FUZZY DIFFERENTIATION OF REAL FUNCTIONS

Mark Burgin

Department of Mathematics
University of California, Los Angeles
405 Hilgard Ave.
Los Angeles, CA 90095

Abstract: The main goal of this work is the further development of the neoclassical analysis. To do this, we utilize the theory of fuzzy limits. It provides for a construction of a fuzzy extension for the classical theory of differentiation. In the second part of this work, going after introduction, elements of the theory of fuzzy limits are presented to make the exposition more complete. The third part is devoted to the construction of fuzzy derivatives of real functions. Two kinds of fuzzy derivatives are introduced: weak and strong ones. Strong fuzzy derivatives are similar to ordinary derivatives of real functions being their fuzzy extensions. Weak fuzzy derivatives generate a new concept of a weak derivative even in a classical case of exact limits.

In the fourth part of this work, fuzzy differentiable functions are studied. Different properties of such functions are obtained. Some of them are the same or at least similar to the properties of the differentiable functions while other properties differ in many aspects from those of the standard differentiable functions. Many classical results are obtained as direct corollaries of propositions for fuzzy derivatives, which are proved in this paper. Some of the classical results are extended and completed. The fifth part of this work contains several interpretations of fuzzy derivatives aiming at application of fuzzy differential calculus to solving practical problems. At the end, some open problems are formulated.

Keywords: Fuzzy limit; fuzzy derivative, real function; fuzzy differential calculus

C o n t e n t

1. Introduction

Mathematics is an efficient tool for modeling real world phenomena. However, in its essence mathematics is opposite to real world because mathematics is exact, rigorous and abstract while are imprecise, vague, and concrete. To lessen this gap, mathematicians elaborate methods that make possible to work with natural vagueness and incompleteness of information using exact mathematical structures. One of the most popular approaches to this problem is fuzzy set theory.

Traditionally, this theory has been developing in two directions (Zimmermann, 1991; Kosko,1993). The first one has been aimed at fuzzification of different classical mathematical structures and studying properties of these fuzzy objects. This development in many aspects has been parallel to the classical mathematics. In such a way, fuzzy sets, fuzzy logic, fuzzy numbers, fuzzy topologies and so on were introduced and studied.

The second direction takes or elaborates fuzzy structures and applies them to different practical problems. In such a way, fuzzy sets were introduced and fuzzy set theory has found many useful applications to problems of Artificial Intelligence, pattern recognition, decision-making, operation research and many others.

Neoclassical analysis (Burgin and Šostak, 1992; 1994; Burgin, 1992; 1995; 1997; 1997a; 1999; 2000; Janiš, 1999) is the third direction in fuzzy set theory. In it, ordinary structures of analysis, that is, functions and operators, are studied by means of fuzzy concepts: fuzzy limits, fuzzy continuity, and fuzzy derivatives. For example, continuous functions, which are studied in the classical analysis, become a part of the set of the fuzzy continuous functions studied in the neoclassical analysis. It extends the scope of analysis making, at the same time, its methods more precise. Consequently, new results are obtained extending and even completing classical theorems. In addition, facilities of analytical methods for various applications also become more broad and efficient.

The principal goal of this work is to extend the range of the classical mathematical analysis by finding means for differentiation of such ordinary functions that have no conventional derivatives. It is done by introducing fuzzy strong and fuzzy weak derivatives.

The necessity to launch investigation and implementation of fuzzy principles in the classical analysis, while studying ordinary functions, is caused by several reasons. One of the most important of them is connected with properties of measurements. Any real measurement provides not absolutely precise but only approximate results. For example, it is impossible to find out if any series of numbers obtained in experiments converges or a function determined by measurements is continuous at a given point. Consequently, constructions and methods developed in the classical analysis are only approximations to what exists in reality. In many situations such approximations has been giving a sufficiently adequate representation of studied phenomena. However, scientists and, especially, engineers have discovered many cases in which such methods did not work because classical approach is too rough (Zimmermann, 1991; Kosko,1993).

Here, we consider only one example from physics. It is the, so-called, Barkhausen effect discovered in 1919 (Vonsovskii, 1974; Burke, 1986). This effect has been thoroughly studied, analyzed, explained, and utilized as a tool for investigation of many properties of ferro- and ferrimagnetic materials. There is a reasonable theoretical model of that effect. Its essence is as follows. If a ferro- or ferrimagnetic sample is being magnetized in an external magnetic field, the magnetization of the sample is increasing, along with the increase of the external magnetizing field. However, even if the magnetizing field is increasing in a continual way, the magnetization of the sample is increasing via thousands of small discontinuities ("Barkhausen jumps"). In other words, an impact of the field the change of which is described by a continuous function produces such changes that may be adequately represented only by a fuzzy continuous function.

Many features of the Barkhausen effect has been studied, including the distributions of Barkhausen discontinuities over their duration, and over their amplitude, and over their shape etc. Much is known about the mechanism of those "jumps" and about their relationship to many other properties of the sample, such as demagnetizing factor, saturation magnetization, remnant magnetization, etc. However, there is no such a continuous function that provides for a sufficiently correct description of the phenomenon. The difficulty is not mathematical but is caused by the physical nature of the process.

There are numerous examples of similar situations. In many cases development of measurement methodology and achieving in such a way higher precision than before helped to discover natural discontinuous processes that seemed continuous for a long period of time.

Besides, when scientists develop models of different phenomena, they encounter similar problems. One of the most popular models in science and engineering is a system of differential equations. Differential equations are used in economics and sociology. These and many other mathematical models utilize limit processes. For example, derivatives in differential equations are constructed as special limits of functions or points. Continuous functions and the calculus, differential equations and topology, all are based on limits and continuity. However, when we perform computations and measurements, we can do only finite number of operations and consequently, achieve only approximate results. At the same time, mathematical technique, e.g., calculus and optimization theory are based on operation of differentiation. This brings us to an unexpected conclusion. Although, it is supposed that numerical computation is a precise methodology in contrast to qualitative methods, this is true only in a very few cases. For limit processes, this is not so and computation adds its uncertainty to the vagueness of initial data. As writes Gregory Chaitin (1999), the fact is that in mathematics, for example, real numbers have infinite precision, but in the computer precision is finite. In some cases, this discrepancy between theoretical schemes and practical actions changes drastically outcomes of a research resulting in uncertainty of knowledge. For example, as remarked the great mathematician Henri Poincare, series convergence is different for mathematicians, who use abstract mathematical procedures, and for astronomers, who utilize numerical computations.

Consequently, new methods and constructions are necessary to take into account such more sophisticated effects in different systems. Such methods and constructions are provided by neoclassical analysis.

The main goal of this work is the further development of neoclassical analysis. To do this, we utilize the theory of fuzzy limits, which is elaborated in (Burgin, 2000). As it is known, theory of limits is the base for differential and integral calculi in classical analysis. Likewise, the theory of fuzzy limits provides for construction of fuzzy extensions for both classical calculi. Here, we are dealing with the first one developing the theory of fuzzy differentiation. It is a natural step after elaboration of the theory of fuzzy limits and the theory of fuzzy continuous functions. For simplicity, only functions on the real line with real values are considered in this work. However, the main constructions and results are valid for a much broader context.

In the second part of the work, going after introduction, elements of the theory of fuzzy limits are presented to make the exposition more complete.

The third part is devoted to the construction of fuzzy derivatives of real functions. Two kinds of fuzzy derivatives are introduced: weak and strong ones. Strong fuzzy derivatives are similar to ordinary derivatives of real functions being their fuzzy extensions. At the same time, weak fuzzy derivatives generate a new concept of a weak derivative even in a classical case of exact limits. Properties of fuzzy derivatives are investigated and compared with the nearness derivative of a function introduced by Janiš (1999).

As in the classical case, a fuzzy derivative of a function represents an approximation of the rate at which the dependent variable changes relative to the independent variable. Strong fuzzy derivatives reflect approximations of all rates, while a weak fuzzy derivative reflects an approximation of a particular rate. Rates of change are highly important in science; for example, velocity is the rate of change of position, and acceleration is the rate of change of velocity. In some cases, the exact rate does not exist. In other cases, it exists but it is impossible to measure such exact rate. For example, if we take the rate of change of position impossibility to measure it is one of the consequences of the Principle of Uncertainty, which was introduced by Heisenberg. Besides, there are cases when exact rate exists, it feasible to measure it, but it is impossible to calculate the value of this exact rate. All these and many other situations imply usefulness and necessity to study fuzzy derivatives.

In the fourth part of this work, fuzzy differentiable functions are studied. Different properties of such functions are obtained. Some of them are the same or at least similar to the properties of the differentiable functions while other properties differ in many aspects from those of the differentiable functions. Many classical results are obtained as direct corollaries of propositions for fuzzy derivatives, which are proved in this paper. Some of the classical results are extended and completed. For example, it is demonstrated (theorem 4.5) that any fuzzy differentiable function f is continuous.

It is necessary to remark that the concept of a weak fuzzy derivative is closely connected to the notion of weakly continuous (Collingwood and Lohwater, 1966) and weakly symmetrically continuous (Ciesielski and Larson, 1993-94; Ciesielski, 1995-96) functions. These connections are also studied in the fourth part of this work. For example, it is demonstrated (Proposition 4.2) that any weakly fuzzy differentiable function f at a point x is weakly continuous at x.

The fifth part of the paper contains several interpretations of fuzzy derivatives. It is aimed at application of fuzzy differential calculus for solving practical problems.

At the end of this paper, some features of neoclassical analysis are discussed, comparison with other related works is presented, and some open problems are formulated.

For simplicity, fuzzy differential calculus is developed for real function on $\boldsymbol{R}$. However, it is possible to develop by the same technique a similar calculus for function having the form f: $\boldsymbol{R}^n \to \boldsymbol{R}^m$ or the form f: $\boldsymbol{C}^n \to \boldsymbol{C}^m$.

Denotations:

$\boldsymbol{N}$ is the set of all natural numbers;

ω is the sequence of all natural numbers;

$\varnothing$ is the empty set;

$\boldsymbol{R}$ is the set of all real numbers;

$\boldsymbol{R}^+$ is the set of all non-negative real numbers;

$\boldsymbol{R}^{++}$ is the set of all positive real numbers;

$\boldsymbol{C}$ is the set of all complex numbers;

If a is a real number, then $|a|$ or $||a||$ denotes its absolute value or modulus;

$\rho(x,y) = | x - y |$ for $x,y \in \boldsymbol{R}$;

if $l = \{a_i \in M; i \in \omega\}$ is a sequence, and $f: M \to L$ is a mapping, then $f(l) = \{f(a_i); i \in \omega\}$;

$a = r\text{-lim } l$ means that a number a is an r-limit of a sequence l;

if $A = \{a_i; i \in I\}$ is an infinite set, then the expression "a predicate $P(x)$ is true for almost all elements from A" means that $P(x)$ can be untrue only for a finite number of elements from A . For example, if $A = \omega$, then almost all elements of A are bigger than 10, or another example is that conventional convergence of a sequence l to x means that any neighborhood of x contains almost all elements from l ;

if X is a subset of a topological space, then $\mathrm{Cl}(X)$ denotes the closure of the set X.

2. Elements of the theory of fuzzy limits

Let $r \in \boldsymbol{R}$ and $l = \{a_i \in \boldsymbol{R}; i \in \omega\}$.

Definition 2.1 (Burgin, 2000). A number a is called an r-limit of a sequence l (it is denoted $a = r\text{-lim}_{i\to\infty} a_i$ or $a = r\text{-lim } l$) if for any $\varepsilon \in \boldsymbol{R}^{++}$ the inequality $\rho(a, a_i) \leq r + \varepsilon$ is valid for almost all a_i , i.e., there is such n that for any i>n, we have $\rho(a, a_i) \leq r + \varepsilon$.

Example 2.1. Let $l = \{1/i; i \in \omega\}$. Then 1 and -1 are 1-limits of l ; 1/2 is a (1/2)-limit of l but 1 is not a (1/2)-limit of l.

Informally, a is an r-limit of a sequence l for an arbitrarily small ε the distance between a and all but a finite number of elements from l is smaller than $r + \varepsilon$.

When $r = 0$, the r-limit coincides with the conventional limit of a sequence as the following result demonstrates.

Lemma 2.1. $a = \lim l$ if and only if $a = 0\text{-}\lim l$.

This result demonstrates that the concept of an r-limit is a natural extension of the concept of conventional limit. However, the concept of an r-limit actually extends the conventional construction of a limit (cf. Example 2.2).

Lemma 2.2. If $a = r\text{-}\lim l$ then $a = q\text{-}\lim l$ for any $q > r$.

Let $r \in \boldsymbol{R}^+$, $l = \{a_i \in \boldsymbol{R};\ i \in \omega\}$, $h = \{b_i \in \boldsymbol{R};\ i \in \omega\}$, $k = \{c_i \in \boldsymbol{R};\ i \in \omega\}$, and l is the disjoint union of h and k .

Lemma 2.3. $a = r\text{-}\lim l$ if and only if $a = r\text{-}\lim h$ and $a = r\text{-}\lim k$.

Definition 2.2 (Burgin, 2000). a) A number a is called a fuzzy limit of a sequence l if it is an r-limit of l for some $r \in \boldsymbol{R}^+$.

b) a sequence l fuzzy converges if it has a fuzzy limit.

Example 2.2. Let us consider sequences $l = \{1 + 1/i;\ i \in \omega\}$, $h = \{1 + (-1)^i;\ i \in \omega\}$, and $k = \{1 + [(1 - i)/i]^i\ ;\ i \in \omega\}$. Sequence l has the conventional limit equal to 1 and many fuzzy limits (e.g., 0, 0.5 , 2 are 1-limits of l). Sequence h does not have the conventional limit but has different fuzzy limits (e.g., 0 is a 1-limit of h, while 1, -1, and 1/2 are 2-limits of h). Sequence k does not have the conventional limit but has a variety of fuzzy limits (e.g., 1 is a 1-limit of k, while 2, 0, 1.5, 1.7, and 0.5 are 2-limits of k).

Thus, we see that many sequences that do not have the conventional limit have lots of fuzzy limits.

Remark 2.1. The measure of convergence of l to points from $\boldsymbol{R}$ defines the normal fuzzy set $\text{Lim}\ l = [\text{L}, \mu(x{=}\lim l)]$ of fuzzy limits of l (Burgin, 2000).

Definition 2.3. A number a is called a partial r-limit of a sequence l (it is denoted $a = r\text{-plim}\ l$) if for any $\varepsilon \in \boldsymbol{R}^{++}$ the inequality $\rho(a, a_i) \leq r + \varepsilon$ is valid for infinitely many elements a_i from l .

In the classical mathematical analysis, some cases of partial limits (such as $\underline{\lim}$ or $\overline{\lim}$) are considered. A general case of partial limits is treated in (Randolph, 1968) where partial limits, i.e., partial 0-limits by Lemma 2.4, are called subsequential limits. These limits make it possible to define by a standard limit construction weakly continuous and uniformly antisymmetric functions studied in (Collingwood and Lohwater, 1966; Ciesielski and Larson, 1993-94; Ciesielski, 1995-96).

Lemma 2.4. A point a is a subsequential (partial) limit of a sequence l if and only if a is a partial 0-limit of l.

Proposition 2.1. *The following conditions are equivalent:*

1) *a sequence l has no fuzzy limits;*

2) *some subsequence of l diverges;*

3) *some subsequence of l has no fuzzy limits;*

4) *the diameter* $\mathrm{d}(\{a_i;\, a_i \in l\})$ *is infinite.*

Theorem 2.1 (Burgin, 2000). *If* $a = r\text{-lim } l$ *and* $a > b+r$, then $a_i > b$ *for almost all* a_i *from l.*

Corollary 2.1 (Ribenboim, 1964; Fihtengoltz, 1955). If $a = \lim l$ and $a > b$, then $a_i > b$ for almost all a_i from l.

Corollary 2.2 (Ribenboim, 1964; Fihtengoltz, 1955). If $a = \lim l$ and $a > 0$, then $a_i > 0$ for almost all a_i from l.

Corollary 2.3. If $a_i \leq q$ for almost all a_i from l and $a = r\text{-lim } l$, then $a \leq q+r$.

Corollary 2.4 (Ribenboim, 1964; Fihtengoltz, 1955). If $a_i \leq q$ for almost all a_i from l and $a = \lim l$, then $a \leq q$.

Theorem 2.2 (Burgin, 2000). *For an arbitrary number* $r \in \boldsymbol{R}^+$, *all r-limits of a sequence l belong to some interval, the length of which is equal to 2r.*

When $r = 0$, this interval shrinks to one point, and we have the following result.

Corollary 2.5 (any course of mathematical analysis, cf., for example, (Ribenboim, 1964; Fihtengoltz, 1955)). *A limit of a sequence is unique* (if this limit exists).

Theorem 2.3 (Burgin, 2000). *A sequence l fuzzy converges if and only if it is bounded.*

It gives a criterion for boundedness of a sequence while classical results give only sufficient conditions (Ribenboim, 1964; Fihtengoltz, 1955).

It is possible to define infinite fuzzy limits.

Definition 2.4. ∞ ($-\infty$) is an r-limit of l if almost all elements a_i are bigger (less) than r ($-r$).

Example 2.3. ∞ is a 10-limit of the sequence $l = \{10 + 1/i;\, i \in \omega\}$, 0-limit of the sequence $h = \{i\, ;\, i \in \omega\}$, and 2-limit of the sequence $k = \{1 + (i - 1)/i\, ;\, i \in \omega\}$.

Lemma 2.5. ∞ ($-\infty$) is the limit of l (in the classical sense (Ribenboim, 1964; Fihtengoltz, 1955) if and only if it is an r-limit of l for any $r > 0$ ($r < 0$)

Lemma 2.6. ∞ ($-\infty$) is an r-limit of $l = \{a_i \in \boldsymbol{R};\ i \in \omega\}$ if and only if 0 is a $1/r$-limit of the inverse sequence $h = \{a_i^{-1};\ i \in \omega\}$.

Lemma 2.7. If ∞ is an r-limit of l, then any q-limit of l is bigger than $r - q$.

Corollary 2.6. If ∞ is an r-limit of l, then any r-limit of l is positive .

Proposition 2.2. *If* ∞ *is an r-limit of l and b is a q-limit of l, then:*

a) $r \leq b\text{-}q$ implies that ∞ is a (b-q)-limit of l ;

b) b) $r > b\text{-}q$ implies that l has a $0.5 \cdot (q+b-r)$-limit.

Remark 2.2. Not all properties of ordinary limits are properties of fuzzy limits. For example, an r-limit may be not unique (Burgin, 2000). In the same way, for ordinary (exact) limits, we have the following result: if $a_i < b_i$ for almost all $i \in \omega$ and $a = \lim a_i$, $b = \lim b_i$, then $a \leq b$ (Ribenboim, 1964; Fihtengoltz, 1955). For fuzzy limits, the resulting inequality is not always true (Burgin, 2000). However, some similar statements for fuzzy limits may be proved (cf. Proposition 2.6).

Proposition 2.3. *If* $q = \delta(l) = \inf \{r;\ \exists\ a = r\text{-lim}\ l\}$ *and* $b = q\text{-lim}\ l$ *then the q-limit of l is unique.*

Theorem 2.4 (Burgin, 2000). *For an arbitrary number* $r \in \boldsymbol{R}^+$, *the set* $\mathrm{L}_r(l) = \{a \in \boldsymbol{R};\ a = r\text{-lim}\ l\}$ *of all r-limits of a sequence l is a convex closed set, i.e., either* $\mathrm{L}_r(l) = [a,b]$ *for some* $a,b \in \boldsymbol{R}$, or $\mathrm{L}_r(l) = \varnothing$ *when l has no r-limits.*

Let $\mathrm{L}_r(l) = [b,c]$.

Proposition 2.4. a) $r \leq c - b \leq 2r$;

b) *the sequence l has a* $0.5 \cdot (2r\text{-}c+b)$*-limit.*

Corollary 2.7. $c - b = 2r$ if and only if the sequence l converges.

Corollary 2.8. $c - b = r$ if and only if c is the smallest and b is the biggest of the partial limits of l.

Let $b < c$ be two r-limits of l .

Proposition 2.5. *If d is an r*-limit of *l and* $b < d < c$, *then*:

a) $c - d < r$ *implies that l has a* $0.5 \cdot (r + c - d)$-limit;

a) $d - b < r$ *implies that l has a* $0.5 \cdot (r + d - b)$-limit.

Proposition 2.6. *If* $a_i < b_i$ *for almost all* $i \in \omega$, $l = \{a_i \in \boldsymbol{R};\ i \in \omega\}$, $h = \{b_i \in \boldsymbol{R};\ i \in \omega\}$, $\mathrm{L}_r(l) = [a,c]$, *and* $\mathrm{L}_r(h) = [b,d]$, *then* $a \leq b$ *and* $c \leq d$.

Corollary 2.9. If h is a subsequence of l, then $L_r(h) \subseteq L_r(l)$ for all $r \in \boldsymbol{R}^+$.

Corollary 2.10 (Ribenboim, 1964). If $a_i < b_i$ for almost all $i \in \omega$ and $a = \lim a_i$, $b = \lim b_i$, then $a \leq b$.

Lemma 2.8. If $r \leq p$, then $L_r(l) \subseteq L_p(l)$ for any sequence l.

Let $l = \{a_i \in \boldsymbol{R}; i \in \omega\}$, $h = \{b_i \in \boldsymbol{R}; i \in \omega\}$, $k = \{c_i \in \boldsymbol{R}; i \in \omega\}$, and $a_i \leq b_i \leq c_i$ for almost all $i \in \omega$.

Proposition 2.7. *If* $L_r(l) = [a,u]$, $L_r(h) = [b,v]$, and $L_r(k) = [c,w]$, *then:*

a) $L_r(h) \subseteq [a,w]$;

b) $L_r(l) = L_r(k) \neq \varnothing$ *implies* $L_r(h) = L_r(l) \neq \varnothing$.

Corollary 2.11 (Ribenboim, 1964; Fihtengoltz, 1955; Randolph, 1968; Goldstein *et al*, 1987; Shenk,1979). If both sequences l and k converge to the same limit a, then h converges and $\lim h = a$.

Proposition 2.8. *If* $a = r\text{-}\lim l$ *and* $\rho(a,b) = p$, *then* $b = (r+p)\text{-}\lim l$.

Theorem 2.5. *If* $a = r\text{-}\lim l$ *and* $b = q\text{-}\lim h$ *then*:

a) $a+b = (r+q)\text{-}\lim(l+h)$ *where* $l+h = \{a + b ; i \in \omega\}$;

b) $a-b = (r+q)\text{-}\lim(l-h)$ *where* $l-h = \{a - b ; i \in \omega\}$;

c) $ka = |k| \cdot r\text{-}\lim (kl)$ *for any* $k \in \boldsymbol{R}$ *where* $kl = \{ka ; i \in \omega\}$.

Corollary 2.12. (any course of the calculus, cf., for example, (Ribenboim, 1964; Fihtengoltz, 1955)). If $a = \lim l$, $b = \lim h$ then:

a) $a+b = \lim (l + h)$;

b) $a-b = \lim (l - h)$;

c) $ka = \lim (kl)$ for any $k \in \boldsymbol{R}$.

Corollary 2.13. If a is a fuzzy limit of l and b is a fuzzy limit of h then $(a+b)$ ($(a - b)$, and ka) is a fuzzy limit of $l + h$ (of $(l - h)$, and kl), respectively.

Definition 2.5. A sequence l is called r-fundamental if for any $\varepsilon \in \boldsymbol{R}^{++}$ there is such $n \in \omega$ that for any $i,j \geq n$ we have $\rho(a_j, a_i) \leq 2r + \varepsilon$.

Definition 2.6. A sequence l is called fuzzy fundamental if it is r-fundamental for some $r \in \boldsymbol{R}^+$.

Lemma 2.9. If $r \leq p$, then any r-fundamental sequence is p-fundamental.

Lemma 2.10. A sequence l is fundamental (in the ordinary sense, i.e., it is a Cauchy sequence) if and only if it is 0-fundamental.

Lemma 2.11. A subsequence of an r-fundamental sequence is r-fundamental.

Theorem 2.6 ***(the Extended Cauchy Criterion).*** *The sequence l has an r-limit if and only if it is r-fundamental.*

Proof. Necessity. Let $a = r$-lim l and $\varepsilon \in \boldsymbol{R}^{++}$. Then by the definition $\exists n \in \omega\ \forall i > n$ ($\rho(a, a_i)$ $\leq r+\varepsilon/2$. Consequently, for any i, j > n, we have $\rho(a_i, a_j) \leq \rho(a, a_i) + \rho(a, a_j) \leq 2r+2(\varepsilon/2) < 2r + \varepsilon$. Thus, l is an r-fundamental sequence.

Sufficiency. Let l be an r-fundamental sequence and $\varepsilon_m=1/m$. Then for each number n(m), which is dependent on m, and for all i, j > n(m), we have $\rho(a_i, a_j) \leq 2r+1/m$, i.e., all points a_i with i > n(m) belong to a closed interval I_m whose length is equal to $2r+1/m$. Really, let $T_m = \{a_i ; i>n(m)\}$, $b = \sup T_m$, and $c = \inf T_m$. Then all $a_i \in [c, b]$ for i > n(m).

Let us estimate the length of I_m . Suppose that $\rho(b, c) > 2r + 1/m$. It means that $\rho(b, c) = 2r+1/m + h$ for some positive number h. At the same time, as l is an r-fundamental sequence, there are such a_i, a_j , for which i, j > n(m) , $\rho(b, a_j) \leq (1/10)\cdot h$, and $\rho(c, a_i) \leq (1/10)\cdot h$ because b is the supremum and c is the infinum of all these elements. Consequently, $\rho(a_i, a_j) > 2r+1/m +(4/5)\cdot h$. It contradicts the choice of the number n(m). Thus, the length of $I_m= [c, b]$ is not bigger than $2r+1/m$.

We can choose these intervals I_m so that the inclusion $I_{m+1} \subseteq I_m$ will be valid for all m . In such a way, we obtain a sequence of imbedded closed intervals $\{ I_m ; n \in \omega\}$. The space $\boldsymbol{R}$ is locally compact. Consequently, the intersection $I = \cap I_m$ is non-void. That is, either the set I consists of one point d or I is a closed interval having the length not bigger than $2r$. When I is a one-point set $\{d\}$, the sequence l converges to d, and thus, it is fundamental (cf. Lemma 2.8 and (Ribenboim, 1964). This means (by Lemma 2.10 and Lemma 2.9) that l is r-fundamental.

When I is a closed interval, its middle e is an r-limit of l because for any $\varepsilon \in \boldsymbol{R}^{++}$ there is some interval $I_m \supseteq I$, for which $\varepsilon > 1/m$, $\rho(e, e_m) < (1/3)\varepsilon$ for the center e_m of I_m and almost all a_i belong to I_m . Consequently, $\rho(e, a_i) \leq \rho(e_m, a_i) + \rho(e, e_m) < (1/3)\varepsilon + r +(1/3)\varepsilon < r + \varepsilon$, i.e., e is an r-limit of l .

Theorem is proved.

From Theorem 2.6, we obtain the following result.

Theorem 2.7 (the ***General Fuzzy Convergence Criterion).*** The sequence l fuzzy converges if and only if it is fuzzy fundamental.

Corollary 2.14 (the **Cauchy Criterion)** (Ribenboim, 1964). The sequence l converges if and only if it is fundamental.

This result and Lemma 2.10 demonstrate that the concept of fuzzy convergence is a natural extension of the concept of conventional convergence.

Proposition 2.9. *The following conditions are equivalent:*

1) *a sequence l is not fuzzy fundamental;*

2) *the sequence l is not bounded;*

3) *some subsequence of l has no fuzzy limits;*

4) ∞ *or* $-\infty$ *is the partial limit of* l.

The neoclassical analysis makes possible not only to extend ordinary concepts obtaining new results for classical structures, but also provides for elaboration of new useful concepts. One of such concepts is given in the definition of fuzzy limits of sets of sequences.

Definition 2.7. A number a is called an r-limit of a set E = $\{ l_j ; j \in \omega \}$ of sequences of real numbers (it is denoted $a = r$-lim E) if a is an r-limit of each sequence l_i from E.

Remark 2.3. If E has a 0-limit a , then this 0-limit is unique and all sequences from E converge to a. In contrast to this sequences from a given set may have different limits but a common fuzzy limit. For example, the set E = $\{ \{1/2^n; n = 1, 2,\dots\}, \{1+ 1/3^n; n = 1, 2,\dots\}, \{2+ 1/5^n; n = 1, 2,\dots\} \}$ has a 1-limit 1.

Lemma 2.1 implies the following result.

Corollary 2.15. If $a = r$-lim E then $a = q$-lim E for any $q>r$.

From Theorem 2.1, we obtain the following result.

Corollary 2.16. If $a = r$-lim E and $a > b+r$, then $a_{ij} > b$ for almost all a_{ij} from each $l \in$ E .

From Theorem 2.2, we obtain the following result.

Corollary 2.17. All r-limits of a set E of sequences belong to some interval the length of which is equal to $2r$.

Since an intersection of closed intervals is a closed interval (Kuratowski, 1966; 1968), from Theorem 2.4 we obtain the following result.

Corollary 2.18. The set $L_r(E) = \{a \in R; a = r\text{-lim } E\}$ is a closed interval.

It is necessary to remark that some of the results contained in this part of the work are proved in (Burgin, 2000) and given here for completeness without proofs, while other results are new.

3. Fuzzy derivatives

Let $X, Y \subseteq \boldsymbol{R}$, $f: X \rightarrow Y$ be a function, $b \in \boldsymbol{R}$, and $r \in \boldsymbol{R}^+$.

Definition 3.1. A number b is called a weak centered (left, right, two-sided) r-derivative of f at a point $x \in X$ if $b = r\text{-}\lim\, (f(x) - f(x_i)) /(x - x_i)$ (and all $x_i < x$; and all $x_i > x$; $b = r\text{-}\lim\, (f(z_i) - f(x_i)) / (z_i - x_i)$, with $z_i < x < x_i$ for all $i \in \omega$, and the sequences $\{x_i\}$, $\{z_i\}$ converging to x) for some sequence $\{x_i\}$ converging to x. It is denoted by $b = w_r^{ct}d/_{dx} f(x)$ ($b = w_r^{l}d/_{dx} f(x)$, $b = w_r^{r}d/_{dx} f(x)$, and $b = w_r^{t}d/_{dx} f(x)$, correspondingly) .

When r is not specified, we call weak centered (left, right, two-sided) r-derivatives of f at a point $x \in X$ fuzzy weak centered (left, right, two-sided, correspondingly) derivatives of f at a point $x \in X$.

Example 3.1. Let us take the membership function $m_Q(x)$ of the set of rational numbers, i.e., $m_Q(x)$ is equal to 1 when x is a rational number and $m_Q(x)$ is equal to 0 when x is an irrational number. This function is not even continuous, consequently it does not have derivatives neither in classical sense (Randolph, 1968; Goldstein *et al*, 1987) nor as a generalized function (Shwartz, 1950-51). However, at any point x from $\boldsymbol{R}$, $m_Q(x)$ has a weak derivative, which is equal to 0.

The construction of a weak centered (left, right, two-sided) r-derivative of f at a point $x \in X$ give birth to a new concept in classical calculus of a weak centered (left, right, two-sided) derivative of f at a point $x \in X$.

Definition 3.2. A number b is called a weak centered (left, right, two-sided) derivative of f at a point $x \in X$ if $b = \lim\, (f(x) - f(x_i)) /(x - x_i)$ (and all $x_i < x$; and all $x_i > x$; $b = r\text{-}\lim\, (f(z_i) - f(x_i)) / (z_i - x_i)$, with $z_i < x < x_i$ for all $i \in \omega$, and the sequences $\{x_i\}$, $\{z_i\}$ converging to x) for some sequence $\{x_i\}$ converging to x. It is denoted by $b = w^{ct}d/_{dx} f(x)$ ($b = w^{l}d/_{dx} f(x)$, $b = w^{r}d/_{dx} f(x)$, and $b = w^{t}\, d/_{dx} f(x)$, correspondingly) .

Remark 3.1. Weak derivatives of functions are special cases of extraderivatives of the same functions (Burgin, 1993).

Definition 3.3. A number b is called a strong centered (left, right, two-sided) r-derivative of f at a point $x \in X$ if $b = r\text{-}\lim\, (f(x) - f(x_i)) /(x - x_i)$ (and all $x_i < x$; and all $x_i > x$; $b = r\text{-}\lim\, (f(z_i) - f(x_i)) / (z_i - x_i)$,, with $z_i < x < x_i$ for all $i \in \omega$, and the sequences $\{x_i\}$, $\{z_i\}$ converging to x) for all sequences $\{x_i ; i \in \omega \}$ converging to x .

A strong centered (left, right, two-sided) r-derivative of f at a point $x \in X$ is denoted by $b = \mathrm{st}_r^{ct} \mathrm{d}/_{\mathrm{d}x} f(x)$ ($b = \mathrm{st}_r^{l} \mathrm{d}/_{\mathrm{d}x} f(x)$, $b = \mathrm{st}_r^{r} \mathrm{d}/_{\mathrm{d}x} f(x)$, and $b = \mathrm{st}_r^{t} \mathrm{d}/_{\mathrm{d}x} f(x)$, correspondingly) .

Remark 3.2. In what follows, $\mathrm{st}_r^{z} \mathrm{d}/_{\mathrm{d}x} f(x)$ denotes one of these four types of strong and $\mathrm{w}_r^{z} \mathrm{d}/_{\mathrm{d}x} f(x)$, denotes one of these four types of weak r-derivatives of $f(x)$. Here $z \in \{ct, l, r, t\}$.

Example 3.2. There exist functions even in mathematics that fail to have a classical derivative at some given point. However, they may have a fuzzy derivative at the same point. A simple example is the following: Let $f(x) = x$ if x is positive or zero and $f(x) = -x$ if x is negative (that is, $f(x) = |x|$, the absolute value of x) and choose x_0 to be 0. It follows that, if h is positive, the difference quotient is the number 1, as shown by calculation; whereas, if h is negative, the quotient is -1. Thus, $f(x)$ does not have a derivative at $x_0 = 0$ because, arbitrarily close to 0, the difference quotient assumes the values 1 and -1; i.e., this difference quotient does not approach one unique number as h approaches zero

However, 0 is the strong two-sided 1-derivative of $f(x)$ at 0 , 1 is a strong right while -1 is a strong left 0-derivatives of $f(x)$ at 0.

Example 3.3. Piecewise linear transformations on the interval have been widely studied in the theory of dynamical systems (Collet and Eckmann, 1980; Marcuard and Visinescu, 1992) and under different names as well: broken linear transformations (Gervois and Mehta, 1977) or weak unimodal maps (Misiurewicz, 1989). An example of such functions if given by a skew tent map $f_{a,b}(x)$ that is equal to $b + ((1-b)/a)x$ when $0 \leq x < a$ and equal to $(1-x)/(1-a)$ when $a \leq x \leq 1$. Piecewise linear transformations do not have conventional derivatives at some points but they have strong fuzzy derivatives at all points. It provides for application of differential methods to these mappings as well as to dynamics generated by them.

Remark 3.3. In contrast to the classical derivative, it is possible that different numbers are strong centered (or left, right, two-sided) r-derivatives of a given function f at a point x .

Remark 3.4. An alternative approach to fuzzy differentiation is suggested by Janiš (1999). His construction for differentiation is based on the concept of fuzzy continuity from (Burgin and Šostak, 1992; 1994). He considers the set $\boldsymbol{R}$ with nearness in the sense of (Kalina, 1997).

Definition 3.4 (Kalina, 1997). A continuous function N: $\boldsymbol{R} \times \boldsymbol{R} \to [0, 1]$ is called a nearness on $\boldsymbol{R}$ if it satisfies the following conditions:

1) for each $x, y \in \boldsymbol{R}$, $\mathrm{N}(x, y) = 1$ if and only if $x = y$;
2) for each $x, y \in \boldsymbol{R}$, $\mathrm{N}(x, y) = \mathrm{N}(y, x)$;

3) if $x < y < z$, then $N(x, y) \geq N(x, z)$;

4) for each $x \in \boldsymbol{R}$, $\lim_{y\to\pm\infty} N(x, y) = 0$;

5) for each $x, y, c \in \boldsymbol{R}$, $N(x, y) = N(x+c, y+c)$.

Remark 3.5. Another kind of nearness spaces was introduced by Herrlich (1974; 1974a).

Let $\alpha \in (0, 1)$, f: $\boldsymbol{R} \to \boldsymbol{R}$, and $a \in \boldsymbol{R}$. Then $D_\alpha(f, a) = \{ (f(x) - f(a)) /(x - a); x \neq a , N(x, a) \geq \alpha \}$.

Definition 3.5. If X is a set of real numbers, then its interval closure $int(X)$ is equal to the least interval that contains X.

Remark 3.6. The interval closure of a set may be infinite. For example, $int(\boldsymbol{N}) = [1, \infty]$.

This makes possible to define the α-nearness derivative $f_\alpha'(a)$ of a function f at a point a by the following formula: $f_\alpha'(a) = int(D_\alpha (f, a)) = [\inf D_\alpha (f, a), \sup D_\alpha (f, a)]$.

Here $int(D_\alpha (f, a))$ is the interval closure of the set $D_\alpha (f, a)$.

In this definition, the α-nearness derivative $f_\alpha'(a)$ of a function f at a point a is a set. In contrast to this, all derivatives that are studied in this work are real numbers, which are the points of the real line. There are interesting relations between nearness derivatives and sets of weak and fuzzy weak derivatives of the same function. These relations are considered further in this section.

Remark 3.7. It is possible to possible to develop all constructions from this work in the space $\boldsymbol{R}$ with nearness.

Let x be an isolated point of X .

Lemma 3.1. Any number $b \in \boldsymbol{R}$ is a strong centered (left, right, two-sided) r-derivative of f at a point $x \in X$ for any $r \in \boldsymbol{R}^+$.

Lemma 3.2. Any strong centered (left, right, two-sided) r-derivative of f at a point $x \in X$ is a weak one-sided (left, right, two-sided) r-derivative of f at the same point for any $r \in \boldsymbol{R}^+$.

Lemma 3.3. $b = st_r^z d/_{dx} f(x)$ if and only if $b = r\text{-}\lim E$ where $E = \{ \{ (f(x) - f(x_i)) /(x - x_i);$ $i \in \omega\}; \{ x_i; i \in \omega\}$ is a corresponding sequence converging to x $\}$.

Lemma 3.4. Any strong (weak) centered r-derivative of f at a point $x \in X$ is both (either) a strong (weak) left and strong (weak) right r-derivative of f at the same point for any $r \in \boldsymbol{R}^+$.

Lemma 3.5. If b is both a strong (weak) left and strong (weak) right r-derivative of f at a point $x \in X$, then strong (weak) centered r-derivative of f at the same point for any $r \in \boldsymbol{R}^+$.

Proof. Let us consider a sequence $\{x_i \in \boldsymbol{R};\ i \in \omega\}$ converging to $x \in X$ and let b be both strong (weak) left and strong (weak) right r-derivatives of f at x. Then the sequence $\{x_i \in \boldsymbol{R};\ i \in \omega\}$ consists of two subsequences $\{v_i \in \boldsymbol{R};\ i \in \omega\}$ and $\{z_i \in \boldsymbol{R};\ i \in \omega\}$ such that $v_i < x$ and $z_i > x$ for all . Each of them either is finite or it converges to x. When one of these subsequences is finite, then $b = r\text{-}\lim\,(f(x) - f(x_i))/(x - x_i)$.

Let both subsequences $\{v_i \in \boldsymbol{R};\ i \in \omega\}$ and $\{z_i \in \boldsymbol{R};\ i \in \omega\}$ be infinite. By the definition of strong r-derivatives $b = r\text{-}\lim\,(f(x) - f(v_i)) /(x - v_i))$ and $b = r\text{-}\lim\,(f(x) - f(x_i)) /(x - x_i)$. Then by Lemma 2.3, $b = r\text{-}\lim\,(f(x) - f(x_i)) /(x - x_i)$. As the sequence $\{x_i \in \boldsymbol{R};\ i \in \omega\}$ is chosen arbitrarily, Lemma 3.5 is proved.

Definition 3.6. A number b is called a complete r-derivative of f at a point $x \in X$ if b is at the same time a strong centered, left, right, and two-sided r-derivative of f at the point x .

Proposition 3.1. *If b is a strong centered r-derivative of f at a point $x \in X$, then b is a strong two-sided r-derivative of f at the point $x \in X$.*

Proof. Let us consider an arbitrary sequence $\{ (f(z_i) - f(x_i)) /(z_i - x_i);\ z_i > x > x_i ,\ i \in \omega \}$. Geometrical considerations demonstrate that either $(f(x) - f(x_i)) /(x - x_i) \le (f(z_i) - f(x_i)) /(z_i - x_i) \le (f(x) - f(z_i)) /(x - z_i)$ or $(f(x) - f(x_i)) /(x - x_i) \ge (f(z_i) - f(x_i)) /(z_i - x_i) \ge (f(x) - f(z_i)) /(x - z_i)$. Consequently, if b is a strong centered r-derivative of f at x , then b is an r-limit of the sequences $\{ (f(x) - f(z_i)) /(x - z_i);\ i \in \omega \}$ and $\{ (f(x) - f(x_i)) /(x - x_i);\ i \in \omega \}$. Properties of r-limits imply that b is an r-limit of the sequence $\{ (f(z_i) - f(x_i)) /(z_i - x_i);\ i \in \omega \}$. As $\{ (f(z_i) - f(x_i)) /(z_i - x_i);\ z_i > x > x_i ,\ i \in \omega \}$ is an arbitrary system, then b is (by the definition) a strong two-sided r-derivative of f at the point $x \in X$.

Proposition 3.1 is proved.

Let f be a continuous function at a point $x \in X$.

Proposition 3.2. *If a strong two-sided r-derivative of f at a point $x \in X$ exists* (*and is equal to b*) *then both one-sided strong r-derivatives of f at a point $x \in X$ exist* (*and coincide with b*) .

Proof. Let us consider a sequence $\{x_i \in \boldsymbol{R};\ i \in \omega\}$, which converges to $x \in X$ and in which all $x_i < x$. As f is a continuous function at x, it is possible to correspond to each x_i such z_i that $x < z_i$ and $|x - z_i| < 1/i$ .Then $| b - (f(x) - f(x_i))/(x - x_i)| < | b - (f(x) - f(x_i))/(z_i - x_i)| + \varepsilon_i < | b - (f(z_i) - f(x_i))/(z_i - x_i)| + \varepsilon_i$. Both sequences $\{| b - (f(z_i) - f(x_i))/(z_i - x_i)| ;\ i \in \omega\}$ and $\{\varepsilon_i;\ i \in \omega\}$ converge to zero. Consequently, number $b \in \boldsymbol{R}^+$ is a strong left r-derivative of f at the point x .

In a similar way, we prove that b is a strong right r-derivative of f at the point x.

Corollary 3.1. If the strong two-sided r-derivative of f at a point $x \in X$ exists (and is equal to b), then a strong centered r-derivative of f at a point $x \in X$ exists (and coincides with b) .

Remark 3.8. Continuity of f is essential for the validity of Proposition 3.2. It is demonstrated by the following example.

Let $f(x) = x$ for all $x>0$, $f(x) = -x$ for all $x < 0$, and $f(0) =1$. Then f has a strong two-sided 3-derivative at 0 having no strong one-sided r-derivatives at 0 for any $r \in \boldsymbol{R}$.

Proposition 2.10 from (Burgin, 2000) and Proposition 3.2 imply the following result.

Corollary 3.2. If b is a strong centered r-derivative of f at a point $x \in X$, then $2b$ is a strong centered $(b+r)$-derivative of f at a point $x \in \mathrm{X}$.

From Lemmas 3.4, 3.5, Proposition 3.2, and Corollary 3.1, we obtain the following result.

Corollary 3.3. If b is a strong centered r-derivative of f at a point $x \in X$, then $2b$ is a strong complete $(b+r)$-derivative of f at a point $x \in X$.

Corollary 3.4. If b is a strong centered r-derivative of f at a point $x \in X$, then b is a strong complete $(b+2r)$-derivative of f at a point $x \in X$.

Proposition 3.3. a) *If a strong centered* 0-*derivative* $\mathrm{st}_0\mathrm{d}^c/_{\mathrm{d}x} f(x)$ *of f at a point $x \in X$ exists, then it is unique and equal to the classical derivative $f'(x)$ of f at x* .

b) *If the classical derivative $f'(x)$ of f at x exists, then it is equal to the strong centered* 0-*derivative* $\mathrm{st}_0\mathrm{d}^c/_{\mathrm{d}x} f(x)$ *of f at x* .

Proof follows from the definition of a strong centered 0-derivative and uniqueness of the classical derivative $f'(x)$.

This result demonstrates that the concept of a fuzzy derivative is a natural extension of the concept of the conventional derivative.

Lemma 3.6. If $b = \mathrm{w}_r\mathrm{d}^z/_{\mathrm{d}x} f(x)$ ($b = \mathrm{st}_r\mathrm{d}^z/_{\mathrm{d}x} f(x)$) then $b = \mathrm{w}_q\mathrm{d}^z/_{\mathrm{d}x} f(x)$ ($b = \mathrm{st}_q\mathrm{d}^z/_{\mathrm{d}x} f(x)$) for any $q>r$.

Proposition 3.4. *If b is a weak* (*strong*) *r-derivative of f at x and* $\rho(\mathrm{b}, \mathrm{e}) < k$ *then e is a weak* (*strong*) ($r+k$)-*derivative of f at x.*

Corollary 3.5. If $b = f'(x)$ and $\rho(b, e) < k$ then e is a strong k-derivative of f at x.

Proposition 3.5. *If b is a strong r-derivative of a function f at at a point x and is not a strong k-derivative of f for any $k < r$, then*:

a) *for any weak p-derivative u of f at x the inequality* $\rho(b, u) < r+p$ *is valid*;

b) *there is exactly one weak* 0*-derivative w of f at x*, *for which* $\rho(b, w)= d$**.**

Definition 3.7. Any r-derivative of f at a point $x\in X$ is called a fuzzy derivative of f at the same point and of the same type (i.e., weak, strong, centered, right, left, or two-sided).

It is denoted $b = \mathrm{wd}^z/_{dx}f(x)$ ($b = \mathrm{std}^z/_{dx}f(x)$) .

From Proposition 3.4., we have the following result.

Corollary 3.6. If $b = f'(x)$, then $b = \mathrm{std}^{ct}/_{dx}f(x)$.

Let x be a non-isolated point of X .

Corollary 3.7. If $b = \mathrm{std}^z/_{dx}f(x)$, then $b = \mathrm{wd}^z/_{dx}f(x)$.

Let $\mathrm{WCFD}_r(f,x)$ ($\mathrm{WLFD}_r(f,x)$, $\mathrm{WRFD}_r(f,x)$, $\mathrm{WTFD}_r(f,x)$, $\mathrm{SCFD}_r(f,x)$, $\mathrm{SLFD}_r(f,x)$, $\mathrm{SRFD}_r(f,x)$, $\mathrm{STFD}_r(f,x)$) be the set of all weak centered (weak left, weak right, weak two-sided, strong centered, strong left, strong right, strong two-sided) r-derivatives of f at a point $x\in X$. In what follows, $\mathrm{YXFD}_r(f,x)$ denotes one of these sets (i.e., Y may be equal to W or S, while X may be equal to C, L, R, or T) and is called the complete r-derivative of f at a point $x\in X$ having type (Y,X) .

Example 3.4. Let $f(x) = |x|$. Then $\mathrm{SCFD}_0(f,1) = \{1\}$, $\mathrm{SCFD}_0(f,0) = [-1, 1]$, and $\mathrm{SCFD}_1(f,1) = [0,2]$.

Theorem 3.1. *Each set* $\mathrm{SXFD}_r(f,x)$ *is a convex closed set, i.e.*, $\mathrm{SXFD}_r(f,x) = [a,b]$ *for some numbers* $a, b \in \boldsymbol{R}$, *or* $\mathrm{SXFD}_r(f,x) = \varnothing$ *if f has no strong r-derivatives of the type* X.

Proof. By the definition and Lemma 3.3, $\mathrm{SXFD}_r(f,x)$ is the set of all r-limits of the sequences having form $(f(x) - f(x_i)) /(x - x_i)$. At the same time, by Corollary 2.18, the set $L_r(E) = \{a\in \boldsymbol{R};\ a = r\text{-}\lim E\}$ of all r-limits for any set E of sequences is a closed interval. Consequently, $\mathrm{SXFD}_r(f,x)$ is a convex closed set.

Theorem 3.2. *The following conditions are equivalent:*

1) *a function f has a strong fuzzy derivative at x of the type* X;

2) *the sets* $\mathrm{WXFD}_r(f,x)$ *are non-empty and bounded for all* $r \geq 0$;

3) *there is such* $t \geq 0$ *that the sets* $\mathrm{SXFD}_r(f,x)$ *are non-empty for all* $r \geq t$;

4) *the set* $\mathrm{WXFD}_0(f,x)$ *is non-empty and bounded.*

Proposition 3.6. *The set* $\mathrm{WCFD}_0(f,x)$ *consists of a single point if and only if the classical derivative* $f'(x)$ *exists*.

Lemma 3.2 implies the following result.

Corollary 3.8. $\mathrm{SXFD}_r(f,x) \subseteq \mathrm{WXFD}_r(f,x)$ when X is equal to C, L, R, or T.

From Theorems 2.2 and 2.3, we obtain the following results.

Proposition 3.7. *The set* $\mathrm{YXFD}_r(f,x)$ *is a union of closed intervals*.

Proposition 3.8. *If* $r \le p$, *then* $\mathrm{YXFD}_r(f,x) \subseteq \mathrm{YXFD}_p(f,x)$.

Proposition 3.9. *If b is a weak* (*strong*) *r-derivative of f at x, then* $\rho(b, \mathrm{WXFD}_0(f,x)) \le r$.

Corollary 3.9. If b is a weak (strong) r-derivative of f at x, then $\rho(b, \mathrm{WXFD}_k(f,x)) \le r - k$ where $r - k = r - k$ when $r \le k$, otherwise $r - k = 0$.

The sets $\mathrm{YXFD}_r(f,x)$ define complete global r-derivatives $\mathrm{YXFD}_r f$ of f on $\boldsymbol{R}$. Each $\mathrm{YXFD}_r f$ is a binary relation on $\boldsymbol{R}$, and namely, $\mathrm{YXFD}_r f = \{ (x,z);\ x\in \boldsymbol{R},\ z\in \mathrm{YXFD}_r(f,x) \}$.

Proposition 3.10. *A set* $\mathrm{YXFD}_r f$ *is closed in* $\boldsymbol{R}$ *for all* $r \ge 0$.

Proposition 3.11. *If the slope of the straight line that connects any point of the graph of a function f and the point* $(x, f(x))$ *is bounded* (*from the left, from the right*) *in some neighborhood of a point x from X, then f has at least one weak centered* (*left, right, correspondingly*) *derivative at this point.*

Remark 3.9. Boundedness is an essential condition for validity of Proposition 3.11 as the following examples show.

Example 3.5. Let us take the function $f(x)$ that is equal to 0 at the points of the form $k\pi$ with $k = 1, 2, 3, \dots$ and equal to the function $\cot x$ at all other points from $\boldsymbol{R}$. At the point 0, this function has no weak derivatives.

Example 3.6. Let us consider the function $f(x)$ that is equal to $1 + \sqrt{1 - x^2}$ at the interval [0, 1] and is equal to $1 + \sqrt{1 - (x - 2)^2}$ at the interval [1, 3] **.** At the point 1, this function has no weak derivatives.

Let us consider the set $\mathrm{WLFD}_r(f,a)$ of all weak left and the set $\mathrm{WRFD}_r(f,a)$ of all weak right r-derivatives of f at a point $a\in X$ and the set $\mathrm{WLD}(f,a)$ of all weak left and the set $\mathrm{WRD}(f,a)$ of all weak right derivatives of f at a point $a\in X$. By There are definite relations between these sets and the set-valued α-nearness derivative $f_\alpha'(a)$ of a function f at a point a, which is defined by Janis (1999).

Proposition 3.11. *For any function* f, *any point* $a \in X$, *any nearness* N, *and any number* α *from the interval* (0, 1) *the following inclusions are valid*: $\mathrm{WLD}(f,a) \subseteq f_\alpha'(a)$ *and* $\mathrm{WRD}(f,a) \subseteq f_\alpha'(a)$.

Proof. Let us take an arbitrary point d from $\mathrm{WLD}(f,a)$ and some number α from the interval (0, 1). Then there is a sequence $\{x_i \in \boldsymbol{R};\ i \in \omega\}$ converging to $a \in X$, for which $\lim_{i\to\infty} (f(a) - f(x_i)) / (a - x_i) = d$. As the nearness N is a continuous mapping, there is some $m \in \omega$ that for all $n > m$, all numbers belong to $D_\alpha(f, a) = \{ (f(x) - f(a)) / (x - a);\ x \neq a,\ N(x, a) \geq \alpha \}$. Consequently, d is an adherent point of the set $D_\alpha(f, a)$.

By the definition, $f_\alpha'(a)$ is a closed set that contains $D_\alpha(f, a)$. Consequently, $f_\alpha'(a)$ contains d. As d is an arbitrary point from $\mathrm{WLD}(f,a)$, $f_\alpha'(a)$ contains $\mathrm{WLD}(f,a)$.

The proof for the set $\mathrm{WRD}(f,a)$ is similar.

Corollary 3.10. For any function f, any point $a \in X$, and number α from the interval (0, 1) the following inclusion is valid: the interval closure $int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a))$ is a subset of the set $f_\alpha'(a)$.

Let us assume that for a given function f and point a from $\boldsymbol{R}$, all closures $\mathrm{Cl}(D_\alpha(f, a))$ of the sets $D_\alpha(f, a)$ are connected sets when $\alpha > \beta$ for some fixed number β from the interval (0, 1).

Proposition 3.12. *For any function* f , *any nearness* N, *and any point* $a \in X$, *the following equality is valid*

$$\bigcap_{\alpha>0} f_\alpha'(a) = int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a)).$$

Proof. As Proposition 3.11 is proved for arbitrary number α from the interval (0, 1), Corollary 3.10 implies that $\bigcap_{\alpha>0} f_\alpha'(a) \supseteq int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a))$.

Let us take some point d that belongs to all closures $\mathrm{Cl}(D_\alpha(f, a))$ of the sets $D_\alpha(f, a)$. By the definition of the nearness derivative $f_\alpha'(a)$, we have three options:

1) for each number 1/n with n = 1, 2, … , there is such an element x_n from X that $N(x_n, a) \geq 1 - 1/n$ and $(f(a) - f(x_i)) / (a - x_i) = d$;
2) $d = \lim_{n\to\infty} d_n$ where $d_n = (f(a) - f(x_i)) / (a - x_i)$, n = 1, 2, … ;
3) in each set $D_\alpha(f, a)$, there is a sequence that converges to d.

It is possible to reduce the third case to second case by standard methods. Besides, we may suppose that either $x_i < a$ or $x_i > a$ for all n = 1, 2, … . At first, we consider the first case.

As the nearness N is a continuous mapping, $a = \lim_{n\to\infty} x_n$. Consequently, d belongs to the set WLD(f,a). If we take the case when $x_i > a$ for all n = 1, 2, … , we come to the conclusion that d belongs to the set WRD(f,a).

As d is an arbitrary point from the set $\bigcap_{\alpha>0} \mathrm{Cl}(D_\alpha(f, a))$, this implies the inclusion

$$\bigcap_{\alpha>0} \mathrm{Cl}(D_\alpha(f, a)) \subseteq int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a)).$$

By the definition, $f_\alpha'(a) = int(D_\alpha(f, a)) = int(\mathrm{Cl}(D_\alpha(f, a))) = \mathrm{Cl}(D_\alpha(f, a))$ as all $\mathrm{Cl}(D_\alpha(f, a))$ are connected sets by the initial condition. Consequently, $\bigcap_{\alpha>0} f_\alpha'(a) = \bigcap_{\alpha>0} \mathrm{Cl}(D_\alpha(f, a)) \subseteq int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a))$.

Thus, $\bigcap_{\alpha>0} f_\alpha'(a) = int(\mathrm{WLD}(f,a), \mathrm{WRD}(f,a))$.

Proposition 3.12 is proved.

However, for all $\alpha > 0$, $f_\alpha'(a)$ may be arbitrarily larger than $int(\mathrm{WLFD}(f,a), \mathrm{WRFD}(f,a))$. This not true for weak fuzzy derivatives as it is proved in the following result.

Corollary 3.11. For any point a from $\boldsymbol{R}$ and any positive number r there is such number α from the interval (0, 1) that $f_\alpha'(a)$ belongs to the set $int(\mathrm{WLFD}_r(f,a), \mathrm{WRFD}_r(f,a))$.

However, it is possible that for this point a and for some number α from the interval (0, 1) there exists no such number r that $f_\alpha'(a)$ belongs to the set $int(\mathrm{WLFD}_r(f,a), \mathrm{WRFD}_r(f,a))$.

From Theorem 2.5, we obtain the following result demonstrating local linearity and additivity of strong fuzzy derivatives.

Theorem 3.3. *a) If b is a strong centered* (*left, right, two-sided*) *a-derivative of f at x and c is a strong d-derivative of g at x, then $b \pm c$ is a strong centered* (*left, right, two-sided*) *$(a + d)$-derivative of $f \pm g$ at x* .

b*) If b is a strong centered* (*left, right, two-sided*) *a-derivative of f at x and $r \in \boldsymbol{R}$, then $r \cdot b$ is a strong centered* (*left, right, two-sided*) *$|r| \cdot a$-derivative of $r \cdot f$ at x* .

Remark 3.10. When the conditions of Theorem 3.3 are satisfied, the point $b - c$ is not necessarily a strong $(a - d)$-derivative of $f - g$ at x. However, in some cases, it might be such a derivative.

As a consequence, Theorem 3.3 gives the well-known result of the classical analysis.

Corollary 3.12 (Goldstein, et al, 1987)**.** a) If $b = f'(x)$ and $c = g'(x)$,then $b \pm c = (f \pm g)'(x)$.

b) If $b = f'(x)$ and $r \in \boldsymbol{R}$, then $rb = rf'(x)$.

Corollary 3.13. The set SXFD(f,x) = $\cup_{r\geq 0}$ SXFD$_r$(f,x) of all fuzzy derivatives of f at x is a real linear space.

Corollary 3.14. a) (Global additivity) If $b = \text{std}_r{}^z/_{dx}f$ and $c = \text{std}_q{}^z/_{dx}g$, then $b \pm c = \text{std}_{r+q}{}^z/_{dx}(f \pm g)$.

b) (Global uniformity) If $b = \text{std}_r{}^z/_{dx}f$ and $a \in \boldsymbol{R}$, then $ab = \text{std}_r{}^z/_{dx}\, af(x)$.

Corollary 3.15. The set SXFD(f) = $\cup_{r\geq 0}$ SXFD$_r$(f) of all fuzzy derivatives of f is a real linear space.

Remark 3.11. For weak fuzzy derivatives and weak derivatives, the result of Theorem 3.3.a is invalid as the following example demonstrates.

Example 3.7. Let $f(x) = 1$ when $x \neq u_n = 1/n$; $f(x) = 1/n$ when $x = u_n$, and $g(x) = 1$ when $x \neq v_n = 1/2n$; $g(x) = 1/n$ when $x = v_n$.

Then 1 is a weak 0-derivative of f and g at 0, but 1+1 =2 is not a weak (0+0)-derivative of f + g at 0.

However, for weak fuzzy derivatives, it is possible to deduce some weaker properties of additivity than those possessed by strong fuzzy derivatives.

Let us assume that:

1) $f: X \to \boldsymbol{R}$ and $g: X \to \boldsymbol{R}$ are arbitrary real functions;

2) the sets $\{ \text{wd}_a{}^z/_{dx} f(x) \}$ of all weak a-derivatives of f at x and $\{ \text{wd}_d{}^z/_{dx}\, g(x) \}$ of all weak d-derivatives of g at x are bounded;

and

3) $\sup \{ \text{wd}_a{}^z/_{dx} f(x)\} = u$, $\sup \{ \text{wd}_d{}^z/_{dx}\, g(x) \} = v$.

Proposition 3.13. *If b is a weak a-derivative of f at x and c is a weak d-derivative of g at x, then there is such number* $e \in \boldsymbol{R}$ *that e is a weak (a+d)-derivative of f + g at x and* $e \leq \min \{b+v; c+u\}$.

4. Fuzzy differentiable functions

Definition 4.1. A real function f is fuzzy differentiable (from the left, from the right, from two sides) at a point x from X if there is some number a such that f has a strong centered (strong left, strong right, strong two-sided) a-derivative at x .

Remark 4.1. There are such functions that have no derivatives at any point of ***R*** but are fuzzy differentiable at all points of ***R***. To demonstrate this, let us consider the function $f(x)$ defined by the following formula:

$$f(x) = \sum_{n=1}^{\infty} g(4^{n-1}x)/4^{n-1} \quad \text{where} \quad g(x + n) = |x| \quad \text{for all } x \text{ with } |x| \leq 1/2 \quad .$$

It is demonstrated in (Gelbaum and Olmsted, 1964) that this function has no derivative at any point of ***R***. At the same time, it is possible to prove that 0 is a strong centered and two-sided 5-derivative of f at any point x from ***R***.

Theorem 3.2 provides for the following criterion for fuzzy differentiability.

Proposition 4.1. *A function f is fuzzy differentiable* (*from the left, from the right*) *at a point x from X if and only if*) *the set* $\mathrm{WXFD}_0(f,x)$ *is non-empty and bounded.*

Theorem 4.1. *If a function f is fuzzy differentiable at a point x from X, then there is such a minimal number a that f has a strong centered a-derivative at x, i.e.,* $a = \min\{r;\ \mathrm{SCFD}_r(f,x) \neq \varnothing\}$.

<u>Proof</u>. Let us consider the set $\mathrm{FD}\,(f,x) = \{\ r;\ f$ has a strong centered r-derivative at x $\}$ and the number $a = \inf \mathrm{FD}\,(f,x)$. If c is a number from $\mathrm{FD}\,(f,x)$, then for any sequence $l = \{\ (f(z_i) - f(x_i))/(z_i - x_i);\ z_i > x > x_i,\ i \in \omega,\ x = \lim_{i\to\infty} x_i = \lim_{i\to\infty} z_i\ \}$ there is such a point u in X that $u = c\text{-}\lim l$. By Proposition 2.9 from (Burgin, 2000) the set $\mathrm{FD}\ l = \{\ r;\ l$ has an r-fuzzy limit$\}$ is a closed ray. By the definition of a strong centered fuzzy derivative, $\mathrm{FD}(f,x) = \cap_t \mathrm{FD}\ l_t$ for all such sequences l_t that have the form that is similar to the form of l .

Any intersection of closed rays is a closed set. It may be void, but in our case $\mathrm{FD}(f,x) \neq \varnothing$. Consequently, it is a closed ray of positive numbers. As a closed set, $\mathrm{FD}(f,x)$ contains the point a , *q.e.d.*

Corollary 4.1. A function f has the classical derivative at x if and only if $\mathrm{FD}(f,x) = 0$.

Corollary 4.2. A function f is differentiable on X if and only if $\mathrm{FD}(f,x) = 0$ for all points x from X .

These results and some others (e.g., Theorems 4.2 - 4.5) demonstrate that the concept of a fuzzy differentiability is a natural extension of the concept of the conventional differentiability.

Remark 4.2. For weak fuzzy derivatives Theorem 4.1 is invalid.

The sets $YXFD_r(f,x)$, introduced in the previous section, define the complete fuzzy derivative $YXFD(f,x)$ of f at a point $x \in X$ having type Y,X . It is called also a complete local fuzzy derivative of f . Each $YXFD(f,x)$ is a fuzzy subset of $\boldsymbol{R}$, and namely, $YXFD(f,x) = (\boldsymbol{R}, \mu_x , [0,1])$ where the membership function μ_x is defined by the equality $\mu_x(z) = 1/(1 + m(x,z))$ where $m(x,z) = \min \{r \in \boldsymbol{R}^+ ;\ z \in YXFD_r(f,x)\}$. This minimum exists by Theorem 4.1.

If we take the join of all complete fuzzy derivatives $YXFD(f,x)$, we obtain the complete global fuzzy derivative $YXFD\, f$ of f on X having type Y,X. Here $YXFD(f,x)$ is a fuzzy binary relation on $\boldsymbol{R}$, and namely, $YXFD\, f = (\boldsymbol{R}^2, \mu , [0,1])$ where the membership function μ is defined by the equality $\mu(x,z) = 1/(1 + m(x,z))$. The result of Theorem 4.1 provides for correctness of the definition of the fuzzy sets $YXFD(f,x)$ for all $x \in \boldsymbol{R}$ as well as of the fuzzy set $YXFD\, f$. By Corollary 4.1 , $\mu_x(z) = \mu(x,z) = 1$ if and only if $z = f'(x)$ at the point x . Consequently fuzzy sets $YXFD(f,x)$ and $YXFD\, f$ are fuzzy set derivatives of crisp (ordinary) functions related to similar constructions from (Kalina, 1997; 1998; 1999).

Here we can see in an explicit form how investigation of ordinary functions involves construction of fuzzy sets and relations.

Remark 4.3. Complete fuzzy derivatives do not possess many properties of ordinary derivatives as well as of other (strong centered, left, right, two-sided etc.) fuzzy derivatives. For example, let us take $f(x) = |x|$ and $g(x) = -|x|$. Then $f + g$ is the function identically equal to zero. All its derivatives are also equal to zero at all points. Consequently, $\mu_0(0) = 1$ for $f + g$.

At the same time, the value of the membership function $\mu_0(0)$ for the sum of any pair of fuzzy sets $YXFD(f,0)$ and $YXFD(g,0)$ is equal to 1/2 . Thus the complete fuzzy derivative of the sum $f + g$ of these functions is not equal to the sum of the complete fuzzy derivatives of f and of g . However, it is well-known that the conventional differentiation is a linear operator (Dieudonné, 1960) and the same is true for all kinds (strong centered, left, right, two-sided etc.) of strong fuzzy derivatives (cf. Corollary 3.12).

Let us investigate interrelations between different types of fuzzy differentiation and continuity of functions.

Theorem 4.2. *If a function f is fuzzy differentiable from the right (left) at a point x from X, then f is continuous at x from the right (left).*

Proof. Let f be a fuzzy differentiable from the right function at a point x from X. Then by Definition 4.1, then there is some number a such that f has a strong right a-derivative b at x. By the definition it means that for any sequence $\{x_n; x_n \in X \text{ and } x_n > x\}$, if $x = \lim_{n\to\infty} x_n$, then $b = a$-lim l where $l = \{ (f(x) - f(x_n))/(x - x_n) ; n \in \omega \}$.

By the definition of fuzzy limits (cf. Section 2), there is some m $\in \omega$ that for all n > m the following inequality is valid: $\rho(b, (f(x) - f(x_n))/(x - x_n)) \leq a$. It implies the inequality $| (f(x) - f(x_n))/(x - x_n) | \leq a + |b|$.

Consequently, we have $| f(x) - f(x_n) | \leq (a + |b|)| x - x_n|$. Thus, convergence of a sequence $\{x_n ; x_n > x , n \in \omega \}$ to the point x implies that $f(x_n) \to f(x)$.

It means that the function f is continuous at x from the right.

Theorem 4.3. *If a function f is fuzzy differentiable from two sides at a point x from X, then f is either continuous at x or has a removable singularity at x.*

Proof. Let f be a fuzzy differentiable from two sides function at a point x from X. Then by Definition 4.1, then there is some number a such that f has a strong two-sided a-derivative b at x. By the definition it means that for any two sequences $\{x_n; x_n \in X \text{ and } x_n < x\}$ and $\{z_n; z_n \in X \text{ and } z_n > x\}$, if $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then $b = a$-lim l where $l = \{ (f(z_n) - f(x_n))/(z_n - x_n) ; n \in \omega \}$.

By the definition of fuzzy limits (cf. Section 2), there is some m $\in \omega$ that for all n > m the following inequality is valid: $\rho(b, (f(x) - f(x_n))/(x - x_n)) \leq a$. It implies the inequality $| (f(x) - f(x_n))/(x - x_n) | \leq a + |b|$.

Consequently, we have $| f(z_n) - f(x_n) | \leq (a + |b|)| z_n - x_n|$. Thus, convergence of the sequences $\{x_n ; x_n > x , n \in \omega \}$ and $\{z_n; z_n \in X \text{ and } z_n > x\}$ to the point x implies that sequences $\{f(x_n) ; n \in \omega \}$ and $\{ f(z_n) ; n \in \omega \}$ have the same limit when $n \to \infty$. As these sequences are taken arbitrarily, this means that the function f is either continuous at x or has a removable singularity at x.

Theorem is proved.

Theorem 4.4. *The following conditions are equivalent:*

1) *a function f is fuzzy differentiable at a point x from X;*

2) *a function f is fuzzy differentiable from the left and from the right at a point x from X;*

3) *a function f is continuous at x and is fuzzy differentiable from two-sides at x.*

Proof. 1) 3) Let f be a fuzzy differentiable function at a point x from X. Then by the definition 4.1, then there is some number a such that f has a strong centered a-derivative b at x. By the definition it means that for any sequence $\{x_n; x_n \in X\}$, if $x = \lim_{n\to\infty} x_n$, then $b = a\text{-}\lim l$ where $l = \{ (f(x) - f(x_n))/(x - x_n) ; n \in \omega \}$.

By the definition of fuzzy limits (cf. Section 2), there is some $m \in \omega$ that for all $n > m$ the following inequality is valid: $\rho(b, (f(x) - f(x_n))/(x - x_n)) \leq a$. It implies the inequality $| (f(x) - f(x_n))/(x - x_n) | \leq a + |b|$.

Consequently, we have $| f(x) - f(x_n) | \leq (a + |b|)| x - x_n|$. Thus, convergence of a sequence $\{x_n ; n \in \omega\}$ to the point x implies that $f(x_n) \to f(x)$.

It means that the function f is continuous at x. Besides, Proposition 3.1 implies that b is a strong two-sided a-derivative of f at x, i.e., f is fuzzy differentiable from two-sides at x.

Implications 3) 2) and 2) 1) follow from Definitions 4.1 and 3.3.

Theorem is proved.

Corollary 4.3. If f is a fuzzy differentiable function at a point $x \in X$, then f is continuous at x.

Corollary 4.4 (any course of the calculus, cf., for example, (Ribenboim, 1964; Fihtengoltz, 1955)). If f is a differentiable function at a point $x \in X$, then f is continuous at x.

From local fuzzy differentiability, we come to a global fuzzy differentiability on some set.

Let $D \subseteq X$ and $f: X \to \boldsymbol{R}$ is a real function.

Definition 4.2. A function f is fuzzy differentiable (from the left, from the right) on D if f is fuzzy differentiable (from the left, from the right) at any point x from D.

As a consequence of Theorem 4.4, we obtain the following results.

Theorem 4.5. *Any fuzzy differentiable on a set D function f is continuous on D.*

Corollary 4.5 (any course of the calculus, cf., for example, (Ribenboim, 1964; Fihtengoltz, 1955)). Any differentiable function f is continuous.

Theorem 4.6. *Any fuzzy differentiable from the left and from the right on a set D function f is continuous on D.*

The concept of a weak fuzzy derivative implies the concept of a weakly fuzzy differentiable function.

Definition 4.3. A function f is called weakly fuzzy differentiable (from the left, from the right) at a point x from X if there is such number a that f has a weak centered (weak left, weak right) a-derivative at x.

Remark 4.4. For weak differentiability, Theorems 4.2 - 4.6 are not true as it is demonstrated by the following example.

Example 4.1. Let $m_Q(x)$ be the membership function of the set of rational numbers, i.e., $m_Q(x)$ is equal to 1 when x is a rational number and $m_Q(x)$ is equal to 0 when x is an irrational number. This function is not continuous at any point from $\boldsymbol{R}$, but it has a weak derivative at any point from $\boldsymbol{R}$, which is equal to 0.

However, existence of weak derivatives implies additional properties for functions.

Theorem 4.7. *If a function f is weakly fuzzy differentiable from the right or from the left at a point x from X, then the point* $(x, f(x))$ *is an adherent point of the graph of the function f.*

Let us consider connections between weak fuzzy differentiation and weak continuity, which is introduced in (Collingwood and Lohwater, 1966).

Definition 4.4. A function f is called weakly continuous at a point x from X if there are such sequences $l = \{ a_i ; a_i > 0, i \in \omega \}$ and $h = \{b_i ; b_i > 0, i \in \omega \}$ that $\lim_{i\to\infty} a_i = 0$, $\lim_{i\to\infty} b_i = 0$, $\lim_{i\to\infty} f(x+ a_i) = f(x) = \lim_{i\to\infty} (x- b_i)$.

Proposition 4.2. *If f is a weakly fuzzy differentiable from the left and from the right function at a point x from X, then f is weakly continuous at x.*

Corollary 4.6. Any weakly fuzzy differentiable from the left and from the right function f is weakly continuous.

Definition 4.5. A function f is called weakly symmetrically fuzzy differentiable at a point x from X if there is such number a that f has a weak centered a-derivative at x such that (cf. Definition 3.1) $x- z_i = x_i - x$ for all $i \in \omega$.

Definition 4.6 (Ciesielski and Larson, 1993-94; Ciesielski, 1995-96). A function f is called weakly symmetrically continuous (or uniformly antisymmetric) at a point x from X if there is such a sequence $l = \{ a_i ; a_i > 0, i \in \omega \}$ that $\lim_{i\to\infty} a_i = 0$, $\lim_{i\to\infty} b_i = 0$, $\lim_{i\to\infty} (f(x+ a_i) - f(x- a_i)) = 0$.

Proposition 4.3. *If f is a weakly symmetrically fuzzy differentiable function at a point x from X, then f is weakly symmetrically continuous at x.*

Corollary 4.7. Any weakly symmetrically fuzzy differentiable function *f* is weakly symmetrically continuous.

Let us consider geometrical aspects of fuzzy differentiation.

Definition 4.7. A pair of direct lines (*l,h*) on a plane XY is called regular if neither of these lines is parallel to the *y*-coordinate line.

Each pair of direct lines (*l,h*) on a plane XY constitutes two pairs of two vertical angles and thus, it divides this plane into two parts. One of these parts is the set of all points between the sides of one pair of these vertical angles while the second part is constituted by the set of all points between the sides of the second pair of these vertical angles. We call this division a vertical division of a plane.

In the Figure 1, the first part a vertical division of a plane consists of the regions 1 and 2, while the second part consists of the regions 3 and 4 .

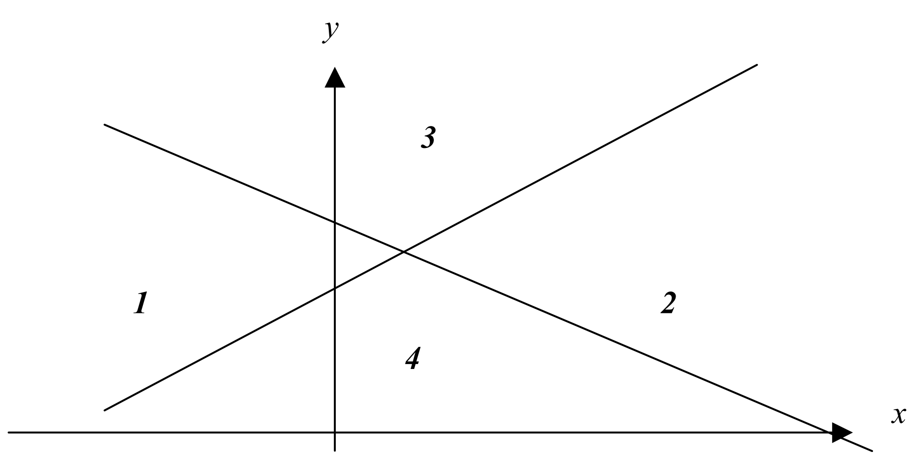

Fig. 1. A vertical division of a plane by two lines

Definition 4.8. The part of the plane which does not contain the direct line that goes through the common vertex of the angles and is parallel to the y-coordinate line of is called regular, while the second part is called irregular.

In the Figure 1 the first part of the plane consisting of areas 1 and 2 is regular while the second part consisting of areas 3 and 4 is irregular.

Theorem 4.8. *A function* f *is fuzzy differentiable at a point* x *from* X *if and only if there is such a regular pair of lines* (l, h) *and such a neighborhood* Ox *of* x *that all points* $f(z)$, *which belong to* Ox *also belong to a regular part determined by* (l, h) .

Theorem 4.9. *If* $f(0) = c$ *and any weak a-derivative* wd $_a{}^z/_{dx} f$ *of f is equal to zero, then the graph of f is situated inside the regular part of the plane* $\boldsymbol{R}^2$, which is *determined by the lines* $y = \pm ax + c$.

When $a = 0$, the pair of direct lines reduces to a single line parallel to the x-axis, and we obtain the following result.

Corollary 4.8. If all weak 0-derivatives of f are equal to zero, then f is the constant function.

Corollary 4.9. If the strong centered a-derivative sd$_a{}^c/_{dx} f$ of f exists and is equal to zero, then graph of f is situated inside the regular part of the plane $\boldsymbol{R}^2$, which is determined by the lines $y = \pm ax + c$.

It implies the classical result.

Corollary 4.10 (Ribenboim, 1964; Randolph, 1968). If the derivative $f\,'$ of f exists and is equal to zero, then f is the constant function.

Remark 4.5. Classical derivatives are used for approximation of arbitrary differentiable functions by linear functions, which are simpler than the initial functions. In the same way, fuzzy derivatives may be used for approximation of arbitrary fuzzy differentiable functions by linear functions. Although fuzzy derivatives do not allow one to achieve infinitely exact approximation like classical derivatives, they provide means, as Theorems 4.8 and 4.9 demonstrate, for arbitrarily precision, are more realistic, and extend the scope of functions, to which such approximation technique is applicable.

Remark 4.6. It is possible to define fuzzy derivatives of all types as linear approximations of a given function, constructing fuzzy Frechet derivatives by means of fuzzy limits. This approach connects fuzzy differential calculus of real functions with interval analysis (Moore, 1966).

5. Interpretations

The classical geometric interpretation (cf., for example, Goldstein *et al*, 1987; Shenk, 1979) of the classical derivative of a function f is that its value $f'(c)$ at a point c is the slope of the tangent line to the graph of f at the point (c,f(c)) . Investigation of curves and their tangent lines gave birth to differential calculus. To specify the tangent line to f at $(c,f(c))$, it is necessary to consider the line through the points $(c,f(c))$ and $(c+h,f(c+h))$. Its slope is given by the formula $m(h) = (f(c+h) - f(c))/((c+h) - c)= (f(c+h) - f(c))/h$. If $\lim_{h\to 0} (f(c+h) - f(c))/ h$ exists, it is said that this limit is the slope of the tangent line to f at the point $(c,f(c))$.

This limit is the classical derivative of $f(x)$ at c. That is, the classical derivative of $f(x)$ at c is equal (in a general case by definition) to the slope of the tangent line through the point $(c,f(c))$.

This geometric interpretation and the argument justifying it extend equally well to other functions, provided that their graphs are sufficiently smooth—*i.e.,* smooth enough to guarantee the existence of a unique tangent line passing through the point in question. Conversely, if the graph of $f(x)$ fails to satisfy this condition at a point, then the derivative fails to exist; it has already been shown that the function $f(x) = |x|$ fails to have a derivative at $x_0 = 0$. This is reflected geometrically by the fact that the graph of $f(x)$ has a "corner" at the point $[x_0, f(x_0)] = (0, 0)$. However, the function $f(x)$ has fuzzy derivatives at this point. A fuzzy derivative represents the angle in which tangent lines to the parts of the graph of $f(x)$, for which the point $[x_0, f(x_0)]$ is an end point. In a general case, fuzzy derivative represents the angle in which tangent lines to the parts of the graph of $f(x)$, for which $[x_0, f(x_0)]$ is a limit point.

In general, when we consider the situation with fuzzy derivatives, we do not have, as a rule, the tangent line because fuzzy derivatives exist for a much wider range of functions than classical derivatives. Instead of the tangent line, the notions of the tangent angle and r-tangent line are introduced.

Let b be a strong centered r-derivative of f at c.

Definition 5.1. Direct lines $y = (b+r)x+c$ and $y = (b-r)x+c$ are called boundary r-tangent lines through the point $(c,f(c))$.

Definition 5.2. The pair of vertical angles defined by the boundary r-tangent lines (cf. Figure 2) is called the r-tangent angle of f at the point $(c,f(c))$.

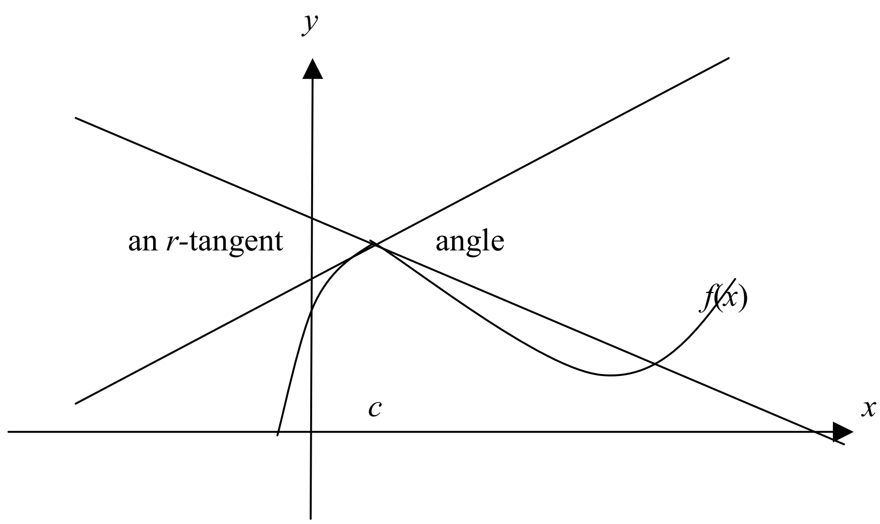

Figure 2. The r-tangent angle of the function *f(x)* at a point (*c, f(c)*) .

Definition 5.3. Any line that is inside the *r*-tangent angle of *f* at the point (*c,f(c)*) is called an *r*-tangent line of *f* through the point (*c,f(c)*).

There are many functions that have no tangent lines but have many *r*-tangent lines for some *r*. As an example, we can take the function from the remark 4.1.

Let us find some properties of these constructions.

Proposition 5.1. *A* 0-*tangent line* (*if it exists*) *is unique and coincides with the classical tangent line.*

From Theorem 3.1, we obtain two following results.

Proposition 5.2. *If at least one r-tangent line of f through the point* (*c,f(c)*) *exists, then for any real function f, any number r from* $\boldsymbol{R}^+$ *and number c from* ***R***, *there two r-tangent lines of f through the point* (*c,f(c)*) *such that one of them has the maximal slope and the second one has the minimal slope.*

Proposition 5.3. *For any function f, numbers r from* $\boldsymbol{R}^+$ *and c from* ***R***, *there are the biggest and the least r-tangent angles of f at the point* (*c,f(c)*) .

Definition 5.4. If α is the angle determined by two direct lines y = nx and y = mx, then the direct measure of α is equal to |n -m| .

From Theorem 4.1, we obtain the following result.

Proposition 5.4. *For any real function f, any number r from* $\boldsymbol{R}^+$ *and any number c from* ***R*** , *there exists the r-tangent angle for f at the point* (*c,f(c)*) *with the least direct measure*.

Taking the definitions, we see that the classical derivatives of functions are idealizations because their values cannot be computed in practice. Really, there is a traditional method to compute the value of the derivative of $f(x)$:

1. First calculate $(f(x+h) - f(x))/h$ for $h \neq 0$.
2. Then let h approach zero.
3. The quantity $(f(x+h) - f(x))/h$ will approach $f'(x)$.

However, this method does not take into consideration that it is possible to perform all calculations only to some precision which is bigger than zero. As a consequence, such a procedure when realized on a computer or calculator provides only for a calculation of a fuzzy derivative. Methods of fuzzy differential calculus make possible to estimate its fuzziness.

An important interpretation of the ordinary derivative of a function is as a rate of change. The same is true for fuzzy derivatives only their estimation is, as a rule more exact. Let us consider an example from (Goldstein *et al*). A weight of an animal may be treated as a function of time, say W(t). Then the average rate of change of W(t) with respect to t from time u to time $u+h$ is equal to (W($u+h$) - W(u)/ h. When h converges to 0, we obtain the ordinary derivative of W(t) at u. But in reality measurements of W(t) may be conducted only from time to time, i.e., at some discrete points of time. Consequently, we can speak realistically only about some fuzzy derivative (a-derivative) of W(t) at u. It is taken with some precision equal to the number a. At the same time the ordinary derivative of W(t) at u gives the rate of change of W(t) with respect to t from time u to time $u+h$ precisely, while no such precision exists in practical situation. Any change of the variable number a reflects precisely the change of precision.

An important physical application of the classical derivative is embodied in the concept of instantaneous velocity. In fact, a precise definition of the velocity of a moving particle at a given time, t, involves exactly the same limiting process that occurred in the definition of the derivative. Consequently, the fuzzy derivative represents an approximation to the exact value of instantaneous velocity. At points of collision, particles do not have the instantaneous velocity, as their trajectory is not differentiable at such points. However, we can correspond to particles an approximate instantaneous velocity at these points as the trajectory may be fuzzy differentiated at these points.

The next example of interpretation is connected with economics. In recent years economic decision making has become more urgent. It caused higher mathematical orientation of the corresponding procedures. Faced with huge masses of statistical data, depending on hundreds or even thousands of different variables, business analysis and economists have increasingly turned to mathematical methods helping them to analyze what is going on and what might happen. In this area, different methods and constructions of classical calculus are utilized. Thus, the theory of the firm utilizes such functions as

C(x) = cost of producing n units of the product,

R(x) = revenue generated by producing n units of the product,

P(x) = R(x) - C(x) = the profit (or loss) generated by producing n units of the product, and so on.

Their derivatives are called marginal function. In such a way, the function C'(x) is called the marginal cost function. Its value, C'(a) for $x = a$ is called the marginal cost of production at level a. In reality, levels are not points but intervals. Consequently, values of C'(x) may be determined computed not exactly but only with some precision giving an approximation of the considered function. It means that such values are fuzzy derivatives. Moreover, the functions C(x), R(x), and P(x) often are defined only for nonnegative integers giving rise to a set of discrete points. In studying these functions, economists usually draw a smooth curve through these points and study this curve. However, such approximation is not the best because a more detailed consideration involves fuzzy continuous functions introduced and studied in (Burgin and Šostak, 1992; 1994; Burgin, 1992; 1995).

6. Conclusions

Thus we have demonstrated that in a broader context of fuzzy limits and derivatives, it is possible to extend the majority of basic results of the classical mathematical analysis. In particular, it is shown that investigation of properties of ordinary functions involves construction of fuzzy sets and relations. Moreover, such a transition to a fuzzy context provides for completion of some basic results of the classical mathematical analysis. For example, it is well known that any convergent sequence of real numbers is bounded. The converse is not true. So, convergence is only sufficient but not necessary condition for boundedness. However, fuzzy convergence makes attainable to prove (cf. Theorem 2.3) a complete criterion of boundedness. Namely, *a sequence of real numbers is bounded if and only if it is fuzzy convergent*. A similar completion of a classical result is obtained in (Burgin and Šostak, 1992; Burgin, 1995) for such basic theorem of analysis that states that a continuous function on a closed interval is bounded.

It is necessary to remark that fuzzy set theory has been developing in many aspects parallel to the classical mathematics. Consequently, differential calculus has been developed for different kinds of conventional fuzzy functions by Zadeh (1978), Dubois and Prade (1980; 1982), Goetshel and Voxman (1986), Puri and Ralescu (1983), Kaleva (1987), Buckley and Yunxia (1991), and Kalina (1997; 1998; 1999). In contrast to all these works, the research of Janiš (1999) belongs to neoclassical analysis and we consider it separately.

The attention of the authors in the first four papers is focused on functions that are not necessarily fuzzy but "carry" the possible fuzziness of their arguments (cf., also (Zimmermann, 1991)). The uncertainty of knowledge about the precise location of the argument induces an uncertainty about the value of the derivative of a function at this point. This is represented by treating functions with fuzzy numbers as their domain and/or range. Kalina (1997; 1999) considers three basic types of vagueness (on the *y*-axis, on the *x*-axis, and on both). It implies three constructions for fuzzy derivatives, which are investigated in this work.

Janiš (1999) introduces and studies a nearness derivative $int(D_\alpha)$ for functions in spaces of real numbers with a nearness. It is a set-valued function in contrast to fuzzy derivatives, which are considered in this work and are point-valued functions. His construction is based on the concept of fuzzy continuity from (Burgin and Šostak, 1992; 1994). His main result gives necessary and sufficient condition for a real function to be increasing/decreasing. As corollaries, Janiš proves generalizations of Rolle and Lagrange mean value theorems for arbitrary real functions.

Neoclassical analysis does not only bring new results, which complete their classical analogues, but also produces deeper insights and a better understanding of the classical theory. In addition to this, the neoclassical analysis makes possible not only to extend ordinary concepts obtaining new results for classical structures, but also provides for elaboration of new useful concepts for the classical mathematical analysis. One of such concepts is introduced in this work. Namely, it is fuzzy limits of sets of sequences.

This does not only bring new mathematical results, but also produces deeper insights and a better understanding of the classical theory. In addition to this, it makes possible to eliminate discrepancies existing in numerical analysis. The problem is that computations are realized on finite machines, while many processes of mathematics, such as differentiation and integration, demand the use of a limit, which is an infinite process. As a consequence, correct algorithms based on classical methods of calculus, when implemented, turn into unreliable programs. Neoclassical analysis treats such processes more adequately. It is demonstrated by applications of neoclassical analysis to problems of numerical computations and control (Burgin and Westman, 2000). Consequently, the new technique provides for a better utilization of numerical computations for artificial intelligence, especially in the case when uncertainty of computation is multiplied by the uncertainty of input information.

Structures from neoclassical analysis are also used for the development of algorithmical tools for computer simulation (Burgin, 2001).

For the further development of neoclassical analysis, it would be useful to consider the following problems.

Problem 1. *Investigate specific properties of the complete local and global fuzzy derivatives of ordinary functions*.

Analysis of function of one variable is only the first step in functional analysis. After making it, it is necessary to consider functions of several variables.

Problem 2. *Develop fuzzy differential calculus for functions in the n-dimensional real vector space* $\boldsymbol{R}^n$.

The next step is to analyze functions on infinite dimensional spaces.

Problem 3. *Develop fuzzy differential calculus for functions in Hilbert spaces.*

After solving this problem, it is natural to extend fuzzy analysis to more general contexts.

Problem 4. *Develop fuzzy differential calculus for extrafunctions* (Burgin, 1993).

Problem 5. *Develop a non-standard fuzzy differential calculus*.

Methods developed by Zadeh (1978), Goetshel and Voxman (1986), Puri and Ralescu (1983), Kaleva (1987), Buckley and Yunxia (1991), and Kalina (1997; 1998; 1999) provide means for differentiation of fuzzy functions. In some sense, it is an exact differentiation of fuzzy functions because it is based on the standard concept of a limit. However, basing on the theory of fuzzy limits, it is possible to elaborate new methods for differentiation of fuzzy relations, in particular, fuzzy functions. Consequently we have the following problem.

Problem 6. *Construct a theory of fuzzy differentiation of fuzzy functions.*

Such a theory might be obtained by synthesizing methods that are developed for differentiation of fuzzy functions with constructions defined in this work. The first stage in doing this is connected with the following problem.

Problem 7. *Elaborate a theory of fuzzy limits for sequences of fuzzy numbers and fuzzy functions.*

As it is demonstrated in Section 3, weakly differentiable functions are linked to weakly continuous functions. Let us consider some problems related to these connections. It is proved in (Collingwood and Lohwater, 1966) that any real function is weakly continuous on the complement of a countable set.

Problem 8. *Are there real functions that are nowhere weakly differentiable (from the left, from the right)?*

The following problem is connected to a similar problem for weakly symmetrically continuous functions (Ciesielski and Larson, 1993-94; Thomson, 1994).

Problem 9. *Does there exist a weakly differentiable function f with the range f(R) being: (a) finite; (b) bounded ?*

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