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Integrable Equations and Motions of Plane Curves

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Motions of plane curves in Klein geometry is studied. It is shown that the KdV, Harry–Dym, Sawada–Kotera, Burgers, the defocusing mKdV hierarchies and the Kaup–Kupershmidt equation naturally arise from motions of plane curves in affine, centro-affine and similarity geometries. These local and nonlocal dynamics conserve global geometric quantities of curves such as perimeter and enclosed area.

1 Introduction

The connection between motion of space or plane curves and integrable equations has drawn wide interest in the past and many results have been obtained. The pioneering work is due to Hasimoto where he showed in [1] that the nonlinear Schrödinger equation describes the motion of an isolated non-stretching thin vortex filament. Lamb [2] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine-Gordon equations. Lakshmanan [3] related the Heisenberg spin model to the motion of space curves in the Euclidean space. Langer and Perline [4] obtained the Schrödinger heirarchy from motions of the non-stretching thin vortex filament. Motions of curves in S^2 and S^3 were considered by Doliwa and Santini [5]. Nakayama [6, 7] investigated motions of curves in Minkowski space and obtained the Regge–Lund equation, a couple of systems of the KdV equations and their hyperbolic type. In contrast to the motions of curves in space, only two types of integrable equations have been shown to be associated to motions of plane curves. In fact, Goldstein and Petrich [8] discovered that the dynamics of a non-stretching string on the plane produces the recursion operator of the mKdV hierarchy. Nakayama, Segur and Wadati [9] obtained the sine-Gordon equation by considering a nonlocal motion. They also pointed out that the Serret–Frenet equations for curves in $\mathbf{E}^2$ and $\mathbf{E}^3$ are equivalent to the AKNS-ZS spectral problem without spectral parameter [10, 11]. It is commonly believed that the KdV equation does not occur in the motion of plane curves.

The purpose of this paper is to study motions of plane curves in Klein geometries. These geometries are characterized by their associated Lie algebras of vector fields in $\mathbf{E}^2$. We shall see that the KdV, Harry–Dym and Sawada–Kotera hierarchies and the Kaup–Kupershmidt equation naturally arise from the motions of plane curves in affine, centro-affine and similarity geometries. The outline of this paper is as follows. In Section 2, we give a brief discussion on the Klein geometry. In Sections 3, 4, and 5, we discuss motion laws of plane curves respectively in affine, centro-affine and similarity geometries. Section 7 is concluding remarks about this work.

2 Klein geometry

In this section, we give an extremely brief account of the Klein geometry. Our basic reference is [12].

Let $\mathcal{G}$ be a Lie transformation group acts locally and effectively on the plane. Its Lie algebra $\mathfrak{g}$ can be identified with a subalgebra of the Lie algebra of all smooth vector fields in $\mathbf{E}^2$ under the usual Poisson bracket. According to the Erlanger Programme, every $\mathcal{G}$ or $\mathfrak{g}$ determines a Klein geometry for plane curves via its invariants. To describe the invariants, let us assume a curve γ and its image γ' under a typical element g in $\mathcal{G}$ are represented locally as graphs $(x, u(x))$ and $(y, v(y))$ over some intervals I and J respectively. A differential invariant of $\mathfrak{g}$ is a n-th smooth function Φ defined on the n-jet space $X \times U^{(n)}$ for some $n \geq 1$ satisfying $\Phi\left(x, u(x), \ldots, u^{(n)}(x)\right) = \Phi\left(y, v(y), \ldots, v^{(n)}(y)\right)$ for all $g \in \mathcal{G}$. An invariant one-form, or, more precisely, a horizontal contact-invariant form, is a one-form defined in the n-jet space $X \times U^{(n)}$, locally in the form $d\sigma = P\left(x, u(x), \ldots, u^{(n)}(x)\right) dx$, satisfying

$$\int_I P\left(x, u(x), \ldots, u^{(n)}(x)\right) dx = \int_J P\left(y, v(y), \ldots, v^{(n)}(y)\right) dy,$$

for all g in $\mathcal{G}$. Let

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u}$$

be an arbitrary vector field in $\mathfrak{g}$. We denote its n-th prolongation vector field on $X \times U^{(n)}$ by $\mathbf{pr}^{(n)}\mathbf{v}$. The infinitesimal criterion for the invariance of Φ and $d\sigma$ are given respectively by

$$\mathbf{pr}^{(n)}\mathbf{v}(\Phi) = 0,$$

and

$$\mathbf{pr}^{(n)}\mathbf{v}(P) + P\mathrm{div}\xi = 0,$$

where $\mathrm{div}\,\xi = \xi_x + \xi_u u_x$. A basic result is

Theorem 1 ([12]). *For any Lie transformation group acting locally and effectively on the plane, there exist an invariant one-form $d\sigma = Pdx$ and a differential invariant Φ, both of lowest order such that every differential invariant can be written as a function of Φ and its derivatives $D\Phi$, $D^2\Phi, \ldots$, where*

$$D = \frac{1}{P}\frac{d}{dx}.$$

Moreover, every invariant one-form is of the form $Id\sigma$ where I is a differential invariant.

Here "order" refers to the highest number of derivatives involved in the local expression for P and Φ.

Definition 1. Invariant one-forms and differential invariant of lowest order of the Lie group are respectively called the *group arclength* and the *group curvature*.

Example 1. We look at the Euclidean geometry which is the Klein geometry associated to the Lie algebra by $\{\partial_x, \partial_u, x\partial_u - u\partial_x\}$. It is readily verified that one can choose its group arclength to be

$$ds = \sqrt{1 + u_x^2}dx$$

and group curvature to be

$$\kappa = \left(1 + u_x^2\right)^{-\frac{3}{2}} u_{xx}.$$

Example 2. Consider the affine geometry which is associated to $SA(2)$ by $\{\partial_x, \partial_u, x\partial_x - u\partial_u, x\partial_u, u\partial_x\}$. Its group arclength and group curvature are

$$d\rho = \kappa^{\frac{1}{3}} ds, \qquad \mu = \kappa^{\frac{4}{3}} + \frac{1}{3}\left(\kappa^{-\frac{5}{3}}\kappa_s\right)_s. \tag{1}$$

One may consult [13] for a discussion on affine geometry.

In the following we shall consider motions of plane curves in affine, centro-affine and similarity geometries. For any parametrized curve γ, we define its *group tangent* and *group normal* to be $\mathbf{T} = \gamma_\sigma$ and $\mathbf{N} = \gamma_{\sigma\sigma}$ respectively, where σ is the group arc-length. A group invariant motion is of the form

$$\frac{\partial\gamma}{\partial t} = f\mathbf{N} + g\mathbf{T}, \tag{2}$$

where f and g are functions of the group curvature. With a given motion law, the equation for its curvature can be obtained in the following four steps. First, we determine the Serret–Frenet formulas for each geometry. It is of the form

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{3}$$

In some occasions this system is the AKNS system without spectral parameter. Second, we compute the first variation for the group perimeter

$$L = \oint_\gamma d\sigma,$$

for a closed curve driven under (2) to obtain

$$\frac{dL}{dt} = \oint_\gamma F d\sigma, \tag{4}$$

where F depends on f and g in (2). By choosing f and g such that F vanishes pointwisely we ensure that $[\partial/\partial t, \partial/\partial\sigma] = 0$, i.e. $\partial/\partial t$ and $\partial/\partial\sigma$ commute. Third, we compute the time evolution of $\mathbf{T}$ and $\mathbf{N}$ to get

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{5}$$

Finally, the compatibility condition between (3) and (5)

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_{t\sigma} = \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_{\sigma t}$$

gives the general equation for the curvature. By choosing f and g suitably we obtain integrable equations. This procedure has been used in [8, 9] to obtain the mKdV and sine-Gordon equations in the Euclidean geometry. Similarly, some other mKdV equations are obtained by Doliwa–Santini [5] in the "restricted conformal" $SO(3)$-geometry.

3 Motion of curves in affine geometry

This is the classical geometry invariant under the unimodular transformations

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix} + B,$$

where $A \in SL(2,\mathbb{R})$, $B \in \mathbb{R}^2$. The affine arc-length $d\rho$ and curvature μ are given in terms of the Euclidean arc-length and curvature by (1), where and hereafter κ and ds always denote the Euclidean curvature and arclength.

The affine Serret–Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_\rho = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{6}$$

The affine tangent and normal are related to the Euclidean tangent $\mathbf{t}$ and normal $\mathbf{n}$ via

$$\mathbf{t} = k^{\frac{1}{3}}\mathbf{T}, \qquad \mathbf{n} = \frac{1}{3}k^{-\frac{5}{3}}k_s\mathbf{T} + k^{-\frac{1}{3}}\mathbf{N}.$$

We relate the motion (2) with the motion in Euclidean geometry

$$\gamma_t = \tilde{f}\mathbf{n} + \tilde{g}\mathbf{t}, \tag{7}$$

where

$$\tilde{f} = k^{\frac{1}{3}}f, \qquad \tilde{g} = k^{-\frac{1}{3}}g - \frac{1}{3}k^{-\frac{5}{3}}k_s f.$$

By a direct computation

$$\tilde{f}_{ss} = \frac{1}{3}\left(k^{-\frac{2}{3}}k_s\right)_s f + k^{-\frac{1}{3}}k_s f_\rho + k f_{\rho\rho},$$
$$\tilde{g}_s - k\tilde{f} = g_\rho - \frac{1}{3}k^{-\frac{4}{3}}k_s(g + f_\rho) - \mu f.$$

Substituting these equations into the evolution equations for s and k [8, 9], we have

$$s_t = s\left[g_\rho - \frac{1}{3}k^{-\frac{4}{3}}k_s(g + f_\rho) - \mu f\right],$$
$$k_t = k\left[f_{\rho\rho} + k^{-\frac{4}{3}}k_s(f_\rho + g) + \mu f\right].$$

Hence, the first variation of the affine perimeter satisfies

$$\frac{dL}{dt} = \oint_\gamma \left(\frac{k_t}{3k} + \frac{s_t}{s}\right) d\rho,$$
$$= \oint \left(\frac{1}{3}f_{\rho\rho} - \frac{2}{3}\mu f + g_\rho\right) d\rho.$$

We impose

$$\oint \mu f d\rho = 0, \tag{8}$$

and

$$g = -\frac{1}{3}f_\rho + \frac{2}{3}\partial_\rho^{-1}(\mu f). \tag{9}$$

On the other hand, we have

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} g_\rho - \mu f & f_\rho + g \\ H_1 & H_2 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}, \tag{10}$$

where $H_1 = g_{\rho\rho} - 2\mu f_\rho - \mu_\rho f - \mu g$ and $H_2 = f_{\rho\rho} + 2g_\rho - \mu f$. Under (8) and (9), $[\partial/\partial\rho, \partial/\partial t] = 0$, and so the compatibility condition between (6) and (10) implies

$$\mu_t = \frac{1}{3}\left(D_\rho^4 + 5\mu D_\rho^2 + 4\mu_\rho D_\rho + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho \partial_\rho^{-1}\mu\right) f, \tag{11}$$

after using (9).

If we take $f = -3\mu_\rho$ in (11), we get the Sawada–Kotera equation [14, 15]

$$\mu_t + \mu_5 + 5\mu\mu_3 + 5\mu_1\mu_2 + 5\mu^2\mu_1 = 0. \tag{12}$$

If we take $f = -3(\mu_3 + 2\mu\mu_1)$, we obtain a seventh-order Sawada–Kotera equation

$$\mu_t + \mu_7 + 7\mu\mu_5 + 14\mu_1\mu_4 + 21\mu_2\mu_3 + 14\mu^2\mu_3 + 42\mu\mu_1\mu_2 + 7\mu_1^3 + \frac{28}{3}\mu^3\mu_1 + a\mu_1 = 0.$$

In general, we take $f = -3\left(D_\rho^2 + \mu + \mu_\rho\partial_\rho^{-1}\right)u$, $u = \Omega^{n-1}(\mu)\mu_\rho$, where

$$\Omega(\mu) = \left(D_\rho^3 + 2\mu D_\rho + 2D_\rho\mu\right)\left(D_\rho^3 + D_\rho^2\mu\partial_\rho^{-1} + \partial_\rho^{-1}\mu D_\rho^2 + \frac{1}{2}\left(\mu^2\partial_\rho^{-1} + \partial_\rho^{-1}\mu^2\right)\right),$$

is the recursion operator of the Sawada–Kotera equation [16]. By a direct computation, the following identity holds

$$\begin{aligned}&\left(D_\rho^4 + 5\mu D_\rho^2 + 4\mu_\rho D_\rho + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho\partial_\rho^{-1}\mu\right)\left(D_\rho^2 + \mu + \mu_\rho\partial_\rho^{-1}\right)\\&\quad = \left(D_\rho^3 + 2\mu D_\rho + 2D_\rho\mu\right)\left(D_\rho^3 + D_\rho^2\mu\partial_\rho^{-1} + \partial_\rho^{-1}\mu D_\rho^2 + \frac{1}{2}\left(\mu^2\partial_\rho^{-1} + \partial_\rho^{-1}\mu^2\right)\right).\end{aligned}$$

Using this identity we see that μ satisfies the Sawada–Kotera hierarchy

$$\mu_t = -\Omega^n(\mu)\mu_\rho. \tag{13}$$

4 Motion of plane curves in centro-affine geometry

The geometrical quantities in centro-affine geometry are invariant under the transformations

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A\begin{pmatrix} x \\ u \end{pmatrix},$$

where $A \in SL(2,\mathbb{R})$. Let $\gamma(p) = (\gamma_1(p), \gamma_2(p))$ be a parametrized curve in $\mathbf{E}^2$. We define its centro-affine arclength $d\tilde{s}$ as

$$d\tilde{s} = (\gamma_1\gamma_2' - \gamma_1'\gamma_2)dp = h ds,$$

where $h = -\gamma\cdot\mathbf{n}$ is the support function of γ [17]. The centro-affine curvature ϕ is given by

$$\phi = \kappa h^{-3}.$$

The centro-affine tangent and normal vectors are given by $\mathbf{T} = \gamma_{\tilde{s}}$ and $\mathbf{N} = \gamma_{\tilde{s}\tilde{s}}$ respectively. They are related to the Euclidean tangent and normal by

$$\mathbf{T} = h^{-1}\mathbf{t}, \qquad \mathbf{N} = \kappa h^{-2}\mathbf{n} - h^{-3}h_s\mathbf{t}.$$

Notice that this frame is centro-affine invariant in the sense that $\mathbf{T}' = A\mathbf{T}$ and $\mathbf{N}' = A\mathbf{N}$. Using the Serret–Frenet formulas in $\mathbf{E}^2$

$$\mathbf{t}_s = \kappa\mathbf{n}, \qquad \mathbf{n}_s = -\kappa\mathbf{t},$$

and the following identities

$$\kappa^{-1}h^2\left(h^{-3}\kappa_s - 3h^{-4}\kappa h_s\right) = \frac{\phi_{\tilde{s}}}{\phi},$$
$$h_{ss} = \kappa^{-1}\kappa_s h_s + \kappa - \kappa^2 h,$$

we obtain the centro-affine Serret–Frenet formulas

$$\mathbf{T}_{\tilde{s}} = \mathbf{N}, \qquad \mathbf{N}_{\tilde{s}} = \frac{\phi_{\tilde{s}}}{\phi}\mathbf{N} - \phi\mathbf{T}. \tag{14}$$

Now we first compute the first variation of the centro-affine perimeter $L = \oint d\tilde{s}$. To this purpose, we express (2) in the form (7), where now

$$\tilde{f} = \kappa h^{-2} f, \qquad \tilde{g} = h^{-1}g - h^{-3}h_s f.$$

By the formulas in $\mathbf{E}^2$ [8, 9]

$$s_t = s\left(\tilde{g}_s - \kappa\tilde{f}\right), \qquad \kappa_t = \tilde{f}_{ss} + \kappa_s\tilde{g} + \kappa^2\tilde{f},$$

and

$$h_t = -\tilde{f} + \left(\tilde{f}_s + \kappa\tilde{g}\right)\gamma\cdot\mathbf{t},$$

we have

$$\begin{aligned} L_t &= \oint\left(h^{-1}h_t + s^{-1}s_t\right)d\tilde{s}, \\ &= \oint\left(g_{\tilde{s}} - 2\phi f\right)d\tilde{s}. \end{aligned}$$

As parallel to the affine case, we require f to satisfy

$$\oint \phi f\, d\tilde{s} = 0, \tag{15}$$

and choose

$$g = 2\partial_{\tilde{s}}^{-1}(\phi f), \tag{16}$$

so that $[\frac{\partial}{\partial t}, \frac{\partial}{\partial \tilde{s}}] = 0$. By (14) and (7) we obtain the time evolution for tangent and normal vectors:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}, \tag{17}$$

where

$$A = g_{\tilde{s}} - \phi f, \qquad B = f_{\tilde{s}} + g + \frac{\phi_{\tilde{s}}}{\phi}f, \qquad C = A_{\tilde{s}} - \phi B, \qquad D = B_{\tilde{s}} + A + \frac{\phi_{\tilde{s}}}{\phi}B.$$

The compatibility condition between (14) and (17) gives the equation for the curvature

$$\phi_t = \phi f_{\tilde{s}\tilde{s}} + 2\phi_{\tilde{s}}f_{\tilde{s}} + (\phi_{\tilde{s}\tilde{s}} + 4\phi^2)f + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}(\phi f), \tag{18}$$

after using (16), where we always assume (15) holds. We now consider several cases:

Case 1. $f = u/\phi$. In this case, (18) becomes

$$\phi_t = \left(D_{\tilde{s}}^2 + 4\phi + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\right) u.$$

Setting $u = -\Omega_1^{n-1}\phi_{\tilde{s}}$. We get the KdV hierarchy

$$\phi_t = -\Omega_1^n \phi_{\tilde{s}}, \qquad n \geq 1,$$

where $\Omega_1 = D_{\tilde{s}}^2 + 4\phi + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}$ is the recursion operator of the KdV equation

$$\phi_t + \phi_{\tilde{s}\tilde{s}\tilde{s}} + 6\phi\phi_{\tilde{s}} = 0. \tag{19}$$

Setting $u = -\phi^{-3/2}\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}(\phi q) + 2\phi^{1/2}q$, $\psi = \phi^{-1/2}$, ψ satisfies

$$\psi_t = -\left[\psi\left(\psi D_{\tilde{s}}^2 - \psi_{\tilde{s}} D_{\tilde{s}} + \psi_{\tilde{s}\tilde{s}} + \psi^2\psi_{\tilde{s}\tilde{s}\tilde{s}}\partial_{\tilde{s}}^{-1}\psi^{-2}\right) + 4\right] q. \tag{20}$$

Taking $q = 0$ in (20), we get the Harry Dym equation

$$\psi_t + \psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}} = 0.$$

Setting $q = \Omega_2^{n-1}\left(\psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\right)$, we get the Harry Dym hierarchy

$$\psi_t = -\Omega_2^n\left(\psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\right),$$

where

$$\Omega_2 = \psi^2 D_{\tilde{s}}^2 - \psi\psi_{\tilde{s}} D_{\tilde{s}} + \psi\psi_{\tilde{s}\tilde{s}} + \psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\partial_{\tilde{s}}^{-1}\psi^{-2} + 4,$$

is a recursion operator of the Harry Dym equation [18].

Case 2. $f = u_{\tilde{s}\tilde{s}\tilde{s}}/\phi + u_{\tilde{s}}$. In this case, (18) becomes

$$\phi_t = \left[D_{\tilde{s}}^5 + 5\phi D_{\tilde{s}}^3 + 4\phi_{\tilde{s}} D_{\tilde{s}}^2 + \left(\phi_{\tilde{s}\tilde{s}} + 4\phi^2\right) D_{\tilde{s}} + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\phi D_{\tilde{s}}\right] u.$$

Taking $u = -\phi$, we get the Sawada–Kotera equation (12). Next, we take $u = -\partial_{\tilde{s}}^{-1}\left(D_{\tilde{s}}^2 + \phi + \phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\right) q$, $q = \Omega^{n-1}(\phi)\phi_{\tilde{s}}$, where $\Omega(\phi)$ is the recursion operator of the Sawada–Kotera equation, we obtain the Sawada–Kotera hierarchy (13).

Case 3. $f = -(\phi_{\tilde{s}\tilde{s}\tilde{s}}/\phi + 16\phi_{\tilde{s}})$. The resulting equation is the Kaup–Kupershmidt equation [19, 20]

$$\phi_t + \phi_5 + 20\phi\phi_3 + 50\phi_1\phi_2 + 80\phi^2\phi_1 = 0.$$

Case 4. $f = \phi^{-4}\phi_{\tilde{s}}$. We have

$$\phi_t = \frac{1}{2}\left(\phi^{-2}\right)_{\tilde{s}\tilde{s}\tilde{s}} + 3\left(\phi^{-1}\right)_{\tilde{s}},$$

which is an integrable equation [21].

It is easy to see that in these four cases the motions also conserve the enclosed area of the curves. Notice that the area does not change under centro-affine action and so it makes sense in the centro-affine geometry. A fuller discussion on the integrable equations can be found in Chou–Qu [22].

5 Motions of curves in similarity geometry

The similarity algebra is obtained by adding the dilatation to $\mathbf{E}^2$. The $Sim(2)$ arc-length is given by $d\theta$, where θ is the angle between the tangent and the x-axis. The curvature in similarity geometry is related to the Euclidean curvature by the Cole–Hopf transformation

$$\chi = (\ln k)_\theta = k^{-2} k_s.$$

Using $\mathbf{T} = k^{-1}\mathbf{t}$ and $\mathbf{N} = k^{-1}\mathbf{n} - k^{-3}k_s\mathbf{t}$, the Serret–Frenet formulas in similarity geometry are given by

$$\mathbf{T}_\theta = \mathbf{N}, \qquad \mathbf{N}_\theta = -2\chi\mathbf{N} - \left(\chi_\theta + \chi^2 + 1\right)\mathbf{T}. \tag{21}$$

We express the motion (2) in the form (7), where now $\tilde{f} = k^{-1}f$ and $\tilde{g} = k^{-1}(g - \chi f)$. The first variation of the similarity perimeter is given by

$$\frac{dL}{dt} = \oint (k_t s + k s_t)dp = \oint \left[\tilde{f}_{ss} + (\chi g)_s\right] ds.$$

Hence dL/dt always vanishes for any closed curve. However, it still makes sense to set

$$g = -f_\theta + 2\chi f + a, \qquad a = \text{const}, \tag{22}$$

so that $[\partial/\partial\theta, \partial/\partial t] = 0$ for any f and g related by (22). The evolution of the similarity tangent and normal are given respectively by

$$\begin{aligned}
\mathbf{T}_t &= -k^{-2}k_t\mathbf{t} + k^{-1}\mathbf{t}_t = -k^{-1}(\mathcal{L}f + a\chi)\mathbf{t} + ak^{-1}\mathbf{n} = -\mathcal{L}f\mathbf{T} + a\mathbf{N},\\
\mathbf{N}_t &= k^{-1}(\mathbf{n} - \chi\mathbf{t})_t - k^{-2}k_t(\mathbf{n} - \chi\mathbf{t})\\
&= -(\mathcal{L}f + 2a\chi)(\mathbf{N} + \chi\mathbf{T}) - [(\mathcal{L}f + a\chi)_\theta - \chi(\mathcal{L}F + a\chi) + a]\mathbf{T}\\
&= -(\mathcal{L}f + 2a\chi)\mathbf{N} - [(\mathcal{L}f + a\chi)_\theta + a(\chi^2 + 1)]\mathbf{T}.
\end{aligned}$$

Hence

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} -\mathcal{L}f & a \\ Q & P \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}, \tag{23}$$

where $Q = -(\mathcal{L}f + a\chi)_\theta - a(\chi^2 + 1)$, $P = -(\mathcal{L}f + 2a\chi)$ and $\mathcal{L} = (\partial_\theta - \chi)^2 + 1$ is a linear operator.

The compatibility condition between (21) and (23) yields the following equation for the $Sim(2)$-curvature χ after using (22),

$$\chi_t = \left[D_\theta^3 - 2\chi D_\theta^2 - \left(3\chi_\theta - \chi^2 - 1\right) D_\theta - (\chi_{\theta\theta} - 2\chi\chi_\theta)\right] f + a\chi_\theta, \tag{24}$$

where f is an arbitrary function.

The simplest choice is $f = -1$. Then (24) becomes the Burgers equation

$$\chi_t = \chi_{\theta\theta} - 2\chi\chi_\theta + a\chi_\theta.$$

The next choice is $f = \chi$, which yields the third order Burgers equation

$$\chi_t = \chi_{\theta\theta\theta} - 3\chi\chi_{\theta\theta} - 3\chi_\theta^2 + 3\chi^2\chi_\theta + (a + 1)\chi_\theta.$$

In general, setting $f = \partial_\theta^{-1}u$, the equation becomes

$$\chi_t = \left[\left(D_\theta - \chi - \chi_\theta\partial_\theta^{-1}\right)^2 + 1\right] u + a\chi_\theta.$$

Setting $u = \Omega_3^{n-2}\chi_\theta$, we obtain the Burgers hierarchy

$$\chi_t = \left(\Omega_3^n + \Omega_3^{n-2} + a\right)\chi_\theta,$$

where $\Omega_3 = D_\theta - \chi - \chi_\theta\partial_\theta^{-1}$ is the recursion operator of the Burgers equation. These equations can be linearized by the Cole–Hopf transformation $\chi = (\ln\eta)_\theta$, where η is the reciprocal of the Euclidean curvature $\eta = 1/k$. Indeed, the hierarchy is transformed to

$$\eta_t = D_\theta^n\eta + D_\theta^{n-2}\eta + a\eta.$$

It is noted that the motion conserves enclosed area of the curve only when n is odd.

6 Concluding remarks

We have shown that many well-known integrable equations including KdV, Sawada–Kotera, Harry Dym, Burgers hierarchies and Kaup–Kupershmidt equation naturally arise from motions of plane curves in affine, centro-affine and similarity geometries. The mKdV equation in the Euclidean space $\mathbf{E}^2$, the KdV equation in the centro-affine geometry and the Sawada–Kotera equation in the affine geometry, are all obtained by choosing the normal velocity to be the derivative of the curvature with respect to the arclength. A further analysis shows that the N-soliton of the mKdV and the Sawada–Kotera equation gives N-loop curves respectively in Euclidean and affine geometries and the N-soliton of the KdV equation gives $N-1$-loop curve in centro-affine geometry [22]. Similar properties also hold for space curves [23]. These analogies suggest that the KdV equation and Sawada–Kotera equation are respectively the centro-affine version and affine version of the mKdV equation.

The vector fields of Lie algebras acting on the plane have been completely classified [12]. Recently we have investigated motions of curves in these geometries and found many associated integrable hierarchies. The reader is referred to as [24] for all details.

Finally we point out that the equivalence between integrable equations for the curvature and invariant motion leads to some new integrable equations. For example, in the Euclidean case, suppose the mKdV flow can be expressed as the graph of $(x, u(x,t))$ of some function u over x-axix, one finds that u satisfies the well-known WKI equation [25]

$$u_t = \left[\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right]_x, \tag{25}$$

which can be solved by the inverse scattering method. Similarly in the affine geometry, Sawada–Kotera flow can be expressed by the following integrable equation

$$u_t = -\left[u_{xx}^{-\frac{5}{3}}u_{xxxx} - \frac{5}{3}u_{xx}^{-\frac{8}{3}}u_{xxx}^2\right]_x. \tag{26}$$

The WKI equation (25) and equation (26) have many similarities, such as they are derived in the same manner, can be solved by the inverse scattering method and have N-loop solitons. A detail analysis to (26) is presented in [24].

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[1] Hasimoto H., A soliton on a vortex filament, *J. Fluid Mech.*, 1972, V.51, N 3, 477–485.

[2] Lamb G.L., Soliton a moving space curve, *J. Math. Phys.*, 1977, V.18, N 8, 1654–1661.

[3] Lakshmanan M., Rigid body motions, space curves, prolongation structures, fiber bundles, and solitons, *J. Math. Phys.*, 1979, V.20, N 8, 1667–1672.

[4] Langer J. and Perline R., Poisson geometry of the filament equation, *J. Nonlinear Sci.*, 1991, V.1, N 1, 71–93.

[5] Doliwa A. and Santini P.M., An elementary geometric characterization of the integrable motions of a curve, *Phys. Lett. A*, 1994, V.185, N 4, 373–384.

[6] Nakayama K., Motion of curves in hyperboloid in the Minkowski space, *J. Phys. Soc. Jpn.*, 1998, V.67, N 9, 3031–3037.

[7] Nakayama K., Motion of curves in hyperboloid in the Minkowski space II, *J. Phys. Soc. Jpn.*, 1999, V.68, N 10, 3214–3218.

[8] Goldstein R.E. and Petrich D.M., The KdV hierarchy as dynamics of closed curves in the plane, *Phys. Rev. Lett.*, 1991, V.67, N 23, 3203–3206.

[9] Nakayama K., Segur H. and Wadati M., Integrability and the motion of curves, *Phys. Rev. Lett.*, 1992, V.69, N 18, 2603–2606.

[10] Ablowitz M.J., Kaup D.J., Newell A.C. and Segur H., The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.*, 1974, V.53, N 4, 249–315.

[11] Zakharov V.E. and Shabat A.B., A plan for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, *Func. Anal. Appl.*, 1974, V.8, N 3, 226–235.

[12] Olver P.J., Equivalence, invariants and symmetry, Cambridge, Cambridge University Press, 1995.

[13] Guggenheimer H.W., Differential geometry, New York, Dover Publications, 1973.

[14] Sawada K. and Kotera T., A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation, *Prog. Theor. Phys.*, 1974, V.51, 1335–1367.

[15] Caudrey P.J., Dodd R.K. and Gibbon J.D., A new hierarchy of Korteweg-de Vries equations, *Proc. R. Soc. London A*, 1976, V.351, N 1666, 407–422.

[16] Fuchssteiner B. and Oevel W., The bi-Hamiltonian structure of some nonlinear fifth- and seventh-order differential equations and recursion formulas for their symmetries and conserved covaria, *J. Math. Phys.*, 1982, V.23, N 3, 358–363.

[17] Chou K.S. and Zhu X.P., The curve shortening problem, Boca Raton, Chapman & Hall/CRC Press, 2001.

[18] Leo M., Leo R.A., Soliani G., Solombrino L. and Martina L., Lie–Bäcklund symmetries for the Harry–Dym equation, *Phys. Rev. D*, 1983, V.27, N 6, 1406–1408.

[19] Kaup D.J., On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx}+6Q\psi_x+6R\psi = \lambda\psi$, *Stud. Appl. Math.*, 1980, V.62, N 3, 189–216.

[20] Kupershmidt B.A., A super Korteweg-de Vries equation: an integrable system, *Phys. Lett. A*, 1984, V.102, N 5–6, 213–215.

[21] Calogero F. and Degasperis A., Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform, *J. Math. Phys.*, 1981, V.22, N 1, 23–31.

[22] Chou K.S. and Qu C.Z., The KdV equation and motion of plane curves, *J. Phys. Soc. Jpn.*, 2001, V.70, N 7, 1912–1916.

[23] Chou K. S. and Qu C.Z., Integrable motions of space curves in affine geometry, *Chaos, Soliton & Fractals*, 2002, V.14, N 1, 29–44.

[24] Chou K.S. and Qu C.Z., Integrable equations arising from motion of plane curves in Klein geometry, *Physica D*, 2002, V.162, N 1–2, 9–33.

[25] Wadati M., Konno K. and Ichikawa Y., New integrable nonlinear evolution equations, *J. Phys. Soc. Jpn.*, 1979, V.47, N 5, 1698–1700.