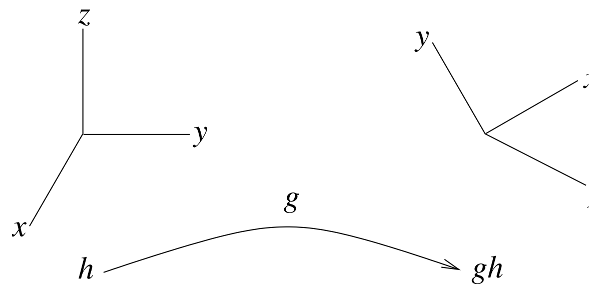
An *inner automorphism* may be defined,

$$I_g : G \to G, \quad I_g(h) = L_g R_{g^{-1}}(h) = R_{g^{-1}} L_g(h) = ghg^{-1}$$

Lie Group: $SO(3)$

$SO(3)$ is the group of rotations in Euclidean space, $\mathbb{R}^3$.
As a matrix Lie group, $g \in SO(3)$ satisfies:

- $gg^T = I$.
- $\det(g) = 1$.



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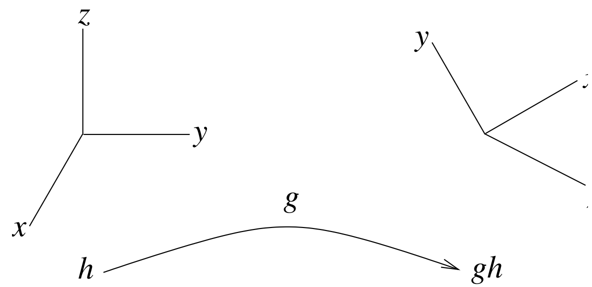
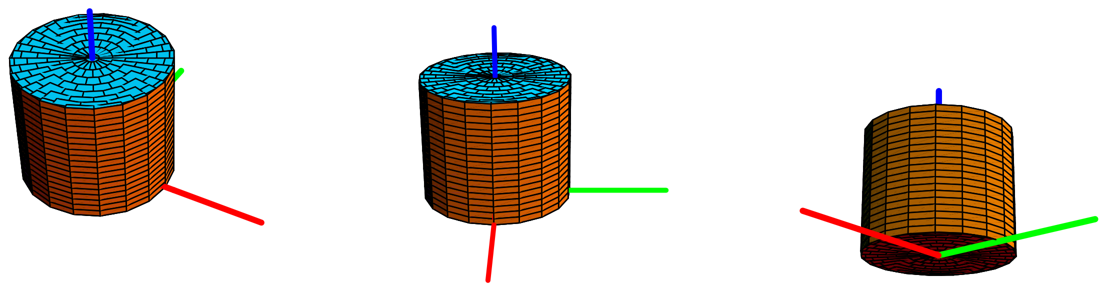
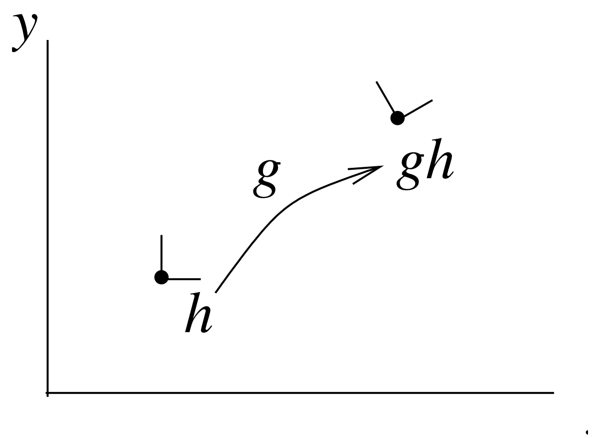




Lie Group: $SE(2)$

$SE(2)$ describes rigid body motions in the Euclidean plane. As a matrix Lie group, $g \in SE(2)$ takes the form:

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$



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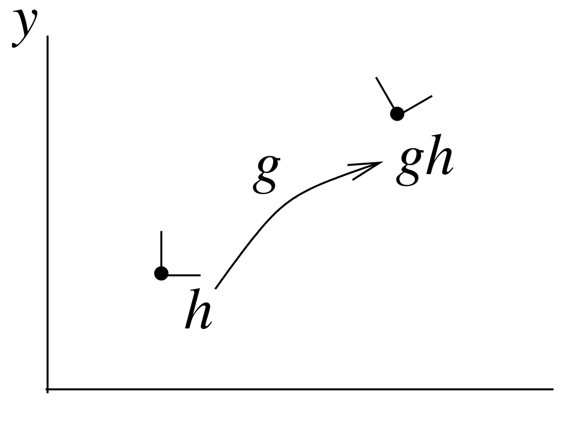
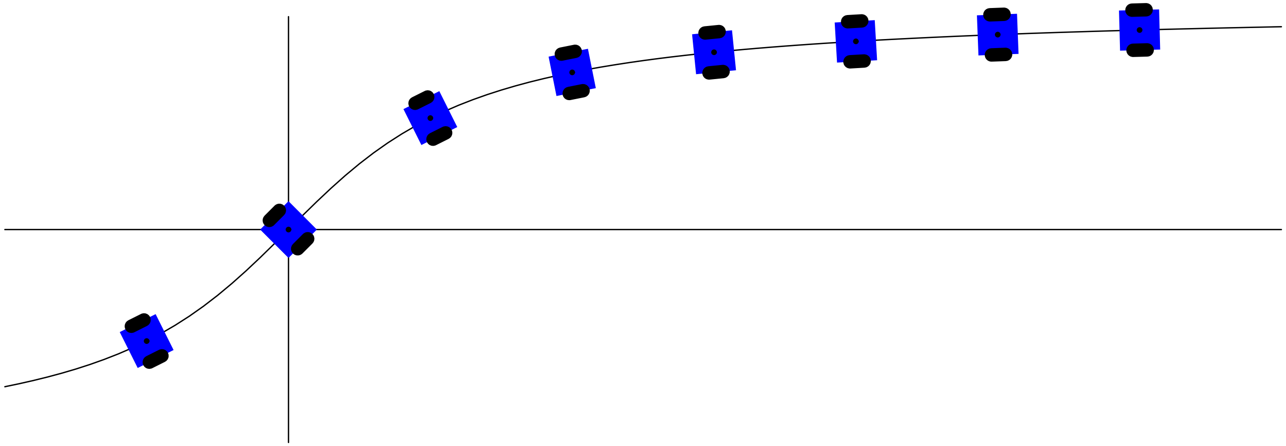


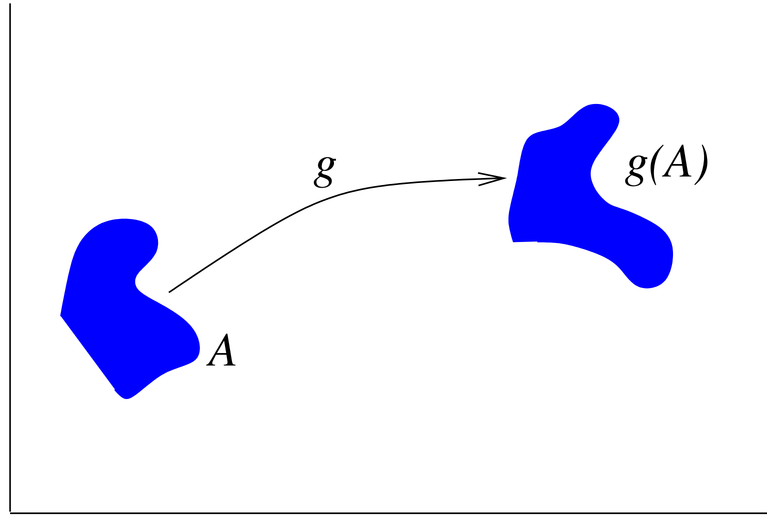


Lie Group: $Diff_{vol}(M)$

$Diff_{vol}(M)$ is the Lie group of volume preserving diffeomorphisms of a manifold M.

An element $g \in Diff_{vol}(M)$ is a mapping

$$g : M \to M$$



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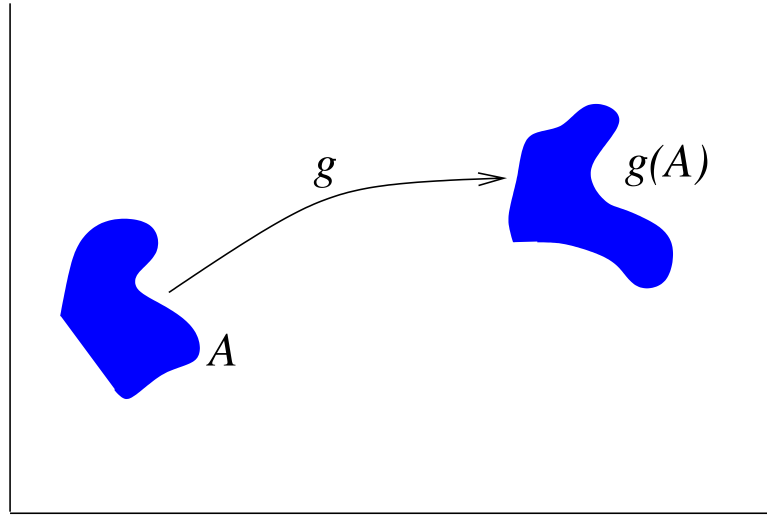
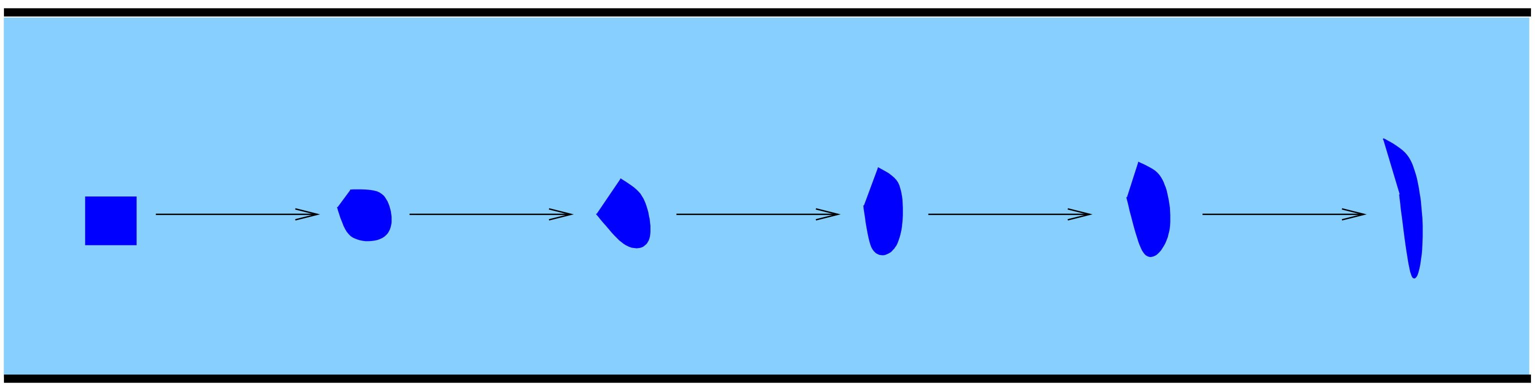
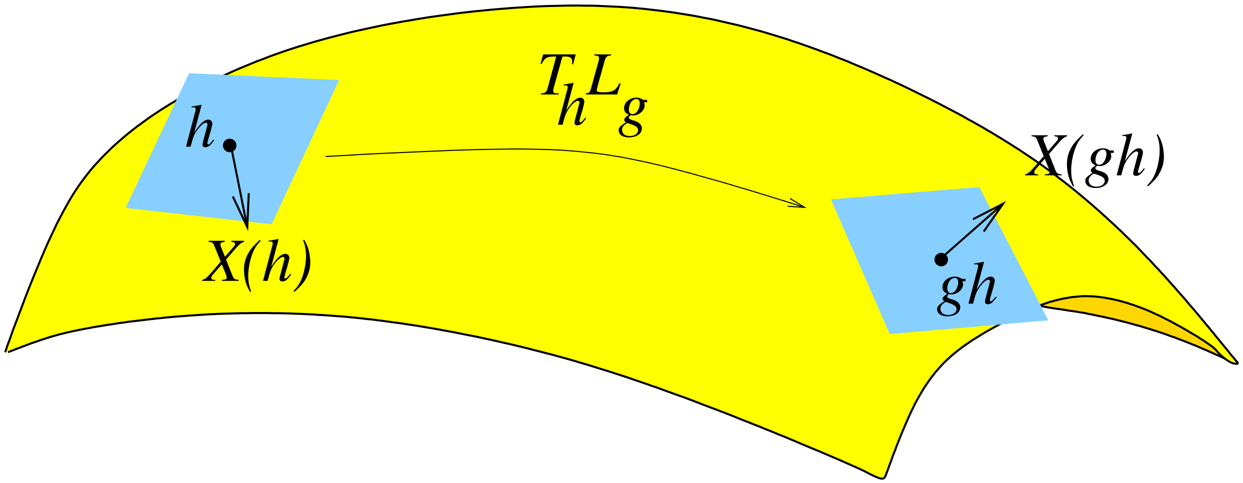




Invariant Vector Fields

A vector field X on G is *left-invariant* if

$$(T_h L_g) X(h) = X(gh)$$



The set of left-invariant vector-fields, $\mathcal{X}_L(G)$, form a Lie sub-algebra since,

$$L_g^* [X, Y] = \left[L_g^* X, L_g^* Y\right] = [X, Y]$$

The Lie Algebra

Elements in $\mathcal{X}_L(G)$ can be identified with T_eG.

$$X(g) = X_\xi(g) = T_eL_g\xi$$

The Jacobi-Lie bracket defined at the point $e \in G$,

$$[\xi, \eta] = \left[X_\xi, X_\eta\right](e)$$

gives the tangent space T_eG a bracket structure.

This bracket is called the *Lie bracket*, and makes T_eG, denoted by $\mathfrak{g}$ into a Lie algebra.

Notes on Lie Algebras

Lie Algebra: A real vector space, V, with a multiplication operation $[\ ,\ ]$ which satisfies for $A, B \in V$:

1. $[A, B] = -[B, A]$;
2. $[A, B + C] = [A, B] + [A, C]$; $[A + B, C] = [A, C] + [B, C]$;
3. for $r \in \mathbb{R}$, $r[A, B] = [rA, B] = [A, rB]$
4. $[A, [B, C]] + [B, [A, C]] + [C, [A, B]] = 0$

The set of smooth vectors fields on a manifold M forms a Lie Algebra under Jacobi-Lie bracket operation.

Examples

Lie Algebra of $GL(n, \mathbb{R})$: Set of all $n \times n$ real matrices

Lie Algebra of $SO(3)$: Set of 3×3 skew-symmetric matrices, denoted by $so(3)$:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The Lie Bracket is the matrix commutator:

$$\hat{\omega}_1, \hat{\omega}_2 = \hat{\omega}_1 \hat{\omega}_2 - \hat{\omega}_2 \hat{\omega}_1$$

Examples Continued

Lie Algebra of $SE(3)$: Matrices in $se(3)$ take the form:

$$\hat{\xi} = \begin{bmatrix} \hat{w} & \vec{v} \\ \vec{0}^T & 1 \end{bmatrix}; \quad \hat{\omega} \in so(3); \;\; \vec{v} \in \mathbb{R}^3$$

The Lie Bracket is given by:

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1 = \begin{bmatrix} [\hat{\omega}_1, \hat{\omega}_2] & \hat{\omega}_1 \vec{v}_2 - \hat{\omega}_2 \vec{v}_1 \\ \vec{0}^T & 0 \end{bmatrix}$$

The Adjoints

Differentiation of the inner automorphism leads to the *adjoint* operator:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_g \eta \equiv T_e I_g \cdot \eta$$

Differentiation of the adjoint operator (with respect to g) leads to the Lie bracket, sometimes denoted by ad,

$$\mathrm{ad}_\xi \eta \equiv T_e(\mathrm{Ad}\eta) \cdot \xi = [\xi, \eta]$$

- Transformation of observer.
- Used for body/spatial transformations.

The Exponential Map

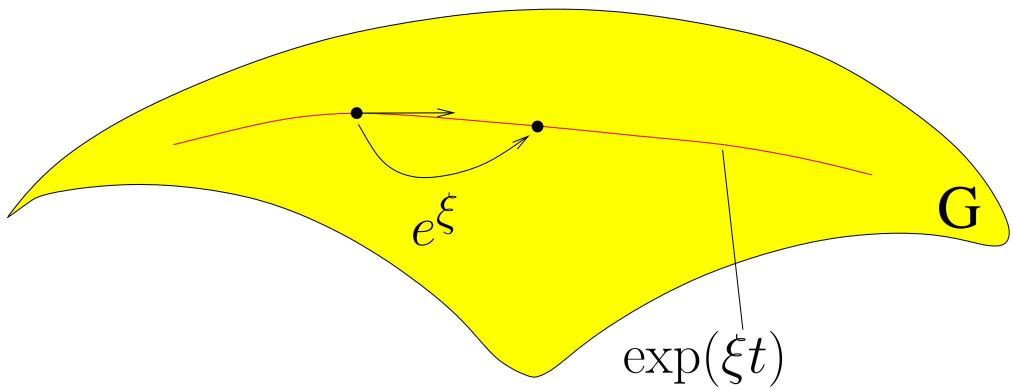
A flow is obtained by solving for the differential equations defined by a left-invariant vector field,

$$\dot{g} = X_\xi(g) = T_e L_g \, \xi = g\xi$$

This flow defines the exponential map,

$$\exp : \mathfrak{g} \to G, \quad \xi \mapsto e^\xi$$

Keeping the time parametrization gives, $\exp(\xi t)$.



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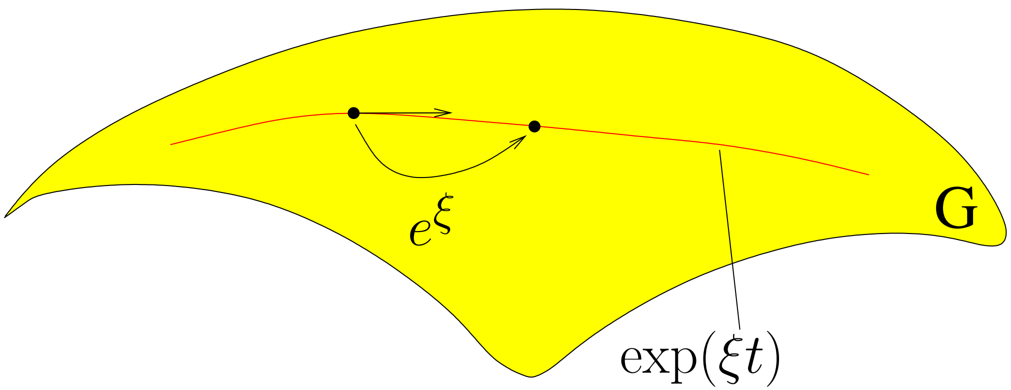


Actions of Lie Groups 2

Definition 5 *Let Q be a manifold and let G be a Lie group. A* left action *of a Lie group G on M is a smooth mapping $\Phi : G \times Q \to Q$ such that:*

1. $\Phi(e, x) = x, \ \forall\, x \in Q$, *and*
2. $\Phi(g, \Phi(h, x)) = \Phi(gh, x), \ \forall\, g, \, h \in G$.

The action of $g \in G$ on $q \in Q$ will typically be written as $g \cdot q$ of simply gq.

1. *free*: for all $x \in Q$, $\Phi_g(x) = x$ implies that $g = e$.
2. *proper*: $W \subset Q$ compact implies $\Phi^{-1}(W) \subset G \times Q$ compact.

Infinitesimal Generators

The action of G on Q induces a vector field on Q.

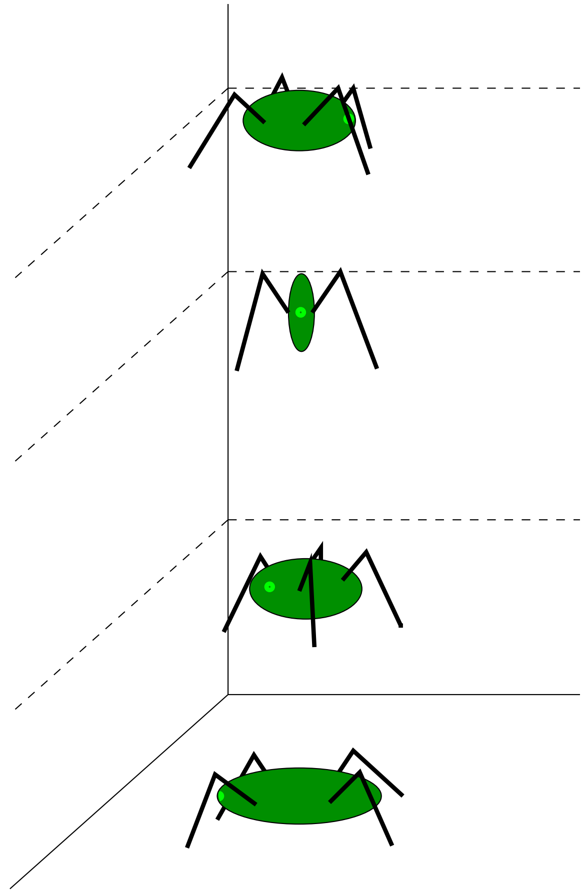
The Lie algebra exponential $\exp$ defines a curve on Q,

$$\Phi_t^{\xi}(q) \equiv \exp(\xi t) \cdot q$$

which after time differentiation,

$$\xi_Q(q) \equiv \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(\xi t) \cdot q = \xi \cdot q$$

gives the *infinitesimal generator*.



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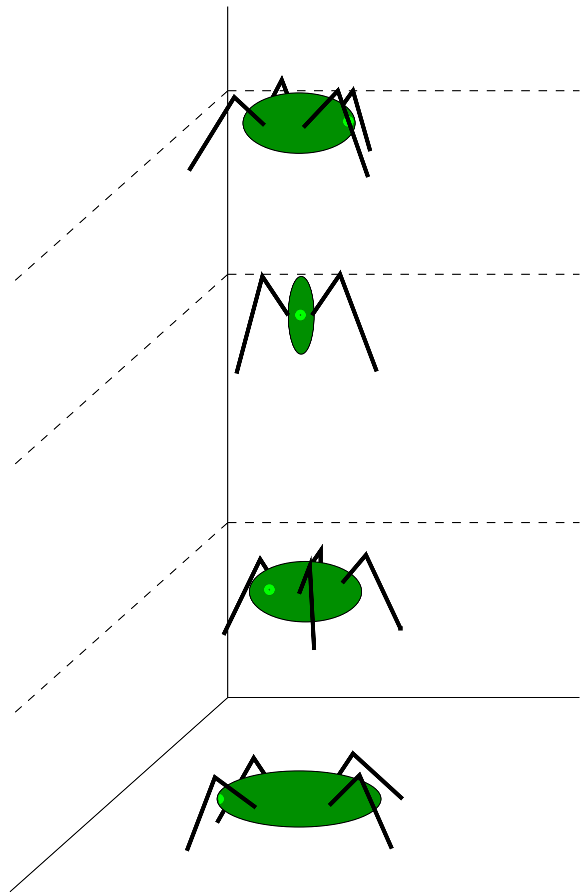


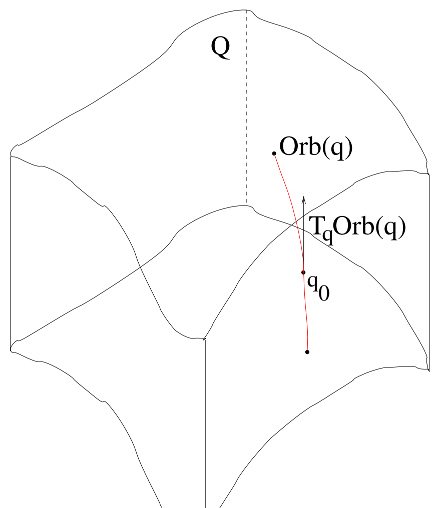
Group Orbits

Definition 5 *Given an action of G on Q and $q \in Q$, the* orbit of q *is defined by*

$$\text{Orb}\,(q) \equiv \{\, \Phi_g(q) \mid g \in G \,\} \subset Q$$

The tangent space at q to the group orbit through q_0 is given by,

$$T_q\text{Orb}\,(q_0) = \{\, \xi_Q(q) \mid \xi \in \mathfrak{g} \,\}$$

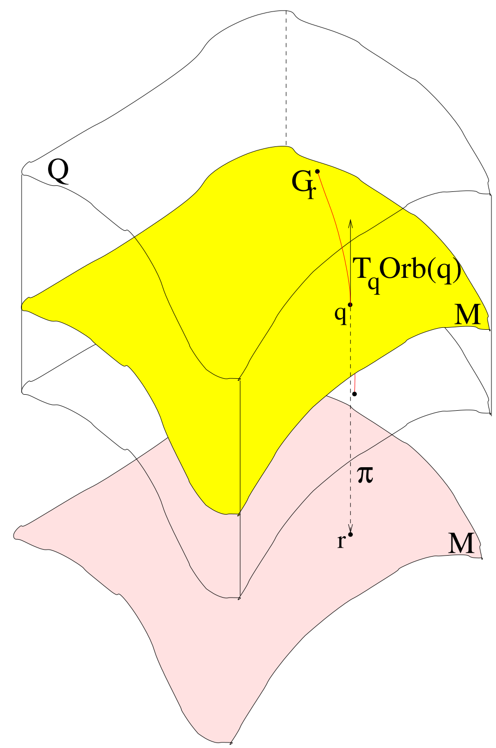


Principal Bundles

Definition 5 *A* principal bundle *is a fiber bundle such that the model fiber is a Lie group,* G.

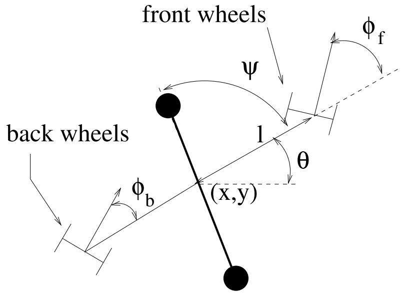
For mechanical systems the *base space*, M, is sometimes called the *shape space*.

- Many control systems decompose this way.
- Shape $\rightarrow$ Directly controlled.
- Group $\rightarrow$ What we want to control (locomote within).
- Inherits all structures discussed.



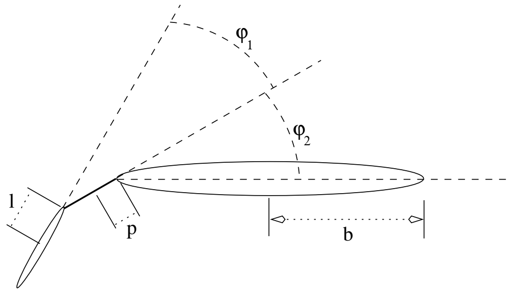
Examples

Snakeboard



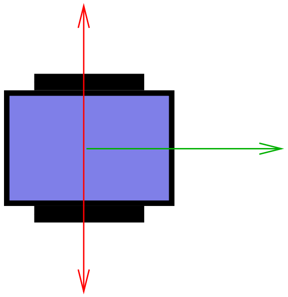
$\mathbb{T}^3 \times SE(2)$

Planar Fish



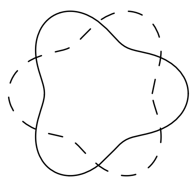
$\mathbb{T}^2 \times SE(2)$

Hilare Robot



$\mathbb{T}^2 \times SE(2)$

Planar Amoeba



$\mathbb{R}^3 \times SE(2)$