A CALCULUS FOR DIFFERENTIAL INVARIANTS OF PARABOLIC GEOMETRIES

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ABSTRACT. The Wünsch's calculus for conformal Riemannian invariants is reformulated and essentially generalized to all parabolic geometries. Our approach is based on the canonical Cartan connections and the Weyl connections underlying all such geometries.

The differential invariants for various geometric structures are the core ingredients for numerous applications both in geometry and geometric analysis. The invariants of conformal Riemannian manifolds themselves attracted a lot of attention in the course of the last 100 years.

In the differential geometry of manifolds equipped with a linear connection, the so called 'first invariant theorem' says that all the invariants are expressions built of the curvature and the covariant derivatives of the connection by means of algebraic tensorial invariants. Let us call them the *affine differential invariants*. The analogous 'first invariant theorem' for Riemannian geometries says that all differential invariants are built of the affine invariants of the canonical Levi–Civita connection with additional help of the orthogonal algebraic invariants. Thus, the construction of all invariants is described easily in principle, but the difficult questions on the relations between the individual invariants remains.

A conformal Riemannian geometry is defined as the class of conformal Riemannian metrics and so the above Riemannian invariant theorem can be reflected in the definition of the conformal invariance. Thus, a conformal invariant is a Riemannian invariant in terms of a metric from the conformal class, such that any change of the metric leaves its values unchanged. Another equivalent definition of the conformal structures treats them in terms of classical G–structures as reductions of the linear frame bundle to the structure group G_0, the group of all conformal Riemannian transformations in the given dimension. This is also equivalent to the choice of a class of linear connections without torsion whose structure group is the conformal Riemannian group. Such a class of connections is always parameterized by one–forms and the Levi–Civita connections coming from the metrics in the conformal class form a subclass parameterized by exact one–forms. The treatment of conformal Riemannian invariants goes back to Cartan, Thomas, Schouten, and others (see e.g. [10, 18, 19]). The broader class of the conformal connections was exploited by Weyl. A lot of spectacular tricks to build invariant expressions have been developed, and some of them were turned into a quite effective calculus for conformal invariants by Wünsch in a series of papers (see e.g. [20]).

Motivated by the rich geometry of conformal Riemannian manifolds and by the recent development of geometries modeled on homogeneous spaces G/P with G semisimple and P parabolic, the Weyl structures and the preferred connections were introduced in this general framework in [7]. In particular, the notions of scales, closed and exact Weyl connections, and (Schouten's) Rho–tensors were extended,

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and straightforward generalizations of classical normal coordinates in affine geometries were discussed. In this setting, the Weyl connections on a parabolic geometry of type G/P correspond to reductions of the parabolic structure group P to its reductive part G_0.

Following the conformal Riemannian example, we consider the differential invariants of parabolic geometries of any fixed type G/P as affine invariants of the Weyl connections, expressed by means of the algebraic G_0–invariant operations, and with values independent of the particular choices of Weyl connections.

The paper is organized as follows: A brief review of the main ingredients of the parabolic geometry theory occupies the first section. Next, an exposition of a calculus for the differential invariants is presented, together with an analogy to the above mentioned 'first invariant theorem'. Exactly as in the case of the conformal Riemannian calculus due to Wünsch, the theorem rather describes a much larger class of expressions distinguished by very special algebraical transformation behavior. All differential invariants are then special cases of the latter expressions and thus the difficulties in their treatment are heavily reduced.

1. Parabolic geometries and Weyl connections

We shall use the terminology and notation of [7] and the reader is also advised to consult the latter paper for more details on all results from this section. The complete procedure deriving the Cartan connection from more elementary data on manifolds was first worked out in [17], a much simpler and slightly stronger version appeared in [6], while a much more transparent construction based on the Weyl connections is also described in [7]. Let us point out that this link is of nearly no importance for our calculus below.

1.1. **The Cartan connections.** In the sequel, a Cartan geometry of type G/P is understood as an absolute parallelism on a principal fiber bundle $\mathcal{G} \to M$ with structure group P, enjoying suitable invariance properties with respect to the principal action of P. Thus, we may view these objects as deformations of the homogeneous space $G \to G/P$ together with the Maurer–Cartan form $\omega \in \Omega^1(G; \mathfrak{g})$. More explicitly, the *Cartan connection* ω on $\mathcal{G}$ is required to obey the following properties

(1) $\omega(\zeta_X)(u) = X$ for all $X \in \mathfrak{p}$, $u \in \mathcal{G}$ (the connection reproduces the fundamental vertical fields)
(2) $(r^b)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ (the connection form is equivariant with respect to the principal action)
(3) $\omega_{|T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$ (the absolute parallelism condition).

The *parabolic geometries* are Cartan connections with a choice of a parabolic subgroup P in a semisimple real Lie group G. The morphisms of the parabolic geometries are principal fiber bundle morphisms $\varphi : \mathcal{G} \to \mathcal{G}'$ with the property $\varphi^*(\omega') = \omega$.

We shall always fix the choice of P and G together with the grading

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$$

on the Lie algebra $\mathfrak{g}$ of the group G, such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ is the Lie algebra of P, and $\mathfrak{g}_0$ is the Lie algebra of the reductive part $G_0 \subset P$. As well known, this represents no loss of generality. We shall also use the notation

$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$$
$$\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k.$$

and we also write $P_+ = \exp \mathfrak{p}_+$ and $G_- = \exp \mathfrak{g}_-$ for the corresponding nilpotent subgroups. Let us recall that for each $g \in P$, there are the unique elements $g_0 \in G_0$

and $\Upsilon_i \in \mathfrak{g}_i$, $i = 1, \dots, k$, such that

$$g = g_0 \cdot \exp \Upsilon_1 \cdot \ldots \cdot \exp \Upsilon_k.$$

This decomposition reflects the fixed splitting of the filtration of $\mathfrak{p}_+$ by P–submodules, i.e. our fixed isomorphism $\operatorname{gr}\mathfrak{p}_+ \to \mathfrak{p}_+$. Finally, let us point out that there is the unique element E in $\mathfrak{g}_0$ (also fixed by our choice of the grading) with the property $\operatorname{ad} E_{|\mathfrak{g}_i} = i \cdot \operatorname{id}_{\mathfrak{g}_i}$ for all $i = -k, \dots, k$. We call E the *grading element.*

The absolute parallelism ω provides the *constant vector fields* $\omega^{-1}(X) \in \mathcal{X}(\mathcal{G})$ defined for all $u \in \mathcal{G}$ and $X \in \mathfrak{g}$ as

$$\omega(\omega^{-1}(X)(u)) = X.$$

In particular due to our fixed splitting of $\mathfrak{g}$, there are the *horizontal vector fields* $\omega^{-1}(X)$ with $X \in \mathfrak{g}_-$. Of course, the equivariance of ω says that the constant vector fields generalize the left invariant vector fields at the homogeneous model $G \to G/P$, i.e.

$$Tr^g \cdot \omega^{-1}(X)(u) = \omega^{-1}(\operatorname{Ad}_{g^{-1}} \cdot X)(u \cdot g),$$

where r^g is the right principal action by the element $g \in P$.

1.2. **The curvature.** The structure equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$ on the homogeneous model makes sense for each Cartan connection ω and provides the two–form

$$K = d\omega + \frac{1}{2}[\omega, \omega]$$

called the *curvature.* The equivariance properties of the Cartan connections imply that K is always a horizontal two–form and so the curvature is completely determined by the *curvature function* $\kappa \in C^\infty(\mathcal{G}, \Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g})$,

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = [X, Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)]).$$

Of course, the values of both the connection and the curvature function split according to the corresponding splitting of $\mathfrak{g}$ into $\omega = \omega_- + \omega_0 + \omega_+$ and $\kappa = \kappa_- + \kappa_0 + \kappa_+$ but the individual components have to be dealt with with care, because only the right filtration by $\mathfrak{g}^i = \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_k$ is P–invariant. Thus κ_- is a well defined object if its values are considered in the quotient $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$. The latter component of the curvature is called the *torsion* of the Cartan connection.

Clearly, the curvature is the obstruction to the integrability of the horizontal distribution $\omega^{-1}(\mathfrak{g}_-)$ in $T\mathcal{G}$ and the Cartan connection is locally isomorphic to its homogeneous model if and only if the curvature vanishes, see e.g. [15].

1.3. **The Weyl structures, connections and Rho tensors.** Consider a fixed parabolic geometry $(\mathcal{G} \to M, \omega)$ together with the principal fiber bundle $\mathcal{G}_0 = \mathcal{G}/P_+$ over M. A Weyl structure is a global smooth G_0–equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the obvious projection $\pi : \mathcal{G} \to \mathcal{G}_0$. Weyl structures always exist and for to such sections $\hat{\sigma}$ and σ, there always is a unique equivariant function $\Upsilon = \Upsilon_1 + \dots + \Upsilon_k :$ $\mathcal{G}_0 \to \mathfrak{p}_+ = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$, such that for all $v \in \mathcal{G}_0$

$$\hat{\sigma}(v) = \sigma(v) \cdot \exp \Upsilon(v) = \sigma(u) \cdot \exp \Upsilon_1(v) \cdots \exp \Upsilon_k(v).$$

For each Weyl structure σ, we shall consider the pullback of the Cartan connection ω split into three parts. Let us notice that while this splitting was not invariant under P, after the reduction to the reductive subgroup G_0, this is a splitting into invariant and thus geometrically well defined objects. We write

$$\begin{aligned}
&\theta = \sigma^*\omega_- = \theta_{-k} + \dots + \theta_{-1} : T\mathcal{G}_0 \to \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \\
&\gamma = \sigma^*\omega_0 : T\mathcal{G}_0 \to \mathfrak{g}_0 \\
&\mathsf{P} = \sigma^*\omega_+ = \mathsf{P}_1 + \dots + \mathsf{P}_k : T\mathcal{G}_0 \to \mathfrak{p}_+ = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k.
\end{aligned}$$

The form θ provides the identification of $\mathcal{G}_0$ with the linear frame bundle on M with structure group G_0 and should be understood as a *soldering form*. In particular, this defines the splitting of the filtration

$$TM = T^{-k}M \supset \cdots \supset T^{-1}M$$

induced by the Cartan connection ω on M. The next component γ is a connection form of a principal connection on $\mathcal{G}_0$. Thus, the first two components provide a reductive Cartan connection $(\theta + \gamma)$ of the type $(G_-{\cdot}G_0)/G_0$ on M. The last component measures the difference between the latter Cartan connection on $\mathcal{G}_0$ and the original Cartan connection ω on $\mathcal{G}$. We call it the *Rho–tensor*. We shall write down these objects with superscript σ if we want to stress the corresponding Weyl structure.

There are two most important points to make. The first one deals with the change of the soldering form, connection form and the Rho–tensor in terms of the difference Υ. The other compares in detail the curvatures (and torsions) of the connections ω and γ. Before we go into these tasks, we shall introduce the geometric objects corresponding to the representations of P and G_0.

1.4. **The natural bundles.** Every P–representation λ on a vector space $\mathbb{V}$ provides the homogeneous vector bundle $G \times_P \mathbb{V}$ and, more generally, the associated vector bundles

$$\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$$

with standard fiber $\mathbb{V}$ over all manifolds with a parabolic geometry of the type G/P. Shortly, we shall talk about P–modules $\mathbb{V}$ and the induced *natural bundles*. Even more generally, we may assume any smooth manifold $\mathbb{S}$ with a smooth action of the Lie group P and all our discussion below can be extended to this case, but we shall restrict ourselves to P–modules.

A quite special class of natural bundles is induced by the G–modules viewed as P–modules. They are called the *tractor bundles*, see [1] for details for some specific geometries and historical links. A very special role is reserved for the *adjoint tractor bundles* coming from the adjoint representation of G on $\mathfrak{g}$. Of course, the Lie bracket itself is Ad–invariant and thus there is the algebraic bracket $\{\ ,\ \}$ on the adjoint tractors.

As well known, the sections of induced bundles correspond to functions s defined on the principal fiber bundle $\mathcal{G}$ and valued in the P–modules $\mathbb{V}$ which are equivariant with respect to the principal action in the following sense (we use bullet to indicate the action λ of P on the P–module)

$$s : \mathcal{G} \to \mathbb{V}, \quad s(u{\cdot}g) = g^{-1} \bullet s(u).$$

We have already seen the curvature function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ representing a section of the adjoint tractor valued form in $\Omega^2(M; \mathcal{A})$, cf 1.2. Of course, the adjoint tractor bundles inherit all the P–invariant objects from $\mathfrak{g}$, including the metric defined by the Cartan–Killing form. Further, the 1–forms on M live in the invariant subbundle $\mathcal{A}^1$ corresponding to $\mathfrak{p}_+$, while the vector fields on M can be viewed (with the help of the Cartan connection on $\mathcal{G}$) as fields in the quotient $\mathcal{A}/\mathcal{A}^1$. In particular, the torsion κ_- of the Cartan connection is a vector valued two–form on M.

Let us also notice that every G_0–invariant object on the P–module $\mathbb{V}$ gives rise to well defined objects on the natural bundles as soon as we fix the choice of a Weyl structure. Moreover, these objects are then always covariantly constant with respect to the induced connection γ. In particular this applies to the algebraic bracket on the adjoint tractors.

1.5. **The transformation rules.** We shall describe the behavior of the soldering form in a more general setting of general P–representations $\mathbb{V}$. In order to write down the formulae, we introduce the following notation: we write $\underline{i}$ for a multiindex $(i_1,\dots,i_k)$ with $i_1,\dots,i_k \geq 0$, and we put $\underline{i}! = i_1!\cdots i_k!$ and $\|\underline{i}\| = i_1+2i_2+\cdots+ki_k$, while $(-1)^{\underline{i}} = (-1)^{i_1+\cdots+i_k}$.

For every representation λ of P on $\mathbb{V}$, the action of the grading element $E \in \mathfrak{g}_0$ decomposes $\mathbb{V}$ into G_0–invariant components

$$\mathbb{V} = \mathbb{V}_0 \oplus \cdots \oplus \mathbb{V}_\ell$$

with the property $\mathfrak{g}_i \bullet \mathbb{V}_j \subset \mathbb{V}_{i+j}$, for all $i = 1,\dots,k$. The Lie algebra $\mathfrak{g}_-$, viewed as $\mathrm{gr}(\mathfrak{g}/\mathfrak{p})$, is a special case. Consider a $\mathbb{V}$–valued function $\tilde{q} : \mathcal{G} \to \mathbb{V}$ and the composition $q = q_1 + \cdots + q_\ell = \tilde{q} \circ \sigma$.

If $\hat{\sigma} = \sigma\cdot \exp \Upsilon_1 \cdots \exp \Upsilon_k$ is a new choice for the Weyl structure with corresponding objects $\hat{\theta}$, $\hat{\gamma}$, $\hat{\mathsf{P}}$, and $\hat{q}$, then the transformation rule for the splitting of the section q reads

$$\hat{q}_\ell = \sum_{\|\underline{i}\|+j=\ell} \frac{1}{\underline{i}!}(-1)^{\underline{i}} \lambda(\Upsilon_1)^{i_1} \circ \ldots \circ \lambda(\Upsilon_k)^{i_k} \circ q_j \tag{1}$$

Of course, the the decomposition of the tangent space into a grading which splits the canonical filtration is a special case of this formula.

Similarly, we may compute the covariant derivative $\hat{\nabla}_\xi$ corresponding to $\hat{\gamma}$ in terms of the covariant derivative ∇ associated to γ and the splitting of the tangent vectors $\xi = \xi_1 + \cdots + \xi$. For every section s of a natural bundle we obtain:

$$\hat{\nabla}_\xi s = \nabla_\xi s + \sum_{\|\underline{i}\|+j=0} \frac{1}{\underline{i}!}(-1)^{\underline{i}} \big(\mathrm{ad}(\Upsilon_1)^{i_1} \circ \ldots \circ \mathrm{ad}(\Upsilon_k)^{i_k}(\xi_j)\big) \bullet s \tag{2}$$

where the bullet denotes the algebraic action of the endomorphisms of the tangent bundle induced by the action of $\mathfrak{g}_0$ on the standard fiber.

Finally, the Rho tensors transform as follows:

$$\begin{aligned}\hat{\mathsf{P}}_i(\xi) = &\sum_{\|\underline{j}\|+\ell=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{j_1}(\xi_\ell) \\ &+ \sum_{m=1}^{k} \sum_{\substack{\|\underline{j}\|+m=i \\ j_1=\cdots=j_{m-1}=0}} \frac{(-1)^{\underline{j}}}{\underline{j}!(j_m+1)} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_m)^{j_m}(\nabla_\xi \Upsilon_m) \\ &+ \sum_{\|\underline{j}\|+\ell=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{j_1}(\mathsf{P}_\ell(\xi)).\end{aligned} \tag{3}$$

Already the simplest case of gradings of the length $k = 1$ provides a good illustration of these formulae. Then the first one is just the identity, while

$$\begin{aligned}\hat{\nabla}_\xi s &= \nabla_\xi s - \{\Upsilon, \xi\} \bullet s \\ \hat{\mathsf{P}}(\xi) &= \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \frac{1}{2}\mathrm{ad}(\Upsilon)^2(\xi).\end{aligned}$$

1.6. **The curvatures of Weyl structures.** As we have seen, each choice of a Weyl structure σ defines a Cartan connection $\theta + \gamma$ on $\mathcal{G}_0$ and the full curvature of this connection is given by the structure equations

$$\begin{aligned}T(\xi,\eta) &= d\theta(\xi,\eta) + [\gamma(\xi),\theta(\eta)]) + [\theta(\xi),\gamma(\eta)] \\ R(\xi,\eta) &= d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)]\end{aligned}$$

where ξ, η are vector fields on $\mathcal{G}_0$. Clearly the torsion T and the curvature R are horizontal two–forms which descend to well defined forms on the underlying manifold. Let us point our, that our definition of the torsion is adjusted by the algebraic structure of $\mathfrak{g}_-$. Since the forms θ and γ are pullbacks of ω_- and ω_0, the above structure equations are easily compared with those of ω. The missing components of curvature coming from the P part can be understood easily as follows. We define the *Cotton–York* tensor on M as

$$Y(\xi, \eta) = d^\nabla \mathsf{P}(\xi, \eta) + \mathsf{P}(\{\xi, \eta\}) + \{\mathsf{P}(\xi), \mathsf{P}(\eta)\},$$

where d^∇ stays for covariant exterior differential with respect to γ. Then we obtain the pleasing formula for the pullback $\kappa^\sigma = \sigma^*\kappa$ of the full Cartan curvature of the connection ω

$$\kappa^\sigma = T + R + Y + \partial\mathsf{P} \tag{1}$$

where $\partial\mathsf{P} = \{\xi, \mathsf{P}(\eta)\} - \{\eta, \mathsf{P}(\xi)\} - \mathsf{P}(\{\xi, \eta\})$ is the Lie algebra cohomology differential.

1.7. **The normal Weyl structures.** As we have seen, the Rho–tensor measures the deviation of the affine connection $\theta + \gamma$ determined by the reduction σ from the given Cartan connection ω. Thus minimizing the values of P and its derivatives in a point should be the best way to invariants. Our idea mimics completely the way how the normal coordinates of affine connections are built.

For each fixed frame $u \in \mathcal{G}$ there is the flow line of the $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ of the horizontal vector field $\omega^{-1}(X)$, and for some neighborhood of the origin in $\mathfrak{g}_-$ these flow lines exist at least up to time $t = 1$. In this way we obtain a horizontal embedding $X \mapsto \varphi_u(X) = \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$ for such u and also local section $p(\varphi_u(X)) \mapsto \varphi_u(X)$ of the Cartan bundle $p : \mathcal{G} \to M$ through u. Consequently there is a unique Weyl structure σ_u through the frame u defined by $\sigma(\pi(\varphi_u(X))) = \varphi_u(X)$. We call these sections normal Weyl structures. The images $c^{u,X}$ of the defining flow lines, i.e. $c^{u,X}(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)})$, are called the *generalized geodesics* of the Cartan connection ω. Thus, each choice of a frame $u \in \mathcal{G}$ over the point $p(u) = x \in M$ determines the uniquely defined geodetical coordinates $T_xM \to M$ on a neighborhood of the center $x \in M$.

The normal Weyl structures are characterized by the property $\mathsf{P}(c(t))(\xi) = 0$ for all generalized geodesics through $x = p(u) \in M$ and vectors ξ tangent to $c(t)$. In particular, this implies that for every number $\ell \geq 0$ of covariant derivatives, and any tangent vectors $\xi_0, \dots, \xi_\ell$, the full symmetrization of the expression

$$(\nabla_{\xi_\ell} \cdots \nabla_{\xi_1} \mathsf{P}(x))(\xi_0) \tag{1}$$

over the ξ's vanishes, see [7] for details.

2. The nearly invariant calculus

2.1. **The basic setup.** There are basically two natural (but rather naive) ideas how to build an invariant differential calculus. The first one suggests to employ the standard calculus for the affine connections and to find (for each particular geometry) some useful building blocks leading to invariantly defined results with respect to the choice of a Weyl connection from the class. This mimics the most usual approach to the conformal Riemannian geometries. The other one views the Cartan connections as real analogies of the affine connections and we hope to find intrinsically invariant procedures in terms of these connections .

We shall follow the second option. Our main goal here is to prove that we essentially cover everything available in the first approach and we provide some technique at the same time. Since the algebra $\mathfrak{g}$ comes split as $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$, the Cartan connection ω at $\mathcal{G}$ automatically is a (generalized) connection in the sense of Ehresmann and thus several concepts of invariant differentials apply. In order to mimic the affine covariant calculus as much as possible, we may define for all vector valued function the operator ∇^ω by differentiating in the direction of the horizontal vector fields

$$\nabla^\omega : C^\infty(\mathcal{G}, \mathbb{V}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{V}), \quad \nabla s(u)(X) = \omega^{-1}(X)(u)\cdot s \in \mathbb{V}.$$

We call this operator the *invariant derivative*. The definition exactly recovers the covariant derivative of the affine connections expressed in terms of horizontal vector fields on the linear frame bundle, and it allows the iteration of the derivative. In the general case, however, we cannot await equivariance of the values $\nabla^\omega s$ even if s itself was equivariant. Thus, this operation does not represent a differential operator on sections of the associated bundles.

2.2. **Two basic invariant operations.** There are two ways out of this problem. The more elegant one is to extend the operator to the mapping

$$D^\omega : C^\infty(\mathcal{G}, \mathbb{V}) \to C^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V}), \quad \nabla s(u)(X) = \omega^{-1}(X)(u)\cdot s \in \mathbb{V}.$$

Although the vertical vector fields $\omega^{-1}(Z) = \zeta_Z$, $Z \in \mathfrak{p}$, act on equivariant functions s just by the negative of the algebraic action of Z, this extension already provides an operation which transforms equivariant functions into equivariant functions. Thus we have obtained a first order differential operators $D^\omega : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{A}^* \otimes \mathcal{V})$. This operator has been often called the *fundamental derivative* in the literature. Although this operator is natural and also allows iterations, there is a large redundancy in its algebraic part. We can see this redundancy by comparing D^ω to the covariant derivative with respect to a fixed Weyl connection ∇ corresponding to a section σ. A straightforward computation reveals that for every vector field ξ and adjoint tractor field $\xi + \zeta$, $\zeta \in \mathcal{A}^0$ on M

$$D_{\xi+\zeta} s = \nabla_\xi s + \mathsf{P}(\xi) \bullet s - \zeta \bullet s, \tag{1}$$

see also [3] for more details on the fundamental derivative. This redundancy can be treated much better for tractor bundles $\mathcal{V}$ and the so called *tractor calculus*, first presented in full generality in [5], has become quite popular and effective tool nowadays. In particular, the Cartan connection induces a linear connection $\nabla^\mathcal{V}$ on each tractor bundle $\mathcal{V}$ which is written down in terms of any Weyl structure, tractor bundle section s, and vector field ξ on M as follows

$$\nabla^\mathcal{V}_\xi s = \nabla_\xi s + \mathsf{P}(\xi) \bullet s + \xi \bullet s, \tag{2}$$

where ξ acts on the tractor as an element in the adjoint tractor bundle by the action inherited from the action of $\mathfrak{g}$ on the G–module $\mathbb{V}$.

Another quite elegant way is to consider the invariant derivative $\nabla^\omega s$ together with the zero's order derivative s. Again, a straightforward computation shows that the mapping

$$C^\infty(\mathcal{G}, \mathbb{V}) \ni s \mapsto (s, \nabla^\omega s) \in C^\infty(\mathcal{G}, \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}))$$

maps equivariant maps into equivariant maps if we equip the target vector space by the P–action induced in the homogeneous model on the first jet prolongation of $\mathbb{V}$. The same is true for every kth order iteration, but we have to be careful about the target space. It turns out that this is the submodule in the iterated jet prolongation consisting of the semiholonomic jets and we call it the *semiholonomic jet module* induced by $\mathcal{V}$. Thus we obtain the kth order *invariant semiholonomic jet* operator, $k = 1, 2, \ldots$

$$C^\infty(\mathcal{G}, \mathbb{V})^P \ni s \mapsto (s, \nabla^\omega s, \ldots, (\nabla^\omega)^k s) \in C^\infty(\mathcal{G}, \mathbb{V} \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{V}))^P. \tag{3}$$

Now, we have come to the tricky and rather confusing point of the whole calculus. The holonomic jet prolongation

$$J^k\mathcal{V} = J^k(G \times_P \mathcal{V})$$

of any homogeneous vector bundle $\mathcal{V}$ over G/P is always a homogeneous vector bundle again, and we shall write $J^k\mathbb{V}$ for its standard fiber (as a P–module) and we call it the *holonomic jet module* induced by $\mathcal{V}$. Of course, we may always build the corresponding induced bundles $\mathcal{G} \times_P J^k\mathbb{V}$ and symmetrize the values of the kth order invariant non–holonomic jet operator in order to obtain the operation

$$(4) \qquad s \mapsto j^k_\omega s = (s, \nabla^\omega s, \dots, \mathrm{Sym}(\nabla^\omega)^k s) \in C^\infty(\mathcal{G}, \mathbb{V} \oplus \dots \oplus (S^k\mathfrak{g}_-^* \otimes \mathbb{V})).$$

We shall write $s \mapsto \bar{j}^k_\omega s$ to distinguish the semi–holonomic jets from the holonomic ones and similarly for the jet modules $\bar{J}^k\mathbb{V}$.

Unfortunately, unlike the $\bar{j}^k_\omega$, the resulting symmetrized operation j^k_ω does not transfer P–equivariant maps into P–equivariant maps unless the curvature of the Cartan connection acts on the target spaces trivially. Thus, not each algebraic P–module homomorphism gives automatically rise to an invariant differential operator, but each homomorphism of the much more complicated semiholonomic jet bundle does. There is a large class of P–module homomorphims which allow a covering by a morphims of the semiholonomic jet modules, in particular the whole BGG–sequences are of this character, see e.g. [11] and [9].

2.3. **The Bianchi and Ricci identities.** Fortunately, we may restrict ourselves to symmetrized invariant derivatives in our quest for invariant expressions on the expense of adding the Cartan curvature and its derivatives. This is due to the analogies of classical Bianchi and Ricci identities which may be written down in terms of the invariant derivatives and the curvature κ from 1.6 as follows.

$$(1) \qquad 0 = \sum_{\text{cycl}} \Big([\kappa(X,Y), Z] + \kappa([X,Y], Z) - \kappa(\kappa(X,Y), Z) - \nabla^\omega_Z \kappa(X,Y) \Big)$$

$$(2) \qquad \nabla^\omega_X \nabla^\omega_Y s - \nabla^\omega_Y \nabla^\omega_X s = \nabla^\omega_{[X,Y]} - \nabla^\omega_{\kappa_-(X,Y)} s + \kappa_{\geq 0}(X,Y) \bullet s$$

Similar identity holds for the fundamental derivatives.

In view of the Ricci identity, every expression in terms of the curvatures κ of the Cartan connection ω and sections s of $\mathcal{V}$ and their invariant derivatives, i.e. expressions in terms of the kth order invariant jets of s and κ, may be also built in terms of symmetrized jets $j^k_\omega s$ and $j^k_\omega \kappa$.

Although this sort of normalization is possible in principle, it is not very handy to write it down explicitly in higher order examples. We shall also use it here for theoretical considerations only.

Of course, the same conclusion is true for the affine invariants of the Weyl connections and their curvatures.

2.4. **Expansion of the iterated differential.** We have seen in 1.5 how a change of the Weyl structure affects the transformations of the Weyl connections, the corresponding splits of the natural bundles, and also the P's. In view of this, it may be hard to imagine, how to build some really invariant expressions in terms of these things. At the same time, we have already seen such instances – the expressions for the fundamental derivative and the tractor connections in 2.2(1) and (2), and actually the same formulae contain also the expansion of the invariant derivative in terms of any underlying Weyl connection. Indeed, at the frames $v \in \mathcal{G}_0$, $\sigma(v) = u \in \mathcal{G}$, and for $X \in \mathfrak{g}_-$ we have $\omega(T\sigma\cdot\gamma^{-1}(X)(v)) = X + \mathsf{P}(v)(X)$ and so

$$\omega^{-1}(X)(u) = T\sigma\cdot\gamma^{-1}(X)(v) - \omega^{-1}(\mathsf{P}(v)(X)).$$

Thus, for each $\tilde{s} : \mathcal{G} \to \mathbb{V}$, $s = \tilde{s} \circ \sigma : \mathcal{G}_0 \to \mathbb{V}$, and $X \in \mathfrak{g}_-$

$$\nabla^\omega_X \tilde{s}(u) = \nabla_X s(v) + \mathsf{P}(v)(X) \bullet s(v). \tag{1}$$

In order to understand the similar expansions of the higher order iterations of the invariant derivative, let us remind that $(\nabla^\omega)^k$ is just the restriction of the iterated fundamental derivative $(D^\omega)^k$ to arguments in $\mathfrak{g}_-$. The important point is that actually the differential operators in

$$D^\omega \circ D^\omega \ldots \circ D^\omega$$

act always on bigger and bigger bundles in the consecutive iterations and so the algebraic action of the vertical part gets important in the second order already, even if the initial P–module has got a trivial P_+ action. For example, in same setting as above for an irreducible P–module $\mathbb{V}$

$$\begin{aligned}\nabla^\omega_X \nabla^\omega_Y \tilde{s}(u) &= (D^\omega)^2 \tilde{s}(u)(X, Y)\\ &= (T\sigma{\cdot}\gamma^{-1}(u)(X) - \omega^{-1}(\mathsf{P}(v)(X))(u)){\cdot}(\nabla^\omega \tilde{s})(u)(Y)\\ &= \nabla_X \nabla_Y s(v) + \nabla_{[\mathsf{P}(v)(X),Y]_{\mathfrak{g}_-}} s(v) - [\mathsf{P}(v)(X), Y]_{\mathfrak{p}} \bullet s(v).\end{aligned}$$

Clearly, continuing inductively such a procedure, we arrive at the following proposition which may be roughly phrased as follows: Whatever expression we build of the invariant semiholonomic jets of sections and the Cartan curvatures in terms of the invariant derivatives (or the fundamental derivatives) will expand in expressions in the iterated covariant derivatives by means of the Weyl connections, the iterated covariant derivatives of the curvature tensor of the Cartan connections and of the iterated covariant derivatives of the P tensors.

Proposition. *Let $\mathbb{V}$ and $\mathbb{W}$ be P–modules, $\tilde{F}$ be any smooth function $\bar{J}^k\mathbb{V} \times \bar{J}^k\mathbb{K} \to \mathbb{W}$. For all P– equivariant maps $s : \mathcal{G} \to \mathbb{V}$ let us consider the mapping*

$$\tilde{F}(\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa) : \mathcal{G} \xrightarrow{(\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa)} \bar{J}^k\mathbb{V} \times \bar{J}^k\mathbb{K} \xrightarrow{\tilde{F}} \mathbb{W}$$

where $\mathbb{K} = \Lambda^2\mathfrak{g}^*_- \otimes \mathfrak{g}$. The G_0–equivariance of $\tilde{F}$ is the necessary and sufficient condition to ensure that the composition of $\tilde{F}(\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa)$ with any Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ will provide affine invariants of the Weyl structure which are expressed by another G_0–equivariant function

$$F(\bar{j}^k_\gamma s, \bar{j}^k_\gamma \rho) = \tilde{F}(\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa) \circ \sigma : \mathcal{G}_0 \xrightarrow{(\bar{j}^k_\gamma s, \bar{j}^k_\gamma \rho)} \bar{J}^k\mathbb{V} \times \bar{J}^k\mathbb{K}_{\leq 0} \xrightarrow{F} \mathbb{W}$$

where $\mathbb{K}_{\leq 0}$ is the subbundle in $\mathbb{K}$ where the curvatures ρ of the Weyl connections live.

Proof. The step we have done above is universal, only the action of the P part will get more and more complicated. The G_0–equivariance is clearly necessary and the expansion procedure shows that it is also sufficient. The construction of the resulting function F also does not depend on the choice of σ.

In the verbal description of the proposition above we further exploited the fact that the Cartan curvature is completely computable from the curvature of the Weyl connection and the P tensor in a quite simple way, see 1.6(1). □

The expression defined in the terms of the affine invariants of the Weyl connections in the latter proposition is called the *expansion* of the operator on $\mathcal{G}$ built of the invariant differentials.

2.5. **The transformation behavior.** Let us summarize our progress so far. The invariant derivatives are not mapping sections to sections, but they are of invariant character. On the other hand, their expansions in terms of the Weyl connections are covariant operations, but they do transform under the change of the Weyl structures in a non-trivial way, in general.

The striking feature of every such expansion it terms of the Weyl connections is that it transforms extremely nicely. Let us state the result in the setting of the latter proposition:

Theorem. *Let $\mathbb{V}$, $\mathbb{W}$ be P–modules, $\tilde{F} : J^k\mathbb{V} \times J^k\mathbb{K} \to \mathbb{W}$ be a G_0–equivariant smooth mapping and let F be the corresponding function from the proposition above. For each Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, the differential operator defined for all $s \in C^\infty(\mathcal{G}, \mathbb{V})^P$ by*

$$D(s) = \tilde{F} \circ (\bar{j}^k_\omega(s \circ \sigma), \bar{j}^k_\omega \kappa) \circ \sigma = F \circ (\bar{j}^k_\gamma(s \circ \sigma), \bar{j}^k_\omega \rho) : \mathcal{G}_0 \to \mathbb{W}$$

transforms algebraically in terms of Υ under the change of the Weyl structures $\hat{\sigma} = \sigma \circ r^{\exp \Upsilon}$.

Proof. We have just to show that the iterated invariant derivative itself transforms algebraically with respect to the principal action of P on the fibers of $\mathcal{G}$ for all functions s which themselves enjoy an algebraic transformation under the right principal action. Then the definition of the expansion itself shows, that two choices of Weyl structures $\hat{\sigma} = \sigma \circ r^{\exp \Upsilon}$ lead to the same operators at a point $x \in M$ if their difference $\Upsilon(x) = 0$.

If $\mathbb{U}$ is a P–module, $f : P \to GL(\mathbb{U})$ is any smooth mapping, and $s : \mathcal{G} \to \mathbb{U}$ is smooth and such that $s \circ r^{\exp \Upsilon} = f(\Upsilon) \circ s$, then we may compute:

$$\begin{aligned}
(\nabla^\omega s)(u{\cdot} \exp \Upsilon)(X) &= \omega^{-1}(X)(u{\cdot} \exp \Upsilon){\cdot} s \\
&= Tr^{\exp \Upsilon}{\cdot}\omega^{-1}(u)(\mathrm{Ad} \exp \Upsilon{\cdot}X){\cdot}s \\
&= \omega^{-1}(u)(\mathrm{Ad}_{\mathfrak{g}_-} \exp \Upsilon){\cdot}(s \circ r^{\exp \Upsilon}) - \lambda(\mathrm{Ad}_{\mathfrak{p}} \exp \Upsilon{\cdot}X) \circ (s \circ r^{\exp \Upsilon}) \\
&= f(\Upsilon) \circ \omega^{-1}(\mathrm{Ad}_{\mathfrak{g}_-} \exp \Upsilon{\cdot}X) - \lambda(\mathrm{Ad}_{\mathfrak{p}} \exp \Upsilon{\cdot}X) \circ f(\Upsilon) \circ s.
\end{aligned}$$

Now, by inductive argument on the order of the iteration, the latter computation shows that indeed the iterated invariant derivative transforms algebraically. Of course, the Cartan curvature itself transforms algebraically and our proof is complete. □

Let us call the affine invariants of Weyl connections which transform algebraically in the Υ's the *nearly invariant operators.* A special case is represented by the invariants of the structure only, where the only arguments are the derivatives of the curvature.

The proof of the latter proposition suggests also the following nice corollary:

Corollary. *Let $\mathbb{V}$, $\mathbb{W}$ be P–modules, $\tilde{F} : J^k\mathbb{V} \times J^k\mathbb{K} \to \mathbb{W}$ be a G_0–equivariant smooth mapping. The differential operation defined for all $s \in C^\infty(\mathcal{G}, \mathbb{V})^P$ by*

$$D(s) = \tilde{F} \circ (\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa) : \mathcal{G} \to \mathbb{W}$$

provides an invariant differential operator if and only if the algebraic mapping $\tilde{F} \circ (\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa)$ is P–equivariant for all s.

Proof. The P–equivariance implies that the values are again sections of the target bundle $\mathcal{W}$ and clearly such an operation is invariant in the categorical sense in the category of Cartan connections. Due to the last theorem, this expression at the same time produces an invariant in the terms of the affine invariants of the Weyl connections and, on the contrary, if the expansion turned out to be independent of the choice of the Weyl structure, the proof of the above theorem shows that the original expression was P-equivariant. □

Of course the expression $\tilde{F} \circ (\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa)$ will be P–equivariant if F itself is equivariant. But let us point out that even for linear operators independent explicitly of the curvature the condition in the corollary does not necessarily mean that F is a P–module homomorphism. The reason is hidden in the Bianchi identities, and the celebrated JGMS operators in the conformal geometry are a good example of such behavior, see [11].

2.6. **The main theorem.** The invariant operators on a manifold equipped by a parabolic geometry of the type $G \to G/P$ are defined as the affine invariants of the Weyl connections which are independent of the choice of a particular Weyl structure σ. Of course, if they do not change at all, they are particular instances of the above nearly invariant operators. But if the search for invariant objects is too difficult, than the study of the nearly invariant ones should help in the first instance.

The transformation formula for the P tensor for conformal geometries starts with the derivative of Υ and so this seems to be the perfect candidate to enter correction terms to the covariant derivatives so that the transformation rule remains algebraic in Υ's. Indeed, this was the original defining property of P and this feature is also in the core of Wünsch's calculus for the conformal Riemannian invariants.

The first look at the transformation formulae of the Weyl connections and P tensors for $|k|$–graded algebras $\mathfrak{g}$ with $k > 1$ does not bring much hope for a reasonable calculus eliminating the non–algebraic appearances of Υ's. However, if we think about the procedure represented by the expansion of the invariant differentials, this is exactly what they do. Nicely enough, we are able to prove the converse assertion as well:

Theorem. *The nearly invariant operators are exactly the expansion of operators D defined for all $s \in C^\infty(\mathcal{G}, \mathbb{V})^P$ in terms of the invariant differential ∇^ω by*

$$D(s) = \tilde{F} \circ (\bar{j}^k_\omega s, \bar{j}^k_\omega \kappa) : \mathcal{G} \to \mathbb{W}$$

where $\tilde{F} : J^k\mathbb{V} \times J^k\mathbb{K} \to \mathbb{W}$ is a G_0–equivariant smooth mapping, and the spaces $\mathbb{V}$, $\mathbb{W}$ are P–modules.

Proof. The proof of the theorem will rely on the normal Weyl structures and will consist of several observations. We have to start with an expression

$$D(s) = F(\bar{j}^k_\gamma s, \bar{j}^k_\gamma \rho) \tag{1}$$

where $s \in C^\infty(\mathcal{G}_0, \mathbb{V})$, and ρ is the curvature of the Weyl connection γ. We assume that D represents a nearly invariant operator. This means that

$$F(\bar{j}^k_\gamma s, \bar{j}^k_\omega \rho) = F(\bar{j}^k_{\hat{\gamma}} s, \bar{j}^k_{\hat{\gamma}} \hat{\rho})$$

over a point $x \in M$, whenever we replace γ by another Weyl connection $\hat{\gamma}$ (and the same with its curvature) such that the difference Υ vanishes at the given point x. Our aim is to find the corresponding expression in terms of invariant derivatives, whose expansion is $D(s)$. The other implication of the theorem was proved above.

First, let us remind that the Ricci identities imply that every expression F as in the equation (1) may be rewritten in the form

$$F(\bar{j}^k_\gamma s, \bar{j}^k_\gamma \rho) = \bar{F}(j^k_\gamma s, j^k_\gamma \rho)$$

valid for all s. This means that we may restrict ourselves to expressions depending on symmetrized jets of the sections and curvatures only. Thus, we shall consider that only symmetrized jets occurred already in the equation (1).

Next observe, that the symmetrized iterated invariant differentials $\mathrm{Sym}(\nabla^\omega)^k\tilde{s}$ expand to expressions with symmetrized iterations of covariant derivatives $\mathrm{Sym}(\nabla)^\ell$ of s and P with leading term $\mathrm{Sym}(\nabla^k)s$. Thus, the vanishing of all symmetrized covariant derivatives of the Rho tensors of the normal Weyl structures at the center $x \in M$ implies that the expansion of $\mathrm{Sym}(\nabla^\omega)^k\tilde{s}$ is just $\mathrm{Sym}(\nabla^k)s$ over the point x. So the remaining problem in our construction of suitable $\tilde{F}$ from F are the curvature parts.

Now, we can exploit the relation between the Cartan curvature on $\mathcal{G}$ and the torsion T, curvature R and the Cotton–York tensor Y in 1.6(1), i.e. $\kappa = T + R + Y + \partial\mathsf{P}$. We should point out here, that we deal with the torsion adjusted to the nilpotent algebra $\mathfrak{g}_-$ (note that the Lie bracket in $\mathfrak{g}_-$ would always appear in T from the classical affine point of view). We are going to use this formula to compute the $j^k_\gamma(T+R+Y)$ from the jet of the Cartan curvature $j^k_\gamma\kappa$ and the $j^k_\gamma\mathsf{P}$.

Claim 1. *For every k, there is an algebraic mapping*

$$\Psi : J^k\mathbb{V} \times J^k(\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}) \to J^k(\Lambda^2\mathfrak{g}_-^* \otimes \mathfrak{g}),$$

such that

$$\Psi(j^k_\gamma\kappa, j^k_\gamma\mathsf{P}) = j^k_\gamma(T+R+Y).$$

In order zero, we have seen this formula already:

$$T + R + Y = \kappa - \partial\mathsf{P}$$

and it is easily seen that $(\nabla)^k(\partial\mathsf{P}) = (\mathrm{id}^{\otimes k}\otimes\partial)(\nabla)^k\mathsf{P}$. So we need to split the symmetric part off the derivatives of this formula and hope that the rest will also be expressed in the the other known symmetric components. Let us formulate the result of the essential step first:

Claim 2. *For every Weyl structure,*

$$\nabla\mathsf{P} = \mathrm{Sym}\,\nabla\mathsf{P} + \frac{1}{2}Y - \frac{1}{2}i_T\mathsf{P} - \frac{1}{2}\{\mathsf{P},\mathsf{P}\}.$$

Let us compute the derivative evaluated on vector fields ξ, η on the manifold M, with the help of the definition of the torsion and the covariant exterior derivative in the Cotton-York, see 1.6. Clearly, $\nabla_\xi\mathsf{P}(\eta) = \mathrm{Sym}(\nabla\mathsf{P})(\xi,\eta) + \frac{1}{2}\mathrm{Alt}(\nabla\mathsf{P})(\xi,\eta)$, so the interesting part is

$$\begin{aligned}\nabla_\xi\mathsf{P}(\eta) - \nabla_\xi\mathsf{P}(\eta) &= \nabla_\xi(\mathsf{P}(\eta)) - \mathsf{P}(\nabla_\xi\eta) - \nabla_\eta(\mathsf{P}(\xi)) + \mathsf{P}(\nabla_\eta\xi)\\ &= \nabla_\xi(\mathsf{P}(\eta)) - \nabla_\eta(\mathsf{P}(\xi)) - \mathsf{P}([\xi,\eta]) + \mathsf{P}(\{\xi,\eta\}) + \{\mathsf{P}(\xi),\mathsf{P}(\eta)\}\\ &\quad - \mathsf{P}(\nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta] + \{\xi,\eta\}) - \{\mathsf{P}(\xi),\mathsf{P}(\eta)\}\\ &= Y(\xi,\eta) - i_{T(\xi,\eta)} - \{\mathsf{P}(\xi),\mathsf{P}(\eta)\}\end{aligned}$$

exactly as required in the Claim 2.

Now, if we assume inductively that our Claim 1 holds fro all $\ell < k$, we obtain its validity for k by applying Claim 2 to all components with P's. Notice, that we the Y components always appear only up to order $k-1$ by our assumption and so they do not represent any problem.

Having proved our Claim 1 we are ready for the final step. Consider our fixed expression

$$D(s) = F(j^k_\gamma s, j^k_\gamma\rho)$$

in the symmetrized jets. Since we know that the symmetrized jets of the P tensors vanish to all orders for normal Weyl structures in the center $x \in M$ we define

$$\tilde{F} = F \circ (\mathrm{id}\times\Psi_0)$$

where we set $\Psi_0(j^k_\gamma\kappa) = \Psi(j^k_\gamma\kappa, 0)$. Now for each normal Weyl structure σ centered at $x \in M$

$$D(s) = F(j^k_\gamma s, j^k_\gamma \rho) = \tilde{F}(j^k_\omega \tilde{s}, j^k_\omega \kappa) = \tilde{D}(\tilde{s}).$$

Finally, for every Weyl structure $\hat{\sigma}$ and point $x \in M$, there is exactly one normal Weyl structure σ centered in x with $\hat{\sigma}(x) = \sigma(x)$. By our assumption, the value of our operator $D(s)$ depends only on $\sigma(x)$ over the point x. Thus our operator $\tilde{D}(\tilde{s})$ agrees with $D(s)$ for all Weyl structures and so $D(s)$ is its expansion as required. □

3. Concluding comments and examples

Our implicit construction of all nearly invariant operators in terms of the iterated invariant derivative itself does not provide good algorithms how to get the invariants explicitly. It can be used efficiently in particular for the description of invariants depending only on the curvatures of low homogeneities. Indeed, in this case we may quite easily list all possibilities of building blocks in terms of the invariant derivatives of of the harmonic curvature. Similarly, we may apply this straightforward approach to low order operators.

In recent decades, there have been many successful attempts using explicitly the representation theory in order to find homomorphims of the invariant semi-holonomic jets, see e.g. [2], [8], [11], [9] and further references therein. The tractor calculus is a very powerful and efficient calculus for invariants for each particular geometry. In fact, the Cartan tractor connection is easily written in terms of the invariant derivative and thus in terms of all the Weyl structures, together with the corresponding splittings.

If we want to expand the expressions in terms of Weyl connections explicitly, a much easier way is to replace the Weyl connection γ by the principal connection on the total bundle $\mathcal{G}$, whose connection form coincides with the Cartan connection ω on the image $\sigma(\mathcal{G}_0)$. The covariant derivative with respect to this connection again exists on all induced bundles to P–representation and in terms of our affine Weyl connections they differ by the algebraic action of P. These expansions were suggested first in [16] and exploited nicely in [4]. Of course, the expansion in terms of these so called *Ricci corrected derivatives* simplifies a lot.

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