Applied and Numerical Harmonic Analysis

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Geometric Mechanics on Riemannian Manifolds

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To My Parents
Marta and Constantin
—O.C.

To My Family
Shian-Chih, Joshua, and Sarah
—D.C.C.

Contents

Preface

Historically, the Fourier transform has been a powerful method for solving linear partial differential equations. This book presents another approach, which shows that many equations are inspired from mechanics and that using geometric methods is the most natural and appropriate treatment. The text is enriched with examples and chapter exercises, which facilitate our understanding.

An Overview for the Reader The goal of this book is to explore some connections between differential geometry and partial differential equations: that is, partial differential equations are linked with a geometric view of classical mechanics in both its Lagrangian and Hamiltonian formulations on Riemannian manifolds. When quantitative solutions cannot be obtain explicitly, the equations of motion are solved qualitatively using conservation laws provided by the geometry of the problem.

Starting with an overview of differential geometry, the book proceeds to a description of topics of current interest such as quantum harmonic oscillators, fundamental solutions for elliptic and parabolic operators, harmonic maps, conservation theorems, Lagrangian and Hamiltonian formalism.

This work is a text for a course or seminar directed at graduate and advanced undergraduate students interested in elliptic and parabolic equations, differential geometry, calculus of variations, quantum mechanics. It is also an ideal resource for pure and applied mathematicians and theoretical physicists working in these areas.

Scientific Outline The subject of calculus of variations is an extension of calculus in which the working space is a manifold. This book deals with an invariant approach to the Lagrangian and Hamiltonian formalism on Riemannian manifolds with applications to constructions of the fundamental solutions for parabolic and elliptic operators.

The construction of some fundamental solutions construction uses the conservations laws and variational formalisms introduced in the first chapter. Fundamental solutions for Schrödinger and heat equations involving linear, quadratic, and quartic potentials are discussed here. Formally, the method works for any potential and represents an application of the variational formalism to partial differential equations. Until now, these fundamental solutions were found using methods of Fourier or Laplace transforms, Feynman's path integrals, or complex analysis techniques. The methods introduced in this text explain why the quartic harmonic oscillator is more difficult to invert than its linear analog model. This approach brings into play differential geometry methods into partial differential equations and quantum mechanics.

It is known that, in general, the coordinate space for a dynamical system is a Riemannian manifold. In order to build a theory of dynamical systems, we need the appropriate tools. Thus, we use a purely geometrical treatment for problems in physics or mechanics. Our approach is done in the context of both local coordinates and invariantly.

The idea is to write down the Euler–Lagrange system of equations for some Lagrangians (with certain physical interpretations) and to characterize the system qualitatively, from the conservation laws point of view, using the symmetry of the coordinate space. Usually these systems cannot be solved explicitly. For simple equations, one may characterize the solutions by finding the first integrals of motion. In the general case, the conservation laws are described by free divergence vector fields, trace free tensor fields, or constant energy functions. The conservation laws in the very simple dynamical systems are those of energy, momentum, or angular momentum. We shall treat these notions in the case of Riemannian manifolds. Principles from classical mechanics such as those of Hamilton, D'Alembert, and Euler, are studied with Noether's theorems and Newton's equations.

The use of conservation laws for the energy-momentum tensor associated with different Lagrangians provides uniqueness for some linear and nonlinear boundary problems (Dirichlet and Neumann) on Riemannian manifolds. Conservation properties of the energy-momentum tensor have interesting applications in geometry, physics, and partial differential equations.

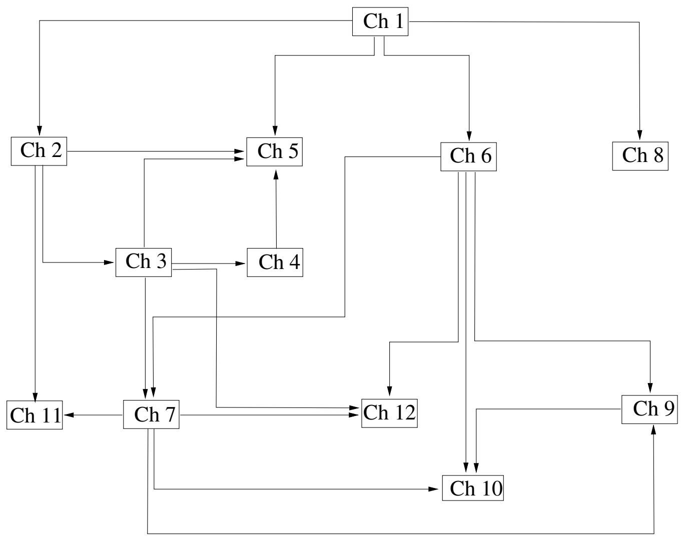
Several chapters of the book discuss the Hamiltonian formalism and the Hamilton–Jacobi equation. Geodesics, harmonic maps, and eiconal equations are approached from this point of view. Another chapter is dedicated to applications for minimal surfaces, minimal waves, and other physical applications, such as the Helmhotz decomposition of vector fields.

Two chapters provide applications of the Lagrangian and Hamiltonian formalism to heat kernels and the fundamental solutions for Laplacians on manifolds. The method uses the concepts of energy and action to describe the fundamental solutions.

A final chapter is dedicated to mechanical curves treated from the energy point of view. We study Lagrangians which generate the motions on these curves. The conservation theorems in these cases provide the first integrals of motion with interesting geometrical interpretations.

Physicists, mathematicians, graduate students in the areas of elliptic and parabolic differential equations, differential geometry, calculus of variations and quantum mechanics, and even well-prepared undergraduates will appreciate this introduction to the beautiful geometric theory of partial differential equations.

Acknowledgments This work owes much to the generous help of many people. First, we would like to thank our teachers P. Greiner and E.M. Stein for their teaching, encouragement, and sharing of their mathematical ideas with us. We would like to thank R. Beals, S. Ianus, T. Luo, Y.T. Siu, J. Tie and S.T. Yau, for their important advice and valuable criticism. Heartful thanks also to R. Smith, K. Klump, and S. Becker for reading the manuscript carefully for typos. We would also like to thank the Mathematics Departments at Eastern Michigan University and Georgetown University for providing excellent research environments for us. Finally, we would like to express our gratitude to Birkhaüser Boston and the ANHA editors, especially J.J. Benedetto in making this endeavor possible.



1

Introductory Chapter

1.1 Manifolds

Roughly speaking, a manifold is essentially a space that is locally similar to the Euclidean space. This resemblance permits differentiation to be defined. On a manifold, we do not distinguish between two different local coordinate systems. Thus, the concepts considered are just those independent of the coordinates chosen. This makes more sense if we consider the situation from the physics point of view. In this interpretation, the systems of coordinates are systems of reference. Physics studies objects like force, matter fields, momenta, and conservation laws, which in the differential geometry point of view are vector fields, tensor fields, one-forms, and first integrals. They are objects independent of the system of coordinates and can be defined globally but may be written locally in a local system of coordinates using local components. For example, the velocity, which is a vector field, may be written in local coordinates as $v = \sum v^i \frac{\partial}{\partial x_i}$,where $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=\overline{1,n}}$ is a basis of the local system of coordinates chosen. This means that the components of velocity measured in this system of reference are $v^1, \ldots, v^n$. Changing the system of coordinates will also modify the components under a certain rule.

A precise definition of the concept of manifold is given in the following. All the manifolds considered in this book are *real*, *i.e.*, the local model is the space $\mathbb{R}^n$.

Definition 1.1 *Let M be a topological space. Then the pair (U, ϕ) is called a chart (coordinate system), if $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a homeomorphism of the open set U in M onto an open set $\phi(U)$ of $\mathbb{R}^n$. The coordinate functions on U are defined as $x^j : U \to \mathbb{R}^n$, and $\phi(p) = (x^1(p), \ldots, x^n(p))$, namely $x^j = u^j \circ \phi$, where $u^j : \mathbb{R}^n \to \mathbb{R}$, $u^j(a_1, \ldots, a_n) = a_j$ is the j^{th} projection. n is called the dimension of the coordinate system.*

Definition 1.2 *A topological space M is called Hausdorff if for every two distinct points $p_1, p_2 \in M$, there are two open sets $U_1, U_2 \subset M$ such that*

$$p_1 \in U_1, \ \ p_2 \in U_2, \ \ \ U_1 \cap U_2 = \varnothing.$$

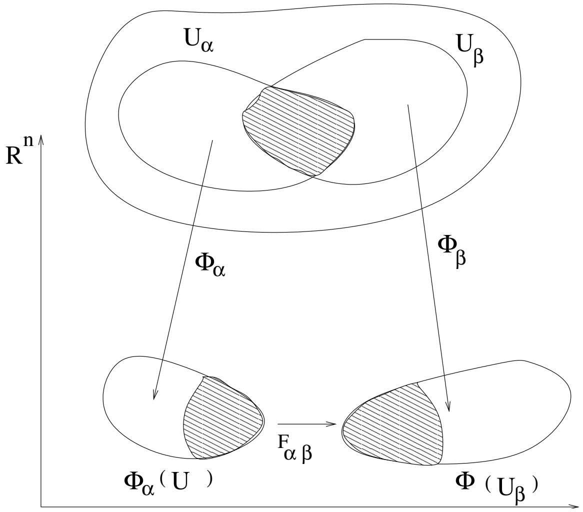


Figure 1.1: The system of coordinates on a manifold overlap smoothly

Definition 1.3 *An atlas $\mathcal{A}$ of dimension n associated with the topological space M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$ such that*
1) $U_\alpha \subset M$, $\bigcup_\alpha U_\alpha = M$ (U_α covers M),
2) if $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$F_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cup U_\beta)$$

is smooth (the systems of coordinates overlap smoothly).

On the topological space M, we may have many atlases. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are called compatible if their union is an atlas on M. The set of compatible atlases with a given atlas can be organized by inclusion. The maximal element is called the *complete atlas* $\mathcal{C}$. It contains all the charts that overlap smoothly with the charts of the given atlas $\mathcal{A}$.

Definition 1.4 *A smooth manifold M is a Hausdorff space endowed with a complete atlas. The dimension n of the atlas is called the dimension of the manifold.*

Examples of manifolds
1) The space $\mathbb{R}^n$ is a smooth manifold of dimension n defined by only one chart, the identity map.
2) A curve $c : (a, b) \to \mathbb{R}^n$ is a one-dimensional manifold, where $M = \Im m(c)$ and the atlas consists of one chart (U, ϕ), where $U = c\big((a, b)\big)$, $\phi : U \to (a, b)$, $\phi = c^{-1}_{|\Im m\, c}$.
3) The sphere $\mathbb{S}^2 = \{a = (a_1, a_2, a_3) \in \mathbb{R}^3 \ ; \ |a| = 1\}$ is a smooth manifold of dimension 2 defined by the atlas $\mathcal{A} = \{U_i, \phi_i\}_{i=\overline{1,3}} \cup \{V_i, \psi_i\}_{i=\overline{1,3}}$

$$U_1 = \{a \ ; \ a_1 > 0\}, \quad \phi_1 : U_1 \to \mathbb{R}^2, \quad \phi_1(a) = (a_2, a_3),$$
$$V_1 = \{a \ ; \ a_1 < 0\}, \quad \psi_1 : V_1 \to \mathbb{R}^2, \quad \psi_1(a) = (a_2, a_3),$$
$$U_2 = \{a \ ; \ a_2 > 0\}, \quad \phi_2 : U_2 \to \mathbb{R}^2, \quad \phi_2(a) = (a_1, a_3),$$
$$V_2 = \{a \ ; \ a_2 < 0\}, \quad \psi_2 : V_2 \to \mathbb{R}^2, \quad \psi_2(a) = (a_1, a_3),$$
$$U_3 = \{a \ ; \ a_3 > 0\}, \quad \phi_3 : U_3 \to \mathbb{R}^2, \quad \phi_3(a) = (a_1, a_2),$$
$$V_3 = \{a \ ; \ a_3 < 0\}, \quad \psi_3 : V_3 \to \mathbb{R}^2, \quad \psi_3(a) = (a_1, a_2).$$

4) If M, N are smooth manifolds, $M \times N$ is a smooth manifold, called the *product manifold*. For example, the cylinder $\mathbb{S}^1 \times [0, 1]$ and the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ are smooth manifolds.
5) The cone $\mathcal{C} = \{x_1^2 + x_2^2 = x_3^2\}$ is not a smooth manifold. This is due to the singularity it has at the origin, where differentiation cannot be performed. Indeed, consider a chart (U, ϕ) around 0. We may assume that there is a ball $\mathbf{B}(0, \epsilon)$ centered at $\phi(0)$ included in $\phi(U)$. Then $U \backslash \{0\}$ has two connected components. Since ϕ is a homeomorphism from U onto $\phi(U)$, $\phi(U) \backslash \{\phi(0)\}$ has two connected components. Then $\mathbf{B}(0, \epsilon) \backslash \phi(0)$ should have the same. This is a contradiction.

1.2 Tangent vectors

Definition 1.5 *A function $f : M \to \mathbb{R}$ is said to be smooth if for every chart (U, ϕ) on M, the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. The set of all smooth functions on the manifold M will be denoted by $\mathcal{F}(M)$.*

Definition 1.6 *A tangent vector at a point $p \in M$ is a map $X_p : \mathcal{F}(M) \to \mathbb{R}$ such that X_p*
i) is $\mathbb{R}$-linear: $X_p(af + bg) = aX_p(f) + bX_p(g)$, $\forall a, b \in \mathbb{R}$, $\forall f, g \in \mathcal{F}(M)$,
ii) satisfies the Leibnitz rule

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g), \quad \forall a, b \in \mathbb{R}, \ \forall f, g \in \mathcal{F}(M). \qquad (1.2.1)$$

The set of all tangent vectors at p to M is denoted by T_pM and is called *the tangent space at p*. It is a vector space of dimension n. A basis in this space is given by the coordinate tangent vectors $\frac{\partial}{\partial x_i}_{|p}$ defined by

$$\frac{\partial}{\partial x_i}_{|p}(f) = \frac{\partial (f \circ \phi^{-1})}{\partial u^i}(\phi(p)), \qquad (1.2.2)$$

where $\phi = (x^1, \ldots, x^n)$ is a system of coordinates around p and $u^1, \ldots, u^n$ are the coordinate functions on $\mathbb{R}^n$.

Every vector $v \in T_pM$ can be written as $v = \sum_i v^i \frac{\partial}{\partial x_i}_{|p}$. $v^i = v(x^i)$ are called the components of v in the system of coordinates $(x^1, \dots, x^n)$. When changing coordinates between two systems $(x^1, \dots, x^n)$ and $(\bar{x}^1, \dots, \bar{x}^n)$, the change of the components of the vector is given by

$$\bar{v}^k = \sum_{i=1}^{n} \frac{\partial \bar{x}^k}{\partial x_i} v^i \tag{1.2.3}$$

where $\{\bar{v}^k\}$ are the components in the second system of coordinates.

If the Jacobian from one chart to another is defined as

$$J = \left(\frac{\partial \bar{x}^k}{\partial x_i} \right)_{i,k=\overline{1,n}} \tag{1.2.4}$$

then $\det J \neq 0$, because ϕ is a diffeomorphism.

The physical notion of velocity corresponds to the geometrical concept of a vector field. The following result states that there is a reference system in which $n - 1$ components of the vector vanish and the n^{th} component is equal to 1.

Definition 1.7 *A smooth map $X : M \to \bigcup_{p \in M} T_pM$ that assigns to each point $p \in M$ a vector X_p in T_pM is called a vector field.*

The set of all vector fields on M will be denoted by $\mathcal{X}(M)$. In a local system of coordinates a vector field is given by $X = \sum X^i \frac{\partial}{\partial x_i}$, where the components $X^i \in \mathcal{F}(M)$ are given by $X^i = X(x^i)$, $i = \overline{1, n}$.

Theorem 1.8. *(Rectification theorem) Let V be a nonzero vector field at a point p on the manifold M. Then there exists a system of coordinates $(\bar{x}^1, \dots, \bar{x}^n)$ about p such that there is $j \in \{1, \dots, n\}$ for which*

$$V = \frac{\partial}{\partial \bar{x}_j}. \tag{1.2.5}$$

Proof. Choose an arbitrary system of coordinates $(x^1, \dots, x^n)$. Then $V = \sum v^i \frac{\partial}{\partial x_i}_{|p}$. Since $V_{|p} \neq 0$, at least one component is not equal to zero. Assuming that $v_n \neq 0$, choose the second system of coordinates $(\bar{x}^1, \dots, \bar{x}^n)$ defined by

$$\bar{x}^j = x^j - \frac{v_j}{v_n} x_n \;, \quad \forall j = \overline{1, n-1},$$
$$\bar{x}^n = \frac{x_n}{v_n}.$$

Then formula (1.2.3) yields (1.2.5) with $j = n$. ∎

Given a vector field X, consider the system

$$\frac{dc^k(t)}{dt} = X^k(c(t)), \quad k = \overline{1, n}. \tag{1.2.6}$$

The next result shows that the system (1.2.6) can be solved locally around the point $x_0 = c(0)$, for $0 < t < \epsilon$. The solution $t \to c(t)$ is called the *integral curve* associated with the vector field X through the point x_0. The local existence and uniqueness of integral curves are given by the following result.

Theorem 1.9. *(Existence and uniqueness) Given $x_0 \in M$ and letting X be a nonzero vector field on an open set $\mathcal{U} \subset M$ of x_0, then there is $\epsilon > 0$ such that the system (1.2.6) has a unique solution $c : [0, \epsilon) \to \mathcal{U}$ such that $c(0) = x_0$.*

Proof. By the rectification theorem, there is a local change of coordinates $\bar{x} = \phi(x)$ such that the system (1.2.6) becomes

$$\frac{d\overline{c}^k(t)}{dt} = \delta_{kn}, \quad k = \overline{1, n}, \tag{1.2.7}$$

where $\overline{c} = \phi(c)$. The system (1.2.7) has a unique solution through the point $\overline{x}_0 = \phi(x_0)$ given by $\overline{c}^k(t) = \overline{x}_0^k$, $k = \overline{1, n-1}$ and $\overline{c}^n(t) = t + \overline{x}_0^n$. Hence this will hold also for the system (1.2.6) in a neighborhood of $x_0 = \phi^{-1}(\overline{x}_0)$. ∎

1.3 The Differential of a Map

Definition 1.10 *A map $F : M \to N$ between two manifolds M and N is smooth about $p \in M$ if for any charts (U, ϕ) on M about p and $(V, \psi) \in N$ about $F(p)$, the application $\psi \circ F \circ \phi^{-1}$ is smooth from $\phi(U) \subset \mathbb{R}^m$ to $\psi(V) \subset \mathbb{R}^n$.*

Definition 1.11 *For every $p \in M$ the differential map dF at p is defined by $dF_p : T_pM \to T_{F(p)}N$ with*

$$(dF_p)(v)(f) = v(f \circ F), \quad \forall v \in T_pM, \ \forall f \in \mathcal{F}(N). \tag{1.3.8}$$

Locally, it is given by

$$dF_p\Big(\frac{\partial}{\partial x_j}_{|p}\Big) = \sum_{k=1}^{n} \frac{\partial F^k}{\partial x_j}_{|p} \frac{\partial}{\partial y^k}_{|F(p)}, \tag{1.3.9}$$

where $F = (F^1, \ldots, F^n)$. The matrix $\left(\frac{\partial F^k}{\partial x_j}\right)_{k,j}$ is the *Jacobian* of F with respect to the charts $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ on M and N respectively.

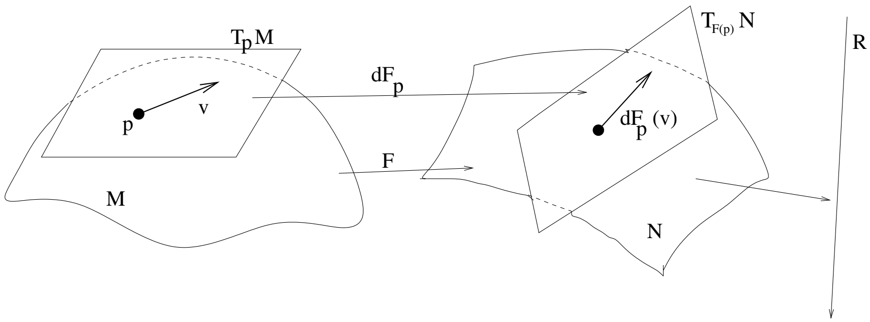


Figure 1.2: The differential of a map

The inverse function theorem on smooth manifolds is stated in the following. For a proof see [43].

Theorem 1.12. *Let $F : M \to N$ be a smooth map. Then the following conditions are equivalent:*
1) $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism;
2) F is a local diffeomorphism in a neighborhood of p;
3) There are two charts $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ on M and N respectively, such that the associated Jacobian is non-degenerate.

1.4 The Lie bracket

An important operation on vector fields is the Lie bracket $[\ ,\ ] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by

$$[V, W] = VW - WV. \tag{1.4.10}$$

In local coordinates,

$$[V, W] = \sum_{i,j=1}^{n} \Big(\frac{\partial W^i}{\partial x_j} V^j - \frac{\partial V^i}{\partial x_j} W^j \Big) \frac{\partial}{\partial x_i}. \tag{1.4.11}$$

The Lie bracket has the following properties:
1) $\mathbb{R}$-bilinearity:

$$[aV + bW, U] = a[V, U] + b[W, U], \qquad \forall a, b \in \mathbb{R},$$

2) skew-symmetry:

$$[U, V] = -[V, U],$$

3) Jacobi identity:

$$[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0,$$

4) $[fV, gW] = fg[V, W] + f(Vg)W - g(Wf)V \ , \quad \forall f, g \in \mathcal{F}(M)$.

If the Lie bracket of two vector fields is zero, $[U, V] = 0$, we say that the vector fields commute. If we start from a point p and go a parameter distance v along the integral curves of V followed by a parameter distance u along the integral curves of U, then we arrive at the same point as if the order of the vector fields is swapped.

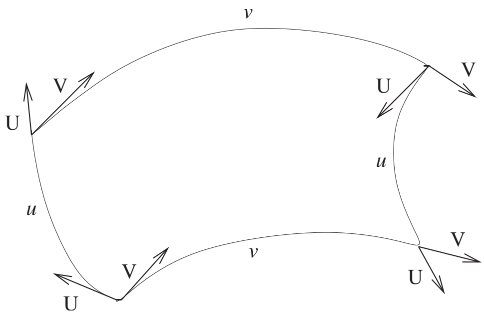


Figure 1.3: Integral curves for commuting vector fields

Example 1.4.1 *Consider on* $\mathbb{R}^3$ *the vector fields* $X = \partial_{x_1} - 2x_2\partial_{x_3}$, $Y = \partial_{x_2} + 2x_1\partial_{x_3}$ *and* $Z = \partial_{x_3}$. *Then* $[X, Y] = -4\partial_t$, $[X, Z] = [Y, Z] = 0$. *X and Y do not commute. Z commutes with both X and Y.*

1.5 One-forms

Let T_p^*M denote the dual space of T_pM which is called the *cotangent space* of M at p. The elements of T_p^*M are called *covectors*. A one-form ω on the manifold M is a function that assigns to each point $p \in M$ a covector $\omega_p \in T_p^*M$.

An example of a one-form is the differential of a function $f \in \mathcal{F}(M)$, which is defined as $(df)_p : T_pM \to \mathbb{R}$,

$$(df)_p(v) = v(f) \ , \quad \forall v \in T_pM. \tag{1.5.12}$$

In local coordinates, $df = \sum_i \frac{\partial f}{\partial x_i} dx^i$, where $\{dx^i\}$ is the basis in the T_p^*M which is dual to the basis $\{\frac{\partial}{\partial x_i}\}$ of T_pM. In general, a one-form in local coordinates can be written as

$$\omega = \sum_{i=1}^{n} \omega^i \, dx^i, \tag{1.5.13}$$

where $\omega^i = \omega(\frac{\partial}{\partial x_i})$. The set of all one-forms on the manifold M will be denoted by $\mathcal{X}^*(M)$. If $\phi : M \to N$ is a smooth function and $\omega \in \mathcal{X}^*(N)$, then the pull-back of the one-form ω is the one-form $\phi^*(\omega) \in \mathcal{X}^*(M)$ defined by

$$\phi^*\omega(V) = \omega(d\phi\ V), \qquad \forall\, V \in \mathcal{X}(N). \tag{1.5.14}$$

For more about differential forms see [12].

1.6 Tensors

A *tensor* of type (r, s) at $p \in M$ is a multi-linear function $T : (T_p^* M)^r \times (T_p M)^s \to \mathbb{R}$. A *tensor field* $\mathcal{T}$ of type (r, s) is a smooth map, which assigns to each point $p \in M$ an (r, s)-tensor $\mathcal{T}_p$ on M at the point p. In local coordinates,

$$\mathcal{T} = \mathcal{T}^{i_1 i_2 \dots i_s}_{j_1 j_2 \dots j_r}\, dx^{j_1} \otimes \cdots \otimes dx^{j_r} \otimes \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_s}}. \tag{1.6.15}$$

$\mathcal{T}$ acts on r one-forms and s vector fields

$$\begin{aligned}
\mathcal{T}(\omega_1, \dots, \omega_r, X_1, \dots, X_s) &= \mathcal{T}^{i_1 .. i_s}_{j_1 .. j_r}\, dx_{j_1}(X_1) \dots dx_{j_r}(X_r)\, \frac{\partial}{\partial x_{i_1}}(\omega_1) \dots \frac{\partial}{\partial x_{i_s}}(\omega_s) \\
&= \mathcal{T}^{i_1 \dots i_s}_{j_1 \dots j_r} X_1^{j_1} \dots X_r^{j_r}\, \omega_1^{i_1} \dots \omega_s^{i_s}.
\end{aligned}$$

We say the tensor $\mathcal{T}$ is s covariant and r contravariant.

If $\mathcal{T}$ is a tensor field of type (r, s) on N, then the pull-back $\phi^*\mathcal{T}$ of $\mathcal{T}$ is a tensor field on M of the same type, defined by

$$(\phi^*\mathcal{T})(X_1, \dots, X_r, \omega_1, \dots, \omega_s) = \mathcal{T}(d\phi\ X_1, \dots, d\phi\ X_r, \phi^*\omega_1, \dots, \phi^*\omega_s), \tag{1.6.16}$$

where $X_i \in \mathcal{X}(M)$, $\omega_i \in \mathcal{X}^*(M)$.

A tensor $\mathcal{T}$ may be Lie differentiated with respect to a vector field $X \in \mathcal{X}(M)$,

$$L_X \mathcal{T}_{|p} = \lim_{t \to 0} \frac{1}{t}(\mathcal{T}_p - (\varphi_t)^* \mathcal{T}_{|\varphi_t(p)}), \tag{1.6.17}$$

where φ_t is the one-parameter group of diffeomorphisms defined by the integral curves of the vector field X. That is $\varphi_t(p) = c(t)$, with $c(t)$ as the unique integral curve of X satisfying $c(0) = p$. The name *one-parameter group* comes from the fact that $\varphi_t \circ \varphi_s = \varphi_{t+s} = \varphi_s \circ \varphi_t$, with $|t|, |s|, |t+s| < \epsilon$.

On coordinates components we have

$$\begin{aligned}
(L_X T)^{ab \dots d}_{ef \dots g} = \frac{\partial T^{ab \dots d}_{ef \dots g}}{\partial x_i} X^i - T^{ib \dots d}_{ef \dots g} \frac{\partial X^a}{\partial x_i} \\
- (\text{all upper indices}) + T^{ab \dots d}_{if \dots g} \frac{\partial X^i}{\partial x_e} + (\text{all lower indices}).
\end{aligned}$$

The $(1, 0)$-tensor fields are in fact vector fields. The $(0, 1)$ tensor fields are one-forms. In this case the Lie derivative is

$$L_X Y = [X, Y],$$
$$L_X(df) = d(Xf), \quad \forall f \in \mathcal{F}(M).$$

Other properties of the Lie derivative are:

$$L_{aX+bY} = aL_X + bL_Y, \quad \forall a, b \in \mathbb{R}, X, Y \in \mathcal{X}(M),$$
$$L_X f = X(f), \quad \forall f \in \mathcal{F}(M),$$
$$L_{[X,Y]} = [L_X, L_Y], \quad \forall X, Y \in \mathcal{X}(M),$$
$$d\Big(L_X\omega\Big) = L_X(d\omega), \qquad \forall \omega \ \ p\text{-form}.$$

If T is an (s, r)-tensor, then $L_X T$ is also an (s, r)-tensor. A vector field is called a Killing vector field if $L_X g = 0$, where g is the Riemannian metric tensor (see next section).
A tensor of type (0, 2) is called *symmetric* if

$$T_{ab} = T_{ba}, \tag{1.6.18}$$

and it is called *antisymmetric* if

$$T_{ab} = -T_{ba}. \tag{1.6.19}$$

1.7 Riemannian Manifolds

There are manifolds on which we may want to measure distances, angles, and lengths of vectors and curves. From the math point of view they represent generalizations of the surfaces of more than two dimensions. From the mechanics point of view, they constitute the models for the coordinate spaces of dynamical systems. Their tangent bundle represents the phase space. The metric they are endowed with allows measuring the energy and constructing Lagrangians on the phase space and Hamiltonians on the cotangent bundle. This way, Riemannian Geometry becomes an elegant frame and proper environment for doing Classical Mechanics.

Definition 1.13 *A Riemannian metric g on a smooth manifold M is a symmetric, positive definite* (0, 2)*-tensor field.*

This means that $\forall p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is a positive definite scalar product. In local coordinates

$$g = g_{ij}\, dx^i \otimes dx^j. \tag{1.7.20}$$

Definition 1.14 *A Riemannian manifold is a smooth manifold M endowed with a Riemannian metric g.*

Let $\mathbb{E}^n = (\mathbb{R}^n, \langle\ ,\ \rangle)$ denote the n-dimensional Euclidean space. For a proof of the next theorem see [4].

Theorem 1.15 (Whitney). *If M is a differentiable manifold of dimension n, then there is a diffeomorphism $\phi : M \to \mathbb{E}^{2n+1}$ such that $\phi(M)$ is closed in $\mathbb{E}^{2n+1}$.*

The existence of a Riemannian metric is given in the next result.

Theorem 1.16. *If M is a smooth manifold, then there is at least one Riemannian metric on M.*

Proof. Denote by $\langle\ ,\ \rangle$ the Euclidean scalar product on $\mathbb{R}^{2n+1}$, and consider the immersion $\phi : M \to \mathbb{E}^{2n+1}$ given by the Whitney theorem. Choose

$$g(X, Y) = \langle \phi_* X, \phi_* Y \rangle\ , \quad \forall X, Y \in \mathcal{X}(M), \tag{1.7.21}$$

Then (M, g) is a Riemannian manifold. ∎

There is a one-to-one, onto correspondence between the one-forms and the vector fields on a Riemannian manifold M. If V is a vector field, then one may associate with it a one-form ω such that

$$\omega(U) = g(V, U), \qquad \forall U \in \mathcal{X}(M). \tag{1.7.22}$$

If in local coordinates $\omega = \omega^i dx_i$ and $V = V^j \frac{\partial}{\partial x_j}$, then

$$\omega^k = g_{jk} V^j.$$

1.8 Linear Connections

The linear connection is an extension of the directional derivative from the Euclidean case.

Definition 1.17 *A linear connection ∇ on a smooth manifold M is a map ∇ : $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ with the following properties:*
1) $\nabla_X Y$ *is $\mathcal{F}(M)$-linear in X,*
2) $\nabla_X Y$ *is $\mathbb{R}$-linear in Y,*
3) *it satisfies the Leibnitz rule:* $\nabla_X (fY) = (Xf)Y + f\,\nabla_X Y\ , \quad \forall f \in \mathcal{F}(M).$

$\nabla_X Y$ is a new vector field which, roughly speaking, is the vector rate change of Y in the direction of X.

Example 1.8.1 *On $\mathbb{R}^n$ a linear connection is*

$$\nabla_U V = \sum_{j=1}^{n} U(V^j) E_j, \tag{1.8.23}$$

where $E_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j^{th} basis vector on $\mathbb{R}^n$ and $V = \sum_j V^j E_j$.

Definition 1.18 *Let ∇ be a linear connection. The torsion is defined as*

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{1.8.24}$$

The curvature of the linear connection is given by

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.8.25}$$

If S is a tensor field of type $(0, r)$, we may differentiate it along a vector field V with respect to the linear connection ∇ as

$$(\nabla_V S)(X_1, \dots, X_r) = V\, S(X_1, \dots, X_r) - \sum_{i=1}^{n} S(X_1, \dots, \nabla_V X_i, \dots, X_r). \tag{1.8.26}$$

If g is the Riemannian metric tensor, the linear connection ∇ is called a *metric connection* if

$$\nabla_V g = 0\,, \quad \forall\, V \in \mathcal{X}(M). \tag{1.8.27}$$

This means that

$$V\, g(X, Y) = g(\nabla_V X, Y) + g(X, \nabla_V Y)\,, \quad \forall\, V, X, Y \in \mathcal{X}(M). \tag{1.8.28}$$

The amazing fact is that there is only one metric connection that has zero torsion. This constitutes the cornerstone of the geometry of Riemannian manifolds. The following theorem can be considered as a definition for the *Levi-Civita connection* and can be found for instance in [35].

Theorem 1.19. *On a Riemannian manifold there is a unique torsion-free, metric connection ∇. Furthermore, ∇ is given by the Koszul formula*

$$\begin{aligned} 2g(\nabla_V X, U) = {} & V\, g(X, U) + X\, g(U, V) - U\, g(V, X) \\ & - g(V, [X, U]) + g(X, [U, V]) + g(U, [V, X]). \end{aligned}$$

One can show that in local coordinates

$$\nabla_X Y = \sum_{i,k} X^i \Big(\frac{\partial Y^k}{\partial x_i} + \sum_j \Gamma_{ij}^k W^j \Big) \frac{\partial}{\partial x_k}, \tag{1.8.29}$$

where $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}$, $Y = \sum_{k=1}^{n} Y^k \frac{\partial}{\partial x_k}$ and Γ_{ij}^k are the Christoffel symbols defined by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} \Big(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \Big) \tag{1.8.30}$$

where (g^{km}) is the inverse matrix of (g_{ij}).

Definition 1.20 *A vector field Y is said to be parallel transported along the curve $c(t)$ if*

$$\nabla_{\dot{c}(t)} Y = 0. \tag{1.8.31}$$

In local coordinates

$$\sum_i \dot{c}^i(t)\Big(\frac{\partial Y^k}{\partial x^i} + \sum_j \Gamma^k_{ij} Y^j\Big)\frac{\partial}{\partial x_k} = 0.$$

The chain rule yields

$$\frac{dY^k}{dt} = \frac{\partial Y^k}{\partial x_i}_{|c(t)} \dot{c}^i(t),$$

so that one obtains that Y is parallel transported along the curve $c(t)$ if and only if

$$\frac{dY^k}{dt} + \sum_{i,j} \Gamma^k_{ij|c(t)} Y^j \dot{c}^i(t) = 0. \tag{1.8.32}$$

Together with the initial condition $Y(0) = v$, by Picard's theorem, equation (1.8.32) has locally a unique solution.

Sometimes we shall use the following shorter notation for the linear connection of a vector field with respect to one of the coordinate vector fields:

$$X^j_{;k} = (\nabla_{\frac{\partial}{\partial x_k}} X)^j. \tag{1.8.33}$$

If f is a function, we write $f_{;k} = \dfrac{\partial}{\partial x_k} f$. In general we shall write $;k$ for $\nabla_{\frac{\partial}{\partial x_k}}$ derivative.

Definition 1.21 *Let $R_{XY}Z = R(X, Y, Z)$ denote the curvature tensor and $\{E_1, \dots, E_n\}$ be an orthonormal system about p. The 2-covariant symmetric tensor defined by*

$$\begin{aligned} Ric(X, Y) &= Trace\Big(V \to R_{XV}Y\Big) \\ &= \sum_{j=1}^n g(R_{YE_j}X, E_j), \end{aligned}$$

is called the Ricci tensor.

1.9 The Volume element

On Riemannian manifolds we can measure not only lengths but also volumes. The *volume form* is an n-form defined locally by

$$dv = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n, \tag{1.9.34}$$

where $|g| = \det(g_{ij})_{i,j}$. As an $(n, 0)$-tensor, dv may be Lie differentiated along the vector field X. As an n-form, $L_X\, dv$ will be proportional to dv,

$$L_X\, dv = f\, dv. \tag{1.9.35}$$

The function f depends on the expansion of X, and it is called the *divergence* of the vector field X,

$$f = div\, X. \tag{1.9.36}$$

If M is a compact manifold, the volume of M is defined as

$$\mathrm{vol}(M) = \int_M dv. \tag{1.9.37}$$

Let(M, g) be a Riemannian manifold and $\iota : M \to \mathbb{R}^n$ be an isometric immersion, *i.e.*, $d\iota$ is one-to-one and g is the pull-back of the flat metric $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ through ι. Let $X \in \mathcal{X}(M)$ be a vector field and ν be the normal vector field to M, *i.e.*, $\nu_p \in T_pM$ and $\langle \nu_p, \nu_p \rangle = 1, \forall p \in M$. Then the divergence theorem takes place,

$$\int_M div X\, dv = \int_{\partial M} \langle X, \nu \rangle\, d\sigma, \tag{1.9.38}$$

where ∂M is the boundary of M and $d\sigma$ is the area element on ∂M.

For more about Calculus on manifolds the reader may consult [43]. For more differential geometry one may see [10], [11], [44].

1.10 Exercises

1. On a domain of a system of coordinates $(x_1, \ldots, x_n)$, if $V = \sum V^i \partial_{x_i}$ and $W = \sum W^j \partial_{x_j}$, then show that

$$[V, W] = \sum_{i,j=1}^{n} \left(\frac{\partial W^i}{\partial x_j} V^j - \frac{\partial V^i}{\partial x_j} V^j \right) \frac{\partial}{\partial x_i}.$$

2. Show that for any three vector fields $U, V, W \in \mathcal{X}(M)$ we have

$$[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0.$$

3. Let $(x_1, \ldots, x_n)$ be a system of coordinates at the point p on the Riemannian manifold (M, g). Consider a new system of coordinates $(x'_1, \ldots, x'_n)$ defined by

$$x'_j = x_j - x_j(p) + \Gamma^j_{ab|p}(x_a - x_a(p))(x_b - x_b(p)).$$

a) Show that in the system of coordinates $(x'_1, \ldots, x'_n)$ the Christoffel symbols $\Gamma^j_{a'b'|p} = 0$.

b) Using $g_{a'b';c'} = 0$ show that in the system of coordinates $(x'_1, \dots, x'_n)$ we have $\dfrac{\partial g_{a'b'}}{\partial x_{c'}}\Big|_p = 0$.

4. Given a point p on the Riemannian manifold (M, g), show that there is a system of coordinates at p in which

$$g_{ij}{}_{|p} = \delta_{ij} \qquad \text{and} \qquad \nabla_{\partial_{x_i}} \partial_{x_j}{}_{|p} = 0.$$

5. Prove or disprove:
Given an open set $\mathcal{U}$ in a differentiable manifold M of dimension n, and $X_1, \dots, X_n$ vector fields on $\mathcal{U}$ such that $[X_i, X_j] = 0$, then there is a system of coordinates $(x_1, \dots, x_n)$ on $\mathcal{U}$ such that $X_j = \dfrac{\partial}{\partial x_j}$.

6. Identify $\mathbb{R}^4$ with the quaternions space

$$\{q = x_0 + ix_1 + jx_2 + kx_3;\ x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

and let $\mathbb{S}^3 = \{q \in \mathbb{R};\ |q| = 1\}$, where $|q|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ be an application defined by $\pi(q) = qiq^{-1}$.

a) Show that

$$\pi(q) = i(x_0^2 + x_1^2 - x_2^2 - x_3^2) + j(2x_0x_3 + 2x_1x_2) + k(2x_1x_3 - 2x_0x_2)$$

and that $\pi(q) \in \mathbb{S}^2$.

b) Show that π is a submersion, *i.e.*, it is differentiable and the differential $d\pi_p$ is onto at each point $p \in \mathbb{S}^3$.

c) Find a nonzero global vector field X on $\mathbb{S}^3$ and calculate $d\pi(X)$.

7. Given a smooth curve $c(s)$ on a differentiable manifold, let $X = \dot{c}(s)$ be its tangent vector field. Show that X can be extended to a vector field on an open neighborhood of the curve $c(s)$.

8. Let $\gamma(s)$ be a curve on the Riemannian manifold (M, g) with the Levi-Civita connection ∇. Denote $V = \dot{\gamma}(s)$ the tangent vector field. The derivative along $\gamma(s)$ is defined as

$$\frac{D}{\partial s} Z = \nabla_Z V,$$

for any vector field Z along $\gamma(s)$. Show that for any $Z, Z_1, Z_2 \in \mathcal{X}(M)$ we have:

i) $\dfrac{D}{\partial s}(aZ_1 + bZ_2) = a\dfrac{D}{\partial s}Z_1 + b\dfrac{D}{\partial s}Z_2, \qquad a, b \in \mathbb{R},$

ii) $\dfrac{D}{\partial s}(hZ) = \dfrac{dh}{ds} Z + h \dfrac{D}{\partial s} Z, \qquad h \in \mathcal{F}(\mathbb{R}),$

iii) $\dfrac{D}{\partial s} g(Z_1, Z_2) = g\Big(\dfrac{D}{\partial s} Z_1, Z_2\Big) + g\Big(Z_1, \dfrac{D}{\partial s} Z_2\Big).$

9. Let $c(s)$ be a curve on the Riemannian manifold (M, g). The Fermi derivative $\dfrac{D_F}{\partial s}$ is a derivative along $c(s)$ defined by

$$\frac{D_F}{\partial s} X = \frac{D}{\partial s} X - g\Big(X, \frac{D}{\partial s} V\Big) V + g(X, V) \frac{D}{\partial s} V,$$

where $V = \dot{c}(s)$ and X is any vector field along $c(s)$. Show that

i) $\dfrac{D_F}{\partial s} V = 0.$

ii) $\dfrac{D_F}{\partial s} = \dfrac{D}{\partial s}$ if $c(s)$ is a geodesic.

iii) Let X, Y be two vector fields along $c(s)$ such that $\dfrac{D_F}{\partial s} X = \dfrac{D_F}{\partial s} Y = 0$. Then $g(X, Y)$ is constant along $c(s)$.

10. Given a curve $\gamma : (-\delta, \delta) \to M$ on the Riemannian manifold (M, g), show that there is a system of coordinates (Fermi coordinates) at $\gamma(0)$ in which $\Gamma^a_{bc} = 0$ along the curve γ.

11. A surface (Σ, g) is called locally conformal to $\mathbb{R}^2$ if there is a local system of coordinates in which

$$g_{ij} = \begin{pmatrix} e^h & 0 \\ 0 & e^h \end{pmatrix}$$

with h a smooth function.

a) Show that any surface is locally conformal to $\mathbb{R}^2$.

b) Is this still true for higher dimensions?

12. Consider Stokes' theorem:

If M is a compact oriented k-dimensional manifold with boundary and ω is a $k-1$ form on M, then

$$\int_M d\omega = \int_{\partial M} \omega,$$

where ∂M denotes the boundary of M.

Let $\omega = \alpha dx + \beta dy$. Show that Stokes' theorem becomes Green's theorem:

Let $M \subset \mathbb{R}^2$ be a compact 2-dimensional manifold with boundary. Suppose that $\alpha, \beta : M \to \mathbb{R}$ are differentiable. Then

$$\int_{\partial M} \alpha dx + \beta dy = \iint_M \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dxdy.$$

13. Let M be a surface and let $\nu(x)$ be the unit outward normal at $x \in M$. Define the area element

$$d\sigma(v, w) = \langle v \times w, \nu(x)\rangle, \qquad \forall v, w \in T_x M,$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{R}^3$.

a) Show that $d\sigma(v, w) = |v \times w|$.

b) Show that

$$d\sigma(v, w) = \det \begin{pmatrix} v \\ w \\ \nu \end{pmatrix}.$$

c) Prove that $d\sigma = \nu^1 dy \wedge dz + \nu^2 dz \wedge dx + \nu^3 dx \wedge dy$, where $\nu = (\nu^1, \nu^2, \nu^3)$.

d) Show that

$$\nu^1 d\sigma = dy \wedge dz, \qquad \nu^2 d\sigma = dz \wedge dx, \qquad \nu^3 d\sigma = dx \wedge dy.$$

14. Let $X = (X^1, X^2, X^3)$ be a vector field on the surface M in $\mathbb{R}^3$ and consider the one-form $\omega = X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy$.

a) Show that $d\omega = div\, X\, dv$.

b) Show that $\omega = \langle X, \nu\rangle d\sigma$.

c) Using Stokes' theorem show that

$$\int_M div X\, dv = \int_{\partial M} \langle X, \nu\rangle\, d\sigma.$$

2

Laplace Operators on Riemannian Manifolds

2.1 Gradient vector field; Divergence and Laplacian

Definition 2.1 *Let (M, g) be a Riemannian manifold and $f \in \mathcal{F}(M)$ be a smooth function. The gradient of f, denoted by ∇f, is a vector field on M metrically equivalent to df:*

$$g(\nabla f,\ X) = df(X), \qquad \forall X \in \mathcal{X}(M). \tag{2.1.1}$$

Remark 2.2 *We note the right-hand side of (2.1.1) can also be written as*

$$df(X) = X(f).$$

Remark 2.3 *Sometimes, to avoid confusion with the Levi-Civita connection, the gradient will be denoted by $grad\ f$.*

In local coordinates the gradient is

$$\nabla f = \sum_{j=1}^{n} (\nabla f)^j \frac{\partial}{\partial x_j}.$$

Using

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i,$$

the equation (2.1.1) yields

$$g_{ij} (\nabla f)^j X^i = \frac{\partial f}{\partial x_j} X^i, \qquad \forall X \in \mathcal{X}(M). \tag{2.1.2}$$

The components of the gradient are

$$(\nabla f)^j = g^{ij} \frac{\partial f}{\partial x_i}, \tag{2.1.3}$$

and then

$$\nabla f = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \tag{2.1.4}$$

with summation over the repeated index.

Example 2.1.1 *On $\mathbb{R}^n$ the gradient of a function f is*

$$\nabla f = \sum_{i=1}^{n} E_i(f)\, E_i, \tag{2.1.5}$$

with $E_i = (0, \ldots, \overbrace{1}^{i^{th}}, \ldots, 0)$.

In physics, a force vector field is called conservative if it is the gradient of a certain potential energy . This definition can be extended for any vector field on manifolds as follows.

Definition 2.4 *Let $X \in \mathcal{X}(M)$ be a vector field on M. We say that X is provided by a potential Φ if there is a differentiable function $\Phi \in \mathcal{F}(M)$ such that $X = \nabla\Phi$.*

In local coordinates

$$X^j = g^{ij} \frac{\partial \Phi}{\partial x_j}. \tag{2.1.6}$$

Definition 2.5 *Let $X \in \mathcal{X}(M)$ be a vector field on M. The divergence of X at the point $p \in M$ is defined as*

$$div(X)_p = \sum_{i=1}^{n} g_p(\nabla_{E_i} X\ ,\ E_i), \tag{2.1.7}$$

where $E_1, \ldots, E_n$ is an orthonormal basis in T_pM and ∇ denotes the Levi-Civita connection on M with respect to g.

Example 2.1.2 *Consider the Newtonian potential $\Phi(x) = \frac{1}{|x|}$, $x \in \mathbb{R}^n \backslash \{0\}$. The force vector field is $F = -\nabla\Big(\frac{1}{|x|}\Big)$ and*

$$div\, F = -\Delta\Big(\frac{1}{|x|}\Big) = 0 \quad on\ \mathbb{R}^n \backslash \{0\}. \tag{2.1.8}$$

The equation (2.1.7) can be written also as

$$div\ X = Trace(Y \to g(\nabla_Y X, Y)\,). \tag{2.1.9}$$

Using the expression in local coordinates

$$div\ (X) = \sum_{i=1}^{n} X^i_{;\,i} = \sum_{i=1}^{n} \Big(\frac{\partial X^i}{\partial x_i} + \sum_j \Gamma^i_{ij} X^j \Big) \tag{2.1.10}$$

we note that $div\ X$ depends not only on X^i, but also on the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \Big(\frac{\partial g_{jl}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} \Big). \tag{2.1.11}$$

The following lemma shows that $div\ X$ depends only on X and $g = det(g_{ij})$.

Lemma 2.6 *In local coordinates we have:*

$$div\ X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} (\sqrt{g} X^j) \tag{2.1.12}$$

with summation over $j = 1, \ldots, n$.

Proof. Using the definition of Γ^i_{jk} and the symmetry of g_{ij},

$$\Gamma^i_{ji} X^j = \frac{1}{2} g^{is} \big(\frac{\partial g_{js}}{\partial x_i} + \frac{\partial g_{is}}{\partial x_j} - \frac{\partial g_{ji}}{\partial x_s} \big) X^j = \frac{1}{2} g^{is} \frac{\partial g_{is}}{\partial x_j} X^j.$$

Then equation (2.1.10) yields

$$div\ X = \frac{\partial X^i}{\partial x_i} + \frac{1}{2} g^{is} \frac{\partial g_{is}}{\partial x_j} X^j. \tag{2.1.13}$$

We compute first the expression $\frac{1}{2} g^{is} \frac{\partial g_{is}}{\partial x_j}$. Let $g = det\ (g_{ij}) = g(g_{11}, g_{12}, ..., g_{ij}, ..., g_{nn})$ denote the determinant.
Then

$$\frac{\partial g}{\partial x_j} = \frac{\partial g}{\partial g_{is}} \frac{\partial g_{is}}{\partial x_j}. \tag{2.1.14}$$

As $\frac{\partial g}{\partial g_{is}}$ is the minor of g_{is},

$$g^{is} = \frac{1}{g} \frac{\partial g}{g_{is}}, \tag{2.1.15}$$

where (g^{is}) is the inverse matrix of (g_{ij}). Then (2.1.14) and (2.1.15) yield

$$\frac{\partial g}{\partial x_j} = g\, g^{is} \frac{\partial g_{is}}{\partial x_j}. \tag{2.1.16}$$

Substitute in (2.1.13) and obtain

$$\begin{aligned} div\ X &= \frac{\partial X^j}{\partial x_j} + \frac{1}{2g}\frac{\partial g}{\partial x_j} X^j \\ &= \frac{1}{\sqrt{g}}\big(\frac{\partial X^j}{\partial x_j}\sqrt{g} + \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x_j} X^j\big) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(\sqrt{g}\, X^j). \end{aligned}$$

■

The definition of the divergence of a vector field given above matches the definition given in the introductory chapter. The equivalence of both definitions is given in the following result.

Proposition 2.7 *If $X \in \mathcal{X}(M)$, then*

$$L_X dv = \Big(divX\Big)\, dv. \tag{2.1.17}$$

Proof. $T = dv = \sqrt{g} dx_1 \wedge \cdots \wedge dx_n$ is an $(n, 0)$- tensor field on M. The Lie derivative L_X of $T = T_{12\ldots n} dx_1 \wedge \cdots \wedge dx_n$ is also an $(n, 0)$- tensor or an n-form

$$L_X T = (L_X T)_{12\ldots n}\, dx_1 \wedge \ldots \wedge dx_n.$$

We shall show that

$$(L_X T)_{12\ldots n} = (divX)\sqrt{g}. \tag{2.1.18}$$

Indeed, using the formula which gives the components of the Lie derivative of a tensor, we have

$$\begin{aligned} (L_X T)_{12\ldots n} &= \frac{\partial T_{12\ldots n}}{\partial x_i} X^i \\ &\quad + T^{j_1 2\ldots n}\frac{\partial X^1}{\partial x_{j_1}} + T^{2 j_2\ldots n}\frac{\partial X^2}{\partial x_{j_2}} + \cdots + T^{12\ldots j_n}\frac{\partial X^n}{\partial x_{j_n}}. \end{aligned}$$

As $T_{1\ldots j_p\ldots n} = \delta_{p,j_p} T_{1\ldots p\ldots n}$, we get

$$\begin{aligned} (L_X T)_{12\ldots n} &= \frac{\partial T_{12\ldots n}}{\partial x_i} X^i + T_{12\ldots n}\Big(\frac{\partial X^1}{\partial x_1} + \cdots + \frac{\partial X^n}{\partial x_n}\Big) \\ &= \frac{\partial T_{12\ldots n}}{\partial x_i} X^i + T_{12\ldots n}\frac{\partial X^i}{\partial x_i} = \frac{\sqrt{g}}{\partial x_i} X^i + \sqrt{g}\frac{\partial X^i}{\partial x_i} \\ &= \frac{\partial}{\partial x_i}\Big(\sqrt{g}\, X^i\Big) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\Big(\sqrt{g} X^i\Big)\sqrt{g} = divX\,\sqrt{g}. \end{aligned}$$

Hence,

$$L_X T = divX\sqrt{g}\, dx_1 \wedge \cdots \wedge dx_n = divX\, dv.$$

■

Remark 2.8 *In the relation $L_X dv = div X \, dv$, the left side is a derivative of a square root of a determinant while the right side is the trace of a derivative (connection). In Linear Algebra this relation is known as*

$$\frac{d}{dt}\det A(t) = Trace\frac{d}{dt}A(t),$$

where $A(t)$ is a matrix, which depends on the parameter t.

Remark 2.9 *If X is a free-divergence vector field, then the volume element is preserved along the integral curves of X,*

$$dv_{|p} = \varphi_t^* \, dv_{|\varphi_t(p)}.$$

Then a free-divergence vector field provides a conservation law.

Lemma 2.10 *Let $f \in \mathcal{F}(M)$ and $X \in \mathcal{X}(M)$. Then*

$$div\,(fX) = f \, div\, X + g(\nabla f, X). \tag{2.1.19}$$

Proof. Using Lemma 2.6, we get

$$\begin{aligned} div\,(fX) &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(\sqrt{g}\, f\, X^j) = \frac{1}{\sqrt{g}}\frac{\partial f}{\partial x_j}\sqrt{g}X^j + f\,\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(\sqrt{g}\,X^j) \\ &= \frac{\partial f}{\partial x_j}X^j + f\, div\, X = g_{kj}(\nabla f)^k\, X^j + f\, div\, X \\ &= g(\nabla f, X) + f\, div\, X. \end{aligned}$$

■

Using Proposition 2.7 yields:

Corollary 2.11 *If $f \in \mathcal{F}(M)$ and $X \in \mathcal{X}(M)$, then*

$$L_{fX}\, dv = f L_X\, dv + X(f)\, dv. \tag{2.1.20}$$

Remark 2.12 *The Lie derivative is not $\mathcal{F}(M)$-linear, i.e., $L_{fX} \neq fL_X$ for any $f \in \mathcal{F}(M)$.*

Definition 2.13 *Let M be a Riemannian manifold and $f \in \mathcal{F}(M)$. Define the Laplacian of f as*

$$\Delta\, f = -div\,(\nabla f), \tag{2.1.21}$$

where ∇ stands here for the gradient.

Proposition 2.14 *For any $\phi,\ f,\ \rho \in \mathcal{F}(M)$, we have:*

$$div(\,f\nabla\phi) = -f\,\Delta\phi + g(\nabla f,\ \nabla\phi). \tag{2.1.22}$$

Proof. The equation (2.1.22) comes from (2.1.19) with the substitution $X = \nabla\phi$. ■

2.2 Applications

Harmonic functions on compact manifolds

The compact manifold M considered in this section will have an empty boundary $\partial M = \varnothing$.

Theorem 2.15. *(Hopf's lemma) Let M be a connected, compact Riemannian manifold and $f \in \mathcal{F}(M)$ such that*

$$\Delta f \geq 0.$$

Then f is constant.

Proof. First, we shall show that

$$\Delta f = 0 \qquad \text{on } M.$$

This is obtained by integrating and applying the divergence theorem

$$0 \leq \int_M \Delta f \, dv = -\int_M div(\nabla f) \, dv = 0,$$

where we used $\partial M = 0$. Substituting $f = \phi$ in (2.1.22), we get

$$div(f \nabla f) = -f \Delta f + g(\nabla f, \nabla f).$$

Integrating and using the divergence theorem again, the

$$0 = \int_M div(f \nabla f) = -\int_M f \Delta f + \int_M |\nabla f|^2 .$$

As the first term on the right-hand side is zero, it follows that

$$\int_M |\nabla f|^2 = 0,$$

which implies

$$|\nabla f| = 0 \qquad \text{on } M.$$

Hence, f is constant on M. ∎

2.2.0.1 Pluri-harmonic functions

Definition 2.16 *Let $k \in \mathbb{N}$. A function $f \in \mathcal{F}(M)$ is called k-pluri-harmonic if $\Delta^k f = 0$ on M, where $\Delta^k = \Delta(\Delta^{k-1})$ and $\Delta^0 = \Delta$.*

Proposition 2.17 *A k-pluri-harmonic function on a compact manifold is constant.*

Proof. There is a $k \in \mathbb{N}$ such that $\Delta^k f = 0$ on M. Then $\Delta(\Delta^{k-1} f) = 0$. Using Hopf's lemma, we get $\Delta^{k-1} f =$ constant. Now we have either $\Delta(\Delta^{k-2} f) \geq 0$ or $\Delta(\Delta^{k-2} f) \leq 0$. Using Hopf's lemma again we obtain

$$\Delta^{k-2} f = \text{constant}.$$

Inductively, after $k - 2$ steps, we end up with f constant. ∎

2.2.0.2 Uniqueness for solution of the Cauchy problem for the heat operator

If $\Delta : \mathcal{C}^2(M) \to \mathcal{C}^0(M)$ is the Laplace operator on the manifold M, then the heat operator $\mathcal{P} : \mathcal{C}^2(M) \times \mathcal{C}^1(\mathbb{R}_t) \to \mathcal{C}^0(M) \times \mathcal{C}^0(\mathbb{R}_t)$ is defined by $\mathcal{P} = \partial_t + \Delta$.

Theorem 2.18. *Let M be a Riemannian, compact manifold, $u \in \mathcal{C}^2(\mathbb{R}_+ \times M)$, $F \in \mathcal{C}^0(M) \times \mathcal{C}^0(\mathbb{R}_t)$, $\phi \in \mathcal{C}^2(M)$ and consider the Cauchy problem*

$$\partial_t u + \Delta u = F(x,t), \qquad (t,x) \in \mathbb{R}_+ \times M,$$

$$u_{|t=0} = \phi \qquad \text{on } M.$$

If u is a solution, then u is unique.

We first state an intermediate result.

Lemma 2.19 *Let w be a solution for $\partial_t w + \Delta w = 0$. Then the potential energy*

$$\int_M w^2(t,x)\,dv$$

is decreasing in time (dissipative process).

Proof. We have

$$w\ \partial_t w = -w\ \Delta w. \tag{2.2.23}$$

Using formula (2.1.22) with $w = f = \phi$, then (2.2.23) yields

$$\frac{1}{2}\,\partial_t w^2 = div\,(w\nabla w) - |\nabla w|^2\,.$$

Using the divergence theorem

$$\frac{1}{2}\,\partial_t \int_M w^2 = \underbrace{\int_M div(w\,\nabla w)}_{=0} - \int_M |\nabla w|^2 \le 0.$$

Hence, $\int_M w^2(t,x)\,dv$ is a decreasing function of t. ■

Proof. (of Theorem 2.18*)* Let u_1, u_2 be two solutions for Cauchy's problem. Denote $w = u_1 - u_2$. We shall prove that

$$\partial_t w = -\Delta w\ , \quad (t,x) \in \mathbb{R}_+ \times M,$$

$$w_{\,|t=0} = 0 \qquad\qquad \text{on } M$$

has the unique solution $w = 0$. Indeed, letting $P(t) = \int_M w^2(t,x)\,dv$ and using Lemma 2.19 we get

$$0 \le P(t) \le P(0) = 0, \qquad \forall\, t \ge 0.$$

Hence, $P(t) = 0$ and $w = 0$. ■

2.3 The Hessian and applications

If we let

$$f_j = \frac{\partial f}{\partial x_j} \quad , \quad f^i = g^{ij} f_j, \tag{2.3.24}$$

the gradient becomes

$$\nabla f = f^i \, \frac{\partial}{\partial x_i} \tag{2.3.25}$$

and then

$$-\Delta f = div\,(f^i \, \frac{\partial}{\partial x_i}) = f^i_{\;;\,i}. \tag{2.3.26}$$

Taking the covariant derivative with respect to $\partial/\partial x_i$ in

$$g^{ij} \; g_{jk} = \delta^i_k,$$

we obtain $g^{ij}_{\;;\,i} = 0$. Then formula (2.3.26) yields

$$-\Delta f = (g^{ij} \; f_j \,)_{\;;\,i} = g^{ij} \; f_{j\;;\,i}.$$

Using the formula for the covariant differentiation

$$f_{j\;;\,i} = \frac{\partial}{\partial x_i} f_j - \Gamma^k_{ji} f_k = \frac{\partial^2 f}{\partial x^j \, \partial x^i} - \Gamma^k_{ji} \; \frac{\partial f}{\partial x_k},$$

we obtain

$$-\Delta f = \; g^{ij} \left(\frac{\partial^2 f}{\partial x^j \, \partial x^i} - \Gamma^k_{ji} \; \frac{\partial f}{\partial x_k} \right). \tag{2.3.27}$$

Formula (2.3.27) can be written globally using the Hessian H^f for a function $f \in \mathcal{F}(M)$.

Definition 2.20 *The Hessian of the function* f *is a symmetric,* 2*-covariant tensor field on* M *given by*

$$H^f \, : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}(M),$$

$$H^f(X, Y) = H^f_{ij} \; X^i Y^j \tag{2.3.28}$$

with

(1.2.7)
$$H^f_{ij} \; = \frac{\partial^2 f}{\partial x^j \, \partial x^i} - \Gamma^k_{ji} \; \frac{\partial f}{\partial x_k}.$$

Formula (2.3.27) can be written using the Hessian H^f,

$$\Delta f = -Trace H^f \; = -g^{ij} H^f_{ij}. \tag{2.3.29}$$

Definition 2.21 *Define the second fundamental form of* $f \in \mathcal{F}(M)$ *as*

$$\nabla df(X, Y) = \nabla_X(df)(Y) = X\big(Y(f)\big) - \big(\nabla_X Y\big)(f) \tag{2.3.30}$$

where ∇ *stands for the Levi-Civita connection.*

As ∇ is a symmetric connection ,

$$\nabla df(X, Y) - \nabla df(Y, X) = [X, Y]\, f + (\nabla_Y X - \nabla_X Y)\, f = 0$$

so that ∇df is a symmetric 2-covariant tensor field. In fact, the second fundamental form is the Hessian.

Proposition 2.22 *The following relations take place:*

(i)
$$H^f = \nabla df,$$

(ii)
$$H^f(X, Y) = g\left(\nabla_X(grad\ f),\ Y\right).$$

Proof. (i) It suffices to check the relation only on the basis.

$$\nabla df\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i\, \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k} = H_{ij}^f = H^f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

(ii) Using that ∇ is a metric connection we obtain

$$\begin{aligned} g\left(\nabla_X(grad\ f),\ Y\right) &= X\ g\,(grad\ f,\ Y) - g\,(grad\ f,\ \nabla_X Y) \\ &= X\big(Y(f)\big) - (\nabla_X Y)\,(f) = H^f\,(X, Y). \end{aligned}$$

■

Thus, we can write

$$\Delta f = -Trace\ \nabla df. \tag{2.3.31}$$

Remark 2.23 *Formula (2.3.30) comes from the definition of the derivation. Indeed, if* $\omega \in T^*M$ *is a one-form, the derivation* $\nabla_X : T^*M \to T^*M$, *is defined as*

$$(\nabla_X \omega)\, Y = X\ \omega(Y) - \omega(\nabla_X Y), \qquad \forall X,\ Y \in \mathcal{X}(M). \tag{2.3.32}$$

In our case $\omega = df$ *and as* $df(Y) = Y(f)$, *we can derive (2.3.30) from (2.3.32).*

Another useful formula for the Laplacian can be obtained if in the formula

$$div\ X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j}(\sqrt{g} X^j)$$

we substitute $X = grad\ f$,

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j}(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x_i}). \tag{2.3.33}$$

As an application we have

Lemma 2.24 *For $f,\ \phi \in \mathcal{F}(M)$, we have*

$$\Delta\ (f\,\phi) = f\ \Delta\phi + \phi\ \Delta f - 2\,g\,(\nabla\phi, \nabla f). \tag{2.3.34}$$

Proof. Applying (2.3.33)

$$\begin{aligned}
\Delta\ (f\,\phi) &= -\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x_j}\big(\sqrt{g}\,g^{ij}\,\frac{\partial(f\phi)}{\partial x_i}\big) \\
&= -\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x_j}\,(\sqrt{g}\,g^{ij}\,f\,\frac{\partial\phi}{\partial x_j}) - \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x_j}\,(\sqrt{g}\,g^{ij}\,\phi\,\frac{\partial f}{\partial x_i}) \\
&= f\ \Delta\phi - g^{ij}\,\frac{\partial f}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \phi\ \Delta f - g^{ij}\frac{\partial\phi}{\partial x_j}\frac{\partial f}{\partial x_i} \\
&= f\ \Delta\phi + \phi\ \Delta f - 2\,g\,(\nabla f,\ \nabla\phi\,).
\end{aligned}$$

■

Making $f = \phi$ yields the following result.

Corollary 2.25 *Let $\phi \in \mathcal{F}(M)$. Then*

$$\Delta(\phi^2) = 2\,\phi\ \Delta\phi - 2\ |\nabla\phi|^2\,. \tag{2.3.35}$$

Proposition 2.26 *Let M be a connected, compact Riemannian manifold and let $\phi \in \mathcal{F}(M)$ such that*

$$\phi\ \Delta\phi = k\ \ |\nabla\phi|^2 \tag{2.3.36}$$

where k is a real constant. Then ϕ is a constant function.

Proof. Suppose first that $k = 1$. Then $\phi\Delta\phi = |\nabla\phi|^2$. Applying (2.3.35) we find $\Delta(\phi^2) = 0$. By Hopf's lemma we get ϕ^2 constant. Suppose now that $k \neq 1$. Substituting $f = \phi$, formula (2.1.22) yields

$$div\,(\phi\ \nabla\phi) = -\phi\ \Delta\phi + |\nabla\phi|^2\,.$$

Using (2.3.36) we conclude

$$div\,(\phi\ \nabla\phi) = (1-k)\ \ |\nabla\phi|^2\,.$$

For $k < 1$, by the divergence theorem we find

$$0 = \int_M div\,(\phi\ \nabla\phi) = (1-k)\ \int_M |\nabla\phi|^2 \geq 0,$$

which implies $|\nabla\phi| = 0$, i.e., ϕ constant. The case $k > 1$ is similar. ■

We can arrive at the same result using the following lemma:

Lemma 2.27 *For any $f \in \mathcal{F}(M)$ and $\alpha \in \mathbb{R}$ we have*

$$\Delta f^{\alpha} = -\alpha f^{\alpha-2}\Big(-f\ \Delta f + (\alpha-1)|\nabla f|^2\Big). \tag{2.3.37}$$

Proof.

$$\begin{aligned}
-\Delta f^{\alpha} &= div(\nabla(f^{\alpha})) = div(\alpha f^{\alpha-1}\,\nabla f)\\
&= -\alpha f^{\alpha-1}\,\Delta f + \alpha\langle \nabla f^{\alpha-1}, \nabla f\rangle\\
&= -\alpha f^{\alpha-1}\,\Delta f + \alpha\langle(\alpha-1) f^{\alpha-2}\,\nabla f, \nabla f\rangle\\
&= -\alpha f^{\alpha-1}\Delta f + \alpha(\alpha-1) f^{\alpha-2}\,|\nabla f|^2\\
&= \alpha f^{\alpha-2}\Big(-f\Delta f + (\alpha-1)|\nabla f|^2\Big).
\end{aligned}$$

■

Corollary 2.28 *Let $f \in \mathcal{F}(M)$ be a nonzero function and $\alpha \in \mathbb{R}$. Then f^{α} is harmonic if and only if*

$$f\Delta f = (\alpha-1)|\nabla f|^2. \tag{2.3.38}$$

Choosing $\alpha = k+1$, we obtain (2.3.36). Then f^{k+1} is harmonic on the compact M and then f is constant, by Hopf's lemma.

The p-Laplacian

The p-Laplacian of a function $f \in \mathcal{F}(M)$ is

$$\Delta_p = -div(|\nabla f|^{2(p-1)}\nabla f),$$

where $p \in \mathbb{N}$. The case $p = 1$ corresponds to the usual Laplacian.

Lemma 2.29 *If $\rho, \phi \in \mathcal{F}(M)$, then*

$$div\Big(\rho\nabla(\phi^2)\Big) = 2\phi\, div(\rho\nabla\phi) + 2\rho\,|\nabla\phi|^2. \tag{2.3.39}$$

Proof. Proposition 2.14 yields

$$\begin{aligned}
div\Big(\rho\,\nabla(\phi^2)\Big) &= -\rho\Delta\phi^2 + g(\nabla\rho, \nabla\phi^2)\\
&= -\rho\Big(2\phi\Delta\phi - 2g(\nabla\phi, \nabla\phi)\Big) + g(\nabla\rho, 2\phi\nabla\phi)\\
&= 2\phi\Big(-\rho\Delta\phi + g(\nabla\rho, \nabla\phi)\Big) + 2\rho\, g(\nabla\phi, \nabla\phi)\\
&= 2\phi\, div(\rho\nabla\rho) + 2\rho|\nabla\phi|^2.
\end{aligned}$$

■

Proposition 2.30 *If $\Delta_p\,\phi = 0$ on a compact, connected Riemannian manifold M, then f is constant.*

Proof. Choose $\rho = |\nabla\phi|^{2(p-1)}$ in Lemma 2.29 and integrate

$$0 = \int_M -div\Big(|\nabla\phi|^{2(p-1)}\Big)\,dv = 2\int_M \phi\,\Delta_p\phi\,dv + 2\int_M |\nabla\phi|^{2p}\,dv \le 0,$$

then $\nabla\phi = 0$ on M and hence $\phi = 0$. ■

2.3.0.3 An application to the heat equation with convection on compact manifolds

Let M be a connected, compact Riemannian manifold without boundary. We define the *heat equation with convection* as

$$\partial_t \phi + \Delta\phi = k \ |\nabla\phi|^2$$

where $k \geq 0$ is a real positive constant. The function $\phi(x, t)$ denotes the temperature at the point x at time t. The goal of this section is to prove the following result.

Theorem 2.31. *Let M be a manifold as above and $k > 0$. If $\phi : [0, T) \times M \to \mathbb{R}$ is a smooth solution for*

$$\partial_t \phi + \Delta\phi = k \ |\nabla\phi|^2 ,$$
$$\phi_{|t=0} = 0,$$

then $\phi \equiv 0$,

We need the following result:

Lemma 2.32 *In the above hypothesis, if ϕ is a solution such that $\phi \leq \dfrac{1}{k}$, then*

$$\phi \equiv 0.$$

Proof. Multiplying by ϕ, we get

$$\phi \ \partial_t\phi + \phi \ \Delta\phi = k \ \phi \ |\nabla\phi|^2 . \tag{2.3.40}$$

Using the fact that $\phi \ \Delta\phi = |\nabla\phi|^2 - div(\phi \ \nabla\phi)$, the relation (2.3.40) becomes

$$\frac{1}{2}\partial_t\phi^2 + |\nabla\phi|^2 - div(\phi\nabla\phi) = k\phi \, |\nabla\phi|^2 .$$

Integrating

$$\frac{1}{2} \ \partial_t \int_M \phi^2 - \int_M div(\phi\nabla\phi) = \int_M (k\phi - 1) \, |\nabla\phi|^2 \leq 0.$$

As the second term on the left-hand side vanishes, it follows that

$$P(t) = \int_M \phi^2(t, x) \, dv$$

is decreasing in t. As $0 \leq F(t) \leq F(0) = 0$, we get $\phi \equiv 0$. ∎

Proof. (of Theorem 2.31*).*
As $\phi_{|t=0} = 0$ and M is compact, there is $\epsilon > 0$ such that

$$\phi(t, x) \leq \frac{1}{k}, \qquad \forall t < \epsilon, \ \forall x \in M.$$

Using Lemma 2.32, we obtain

$$\phi(t, x) = 0, \qquad (t, x) \in [0, \epsilon) \times M.$$

Let ϵ^* be the maximal ϵ with the above property,

$$\epsilon^* = \sup\{\epsilon \ ; \ \phi(t, x) = 0, \forall (t, x) \in [0, \epsilon) \times M\}.$$

If $\epsilon^* = T$, the proof is finished.

Suppose $\epsilon^* < T$. By continuity, $\phi_{|t=\epsilon^*} = 0$. Applying the above argument, we can find $\epsilon' > 0$ such that $\phi(t, x) = 0$, $\forall x \in M$ and $\forall t \in [0, \epsilon^* + \epsilon')$ which contradicts the definition of ϵ^*. ■

2.4 Exercises

1. Let M be a Riemannian manifold and $p \in M$ be a point. Consider an orthonormal basis $\{E_1, \ldots, E_n\}$ in T_pM. Let γ_i be the geodesic that verifies $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = E_i$ and is parametrized by the arc length.

a) Show that for any function $f \in \mathcal{F}(M)$ we have

$$(\Delta f)_p = -\sum_{i=1}^{n} \frac{d^2(f \circ \gamma_i)}{ds^2}(0).$$

b) Show that in the case when M is the Euclidean space we obtain the usual Laplacian.

2. A nonconstant harmonic function defined on an open set of a Riemannian manifold does not have interior maximum points.

3. The motion of an ideal fluid is described by the continuity equation

$$\frac{\partial \rho}{\partial t} + div(\rho V) = f,$$

where $V(x, t)$ is the velocity vector field, $\rho(x, t)$ is the density function, and $f(x, t)$ is the source intensity function. Solve the continuity equation in the case of a homogeneous density function $\rho = \rho(t)$ with the initial condition $\rho(0) = \rho_0$.

4. Let Δ be the Laplace operator on $\mathbb{R}^2$ and let ϕ be a solution of

$$\Delta\phi + f(\phi^2)\phi = 0, \tag{2.4.41}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

a) Show that for any $v \in \mathbb{R}^2$, the function $\psi_v(x) = \phi(x + v)$ is a solution of (2.4.41).

b) Show that for any $s \in \mathbb{R}$, the function $\rho_s(x) = \phi(R_s(x))$ is a solution of (2.4.41), where

$$R_s = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$

is the rotation of angle s.

5. Let $\Omega = (0, 1) \times (0, 1)$ and $\varphi : \Omega \to \mathbb{R}$, given by

$$\varphi(x_1, x_2) = \begin{cases} 1 + x_1^2 & \text{for } x_2 = 1, \\ 0 & \text{for } x_2 = 0, \\ 0 & \text{for } x_1 = 0, \\ 0 & \text{for } x_1 = 1. \end{cases}$$

Show that the boundary value problem

$$\partial_t u - \partial_x^2 u = -1,$$
$$u_{|\partial\Omega} = \varphi$$

does not have solutions in the space

$$\{u : \overline{\Omega} \to \mathbb{R};\ u \in \mathcal{C}(\overline{\Omega}),\ \partial_t u, \partial_x^2 u \in \mathcal{C}\big((0, 1) \times (0, 1]\big)\}.$$

6. Consider the n-dimensional unit sphere endowed with the Riemannian metric induced by the inclusion $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$. Show that for any function $f \in \mathcal{F}(\mathbb{R}^{n+1})$ we have

$$\left(\Delta^{R^{n+1}} f\right)_{|\mathbb{S}^n} = \Delta^{\mathbb{S}^n}(f_{|\mathbb{S}^n}) - \frac{\partial^2 f}{\partial r^2}_{|\mathbb{S}^n} - n\frac{\partial f}{\partial r}_{|\mathbb{S}^n},$$

where $\Delta^{R^{n+1}}$, $\Delta^{\mathbb{S}^n}$ and $\dfrac{\partial}{\partial r}$ are the Laplace operators on $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$, and the radial derivative, respectively.

7. Let $\mathbb{S}^n$ be the unit sphere endowed with the usual Riemannian structure from $\mathbb{R}^{n+1}$. Denote by $\mathcal{H}_k$ the vector space of the harmonic polynomials of degree $k \geq 0$ defined on $\mathbb{R}^{n+1}$. Let $\widetilde{H}_k = \{f_{|\mathbb{S}^n};\ f \in \mathcal{H}_k\}$.

a) Show that

$$\Delta^{\mathbb{S}^n} f = k(n+k-1)f, \qquad \text{for all } \; f \in \widetilde{H}_k,$$

and hence $k(n+k-1)$ is an eigenvalue of the Laplaceian $\Delta^{\mathbb{S}^n}$.

b) $\widetilde{H}_k$ is the eigenspace corresponding to the eigenvalue $\lambda_k = k(n+k-1)$.

c) The set $\{k(n+k-1); k \in \mathbb{N}\}$ is the set of eigenvalues (the spectrum) of $\Delta^{\mathbb{S}^n}$.

3

Lagrangian Formalism on Riemannian Manifolds

3.1 A simple example

It is natural to study a Physics problem using the following steps:

• First, find a suitable Lagrangian, which in the simplest case is the difference between the kinetic and the potential energy involved in the phenomenon.
• Write down the Euler–Lagrange equations, the Hamilton equations, and the Hamilton–Jacobi equation.
• Choose one of the above equations which can be studied from the point of view of existence, uniqueness, and regularity of solutions. Since the equation comes from a real physical problem, all of these conditions should be satisfied. This is a step which sometimes is skipped by physicists but is challenging for the mathematicians.
• If for the above equations an exact solution cannot be found, try numerical methods.

To demonstrate this, we shall consider a simple example from Classical Mechanics. Suppose that a body is launched obliquely in space. Neglecting the friction forces, the Lagrangian is the difference between kinetic and potential energy

$$L = \frac{m\,v^2}{2} - mgy,$$

where v is the speed, given by $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, m is the body mass, which can be assumed to equal 1, and g is the gravitational acceleration.

Euler–Lagrange equations for the Lagrangian $L = L(x, y, \dot{x}, \dot{y})$ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}.$$

For the above Lagrangian, we have

$$\ddot{x} = 0, \qquad \ddot{y} = -g.$$

This is a uniform motion along the x-axis

$$x = v_x t + x_0,$$

and an accelerated motion along the y-axis

$$y = -\frac{1}{2}gt^2 + v_0 t + x_0.$$

The first Euler–Lagrange equation is the Laplace equation and the latter is the Poisson equation, both in dimension 1.

It is not always easy to solve the Euler–Lagrange equations. The next section provides a more complicated example.

3.2 The pendulum equation

In this section we shall discuss the case of a simple pendulum. This is a dynamical system which can be described by the parameter θ, which is the angle between the string and the vertical direction. Denote by m the mass of the pendulum weight, by ℓ the length of the pendulum string, and by g the gravitational acceleration.

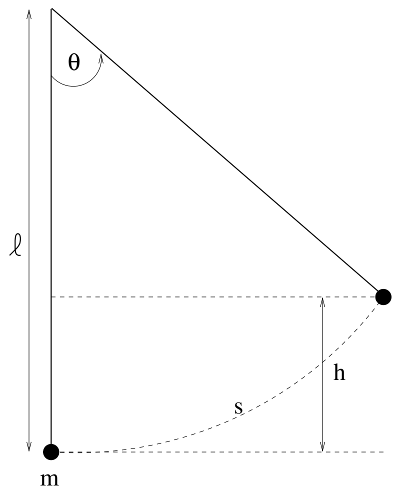


Figure 3.1: The pendulum.

The Lagrangian is given by the difference between the kinetic energy and the potential energy

$$L = K - U.$$

The kinetic energy is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m\ell^2\left(\frac{d\theta}{dt}\right)^2,$$

where $s = \ell\theta$ is the arc length, v is the tangential speed, and t is the time parameter.

The potential energy is

$$U = mgh = mg\ell(1 - \cos\theta),$$

where h is the height. The Lagrangian becomes

$$L(\theta, \dot{\theta}) = m\ell\Big(\frac{1}{2}\ell\dot{\theta}^2 + g\cos\theta\Big) - mg\ell.$$

Using that

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\theta}}\Big) = m\ell^2\ddot{\theta}, \qquad \frac{\partial L}{\partial\theta} = -m\ell g\sin\theta,$$

the Euler–Lagrange equation is

$$\ddot{\theta} = -\kappa\sin\theta, \tag{3.2.1}$$

where $\kappa = g/\ell > 0$ is a constant. Equation (3.2.1) is called the *pendulum equation*.

We shall show that the total energy $E = K + U$ of the pendulum is conserved.

$$\begin{aligned} E = K + U &= \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta) \\ &= m\ell\Big(\frac{1}{2}\ell\dot{\theta}^2 - g\cos\theta\Big) + m\ell g. \end{aligned} \tag{3.2.2}$$

Differentiating with respect to time yields

$$\frac{dE}{dt} = m\ell^2\dot{\theta}(\ddot{\theta} + \frac{g}{\ell}\sin\theta) = 0,$$

where we used the pendulum equation (3.2.1).

In the following we shall integrate the pendulum equation (3.2.1) subject to the initial conditions

$$\theta(0) = \frac{\pi}{2}, \qquad \dot{\theta}(0) = 0, \tag{3.2.3}$$

which corresponds to a free falling of the pendulum from a direction parallel to the horizontal axis. The equation (3.2.2) can be written as

$$\frac{E - m\ell g}{mL} = \frac{1}{2}\ell\dot{\theta}^2 - g\cos\theta.$$

Separating $\dot{\theta}$, we get

$$\dot{\theta}^2 = 2\kappa\cos\theta + C, \tag{3.2.4}$$

where

$$C = \frac{2}{m\ell^2}(E - m\ell g).$$

From (3.2.3)

$$C = \dot{\theta}(0)^2 - 2\kappa \cos \frac{\pi}{2} = 0.$$

Hence the equation (3.2.4) yields

$$\frac{d\theta}{dt} = -\sqrt{2\kappa \cos \theta},$$

where the negative sign means that the angle $\theta = \theta(t)$ decreases from $\pi/2$ to 0. Separating and integrating between $\theta_0 = \pi/2$ and $\theta(t)$ yields

$$\int_{\pi/2}^{\theta(t)} \frac{d\theta}{\sqrt{\cos \theta}} = -\sqrt{2\kappa}\, t. \tag{3.2.5}$$

With the substitution $\theta = \arccos u$ on the left-hand side, (3.2.5) becomes

$$\int_0^{\cos \theta(t)} \frac{du}{\sqrt{u(1-u^2)}} = \sqrt{2\kappa}\, t. \tag{3.2.6}$$

We need the following:

Lemma 3.1

(i)
$$\int_1^z \frac{du}{\sqrt{u(1-u^2)}} = 2 \int_{\sqrt{2}}^{\sqrt{z+1}} \frac{du}{\sqrt{(u^2-1)(2-u^2)}},$$

(ii)
$$\int_1^z \frac{du}{\sqrt{u(1-u^2)}} = -\sqrt{2}\, dn^{-1}\Big(\sqrt{\frac{z+1}{2}}, \frac{1}{\sqrt{2}}\Big),$$

(iii)
$$\int_0^1 \frac{du}{\sqrt{u(1-u^2)}} = \sqrt{2} K(\frac{1}{\sqrt{2}}) \approx 2.62,$$

where K is a complete elliptic integral.

Proof. (i) Consider the functions

$$\phi = \int_1^z \frac{du}{\sqrt{u(1-u^2)}}, \qquad \psi = 2 \int_{\sqrt{2}}^{\sqrt{z+1}} \frac{du}{\sqrt{(u^2-1)(2-u^2)}}.$$

From the Fundamental Theorem of Calculus,

$$\phi'(z) = \psi'(z) = \frac{1}{\sqrt{u(1-u^2)}},$$

and hence

$$\phi(z) = \psi(z) + C_0.$$

As $\phi(1) = \psi(1) = 0$, it follows that $C_0 = 0$. Hence, $\phi(z) = \psi(z)$.

(ii) From Lawden [23], equation (3.2.11) we have

$$\int_x^a \frac{du}{\sqrt{(a^2-u^2)(u^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\Big(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\Big), \quad b \le x \le a.$$

Substitute $a = \sqrt{2}$, $b = 1$ and $x = \sqrt{z+1}$ and we get

$$\int_{\sqrt{z+1}}^{\sqrt{2}} \frac{du}{\sqrt{(2-u^2)(u^2-1)}} = \frac{1}{\sqrt{2}}\,\mathrm{dn}^{-1}\Big(\sqrt{\frac{z+1}{2}}, \frac{1}{\sqrt{2}}\Big).$$

Swapping the limits of integration and using (i), we arrive at formula (ii).
(iii) From Lawden [23], equation (3.8.1) we have

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Then

$$\begin{aligned}
K(1/\sqrt{2}) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{2-\sin^2\theta}} \\
&= \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\cos^2\theta}} \overset{t=\cos\theta}{=} \sqrt{2}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1+t^2)}} \\
&\overset{u=t^2}{=} \frac{1}{\sqrt{2}}\int_0^1 \frac{du}{\sqrt{u(1-u^2)}},
\end{aligned}$$

i.e. (iii). ■

Using Lemma 3.1 the equation (3.2.6) can be written as

$$\begin{aligned}
&\int_0^1 \frac{du}{\sqrt{u(1-u^2)}} + \int_1^{\cos\theta} \frac{du}{\sqrt{u(1-u^2)}} = \sqrt{2\kappa}\,t \\
\iff\ & K(1/\sqrt{2}) - \sqrt{\kappa}\,t = \mathrm{dn}^{-1}\Big(\sqrt{\frac{\cos\theta+1}{2}}, 1/\sqrt{2}\Big) \\
\iff\ & \mathrm{dn}(K(1/\sqrt{2}) - \sqrt{\kappa}\,t) = \cos\frac{\theta}{2} \\
\iff\ & \theta(t) = 2\arccos\big(\mathrm{dn}(K(1/\sqrt{2}) - \sqrt{\kappa}\,t)\big). \qquad (3.2.7)
\end{aligned}$$

From Lawden [23], equation (2.2.19) we have

$$\mathrm{dn}(u+K) = k'\,\mathrm{nd}\,u = k'/\mathrm{dn}\,u.$$

As dn is an even function, equation (3.2.7) yields

$$\theta(t) = 2 \arccos \frac{1}{\sqrt{2}\,\mathrm{dn}(\sqrt{\kappa}\, t)}. \tag{3.2.8}$$

The dynamical system discussed above is one dimensional. However, it was not easy to integrate the Euler–Lagrange equation, even in the particular case $C = 0$. The solution required the use of elliptic functions. In other cases, even elliptic functions are not enough to solve the Euler–Lagrange equation. We may say that for some equations, it is not possible to obtain explicit formulas. This is also the case for an Euler–Lagrange equation on manifolds, where we encounter more than one parameter. In this case, the best we can do is to perform a qualitative analysis of the solutions. This will consist of finding first integrals of motion, currents, and free divergence tensors. An important part of the next chapters will be dedicated to conservation laws on Riemannian manifolds.

Using Lagrangians on Riemannian manifolds, we shall be able to get the above equations in a more general case. Some solutions of these two equations are already known. For instance, on compact manifolds the Laplace equation has only constant solutions.

3.3 Euler–Lagrange equations on Riemannian manifolds

Unlike in Quantum Mechanics, where there exists the Heisenberg principle of uncertainty, in Classical Mechanics the moving particle is completely described by its position $\mathbf{x}$ and its speed $\mathbf{v}$. The position $\mathbf{x}$ belongs to a space called *the coordinate space* which is, in general, a Riemannian manifold with the metric defined by the kinetic energy. The space of the positions and velocities $(\mathbf{x}, \mathbf{v})$ is called *phase space*, and it is identified with the tangent bundle TM of the coordinate space M. The pair $(\mathbf{x}, \mathbf{v})$ is called the *state* of the particle.

For instance, in the previous example of a body launched in space, we have $\mathbf{x} = (x, y)$ and $(\mathbf{x}, \mathbf{v}) = (x, y, \dot{x}, \dot{y}) \in TM \simeq \mathbb{R}^4$.

The coordinates and velocities depend on the time t. The trajectory in the coordinate space is a curve parameterized by t, which is a solution of the Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{x}} = \frac{\partial L}{\partial \dot{\mathbf{x}}}.$$

This holds for particles that depend on only one parameter, time. But there are a lot of phenomena that depend on several parameters. Furthermore, these new parameters can change in time and can be related to each other, so that we can speak about a parameter space. This is a manifold endowed with a Lorentzian metric $(+, ..., +, -)$, where $(-)$ corresponds to the time coordinate. This is also the basic idea of sigma-models or chiral fields introduced first by M. Gell-Mann and M. Levi in 1960 for describing pion-nucleon physics in a low energy approximation, see [30]. We shall discuss this idea later in the context of harmonic map theory, see chapter 4.

Let (M, g) be a Riemannian manifold and $\phi \in \mathcal{F}(M)$. Denote by $\phi_{;\,j}$ the derivative of ϕ in $\frac{\partial}{\partial x_j}$ direction, where $\{\frac{\partial}{\partial x_1}_{|p}, ..., \frac{\partial}{\partial x_m}_{|p}\}$ is a basis of T_pM,

$$\phi_{;\,j} = \frac{\partial \phi}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \phi, \tag{3.3.9}$$

where ∇ is the Levi-Civita connection on M.

Consider a map $\Psi : M \to N$, where M and N are Riemannian manifolds. The first is the space of parameters and the second the space of coordinates. If $(x_1, ..., x_m)$ are local coordinates around $p \in M$, and $(y_1, ..., y_n)$ are coordinates around $\Psi(p) \in N$, we define the vector field $\Psi_{;\,i} \in \mathcal{X}\big(\Psi(M)\big)$ by

$$\Psi_{;\,i} = \Psi_*\big(\frac{\partial}{\partial x_i}\big) = \Psi^j_{\;;\,i}\,\frac{\partial}{\partial y_j}. \tag{3.3.10}$$

In the particular case when $M = \mathbb{R}_t$, we obtain the tangent vector field along Ψ,

$$\dot{\Psi}(t) = \Psi_*\big(\frac{d}{dt}\big). \tag{3.3.11}$$

Definition 3.2 *A Lagrangian is a function $L : TN \to \mathbb{R}$, where N is the coordinate space. The Lagrangian L associated with $\Psi : M \to N$ is a scalar function of Ψ and $\Psi_{;\,i}$. The expression of the Lagrangian may contain the metrics g_{ij} and h_{ij} of M and N, respectively.*

Definition 3.3 *Let $\mathcal{D} \subset M$ be a bounded, closed set. A variation of Ψ in $\mathcal{D}$ is a one-parameter family of functions $\Psi(s, x)$, where $s \in (-\epsilon, \epsilon)$ and $x \in M$ such that*

(i) $$\Psi(0, x) = \Psi(x);$$

(ii) $$\Psi(s, x) = \Psi(x), \quad \forall x \in M \backslash \mathcal{D}.$$

Denote

(2.2.4) $$\delta\Psi^i_{(x)} = \frac{\partial \Psi^i(s, x)}{\partial s}_{|s=0}, \quad i = \overline{1, n}.$$

Definition 3.4 *The integral*

$$I = \int_{\mathcal{D}} L\, dv_g \tag{3.3.12}$$

is called stationary under the above variation if

$$\frac{dI}{ds}_{|s=0} = 0. \tag{3.3.13}$$

We denote the volume element

$$dv_g = \sqrt{|g|}\, dx^1 ... dx^m, \tag{3.3.14}$$

where $|g| = \left| det\, g_{ij} \right|$.

Theorem 3.5. *The integral (3.3.12) is stationary under any variation of* Ψ *iff the following Euler–Lagrange equations are satisfied*

$$\sum_{k=1}^{m} \left(\frac{\partial L}{\partial (\Psi^i_{\ ;\, k})} \right)_{;\, k} = \frac{\partial L}{\partial \Psi^i}, \qquad \forall i = \overline{1, n}. \tag{3.3.15}$$

Proof. Applying the chain rule

$$\frac{dI}{du}_{\,|u=0} = \sum_i \int_{\mathcal{D}} \left[\frac{\partial L}{\partial \Psi^i}\, \delta \Psi^i + \frac{\partial L}{\partial (\Psi^i_{\ ;\, e})}\, \delta (\Psi^i_{\ ;\, e}) \right] dv_g.$$

As $\delta\, (\Psi^i)_{;\, e} = (\delta \Psi^i)_{;\, e}$, the second term in the right hand side can be expressed as

$$\sum_i \int_{\mathcal{D}} \left(\left[\frac{\partial L}{\partial (\Psi^i_{\ ;\, e})}\, \delta \Psi^i \right]_{;\, e} - \left(\frac{\partial L}{\partial (\Psi^i_{\ ;\, e})} \right)_{;\, e} \delta \Psi^i \right) dv_g.$$

Let

$$X = X^e \frac{\partial}{\partial x_e},$$

where

$$X^e = \sum_i \frac{\partial L}{\partial (\Psi^i_{\ ;\, e})} \delta \Psi^i,$$

and by the divergence theorem

$$\int_{\mathcal{D}} X^e_{\ ;\, e}\, dv = 0,$$

as X vanishes on $\partial \mathcal{D}$.

Thus,

$$\frac{dI}{ds}_{\,|s=0} = \int_{\mathcal{D}} \left[\frac{\partial L}{\partial \Psi^i} - \left(\frac{\partial L}{\partial \Psi^i_{;e}} \right)_{;\, e} \right] \delta \Psi^i\, dv = 0,$$

for all variations of Ψ, which means that (3.3.15) is satisfied. Indeed, if we take the variation $\Psi(s, x) = exp(s\, V_{\Psi(x)})$, where $V_{\Psi(x)} \in T_{\Psi(x)} N$, we have $\Psi(0, x) = \Psi(x)$ and

$$\delta \Psi = \frac{\partial \Psi(s, x)}{\partial s}_{\,|\, s=0} = V_{\Psi(x)},$$

for any arbitrary V. ∎

3.4 Laplace's Equation $\Delta f = 0$

The Laplace equation describes stationary processes in physics such as the displacement of a membrane or soap film with a prescribed contour, the gravitational potential in the absence of mass, the steady-state flow of heat in the absence of sources of heat, the velocity potential for some fluids, the electrostatic potential in the absence of charge, and many other static processes.

Let (M, g) be a compact Riemannian manifold and $f \in \mathcal{F}(M)$. Define the *kinetic energy* of f as

$$E(f) = \int_M \frac{1}{2} |\nabla f|^2 \, dv, \tag{3.4.16}$$

where $|\nabla f|^2 = g(\nabla f, \nabla f)$, and $\nabla f = grad\ f$. As M is compact, $0 < E(M) < \infty$. The Lagrangian is

$$L = \frac{1}{2} |\nabla f|^2 . \tag{3.4.17}$$

Theorem 3.6. *The Euler–Lagrange equation for the Lagrangian (3.4.17) is*

$$\Delta f = 0. \tag{3.4.18}$$

Proof. In local coordinates,

(2.3.4) $$L = \frac{1}{2} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{1}{2} g^{ij} f_{;i} f_{;j}.$$

As L does not depend on f, the right side of (3.3.15) is zero. For the expression on the left side, we have

(2.3.5) $$\frac{\partial L}{\partial f_{;k}} = g^{kj} f_{;j} = (\nabla f)^k.$$

Hence, (3.3.15) becomes $(\nabla f)^k_{;k} = 0$ or $div\ (\nabla f) = 0$, *i.e.*, (3.4.18). ■

In the case when M has a nonzero boundary, Hopf's lemma becomes the uniqueness theorem for the Dirichlet problem.

Theorem 3.7. *Let M be a connected, compact manifold and $f \in \mathcal{F}(M)$ such that*

$$\Delta f = 0, \qquad on\ M,$$
$$f_{|\partial M} = 0.$$

Then $f \equiv 0$.

Proof. Integrate the expression

$$div\ (f\ \nabla f) = -f\ \Delta f + |\nabla f|^2$$

and use the divergence theorem

$$\int_M div\ X\ dv = \int_{\partial M} (X, N)\, d\sigma,$$

with $X = f \nabla f$. ■

3.5 A geometrical interpretation for a Δ operator

Let M be a manifold of dimension m and $f : M \to \mathbb{R}^n$ an immersion, *i.e.*, df is one-to-one. Consider M as a Riemannian manifold with the induced metric by the immersion f,

$$g_{ij} = f^*(\delta_{ij}),$$

where δ_{ij} is the canonical metric on $\mathbb{R}^n$. Such an immersion is called *isometric*. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\mathbb{R}^n$,

$$\tilde{\nabla}_X Y = \sum_{i=1}^{n} X(Y^i)\, e_i, \tag{3.5.19}$$

where $Y = Y^i e_i, \quad X = X^i e_i, \quad \text{and} \quad e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$.

If ∇ is the Levi-Civita connection on M, the second fundamental form of the immersion f is the two-covariant, symmetric tensor field on M

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.5.20}$$

The equation (3.5.20) is called Gauss's formula, and we have

$$h(X, Y) = nor\ (\tilde{\nabla}_X Y),$$
$$\nabla_X Y = tan\ (\tilde{\nabla}_X Y),$$

where *nor (tan)* represents the normal (tangential) component with respect to M.

Definition 3.8 *The mean curvature vector field of the submanifold M of $\mathbb{R}^n$ is*

$$H = \frac{1}{m}\, Trace_g\ h. \tag{3.5.21}$$

Thus, H_x is always normal to $T_x M$.

In the particular case when M is a hypersurface ($n = m + 1$), the vector fields H and N (the unit normal field) are proportional,

$$H = \alpha\, N. \tag{3.5.22}$$

The function $\alpha \in \mathcal{F}(M)$ is called the *scalar mean curvature*.

The geometry contained in the Δ operator is illustrated in the following result.

Lemma 3.9 *Let $f : M \to \mathbb{R}^n$ be an isometric immersion. Then*

$$\Delta f = -m\, H. \tag{3.5.23}$$

Proof. As

$$(\nabla df)(X, Y) = h(X, Y),$$

we obtain

$$\Delta f = -Trace_g(\nabla df) = -m\, H.$$

∎

Corollary 3.10 *Under the above hypothesis, Δf is a vector field normal to M.*

Corollary 3.11 *Under the above hypothesis, M is a minimal submanifold (i.e., $H = 0$) iff f is harmonic.*

Corollary 3.12 *There are no compact minimal submanifolds in $\mathbb{R}^n$.*

Proof. If M is a minimal submanifold, there is an isometric immersion $f : M \to \mathbb{R}^n$ such that $\Delta f^i = 0$, for $i = \overline{1, n}$. Applying Hopf's lemma, we find that $f(M)$ is reduced to a point. This is a contradiction. ■

3.6 Poisson's equation

There are many situations when physical problems are described by a Poisson equation. A few examples are: the equilibrium displacement of a membrane under exterior forces, the gravitational potential in the presence of mass, the electrostatic potential in the presence of distributed charge, the steady-state temperature in the presence of sinks or sources of heat, and the velocity potential for an incompressible, irrotational, homogeneous fluid in the presence of distributed sources or sinks.

Let $f, \rho \in \mathcal{F}(M)$, where (M, g) is a Riemannian manifold, and consider the Lagrangian

$$L = \frac{1}{2}\ |\nabla f|^2 - \rho f. \tag{3.6.24}$$

The Euler–Lagrange equation is obtained from relation (3.3.15) with the right-hand side $\dfrac{\partial L}{\partial f} = -\rho$. Then equation (3.3.15) becomes Poisson's equation

$$\Delta f = \rho. \tag{3.6.25}$$

Proposition 3.13 *Let $k \in \mathbb{R}$. The equation on the sphere $\mathbb{S}^n$,*

$$\Delta f = k$$

has solutions $f \in \mathcal{F}(S^n)$ iff $k = 0$. In this case, solutions are constants.

Proof. Apply Hopf's lemma. ■

One of the physical applications of equation (3.6.25) is in gravitation. The function ρ denotes matter density and f denotes gravitational potential. Since the gravitational force is defined as $F = -\nabla f$, the equation (3.6.25) can be written

$$div\ F = \rho. \tag{3.6.26}$$

In an empty space, $\rho = 0$ and F is a divergence-free vector field, which means that the volume element is preserved along the integral curves of F.

3.7 Geodesics

Let $I \subseteq \mathbb{R}$ be an interval and (M, g) be a Riemannian manifold. Consider the curve $\phi : I \to (M, g)$ and take the Lagrangian

$$L(\phi, \dot{\phi}) = \frac{1}{2}\, |\dot{\phi}|_g^2 = \frac{1}{2}\, g_{ij\,|\phi}\, \dot{\phi}^i \dot{\phi}^j \tag{3.7.27}$$

as the kinetic energy along the curve $\phi(t)$. Denote the tangent field along the curve $\phi(t)$ by

$$\dot{\phi} = \phi_*(\frac{d}{dt}). \tag{3.7.28}$$

Theorem 3.14. *The extremizers of the integral*

$$J(\phi) = \int_I \frac{1}{2}\, \left|\dot{\phi}\right|_g^2\, dt \tag{3.7.29}$$

are solutions for the equation

$$\ddot{\phi}^l + \Gamma^l_{is\,|\phi}\, \dot{\phi}^i \dot{\phi}^s = 0, \qquad l = \overline{1, n}. \tag{3.7.30}$$

Proof. We shall show that the above equation is the Euler–Lagrange equation for the Lagrangian (3.7.29). Indeed, computing both sides of the equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\phi}^k}\Big) = \frac{\partial L}{\partial \phi^k}, \tag{3.7.31}$$

we conclude

$$\begin{aligned} \frac{\partial L}{\partial \phi^k} &= \frac{1}{2}\, \frac{\partial g_{ij}}{\partial x_k}{}_{|\phi(t)}\, \dot{\phi}^i(t)\dot{\phi}^j(t) \\ \Longleftrightarrow \frac{\partial L}{\partial \dot{\phi}^k} &= g_{ik\,|\phi(t)}\, \dot{\phi}(t). \end{aligned}$$

So that

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\phi}^k}\Big) &= \frac{d}{dt}\, \Big(g_{ik\,|\phi(t)}\, \dot{\phi}(t)\Big) \\ &= \frac{\partial g_{ik}}{\partial x_s}\, \dot{\phi}^s(t)\dot{\phi}^i(t) + g_{ik\,\phi(t)}\, \ddot{\phi}^i(t). \end{aligned}$$

Equation (3.7.31) becomes

$$\begin{aligned} \ddot{\phi}^i\, g_{ik} + \frac{\partial g_{ik}}{\partial x_s}\, \dot{\phi}^s \dot{\phi}^i &= \frac{1}{2}\, \frac{\partial g_{ij}}{\partial x_k}\, \dot{\phi}^i \dot{\phi}^j \\ \Longleftrightarrow \ddot{\phi}^i\, g_{ik} + \frac{1}{2}\Big[\frac{\partial g_{ik}}{\partial x_s}\, \dot{\phi}^s \dot{\phi}^i + \frac{\partial g_{ki}}{\partial x_s}\, \dot{\phi}^i \dot{\phi}^s - \frac{\partial g_{ij}}{\partial x_k}\, \dot{\phi}^i \dot{\phi}^j\Big] &= 0 \\ \Longleftrightarrow \ddot{\phi}^i\, g_{ik} + \frac{1}{2}\Big[\frac{\partial g_{ik}}{\partial x_s} + \frac{\partial g_{ks}}{\partial x_i} - \frac{\partial g_{is}}{\partial x_k}\Big]\dot{\phi}^i \dot{\phi}^s &= 0. \end{aligned} \tag{3.7.32}$$

Multiply by g^{kl} and sum over k to yield

$$\ddot{\phi}^l + \frac{1}{2} g^{kl} \Big(\frac{\partial g_{ik}}{\partial x_s} + \frac{\partial g_{ik}}{\partial x_i} - \frac{\partial g_{is}}{\partial x_k} \Big) \dot{\phi}^i \dot{\phi}^s = 0$$
$$\Longleftrightarrow \ddot{\phi}^l + \Gamma^l_{is\,|\,\phi(t)} \dot{\phi}^i \dot{\phi}^s = 0.$$

■

The equation (3.7.30) is written in local coordinates. A global expression for this equation is given in the following result.

Proposition 3.15 *Let $\dot{\phi}(t)$ be given by (3.7.28). Then the following relation takes place:*

$$\nabla_{\dot{\phi}} \dot{\phi} = (\ddot{\phi}^s + \Gamma^s_{ij} \dot{\phi}^i \dot{\phi}^j) \partial_s. \tag{3.7.33}$$

Proof. Using the properties of the linear connection, we write

$$\nabla_{\dot{\phi}} \dot{\phi} = \nabla_{\dot{\phi}^k \partial_k} \dot{\phi}^j \partial_j = \dot{\phi}^k \nabla_{\partial_k} (\dot{\phi}^j \partial_j)$$
$$= \dot{\phi}^k (\dot{\phi}^j_{\ ;\,k} \partial_j + \dot{\phi}^j \Gamma^s_{kj} \partial_s).$$

Using

$$\ddot{\phi}^j = (\partial_k \dot{\phi}^j) \dot{\phi}^k,$$

we obtain equation (3.7.33). ■

The expression $\nabla_{\dot{\phi}} \dot{\phi}$ is interpreted as *acceleration* along the curve $\phi(t)$. Then the Euler–Lagrange equation for the Lagrangian $L = \frac{1}{2} |\dot{\phi}|^2$ is

$$\nabla_{\dot{\phi}} \dot{\phi} = 0 \qquad \text{(zero acceleration).} \tag{3.7.34}$$

The curves that satisfy (3.7.34) are called *geodesics* on the Riemannian manifold (M, g).

Remark 3.16 *The equation (3.7.34) is Newton's equation on the manifold (M, g) when the force is zero. Later, we shall consider the equation $\nabla_{\dot{\phi}} \dot{\phi} = F$, where F is the force vector field.*

3.8 The natural Lagrangian on manifolds

Let $\phi : I \subseteq \mathbb{R} \to (M, g)$ be a curve on a Riemannian manifold M. Define the natural Lagrangian associated with the curve ϕ and the potential $U : M \to \mathbb{R}$ as the difference between the kinetic energy K and the potential energy U. We consider a unit mass particle moving along the curve ϕ situated at the moment t at the point $\phi(t)$, with the speed $\dot{\phi}(t)$. Then,

$$L(\phi, \dot{\phi}) = \frac{1}{2} g(\dot{\phi}, \dot{\phi}) - U(\phi). \tag{3.8.35}$$

3.8.0.4 Momentum and Work

Define two one-forms $\omega_\phi,\ w_\phi \in T^*M$ associated with ϕ as

$$\omega_\phi\,(V) = g(\dot\phi, V) \qquad momentum \quad in \;\; the\ V - direction, \tag{3.8.36}$$

$$w_\phi\,(V) = g(\nabla_{\dot\phi}\dot\phi, V) \qquad work \quad in \;\; the\ V - direction, \tag{3.8.37}$$

where $V \in \mathcal{X}(M)$ and ∇ is the Levi-Civita connection. Using that ∇ is a metric connection

$$\dot\phi\ g(\dot\phi, V) = g(\nabla_{\dot\phi}\dot\phi, V) + g(\dot\phi, \nabla_{\dot\phi}V),$$

we obtain a formula which gives the work in terms of momentum

$$w_\phi\,(V) = \dot\phi\,\omega_\phi(V) - \omega_\phi\,(\nabla_{\dot\phi}V)\ , \qquad \forall\ V \in \mathcal{X}(M). \tag{3.8.38}$$

Proposition 3.17 *Let $\phi(t)$ be a geodesic. Then*

1) $w_\phi(V) = 0,\ \forall V \in \mathcal{X}(M)$ *(the work is zero);*

2) *The momentum $\omega_\phi(\dot\phi)$ in the $\dot\phi$-direction is preserved along the geodesic.*

Proof. 1) Use the equations (3.7.34) and (3.8.37).

2) Using 1), formula (3.8.38) becomes

$$\dot\phi\ \omega_\phi(V) = \omega_\phi(\nabla_{\dot\phi}V), \tag{3.8.39}$$

and taking $V = \dot\phi$ and using (3.7.34), we get

$$\dot\phi\ \omega_\phi(\dot\phi) = \omega_\phi(\nabla_{\dot\phi}\dot\phi) = 0.$$

Hence, $\omega_\phi(\dot\phi)$ is constant along the geodesic. ■

Remark 3.18 *i) A curve is a geodesic if and only if the work is zero.*
ii) As $\omega_\phi(V)$ is a function on M, we can write

$$\nabla_{\dot\phi}\,\omega_\phi(V) = \dot\phi\,\omega_\phi(V),$$

and then (3.8.38) becomes

$$\omega_\phi(V) = \nabla_{\dot\phi}\omega_\phi(V) - \omega_\phi(\nabla_{\dot\phi}V),$$

which shows that the work w_ϕ measures the non-commutativity between ω and $\nabla_{\dot\phi}$.

3.8.0.5 Force and Newton's Equation

Definition 3.19 *Consider the potential function $U \in \mathcal{F}(M)$. The vector field F defined as*

$$F = -\nabla U \tag{3.8.40}$$

is called the force vector field.

Theorem 3.20. *The curve ϕ is an extremizer for the integral*

$$\int_{t_1}^{t_2} L(\phi, \dot{\phi})\, dt, \tag{3.8.41}$$

with L given by (3.8.35)*, iff ϕ verifies Newton's equation*

$$\nabla_{\dot{\phi}}\dot{\phi} = -\nabla U. \tag{3.8.42}$$

Proof. As the Lagrangian is $L = K - U$, Euler–Lagrange equations are obtained by subtracting the equations

$$\frac{d}{dt}\Big(\frac{\partial K}{\partial \dot{\phi}^k}\Big) - \frac{\partial K}{\partial \phi^k} = 0 \quad \text{and} \tag{3.8.43}$$

$$\frac{d}{dt}\Big(\frac{\partial U}{\partial \dot{\phi}^k}\Big) - \frac{\partial U}{\partial \phi^k} = 0, \qquad \forall k = \overline{1, n} \tag{3.8.44}$$

where $K = \frac{1}{2}\, g(\dot{\phi}, \dot{\phi})$.

As we know from Theorem 3.14, equation (3.8.43) is given by (3.7.32), while (3.8.44) becomes

$$-\frac{\partial U}{\partial x_k} = 0.$$

Multiplying by g^{kl}, summing over k, and adding the last two equations, we find

$$\ddot{\phi}^l + \Gamma^l_{is\,|\phi(t)}\, \dot{\phi}^i \dot{\phi}^s = -g_{kl}\, \frac{\partial U}{\partial x_k}, \tag{3.8.45}$$

which is the Euler–Lagrange equation for L.

Using that

$$(\nabla_{\dot{\phi}}\dot{\phi}\,)^l = \ddot{\phi}^l + \Gamma^l_{is}\dot{\phi}^i\dot{\phi}^s,$$

and

$$(\nabla U)^l = g^{lk}\, \frac{\partial U}{\partial x_k},$$

we obtain

$$(\nabla_{\dot{\phi}}\dot{\phi}\,)^l = -(\nabla U)^l, \quad \forall l = \overline{1, n}$$

which is (3.8.42) on components. ∎

The above theorem enables us to write the work as

$$w_\phi(V) = g(-\nabla U, V) = g(F, V), \tag{3.8.46}$$

namely, the work is the *scalar product between the force and direction vector*. This is the definition for work known from Classical Mechanics.

Using the definition of the gradient,

$$w_\phi(V) = -dU(V), \quad \forall\, V \in \mathcal{X}(M).$$

Written as a one-form, the work is

$$w_\phi = -dU. \tag{3.8.47}$$

This can be taken as another definition for the work, involving the potential U, where ϕ is an extremizer.

Theorem 3.21. *(Momentum conservation theorem) Let ϕ be an extremizer for the integral* (3.8.41)*, and V be a Killing vector field on M such that*

$$w_\phi(V) = 0.$$

Then: 1) $\omega_\phi(V)$ *is constant along* ϕ*,*

2) $w_\phi(\nabla_{\dot\phi} V) = 0$.

Proof. 1) Let $(h^s)_s$ be the 1-parameter group of diffeomorphisms associated with the Killing vector field V. As $(h^s)_s$ are local isometries, each h^s will preserve the Lagrangian, *i.e.*,

$$L(\phi, \dot\phi) = L(h^s(\phi), h^s_*(\dot\phi)\,). \tag{3.8.48}$$

Indeed, as h^s_* is an isometry,

$$g(\dot\phi, \dot\phi) = g(h^s_*(\dot\phi), h^s_*(\dot\phi)\,),$$

so that the kinetic energy is preserved. As $w_\phi(V) = 0$, we get $dU(V) = 0$, *i.e.*, U is constant along the integral curves of V, and

$$U(x) = U(\,h^s(x)\,), \quad \forall s. \tag{3.8.49}$$

Hence, we get the equation (3.8.48). Applying Noether's Theorem (see chapter 5, Theorem 5.13), a first integral of motion is the momentum

$$\omega_\phi(V) = g(\dot\phi, V),$$

which will be constant along ϕ.

2) From 1), we have $\dot\phi\, \omega_\phi(V) = 0$ and using (3.8.38) we get the result. ∎

Exercise 3.22 *In local coordinates, $w_\phi = w_j\, dx^j$, where*

$$w_j = g_{ik}\,(\ddot\phi^k + \Gamma^k_{ab}\dot\phi^a\dot\phi^b\,). \tag{3.8.50}$$

Proposition 3.23 *Let ϕ be an extremizer for the integral* (3.8.41). *Then $|\dot{\phi}|$ is constant along ϕ iff U is constant along ϕ.*

Proof. It follows from

$$\nabla_{\dot{\phi}}\, g(\dot{\phi}, \dot{\phi}) = 2\, g(\nabla_{\dot{\phi}}\dot{\phi}, \dot{\phi}) = 2\, w_{\phi}(\dot{\phi}) = -2\, \dot{\phi}(U).$$

■

Corollary 3.24 *If U is constant on M, we get the well-known result that the vector tangent to a geodesic has a constant length.*

The Total Energy

Even when there are no Killing vectors on M, we can always find another first integral of motion, called *total energy*:

$$E(\phi) = \frac{1}{2}\, g(\dot{\phi}(t), \dot{\phi}(t)\,) + U\Big(\phi(t)\Big). \tag{3.8.51}$$

E is the sum of the kinetic and the potential energy, while the Lagrangian is the difference between them.

Theorem 3.25. *E is constant along the extremizers of integral (3.8.41).*

Proof. A direct computation shows

$$\begin{aligned} \frac{d}{dt}E(\phi(t)\,) &= \frac{d}{dt}\left[\frac{1}{2}\, g_{ij}(\phi(t)\,)\dot{\phi}^i(t)\dot{\phi}^j(t) + U(\phi(t)\,)\right] \\ &= \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_k}\dot{\phi}^k\dot{\phi}^i\dot{\phi}^j + g_{ij}\ddot{\phi}^i\dot{\phi}^j + \frac{\partial U}{\partial x_s}\dot{\phi}^s. \end{aligned} \tag{3.8.52}$$

As ϕ is an extremizer, from (3.8.48)

$$\frac{\partial U}{\partial x_s} = -g_{ks}\,(\ddot{\phi}^k + \Gamma^k_{ij}\,\dot{\phi}^i\dot{\phi}^j\,). \tag{3.8.53}$$

Substituting (3.8.53) in (3.8.52), we get

$$\begin{aligned} \frac{d}{dt}E(\phi(t)\,) &= \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_k}\,\dot{\phi}^k\dot{\phi}^i\dot{\phi}^j + g_{ij}\ddot{\phi}^i\,\dot{\phi}^j - g_{ks}\,(\ddot{\phi}^k + \Gamma^k_{ij}\,\dot{\phi}^i\dot{\phi}^j\,)\dot{\phi}^s \\ &= \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_k}\,\dot{\phi}^k\dot{\phi}^i\dot{\phi}^j - g_{ks}\Gamma^k_{ij}\,\dot{\phi}^i\dot{\phi}^j\dot{\phi}^s \\ &= \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_k}\,\dot{\phi}^k\dot{\phi}^i\dot{\phi}^j - \frac{1}{2}\left(\frac{\partial g_{is}}{\partial x_j} + \frac{\partial g_{js}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_s}\right)\dot{\phi}^i\dot{\phi}^j\dot{\phi}^s \\ &= 0, \end{aligned}$$

so that $E(\phi)$ is a first integral. ■

3.9 A geometrical interpretation for the potential U

Let $\phi : M \to \mathbb{R}^n$ be an isometric immersion of a Riemannian manifold M of dimension $m = n - 1$. If α is the mean scalar curvature of M, from Lemma 3.9 we have

$$\Delta\phi = -m\alpha N, \tag{3.9.54}$$

where Δ is written in the metric of M. If $\alpha = 0$, ϕ is a harmonic map and it is a critical point for the Dirichlet integral

$$\int_M \frac{1}{2} |\nabla\phi|^2 \, dv = \int_M \frac{1}{2} \sum_{k=1}^{n} \left|\nabla\phi^k\right|^2 dv, \tag{3.9.55}$$

where M is considered bounded with nonzero boundary. If $\alpha \neq 0$, we consider the Dirichlet integral perturbed by some potential $U : \mathbb{R}^n \to \mathbb{R}$, such that the immersion ϕ becomes a critical point for

$$I_U(\phi) = \int_M \Big(\frac{1}{2} |\nabla\phi|^2 - U(\phi)\Big) \, dv. \tag{3.9.56}$$

As ϕ is a critical point for $I_U(\phi)$, then

$$\Delta\phi = -\nabla U.$$

Comparing with (3.9.54) we get the following result.

Proposition 3.26 *Let $\phi : M \to \mathbb{R}^n$ be an isometric immersion of the hypersurface M. Then ϕ is a critical point for $I_U(\phi)$ iff the following two conditions are satisfied:*
1) the force $F = -\nabla U$ is normal to $\phi(M)$,
2) $|\alpha| = \dfrac{1}{n-1} \, |F|$.

Thus, from the geometrical point of view, force signifies mean curvature. No force situation corresponds to $\alpha = 0$, *i.e.*, M is a minimal hypersurface.

We can now address the following natural problem:

Given a hypersurface in $\mathbb{R}^n$, find a natural Lagrangian for which the hypersurface immersion is a critical point.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a function that defines M locally as $M = \{x \in \mathbb{R}^3; \psi(x) = 0\}$. As the normal is $N = \dfrac{\nabla\psi}{|\nabla\psi|}$, where $\nabla\psi = (\partial_1\psi, \dots, \partial_n\psi)$, we get

$$\frac{(n-1)\nabla\psi}{|\nabla\psi|}\alpha = \nabla U, \tag{3.9.57}$$

or

$$\partial_j U = \frac{(n-1)\partial_j\psi}{|\nabla\psi|}\alpha,$$

which provides the potential U up to an additive constant.

Example 3.9.1 *Let $\phi : \mathbb{S}^{n-1} \to \mathbb{R}^n$, where ϕ is the natural inclusion of the unit sphere. Choose $\psi(x) = |x|^2 - 1$ and get $\nabla\psi = 2x$, $\alpha = 1$. Then (3.9.57) becomes*

$$\partial_j U = \frac{(n-1)x_j}{|x|},$$

so that we can write

$$U(x) = (n-1)\,|x|\,,$$

up to a multiplicative constant. The Lagrangian is $L = \dfrac{1}{2}|\nabla\phi(x)|^2 - (n-1)|\phi(x)|$.

The following well-known result in geometry is approached here using equipotential surfaces.

Proposition 3.27 *Let $\phi : [0,1] \to \mathbb{R}^3$ be a unit speed curve. Then there exists a surface $\Sigma \subset \mathbb{R}^3$ that contains $\phi([0,1])$, and $\phi : [0,1] \to \Sigma$ is a geodesic.*

Proof. Let $p = \phi(0)$, $q = \phi(1)$. It is obvious from the physical point of view that there exists a force which perturbs the straight segment $[p, q]$ into $\phi([0,1])$. Let U be the potential for this force. Then ϕ will minimize

$$\int_0^1 \frac{1}{2}|\dot{\phi}|^2 - U(\phi). \tag{3.9.58}$$

As ϕ is a unit speed curve, using Proposition 3.23 we get $U_{\,|\phi}$ constant. Let $k = U_{\,|\phi}$. Consider the equipotential surface

$$\Sigma = \{x \in \mathbb{R}^3\,;\; U(x) = k\},$$

which contains $\phi([0,1])$. The Euler–Lagrange equation associated with (3.9.58) provides

$$\ddot{\phi}(t) = -\nabla U\left(\phi(t)\right).$$

As ∇U is normal to Σ, it follows that $\ddot{\phi}$ is normal to Σ, which means that ϕ is a geodesic on Σ. ∎

Example 3.9.2 *Let $\phi(t) = (\cos t, \sin t, 0)$ be a circle. Using the above method, we shall find a surface that contains the circle as a geodesic. The Euler–Lagrange equation is*

$$\ddot{\phi} = (-\cos t, -\sin t, 0) = (-\partial_1 U_{|\phi}, -\partial_2 U_{|\phi}, -\partial_3 U_{|\phi})$$

so that we can choose $U(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and $U_{|\phi} = \frac{1}{2}$. Then

$$\Sigma = U^{-1}\Big(\frac{1}{2}\Big) = \{x_1^2 + x_2^2 = 1\}$$

is a cylinder. If we choose $U(x) = \dfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, we find that Σ is a sphere.

3.10 Exercises

1. Let $\varphi : M \to \mathbb{R}^m$ be an isometric immersion of the compact manifold M and let $\varphi_t(x) = f(t)\varphi(x)$ be a smooth conformal variation of the immersion φ, with $f : (-\epsilon, \epsilon) \to (0, \infty)$, $f(0) = 1$. Let $g = \varphi^*(\delta)$ and $g(t) = \varphi_t^*(\delta)$ be the induced Riemannian metrics on M by φ and φ_t, respectively. Show the following:

a) $g_{ab}(t) = f^2(t)g_{ab}$

b) $g^{ab}(t) = f^{-2}(t)g^{ab}$

c) $\dfrac{\partial g_{ab}(t)}{\partial t}_{|t=0} = 2f'(0)g_{ab}$

d) $\Delta_{g(t)}\varphi = f^2(t)\Delta\varphi$

e) $\Delta_{g(t)}\varphi_t = f^3(t)\Delta\varphi$

f) Show that $\varphi_t = f(t)\varphi$ is a solution for $\big(\partial_t - \Delta_{g(t)}\big)\varphi_t = 0$ if and only if $f(t)$ verifies

$$f'(t) = \lambda_j f^3(t),$$
$$f(0) = 1,$$

where λ_j is an eigenvalue of Δ (Laplacian in the g-metric).

g) Show that

$$\varphi_t(x) = \frac{1}{\sqrt{1 - 2\lambda_j t}}\varphi_j(x),$$

with $\Delta\varphi_j = \lambda_j\varphi_j$.

h) The manifold $\varphi_t(M)$ blows up in finite time:

$$\lim_{t \nearrow \frac{1}{2\lambda_1}} | \varphi_t(x) |= \infty,$$

where $0 < \lambda_1$ is the smallest eigenvalue of the Laplacian on (M, g).

2. Let (M, g) be a compact manifold and $\varphi : (M, g) \to \mathbb{R}^m$ be an isometric immersion. Let $(\varphi_t)_{t\in[0,\epsilon)}$ be a smooth variation of φ such that

$$(\partial_t + \Delta_g)\varphi_t(x) = 0, \tag{3.10.59}$$
$$\varphi_t(x)_{|t=0} = \varphi(x),$$

where Δ_g is the Laplace operator with respect to the metric g.

a) Let $(\phi_j)_{j\geq 1}$ be a set of eigenfunctions of Δ_g, *i.e.*, $\Delta_g\phi_j = \lambda_j\phi_j, \lambda_j \in (0, +\infty)$, $j \geq 1$. Show that there are constants $c_j \in \mathbb{R}$ such that φ can be written in the unique representation

$$\varphi = \sum_{j \geq 1} c_j \phi_j.$$

b) Consider the smooth variation

$$\varphi_t(x) = \sum_{j \geq 1} c_j(t)\phi_j(x) \tag{3.10.60}$$

with $c_j(0) = c_j$. Show that (3.10.60) is a solution of problem (3.10.59) if and only if the functions $c_j(t)$ satisfy the initial value problem

$$c_j'(t) + \lambda_j c_j(t) = 0,$$
$$c_j(0) = c_j,$$

where λ_j is the j-th eigenvalue of Δ_g.

c) Show that any smooth variation $(\varphi)_t$ of φ which is a solution of the problem (3.10.59) can be represented as

$$\varphi_t(x) = \sum_{j \geq 1} \gamma_j e^{-\lambda_j t}\phi_j(x), \qquad \gamma_j \in \mathbb{R}.$$

d) If φ_t is a solution of the problem (3.10.59), then

$$\lim_{t \to \infty} \varphi_t(x) = 0_{\mathbb{R}^m}, \qquad \forall x \in M,$$

i.e., the manifold $\varphi_t(M)$ shrinks to a point as $t \to \infty$.

3. Let (M, g) be a Riemannian manifold and $p_0 \in M$ be a point. For any $v \in T_{p_0}M$ with $|v| = 1$, let c_v denote the maximal geodesic defined by $c_v(0) = p_0$, $\dot{c}_v(0) = v$ and parametrized by arc length. If $p = c_v(r)$, then let $(r, v_1, v_2, \ldots, v_n)$ be the coordinates of p, called the polar coordinates at p_0.

a) Show that the length element with respect to polar coordinates can be written as

$$ds^2 = dr^2 + \sum_{i,j=1}^{n-1} G_{ij}(r, v)dv_i dv_j.$$

b) Show that the Laplacian in polar coordinates is given by

$$\Delta = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial r}\Big(\sqrt{G}\frac{\partial}{\partial r}\Big) - \sum_{i,j=0}^{n-1} \frac{1}{\sqrt{G}}\frac{\partial}{\partial v_i}\Big(\sqrt{G}G^{ij}\frac{\partial}{\partial v_j}\Big).$$

c) Show that if $f \in \mathcal{F}(M)$ is a function such that $f(p)$ depends only on the Riemannian distance between p and p_0, then

$$\Delta f = -\frac{d^2 f}{d^2 r^2} - \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial r} \frac{df}{dr}.$$

4. Let (M, g) be a Riemannian manifold. Consider the Lagrangian

$$L(x, \phi, \nabla\phi) = \frac{1}{2}|\nabla\phi|^2 \rho(x),$$

where $\phi : M \to \mathbb{R}$ and $\rho : M \to (0, \infty)$ is a density function.

a) Show that the Euler–Lagrange equation is

$$div(\rho(x)\nabla\phi) = 0.$$

b) Show that the Euler–Lagrange equation can be written as $\Delta\phi = F(\phi, \rho)$, with $F(\phi, \rho) = \frac{\langle \nabla\phi, \nabla\rho \rangle}{\rho}$.

c) Let $M = \mathbb{R}$ and $\rho = 1 + x^2$. Solve the Euler–Lagrange equation in this case. Find the solution $\phi(x)$ which satisfies $\phi(0) = 1$, $\dot{\phi}(0) = 1$.

5. Let $\varphi : (M, g) \to \mathbb{R}$ and consider the Lagrangian

$$L(\varphi, \nabla\varphi) = \frac{1}{2}|\nabla\varphi|^2 \cdot \varphi^2.$$

a) Write the Euler–Lagrange equation as $\Delta\varphi = F(\varphi, \nabla\varphi)$ and find the function F.

b) Solve the Euler–Lagrange equation in the case $M = \mathbb{R}$.

6. Let (M, g) be a Riemannian manifold and $p \in M$ be a point. Let $v_i \in T_pM$ such that $g(v_i, v_j) = \delta_{ij}$. Show that there is an open neighborhood $\mathcal{U}$ of p and the vector fields V_i on $\mathcal{U}$ such that $V_i(p) = v_i$, $i = 1, \ldots, n$ and $g(V_i, V_j) = \delta_{ij}$ on $\mathcal{U}$. (Hint: Use the parallel transport with respect to the geodesics starting at p).

4

Harmonic Maps from a Lagrangian Viewpoint

4.1 Introduction to harmonic maps

Harmonic maps are mappings between Riemannian or pseudo-Riemannian manifolds which extremize a certain action, namely a natural energy integral that generalizes the classical Dirichlet's integral $\int |\nabla\phi|^2 \, dv$. Harmonic maps are generalizations of geodesics and harmonic functions as well.

In fact, harmonic maps come from theoretical physics, where they are known under the name of *nonlinear sigma models* or *chiral fields*. Nonlinear sigma models were introduced by Gell-Mann and Levi [30]. Their aim was to describe pion-nucleon physics in a low energy approximation, using Lagrangian theory for some self-interacting scalar fields. These fields can be assembled into a single map Ψ from the n-dimensional Minkowski space $(\mathbb{R}^n, \eta)$, where $\eta_{ij} = diag \ (-1, 1, \ldots, 1)$, into some real finite dimensional vector space E with a positive definite scalar product "$\centerdot$" and with the Lagrangian given by

$$L(\Psi) = \frac{1}{2} \eta^{\alpha\beta} \, \partial_\alpha \Psi \centerdot \partial_\beta \Psi - V(\Psi). \tag{4.1.1}$$

Here $V : E \to \mathbb{R}_+$ is a smooth function called potential and describes the self-interactions of the system.

In the low energy approximation, the Lagrangian L is modified by requiring the original fields to be constrained to the set of the minima M of the potential V

$$M = V^{-1}\Big(\{c\}\Big), \tag{4.1.2}$$

where $c = min \ V$.

Under certain conditions M is supposed to be a connected submanifold of E, so that the scalar product $\centerdot : E \times E \to \mathbb{R}$ induces a Riemannian metric g on M. The Lagrangian becomes

$$\mathcal{L}(\Psi) = \frac{1}{2} \eta^{\alpha\beta} \, \partial_\alpha \Psi^i \partial_\beta \Psi^j \, g_{ij}, \tag{4.1.3}$$

which will be the Lagrangian for the harmonic maps and will be considered later.

In geometry the notion was introduced by J. Eells and J.H. Sampson, see [13].

4.1.1 The energy density

Definition 4.1 *Let (M, g) and (N, h) be two Riemannian manifolds and $f : (M, g) \to (N, h)$ be a differentiable map. Define the energy density of f as*

$$e(f) = \frac{1}{2} Trace_g\,(f^* h), \tag{4.1.4}$$

where f^* is the pull-back of f and Trace is taken in the g-metric.

Proposition 4.2 *In local coordinates we have*

$$e(f)_x = \frac{1}{2} g^{ij}(x) f^{\alpha}_{\,;\,i} f^{\beta}_{\,;\,j} h_{\alpha\beta\,|\,f(x)}. \tag{4.1.5}$$

Proof. As we have

$$Trace_g(f^* h) = g^{ij}(f^* h)_{ij},$$

and

$$\begin{aligned}(f^* h)_{ij} &= (f^* h)(\partial_i, \partial_j) = h\Big(df(\partial_i), df(\partial_j)\Big)\\ &= h(f^{k}_{\,;\,i}\partial_k, f^{l}_{\,;\,j}\partial_l) = f^{k}_{\,;\,i}\, f^{l}_{\,;\,j}\, h_{kl},\end{aligned}$$

we get (4.1.5). ∎

Remark 4.3 *If (M, g) is the Minkowski space $(\mathbb{R}^n, \eta)$, $e(f)$ is exactly the Lagrangian (4.1.3).*

Another way of writing the energy density $e(f)$ is the following.

Proposition 4.4 *If $\{e_1, \ldots, e_n\} \subset T_x M$ is an orthonormal basis, then*

$$e(f)_x = \frac{1}{2} \sum_{i=1}^{m} |df_x(e_i)\,|_h^2, \tag{4.1.6}$$

where we denote each $X \in \mathcal{X}(N)$ by

$$|X|_h = \sqrt{h(X, X)},$$

the magnitude of X in h-metric.

Proof. Because of the orthornormality,

$$g_{ij}(x) = g^{ij}(x) = \delta_{ij},$$

and (4.1.5) becomes

$$e(f)_x = \frac{1}{2} \sum_{i,\alpha,\beta} f^{\alpha}_{\,;\,i} f^{\beta}_{\,;\,i} h_{\alpha\beta\,|\,f(x)}. \tag{4.1.7}$$

On the other side we have

$$\frac{1}{2}\sum_i |df_x(e_i)\,|_h^2 = \frac{1}{2}\sum_i h(f^{\alpha}_{\,;\,i}\,\partial_\alpha,\ f^{\beta}_{\,;\,i}\partial_\beta\,) = \frac{1}{2} f^{\alpha}_{\,;\,i}\, f^{\beta}_{\,;\,i}\, h_{\alpha\beta},$$

which is exactly (4.1.7). ∎

Remark 4.5 *1) Sometimes $e(f)$ is called the Hilbert–Schmidt norm of f and is denoted by $\|df\|^2$.*
2) The above norm depends on both metrics of M and N, on f, and the first covariant derivative of f.

4.1.2 Harmonic maps using Lagrangian formalism

Definition 4.6 *Let (M, g) be a compact manifold and $f : (M, g) \to (N, h)$ be a smooth map. Define the energy of f by*

$$\mathcal{E}(f) = \int_M e(f)\, dv_g, \tag{4.1.8}$$

where $dv_g = \sqrt{|g|}\, dx_1, \ldots, dx_n$.

Definition 4.7 *A map $f : (M, g) \to (N, h)$ is called harmonic if it is an extremizer for the energy functional*

$$f \to \mathcal{E}(f). \tag{4.1.9}$$

If M is not compact, define the harmonic map f as an extremizer for the energy $\mathcal{E}_{M'}(f)$ relative to every compact subdomain M' of M, where

$$\mathcal{E}_{M'}(f) = \int_{M'} e(f)\, dv_g.$$

The following theorem provides an equation in local coordinates for harmonic maps.

Theorem 4.8. *$f : (M, g) \to (N, h)$ is a harmonic map iff*

$$-\Delta(f^i) + g^{\alpha\beta}\ {}^N\Gamma^i_{pj}\, f^p_{;\alpha} f^j_{;\beta} = 0, \quad \forall i = \overline{1, n}. \tag{4.1.10}$$

Proof. f is a harmonic map iff the Euler-Lagrange equations provided by Theorem 3.5 hold

$$\left(\frac{\partial e(f)}{\partial f^\gamma_{;k}}\right)_{;k} = \frac{\partial(f)}{\partial e(f)^\gamma}, \qquad \forall \gamma = \overline{1, n}.$$

We have

$$\begin{aligned}
\frac{\partial e(f)}{\partial f^\gamma_{;k}} &= \frac{\partial}{\partial f^\gamma_{;k}}\Big(\frac{1}{2}\, g^{\alpha\beta}\, f^i_{;\alpha} f^j_{;\beta}\, h_{ij}\Big) \\
&= \frac{1}{2} g^{\alpha\beta}\, h_{ij} \left(\frac{\partial f^i_{;\alpha}}{\partial f^\gamma_{;k}}\, f^j_{;\beta} + f^i_{;\alpha}\, \frac{\partial f^j_{;\beta}}{\partial f^\gamma_{;k}}\right) \\
&= \frac{1}{2} g^{k\beta} h_{\gamma j} f^j_{;\beta} + \frac{1}{2} g^{k\alpha} h_{\gamma i} f^i_{;\alpha} = g^{k\beta} h_{\gamma j} f^j_{;\beta}.
\end{aligned}$$

Therefore,

$$\frac{\partial e}{\partial (f^{\gamma}_{;k})} = g^{k\beta} h_{\gamma j \,|\, f} f^{j}_{;\beta}.$$

Define the Euler-operator by

$$\tau(f)_{\gamma} = \left(\frac{\partial e(f)}{\partial f^{\gamma}_{;k}}\right)_{;k} - \frac{\partial e(f)}{\partial f^{\gamma}}, \quad \gamma = \overline{1,n}. \tag{4.1.11}$$

We have the following computation:

$$\begin{aligned}
\tau(f)_{\gamma} &= (g^{k\beta} h_{\gamma j} f^{j}_{;\beta})_{;k} - \frac{1}{2} g^{\alpha\beta} f^{i}_{;\alpha} f^{j}_{;\beta} \frac{\partial h_{ij}}{\partial y^{r}} \frac{\partial f^{r}}{\partial f^{\gamma}} \\
&= g^{k\beta}_{;k} h_{\gamma j} f^{j}_{;\beta} + g^{k\beta} \frac{\partial h_{\gamma j}}{\partial y_{p}} f^{p}_{;k} f^{j}_{;\beta} \\
&\quad + g^{k\beta} h_{\gamma j} f^{j}_{;\beta k} - \frac{1}{2} g^{\alpha\beta} f^{i}_{;\alpha} f^{j}_{;\beta} \frac{\partial h_{ij}}{\partial y^{\gamma}}.
\end{aligned}$$

As $g^{k\beta}_{;k} = 0$, if we define

$$\tau(f)^{i} = \tau(f)_{\gamma}\, h^{\gamma i},$$

we obtain

$$\begin{aligned}
\tau(f)^{i} &= g^{k\beta} h^{\gamma i} \frac{\partial h_{\gamma j}}{\partial y_{p}} f^{p}_{;k} f^{j}_{;\beta} \\
&= g^{k\beta} f^{i}_{;\beta k} - \frac{1}{2} g^{\alpha\beta} h^{\gamma i} f^{i}_{;\alpha} f^{j}_{;\beta} \frac{\partial h_{ij}}{\partial y_{\gamma}}.
\end{aligned}$$

As

$$-\Delta(f^{i}) = g^{k\beta}\, f^{i}_{;\beta k},$$

we get

$$\begin{aligned}
\tau(f)^{i} &= -\Delta(f^{i}) + g^{\alpha\beta} h^{\gamma i} \left(\frac{\partial h_{\gamma j}}{\partial y_{p}} f^{p}_{;\alpha} f^{j}_{;\beta} - \frac{1}{2} f^{i}_{;\alpha} f^{j}_{;\beta} \frac{\partial h_{ij}}{\partial y_{\gamma}} \right) \\
&= -\Delta(f^{i}) + g^{\alpha\beta}\, {}^{N}\Gamma^{i}_{pj} f^{p}_{;\alpha} f^{j}_{;\beta},
\end{aligned}$$

and the Euler-Lagrange equation is equivalent to

$$\tau(f)^{i} = 0, \qquad i = \overline{1,n}.$$

∎

In the particular case when $M = (a, b) \subset \mathbb{R}$, equation (4.1.10) becomes the familiar equation of a geodesic in local coordinates

$$\ddot{f}^{i} + \Gamma^{i}_{pj} \dot{f}^{p} \dot{f}^{j} = 0, \qquad i = \overline{1,n}. \tag{4.1.12}$$

We had shown before that the above equation can be written globally as

$$\nabla_{\dot{f}} \dot{f} = 0. \tag{4.1.13}$$

Such a global characterization also takes place for harmonic maps. This will be shown in the following.

Let $f : (M, g) \to (N, h)$ be a map and ∇^M, ∇^N be the Levi-Civita connections on (M, g), and (N, h), respectively. Define the second fundamental form of f as the 2-covariant symmetric tensor field

$$(\nabla df)(X, Y) = \nabla^N_{df(X)} df(Y) - df(\nabla^M_X Y), \qquad \forall X, Y \in \mathcal{X}(M). \tag{4.1.14}$$

Proposition 4.9 *In local coordinates we have*

$$(\nabla df)^s_{ij} = H^{f^s}_{ij} + {}^N\Gamma^s_{\alpha\beta} f^\alpha_{;i} f^\beta_{;j}. \tag{4.1.15}$$

Proof. A computation shows

$$\begin{aligned}
(\nabla df)_{ij} &= \nabla^N_{df(\frac{\partial}{\partial x_i})} df\Big(\frac{\partial}{\partial x_j}\Big) - df\Big(\nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\Big) \\
&= \nabla^N_{f^l_{;i}\frac{\partial}{\partial y_l}} f^p_{;j} \frac{\partial}{\partial y_p} - df\left({}^M\Gamma^p_{ij} \frac{\partial}{\partial x_p}\right) \\
&= f^l_{;i}\left(f^p_{;j}\, {}^N\Gamma^s_{lp} \frac{\partial}{\partial y_s} + \frac{\partial f^p_{;j}}{\partial y_l} \frac{\partial}{\partial y_p}\right) - {}^M\Gamma^p_{ij} f^s_{;p} \frac{\partial}{\partial x_s} \\
&= f^l_{;i} f^p_{;j}\, {}^N\Gamma^s_{lp} \frac{\partial}{\partial y_s} + \frac{\partial f^p_{;j}}{\partial y_l} f^l_{;i} \frac{\partial}{\partial y_p} - {}^M\Gamma^p_{ij} f^s_{;p} \frac{\partial}{\partial x_s} \\
&= \left(f^s_{;ij} - {}^M\Gamma^p_{ij} f^s_{;p} + f^l_{;i} f^p_{;j}\, {}^N\Gamma^s_{lp}\right) \frac{\partial}{\partial y_s} \\
&= \left(H^{f^s}_{ij} + f^l_{;i} f^p_{;j}\, {}^N\Gamma^s_{lp}\right) \frac{\partial}{\partial y_s}.
\end{aligned}$$

■

Definition 4.10 *The tension field of the map $f : (M, g) \to (N, h)$ is defined by*

$$\tau(f) = Trace_g(\nabla df). \tag{4.1.16}$$

This can be written locally as

$$\tau(f)^s = g^{ij} (\nabla df)^s_{ij} = -\Delta(f^s) + g^{ij\,N}\Gamma^s_{\alpha\beta} f^\alpha_{;i} f^\beta_{;j}.$$

Therefore, the Euler-Lagrange equations (4.1.10) can be written globally as

$$Trace_g(\nabla df) = 0, \tag{4.1.17}$$

or

$$\tau(f) = 0. \tag{4.1.18}$$

Remark 4.11 *i) $\tau(f)$ is not a vector field on N (as a section of $TN \to N$). It is a section in $f^{-1}(TN) \to M$.*
ii) Another way for finding Euler-Lagrange equations is to prove the first variation formula

$$\frac{d\mathcal{E}(f_t)}{dt}_{|t=0} = -\int_M h(\tau(f), V)\, dv_g, \tag{4.1.19}$$

where

$$V_x = \frac{df_t(x)}{dt}_{|t=0}$$

is the deformation vector field and $(f_t)_{t\in(-\epsilon,\epsilon)}$ is a variation for f.

Example 4.1.1 *Let $M = \mathbb{S}^1$ and $\phi : M \to N$. Then the energy is*

$$\mathcal{E}(\phi) = \frac{1}{2}\int_{S^1} \left|\dot{\phi}(s)\right|^2 ds$$

and the Euler-Lagrange operator is

$$\tau(\phi) = \nabla^N_{d\phi(\dot{c})} d\phi(\dot{c}),$$

(where $\dot{c}$ is the tangent to the circle $\mathbb{S}^1$).

Since

$$\tau(\phi) = \nabla^N_{d\phi(\dot{c})} d\phi(\dot{c}) - d\phi \nabla^{S^1}_{\dot{c}} \dot{c},$$

and

$$\nabla^{\mathbb{S}^1}_{\dot{c}} \dot{c} = 0,$$

the Euler-Lagrange equation becomes

$$\nabla^N_{d\phi(\dot{c})} d\phi(\dot{c}) = 0,$$

which means that $\phi(S^1)$ is a closed geodesic in N.

Example 4.1.2 *$\phi : \mathbb{R} \to N$ is a harmonic map if and only if ϕ is a geodesic on N. This example is related to Classical Mechanics, where N is the coordinate space and ϕ is the trajectory of a dynamical system with the Lagrangian*

$$L = \frac{1}{2}\left|\dot{\phi}(t)\right|^2.$$

Example 4.1.3 *$\phi : M \to \mathbb{R}^n$ is a harmonic map iff*

$$\Delta\phi^j = 0, \quad \forall j = \overline{1, n}.$$

In general, this takes place if the manifold $\mathbb{R}^n$ is replaced with a flat one ($\Gamma^i_{jk} = 0$).

Example 4.1.4 *Let $\phi : M \to N$ be a geodesic map, namely the second fundamental form is zero. Then ϕ is a harmonic map.*

4.2 D'Alembert principle on Riemannian manifolds

In Classical Mechanics, there is a principle stated by D'Alembert which is equivalent to the Lagrangian variational principle. We shall illustrate this principle briefly below.

Suppose that $\mathcal{M}$ is a surface in $\mathbb{R}^3$ and a material point is required to move on the surface $\mathcal{M}$. If U denotes the potential, Newton's equation should give the equation of motion $m\ddot{x} + \nabla U = 0$. If $U = 0$, which means that exterior forces are neglected, then $m\ddot{x} = 0$, with the solution $x(t) = At + B$. However, a line cannot be contained by an arbitrary surface $\mathcal{M}$. That means there is another force that requires the material point to lie on the surface $\mathcal{M}$. This is the *reaction force* denoted by $\mathcal{R}$ and is given by

$$\mathcal{R} = m\ddot{x} + \nabla U. \tag{4.2.20}$$

The *D'Alembert principle* states that the reaction force $\mathcal{R}$ is normal to the surface M, *i.e.*,

$$\langle m\ddot{x} + \nabla U, \xi \rangle = 0, \qquad \forall \xi \in TM. \tag{4.2.21}$$

Now we shall extend D'Alembert's principle on Riemannian manifolds, replacing $\mathbb{R}^3$ by an arbitrary Riemannian manifold P. The surface $\mathcal{M}$ and the space $\mathbb{R}$ of the t-variable are replaced by two other Riemannian spaces N and M, respectively.

The following result is an extension of Theorem 3.20 for harmonic maps.

Theorem 4.12. *Let $\phi : M \to N$ and $U \in \mathcal{F}(N)$ be the potential. Then ϕ is an extremizer for the integral*

$$\int_M [e(\phi) - U(\phi)]\, dv \tag{4.2.22}$$

if and only if

$$\tau(\phi) = -\nabla U. \tag{4.2.23}$$

Proof. The proof is the same as in the case of Theorem 3.20. Using the computations made in the proof of Theorem 4.9, the tension field $\tau(\phi)$ is obtained on the left-hand side. ∎

The equation (4.2.23) shows that the external force $F = -\nabla U$ is equal to the tension field of the map ϕ.

Theorem 4.13. *Let M, N, P be Riemannian manifolds and $\phi : M \to N$, and $\psi : N \to P$, with ψ immersion. Let $U \in \mathcal{F}(N)$ be a potential, and $\Phi = \psi \circ \phi$. The following are equivalent:*

(i) $$\tau(\phi) = -\nabla U,$$

(ii) $$\tau(\Phi) + d\psi(\nabla U) \ \text{is normal to} \ \psi(N).$$

To prove the above theorem we need the following:

Lemma 4.14

$$\nabla d(\psi \circ \phi) = d\psi \, \nabla d\phi + \nabla d\psi(d\phi, d\phi). \tag{4.2.24}$$

Proof.

$$\begin{aligned}
\nabla d(\psi \circ \phi)(X, Y) &= \nabla^P_{d(\psi \circ \phi)X} d(\psi \circ \phi)Y - d(\psi \circ \phi)\nabla^M_X Y \\
&= \nabla^P_{d\psi(d\phi X)} \, d\psi(d\phi Y) - d\psi \, d\phi \, \nabla^M_X Y = \nabla^P_{d\psi(d\phi X)} \, d\psi(d\phi \, Y) \\
&\quad -d\psi \, \nabla^N_{d\phi \, X} \, d\phi(Y) + d\psi \, \nabla^N_{d\phi \, X} \, d\phi(Y) - d\psi \, d\phi \, \nabla^M_X Y \\
&= d\psi \, \nabla d\phi + \nabla d\psi(d\phi X, d\phi Y).
\end{aligned}$$

■

Proof. (of Theorem 4.13*)* Take *Trace* in both sides of the relation (4.2.24) and use the definition of the torsion field to obtain

$$\tau(\psi \circ \phi) = d\psi\Big(\tau(\phi)\Big) + Trace \, \nabla d\psi(d\phi, d\phi). \tag{4.2.25}$$

As $\tau(\phi) = -\nabla U$, the relation (4.2.25) becomes

$$\tau(\Phi) + d\psi(\nabla U) = Trace \, \nabla d\psi(d\phi, d\phi).$$

Since $Trace \, \nabla d\psi(d\phi, d\phi) = nor \, \Big(\tau(\Phi)\Big)$, we get $\tau(\Phi) + d\psi(\nabla U)$ normal to $\psi(N)$.

The reverse can be proved using the same equivalences and the fact that $d\psi$ is one-to-one. ■

Corollary 4.15 *If M, N, P, ϕ, ψ and Φ are as above, then the following are equivalent:*

(i) $\quad \phi \; is \; a \; harmonic \; map,$

(ii) $\quad \tau(\Phi) \; is \; normal \; to \; \psi(N).$

Remark 4.16 *Theorem 4.13 states the equivalence between the Euler–Lagrange equation* (i) *and D'Alembert principle given in (ii). In this case the reaction force is*

$$\mathcal{R} = \tau(\Phi) + d\psi(\nabla U).$$

Corollary 4.17 *ϕ is an extremizer for the integral (4.2.22) if and only if*

$$\tau(\Phi) + d\psi(\nabla U) \;\; is \;\; normal \;\; on \;\; \psi(N).$$

Application 4.18 *Let $\Phi : M^{n-2} \to \mathbb{R}^n$ be an isometric immersion. Then there exists $\mathcal{S} \subset \mathbb{R}^n$, a hypersurface such that $M \subset \mathcal{S}$ and M is minimal in $\mathcal{S}$.*

Indeed, as Φ is an isometric immersion, the energy density of Φ is constant, $|\nabla\Phi|^2 = k$. In section 3.9, we constructed a potential U such that Φ is a critical point for

$$\int_M |\nabla\Phi|^2 - U(\Phi).$$

Take $\mathcal{S} = U^{-1}(\{k\})$. Then $M \subset \mathcal{S}$ and ∇U is normal to $\mathcal{S}$. As $\Delta\Phi = -\nabla U$, we get $\Delta\Phi$ normal to $\mathcal{S}$. Applying D'Alembert's principle, we find that M is minimal in $\mathcal{S}$.

Application 4.19 *(Harmonic maps into $\mathbb{S}^n$) Let $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the inclusion, and $\phi : M \to \mathbb{S}^n$ be a map, and $\Phi = i \circ \phi$. Applying D'Alembert's principle, ϕ is harmonic if and only if $\Delta\Phi$ is normal to $\mathbb{S}^n$. Therefore, there exists a proportionality function $A \in \mathcal{F}(M)$ such that $\Delta\Phi = A\Phi$. As $|\Phi(x)|^2 = 1$, we get*

$$\begin{aligned} 0 &= \frac{1}{2}\Delta\,|\Phi(x)|^2 = \frac{1}{2}\Delta \sum_j \left(\Phi^j(x)\right)^2 \\ &= \frac{1}{2}\sum_j \left[2\Phi^j(x)\,\Delta\Phi^j(x) - 2\left|\nabla\Phi^j\right|^2\right] \\ &= \langle\Phi, A\Phi\rangle - 2e(\Phi) = A - 2e(\Phi). \end{aligned}$$

So ϕ is harmonic if and only if

$$\Delta\Phi = 2e(\Phi)\,\Phi.$$

Application 4.20 *Let $c : [0,1] \to \mathcal{S} \subset \mathbb{R}^3$ be a curve on a surface $\mathcal{S}$. Then c is a geodesic if and only if $\ddot{c}(t)$ is normal to the surface $\mathcal{S}$.*

Indeed, c is harmonic if and only if it is geodesic. Using $\tau(c) = \ddot{c}$ and D'Alembert's principle we get $\ddot{c}$ normal to the surface $\mathcal{S}$.

In general, c is a geodesic perturbed by a potential U, where $U \in \mathcal{F}(\mathcal{S})$, if and only if

$$\ddot{c}(s) + \nabla U_{c(s)}$$

is normal to the surface $\mathcal{S}$, see Figure 4.1.

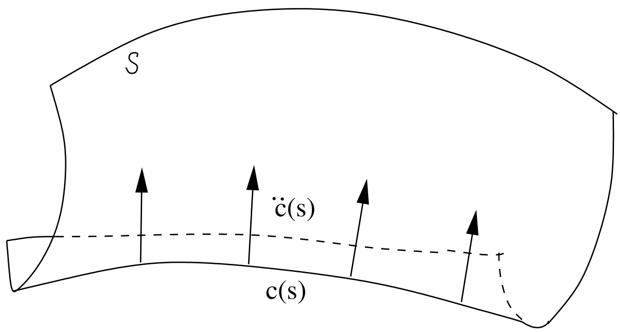


Figure 4.1: A curve $c(s)$ with $\ddot{c}(s)$ normal to the surface $\mathcal{S}$.

For more details on harmonic maps the reader may consult [14], [15], [16]. For a study of harmonic maps between spheres see [38]. For other advanced topics see [36], [39], [40].

4.3 Exercises

1. (Takahashi) Let $F : (M, g) \to \mathbb{R}^m$ be an isometric immersion of a compact manifold M of dimension n, with $1 \leq n \leq m - 1$. If $\Delta F = \lambda F$ with $\lambda > 0$, then show that

   a) $F(M) \subset S^{n-1}\Big(0, \frac{\sqrt{n}}{\lambda}\Big)$,

   b) F is a harmonic map from (M, g) to $S^{n-1}\Big(0, \frac{\sqrt{n}}{\lambda}\Big)$.

2. (Ferandez and Lucas) If $\varphi : M \to \mathbb{R}^3$ is an isometric immersion of the surface M into the Euclidean space, and $\Delta H = \lambda H$, where H denotes the mean curvature vector field, then show that

   a) M is minimal,

   b) $\varphi(M)$ is an open set in the sphere $\mathbb{S}^2(r)$ or the cylinder $\mathbb{S}^1 \times \mathbb{R}$.

3. Let e denote the energy density function of the map $\phi : (M, g) \to (N, h)$ and let $X \in \mathcal{X}(M)$ be a vector field. Show that

$$L_X e = \langle d\phi, \nabla(d\phi \cdot X)\rangle - \frac{1}{2}\langle L_X g, \phi^* h\rangle.$$

4. Let e denote the energy density function of the map $\phi : (M, g) \to (N, h)$. Let $X \in \mathcal{X}(M)$ and denote $v_g = \sqrt{\det g}\, dx_1 \wedge \cdots \wedge dx_n$ the volume element on (M, g). Show that

$$L_X(e \cdot v_g) = \langle d\phi, \nabla(d\phi \cdot X)\rangle v_g + \frac{1}{2}\langle L_X g, S_\phi\rangle v_g,$$

where $S_\phi = e \cdot g - \phi^* h$ and L_X denotes the Lie derivative with respect to X.

5. Define the stress-energy tensor of $\phi : (M, g) \to (N, h)$ by

$$S_\phi = e \cdot g - \phi^* h.$$

   a) Show that $div\, S_\phi = -\langle \tau(\phi), d\phi\rangle$, where $(div\, S_\phi)_i = g^{jk}\nabla_{\partial_{x_j}} S_{ki}$.

   b) Show that if the map ϕ is harmonic, then $div\, S_\phi = 0$.

   c) Find a counterexample when $div\, S_\phi = 0$ and ϕ is not harmonic.

6. Let $\varphi : \mathbb{R}^m \to (N, h)$ be a harmonic map of finite energy. Show that if $m \geq 3$, φ is constant.

7. Let $\phi : (M, g) \to (N, h)$ be a mapping between Riemannian manifolds. ϕ is called a totally geodesic map if $\nabla d\phi = 0$.

a) Show that ϕ is totally geodesic map if and only if ϕ maps geodesics of M linearly into geodesics of N.

b) Prove that any totally geodesic map is harmonic.

c) Find a counterexample of a harmonic map that is not totally geodesic.

8. The mean curvature of an immersion $\varphi : (M, g) \to (N, h)$ is the trace of the second fundamental form divided by $m = dim(M)$.

a) Show that a totally geodesic immersion has zero curvature.

b) Let $\varphi : (M, g) \to \mathbb{S}^n$ be an isometric immersion of constant mean curvature of M into the Euclidean sphere. Let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the canonical imbedding. Then $\iota \circ \varphi$ has constant mean curvature.

5

Conservation Theorems

5.1 Noether's Theorem

In Classical Mechanics, most of the conservation laws such as the conservation of momentum, angular momentum, etc, are particular cases of a single theorem due to E. Noether:

To every one-parameter group of diffeomorphisms of the coordinate space of a Lagrangian system which preserves the Lagrangian, corresponds a first integral of the Euler-Lagrange equation of motion.

In our work, the *space of parameters* is multidimensional. Therefore, we need to deal with objects that are more general than a first integral. A natural generalization of the first integral is the notion of *current*.

Definition 5.1 *A current is a free-divergence vector field which depends on the solution of the Euler–Lagrange equation.*

In particular, when the space of parameters is one-dimensional (just the time parameter), a current becomes a usual first integral, *i.e.*, a function constant along the solutions of the Euler–Lagrange system.

Theorem 5.2. *Let $\phi : (M, g) \to (N, h)$ be a harmonic map between two Riemannian manifolds and $(h_s)_s$ a one-parameter group of diffeomeorphisms on M that preserves energy density*

$$e(\phi \circ h_s) = e(\phi), \qquad \forall s \in \mathbb{R}. \tag{5.1.1}$$

Let V be the vector field induced by $(h_s)_s$. Then the vector field

$$X = \left(g^{kj} \phi^{\beta}_{;\,j} h_{p\beta} V(\phi^p) \right) \frac{\partial}{\partial x_k} \tag{5.1.2}$$

is a current.

Proof. As ϕ is a harmonic map, then $\tau(\phi) = 0$. The Euler–Lagrange equations can be written as

$$\left(\frac{\partial e}{\partial(\phi^p_{;k})}\right)_{;k} = \frac{\partial e}{\partial \phi^p}, \qquad \forall\, p = \overline{1,n}. \tag{5.1.3}$$

Let $\Phi : \mathbb{R} \times M \to N$ be defined by $\Phi(s,x) = \phi\Big(h_s(x)\Big)$. As $e(\phi) = e(\phi \circ h_s)$, the chain rule yields

$$0 = \frac{\partial e(\Phi)}{\partial s} = \frac{\partial e(\Phi)}{\partial \Phi^p}\frac{\partial \Phi^p}{\partial s} + \frac{\partial e(\Phi)}{\partial(\Phi^\alpha_{;k})}\frac{\partial(\Phi^\alpha_{;k})}{\partial s}. \tag{5.1.4}$$

Applying the commutativity of the partial derivatives,

$$\frac{\partial(\Phi^\alpha_{;k})}{\partial s} = \left(\frac{\partial \Phi^\alpha}{\partial s}\right)_{;k} \tag{5.1.5}$$

and substituting the relation (5.1.3) in (5.1.4), we obtain

$$0 = \left(\frac{\partial e(\Phi)}{\partial(\Phi^p_{;k})}\right)_{;k}\frac{\partial \Phi^p}{\partial s} + \frac{\partial e(\Phi)}{\partial(\Phi^\alpha_{;k})}\left(\frac{\partial \Phi^\alpha}{\partial s}\right)_{;k} \tag{5.1.6}$$

$$= \left(\frac{\partial e(\Phi)}{\partial(\Phi^p_{;k})}\frac{\partial \Phi^p}{\partial s}\right)_{;k}. \tag{5.1.7}$$

Taking $s = 0$,

$$0 = \left(\frac{\partial e(\phi)}{\partial(\phi^p_{;k})}\frac{\partial \Phi^p}{\partial s}_{\,|\,s=0}\right)_{;k} = \left(\frac{\partial e(\phi)}{\partial(\phi^p_{;k})}\,V(\phi^p)\right)_{;k} = X^k_{\;;k} \tag{5.1.8}$$

where

$$X^k = \frac{\partial e(\phi)}{\partial(\phi^p_{;k})}\,V(\phi^p),$$

and the induced vector field by $(h_s)_s$ is defined by

$$V(f) = \frac{d(f \circ h_s)}{ds}_{\,|\,s=0}, \qquad \forall f \in \mathcal{F}(M). \tag{5.1.9}$$

As computation shows that

$$\frac{\partial e(\phi)}{\partial(\phi^p_{;k})} = g^{kj}\phi^\beta_{;j}h_{p\beta}, \tag{5.1.10}$$

Equation (5.1.8) yields

$$X^k = g^{kj}\phi^\beta_{;j}h_{p\beta}\,V(\phi^p). \tag{5.1.11}$$

■

In the case when the right-hand side manifold N is the real line $\mathbb{R}$, we obtain the following:

Corollary 5.3 *Let $\phi : (M, g) \to R$ be a harmonic function. The vector field on M,*

$$X = V(\phi)\,\nabla\phi, \tag{5.1.12}$$

is a current. This provides a conservation along the normal direction to the equipotential surfaces of ϕ.

Proof. If we substitute $h_{pk} = 1$ in relation (5.1.2), we obtain $X^k = (\nabla\phi)^k\, V(\phi)$. Furthermore, $\nabla\phi$ is normal to the surfaces $\{\phi = constant\}$. ■

Corollary 5.4 *Let $\phi : (M, g) \to \mathbf{R}$ be a harmonic function. Then*

$$g\Big(\nabla\phi, \nabla(V(\phi))\Big) = 0. \tag{5.1.13}$$

Proof. Applying Lemma 2.10 yields

$$div\,\Big(V(\phi)\,\nabla\phi\Big) = -V(\phi)\Delta\phi + g\Big(\nabla\phi, \nabla(V(\phi))\Big). \tag{5.1.14}$$

Using $\Delta\phi = 0$ and Corollary 5.3, we get the desired result. ■

Remark 5.5 *Corollary* 5.4 *says that the vector field $\nabla\big(V(\phi)\big)$ is tangent to the constant level surfaces of ϕ (equipotential surfaces).*

When the space of parameters M is the real line $\mathbb{R}$ (just time parameter), Theorem 5.2 will provide the conservation of energy along the geodesic $\phi : \mathbb{R} \to N$.

Corollary 5.6 *$h(\dot\phi, \dot\phi)$ is preserved along the geodesic $\phi : \mathbb{R} \to N$.*

Proof. In one dimension the div becomes the derivation in t and $V(\phi) = \dot\phi$. ■

Other conservation laws can be obtained if the one-parameter group of diffeomorphisms, which preserves the Lagrangian, is considered on the target manifold.

Theorem 5.7. *Let (M, G) be a Riemannian manifold and $(h_s)_s$ a one-parameter group of diffeomorphisms on M that preserves the energy density for the geodesic $\phi : \mathbb{R} \to M$, i.e., $e(h_s \circ \phi) = e(\phi)$, $\forall s \in \mathbb{R}$. Then*

$$g(\dot\phi(t), V_{\mid\,\phi(t)}) = constant\ , \quad \forall t \in \mathbb{R}, \tag{5.1.15}$$

where $\dot\phi(t)$ is the tangent vector to the curve $\phi(t)$ and V is the vector field induced by $(h_s)_s$ on M.

Proof. Take $\Phi : \mathbb{R} \times \mathbb{R} \to M$ given by $\Phi(t, s) = h_s(\phi(t))$. As $(h_s)_s$ preserves the energy density, we have

$$\begin{aligned} 0 = \frac{\partial e(\Phi)}{\partial s} &= \frac{\partial e(\Phi)}{\partial \Phi^\gamma}\,\frac{\partial \Phi^\gamma}{\partial s} + \frac{\partial e(\Phi)}{\partial \dot\Phi^\gamma}\,\frac{\partial \dot\Phi^\gamma}{\partial s} \\ &= \frac{d}{dt}\,\frac{\partial e(\Phi)}{\partial \dot\Phi^\gamma}\,\frac{\partial \Phi^\gamma}{\partial s} + \frac{\partial e(\Phi)}{\partial \dot\Phi^\gamma}\,\frac{\partial}{\partial s}\Big(\frac{\partial}{\partial t}\Phi^\gamma\Big) \\ &= \frac{d}{dt}\Big(\frac{\partial e(\Phi)}{\partial \dot\Phi^\gamma}\,\frac{\partial \Phi^\gamma}{\partial s}\Big). \end{aligned}$$

Recall that $\Phi_{\mid s=const.} : \mathbb{R} \to M$ is harmonic and apply the Euler–Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial e(\Phi)}{\partial \dot{\Phi}^\gamma} = \frac{\partial e(\Phi)}{\partial \Phi^\gamma}.$$

Taking the value at $s = 0$ and applying the formula

$$\frac{\partial e(\phi)}{\partial \dot{\phi}^\gamma} = g_{\gamma\alpha}\dot{\phi}^\alpha$$

yields

$$0 = \frac{d}{dt}\left(g_{\gamma\alpha}\dot{\phi}^\alpha\left(\frac{\partial(h_s \circ \phi)}{\partial s}_{\mid s=0}\right)^\gamma\right) = \frac{d}{dt}g(\dot{\phi}, V_{\mid \phi}).$$

■

Remark 5.8 *The above theorem states that the momentum in the V-direction is constant.*

Using the Euler–Lagrange equation in general form and the same idea of proof, one can get the following theorem.

Theorem 5.9. *Let $f : (M, G) \to (N, h)$ be a harmonic map and $(\xi_s)_s$ a one-parameter group of diffeomorphisms on N such that $e(\xi_s \circ f) = e(f), \quad \forall s \in \mathbb{R}$. Let*

$$V^\gamma_{\mid f} := \frac{d(\xi_s \circ f)^\gamma}{ds}_{\mid s=0} \tag{5.1.16}$$

be the vector field generated by ξ_s along f. Then the vector field on N,

$$Y = \left(g^{kj}\, f^\beta_{;\,j}\, h_{\gamma\beta}\, V^\gamma_{\mid f}\right)\frac{\partial}{\partial y_k}, \tag{5.1.17}$$

is a current on N, i.e., $div\ Y = 0$.

5.2 The role of Killing vector fields

The theorems proved in Section 5.1 are general. In this chapter, we deal with some particular 1-parameter groups of diffeomorphisms generated by special vector fields called *Killing vector fields*.

Definition 5.10 *A vector field X on a Riemannian manifold (M, g) is a Killing vector field if*

$$L_X g = 0, \tag{5.2.18}$$

where L_X is the Lie derivation in the X direction.

Relation (5.2.18) says that the metric is preserved along the integral lines of X,

$$h_s^*(g_{ij}) = g_{ij}, \qquad \forall s \in \mathbb{R}, \tag{5.2.19}$$

where $(h_s)_s$ is the 1-parameter group of diffeomorphisms generated by the vector field X.

Proposition 5.11 *Let $f : (M, g) \to (N, h)$ be a map, V be a Killing vector field on N, and $(\xi_s)_s$ the one-parameter group of diffeomorphisms definite by V. Then*

$$e(f) = e(\xi_s \circ f), \qquad \forall s. \tag{5.2.20}$$

Proof. As V is Killing, $\xi_s^*(h) = h, \ \forall s$. Then

$$\begin{aligned} f^*\Big(h - \xi_s^*(h)\Big) &= 0 \\ \iff f^*(h) &= f^* \ \xi_s^*(h) \\ \iff f^*(h) &= (\xi_s \circ f)^*(h). \end{aligned}$$

Taking the $Trace$ in metric g and using formula (4.1.4) we get

$$\begin{aligned} Trace_g \ f^*(h) &= Trace_g \ (\xi_s \circ f)^*(h) \\ \iff e(f) &= e(\xi_s \circ f), \quad \forall s. \end{aligned}$$

∎

Using Proposition 5.11, Theorem 5.9 becomes:

Theorem 5.12. *Let $\phi : (M, g) \to (N, h)$ be a harmonic map between two Riemannian manifolds and $V \in \mathcal{X}(N)$ be a Killing vector field. The vector field*

$$Y = \left(g^{kj} f^{\beta}_{\ ;\, j} h_{\gamma\beta} V^{\gamma}_{\ |\, f}\right) \frac{\partial}{\partial y_k} \tag{5.2.21}$$

is a current on N.

Theorem 5.7 becomes:

Theorem 5.13. *Let $\phi : \mathbb{R} \to (M, g)$ be a geodesic and V be a Killing vector field on M. Then*

$$g(\dot{\phi}(t), V_{|\,\phi(t)}) = constant \ , \quad \forall t \in \mathbb{R}, \tag{5.2.22}$$

which means the momentum in the direction of a Killing vector field along a geodesic is preserved.

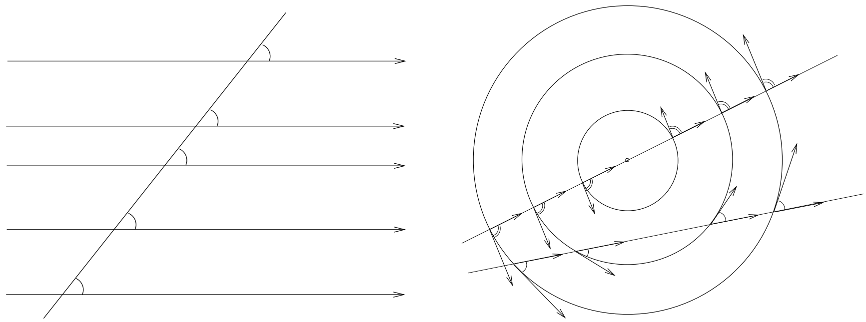


Figure 5.1: Geodesics and Killing vector fields in the plane; see example 5.2.1.

Example 5.2.1 *In the Euclidean plane, the Killing vector fields correspond to translations and rotations and the geodesics are lines. We find that at intersection points between a fixed line and variable circles centered at the origin, the scalar product between their tangent vectors is constant (is not dependent on the circle).*

Example 5.2.2 *On a surface of revolution, we have the Killing vector field of rotation. Let $\theta_{\mid \phi(t)}$ be the angle between a fixed geodesic $\phi(t)$ and the latitude circles at the point $\phi(t)$. Since the length of the tangent to the circle is the radius r of the circle, using the above theorem we conclude that $\langle \dot{\phi}(t), V_{\mid \phi(t)} \rangle = |\dot{\phi}| r \, \cos\theta_{\mid \phi(t)}$ is constant, or equivalently, $r \, \cos\theta = constant$. If the inclination angle α of a geodesic with respect to its meridian is defined by $\alpha = \pi/2 - \beta$, we arrive at the result known as Clairaut's theorem (see [31]).*

Theorem 5.14. *Let $\phi(t)$ be a geodesic on a smooth surface of revolution $\mathcal{S}$. Then at any point P of $\phi(t)$ the radius $r(P)$ of the circle of latitude at P multiplied by the sine of the inclination angle $\alpha(P)$ of $\phi(t)$ with respect to the meridian through P is a constant, i.e. $r \sin\alpha$ =constant.*

Another necessary condition for preserving energy density is given by the following:

Proposition 5.15 *Let $f : (M, g) \to (N, h)$ be an immersion. Let $\tilde{g}$ be the induced metric on M by f, i.e. $\tilde{g} = f^*(h)$. If V is a Killing vector field on $(M, \tilde{g})$, then*

$$e(f \circ \xi_s) = e(f), \quad \forall s \in \mathbb{R}, \tag{5.2.23}$$

where $(\xi_s)_s$ is the one-parameter group generated by V.

Proof. As V is Killing on $(M, \tilde{g})$, we have

$$\begin{aligned} \xi_s^*(\tilde{g}) &= \tilde{g}, \Longleftrightarrow \\ \xi_s^* \circ f^*(h) &= f^*(h) \Longleftrightarrow \\ (f \circ \xi_s)^*(h) &= f^*(h). \end{aligned}$$

Taking the $Trace$ in metric g yields

$$Trace_g(f \circ \xi_s)^*(h) = Trace_g\, f^*(h),$$
$$\Longleftrightarrow e(f \circ \xi_s) = e(f), \quad \forall s \in \mathbb{R}.$$

∎

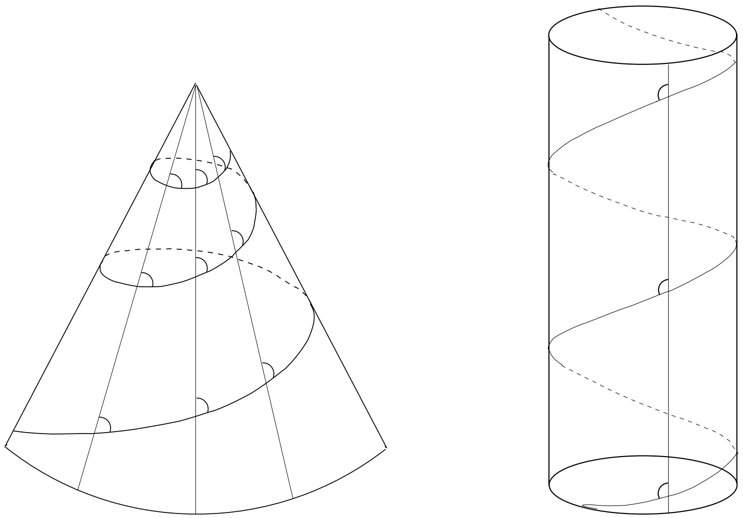


Figure 5.2: Geodesics on a cone and on a cylinder and Clairaut's theorem.

Using Theorem 5.2 and Proposition 5.15, we get the following:

Proposition 5.16 *Under the hypothesis of Proposition 5.15, if f is a harmonic immersion, then the vector field with the components*

$$X^k = g^{kj} f^{\beta}_{;\,j} h_{p\beta} V(f^p) = (\nabla f^{\beta})^k\, h_{p\beta} V(f^p) \tag{5.2.24}$$

is a current.

Proposition 5.17 *Let $f : (M, g) \to (N, h)$ be an isometric harmonic immersion and let V be a Killing vector field on M. Then $\xi_s : M \to M$ is a harmonic diffeomorphism for every s.*

Proof. Applying $Trace$ in metric g in the relation of Lemma 4.14, we get

$$\tau(f \circ \xi_s) = df\, \tau(\xi_s) + Trace\, \nabla df(d\xi_s, d\xi_s).$$

From Proposition 5.15 the Lagrangian $e(f)$ is preserved by ξ_s. Hence, the Euler–Lagrange equation will be the same

$$\tau(f \circ \xi_s) = \tau(f), \quad \forall s.$$

Since f is harmonic, $\tau(f) = 0$ and so $\tau(f \circ \xi_s) = 0$. As the normal component $nor\ \tau(f \circ \xi_s) = df\ \tau(\xi_s)$, then

$$df\ \tau(\xi_s) = 0.$$

As df is one-to-one (f immersion), we get $\tau(\xi_s) = 0$ for every s, *i.e.*, ξ_s is harmonic. ∎

5.3 The Energy-Momentum tensor

The energy-momentum tensor comes from Physics where it describes the matter fields equations. It depends on the field, their covariant derivatives, and the metric. The energy-momentum tensor mainly describes two things:

(i) The principle that all fields have energy. That, the energy-momentum vanishes on an open set U if and only if all the matter fields vanish on U. From the Physics point of view one should not distinguish between two different matter fields that have the same energy-momentum tensor.

(ii) The total flux over a closed surface of the K-component of the energy-momentum tensor is zero, where K is a Killing vector field.

The last property provides conservation of angular momentum by means of rotation vector fields for the Euclidian flat space (see [21]). Knowledge of the energy-momentum tensor was used in the Brans-Dicke theory for determination of the conformal factor of the metric (see [21]).

The energy-momentum tensor was successfully used in the general theory of relativity to describe gravitational effects. In this case it equals a certain free-divergence tensor which depends only on the metric of the space. There is a standard procedure to obtain the energy-momentum tensor from the associated Lagrangian of a matter field.

Returning to PDEs, we note that in the particular case when the Lagrangian depends only on a scalar field and its first derivative, we may associate the Euler–Lagrange system of equations, which is the equation for the first variation of the action. A classical minimum action principle states that the scalar field satisfies the Euler–Lagrange equation. In general, this equation is a second order partial differential equation.

On the other hand, the scalar field is characterized by its energy-momentum tensor. The conservation properties of the energy-momentum tensor may help to obtain information about the solutions of the Euler–Lagrange equations. Used together with the boundary conditions, this is a useful tool to prove uniqueness for linear homogeneous boundary value problems. It is important to obtain such results when the background metric is Riemannian and the Euler–Lagrange equations are elliptic.

This section deals with a geometric approach for some linear partial differential equations derived as Euler–Lagrange equations from certain Lagrangians. One may associate the energy-momentum tensor with these Lagrangians, which satisfies some conservation properties. The goal of this section is to exploit the conservation properties of the energy-momentum tensor and to obtain information about the solutions of the Euler–Lagrange equation. For an approach of harmonic maps between semi-Riemannian manifolds from the conservation property point of view, see [33]. An extension of the variational methods to subRiemannian is done in [34].

5.3.1 Definition of Energy-Momentum

A physical field is given by its Lagrangian and its dynamic is described by the Euler–Lagrange equations, called the field equations. An important problem is to determine the flow energy along a given direction for a given physical field. This description uses a 2-covariant symmetric tensor field T_{ij}, called *the energy-momentum tensor*. The energy flow in the X-direction is given by the expression

$$T(X, X) = T_{ij} X^i X^j. \tag{5.3.25}$$

Let L be a Lagrangian which depends on the field ϕ, on its first derivatives $\phi_{;k}$, and on the metric g_{ij} of the Riemannian manifold M. Consider the integral

$$I = \int_{\mathcal{D}} L\, dv, \tag{5.3.26}$$

where $\mathcal{D} \subset \mathcal{M}$ is a compact domain. Consider the variations of the metric $g_{ij}(s, x)$ given by $g_{ij}(0, x) = g_{ij}(x)$, with the variation field

$$\delta g_{ij}(x) = \frac{\partial g_{ij}(s, x)}{\partial s}_{|s=0}.$$

Definition 5.18 *The energy-momentum tensor T_{ij} is defined by*

$$\frac{dI}{ds}_{|s=0} = \int_{\mathcal{D}} T^{ab}\, \delta g_{ab}\, dv.$$

Lemma 5.19 *On the Riemannian manifold with volume element dv we have*

(i)
$$\frac{\partial(dv)}{\partial g_{ab}} = \frac{1}{2} g^{ab} dv,$$

(ii)
$$\delta(dv) = \frac{1}{2} g^{ab}\, \delta g_{ab}\, dv.$$

If the Lagrangian L depends only on ϕ, $\phi_{;i}$ and the metric g_{ab}, then

(iii)
$$\delta L = \frac{\partial L}{\partial g_{ab}}_{|s=0} \delta g_{ab}.$$

Proof.

(i)
$$\frac{\partial(dv)}{\partial g_{ab}} = \frac{\partial}{\partial g_{ab}}(\sqrt{g}\,dx^1\ldots dx^n) = \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial g_{ab}}\,dx^1\ldots dx^n.$$

As $\partial g/\partial g_{ab}$ is the minor of g_{ab}, then

$$g^{ab} = \frac{1}{g}\frac{\partial g}{\partial g_{ab}}, \quad \text{or} \quad \frac{\partial g}{\partial g_{ab}} = g\,g^{ab}.$$

It follows that

$$\begin{aligned}\frac{\partial(dv)}{\partial g_{ab}} &= \frac{1}{2\sqrt{g}}\,g\,g^{ab}\,dx^1\ldots dx^n \\ &= \frac{1}{2}g^{ab}\sqrt{g}\,dx^1\ldots dx^n = \frac{1}{2}g^{ab}\,dv.\end{aligned}$$

(ii)
$$\delta(dv) = \frac{\partial\,dv}{\partial s}_{|s=0} = \frac{\partial\,dv}{\partial g_{ab}}\frac{\partial g_{ab}}{\partial s}_{|s=0} = \frac{1}{2}g^{ab}\delta g_{ab}\,dv$$

by (i).
(iii)

$$\begin{aligned}\delta L &= \frac{\partial L(\phi,\phi_{;i},g_{ab})}{\partial s}_{|s=0} \\ &= \frac{\partial L}{\partial\phi}\underbrace{\frac{\partial\phi}{\partial s}}_{=0} + \frac{\partial L}{\partial\phi_{;i}}\underbrace{\frac{\partial\phi_{;i}}{\partial s}}_{=0} + \frac{\partial L}{\partial g_{ab}}\frac{\partial g_{ab}}{\partial s}_{|s=0} \\ &= \frac{\partial L}{\partial g_{ab}}_{|s=0}\delta g_{ab},\end{aligned}$$

where we used the fact that the variation in s does not affect the function ϕ and its derivatives $\phi_{;i}$. ∎

Theorem 5.20. *(Existence of energy-momentum tensor)*
Let L be a Lagrangian which depends on ϕ, $\phi_{;i}$, and the metric g_{ab}. Then the energy-momentum tensor is given by

$$T^{ab} = \frac{\partial L}{\partial g_{ab}} + \frac{1}{2}g^{ab}L. \tag{5.3.27}$$

Proof. Using the above lemma we have

$$\begin{aligned}\delta I &= \int_D \delta L\,dv + L\,\delta(dv) \\ &= \int_D \Big[\frac{\partial L}{\partial g_{ab}}\delta g_{ab}\,dv + \frac{1}{2}Lg^{ab}\delta g_{ab}\,dv\Big] \\ &= \int_D \underbrace{\Big[\frac{\partial L}{\partial g_{ab}} + \frac{1}{2}Lg^{ab}\Big]}_{=T^{ab}}\delta g_{ab}\,dv.\end{aligned}$$

∎

5.3.2 Einstein tensor

Let (M, g) be a Riemannian manifold and let T be a symmetric 2-covariant tensor field on M.

Definition 5.21 *The divergence of the tensor field T is a vector field denoted by $div\, T$ given by $div\, T = \big(div\, T\big)^i \partial_{x_i}$ with the components*

$$\big(div\, T\big)^i = T^{ji}_{;j} = \nabla_{\partial_{x_j}} T^{ji}.$$

The tensor T is divergence-free if $T^{ij}_{;j} = 0$.

Example 5.3.1 *The metric tensor g is divergence-free. The identity $g^{ij}_{;j} = 0$ is called the Ricci identity and it is equivalent with the fact that the Levi-Civita connection is a metric connection.*

Definition 5.22 *Let Ric denote the Ricci tensor and R the scalar curvature. The symmetric tensor*

$$T = Ric - \frac{1}{2} R g_{ij} \tag{5.3.28}$$

is called the Einstein tensor. On components we have $T_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$.

The following results will be useful in the study of the Einstein tensor divergence. The next lemma can also be found in [35].

Lemma 5.23 *Let R be the Ricci scalar curvature. Then*

$$\nabla R = 2 div\, Ric. \tag{5.3.29}$$

Proof. The second Bianchi identity in local coordinates can be expressed as

$$R^i_{jkl;r} + R^i_{jlr;k} + R^i_{jrk;l} = 0.$$

Swapping r and k with the change of sign yields

$$R^i_{jkl;r} + R^i_{jlr;k} - R^i_{jkr;l} = 0.$$

Contracting on i and r yields

$$\sum_r R^r_{jkl;r} + \sum_r R^r_{jlr;k} - \sum_r R^r_{jkr;l} = 0,$$

which becomes

$$\sum_r R^r_{jkl;r} + + R_{jl;k} - R_{jk;l} = 0.$$

Contract multiplying by g^{jk},

$$\sum_{r,j,k} g^{jk} R^r_{jkl;r} + g^{jk} R_{jl;k} - g^{jk} R_{jk;l} = 0$$

$$\sum_r R^r_{l;r} + \sum_k R^k_{l;k} = R_{;l}$$

$$2\sum_r R^r_{l;r} = R_{;l}.$$

Multiplying by g^{lj} yields

$$\begin{aligned} 2g^{lj} R^r_{l;r} &= g^{lj} R_{;l}, \\ 2R^{jr}_{;r} &= \big(\nabla R\big)^j, \\ 2div\, Ric &= \nabla R. \end{aligned}$$

■

The following result is an analog of Lemma 2.10 for tensor fields.

Lemma 5.24 *Let $f \in \mathcal{F}(M)$ be a function and S be a symmetric 2-covariant tensor. Then*

$$div\big(fS\big)^i = f(div S)^i + g_{pk}(\nabla f)^k S^{ip}.$$

Proof. A computation involving derivation yields

$$\begin{aligned} div(fS)^i &= (fS)^{ji}_{;j} = f_{;j} S^{ji} + f S^{ji}_{;j} \\ &= f_{;j} S^{ji} + f(div S)^i \\ &= f_{;j} g^{jk} S^i_k + f(div S)^i \\ &= (\nabla f)^k S^i_k + f(div S)^i \\ &= (\nabla f)^k S^{ip} g_{kp} + f(div S)^i. \end{aligned}$$

■

Theorem 5.25. *The Einstein tensor is divergence free.*

Proof. Making $f = R$ and $S = g$ in Lemma 5.24 yields

$$\begin{aligned} div(Rg)^i &= R(div\, g)^i + g_{pk}(\nabla R)^k g^{ip} \\ &= 0 + (\nabla R)^k \delta^i_k = (\nabla R)^i, \end{aligned}$$

where we used the fact that the metric tensor g is divergence free. Lemma 5.23 yields

$$\begin{aligned} div(Rg) \;&=\; \nabla R = 2\, div\, Ric \\ &\Longrightarrow div(2Ric - Rg) = 0, \end{aligned}$$

which yields $div\, T = div(Ric - \frac{1}{2} Rg) = 0.$ ■

Remark 5.26 *The above theorem will be proved in a more general framework in a next section of this chapter.*

5.3.3 Field equations

The field equations for Einstein's gravitational potential
The goal of this section is to show that the Einstein tensor can be derived as an energy-momentum tensor for a certain action integral. We shall apply it to the surface and curve theory. From the definition of the energy-momentum tensor we have:

Proposition 5.27 *The integral*

$$I = \int_{\mathcal{D}} L\, dv$$

is stationary under the variations of the metric which leaves ϕ unchanged iff $T_{ij} = 0$.

The tensorial equation

$$T_{ij} = 0 \tag{5.3.30}$$

is called a *field equation*. If the Lagrangian depends on ϕ, $\phi_{;i}$, and the metric g_{ab}, then the equation (5.3.30) can be written as

$$\frac{\partial L}{\partial g_{ab}} = -\frac{1}{2} L g^{ab}$$

or, after multiplying by g_{ab},

$$\frac{n}{2} L = -g_{ab} \frac{\partial L}{\partial g_{ab}},$$

where $n =$dim(M).

We shall consider some examples where the Lagrangian depends only on the Riemannian metric and its derivatives and there is no function ϕ.

The following two lemmas will be useful in the future. See also [21].

Lemma 5.28 *If M is a compact, orientable, without boundary Riemannian manifold, then*

$$\int_M g^{ab} \delta R_{ab}\, dv = 0.$$

Proof. We shall write the integrand as the divergence of a vector field. The divergence theorem will lead to the desired relation. A computation in tensors yields

$$\begin{aligned} g^{ab}\delta R_{ab} &= g^{ab}\Big[\left(\delta\Gamma^c_{ab}\right)_{;c} - \left(\delta\Gamma^c_{ac}\right)_{;b}\Big] \\ &= \left(g^{ab}\delta\Gamma^c_{ab}\right)_{;c} - \left(g^{ab}\Gamma^c_{ac}\right)_{;b} \\ &= \left(g^{ab}\delta\Gamma^c_{ab}\right)_{;c} - \left(g^{ac}\delta\Gamma^d_{ad}\right)_{;c} \\ &= \left(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^d_{ad}\right)_{;c} = V^c_{;c} = div\, V, \end{aligned}$$

with $V^c = g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^d_{ad}$. ■

Lemma 5.29 *We have* $g_{ab}\,\delta g^{ab} = -g^{ab}\delta g_{ab}$.

Proof. Apply δ to $g^{ab}g_{ab} = 1$. ∎

Proposition 5.30 *Consider the Lagrangian equal to the scalar curvature, i.e.,* $L = R = g^{ij}R_{ij}$*, on a compact, orientable Riemannian manifold* M*, without boundary. Then*

$$I(g) = \int_M R\,dv \tag{5.3.31}$$

is stationary under variations of the metric iff g_{ij} *obeys the field equations*

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0. \tag{5.3.32}$$

Proof. We shall show that the energy-momentum tensor is $T_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$. We have

$$\begin{aligned}
\delta I(g) &= \delta\int_M R\,dv = \int_M \delta\big(R\,dv\big) \\
&= \int_M \delta R\,dv + \int_M R\,\delta(dv) \\
&= \int_M \delta\Big(R_{ab}g^{ab}\Big)\,dv + \int_M R\frac{1}{2}g^{ab}\,\delta g_{ab}\,dv \\
&= \int_M \Big(g^{ab}\delta R_{ab} + R_{ab}\delta g^{ab} + \frac{1}{2}Rg^{ab}\delta g_{ab}\Big)\,dv \\
&= \underbrace{\int_M g^{ab}\,\delta R_{ab}\,dv}_{=0} + \int_M \Big(R_{ab}\delta g^{ab} + \frac{1}{2}Rg^{ab}\delta g_{ab}\Big)\,dv \\
&= \int_M \Big(R_{ab}\delta g^{ab} - \frac{1}{2}Rg_{ab}\delta g^{ab}\Big)\,dv \qquad (5.3.33) \\
&= \int_M \Big(R_{ab} - \frac{1}{2}Rg_{ab}\Big)\delta g^{ab}\,dv = \int_M T_{ab}\delta g^{ab}\,dv,
\end{aligned}$$

where in order to get (5.3.33) we have used Lemmas 5.28 and 5.29. ∎

The equation (5.3.32) is called the *Einstein equation* and the integral $I(g)$ given by (5.3.31) is called *Einstein's gravitational potential.*

Solving the Einstein equation. We distinguish two cases depending on the dimension of the manifold: $n = 2$ and $n \neq 2$.

The case $n \neq 2$*:* The Einstein equation

$$R_{ij} = \frac{1}{2}Rg_{ij} \tag{5.3.34}$$

yields

$$R_j^k = g^{ik}R_{ij} = \frac{1}{2}g^{ik}Rg_{ij} = \frac{1}{2}R\delta_j^k.$$

In particular, $R_j^j = \frac{1}{2}R\delta_j^j = \frac{1}{2}R$. Then summing over j yields

$$R = R_j^j = \frac{1}{2}R\delta_j^j = \frac{n}{2}R,$$

and hence $\left(\frac{n}{2} - 1\right)R = 0$. As $n \neq 2$ it follows that $R = 0$. Using (5.3.34) yields $R_{ij} = 0$.

The case $n = 2$: This is a special case which leads to the following well-known theorem:

Theorem 5.31. *(Gauss–Bonnet theorem)*
Let M be a compact surface in $\mathbb{R}^3$ and K the Gaussian curvature. Then

$$\int_M K\, d\sigma \tag{5.3.35}$$

does not depend on the Riemannian metric considered on M.

Proof. In the 2-dimensional case, $K = R/2$, and

$$R_{ij} = \frac{1}{2}Rg_{ij} = Kg_{ij}.$$

Using Proposition 5.30 we prove

$$\delta I = \int_M T_{ij}\delta g^{ij}\, d\sigma = \int_M \big(R_{ij} - \frac{1}{2}Rg_{ij}\big)\delta g^{ij}\, d\sigma = 0.$$

Let $\mathcal{RM}(M)$ denote the space of Riemannian metrics on M. I is a functional on $\mathcal{RM}(M)$ such that $\delta I_{|g} = 0$, for any metric g. Hence I is constant on $\mathcal{RM}(M)$ and does not depend on g. ■

In fact $\int K\, d\sigma$ is a topological invariant equal to $2\pi\chi(M)$, where $\chi(M)$ denotes the Euler–Poincare characteristic of M, which is a positive integer. Lagrangians that provide integral invariants are called *null Lagrangians*, see [31]. The following proposition deals with integral invariants.

Proposition 5.32 *Let f be a smooth function that depends on the metric tensor g_{ab}. Then the integral*

$$\int_M f(g_{ab})\, dv$$

is an integral invariant (not changing with variations of the metric) iff f satisfies the equation

$$\frac{\partial f}{\partial g_{ab}} + \frac{1}{2}g^{ab}f = 0.$$

Proof. We use the fact that the energy-momentum tensor is $T^{ab} = \dfrac{\partial f}{\partial g_{ab}} + \dfrac{1}{2} g^{ab} f$ and it is zero for any metric g_{ab}. ■

The field equations for the volume functional
Let (M, g) be a compact, orientable Riemannian manifold. Consider the volume functional

$$V(g) = \int_M dv = \int_M \sqrt{|g|} dx_1 \wedge \cdots \wedge dx_n.$$

A variation with respect to g yields

$$\begin{aligned} \delta V(g) &= \int_M \delta(dv) = \int_M \frac{1}{2} g^{ab} \delta g_{ab} \\ &= \int_M T^{ab} \delta g_{ab}\, dv. \end{aligned}$$

Hence the energy-momentum in this case is $T = \dfrac{1}{2} g$, and hence T is divergence free and the field equations are $g_{ij} = 0$.

The energy-momentum for the Newtonian potential
We shall compute the energy-momentum in the case of Newtonian potential in dimensions $n = 2, 3$.

Case n=2: Consider the Newtonian potential in two dimensions $\phi(x) = \ln |x|$, where $x = (x_1, x_2)$. As $\Delta\phi(x) = 0, \ \forall x \neq 0$, then $\phi(x)$ is an extremizer for the Dirichlet functional

$$\int_{\mathcal{D}} \frac{1}{2} |\nabla\phi|^2 \ dx_1 dx_2, \quad \text{if } 0 \notin \mathcal{D}. \tag{5.3.36}$$

The energy-momentum tensor is

$$\begin{aligned} T^{ab} &= \frac{\partial L}{g_{ab}} + \frac{1}{2} g^{ab} L = \frac{\partial \big[\frac{1}{2} g_{ab} (\nabla\phi)^a (\nabla\phi)^b\big]}{\partial g_{ab}} + \frac{1}{2} g^{ab} \ \frac{1}{2} |\nabla\phi|^2 \\ &= \frac{1}{2}\big[(\nabla\phi)^a (\nabla\phi)^b + \frac{1}{2} g^{ab} \, |\nabla\phi|^2\big]. \end{aligned}$$

Then

$$\begin{aligned} T_{ij} &= g_{ia} g_{jb} T^{ab} = \frac{1}{2}\Big(g_{ia} g^{ak} \phi_{;k} g_{jb} g^{br} \phi_{;r} + \frac{1}{2} g_{ij} |\nabla\phi|^2\Big) \\ &= \frac{1}{2}\Big(\phi_{;i}\phi_{;j} + \frac{1}{2} g_{ij} |\nabla\phi|^2\Big). \end{aligned}$$

In our case, the metric on $\mathbb{R}^2$ is the standard one, so that

$$T_{ab} = \frac{1}{2}\big[\phi_{;a}\phi_{;b} + \frac{1}{2}\delta_{ab} \, |\nabla\phi|^2\big], \quad a, b \in \{1, 2\}. \tag{5.3.37}$$

The energy-momentum tensor corresponding to $\phi = \ln |x|$ is

$$T_{ab} = \frac{1}{2|x|^2} \begin{pmatrix} \frac{x_1^2}{|x|^2} + \frac{1}{2} & \frac{x_1 x_2}{|x|^2} \\ \frac{x_1 x_2}{|x|^2} & \frac{x_2^2}{|x|^2} + \frac{1}{2} \end{pmatrix},$$

which can be written in polar coordinates as

$$\begin{aligned} T_{ab} &= \frac{1}{2r^2} \begin{pmatrix} \cos\phi^2 + \frac{1}{2} & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin\phi^2 + \frac{1}{2} \end{pmatrix} \\ &= \frac{1}{4r^2} \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} + \frac{1}{2r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Case n=3: Consider the Newtonian potential in three dimensions $\phi(x) = \dfrac{1}{|x|}$, where $x = (x_1, x_2, x_3)$. As $\Delta\phi(x) = 0, \ \forall x \neq 0$, then $\phi(x)$ is an extremizer for the Dirichlet functional. The energy-momentum tensor has the components given by (5.3.37). A computation provides

$$T_{ab} = \frac{1}{2|x|^2} \begin{pmatrix} \frac{x_1^2}{|x|^4} + \frac{1}{2} & \frac{x_1 x_2}{|x|^4} & \frac{x_1 x_3}{|x|^4} \\ \frac{x_2 x_1}{|x|^4} & \frac{x_2^2}{|x|^4} + \frac{1}{2} & \frac{x_2 x_3}{|x|^4} \\ \frac{x_3 x_1}{|x|^4} & \frac{x_3 x_2}{|x|^4} & \frac{x_3^2}{|x|^4} + \frac{1}{2} \end{pmatrix}.$$

5.3.4 Divergence of the energy-momentum tensor

We have already shown that the Einstein tensor has divergence zero. The goal of this section is to prove that, in general, an energy-momentum tensor is divergence free. This result will be used later in the proof of the conservation theorems. We shall use L for the Lagrangian and $\mathcal{L}$ for the Lie derivative.

Lemma 5.33 *If $\mathcal{L}_X$ denotes the Lie derivative with respect to vector field X, then*

$$\mathcal{L}_X g_{ab} = X_{a;b} + X_{b;a}. \tag{5.3.38}$$

Proof. Applying the formula for the Lie derivative in local coordinates, we get

$$\begin{aligned} \mathcal{L}_X g_{ab} &= \frac{\partial g_{ab}}{\partial x^i} X^i + g_{\alpha b} \frac{\partial X^\alpha}{\partial x^a} + g_{a\beta} \frac{\partial X^\beta}{\partial x^b} \\ &= \frac{\partial g_{ab}}{\partial x^i} X^i + g_{\alpha b}\Big(X^\alpha_{;a} - X^i \Gamma^\alpha_{ia}\Big) + g_{a\beta}\Big(X^\beta_{;b} - X^i \Gamma^\beta_{ib}\Big) \end{aligned}$$

$$= X^i\left(\frac{\partial g_{ab}}{\partial x^i} - g_{\alpha b}\Gamma^\alpha_{ia} - g_{a\beta}\Gamma^\beta_{ib}\right) + g_{\alpha b}X^\alpha_{;a} + g_{a\beta}X^\beta_{;b}$$
$$= X^i g_{ab;i} + g_{\alpha b}X^\alpha_{;a} + g_{a\beta}X^\beta_{;b}.$$

Using $g_{ab;i} = 0$ we obtain the desired result.

■

Lemma 5.34 *If*

$$\int_\Omega T^{ab}\,\mathcal{L}_X g_{ab}\,dv = 0, \quad \forall X \in \mathcal{X}(M), \tag{5.3.39}$$

then $T^{ab}_{;\,b} = 0$.

Proof. Using Lemma 5.33 and the divergence theorem yields

$$\begin{aligned} 0 &= \int_\Omega T^{ab}\,\mathcal{L}_X g_{ab}\,dv = 2\int_\Omega T^{ab}\,X_{a;b}\,dv \\ &= 2\int_\Omega \left(T^{ab}X_a\right)_{;b} dv - 2\int_\Omega T^{ab}_{;b}X_a\,dv \\ &= -2\int_\Omega T^{ab}_{;b}X_a\,dv \end{aligned}$$

for every field X, so that $T^{ab}_{;b} = 0$. ■

Theorem 5.35. *If L is a Lagrangian on M, which depends on ϕ^k, $\phi^k_{;i}$, and g_{ij}, where ϕ satisfies the Euler–Lagrange equations, then the energy-momentum tensor T_{ij} associated with the Lagrangian L is divergence free, i.e., $T^{ij}_{;j} = 0$.*

Proof. Consider $f : M \to M$, a diffeomorphism such that $f(\Omega) = \Omega$ and $f_{|M\setminus\Omega}$ is the identity. As the integral is not affected by a coordinate transformation,

$$\int_\Omega L\,dv = \int_{f(\Omega)} L\,dv = \int_\Omega f^*\left(Ldv\right),$$

and then

$$\int_\Omega Ldv - f^*(Ldv) = 0.$$

Using the definition of the Lie derivative,

$$\int_\Omega \mathcal{L}_X(Ldv) = 0,$$

where X is the vector field associated with the diffeomorphism f. The chain rule yields

$$
\begin{aligned}
0 = \mathcal{L}_X \int_\Omega L\, dv &= \int_\Omega T^{ab} \mathcal{L}_X g_{ab}\, dv + \int_\Omega \Big[\frac{\partial L}{\partial \phi^k}\, \mathcal{L}_X \phi^k + \frac{\partial L}{\partial \phi^k_{;i}} \mathcal{L}_X \phi^k_{;i} \Big]\, dv \\
&= \int_\Omega T^{ab} \mathcal{L}_X g_{ab}\, dv + \int_\Omega \Big[\frac{\partial L}{\partial \phi^k} - \Big(\frac{\partial L}{\partial \phi^k_{;i}} \Big)_{;i} \Big] \mathcal{L}_X \phi^k\, dv \\
&\quad + \int_\Omega \Big(\frac{\partial L}{\partial \phi^k_{;i}} \mathcal{L}_X \phi^k \Big)_{;i}\, dv.
\end{aligned}
\tag{5.3.40}
$$

The second integral vanishes because of the Euler–Lagrange equations. The last integral vanishes due to the divergence theorem. Then equation (5.3.40) yields

$$\int_\Omega T^{ab} \mathcal{L}_X g_{ab}\, dv = 0.$$

By Lemma 5.34 we obtain that T^{ab} is divergence free. ■

Remark 5.36 *The fact that T_{ij} is divergence free is a consequence of the Euler-Lagrange equations. If ϕ is not an extremizer for $\int L\, dv$, then $T^{ij}_{;j} = 0$ is not necessarily true.*

5.3.5 Conservation Theorems

This section presents two conservation theorems. The first uses a global unit Killing vector field. The second theorem doesn't need a Killing vector field but has only a local behavior.

The second conservation theorem has a nice intuitive interpretation. If the manifold is a disk D in the plane and the Lagrangian is $L = \frac{1}{2}\, |\nabla\phi|^2$, ϕ will be a harmonic potential. The physical model is a drum where ϕ is interpreted as the elastic potential and T_{ij} is the strength tensor in the drum. As the drum is strengthened in all directions (no compression), the tensor T_{ij} is positive definite, *i.e.*, $T(X, X) \ge 0$, for all directions X. When the strength on the boundary of the drum is zero, then the strength is vanishing everywhere in the drum. This resembles the min-max theorem for the Laplacian.

Lemma 5.37 *If K is a Killing vector field, then the vector F whose components are $F^a = T^{ab} K_b$ is divergence free.*

Proof.

$$div\, F = F^a_{;a} = T^{ab}_{;a} F_b + T^{ab} K_{b;a}.$$

Both terms on the right-hand side are zero. The first vanishes because T^{ab} is free divergence and the second because of the symmetry of T^{ab} and the property of K,

$$
\begin{aligned}
T^{ab} K_{b;a} &= \frac{1}{2} \Big(T^{ab} K_{b;a} + T^{ba} K_{a;b} \Big) \\
&= \frac{1}{2} T^{ab} \big(K_{b;a} + K_{b;a} \big) = \frac{1}{2} T^{ab}\, \mathcal{L}_K\, g_{ab} = 0.
\end{aligned}
$$

■

Theorem 5.38. *Let $\mathcal{U}$ be a compact, orientable region of a Riemannian manifold M, which can be written as a direct product $\mathcal{U} = [a, b] \times V$, where* $\dim V = \dim M - 1$. *Consider that the tangent vector field to the one-dimensional fibres $[a, b] \times \{u\}$, $u \in V$ is a unitary Killing vector field K normal to $\{t\} \times V$, $\forall t \in [a, b]$. If $T_{ij\,|\partial\mathcal{U}} = 0$ and $T(K, K) \geq 0$, then $T(K, K) = 0$.*

Proof. K is the unit normal vector to the surfaces $\mathcal{H}(t) = \{t\} \times V$, see Figure 5.3. Let $\mathcal{U}(t) = \bigcup_{t' \leq t} \mathcal{H}(t) \cap \mathcal{U} = [a, t] \times V$. Let $F^a = T^{ab} K_b$, Fubini's and divergence theorem yield

$$\begin{aligned} 0 \leq \int_{\mathcal{U}(t)} T(K, K)\, dv &= \int_{\mathcal{U}(t)} T^{ab} K_b K_a\, dv \\ &= \int_a^t \left(\int_{\mathcal{H}(t')} F^a K_a\, d\sigma \right) dt' = \int_a^t \left(\int_{\mathcal{H}(t')} F^a\, d\sigma_a \right) dt' \\ &= \int_a^t \left(\int_{\partial\mathcal{U}(t')} F^a\, d\sigma_a \right) dt' = \int_a^t \left(\int_{\mathcal{U}(t')} div\, F\, dv \right) dt' = 0, \end{aligned}$$

as F is divergence free and F vanishes on $\partial\mathcal{U}(t') \backslash \mathcal{H}(t')$. Therefore,

$$\int_{\mathcal{U}(t)} T(K, K)\, dv = 0, \quad \text{and hence,} \quad T(K, K) = 0.$$

■

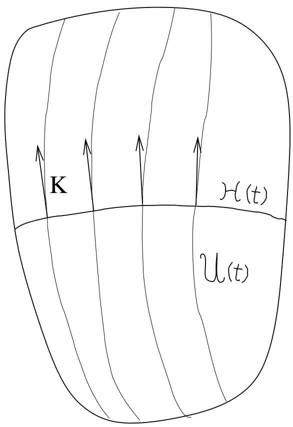


Figure 5.3: The space $\mathcal{U} = [a, b] \times V$.

Definition 5.39 *T_{ij} is called positive definite if $T(X, X) \geq 0$, $\forall X$. T_{ij} is called non-degenerate if $T(X, X) \neq 0$, $\forall X \neq 0$.*

Corollary 5.40 *Assume that the energy-momentum tensor T_{ij} in the hypothesis of Theorem 5.38 is positive and non-degenerate on $\mathcal{U}$. Then $T_{ij} = 0$ on $\mathcal{U}$.*

In order to prove the second conservation theorem we need:

Lemma 5.41 *If $\mathcal{U}$ is an orientable compact region of a Riemannian manifold, and T^{ab} denotes the energy-momentum tensor, then for any vector field X,*

$$\int_{\partial\mathcal{U}} T^{ab} X_a \, d\sigma_a = \int_U T^{ab} X_{a;b} \, dv.$$

Proof. By the divergence theorem

$$\begin{aligned}\int_{\partial\mathcal{U}} T^{ab} X_a \, d\sigma_b &= \int_U \left(T^{ab} X_a\right)_{;b} dv \\ &= \int_U \big(\underbrace{T^{ab}_{;b}}_{=0} X_a + T^{ab} X_{a;b}\big) \, dv = \int_U T^{ab} X_{a;b} \, dv.\end{aligned}$$

■

Lemma 5.42 *(Gronwall) Let f and g be continuous and nonnegative functions on $[a, b]$, and let $C \geq 0$. Suppose that*

$$f(x) \leq C + \int_a^x f(u) g(u) \, du, \qquad a \leq x \leq b.$$

Then

$$f(x) \leq C e^{\int_a^x g(u) \, du}.$$

In particular, when $C = 0$, then $f = 0$.

Proof: See, for instance, Hartman [20].

Lemma 5.43 *Let T^{ab} be a positive definite, non-degenerate energy-momentum tensor defined on $\mathcal{U}$, such that $T^{ab}_{|\partial\mathcal{U}} = 0$. Then for any vector field X, there is a constant $M > 0$ such that*

$$T^{ab} X_{a;b} \leq M \, T^{ab} X_a X_b.$$

Proof. The functions $f_1 = T^{ab} X_{a;b}$ and $f_2 = T^{ab} X_a X_b$ are continuous on $\mathcal{U}$ and vanish on $\partial\mathcal{U}$. The functions $|f_1|$ and f_2 are bounded and nonnegative on $\mathcal{U}$. The zeros of f_2 are among the zeros of $|f_1|$. Hence, there is a continuous positive function g such that $|f_1| \leq g \cdot f_2$. Take $M = \max g$. ■

Theorem 5.44. *(Conservation theorem) Let M be an orientable Riemannian manifold and T_{ij} a positive definite, non-degenerate energy-momentum tensor. Then $\forall x \in M$, there is a compact neighborhood $\mathcal{U}$ of x such that*

$$\text{if} \quad T_{ij\,|\partial\mathcal{U}} = 0, \qquad \text{then} \quad T_{ij} = 0 \text{ on } \mathcal{U}.$$

Proof. Consider all the unit speed geodesics c_v starting at the point x and define the surfaces

$$\mathcal{H}(t) = \{c_v(t);\, v \in T_xM\}, \quad 0 \le t \le \tau < \tau_1,$$

where $\tau_1 := \inf\{t;\, c_v(t) \text{ is conjugate to } x = c_v(0), \forall v \in T_xM\}$. Define $\mathcal{U}(t) = \bigcup_{0\le t'\le t} \mathcal{H}(t')$. Let X be the geodesic vector field along the above geodesic flow. X is the normal vector field to $\mathcal{H}(t)$. Denote

$$f(t) = \int_{\mathcal{U}(t)} T(X,X)\,dv \ge 0.$$

Applying Fubini's theorem and Lemma 5.41 yields

$$\begin{aligned} f(t) &= \int_{\mathcal{U}(t)} T^{ab}X_aX_b\,dv = \int_0^t \int_{\mathcal{H}(t')} \big(T^{ab}X_aX_b\,d\sigma\big)\,dt' \\ &= \int_0^t \Big(\int_{\mathcal{H}(t')} T^{ab}X_a\,d\sigma_b\Big)\,dt' = \int_0^t \Big(\int_{\mathcal{U}(t')} T^{ab}X_{a;b}\,dv\Big)\,dt'. \end{aligned}$$

By Lemma 5.43, there is a constant $M > 0$ such that

$$T^{ab}X_{a;b} \le M\,T(X,X) \qquad \text{on } \mathcal{U},$$

and hence (5.3.41) yields

$$f(t) \le M \int_0^t f(t')\,dt'.$$

By Lemma 5.42, we obtain $f(t) = 0$ and since $X \neq 0$, it follows that $T_{ij} = 0$ on $\mathcal{U}$. ■

Remark 5.45 *If the manifold M has negative curvature, the above local property becomes a global one.*

From the physical point of view, the vanishing of T_{ij} in a region $\mathcal{U}$ means the absence of the matter field in that region. The last theorem states that if there is no matter field on the boundary, then there is no matter field in the interior. This can be interpreted saying that the matter field cannot have a compact support, being surrounded by a vacuum (see [21]).

5.3.6 Applications of the conservation theorems

We shall consider in this section a few Lagrangians which depend on the scalar field, its first derivative, and the Riemannian metric. The scalar field satisfies the Euler–Lagrange equation. The conservation properties of the energy-momentum tensor can help to obtain information about the solutions of the Euler–Lagrange equations.

In the following theorems, $\mathcal{U}$ denotes

- a small enough, connected neighborhood of the given point $x \in M$,
- any connected neighborhood of the given point $x \in M$, provided M has negative curvature,
- any connected neighborhood of the given point $x \in \mathbb{R}^n$.

1. Laplace equation. Consider the Lagrangian $L = \frac{1}{2} |\nabla\phi|^2$, where $\phi : (M, g) \to \mathbb{R}$ satisfies the Euler–Lagrange equation $\Delta\phi = 0$. The energy-momentum tensor

$$T_{ab} = \frac{1}{2}\big[\phi_{;a}\phi_{;b} + \frac{1}{2} |\nabla\phi|^2\big]$$

is positive definite because

$$\begin{aligned} T_{ab}X^aX^b &= \frac{1}{2}\big[X^a\phi_{;a}X^b\phi_{;b} + \frac{1}{2}g_{ab}X^aX^b |\nabla\phi|^2\big] \\ &= \frac{1}{2}\big[X(\phi)^2 + \frac{1}{2} |\nabla\phi|^2\big] \geq 0. \end{aligned}$$

Theorem 5.46. *The boundary problem*

$$\begin{aligned} \Delta\phi &= 0 \text{ on } U, \\ \frac{\partial\phi}{\partial x_i} &= 0 \text{ on } \partial U, \end{aligned}$$

has the solution $\phi = constant$.

Proof. Applying the conservation theorem, we get $T(X, X) = 0$ and hence $\phi =$ constant. ∎

2. Nonlinear Poisson equation. For the Lagrangian

$$L = \frac{1}{2} |\nabla\phi|^2 + \frac{\lambda^2}{2p}\phi^{2p},$$

with $p \in \mathbb{N}$, the Euler–Lagrange equation is

$$\Delta\phi = -\lambda^2\phi^{2p-1}.$$

The energy-momentum tensor

$$T_{ab} = \frac{1}{2}\big[\phi_{;a}\phi_{;b} + \frac{1}{2}g_{ab}(|\nabla\phi|^2 + \frac{\lambda^2}{p}\phi^{2p})\big]$$

is positive definite and non-degenerate. Using the conservation theorem, we get the following:

Theorem 5.47. *The boundary problem*

$$\Delta\phi = -\lambda^2\phi^{2p-1} \text{ on } U,$$
$$\phi = 0 \text{ on } \partial U,$$
$$\frac{\partial}{\partial x_i}\phi = 0 \text{ on } \partial U,$$

has the solution $\phi = 0$.

3. Harmonic maps. The Lagrangian for a harmonic map $\phi : (M, g) \to (N, h)$ is the energy density

$$e(\phi) = \frac{1}{2} g^{ab}\phi^i_{;a}\phi^j_{;b}h_{ij} = \frac{1}{2}(\nabla\phi^i)^a\,(\nabla\phi^j)^b g_{ab}h_{ij}.$$

The energy-momentum tensor is given by

$$\begin{aligned} T^{ab} &= \frac{\partial e(\phi)}{\partial g_{ab}} + \frac{1}{2}g^{ab}e(\phi) \\ &= \frac{1}{2}(\nabla\phi^i)^a(\nabla\phi^j)^b h_{ij} + \frac{1}{2}g^{ab}e(\phi) \\ &= \frac{1}{2}g^{ka}g^{rb}\phi^i_{;k}\phi^j_{;r}h_{ij} + \frac{1}{2}g^{ab}e(\phi) \\ &= \frac{1}{2}g^{ka}g^{rb}\left(\phi^*h\right)_{kr} + \frac{1}{2}g^{ab}e(\phi) \\ &= \frac{1}{2}\left(\phi^*h\right)^{ab} + \frac{1}{2}g^{ab}e(\phi). \end{aligned}$$

Hence the energy-momentum tensor can be expressed invariantly as

$$T = \frac{1}{2}(\phi^*h + g\,e(\phi)).$$

For every vector field X we have

$$T(X, X) = \frac{1}{2}\,|\phi_* X|^2_h + \frac{1}{2}\,|X|^2_g\,e(\phi) \geq 0.$$

The conservation theorem yields:

Theorem 5.48. *Let* $\phi : M \to N$ *be a harmonic map such that* $\phi_{;k} = 0$ *on* $\partial\mathcal{U}$. *Then* ϕ *is constant on* $\mathcal{U}$.

Proof. From the conservation theorem, $T(X, X) = 0$, then $e(\phi) = 0$ and hence ϕ is constant on $\mathcal{U}$. ∎

In the following we shall provide more applications of the conservation theorems for some special cases of harmonic maps.

Lemma 5.49 *Let $\phi : (M, g) \to (N, h)$ be a map, with M connected manifold. Then ϕ is constant iff the associated energy-momentum tensor T is trace free, i.e., $g^{ij} T_{ij} = 0$.*

Proof. We shall prove only the non-obvious implication. Let m =dim M.

$$\begin{aligned} Trace_g\, T &= \frac{1}{2} Trace_g \left(\phi^* h + g e(\phi)\right) \\ &= \frac{1}{2} Trace_g \left(\phi^* h\right) + \frac{1}{2} g^{ij} g_{ij} e(\phi) \\ &= e(\phi) + \frac{1}{2} m e(\phi) = \frac{m+2}{2} e(\phi). \end{aligned}$$

Let $p \in M$ and $\{e_1, e_2, \ldots, e_m\} \subset T_p M$ be an orthonormal basis. Then

$$0 = Trace_g\, T = \frac{m+2}{2} e(\phi) = \frac{m+2}{4} \sum_{k=1}^{m} |\phi_*(e_i)|_h^2,$$

and hence $\phi_*(e_i) = 0$, for any $p \in M$ and $i = 1, \ldots, m$. As M is connected, ϕ is constant. ■

Corollary 5.50 *If the energy-momentum $T = 0$, then ϕ is constant.*

Conformal maps.

Definition 5.51 *A map $\phi : (M, g) \to (N, h)$ is called (weakly) conformal if there is a function $\rho \in \mathcal{F}(M)$, $\rho \geq 0$ such that $\phi^* h = \rho \cdot g$. If the function ρ is constant, the map ϕ is called homothetic.*

The following result can be found also in [16].

Theorem 5.52. *Let $\phi : (M, g) \to (N, h)$ be a harmonic conformal map. Then ϕ is homothetic.*

Proof. Taking trace yields

$$\begin{aligned} e(\phi) &= \frac{1}{2} Trace_g \phi^* h = \frac{1}{2} Trace_g (\rho g) \\ &= \frac{\rho}{2} g^{ij} g_{ij} = \frac{m}{2} \rho. \end{aligned}$$

The energy-momentum tensor becomes

$$\begin{aligned} T &= \frac{1}{2}\big(\phi^* h + g e(\phi)\big) = \frac{1}{2}\big(\rho g + g \frac{m}{2} \rho\big) \\ &= \frac{1}{2}\big(1 + \frac{m}{2}\big) \rho g. \end{aligned}$$

As ϕ is harmonic, the tensor T is divergence free $T^{ij}_{;j} = 0$. Then

$$0 = \big(\rho g\big)^{ij}_{;j} = \rho_j g^{ij} + \rho \underbrace{g^{ij}_{;j}}_{=0} = \big(\nabla \rho\big)^i.$$

Hence $\nabla \rho = 0$ on M. As M is a connected manifold, it follows that ρ is constant and hence ϕ is homothetic. ■

Isometric immersions. Let $\phi : (M, g) \to (N, h)$ be an isometric immersion. Then $g = \phi^* h$ and hence

$$e(\phi) = \frac{1}{2} Trace_g \phi^* h = \frac{1}{2} g^{ij} g_{ij} = \frac{m}{2}.$$

The energy-momentum tensor becomes

$$T = \frac{1}{2}\Big(g + g\frac{m}{2}\Big) = \frac{1}{2}\Big(1 + \frac{m}{2}\Big)g.$$

The conservation theorem is satisfied

$$div\, T = \frac{1}{2}\Big(1 + \frac{m}{2}\Big) g^{ij}_{;j} = 0.$$

Geodesic curves. Consider $\dim M = 1$. Then $\phi : (M, g) \to (N, h)$ is a curve. The energy density in this case is

$$e(\phi) = \frac{1}{2}\phi^i_{;1}\phi^j_{;1}h_{ij} = \frac{1}{2}\dot{\phi}^i\dot{\phi}^j h_{ij} = \frac{1}{2}|\dot{\phi}|^2_h.$$

We also have $g = g_{11} = 1$ and

$$\begin{aligned}\phi^* h = \big(\phi^* h\big)_{11} &= \phi^i_{;1}\phi^j_{;1}h_{ij} \\ &= \dot{\phi}^i\dot{\phi}^j h_{ij} = |\dot{\phi}|^2_h.\end{aligned}$$

The energy-momentum becomes

$$T = \frac{1}{2}\Big(|\dot{\phi}|^2_h + \frac{1}{2}|\dot{\phi}|^2_h\Big) = \frac{3}{4}|\dot{\phi}|^2_h. \tag{5.3.41}$$

If ϕ is a geodesic, then $|\dot{\phi}|_h$ is constant and the conservation theorem div $T = 0$ is obviously satisfied. Let $c : (0, +\infty) \to N$ be a geodesic. If $T_{\phi(0)} = 0$, then $T_{\phi(t)} = 0$, for any $t \geq 0$. This is a consequence of (5.3.41).

Totally geodesic maps.

Definition 5.53 *A map $\phi : (M, g) \to (N, h)$ is called totally geodesic if its second fundamental form is zero, i.e., $\nabla d\phi = 0$, where $\nabla d\phi$ is the symmetric 2-covariant tensor field defined by*

$$\nabla d\phi(X, Y) = \nabla_X(d\phi)(Y) = \nabla_X d\phi(Y) - \big(\nabla_X Y\big)(\phi), \qquad \forall X, Y \in \mathcal{X}(M).$$

The following three results can be found in [16].

Lemma 5.54 *Let $\phi : (M, g) \to (N, h)$ be a totally geodesic map. Then for any $X \in \mathcal{X}(M)$ we have $\nabla_X(\phi^*h) = 0$.*

Proof. Let $p \in M$ and $X, Y, Z \in T_pM$. Extend Y and Z around p such that $\nabla_X Y = 0 = \nabla_X Z$ at p. Then at p we have

$$\begin{aligned}
\big(\nabla_X\phi^*h\big)(Y, Z) &= \nabla_X\phi^*h(Y, Z) - \phi^*h(\underbrace{\nabla_X Y}_{=0}, Z) - \phi^*h(Y, \underbrace{\nabla_X Z}_{=0}) \\
&= \nabla_X\phi^*h(Y, Z) = \nabla_X h(d\phi(Y), d\phi(Z)) \\
&= h\big(\nabla_X d\phi(Y), d\phi(Z)\big) + h\big(d\phi(Y), \nabla_X d\phi(Z)\big) \\
&= h\big(\nabla_X d\phi(Y) - \underbrace{(\nabla_X Y)}_{=0}\phi, d\phi(Z)\big) + h\big(d\phi(Y), \nabla_X d\phi(Z) - \underbrace{(\nabla_X Z)}_{=0}\phi\big) \\
&= h\big(\underbrace{\nabla d\phi(X, Y)}_{=0}, d\phi(Z)\big) + h\big(d\phi(Y), \underbrace{\nabla d\phi(X, Z)}_{=0}\big) = 0.
\end{aligned}$$

■

Proposition 5.55 *Let $\phi : (M, g) \to (N, h)$ be a totally geodesic map, with M a connected manifold. Then the energy density $e(\phi)$ is a constant function.*

Proof. Differentiating covariantly in the expression $e(\phi) = \frac{1}{2}g^{ij}(h^*\phi)_{ij}$ yields

$$e(\phi)_{;k} = \frac{1}{2}g^{ij}_{;k}(h^*\phi)_{ij} + \frac{1}{2}g^{ij}(h^*\phi)_{ij;k} = 0.$$

The first term in the right side is zero because g is a metric connection and the second term vanishes because of Lemma 5.54 written in local coordinates. ■

Theorem 5.56. *Let $\phi : (M, g) \to (N, h)$ be a totally geodesic map, with M a connected manifold. Then the energy-momentum tensor is divergence free.*

Proof. Lemma 5.54 and Proposition 5.55 yield

$$T^{ij}_{;j} = \frac{1}{2}\big((\phi^*h)^{ij} + e(\phi)g^{ij}\big)_{;j} = \frac{1}{2}\big((\phi^*h)^{ij}_{;j} + e(\phi)g^{ij}_{;j}\big) = 0.$$

■

4. p-harmonic functions

Definition 5.57 *Let (M, g) be a Riemannian manifold and $\phi : M \to \mathbb{R}$ be a differentiable function. For each $p > 0$, define the p-energy of ϕ with respect to a compact set $U \subset M$ by*

$$E_p(\phi, U) = \int_U \Big(\frac{1}{2}|\nabla\phi|^2\Big)^p dv = \frac{1}{2^p}\int_U |\nabla\phi|^{2p}\, dv.$$

The extremizers for the energy E_p are called p-harmonic functions on (M, g).

The associated Euler–Lagrange equation is

$$div\,(\,|\nabla\phi|^{2(p-1)}\ \nabla\phi\,) = 0. \tag{5.3.42}$$

This can be checked by taking the Lagrangian

$$L = \frac{1}{2^p}\,|\nabla\phi|^{2p} = \frac{1}{2^p}(g^{ij}\phi_{\,;i}\phi_{\,;j})^p$$

and differentiating

$$\begin{aligned}\frac{\partial L}{\partial\phi_{;k}} &= \frac{p}{2^{(p-1)}}\,|\nabla\phi|^{2(p-1)}\,(\nabla\phi)^k = (|\nabla\phi|^{2p-2}\ \nabla\phi)^k\frac{p}{2^{p-1}},\\ \left(\frac{\partial L}{\partial\phi_{;k}}\right)_{;k} &= \frac{p}{2^{p-1}}\Big(|\nabla\phi|^{2p-2}\,\nabla\phi\Big)^k_{;k} = \frac{p}{2^{p-1}}div\,(|\nabla\phi|^{2p-2}\,\nabla\phi),\end{aligned}$$

and applying the Euler–Lagrange equation

$$\left(\frac{\partial L}{\partial\phi_{;k}}\right)_{;k} = \frac{\partial L}{\partial\phi}$$

we get the equation (5.3.42).

Remark 5.58 *For $p \neq 1$, equation (5.3.42) is nonlinear. The left side is called p-Laplacian.*

Proposition 5.59 *The energy-momentum tensor for $E_p(\phi)$ is given by*

$$T_{ij} = \frac{p}{2^p}\,|\nabla\phi|^{2(p-1)}\ \ (\phi_{;i}\phi_{;j} + \frac{1}{2p}\,|\nabla\phi|^2\,g_{ij}). \tag{5.3.43}$$

Proof. Let $e(\phi) = \frac{1}{2}\,|\nabla\phi|^2$. Then

$$\delta E_p(\phi) = \int_U \delta(e(\phi)^p\sqrt{g})\,dx = \int_U\Big[\delta(e\big(\phi)^p\big)\sqrt{g} + e(\phi)^p\ \delta(\sqrt{g})\Big]\,dx.$$

Using

$$\delta\big(e(\phi)^p\big) = p\,e(\phi)^{p-1}\ \delta e(\phi) = pe(\phi)^{p-1}\frac{1}{2}\phi_{;i}\phi_{;j}\delta g^{ij},$$

and

$$\delta(\sqrt{g}) = \frac{1}{2}g_{ij}\,\sqrt{g}\,\delta g^{ij},$$

yields

$$\begin{aligned}\delta E_p(\phi) &= \int_U\left(\frac{p}{2}e(\phi)^{p-1}\,\phi_{;i}\phi_{;j} + \frac{1}{2}e(\phi)^p g_{ij}\right)\sqrt{g}\ \delta g^{ij}\,dx\\ &= \frac{p}{2}\int_U e(\phi)^{p-1}(\phi_{;i}\phi_{;j} + \frac{1}{p}e(\phi)g_{ij})\,\delta g^{ij}\,dv.\end{aligned}$$

Replacing $e(\phi)$, we get the desired result. ■

As

$$T(X, X) = \frac{p}{2^p} |\nabla\phi|^{2(p-1)} \big(X(\phi)^2 + \frac{1}{2p} |\nabla\phi|^2 \ |X|^2\big),$$

T_{ij} is positive definite and non-degenerate. The conservation theorem yields:

Theorem 5.60. *If $p > 0$, the following boundary problem for the p-Laplacian*

$$div\,(|\nabla\phi|^{2(p-1)}\,\nabla\phi) = 0 \ on\ U,$$
$$\frac{\partial\phi}{\partial x_i} = 0 \ on\ \partial U,$$

has only constant solutions.

5. A nonlinear elliptic equation. For $\phi : M \to \mathbb{R}_+$ consider the Lagrangian

$$L = \frac{1}{2}\frac{|\nabla\phi|^2}{\phi^{2k}}$$

where $k \in \mathbb{N}$. One can verify that the Euler–Lagrange equation is

$$\phi\,\Delta\phi = k|\nabla\phi|^2. \tag{5.3.44}$$

Consider the equation (10.6.40) on the domain $\mathcal{U}$, subject to the boundary condition

$$\frac{\partial\phi}{\partial x_i}_{|\partial\mathcal{U}} = 0. \tag{5.3.45}$$

We have the following result.

Proposition 5.61 *The equation (5.3.44) with the boundary condition (5.3.45) has only constant solutions.*

Proof. The energy-momentum tensor is

$$\begin{aligned} T^{ab} &= \frac{\partial L}{\partial g_{ab}} + \frac{1}{2} g^{ab} L \\ &= \frac{1}{2}\frac{1}{\phi^{2k}}\Big((\nabla\phi)^a(\nabla\phi)^b + \frac{1}{2}g^{ab}|\nabla\phi|^2\Big). \end{aligned}$$

The tension in the X-direction is positive

$$T(X, X) = T^{ab}X_aX_b = \frac{1}{2}\frac{1}{\phi^{2k}}\Big(X(\phi)^2 + \frac{1}{2}|X|^2\,|\nabla\phi|^2\Big) \geq 0.$$

Using the conservation theorem, we get $T(X, X) = 0$. Hence, $|\nabla\phi| = 0$ and then ϕ is constant on $\mathcal{U}$. ■

For further readings about conservation laws and applications to physics, see [24], [42], [45]. For ordinary differential equations see [2] and [21].

5.4 Exercises

1. Consider the Lagrangian on $\mathbb{R}^2$ given by $L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$.
(i) Show that L is invariant by translations and rotations.
(ii) Derive conservation laws associated with each vector field in (i). They are first integrals of motion for the geodesics defined by L.

2. Consider the Lagrangian $L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \lambda(x\dot{y} - y\dot{x})$.
(i) Show that L is invariant by rotations.
(ii) Derive a first integral of motion associated with the above invariance.

3. Consider the Lagrangian that describes the dynamics on the Poincaré upper half-plane $L(x, y, \dot{x}, \dot{y}) = \dfrac{1}{2y^2}(\dot{x}^2 + \dot{y}^2)$.
(i) Show that L is invariant with respect to translations along the x-axis.
(ii) Derive the correspondent conservation law.

4. Prove the Gronwall lemma.

5. Consider the Lagrangian $L = \dfrac{1}{2}\dfrac{|\nabla\phi|^2}{\phi^{2k}}$ with $k \in \mathbb{N}$ on M. Show that the Euler–Lagrange equation is $\phi\Delta\phi = k|\nabla\phi|^2$. Solve it in the case when M is a compact manifold, without boundary and $k \geq 2$.

6. Prove the second Bianchi identity in local coordinates

$$R^i_{jkl;r} + R^i_{jlr;k} + R^i_{jrk;l} = 0.$$

7. Let (M, g) be a connected Riemannian manifold. If there is a function $f \in \mathcal{F}(M)$ such that $Ric = f \cdot g$, then the function f is constant on M.

6

Hamiltonian Formalism

This chapter deals with Hamiltonian formalism on differentiable manifolds. This is a different way to look at variational problems, using a Hamiltonian function instead of a Lagrangian. Both theories (Hamiltonian and Lagrangian) are equivalent, but in some practical problems it is easier to use one or the other. The equations for the harmonic maps, geodesics, and other applications are provided.

6.1 Momenta vector fields. Hamiltonian

Let (M, g), (N, h) be two Riemannian manifolds of dimension m and n. Consider a Lagrangian $L(\phi, \phi^k_{;j})$ associated with a map $\phi : M \to N$.

Definition 6.1 *Define a momenta matrix as*

$$p^k_j = \frac{\partial L}{\partial \phi^j_{;k}}, \qquad \textit{where} \;\; j = \overline{1, n}, \;\; k = \overline{1, m}. \tag{6.1.1}$$

Proposition 6.2 *Under a change of coordinates, momenta behave as*

$$p^s_l = \bar{p}^k_l \frac{\partial x^s}{\partial \bar{x}^k}, \tag{6.1.2}$$

where $x = (x^1 \dots x^m)$, $\bar{x} = (\bar{x}^1 \dots \bar{x}^m)$ *are two local coordinate systems on* M. *Then*

$$p_j = p^k_j \frac{\partial}{\partial x^k}, \quad j = \overline{1, n} \tag{6.1.3}$$

can be considered as vector fields on M.

Proof. Denote $\phi = \bar{\phi} \circ \chi$, where $\chi(x) = \bar{x}$. Applying the chain rule yields

$$\bar{\phi}^l_{;k} = \frac{\partial \phi^l}{\partial \bar{x}^k} = \frac{\partial \phi^l}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^k} = \phi^l_{;j} \frac{\partial x^j}{\partial \bar{x}^k}.$$

The Lagrangian becomes

$$L(\bar{\phi}, \bar{\phi}^l_{;k})_{(\bar{x})} = L(\phi, \phi^l_{;s}\frac{\partial x^s}{\partial \bar{x}^k})_{(x)},$$

and hence the momenta behave as

$$p^s_l = \frac{\partial L}{\partial \phi^l_{;s}} = \frac{\partial L}{\partial \bar{\phi}^l_{;k}}\frac{\partial x^s}{\partial \bar{x}^k} = \bar{p}^k_l\frac{\partial x^s}{\partial \bar{x}^k}.$$

■

The vector fields $p_1, \ldots, p_n$ are called *momenta vector fields*. Using momenta vector fields, the Euler–Lagrange equations

$$\left(\frac{\partial L}{\partial \phi^j_{;k}}\right)_{;k} = \frac{\partial L}{\partial \phi^j}, \qquad j = \overline{1, n} \tag{6.1.4}$$

can be written as

$$div\ p_j = \frac{\partial L}{\partial \phi^j}, \qquad \forall j = \overline{1, n}. \tag{6.1.5}$$

Suppose that L is convex in $\phi^i_{;k}$. Define the Hamiltonian

$$H : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \times \mathcal{F}(M, N) \to \mathcal{F}(M)$$

using the Legendre transform

$$H(p, \phi) = \sum_{j,k} p^j_k \phi^k_{;j} - L(\phi, \phi^i_{;k}), \tag{6.1.6}$$

where $\phi^j_{;k}$ satisfies the equation

$$p^k_j = \frac{\partial L}{\partial \phi^j_{;k}}. \tag{6.1.7}$$

Example 6.1.1 *In the particular case when $M = \mathbb{R}$, $N = \mathbb{R}^n$, $\phi : \mathbb{R} \to \mathbb{R}^n$, $\phi = (\phi^1, \ldots, \phi^n)$, the momenta are*

$$p_k = p^1_k = \frac{\partial L}{\partial \dot{\phi}^k} \tag{6.1.8}$$

and the Hamiltonian is

$$H(p, \phi) = p_k \dot{\phi}^k - L(\phi, \dot{\phi}), \tag{6.1.9}$$

where $\dot{\phi}$ verifies

$$p = \frac{\partial L}{\partial \dot{\phi}}.$$

Example 6.1.2 *When $\phi : M \to \mathbb{R}$, the momenta are*

$$p^j = p_1^j = \frac{\partial L}{\partial \phi_{;j}}, \tag{6.1.10}$$

and the Hamiltonian is

$$H(p, \phi) = \sum_j p^j \phi_{;j} - L(\phi, \phi_{;j}), \tag{6.1.11}$$

where $\phi_{;j}$ satisfy (6.1.10).

6.2 Hamilton's system of equations

Consider a map $\phi : M \to N$. Computing dH for $H(p, \phi)$ in local coordinates in two ways, we shall identify the coefficients of similar forms in these expressions. Differentiating

$$dH = \frac{\partial H}{\partial p_j^i} dp_j^i + \frac{\partial H}{\partial \phi^p} d\phi^p. \tag{6.2.12}$$

Differentiating the expression of the Hamiltonian given in (6.1.6) yields

$$dH = dp_k^j \, \phi^k_{;j} + p_k^j \, d\phi^k_{;j} - \frac{\partial L}{\partial \phi^p} d\phi^p - \frac{\partial L}{\partial \phi^k_{;j}} d\phi^k_{;j}. \tag{6.2.13}$$

Applying the definition of the momentum (6.1.1), equation (6.2.13) becomes

$$dH = \phi^k_{;j} dp_k^j - \frac{\partial L}{\partial \phi^p} d\phi^p. \tag{6.2.14}$$

Identifying the coefficients of similar form in (6.2.12) and (6.2.14) yields

$$\phi^k_{;j} = \frac{\partial H}{\partial p_k^j} \qquad \text{and} \qquad -\frac{\partial L}{\partial \phi^k} = \frac{\partial H}{\partial \phi^k}. \tag{6.2.15}$$

Applying (6.1.5), we get the system of equations

$$\begin{cases} \phi^k_{;j} = \dfrac{\partial H}{\partial p_k^j}, \\ div\ p_k = -\dfrac{\partial H}{\partial \phi^k}. \end{cases} \tag{6.2.16}$$

When $M = \mathbb{R}^n$ and $N = \mathbb{R}$, the system (6.2.16) becomes

$$\begin{cases} \nabla \phi = \nabla_p H, \\ div\ p = -\nabla_\phi H. \end{cases} \tag{6.2.17}$$

When $M = \mathbb{R}$ and $N = \mathbb{R}^n$, the system (6.2.16) can be written as

$$\begin{cases} \dot{p}_k = -\dfrac{\partial H}{\partial \phi^k}, \\ \dot{\phi}^j = \dfrac{\partial H}{\partial p_j}, \end{cases} \tag{6.2.18}$$

which is usually called Hamilton's system of equations.

Remark 6.3 *If H does not depend on ϕ, the second equation in (6.2.16) provides a conservation law of momentum*

$$div\ p_k = 0, \tag{6.2.19}$$

which says that p_k is a momentum current.

Example 6.2.1 *For $\phi : M \to \mathbb{R}$, consider the Lagrangian*

$$L(\phi, \nabla\phi) = \frac{1}{2}\,|\nabla\phi|^2 = \frac{1}{2} g^{kl} \phi_{;k} \phi_{;l}. \tag{6.2.20}$$

The associated Hamiltonian is

$$H(p, \phi) = p^j \phi_{;j} - \frac{1}{2} g^{kl} \phi_{;k} \phi_{;l},$$

where

$$p^j = \frac{\partial L}{\partial \phi_{;j}} = g^{kj} \phi_{;k} \quad and \quad \phi_{;k} = g_{kr} p^r. \tag{6.2.21}$$

Hence,

$$\begin{aligned} H(p, \phi) &= g^{kj} \phi_{;k} \phi_{;j} - \frac{1}{2} g^{kj} \phi_{;k} \phi_{;j} = \frac{1}{2} g^{kj} \phi_{;k} \phi_{;j} \\ &= \frac{1}{2} g^{kj} g_{ks} p^s g_{jr} p^r = \frac{1}{2} g_{sr} p^s p^r = \frac{1}{2} |p|^2, \end{aligned}$$

where

$$p = p^s \frac{\partial}{\partial x^s}.$$

Hence,

$$H(p, \phi) = \frac{1}{2} g(p, p) = \frac{1}{2}\,|p|^2\,. \tag{6.2.22}$$

6.3 Harmonic functions

Now we shall find the harmonic functions equation using Hamiltonian formalism. Consider the Hamiltonian (6.2.22). As H does not depend on ϕ, $div\ p = 0$. Using (6.2.18), we have

$$div\ p = p^j_{;j} = (g^{kj} \phi_{;k})_{;j} = \underbrace{g^{kj}_{;j}}_{=0} \phi_{;k} + g^{kj} \phi_{;kj}. \tag{6.3.23}$$

Since

$$\phi_{;kj} = \frac{\partial^2 \phi}{\partial x^k \partial x^j} - \Gamma^r_{kj} \frac{\partial \phi}{\partial x^r},$$

equation (6.3.23) yields

$$g^{kj}\left(\frac{\partial^2 \phi}{\partial x^k \partial x^j} - \Gamma^r_{kj} \frac{\partial \phi}{\partial x^r}\right) = 0,$$

which is

$$\Delta \phi = 0.$$

6.4 Geodesics

Consider the interval $I \subset \mathbb{R}$ and let $\phi : I \to (M, g)$ be a smooth curve. Let the Hamiltonian be

$$H(p, \phi) = \frac{1}{2} g^{ij}_{|\phi} p_i p_j. \tag{6.4.24}$$

Theorem 6.4. *ϕ is a solution for the Hamiltonian system $\dot{\phi} = \dfrac{\partial H}{\partial p}$, $\dot{p} = -\dfrac{\partial H}{\partial p}$ if and only if $\nabla_{\dot{\phi}} \dot{\phi} = 0$, where ∇ stands for the Levi-Civita connection on (M, g).*

Proof. We have

$$\dot{p}_k = -\frac{\partial H}{\partial \phi^k} = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} p_i p_j, \tag{6.4.25}$$

$$\dot{\phi}^k = \frac{\partial H}{\partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{1}{2} g^{ij}_{|\phi} p_i p_j\right) = g^{ik} p_i,$$

therefore

$$p_k = \dot{\phi}^i g_{ik}. \tag{6.4.26}$$

We shall compute $\partial g^{ij}/\partial x^k$ which appears in (6.4.25). Using $g^{ip} g_{ps} = \delta^i_s$, we get

$$\frac{\partial g^{ip}}{\partial x^k} g_{ps} = -g^{ip} \frac{\partial g_{ps}}{\partial x^k}.$$

Multiplying by g^{sj} and summing over s,

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{ip} g^{sj} \frac{\partial g_{ps}}{\partial x^k}. \tag{6.4.27}$$

Differentiating in (6.4.26) yields

$$\begin{aligned} \dot{p}_k &= \ddot{\phi}^i g_{ik} + \dot{\phi}^i \frac{\partial g_{ik}}{\partial x_s} \dot{\phi}^s \\ &= \ddot{\phi}^b g_{kb} + \dot{\phi}^b \frac{\partial g_{kb}}{\partial x^r} \dot{\phi}^r. \end{aligned} \tag{6.4.28}$$

Substitute (6.4.26), (6.4.27), (6.4.28) in (6.4.25) and obtain

$$\ddot{\phi}^b g_{kb} + \frac{\partial g_{kb}}{\partial x^r}\dot{\phi}^b\dot{\phi}^r = \frac{1}{2} g^{ip} g^{sj} \frac{\partial g_{ps}}{\partial x^k}\dot{\phi}^c g_{ic}\dot{\phi}^d g_{jd}$$

$$\Longleftrightarrow \ddot{\phi}^b g_{kb} + \frac{\partial g_{kb}}{\partial x^r}\dot{\phi}^b\dot{\phi}^r = \frac{1}{2} g^{ip} g_{ic} g^{sj} g_{jd} \frac{\partial g_{ps}}{\partial x^k}\dot{\phi}^c\dot{\phi}^d$$

$$\Longleftrightarrow \ddot{\phi}^b g_{kb} + \frac{1}{2}\left[\frac{\partial g_{kb}}{\partial x^r}\dot{\phi}^b\dot{\phi}^r + \frac{\partial g_{kr}}{\partial x^b}\dot{\phi}^r\dot{\phi}^b\right] = \frac{1}{2}\frac{\partial g_{cd}}{\partial x^k}\dot{\phi}^c\dot{\phi}^d.$$

On the right-hand side let $c = b$, $d = r$, and we get

$$\ddot{\phi}^b g_{kb} + \frac{1}{2}\left[\frac{\partial g_{kb}}{\partial x^r} + \frac{\partial g_{kr}}{\partial x^b} - \frac{\partial g_{rb}}{\partial x^k}\right]\dot{\phi}^b\dot{\phi}^r = 0$$

$$\Longleftrightarrow \ddot{\phi}^b g_{kb} + \Gamma_{rbk}\dot{\phi}^b\dot{\phi}^r = 0.$$

Multiplying by g^{ks} and using $\Gamma^s_{rb} = g^{ks}\Gamma_{rbk}$ yields

$$\ddot{\phi}^s + \Gamma^s_{rb}\,\dot{\phi}^b\,\dot{\phi}^r = 0, \tag{6.4.29}$$

which can be written invariantly as $\nabla_{\dot{\phi}}\dot{\phi} = 0$, where $\nabla_{\partial_{x_k}}\partial_{x_j} = \Gamma^s_{kj}\partial_j$. ■

Hence, one may avoid the Christoffel symbols, defining the geodesics using the Hamiltonian formalism.

Definition 6.5 *A geodesic is the projection on M space of a solution of the Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x},$$

with the Hamiltonian

$$H(x, p) = \frac{1}{2}|p|^2.$$

Geodesic lift

Let $\phi : [0, 1] \to (M, g)$ be a Riemannian geodesic. Define $\nabla H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p}\right)$ and denote by J the matrix $J \in \mathcal{M}_{2n}(\mathbb{R})$ such that $J^2 = -I_{2n}$.

Definition 6.6 $z : [0, 1] \to M \times T^*M$ *is a geodesic lift of* ϕ *if there is a function* $p : [0, 1] \to T^*M$ *such that* $z(s) = (\phi(s), p(s))$ *is a solution for the Hamiltonian system* $\dot{z}(s) = J\nabla H(z(s))$.

Proposition 6.7 *If* ϕ *is a Riemannian geodesic on* (M, g)*, there is a unique geodesic lift* $z(s) = (\phi(s), p(s))$ *with* $p = (p_1, \dots, p_n)$ *and*

$$p_k(s) = \sum_{r=1}^{n} g_{kr}(\phi(s))\,\dot{\phi}^r(s).$$

Proof. From the Hamiltonian equation $\dot{\phi}^k = \dfrac{\partial H}{\partial p_k}$, we get $p_k = \sum_{r=1}^n g_{kr}\,\dot{\phi}^r$, see formula (6.4.26). ■

Proposition 6.8 *Consider the natural Lagrangian* $L(\phi, \dot{\phi}) = \frac{1}{2} g(\dot{\phi}, \dot{\phi}) - U(\phi)$. *Then the associated Hamiltonian is*

$$H(p, \phi) = \frac{1}{2} g^{ij} p_i p_j + U(\phi). \tag{6.4.30}$$

Proof. As $p_k = \dfrac{\partial L}{\partial \dot{\phi}^k} = \dot{\phi}^i g_{ik}$, then $\dot{\phi}^k = p_r g^{rk}$ and $g_{rk}\dot{\phi}^r \dot{\phi}^k = g^{rk} p_r p_k$. The Legendre transform yields

$$\begin{aligned} H(p, \phi) &= p_k \dot{\phi}^k - L = p_k p_r g^{rk} - \frac{1}{2} p_k p_r g^{rk} + U(\phi) \\ &= \frac{1}{2} p_k p_r g^{rk} + U(\phi). \end{aligned}$$

■

Corollary 6.9 *The Hamiltonian (6.4.30) is constant along the solutions of Hamilton's system.*

Proof. Using Hamilton's equations

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial \phi^k} \dot{\phi}^k \\ &= \frac{\partial H}{\partial p_k}\Big(- \frac{\partial H}{\partial \phi^k}\Big) + \frac{\partial H}{\partial \phi^k}\frac{\partial H}{\partial p_k} = 0. \end{aligned}$$

■

6.5 Harmonic maps

Consider the Hamiltonian

$$H(p, \phi) = \frac{1}{2}\, p^j_\beta \; p^l_\rho \; g_{jl} \; h^{\beta\rho}_{\;|\phi}, \tag{6.5.31}$$

where $\phi : (M, g) \to (N, h)$ is a map between two Riemannian manifolds. From the Hamiltonian equation

$$\phi^j_{;i} = \frac{\partial H}{\partial p^i_j} = p^k_l \; g_{ik} \; h^{jl},$$

and hence

$$p^\alpha_k = g^{\alpha\beta} \; h_{kj} \; \phi^j_{;\beta}. \tag{6.5.32}$$

The second Hamiltonian equation provides

$$div\ p_k = -\frac{\partial H}{\partial \phi^k} = -\frac{1}{2}\ p^i_j\ p^s_l\ g_{is}\ \frac{\partial h^{jl}}{\partial y^k}. \tag{6.5.33}$$

Now we shall compute $div\ p_k$ in another way using (6.5.32),

$$\begin{aligned} div\ p_k = p^{\alpha}_{k\ ;\alpha} &= \left(g^{\alpha\beta}\ h_{kj}\ \phi^j_{\ ;\beta}\right)_{;\alpha} \\ &= g^{\alpha\beta}_{\ ;\alpha}\ h_{kj}\ \phi^j_{\ ;\beta} + g^{\alpha\beta}\ h_{kj\ ;\alpha}\ \phi^j_{\ ;\beta} + g^{\alpha\beta}\ h_{kj}\ \phi^j_{\ ;\beta\alpha} \\ &= g^{\alpha\beta}\ \frac{\partial h_{kj}}{\partial y^s}\ \phi^s_{\ ;\alpha}\ \phi^j_{\ ;\beta} + h_{kj}\ \Delta\phi^j. \end{aligned}$$

Hence,

$$div\ p_k = h_{jk}\ \Delta\phi^j + g^{\alpha\beta}\ \frac{\partial h_{kj}}{\partial y^s}\ \phi^s_{\ ;\alpha}\phi^j_{\ ;\beta}. \tag{6.5.34}$$

As

$$\frac{\partial h^{jl}}{\partial y^k} = -h^{pl}\ hsj\,\frac{\partial h_{sp}}{\partial y^k},$$

using (6.5.32), the right-hand side of (6.5.33) becomes

$$\begin{aligned} &-\frac{1}{2}g^{ia}\ h_{jb}\ \phi^b_{\ ;a}\ g^{s\alpha}\ h_{l\beta}\ \phi^{\beta}_{\ ;\alpha}\ g_{is}(-1)\ h^{mj}\ h^{nl}\ \frac{\partial h_{mn}}{\partial y^k} \\ &= \frac{1}{2}g^{ia}\ g^{s\alpha}\ g_{is}\ h_{jb}\ h_{l\beta}\ h^{mj}\ h^{nl}\ \phi^b_{\ ;a}\ \phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{mn}}{\partial y^k} \\ &= \frac{1}{2}g^{ia}\ g^{s\alpha}\ g_{is}\ \delta_{bm}\ \delta_{\beta n}\ \phi^b_{\ ;a}\phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{mn}}{\partial y^k} \\ &= \frac{1}{2}\ g^{\alpha a}\ \phi^b_{\ ;a}\ \phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{b\beta}}{\partial y^k}. \end{aligned} \tag{6.5.35}$$

So (6.5.33) becomes

$$div\ p_k = \frac{1}{2}\ g^{\alpha a}\ \phi^b_{\ ;a}\ \phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{b\beta}}{\partial y^k}. \tag{6.5.36}$$

From relations (6.5.34) and (6.5.35) we obtain

$$h_{kj}\ \Delta\phi^j + g^{\alpha\beta}\ \frac{\partial h_{kj}}{\partial y^s}\ \phi^s_{\ ;\alpha}\ \phi^j_{\ ;\beta} = \frac{1}{2}\ g^{\alpha a}\ \phi^b_{\ ;a}\ \phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{b\beta}}{\partial y^k}$$

$$\Longleftrightarrow h_{kj}\ \Delta\phi^j + \frac{1}{2}\left[2g^{\alpha\beta}\ \frac{\partial h_{kj}}{\partial y^s}\ \phi^s_{\ ;\alpha}\ \phi^j_{\ ;\beta} - g^{\alpha a}\ \phi^b_{\ ;a}\ \phi^{\beta}_{\ ;\alpha}\ \frac{\partial h_{b\beta}}{\partial y^k}\right] = 0$$

$$\Longleftrightarrow h_{kj}\ \Delta\phi^j + \frac{1}{2}\left[g^{\alpha\beta}\,\frac{\partial h_{kj}}{\partial y^s}\,\phi^s_{;\alpha}\,\phi^j_{;\beta} + g^{\alpha\beta}\,\frac{\partial h_{ks}}{\partial y^j}\,\phi^j_{;\alpha}\,\phi^s_{;\beta} - g^{\alpha\beta}\,\frac{\partial h_{js}}{\partial y^k}\,\phi^j_{;\beta}\,\phi^s_{;\alpha}\right] = 0,$$

$$\Longleftrightarrow h_{kj}\ \Delta\phi^j + \frac{1}{2}\,g^{\alpha\beta}\left[\frac{\partial h_{kj}}{\partial y^s} + \frac{\partial h_{ks}}{\partial y^j} - \frac{\partial h_{js}}{\partial y^k}\right]\phi^j_{;\beta}\,\phi^s_{;\alpha} = 0,$$

$$\Longleftrightarrow h_{kj}\ \Delta\phi^j + g^{\alpha\beta}\,\Gamma_{jsk}\,\phi^j_{;\beta}\,\phi^s_{;\alpha} = 0.$$

Multiplying by h^{kr}, we get

$$\Delta\phi^r + g^{\alpha\beta}\,\Gamma^r_{js}\,\phi^j_{;\beta}\,\phi^s_{;\alpha} = 0, \qquad r = \overline{1,n}, \tag{6.5.37}$$

which is the equation for the harmonic maps $\phi : (M, g) \to (N, h)$.

Remark 6.10 *In Chapter* 4 *we arrived at equation* (6.5.37) *using Lagrangian formalism with the Lagrangian* $L = e(\phi) = 1/2\, g^{ij}\,\phi^\alpha_{;i}\,\phi^\beta_{;j}\,h_{\alpha\beta\,|\phi}$, *called density energy.*

The Hamiltonian (6.5.31) is related to the energy density by

$$H(p,\phi) = p^k_\gamma\,\phi^\gamma_{;k} - \frac{1}{2}\,g^{ij}\,\phi^\alpha_{;i}\,\phi^\beta_{;j}\,h_{\alpha\beta\,|\phi}, \tag{6.5.38}$$

where $\phi^\gamma_{;k}$ is given from the momenta expression

$$p^k_\gamma = \frac{\partial e}{\partial \phi^\gamma_{;k}} = g^{k\beta}\,h_{\gamma j}\,\phi^j_{;\beta}. \tag{6.5.39}$$

Substituting (6.5.39) in (6.5.38), yields

$$\begin{aligned} H(p,\phi) &= g^{k\beta}\,h_{\gamma j}\,\phi^j_{;\beta}\,\phi^\gamma_{;k} - \frac{1}{2}\,g^{ij}\,\phi^\alpha_{;i}\,\phi^\beta_{;j}\,h_{\alpha\beta} \\ &= \frac{1}{2}\,g^{ij}\,\phi^\alpha_{;i}\,\phi^\beta_{;j}\,h_{\alpha\beta} = \frac{1}{2}\,p^j_\beta\,\phi^\beta_{;j}. \end{aligned} \tag{6.5.40}$$

From (6.5.39), we obtain

$$\phi^\beta_{;j} = p^l_\rho\,g_{lj}\,h^{\rho\beta}. \tag{6.5.41}$$

Substitute (6.5.41) in (6.5.40) and get

$$H(p,\phi) = \frac{1}{2}\,p^j_\beta\,p^l_\rho\,g_{jl}\,h^{\rho\beta},$$

i.e., the Hamiltonian (6.5.31).

6.6 Poincaré half-plane

Consider $\mathbb{H}^2 = \{(x, y)|y > 0\} \subset \mathbb{R}^2$ endowed with the Riemannian metric $g = \dfrac{dx^2 + dy^2}{y^2}$. $(\mathbb{H}^2, g)$ is called the real hyperbolic plane or Poincaré half-plane. We are interested in finding the geodesics on $\mathbb{H}^2$ using Hamiltonian and Lagrangian formalism. The Lagrangian is

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2), \tag{6.6.42}$$

with the associated Hamiltonian

$$H(p_1, p_2, x, y) = \frac{1}{2}y^2(p_1^2 + p_2^2). \tag{6.6.43}$$

As the Hamiltonian does not depend on the variable x, one of Hamilton's equations yields

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \Longrightarrow p_1 = k \text{ (constant)}. \tag{6.6.44}$$

On the other hand, the momentum p_2 is given by

$$p_2 = \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{y^2}. \tag{6.6.45}$$

As the Hamiltonian does not depend explicitly on the parameter t, a consequence of Hamilton's equations and the chain rule is

$$\frac{dH}{dt} = 0 \Longrightarrow H = \frac{1}{2}C^2 \text{ (constant)}.$$

Case $k \neq 0$
Substituting in formula (6.6.43) yields

$$y^2\Big(k^2 + \frac{\dot{y}^2}{y^4}\Big) = C^2, \tag{6.6.46}$$

which is an equation in the variable y. The equation can be written as

$$\dot{y}^2 = y^2\Big(C^2 - k^2y^2\Big), \tag{6.6.47}$$

which becomes $\dot{y} = \pm y\sqrt{C^2 - k^2y^2}$. Separating

$$\frac{dy}{y\sqrt{C^2 - k^2y^2}} = \pm dt,$$

and integrating

$$\int \frac{dy}{y\sqrt{1 - \alpha^2y^2}} = \pm|C|t + b,$$

where $\alpha = k/C$. Using Exercise 4, we get

$$-\ln\left|\frac{1+\sqrt{1-\alpha^2 y^2}}{y}\right| = \pm|C|t - b.$$

Using Exercise 3, we find

$$-\operatorname{sech}^{-1}(\alpha y) - \ln|\alpha| = \pm|C|t - b,$$

which yields

$$y(t)_{\pm} = \frac{1}{\alpha}\operatorname{sech}(\pm|C|t - b - \ln|\alpha|). \tag{6.6.48}$$

We can drop the $\pm$ sign because $t \in \mathbb{R}$ can be considered taking all positive and negative values. Hence

$$y(t) = \frac{1}{\alpha}\operatorname{sech}(|C|t - a), \tag{6.6.49}$$

where $a = b + \ln|\alpha|$. We have $\lim_{t\to\pm\infty} y(t) = 0$, which means that the geodesics never reach the line $\{y = 0\}$.

To find the x-component, we use $p_1 = k$ and write

$$p_1 = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}.$$

This yields $\dot{x} = ky^2$. Integrating, we find

$$x(t) = k\int y^2(t)\,dt = \frac{k}{\alpha^2}\int \frac{dt}{\cosh^2(|C|t - a)}.$$

Using Exercise 5 yields

$$x(t) = \frac{1}{\alpha}\tanh(|C|t - a) + K. \tag{6.6.50}$$

The formulas (6.6.50) and (6.6.49) describe a semicircle with $y > 0$ centered at $(K, 0)$ with radius $r = 1/\alpha$:

$$(x(t) - K)^2 + (y(t) - 0)^2 = \frac{1}{\alpha^2}\left(\tanh^2(|C|t - a) + \frac{1}{\cosh^2(|C|t - a)}\right) = \frac{1}{\alpha^2}.$$

Case $k = 0$

In this case, $p_1 = 0$ and then $\dot{x} = 0$. Hence, $x(t) = x(0)$ is constant. Equation (6.6.46) becomes $\dot{y}^2 = C^2 y^2$ with solution $y(t) = y(0)e^{\pm|C|t}$. These solutions correspond to lines perpendicular to the x-axis.

The distance

In this section we shall find the distance $d = d\big((x_0, y_0), (x, y)\big)$ computed with respect to the metric on the Poincaré plane.

Substituting $t = 0$ in the formulas

$$x(t) = \frac{1}{\alpha}\tanh(Ct - a) + K, \quad y(t) = \frac{1}{\alpha}\operatorname{sech}(Ct - a), \qquad C > 0, \tag{6.6.51}$$

yields

$$\begin{aligned} y_0 &= \frac{1}{\alpha}\operatorname{sech}(-a) = \frac{1}{\alpha}\operatorname{sech}(a), \\ x_0 &= \frac{1}{\alpha}\tanh(-a) + K = -\frac{1}{\alpha}\tanh(a) + K \\ &= -\sinh(a)\frac{\operatorname{sech}(a)}{\alpha} + K \\ &= -\sinh(a)\, y_0 + K. \end{aligned}$$

$$x_0 - K = -y_0 \sinh(a) \Longrightarrow a = \sinh^{-1}\left(\frac{K - x_0}{y_0}\right).$$

Let $(x, y) = (x(\tau), y(\tau))$. Substituting $t = \tau$ in (6.6.51) yields

$$\begin{aligned} x &= \frac{1}{\alpha}\tanh(C\tau - a) + K, \\ y &= \frac{1}{\alpha}\operatorname{sech}(C\tau - a). \end{aligned}$$

The product $C\tau$ can be evaluated as follows. It is known that the energy along a geodesic joining the points (x_0, y_0) and (x, y) is given by $E = \dfrac{d^2}{2\tau^2}$. Then $\sqrt{2E} = \dfrac{d}{\tau}$. Using that $C = \sqrt{2E} = H$ we find that

$$C\tau = d.$$

Hence the above formulas become

$$\begin{aligned} x &= \frac{1}{\alpha}\tanh(d - a) + K = \sinh(d - a)y + K, \\ y &= \frac{1}{\alpha}\operatorname{sech}(d - a). \end{aligned}$$

From the first formula we obtain

$$\frac{x - K}{y} = \sinh(d - a) \Longrightarrow \sinh^{-1}\left(\frac{x - K}{y}\right) = d - a$$

and hence

$$
\begin{aligned}
d &= a + \sinh^{-1}\left(\frac{x-K}{y}\right) = \sinh^{-1}\left(\frac{K-x_0}{y_0}\right) + \sinh^{-1}\left(\frac{x-K}{y}\right) \\
&= \ln\left(\frac{K-x_0}{y_0} + \sqrt{1+\left(\frac{K-x_0}{y_0}\right)^2}\right) + \ln\left(\frac{x-K}{y} + \sqrt{1+\left(\frac{x-K}{y}\right)^2}\right) \\
&= \ln\left(\frac{K-x_0+\sqrt{y_0^2+(K-x_0)^2}}{y_0}\right) + \ln\left(\frac{x-K+\sqrt{y^2+(x-K)^2}}{y}\right) \\
&= \ln\left(\frac{K-x_0+r}{y_0}\right) + \ln\left(\frac{x-K+r}{y}\right) = \ln\left(\frac{K-x_0+r}{y_0}\cdot\frac{x-K+r}{y}\right),
\end{aligned}
$$

where r is the radius. Hence

$$
d = \ln\left(\frac{A'M \cdot NB'}{AA' \cdot BB'}\right) = \ln\left(\tan\widehat{A'AM}\cdot\tan\widehat{NBB'}\right),
$$

see Figure 6.1.

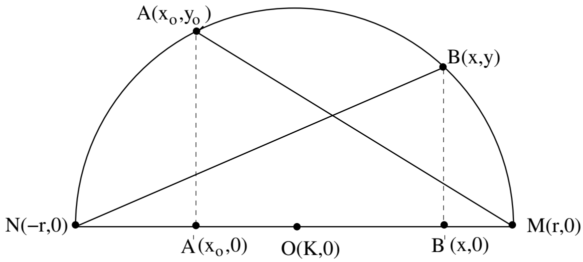


Figure 6.1: The points $A(x_0, y_0)$, $A'(x_0, 0)$, $B(x, y)$ and $B(x, y)$.

A formula for the distance d depending only on the coordinates of the boundary points can be obtained if we use

$$
\begin{aligned}
r^2 &= \frac{1}{\alpha^2} = y_0^2 + (K-x_0)^2, \\
K &= \frac{1}{2}\frac{(x-x_0)^2 + y^2 - y_0^2}{x-x_0},
\end{aligned}
$$

see Exercise 8. For more applications of the Hamiltonian formalism the reader may consult [3].

6.7 Exercises

1. Let $\sinh x = \dfrac{e^x - e^{-x}}{2}$ be the hyperbolic sine function.

(i) Show that the inverse function is given by $\sinh^{-1} y = \ln|y + \sqrt{y^2+1}|$, for any $y \in \mathbb{R}$.

(ii) Show that the solution of the equation $\dfrac{1}{\sinh x} = y$ is $x = \ln\left|\dfrac{1+\sqrt{1+y^2}}{y}\right|$. Find a formula for the inverse function of $\operatorname{csch} x$.

2. Consider the hyperbolic cos function $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

(i) Show that the inverse function is given by $\cosh^{-1} y = \ln|y + \sqrt{y^2 - 1}|$.

(ii) Show that the solution of the equation $\dfrac{1}{\cosh x} = y$ is $x = \ln\left|\dfrac{1+\sqrt{1-y^2}}{y}\right|$. Find a formula for the inverse function of $\operatorname{sech} x$.

3. Using Exercise 2, show that

$$\ln\left|\frac{1+\sqrt{1-\alpha^2 y^2}}{y}\right| = \operatorname{sech}^{-1}(\alpha y) + \ln|\alpha|.$$

4. Show

$$\int \frac{dy}{y\sqrt{1-\alpha^2 y^2}} = -\ln\left|\frac{1+\sqrt{1-\alpha^2 y^2}}{y}\right|$$

following the steps:

(i) making the substitution $u = \dfrac{1}{y}$, show that the integral is equal to $-\displaystyle\int \frac{du}{\sqrt{u^2-\alpha^2}}$.

(ii) Use the fact that $\displaystyle\int \frac{du}{\sqrt{u^2-\alpha^2}} = \ln|u + \sqrt{u^2-\alpha^2}|$.

5. Show that $\displaystyle\int \frac{1}{\cosh^2 u} = \tanh u$, where $\tanh u = \sinh x / \cosh x$.

6. Consider the sphere $\mathbb{S}^2$ endowed with the Riemannian metric $g^{11} = 1 - x^2$, $g^{22} = 1 - y^2$, $g^{12} = g^{21} = -xy$.

(i) Show that the Hamiltonian is $H = \dfrac{1}{2}(p_1^2 + p_2^2) - \dfrac{1}{2}(xp_1 + yp_2)^2$.

(ii) Show that the Lagrangian is $L = \dfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dfrac{1}{2}(x\dot{x} + y\dot{y})^2$.

(iii) Show that the geodesics are great circles.

7. (Poincaré Disk.) Consider $\mathbb{B} = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ endowed with the Riemannian metric $g_{ij} = \dfrac{4}{(1 - x^2 - y^2)^2}\,\delta_{ij}$.

(i) Write the Lagrangian and the Hamiltonian in polar coordinates.

(ii) Find the geodesics of $(\mathbb{B}, g_{ij})$.

8. Let $A(x_0, y_0)$ and $B(x, y)$ be two points in the upper half-plane.

(i) The equation of the perpendicular bisector of the segment AB is

$$y = -\frac{x - x_0}{y - y_0}x + \frac{1}{2}\frac{(x - x_0)^2 + y^2 - y_0^2}{y - y_0}.$$

(ii) Using that the intersection point between the above segment bisector and the x-axis is the center of the circle $(K, 0)$, find K.

9. Let $X = X_1(x, y)\partial_x + X_2(x, y)\partial_y$ be a vector field on the Poincaré half-plane. Show that

$$\text{div}X = \partial_x X_1 + y^2\partial_y\Big(\frac{1}{y^2}X_2\Big).$$

10. Consider the relativistic Hamiltonian for a free particle of mass m_0,

$$H(p, q) = (p_1^2 + p_2^2 + p_3^2 + m_0^2)^{1/2}.$$

a) Write the Hamiltonian system.

b) Find the associate Lagrangian.

c) Give a characterization of the solutions of the Hamiltonian system.

7

Hamilton–Jacobi Theory

7.1 Hamilton–Jacobi equation in the case of natural Lagrangian

Consider a curve $\phi : (t_1, t_2) \to (M, g)$ on a Riemannian manifold. Denote by $U : M \to \mathbb{R}$ the potential and let L be the natural Lagrangian

$$L(\phi, \dot{\phi}) = \frac{1}{2}|\dot{\phi}(t)|_g^2 - U(\phi(t)). \tag{7.1.1}$$

The extremizers of the integral

$$I = \int_{t_1}^{t_2} L(\phi, \dot{\phi})\, dt \tag{7.1.2}$$

satisfy the Euler–Lagrange equation

$$\nabla_{\dot{\phi}}\dot{\phi} = -\nabla U_{|\phi}. \tag{7.1.3}$$

The total energy is

$$H = \frac{1}{2}|\dot{\phi}(t)|_g^2 + U(\phi(t)), \tag{7.1.4}$$

i.e., the sum of the kinetic and the potential energy.

Lemma 7.1 *Let $S : \mathbb{R} \times M \to \mathbb{R}$ be a function. Then*

$$dS_{|\phi} = \Big(\frac{\partial S}{\partial t}_{|\phi} + g(\nabla S, \dot{\phi})\Big)dt.$$

Proof. A computation shows

$$dS = \frac{\partial S}{\partial t}dt + \sum_r \frac{\partial S}{\partial x^r}dx^r = \frac{\partial S}{\partial t}dt + \Big(\sum_r \frac{\partial S}{\partial x^r}\dot{x}^r\Big)dt,$$

so that

$$dS_{|\phi} = \left(\frac{\partial S}{\partial t}_{|\phi} + \sum_r \frac{\partial S}{\partial x^r} \dot{\phi}^r \right) dt.$$

As the gradient is given by

$$\nabla S = g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial}{\partial x^j},$$

we have

$$g(\nabla S, \dot{\phi}) = g_{ij} (\nabla S)^i \, \dot{\phi}^j = g_{ij} g^{ki} \frac{\partial S}{\partial x^k} \dot{\phi}^j = \frac{\partial S}{\partial x^j} \dot{\phi}^j.$$

Hence,

$$dS_{|\phi} = \left(\frac{\partial S}{\partial t}_{|\phi} + g(\nabla S, \dot{\phi}) \right) dt.$$

∎

The integrals

$$I = \int_{t_1}^{t_2} L \, dt \quad \text{and} \quad J = \int_{t_1}^{t_2} (L \, dt - dS)$$

reach the extremum for the same curve $\phi : (t_1, t_2) \to (M, g)$, because

$$J = I - S(t_2, \phi(t_2)) + S(t_1, \phi(t_1)).$$

Lemma 7.2 *The integrand of the integral J is equal to*

$$\frac{1}{2} |\dot{\phi} - \nabla S|_g^2 - \left(\frac{\partial S}{\partial t} + \frac{1}{2} \, |\nabla S|^2 + U \right).$$

Proof. The integrand of J is $L - dS/dt$. Using Lemma 7.1, we obtain

$$\begin{aligned} L - \frac{dS}{dt} &= L - \left(\frac{\partial S}{\partial t} + g(\nabla S, \dot{\phi}) \right) \\ &= \frac{1}{2} |\dot{\phi}|_g^2 - U - \frac{\partial S}{\partial t} - g(\nabla S, \dot{\phi}) \\ &= \frac{1}{2} |\dot{\phi}|_g^2 - g(\nabla S, \dot{\phi}) + \frac{1}{2} |\nabla S|^2 - \frac{1}{2} |\nabla S|^2 - \frac{\partial S}{\partial t} - U \\ &= \frac{1}{2} |\dot{\phi} - \nabla S|_g^2 - \left(\frac{\partial S}{\partial t} + \frac{1}{2} |\nabla S|^2 + U \right). \end{aligned}$$

∎

Therefore, the integrals I and

$$J = \int_{t_1}^{t_2} \left[\frac{1}{2} \left| \dot{\phi} - \nabla S \right|^2 - \left(\frac{\partial S}{\partial t} + \frac{1}{2} \, |\nabla S|^2 + U \right) \right] dt \tag{7.1.5}$$

reach the extremum for the same curve $\phi : (t_1, t_2) \to M$, where S is an arbitrary function $S : \mathbb{R} \times M \to \mathbb{R}$.

To simplify (7.1.5), we can choose S such that

$$\frac{\partial S}{\partial t} + \frac{1}{2}\,|\nabla S|^2 + U = 0. \tag{7.1.6}$$

Then the integral J becomes

$$J = \int_{t_1}^{t_2} \frac{1}{2}|\dot{\phi} - \nabla S|_g^2\, dt. \tag{7.1.7}$$

Hence, J is minimal if and only if

$$\dot{\phi} = \nabla S, \tag{7.1.8}$$

where S is a solution of (7.1.6).

Definition 7.3 *The equation (7.1.6) is called a Hamilton–Jacobi equation. It can be also written as*

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial x}, x) = 0,$$

or

$$\frac{\partial S}{\partial t} + H(\nabla S) = 0, \tag{7.1.9}$$

where H denotes the Hamiltonian.

Theorem 7.4. *Along the solution $\phi(t)$ of the Euler–Lagrange equation, we have*

$$\dot{\phi}(t) = \nabla_{\phi} S(t, \phi(t)), \tag{7.1.10}$$

where S is a solution of the Hamilton–Jacobi equation (7.1.6). Conversely, any curve which satisfies (7.1.10) is a solution of Euler–Lagrange equations, up to a reparametrization.

Singularities of the action S

Let X be the vector field generated by a flow of solutions of Euler–Lagrange equations, *i.e.*,

$$X_{\phi(t)} = \dot{\phi}(t).$$

Applying the divergence and using Theorem 7.4,

$$div\, X = div\, \nabla S = \Delta S,$$

where Δ denotes the Laplacian.

As long as the flow of solutions X does not have conjugate points, div X doesn't have singularities. Using that Δ is a hypoelliptic operator (preserves the singular support of functions), it follows that the action S does not have singularities.

Proposition 7.5 *The action S is singular at the conjugate points of the solutions flow.*

The case of geodesics

In this case, $U = 0$ and the Hamilton–Jacobi equation becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2} |\nabla S|^2 = 0 \tag{7.1.11}$$

and

$$\dot{\phi} = \nabla S.$$

We shall look for solutions with separate variables

$$S(t, x) = a(t) + b(x).$$

Then (7.1.11) becomes

$$a'(t) + \frac{1}{2} |\nabla b(x)|^2 = 0.$$

There is a constant $E > 0$ such that

$$-a'(t) = \frac{1}{2} |\nabla b(x)|^2 = E.$$

In fact, E is the energy because

$$E = \frac{1}{2} |\nabla b\,(x)|^2 = \frac{1}{2} |\nabla(a+b)|^2 = \frac{1}{2} |\nabla S|^2 = \frac{1}{2} |\dot{\phi}|^2.$$

It follows that

$$a(t) = -Et + a(0)$$

and

$$\frac{1}{2} |\nabla b|^2 = E.$$

Let

$$\beta(x) = \frac{1}{\sqrt{2E}} \big(b(x) - b(x_0)\big).$$

Then β satisfies the eiconal equation (see section 7.3)

$$|\nabla \beta|^2 = 1,$$

$$\beta(x_0) = 0,$$

so that $\beta(x) = d(x_0, x)$, see Theorem 7.15. Thus,

$$b\,(x) = b\,(x_0) + \sqrt{2E}\, d(x_0, x).$$

Hence,

$$\begin{aligned} & S(t, x) = -Et + \sqrt{2E}\, d(x_0, x) + a(0) + b\,(x_0) \\ \Longleftrightarrow\ & S(t, x) = -Et + \sqrt{2E}\, d(x_0, x) + S(0, x_0). \end{aligned}$$

Remark 7.6 *We have*

$$\lim_{t\to\infty} \frac{S(t,x)}{t} = -E.$$

Remark 7.7 *For general conditions* t_0, x_0*, we get*

$$S(t,x) = S(t_0,x_0) - (t-t_0)\,E + \sqrt{2E}\,d(x_0,x)$$

and thus,

$$S(t_0,x) - S(t_0,x_0) = \sqrt{2E}\,d(x_0,x) - (t-t_0)E.$$

7.2 The action function on Riemannian manifolds

Consider a Riemannian manifold M and let $\phi : (t_0,t_1) \to M$ be a smooth map. Suppose the Lagrangian is nonnegative, $L(\phi,\dot\phi) \geq 0$.

Definition 7.8 *The action function with the initial condition*

$$S(t_0,\phi(t_0)) = S_0 \tag{7.2.12}$$

is defined as

$$S(t,\phi(t)) = S_0 + \int_{t_0}^{t} L(\phi(s),\dot\phi(s))\,ds, \tag{7.2.13}$$

where ϕ *is a solution of the Euler–Lagrange equation which connects* $\phi(t_0)$ *and* $\phi(t)$*.*

The relation between the action and the Hamilton–Jacobi equation is given in the following:

Theorem 7.9. *The action defined by (7.2.13) verifies the Hamilton–Jacobi equation*

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial \phi},\phi) = 0 \tag{7.2.14}$$

with the initial condition $S(t_0,\phi(t_0)) = S_0$*, where* H *is the Hamiltonian associated with the Lagrangian* L*.*

Proof. Applying the chain rule

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \phi_k}\dot\phi_k = \frac{\partial S}{\partial t} + \langle \frac{\partial S}{\partial \phi},\dot\phi\rangle.$$

Using the definition of S yields

$$\frac{\partial S}{\partial t} = \frac{dS}{dt} - \langle \frac{\partial S}{\partial \phi},\dot\phi\rangle = L(\phi(t),\dot\phi(t)) - \langle \frac{\partial S}{\partial \phi},\dot\phi\rangle. \tag{7.2.15}$$

Using the Legendre transform,

$$H(\frac{\partial S}{\partial \phi}, \phi) = \langle \frac{\partial S}{\partial \phi}, \dot{\phi} \rangle - L(\phi(t), \dot{\phi}(t)). \tag{7.2.16}$$

Adding equations (7.2.15) and (7.2.16), we obtain the Hamilton–Jacobi equation (7.2.14). ■

As a nonlinear equation, the Hamilton–Jacobi equation (7.2.14) with the initial condition (7.2.12) may have more than one solution. Such a situation is described by the following example.

Consider the Lagrangian $L(x, \dot{x}) = \frac{1}{2}\dot{x}^2$ with the Euler–Lagrange equation $\ddot{x} = 0$ and the solution $x = x(t) = ct + x_0$. The associated Hamiltonian is $H(p, x) = \frac{1}{2}p^2$. The function $f(t, x) = \sqrt{2}x - t$ is a solution for the Hamilton–Jacobi equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\Big(\frac{\partial f}{\partial x}\Big)^2 = 0$$

with the initial condition $f(0, 0) = 0$, where $x_0 = x(0) = 0$.

A different solution is given by the action $S(t, x)$,

$$S(t, x(t)) = \underbrace{S(0, 0)}_{=0} + \int_0^t \frac{1}{2}\dot{x}(s)^2\, ds = \frac{1}{2}c^2 t = \frac{1}{2}\frac{(ct)^2}{t} = \frac{x(t)^2}{2t}.$$

Now we can address the following natural question:

What condition should a solution of the Hamilton–Jacobi equation satisfy in order to be the action?

We start by observing that the momentum in the above problem is

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} = c.$$

On the other hand,

$$\frac{\partial S}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{x^2}{2t}\Big) = \frac{x}{t} = c.$$

Hence,

$$p = \frac{\partial S}{\partial x},$$

for any solution of the Euler–Lagrange equation which passes through the origin. The following theorem will show that this is a sufficient condition for a solution of the Hamilton–Jacobi equation to be the action.

Theorem 7.10. *Let $S = S(t, \phi)$ be a solution for the Hamilton–Jacobi equation*

$$\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial \phi}, \phi) = 0,$$
$$S(t_0, \phi(t_0)) = S_0,$$

such that

$$p = \frac{\partial S}{\partial \phi}, \tag{7.2.17}$$

where the momentum $p = \partial L/\partial \dot{\phi}$. *Then* S *is given by*

$$S(t, \phi(t)) = S_0 + \int_{t_0}^{t} L(\phi(s)\dot{\phi}(s))ds, \tag{7.2.18}$$

where L *is the Lagrangian associated with the Hamiltonian* H *and* ϕ *is a solution of the Euler–Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}^k} = \frac{\partial L}{\partial \phi^k}, \qquad \forall k = \overline{1, n}$$

for small enough $|t - t_0|$.

Proof. Consider a solution ϕ for the Euler–Lagrange equation that connects $\phi(t_0)$ and $\phi(t_1)$, with small enough $|t_1 - t_0|$. Fix $t \in [t_0, t_1]$. We may assume $t = t_1$. Let

$$I(\phi) = \int_{t_0}^{t_1} L(\phi(t), \dot{\phi}(t))\, dt,$$
$$J(\phi) = \int_{t_0}^{t_1} \big(L - \frac{dS}{dt}\big)\, dt.$$

We have

$$I(\phi) = J(\phi) + S(t_1, \phi(t_1)) - S(t_0, \phi(t_0)). \tag{7.2.19}$$

The chain rule yields

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \langle \frac{\partial S}{\partial \phi}, \dot{\phi} \rangle,$$

while the Legendre transform is

$$L(\phi, \dot{\phi}) = \langle p, \dot{\phi} \rangle - H(p, \phi)$$

where $p = \partial L/\partial \phi$. Substituting in the integral $J(\phi)$, we get

$$J(\phi) = \int_{t_0}^{t_1} \Big(\langle p, \dot{\phi} \rangle - H(p, \phi) - \frac{\partial S}{\partial t} - \langle \frac{\partial S}{\partial \phi}, \dot{\phi} \rangle \Big)\, dt$$
$$= \int_{t_0}^{t_1} \Big(\langle p - \frac{\partial S}{\partial \phi}, \dot{\phi} \rangle - \Big(\frac{\partial S}{\partial t} + H\big(\frac{\partial S}{\partial \phi}, \phi\big) \Big) \Big)\, dt = 0,$$

because $p = \dfrac{\partial S}{\partial \phi}$ and S satisfies the Hamilton–Jacobi equation. Hence, (7.2.19) becomes

$$I(\phi) = S(t_1, \phi(t_1)) - S(t_0, \phi(t_0)).$$

Replacing t_1 by an arbitrary $0 \le t \le t_1$, we get the action (7.2.18). ∎

We now examine if the momentum condition is also necessary. Differentiating with respect to ϕ in

$$S(\phi) - S_0(\phi) = \int_{t_0}^{t_1} L(\phi, \dot{\phi})\, ds,$$

and using Euler–Lagrange equations, we get

$$\begin{aligned}\frac{\partial S}{\partial \phi} - \frac{\partial S_0}{\partial \phi} &= \int_{t_0}^{t_1} \frac{\partial L}{\partial \phi}\, ds = \int_{t_0}^{t_1} \frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{\phi}}\Big)\, ds \\ &= \int_{t_0}^{t_1} \frac{dp}{ds}\, ds = p(t_1) - p(t_0).\end{aligned}$$

Hence, with the additional hypotheses $p(t_0) = 0$ and $\dfrac{\partial S_0}{\partial \phi} = 0$, the momentum condition is necessary.

7.2.0.1 Hamilton–Jacobi for conservative systems

In the case when the Hamiltonian H does not depend explicitly on time t, using Hamilton's equations:

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial \phi}\dot{\phi} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0,$$

so that $H(p, \phi)$ is constant along the solutions of Hamilton's system and equal to the constant of energy E. Therefore, the Hamilton–Jacobi system becomes

$$\begin{aligned}\frac{\partial S}{\partial t} + E &= 0, \\ S(t_0, \phi(t_0)) &= S_0\end{aligned}$$

with the solution

$$S(t, \phi(t)) = S_0 - Et.$$

The energy E depends on the end points $\phi(0)$ and $\phi(t)$ as well as on t.

7.2.1 Action for an arbitrary Lagrangian

The main result of this section is the following theorem.

Theorem 7.11. *Let $L = L(x, \dot{x}, t)$ be a Lagrangian function. There is a function $S = S(x, t)$ such that along the solutions of the Euler–Lagrange system of equations we have*

$$L\, dt = dS.$$

Proof. Let $x = x(t)$ be a solution for the Euler–Lagrange system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} = \frac{\partial L}{\partial x_k}. \tag{7.2.20}$$

Let

$$p_k = \frac{\partial L}{\partial \dot{x}_k} \tag{7.2.21}$$

be the k-th momentum. Expanding in (7.2.20) yields

$$\sum_k \Big[\frac{\partial p_k}{\partial x_r}\dot{x}_r + \frac{\partial p_k}{\partial \dot{x}_r}\ddot{x}_r + \frac{\partial p_k}{\partial t}\Big] = \frac{\partial L}{\partial x_k}. \tag{7.2.22}$$

As the Lagrangian $L = L(x, \dot{x}, t)$ does not depend on $\ddot{x}$, the coefficient of $\ddot{x}_r$ in (7.2.22) vanishes

$$\frac{\partial p_k}{\partial \dot{x}_r} = 0. \tag{7.2.23}$$

Substituting (7.2.21) in (7.2.23) yields

$$\frac{\partial^2 L}{\partial \dot{x}_r \partial \dot{x}_k} = 0,$$

and hence L is a linear function of velocities $L = L_0(x, t) + \sum_r a_r \dot{x}_r$. Using (7.2.21) yields $a_r = \frac{\partial L}{\partial \dot{x}_r} = p_r$. Then

$$L = L_0(x, t) + \sum_r p_r \dot{x}_r. \tag{7.2.24}$$

The Euler–Lagrange system $\frac{\partial L}{\partial x_k} = \dot{p}_k$ can be expanded as

$$\frac{\partial L_0}{\partial x_k} + \sum_k \frac{\partial p_r}{\partial x_k}\dot{x}_r = \sum_r \frac{\partial p_k}{\partial x_r}\dot{x}_r + \frac{\partial p_k}{\partial t},$$

where in the left side we used (7.2.24) and in the right side we used $p_k = p_k(x, t)$. Identifying the coefficients yields

$$\frac{\partial p_r}{\partial x_k} = \frac{\partial p_k}{\partial x_r}, \qquad \frac{\partial p_k}{\partial t} = \frac{\partial L_0}{\partial x_k}, \tag{7.2.25}$$

which shows the one-form

$$L\,dt = L_0\,dt + \sum_r p_r\,dx_r$$

is exact. This means there is a function $S = S(x, t)$ such that $L\,dt = dS$ along the solutions. ■

Corollary 7.12 *Let S be the function given by Theorem 7.11. Then*

$$\int_0^\tau L\,dt = S(\tau) - S(0).$$

The function S is the action associated with the Lagrangian L.

7.2.2 Examples

Example 7.2.1 *A unit mass particle in a uniform circular motion*

Consider the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - y\dot{x}). \tag{7.2.26}$$

In polar coordinates,

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

The Lagrangian becomes

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + r^2\dot{\phi}. \tag{7.2.27}$$

The Euler–Lagrange system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

yields

$$\ddot{r} = r\dot{\phi}^2 + 2r\dot{\phi}, \quad \frac{d}{dt}(r^2\dot{\phi} + r^2) = 0. \tag{7.2.28}$$

The second equation gives a first integral $r^2(\dot{\phi} + 1) = C$(constant). Considering the initial condition $r(0) = 0$, we get $C = 0$ and $\dot{\phi} = -1$. Hence, the first equation of (7.2.28) becomes $\ddot{r} = -r$. The solution corresponding to the boundary conditions

$$r(0) = 0\,, \qquad r(\tau) = R$$

is

$$r(t) = \frac{R\sin t}{\sin\tau}, \qquad t \in [0, \tau]. \tag{7.2.29}$$

The Lagrangian along the solution is

$$L(r(t), \dot{r}(t), \dot{\phi}(t)) = \frac{1}{2}\frac{R^2}{\sin^2\tau} - \frac{R^2\sin^2 t}{\sin^2\tau} = \frac{R^2}{2\sin^2\tau}\cos 2t.$$

And the action starting at the origin at the moment $t_0 = 0$ is

$$S(\tau, x(\tau), y(\tau)) = \int_0^\tau L(r(t), \dot r(t), \dot\phi(t))\, dt$$
$$= \frac{1}{2}\frac{R^2}{\sin^2\tau}\int_0^\tau \cos 2t\, dt = R^2 \cot\tau$$
$$= \frac{1}{2}\big(x^2(\tau) + y^2(\tau)\big)\cot\tau.$$

Thus S behaves like a Euclidean distance from the origin. The action starting outside of the origin is treated in Chapter 12.

Proposition 7.13 *The action*

$$S(\tau, x, y) = \frac{1}{2}(x^2 + y^2)\cot\tau$$

is a solution for the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial\tau} + \frac{1}{2}\Big[\Big(\frac{\partial S}{\partial x}\Big)^2 + \Big(\frac{\partial S}{\partial y}\Big)^2\Big] + \frac{1}{2}\Big(\frac{\partial S}{\partial x}y - \frac{\partial S}{\partial y}x\Big) + \frac{1}{8}(x^2 + y^2) = 0,$$
$$S(0, (0, 0)) = 0.$$

Proof. The Hamiltonian associated with the above Lagrangian is

$$H = p_1\dot x + p_2\dot y - L$$

where

$$p_1 = \frac{\partial L}{\partial\dot x} = \dot x - \frac{1}{2}y, \quad \dot x = p_1 + \frac{1}{2}y,$$
$$p_2 = \frac{\partial L}{\partial\dot y} = \dot y - \frac{1}{2}x, \quad \dot y = p_2 + \frac{1}{2}x.$$

Performing the computation, we obtain

$$H(p, x, y) = \frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_1 y - p_2 x) + \frac{1}{8}(x^2 + y^2).$$

∎

Example 7.2.2 *A unit mass particle under the influence of an inverse quadratic potential*

Consider the Lagrangian

$$L(x, \dot x) = \frac{1}{2}(\dot x_1^2 + \dot x_2^2) + \frac{1}{2}\frac{k^2}{x_1^2 + x_2^2}, \tag{7.2.30}$$

which describes the trajectory of a particle in the x-plane under the influence of the potential

$$U(x) = -\frac{1}{2}\frac{k^2}{|x|^2} \tag{7.2.31}$$

where k is a constant. The Lagrangian is rotational invariant, therefore, polar coordinates (r, ϕ) are more suitable:

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}\frac{k^2}{r^2}. \tag{7.2.32}$$

In order to find the action, we shall use the Hamiltonian formalism. The momenta are

$$p_1 = \frac{\partial L}{\partial \dot{r}} = \dot{r}, \quad p_2 = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} \tag{7.2.33}$$

and hence the Hamiltonian is

$$H(p, r) = p_1\dot{r} + p_2\dot{\phi} - L = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{r^2}\right) - \frac{1}{2}\frac{k^2}{r^2}. \tag{7.2.34}$$

As $\dot{p}_2 = \dfrac{\partial H}{\partial \phi} = 0$, the momentum p_2 is a constant of motion (called areal velocity). Another constant of motion is the total energy

$$\frac{dH}{dt} = 0 \quad \Longrightarrow \quad H \quad \text{is constant along solutions.}$$

From equations (7.2.32), (7.2.33) and (7.2.34) we find that along a solution,

$$L = H + \frac{k^2}{r^2} = H + \frac{k^2}{p_2}\dot{\phi},$$

and hence the action is

$$S(\tau) = S(0) + \int_0^\tau L = S(0) + H\tau + \frac{k^2}{p_2}(\phi(\tau) - \phi(0)). \tag{7.2.35}$$

The constants H and p_2 should be written in terms of the boundary conditions

$$R = r(\tau), \quad r_0 = r(0), \quad \Phi = \phi(\tau), \quad \phi_0 = \phi(0).$$

In general, this cannot be done explicitly. From (7.2.33) and (7.2.34),

$$E = \dot{r}^2 + \frac{p_2^2 - k^2}{r^2},$$

where $E = 2H$. Let $\alpha = p_2^2 - k^2$ and write

$$\dot{r} = \pm\frac{\sqrt{r^2E - \alpha}}{r}. \tag{7.2.36}$$

There are three cases to investigate:
i) $\alpha = 0$**:** Then $r(t) = \pm\sqrt{E}t + r_0$, and $r_0 < R$ yields

$$r(t) = \frac{R - r_0}{\tau} t + r_0.$$

Integrating the Hamilton's equation

$$\dot{\phi} = \frac{\partial H}{\partial p_2} = \frac{p_2}{r^2} = \frac{k}{r^2},$$

yields

$$\phi(\tau) - \phi_0 = k \int_0^\tau \frac{dt}{(\sqrt{E}t + r_0)^2} = \frac{k}{\sqrt{E}} \Big(\frac{1}{r_0} - \frac{1}{\sqrt{E}\tau + r_0} \Big) = \frac{k}{\sqrt{E}} \Big(\frac{1}{r_0} - \frac{1}{R} \Big).$$

Using the expression for E,

$$\Phi - \phi_0 = \frac{k\tau}{R - r_0} \Big(\frac{1}{r_0} - \frac{1}{R} \Big) = \frac{k\tau}{r_0 R}. \tag{7.2.37}$$

Substituting $H\tau = \dfrac{E\tau}{2} = \dfrac{(R - r_0)^2}{\tau}$ and (7.2.37) in equation (7.2.35) yields

$$S(\tau) = S(0) + \frac{(R - r_0)^2}{2\tau} + \frac{k^2 \tau}{r_0 R}. \tag{7.2.38}$$

ii) $\alpha > 0$**:** Integrating (7.2.36), where we consider a positive sign, yields

$$\int_{r_0}^{r(t)} \frac{r\, dr}{\sqrt{Er^2 - \alpha}} = t,$$

$$\sqrt{Er^2(t) - \alpha} = \sqrt{Er_0^2 - \alpha} + Et,$$

$$r^2(t) = \frac{1}{E} \Big(\alpha + \big(Et + \sqrt{Er_0^2 - \alpha} \big)^2 \Big). \tag{7.2.39}$$

Integrating the Hamilton's equation $\dot{\phi} = \dfrac{\partial H}{p_2} = \dfrac{p_2}{r^2}$ yields

$$\phi(t) - \phi_0 = p_2 \int_0^t \frac{ds}{r^2(s)} = \frac{p_2}{\sqrt{\alpha}} \Big(\tan^{-1} \frac{Et + \sqrt{Er_0^2 - \alpha}}{\sqrt{\alpha}} - \tan^{-1} \frac{\sqrt{Er_0^2 - \alpha}}{\sqrt{\alpha}} \Big).$$

iii) $\alpha < 0$**:** Consider $\alpha = -a^2$. The function $r(t)$ is still given by the equation (7.2.39), but ϕ is given by

$$\phi(t) - \phi_0 = p_2 \int_0^t \frac{ds}{r^2(s)} = \frac{p_2}{2a} \ln \Big| \frac{Et + \sqrt{Er_0^2 + a^2} - a}{\sqrt{Er_0^2 + a^2 - a}} \cdot \frac{\sqrt{Er_0^2 + a^2} + a}{\sqrt{Et + Er_0^2 + a^2} + a} \Big|.$$

In the case $\alpha \neq 0$, the constants E and p_2 cannot be written explicitly as a function of the boundary conditions, as we did in the case $\alpha = 0$. Finding an explicit formula for the action function is equivalent with solving the nonlinear Hamilton–Jacobi equation

$$2\frac{\partial S}{\partial \tau} + \Big(\frac{\partial S}{\partial r} \Big)^2 + \frac{1}{r^2} \Big(\frac{\partial S}{\partial \phi} \Big)^2 = \frac{k^2}{r^2}.$$

Example 7.2.3 *Kepler's problem*

Consider the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{M}{\sqrt{x_1^2 + x_2^2}}, \tag{7.2.40}$$

which describes the motion of a unit mass particle under the influence of gravitational potential (inverse proportional to distance). The Euler–Lagrange equation is $\ddot{x} = -\frac{M}{|x|^3}x$. In polar coordinates (r, ϕ), the Euler–Lagrange equations are

$$\ddot{r} - r\dot{\phi}^2 = -\frac{M}{r^2},$$
$$\frac{d}{dt}(r^2\dot{\phi}) = 0,$$

which yields $r^2\dot{\phi}$ =constant. This is the second of Kepler's laws, which says that areal velocity is constant. The Hamiltonian is

$$H(p; r, \phi) = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{r^2}\right) - \frac{M}{r}$$

and it is preserved along the solutions. As $\dot{p}_2 = \frac{\partial H}{\partial \phi} = 0$, p_2 is constant. On the other hand, $p_2 = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}r^2$, and hence the momentum p_2 is the areal velocity. Let $E = 2H$, and using $p_1 = \frac{\partial L}{\partial \dot{r}} = \dot{r}$, we obtain

$$\frac{dr}{dt} = \pm\sqrt{E - \frac{p_2^2}{r^2} + \frac{2M}{r}}. \tag{7.2.41}$$

As the areal velocity is constant,

$$\frac{d\phi}{dt} = \frac{p_2}{r^2}. \tag{7.2.42}$$

Divide equations (7.2.41) and (7.2.42), separate the variables and integrate to yield,

$$\int_{r_0}^{r(t)} \frac{dr}{r\sqrt{Er^2 + 2Mr - p_2^2}} = p_2 \int_{\phi_0}^{\phi(t)} d\phi.$$

The substitution $u = 1/r$ yields

$$-\int_{1/r_0}^{1/r(t)} \frac{du}{\sqrt{E + 2Mu - p_2^2u^2}} = p_2(\phi(t) - \phi_0).$$

With $A = E/p_2^2$ and $B = M/p_2^2$ we have

$$-\int_{1/r_0}^{1/r(t)} \frac{du}{\sqrt{A + 2Bu - u^2}} = p_2^2(\phi(t) - \phi_0).$$

Using $A + 2Bu - u^2 = A + B^2 - (u - B)^2$, we get

$$\arccos\Big(\frac{u - B}{\sqrt{A + B^2}}\Big)\Big|_{1/r_0}^{1/r(t)} = p_2^2(\phi(t) - \phi_0).$$

This can be written as

$$r(t) = \frac{1}{B + \sqrt{A + B^2}\,\cos\big(p_2^2(\phi(t) - \phi_0) + C\big)}, \tag{7.2.43}$$

with

$$C = \arccos\left(\frac{\frac{1}{r_0} - B}{\sqrt{A + B^2}}\right),$$

which is an equation for a conic in polar coordinates.

7.3 The Eiconal Equation on Riemannian Manifolds

Let $\phi(s)$ be a solution for the Euler–Lagrange system with Lagrangian $L(x, \dot{x})$, which joins the points $x_0 = \phi(0)$ and $x = \phi(\tau)$ on the Riemannian manifold (M, g). In this section, the *action* $S(\tau) = S(x_0, x, \tau)$ will be considered as the integral of the Lagrangian along the solution

$$S(\tau) = \int_0^{\tau} L(\phi(s), \dot{\phi}(s))\, ds. \tag{7.3.44}$$

Example 7.3.1 *Consider the Lagrangian* $L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2)$ *on* $\mathbb{R}^2$ *with Euler–Lagrange equations* $\ddot{x}_i = 0$, $i = \overline{1,2}$. *The solutions are lines*

$$x_i(s) = k_i s + x_i(0) = (x_i - x_i(0)\,)\frac{s}{\tau} + x_i(0), \quad i = \overline{1,2}.$$

The action becomes

$$\begin{aligned} S(\tau) &= \int_0^{\tau} L(x(s), \dot{x}(s)\,) = \frac{1}{2}\int_0^{\tau} \sum_i \Big(\frac{x_i - x_i(0)}{\tau}\Big)^2 \\ &= \frac{1}{2}\tau \frac{\sum (x_i - x_i(0)\,)^2}{\tau^2} = \frac{d^2(x(0), x)}{2\tau}. \end{aligned}$$

The above formula relates the action and the Euclidian distance. One of the goals of this section is to show that a similar relation holds on Riemannian manifolds. However, in general, the Euler–Lagrange equations cannot be solved explicitly, so we need to find the action working around the solutions.

Consider the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{7.3.45}$$

on the Riemannian manifold (M, g). It is known that the Euler-Lagrange system is

$$\ddot{\phi}^k(s) + \Gamma^k_{ij_{\phi(s)}} \dot{\phi}^i(s)\dot{\phi}^j(s) = 0, \quad k = \overline{1, n}, \tag{7.3.46}$$

which are the geodesic equations. The action $S(\tau)$ corresponding to the initial point x_0 and the final point x is

$$S(\tau) = \int_0^\tau \frac{1}{2} g_{ij}\dot{\phi}^i(s)\dot{\phi}^j(s)\, ds = \frac{1}{2}\int_0^\tau |\dot{\phi}(s)|^2\, ds,$$

where $\phi(s)$ is a solution of (7.3.46) with the boundary conditions $\phi(0) = x_0$, $\phi(\tau) = x$.

The system (7.3.46) can be written globally as $\nabla_{\dot{\phi}(s)}\dot{\phi}(s) = 0$, where ∇ denotes the Levi-Civita connection. The fact that $|\dot{\phi}(s)|^2$ is constant along the geodesic is a consequence of the metric property of the Levi-Civita connection,

$$\dot{\phi}(s)\ g\big(\dot{\phi}(s), \dot{\phi}(s)\big) = 2\, g(\nabla_{\dot{\phi}(s)}\dot{\phi}(s), \dot{\phi}(s)) = 0.$$

It follows that the Holder inequality

$$\int_0^\tau |\dot{\phi}(s)|\, ds \le \Big(\int_0^\tau |\dot{\phi}(s)|^2\, ds\Big)^{\frac{1}{2}} \Big(\int_0^\tau\Big)^{\frac{1}{2}} \tag{7.3.47}$$

can be replaced by the identity

$$\int_0^\tau |\dot{\phi}(s)|\, ds = \Big(\int_0^\tau |\dot{\phi}(s)|^2\, ds\Big)^{\frac{1}{2}} \tau^{\frac{1}{2}}. \tag{7.3.48}$$

If $\phi(s)$ is the geodesic joining the points x_0 and x, the Riemannian distance between them is

$$d(x_0, x) = \int_0^\tau |\dot{\phi}(s)|\, ds. \tag{7.3.49}$$

Hence,

$$\int_0^\tau |\dot{x}(s)|^2\, ds = \frac{d^2(x_0, x)}{\tau},$$

and the action is

$$S(\tau) = \frac{d^2(x_0, x)}{2\tau}. \tag{7.3.50}$$

In the following we shall denote the gradient vector field of a function $f \in \mathcal{F}(M)$ by $\nabla f = g^{ij} f_{;i} \frac{\partial}{\partial x_j}$.

Definition 7.14 *The equation* $|\nabla f|_g^2 = 1$ *is called the* **eiconal equation** *on the Riemannian manifold* (M, g).

The next result shows that the Riemannian distance solves the eiconal equation.

Theorem 7.15. $f(x) = d(x_0, x)$ *is a solution for the eiconal equation* $|\nabla f|_g^2 = 1$ *with the initial condition* $f(x_0) = 0$.

Proof. The Hamiltonian associated with the Lagrangian (7.3.45) is

$$H(p, x) = \frac{1}{2}|p|_g^2 = \frac{1}{2}g_{jk}p^j p^k.$$

Substitute the action (7.3.50) in the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}\,|\nabla S|^2 = 0 \tag{7.3.51}$$

and obtain

$$\begin{aligned}
-\frac{1}{2\tau^2}\,d^2(x) + \frac{1}{2}\,\frac{1}{\tau^2}\left|\nabla(\frac{1}{2}d^2)\right|^2 &= 0 \\
\Longleftrightarrow -d^2(x) + \frac{1}{4}\left|\nabla(d^2(x))\right|^2 &= 0 \\
\Longleftrightarrow |2d\nabla d(x)|^2 &= 4\,d^2(x) \\
\Longleftrightarrow |\nabla d(x)|^2 &= 1,
\end{aligned} \tag{7.3.52}$$

where $d(x) = d(x_0, x)$. ■

Corollary 7.16 *The function* $\Phi(x) = d^2(x_0, x)$ *satisfies the equation*

$$|\nabla\Phi|^2 = 4\,\Phi$$

with the initial condition $\Phi(x_0) = 0$.

Proof. It follows from the equation (7.3.52). ■

The above theorem proves the existence of solutions for the eiconal equation. Unfortunately, the uniqueness does not hold in general. A counterexample is provided below.

The eiconal equation on $\mathbb{R}^2$ takes the form

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1. \tag{7.3.53}$$

For any constant $\lambda \in \mathbb{R}$, the function

$$f_\lambda(x, y) = (x - x_0)\cos\lambda + (y - y_0)\sin\lambda \tag{7.3.54}$$

is a solution of (7.3.53) satisfying the initial condition

$$f_\lambda(x_0, y_0) = 0.$$

The same eiconal equation and initial condition is verified also by the Euclidian distance

$$d(x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}. \tag{7.3.55}$$

Remark 7.17 *The solutions given by (7.3.54) and (7.3.55) are related by*

$$f_\lambda(x, y) = d(x, y) \cdot \cos(\lambda - \theta),$$

where $\theta = \tan^{-1} \dfrac{y - y_0}{x - x_0}$.

7.4 Applications of Eiconal equation

7.4.1 Fundamental solution for the Laplace–Beltrami operator

Consider the Laplacian on $\mathbb{R}^n$, $n \geq 3$,

$$\Delta = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

From Lemma 2.27,

$$\Delta f^\alpha = -\alpha f^{\alpha-2}\Big(- f \Delta f + (\alpha - 1)|\nabla f|^2\Big).$$

Substituting $f(x) = d(x)$ and using the eiconal equation yields

$$\Delta(d^\alpha) = \alpha d^{\alpha-2}\Big(- d\, \Delta d + (\alpha - 1)\Big). \tag{7.4.56}$$

From Corollary 2.25,

$$\Delta d^2 = 2d\Delta d - 2|\nabla d|^2. \tag{7.4.57}$$

Using $\Delta d^2 = -2n$ and $|\nabla d|^2 = 1$, (7.4.57) yields

$$d\Delta d = 1 - n.$$

Substituting in (7.4.56),

$$\Delta(d^\alpha) = -\alpha d^{\alpha-2}(n - 2 + \alpha).$$

Hence, choosing $\alpha = 2 - n$,

$$\Delta\Big(\frac{1}{d^{n-2}(x)}\Big) = 0, \qquad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

7.4.2 Fundamental Singularity for the Laplacian

Consider the Laplacian

$$\Delta = -\sum_{j,k=1}^{n} g^{jk}\Big(\frac{\partial^2}{\partial x_j \partial x_k} - \Gamma_{jk}^r \frac{\partial}{\partial x_r}\Big)$$

on a Riemannian manifold (M, g). Given a fixed point $y \in M$, we cannot calculate in general a fundamental solution for Δ, but we can find a fundamental singularity $G(y, x)$:

$$\Delta G(y, x) = R(y, x), \quad \text{for } y \neq x,$$

with

$$R(y, x) = O\Big(\frac{1}{|y - x|^{n-1}}\Big).$$

For x and y nearby points, the distance is given by

$$d^2(y, x) = D(y, x) + O(|x - y|^3)$$

with $D(y, x) = \sum_{j,k} g_{jk}(y)(x_j - y_j)(x_k - y_k)$. In order to compute $\Delta D(y, x)$, substitute $u = x - y$ and get $D(y, u) = \sum_{j,k} g_{jk}(y)u^j u^k$. The Laplacian becomes $\Delta = -(P + L)$ with the principal part $P = \sum g^{ik} \frac{\partial^2}{\partial u^j \partial u^k}$ and the linear part $L = \sum g^{jk}\Gamma_{jk}^r \frac{\partial}{\partial u^r}$. One may show that $LD(y, u) = O(|u|)$, while a computation shows $PD(y, u) = 2\sum g^{jk} g_{jk} = 2n$. Hence,

$$\Delta d(y, x)^2 = -2n + O(|y - x|). \tag{7.4.58}$$

Using the eiconal equation, (7.4.57) yields

$$d\Delta d = 1 - n + O(|y - x|).$$

Substituting in (7.4.56),

$$\Delta d^\alpha = \alpha d^{\alpha-2}\big(n - 1 + \alpha - 1 + O(|y - x|)\big).$$

Choosing $\alpha = 2 - n$, as $d(y, x) = O(|y - x|)$, we get

$$\Delta\Big(\frac{1}{d(y, x)^{n-2}}\Big) = O\Big(\frac{1}{|x - y|^{n-1}}\Big).$$

7.4.3 Laplacian momenta on a compact manifold

Consider a compact Riemannian manifold (M, g), without boundary. Let $x_0 \in M$ be a fixed point. Define the Laplacian momenta with respect to x_0 by

$$\mu_k(x_0) = \int_M d^k(x_0, x)\, \Delta d(x_0, x) \sqrt{|g|}dx_1 \wedge \cdots \wedge dx_n, \quad k \in \mathbb{N},$$

where $d(x_0, x)$ is the Riemannian distance starting from x_0.

By the divergence theorem, $\mu_0 = 0$. Integrating in formula (7.4.57) and applying the eiconal equation for d, we have $\mu_1 = \mathrm{vol}(M)$. The first two momenta do not depend on the point x_0.

Proposition 7.18 *For any $x_0 \in M$,*

$$0 < \mu_k(x_0) \leq kD^{k-1}vol(M), \quad k \geq 1, \tag{7.4.59}$$

where $D = dia(M)$.

Proof. Integrate in equation (7.4.56) and apply the divergence theorem

$$\mu_{\alpha-1} = (\alpha - 1) \int_M d^{\alpha-2} > 0.$$

Using $d \leq D$ yields (7.4.59). ∎

7.4.4 Minimizing geodesics

The goal of this section is to show that locally, geodesics are length minimizing. This will be done using the eiconal equation and the action defined in the previous sections. We shall use that the geodesics are the projections on M space of solutions of Hamilton's system of equations with Hamiltonian $H(p, x) = \frac{1}{2}|p|^2$. By the length of a curve $c : [0, 1] \to M$ we mean $\ell(c) = \int_0^1 |\dot{c}(s)|\, ds = \int_0^1 \sqrt{g(\dot{c}(s), \dot{c}(s)}\, ds$. We shall show that locally, among all the curves that join any two given points, the geodesic is the shortest curve.

For this, it is useful to use a special frame in which the formulas involved look simpler.

Lemma 7.19 *(Existence of a local orthonormal frame of vector fields)*
For a given point $p \in M$, there is a neighborhood $\mathcal{U}$ of p and n vector fields $X_1, \ldots, X_n$ on $\mathcal{U}$ such that

$$g_x(X_i, X_j) = \delta_{ij}, \qquad \forall x \in \mathcal{U}.$$

Proof. Consider an orthonormal frame $\{E_1, \dots, E_n\} \subset T_pM$, i.e., $g_p(E_i, E_j) = \delta_{ij}$. Let γ_v be the geodesic such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$, with $\gamma_v : [0, s_1] \to M$ such that there are no conjugate points between $\gamma(0)$ and $\gamma(s_1)$. Denote $\mathcal{U} = \{\gamma_v(s); s \in [0, s_1], v \in T_pM\}$. The parallel transport of E_k along all geodesics γ_v, $v \in T_pM$ yields a local vector field X_k on $\mathcal{U}$ with $X_{k(p)} = E_k$, $\forall k = \overline{1, n}$. As the parallel transport preserves the lengths and the angles, we get $g_x(X_i, X_j) = \delta_{ij}, \forall x \in \mathcal{U}$. ■

Proposition 7.20 *If $\{X_1, \dots, X_n\}$ is a local orthonormal frame of vector fields, then the gradient of a function f is given by*

$$\nabla f = \sum_{k=1}^{n} X_k(f)\, X_k. \tag{7.4.60}$$

Proof. Using the definition of the gradient,

$$\langle \nabla f, X_k \rangle = X_k(f).$$

Then,

$$\nabla f = \sum_{k=1}^{n} (\nabla f)^k\, X_k = \sum_{k=1}^{n} \langle \nabla f, X_k \rangle\, X_k = \sum_{k=1}^{n} X_k(f)\, X_k.$$

■

The Hamiltonian in a local orthonormal frame can be written as

$$H(p, x) = \frac{1}{2} \sum_{k=1}^{n} p(X_k)^2. \tag{7.4.61}$$

If $p = df$,

$$H(df, x) = \frac{1}{2} \sum_{k=1}^{n} df(X_k)^2 = \frac{1}{2} \sum_{k=1}^{n} X_k(f)^2 = \frac{1}{2} |\nabla f|^2.$$

For $f = S$, where S is the action along a geodesic $c(s)$ parametrized by arc length, we have

$$H(dS, x) = \frac{1}{2} |\nabla S|^2 = \frac{1}{2} |\dot{c}|^2 = \frac{1}{2}.$$

We may rewrite this as the fact that the action S satisfies the eiconal equation

$$|\nabla S|^2 = (X_1 S)^2 + (X_2 S)^2 = 1. \tag{7.4.62}$$

Lemma 7.21 *Given a point $p \in M$, there is a neighborhood $\mathcal{U}$ of p, such that for any vector v tangent at $\mathcal{U}$,*

$$|dS(v)| \le |v|. \tag{7.4.63}$$

Proof. Using an orthonormal frame of vector fields in a neighborhood of p,

$$dS(v) = dS(\sum v^k X_k) = \sum v^k dS(X_k) = \sum v^k X_k(S). \tag{7.4.64}$$

Cauchy's inequality yields

$$|dS(v)| \leq \sqrt{\sum (v^k)^2}\sqrt{\sum X_k(S)^2} = |v| \cdot |\nabla S| = |v|, \tag{7.4.65}$$

where we used (7.4.62). ∎

Theorem 7.22. *Given two points p and q that are close enough, the geodesic is the shortest curve connecting p and q.*

Proof. Let c be a geodesic joining p and q. We shall assume that c is parametrized by arc length, *i.e.*, $c : [0, L] \to M$, where $L = \ell(c)$ is the length of c. Consider an arbitrary curve γ with the same endpoints as c and parametrized by the same interval $[0, L]$. Then

$$\int_c dS = \int_\gamma dS. \tag{7.4.66}$$

The left side is

$$\int_c dS = \int_0^L dS(\dot{c}(s))\, ds = \int_0^L \langle \nabla S, \dot{c} \rangle\, ds = |\dot{c}|^2 L = L = \ell(c),$$

where we used $\nabla S = \dot{c}$. Using Lemma 7.21, the right side becomes

$$\int_\gamma dS = \int_0^L dS(\dot{\gamma}(s))\, ds \leq \int_0^L |\dot{\gamma}|\, ds = \ell(\gamma).$$

Hence $\ell(c) \leq \ell(\gamma)$. The identity holds when Cauchy's inequality becomes the identity, *i.e.*, when $\dot{\gamma}$ and $\dot{c} = \nabla S$ are proportional. This means that the curves c and γ coincide up to a reparametrization. ∎

7.5 Exercises

1. Consider $X_1, \ldots, X_n$ a frame of orthonormal vector fields on the manifold (M, g). Define $D : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by $D_V W = \sum_k V g(W, X_k) X_k$. Show:

(i) D is a metric linear connection.
(ii) D is a symmetric connection iff $[X_i, X_j] = 0, \forall i, j = \overline{1, n}$.

2. Define the divergence with respect to connection D by $\text{div} Z = \text{Trace}_g(V \to D_V Z) = \sum_k g(X_k, D_{X_k} Z)$. Show that:
(i) $\text{div} Z = \sum_k X_k(Z^k)$, where $Z = \sum_k Z^k X_k$.

(ii) For any smooth function f on M, we have $\mathrm{div} \nabla f = \sum_k X_k^2 f$.

3. Define $\mathcal{D} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{D}_V W = \sum_k V g(W, X_k) X_k + \frac{1}{2} \sum_{k,j} g(W, X_k) g(V, X_j) [X_k, X_j].$$

(i) Show that $\mathcal{D}$ is a linear connection.
(ii) Prove that $\mathcal{D}$ has free torsion: $\mathcal{D}_V W - \mathcal{D}_W V = [V, W]$.
(iii) Is $\mathcal{D}$ a metric connection?
(iv) Compute the divergence with respect to $\mathcal{D}$.

4. Show that for every $x_0 \in M$, the series $\sum \mu_k(x_0)$ is convergent, where M is a Riemannian manifold with $dia(M) < 1$.

5. Do the momenta μ_k depend on the choice of x_0?

6. Prove or disprove: *Two manifolds of the same dimension with the same momenta are isometric.*

7. Find the action in the case of the Kepler problem defined by the Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{M}{\sqrt{x_1^2 + x_2^2}},$$

where $M > 0$ is a constant.

8

Minimal Hypersurfaces

8.1 The Curl tensor

In Classical Mechanics the dynamics of a flow are described by its rotation and expansion. The rotation component is given by the *curl* vector, while the expansion is described by the *divergence* function. The classical formulas involving rotation and expansion in the case of a function $\phi \in \mathcal{F}(\mathbb{R}^3)$ and a vector field $V \in \mathcal{X}(\mathbb{R}^3)$ are

$$curl(grad\,\phi) = 0 \quad \text{and} \quad div(curl\,V) = 0. \tag{8.1.1}$$

The first of the above formulas shows that gradient vector fields do not have rotation and the latter says that the curl vector field is incompressible (zero expansion). On Riemannian manifolds the *curl* of a vector field is not a vector field, but a tensor.

Definition 8.1 *The curl of a vector field X on a Riemannian manifold (M, g) is defined as a 2-covariant antisymmetric tensor A with the components A_{ij} given by*

$$A_{ij} = X_{i;j} - X_{j;i}. \tag{8.1.2}$$

Using the definition of the covariant derivative one may show that (see Exercise 1)

$$A_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}. \tag{8.1.3}$$

The next proposition shows that the first formula of (8.1.1) takes place on manifolds.

Proposition 8.2 *If $X \in \mathcal{X}(M)$ is a vector field,*

$$X = grad\,\phi \iff curl\,X = 0.$$

Proof. Let $X = grad\,\phi$. Then $X^k = g^{kj}\dfrac{\partial \phi}{\partial x_j}$ or $X_i = \dfrac{\partial \phi}{\partial x_i}$. Equation (8.1.3) yields

$$(curl\,X)_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Reciprocally, consider a vector field X such that $curl(X) = 0$. Then $\dfrac{\partial X_k}{\partial x_j} = \dfrac{\partial X_j}{\partial x_k}$. Hence the one-form $\omega = \sum X_k\, dx_k$ is exact. This means there is a function f, defined locally, such that $\omega = df = \sum \frac{\partial f}{\partial x_k} dx_k$. Therefore $X_k = \dfrac{\partial f}{\partial x_k}$ or $X^j = \sum g^{kj} \frac{\partial f}{\partial x_k}$, *i.e.*, $X = grad\ f$. ■

The following result is an analog of the second formula of (8.1.1).

Proposition 8.3 *We have:*

$$Trace\ curl\ X = 0, \quad \forall X \in \mathcal{X}(M).$$

Proof.

$$Trace\ curl\ X = g^{ij}(X_{i;j} - X_{j;i}) = X^j_{;j} - X^i_{;i} = 0.$$

■

The following result deals with a Bianchi type identity.

Proposition 8.4 *The cyclic covariant derivative of $A = curl\ X$ is zero,*

$$A_{ij;k} + A_{jk;i} + A_{ki;j} = 0. \tag{8.1.4}$$

Proof. Use the definition of the *curl* and cancel the terms in pairs. ■

The following proposition provides a global, invariant written formula for *curl*. The Riemannian metric is denoted by $\langle\ \ \rangle$, and its associated Levi-Civita connection by ∇.

Proposition 8.5 *If $A = curl\ X$, we have*

$$A(U, V) = \langle \nabla_V X, U\rangle - \langle \nabla_U X, V\rangle \quad \forall\, U, V \in \mathcal{X}(M). \tag{8.1.5}$$

Proof. For every $U, V \in \mathcal{X}(M)$,

$$\begin{aligned} A(U, V) &= A_{ij}U^i V^j = (X_{i;j} - X_{j;i})U^i V^j = (\nabla_{\partial_j} X)_i U^i V^j - (\nabla_{\partial_i} X)_j U^i V^j \\ &= \langle \nabla_{\partial_j} X, U\rangle V^j - \langle \nabla_{\partial_i} X, U\rangle U^i = \langle \nabla_{V^j \partial_j} X, U\rangle - \langle \nabla_{U^i \partial_i} X, V\rangle \\ &= \langle \nabla_V X, U\rangle - \langle \nabla_U X, V\rangle. \end{aligned}$$

■

Lemma 8.6 *Let $A = curl\ X$, where $X \in \mathcal{X}(M)$. Then we have*

$$A(U, V) = V\langle X, U\rangle - U\langle X, V\rangle + \langle X, [U, V]\rangle. \tag{8.1.6}$$

Proof. Since ∇ is a metric connection

$$V\langle X, U\rangle = \langle \nabla_V X, U\rangle + \langle X, \nabla_V U\rangle,$$

$$U\langle X, V\rangle = \langle \nabla_U X, V\rangle + \langle X, \nabla_U V\rangle.$$

Using the symmetry of ∇, subtracting we obtain

$$V\langle X, U\rangle - U\langle X, V\rangle = A(U, V) + \langle X, [V, U]\rangle,$$

which is equivalent to (8.1.6). ∎

The following result makes the relation between the *curl*, Levi-Civita connection, and the Lie derivative.

Theorem 8.7. *If $A = curl\ X$ and ∇ is the Levi-Civita connection on (M, g),*

$$A(U, V) = 2\langle\nabla_V X, U\rangle - (L_X g)(U, V). \tag{8.1.7}$$

Proof. From the Koszul formula for Levi-Civita connection, we have

$$2\langle\nabla_V X, U\rangle = V\langle X, U\rangle + X\langle U, V\rangle - U\langle V, X\rangle - \langle V, [X, U]\rangle + \langle X, [U, V]\rangle + \langle U, [V, X]\rangle.$$

Lemma 8.6 yields

$$\begin{aligned} 2\langle\nabla_V X, U\rangle &= A(U, V) + X\langle U, V\rangle - \langle V, [X, U]\rangle + \langle U, [V, X]\rangle \\ &= A(U, V) + X\langle U, V\rangle - \langle V, L_X U\rangle - \langle U, L_X V\rangle. \end{aligned}$$

Using

$$(L_X g)(U, V) = X\langle U, V\rangle - \langle L_X U, V\rangle - \langle U, L_X V\rangle$$

yields

$$2\langle\nabla_V X, U\rangle = A(U, V) + (L_X g)(U, V).$$

∎

Corollary 8.8 *If X is a Killing vector field (i.e., $L_X g = 0$), then*

$$(curl\ X)(U, V) = 2\langle\nabla_V X, U\rangle.$$

Corollary 8.9 *If X is a vector field provided by a gradient (i.e., $X = grad\ \phi$), then*

$$(L_X g)(U, V) = 2\langle\nabla_V X, U\rangle.$$

Definition 8.10 *Let $f \in \mathcal{F}(M)$ be a function. Define the torsion of f by $T_f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$,*

$$T_f(U, V) = V(f)U - U(f)V. \tag{8.1.8}$$

As T_f is $\mathcal{F}(M)$-linear in each argument, it follows that T_f is a 2-covariant tensor.

Proposition 8.11 *The torsion has the following properties:*

$$\begin{aligned} &(i) && T_f(U, V) = -T_f(V, U). \\ &(ii) && Trace\ T_f = 0. \\ &(iii) && T_{fh} = fT_h + hT_f, \quad \forall f, h \in \mathcal{F}(M). \\ &(iv) && T_f(U, V) = 0, \forall U, V \Longrightarrow f \text{ is constant.} \end{aligned}$$

Proof.

$$(i) \qquad T_f(U,V) = -\big(U(f)V - V(f)U\big) = T_f(V,U).$$

$$\begin{aligned}(ii) \qquad Trace\, T_f &= g^{ij} T_f(\partial_i, \partial_j) = g^{ij}\big((\partial_i f)\partial_j - (\partial_j f)\partial_i\big)\\ &= grad\, f - grad\, f = 0.\end{aligned}$$

$$\begin{aligned}(iii) \qquad T_{fh}(U,V) &= V(fh)U - U(fh)V\\ &= fV(h)U + hV(f)U - fU(h)V - hU(f)V\\ &= f\Big(V(h)U - U(h)V\Big) + h\Big(V(f)U - U(f)V\Big)\\ &= fT_h(U,V) + hT_f(U,V).\end{aligned}$$

(iv) Taking U and V linear independent vector fields, yields $V(f) = U(f) = 0$, for any vector fields U and V. Hence f is constant. ■

The following result shows that $curl$ is not $\mathcal{F}(M)$-linear in X. However it is still a tensor, because it is $\mathcal{F}(M)$-linear in the arguments of U and V, when considering $curl(X)(U,V)$.

Proposition 8.12 *Let $f \in \mathcal{F}(M)$ and $X \in \mathcal{X}(M)$. Then*

$$curl(fX) = f\, curl(X) + \langle T_f, X\rangle. \tag{8.1.9}$$

Proof. Denote $A = curl(X)$ and $A_f = curl(fX)$. Applying Lemma 8.6 yields

$$\begin{aligned}A_f(U,V) &= V\langle fX,U\rangle - U\langle fX,V\rangle + \langle fX,[U,V]\rangle\\ &= V(f)\langle X,U\rangle + fV\langle X,U\rangle - fU\langle X,V\rangle - U(f)\langle X,V\rangle + f\langle X,[U,V]\rangle\\ &= f\Big(V\langle X,U\rangle - U\langle X,V\rangle + \langle X,[U,V]\rangle\Big) + V(f)\langle X,U\rangle - U(f)\langle X,V\rangle\\ &= fA(U,V) + \langle X, V(f)U - U(f)V\rangle = fA(U,V) + \langle X, T_f(U,V)\rangle.\end{aligned}$$

■

Proposition 8.13 *For any vector field X on a Riemannian manifold (M, g),*

$$Trace\,(L_X g) = 2\, div\, X. \tag{8.1.10}$$

Proof. Taking the trace in Theorem 8.7,

$$Trace\, A = 2\, Trace\big(V \to \langle \nabla_V X, V\rangle\big) - Trace(L_X g).$$

Proposition 8.3 yields $Trace\, A = 0$. Using the definition of the divergence as a trace, we obtain (8.1.10). ■

8.2 Application to minimal hypersurfaces

Let $\mathcal{H} \subset M$ be a hypersurface given locally by $\phi^{-1}\{0\} = \{x \in M | \phi(x) = 0\}$. Denote the gradient vector field by $X = \nabla\phi$. The unit normal vector is

$$N = \frac{X}{\|X\|} = \frac{\nabla\phi}{\|\nabla\phi\|}.$$

Denote $f = \dfrac{1}{\|X\|}$. Then $N = fX$, and for any vector field V tangent to $\mathcal{H}$,

$$\nabla_V N = \nabla_V(fX) = f\nabla_V X + V(f)X,$$

where ∇ is the Levi-Civita connection on (M, g). Therefore, for any $U \in \mathcal{X}(\mathcal{H})$,

$$\langle \nabla_V N, U\rangle = f\langle \nabla_V X, U\rangle + V(f)\langle X, U\rangle.$$

As $X = \nabla\phi$ is normal to $\mathcal{H}$, then $\langle X, U\rangle = 0$. Hence

$$\langle \nabla_V N, U\rangle = f\langle \nabla_V X, U\rangle, \quad \forall U, V \in \mathcal{X}(\mathcal{H}).$$

Corollary 8.9 yields

$$(L_X g)(U, V) = 2\|X\|\, \langle \nabla_V N, U\rangle, \quad \forall U, V \in \mathcal{X}(\mathcal{H}). \tag{8.2.11}$$

Recall the Weingarten map, which is a tensor $S \in \mathcal{T}^{1,1}(\mathcal{H})$ defined as

$$\langle S(V), U\rangle = -\langle \nabla_V N, U\rangle, \quad \forall U, V \in \mathcal{X}(\mathcal{H}). \tag{8.2.12}$$

Then (8.2.11) yields

$$-2\,\|X\|\,\langle S(V), U\rangle = (L_X g)(U, V). \tag{8.2.13}$$

Definition 8.14 *If $\{e_1, \ldots, e_{n-1}\} \subset T_p\mathcal{H}$ is an orthonormal frame, the mean scalar curvature of $\mathcal{H}$ at point p is given by:*

$$\alpha_p = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle S(e_i), e_i\rangle = \frac{1}{n-1}\, Trace\, S. \tag{8.2.14}$$

Using (8.2.13) we get

$$\alpha_p = \frac{-1}{2(n-1)}\,\frac{1}{\|X\|_p}\sum_{i=1}^{n-1}(L_X g)(e_i, e_i). \tag{8.2.15}$$

In order to find a formula for the right-hand side of (8.2.15), we shall complete $\sum_{i=1}^{n-1}(L_X g)(e_i, e_i)$ up to $Trace\, L_X g$ on the manifold (M, g). In order to perform that, we need the following result.

Lemma 8.15 *If $N = fX$ and $f = \|X\|^{-1}$, then*

$$(L_X g)(N, N) = -2\frac{X(f)}{f}. \tag{8.2.16}$$

Proof. Using $L_X(fX) = [X, fX] = X(f)X$, we have

$$\begin{aligned}(L_X g)(N, N) &= X\langle N, N\rangle - 2\langle L_X N, N\rangle = -2\langle L_X N, N\rangle \\ &= -2\langle L_X(fX), fX\rangle = -2\langle X(f)X, fX\rangle \\ &= -2f\,X(f)\|X\|^2 = -2\frac{X(f)}{f}.\end{aligned}$$

■

Theorem 8.16. *The following relation takes place:*

$$\alpha_p = -\frac{1}{n-1}\,div\,N_{|p}.$$

Proof. Let $\{e_1, \ldots, e_{n-1}\} \subset T_p\mathcal{H}$ be an orthonormal basis. Choose $e_n = N_p$. Then $\{e_1, \ldots, e_{n-1}, e_n\}$ is an orthonormal basis in T_pM. Then at point p,

$$Trace\,(L_X g) = \sum_{i=1}^{n}(L_X g)(e_i, e_i) = \sum_{i=1}^{n-1}(L_X g)(e_i, e_i) + (L_X g)(N, N).$$

Using Lemma 8.15 and Proposition 8.13, we have

$$2\,div\,X = -2(n-1)\alpha_p\|X\| - 2\|X\|X(f).$$

This can be written also as

$$-\big(f\,div\,X + X(f)\big) = (n-1)\alpha_p.$$

As the left side is equal to $-div(fX) = -div\,N$, we get

$$\alpha_p = -\frac{div\,N}{n-1}.$$

■

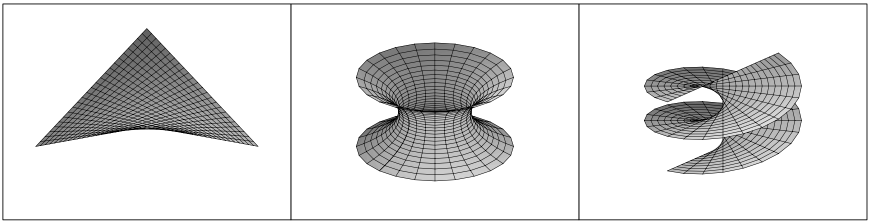
Proposition 8.17 *Let (M, g) be a Riemannian manifold and $\mathcal{H} \subset M$ be a hypersurface with the unit normal vector field N. The following statements are equivalent:*
1) $\mathcal{H}$ is a minimal hypersurface of M,

2) $div\,N_{|\mathcal{H}} = 0$.

In the following we provide a few examples.

Example 8.2.1 *Consider* $M = \mathbb{R}^n$ *and* $\mathcal{H} = \{x_n = 0\}$. *The normal vector field is* $N = e_n = (0, \ldots, 0, 1)$ *and* $div\, N = 0$. *Hence* $\mathcal{H}$ *is a minimal hypersurface in* $\mathbb{R}^3$.

Example 8.2.2 *Let* $\mathbb{S}^{n-1}$ *be the* $n-1$*-dimensional sphere in* $\mathbb{R}^n$. *The unit vector field* $N_x = \sum_{i=1}^{n} \frac{x_i}{|x|} \partial_{x_i}$ *is normal to* $\mathbb{S}^{n-1}$ *and has* $div\, N = \frac{n-1}{|x|}$. *(See Exercise 5.) Hence the mean scalar curvature of* $\mathbb{S}^{n-1}$ *is* $|\alpha| = \frac{n-1}{n-1} = 1$.



Saddle Catenoid Helicoid

Figure 8.1: Examples of surfaces.

Example 8.2.3 *Consider the saddle surface* $\mathcal{H} = \phi^{-1}\{0\}$, $\phi(x, y, z) = xy - z$. *The unit normal vector field is*

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \Big(\frac{y}{\sqrt{x^2 + y^2 + 1}}, \frac{x}{\sqrt{x^2 + y^2 + 1}}, \frac{-1}{\sqrt{x^2 + y^2 + 1}}\Big).$$

Then

$$\begin{aligned} div\, N &= \frac{\partial}{\partial x}\frac{y}{\sqrt{x^2 + y^2 + 1}} + \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2 + y^2 + 1}} + \frac{\partial}{\partial z}\frac{-1}{\sqrt{x^2 + y^2 + 1}} \\ &= \frac{-2xy}{(1 + x^2 + y^2)^{3/2}} = \frac{-2z}{(1 + x^2 + y^2)^{3/2}}. \end{aligned}$$

Hence the mean scalar curvature is

$$\alpha = \frac{z}{(1 + x^2 + y^2)^{3/2}}.$$

Example 8.2.4 *Consider the catenoid parametrized by* $x = \cosh u \cos\theta$, $y = \cosh u \sin\theta$, $z = u$, *for* $0 < u < \sinh^{-1}(1)$ *and* $0 < \theta < 2\pi$. *The coordinate tangent vector fields are*

$$X_1 = (\sinh u \cos\theta, \sinh u \sin\theta, 1), \quad X_2 = (-\cosh u \sin\theta, \cosh u \cos\theta, 0).$$

The unit normal vector field is

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|} = -\frac{(\cosh u \cos\theta, \cosh u \sin\theta, -\sinh u \cosh u)}{\cosh^2 u}.$$

Using $x = \cosh u \cos\theta$, $y = \cosh u \sin\theta$, $z = u$, $x^2 + y^2 = \cosh^2 u$, $\sinh u \cosh u = \cosh u\sqrt{1 + \cosh^2 u} = \sqrt{(x^2 + y^2)(1 + x^2 + y^2)}$*, we obtain*

$$N = \frac{(x, y, -\sqrt{(x^2 + y^2)(1 + x^2 + y^2)}}{x^2 + y^2}.$$

A computation shows that $div\, N = 0$ *(See Exercise 6). Hence the catenoid is a minimal surface in* $\mathbb{R}^3$.

Example 8.2.5 *Consider the helicoid parametrized by* $x = v\cos\phi$, $y = v\sin\phi$, $z = \phi$*, for* $|v| < 1$ *and* $0 < \phi < 2\pi$*. Using the tangent vector fields* $X_1 = (\cos\phi, \sin\phi, 0)$, $X_2 = (-v\sin\phi, v\cos\phi, 1)$ *we construct the unit normal*

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \frac{(\sin\phi, -\cos\phi, v)}{\sqrt{1 + v^2}} = \frac{(y, -x, x^2 + y^2)}{\sqrt{(1 + x^2 + y^2)(x^2 + y^2)}}.$$

By computation $divN = 0$*, see Exercise 7. Hence the helicoid is a minimal surface in* $\mathbb{R}^3$.

Proposition 8.18 *Consider the surface given as a Monge patch* $(x, y) \to (x, y, f(x, y))$*. The surface is minimal in* $\mathbb{R}^3$ *if and only if* f *satisfies the equation*

$$\frac{1}{2}\big(\partial_x^2 f + \partial_y^2 f\big)\sqrt{(\partial_x f)^2 + (\partial_y f)^2 + 1} = (\partial_x f)^2 \cdot \partial_x^2 f + (\partial_y f)^2 \cdot \partial_y^2 f + 2\partial_x f \cdot \partial_y f \cdot \partial_{xy} f. \tag{8.2.17}$$

Proof. The surface is given by $\phi^{-1}(0)$, for $\phi(x, y, z) = f(x, y) - z$. We have $\nabla\phi = (\partial_x f, \partial_y f, -1)$ and $|\nabla\phi| = \sqrt{(\partial_x f)^2 + (\partial_y f)^2 + 1}$. The surface is minimal if and only if $div\, N = 0$, where

$$div\, N = div\Big(\frac{1}{|\nabla\phi|}\nabla\phi\Big) = \frac{1}{|\nabla\phi|} div \nabla\phi + \nabla\phi\Big(\frac{1}{|\nabla\phi|}\Big). \tag{8.2.18}$$

A computation shows

$$\partial_x\Big(\frac{1}{|\nabla\phi|}\Big) = \frac{-2}{|\nabla\phi|^2}\big(\partial_x f \cdot \partial_x^2 f + \partial_y f \cdot \partial_{xy} f\big),$$

$$\partial_y\Big(\frac{1}{|\nabla\phi|}\Big) = \frac{-2}{|\nabla\phi|^2}\big(\partial_y f \cdot \partial_y^2 f + \partial_x f \cdot \partial_{xy} f\big).$$

Therefore

$$\begin{aligned}\nabla\phi\Big(\frac{1}{|\nabla\phi|}\Big) &= \partial_x f \cdot \partial_x\Big(\frac{1}{|\nabla\phi|}\Big) + \partial_y f \cdot \partial_y\Big(\frac{1}{|\nabla\phi|}\Big) - \partial_z\Big(\frac{1}{|\nabla\phi|}\Big)\\ &= \frac{-2}{|\nabla\phi|^2}\Big((\partial_x f)^2 \cdot \partial_x^2 f + (\partial_y f)^2 \cdot \partial_y^2 f + 2\partial_x f \cdot \partial_y f \cdot \partial_{xy} f\Big).\end{aligned}$$

Substituting in (8.2.18) and using $div\nabla\phi = \partial_x^2 f + \partial_y^2 f$, we get (8.2.17). ■

Corollary 8.19 *Consider the function* $f(x, y) = \sum_{k=0}^{m} a_k x^k y^{m-k}$ *with* $a_m, a_0 \neq 0$. *Then the surface* $(x, y) \to (x, y, f(x, y))$ *is minimal in* $\mathbb{R}^3$ *if and only if* $m = 1$. *In this case* $f(x, y) = a_0 y + a_2 x$ *and corresponds to a plane.*

Proof. We shall investigate the order of magnitude of both sides of equation (8.2.17). Using $\partial_x f = O(|x|^{m-1})$, $\partial_y f = O(|y|^{m-1})$, $\partial_x^2 f = O(|x|^{m-2})$, $\partial_y^2 f = O(|y|^{m-2})$ we get

$$\sqrt{(\partial_x f)^2 + (\partial_y)^2 + 1} = O(|x|^{m-1}, |y|^{m-1}),$$

and the left side of (8.2.17) is $O(|x|^{2m-3}, |y|^{2m-3})$.
Using

$$(\partial_x f)^2 \cdot \partial_x^2 f = O(|x|^{2(m-1)})O(|x|^{m-2)}) = O(|x|^{3m-4}),$$
$$(\partial_y f)^2 \cdot \partial_y^2 f = O(|y|^{2(m-1)})O(|y|^{m-2)}) = O(|y|^{3m-4})$$

the right side is $O(|x|^{3m-4}, |y|^{3m-4})$. For $m = 1$ the left and the right sides have the same order of magnitude. Using Exercise 8, one obtains that the surface is a plane. ∎

8.3 Helmholtz decomposition

This section is an application of the formulas regarding *curl* and *div*. We shall show that a vector field X on a compact Riemannian manifold can be uniquely decomposed as a sum of two vectors Y and Z, where Y is the rotation component and Z the expansion component.

Theorem 8.20. *If X is a vector field on a compact Riemannian manifold* (M, g)*, there are two vector fields Y, Z on M such that*

$$X = Y + Z,$$

with $div\, Y = 0$ *and* $curl\, Z = 0$. *Moreover, the decomposition is unique.*

Proof. **Existence:** Denote $\omega = div\, X$ and let ϕ be the solution of the elliptic equation

$$\Delta\phi = \omega \text{ on } (M, g).$$

Take $Z = \nabla\phi$ and $Y = X - \nabla\phi$. Then $curl\, Z = curl\, \nabla\phi = 0$ and $div\, Y = \omega - \Delta\phi = 0$.
Uniqueness: Consider two decompositions:

$$X = Y_1 + Z_1 = Y_2 + Z_2.$$

As $curl\, Z_i = 0$, it follows that there are two functions ϕ_i such that $Z_i = \nabla\phi_i$, $i = 1, 2$. Subtracting, we get

$$Y_2 - Y_1 = \nabla(\phi_1 - \phi_2).$$

Denoting $U = Y_2 - Y_1$ and $\phi = \phi_1 - \phi_2$, we obtain

$$div\, U = div \nabla\phi.$$

As $div\, U = div\, Y_2 - div\, Y_1 = 0$, we get $\Delta\phi = 0$. By Hopf's lemma we have $\phi =$ constant, or $\phi_1 - \phi_2 =$constant. Taking the gradient yields $Z_1 - Z_2 = 0$. Then we have also $Y_1 = Y_2$ and the decomposition is unique. ∎

We note that $div\, X = div\, Z$ and $curl\, X = curl\, Y$. This can be interpreted as a decomposition in two vector fields Y, Z, where Y contains the rotation and Z contains the expansion.

Example 8.3.1 *Let $X = (x_1 - x_2)\partial_{x_1} + (x_1 + x_2)\partial_{x_2}$. Then the Helmholtz decomposition is $X = Y + Z$, with $Z = x_1\partial_{x_1} + x_2\partial_{x_2}$ and $Y = -x_2\partial_{x_1} + x_1\partial_{x_2}$.*

8.3.0.1 The non-compact case

If the manifold is not compact, the Helmholtz decomposition is not unique. Let $a_1(x_1), a_2(x_2), b_1(x_1), b_2(x_2)$ be smooth functions. Consider the vector field

$$X = \Big(a_2(x_2)\int b_1(x_1)\, dx_1\Big)\partial_{x_1} + \Big(b_1(x_1)\int a_2(x_2)\, dx_2\Big)\partial x_2.$$

Then $div\, X = a_2 b_1 - b_1 a_2 = 0$. Let ϕ be a harmonic function on $\mathbb{R}^2$, for instance

$$\phi(x_1, x_2) = \alpha x_1 + \beta x_2 + \gamma x_1 x_2 + \delta,$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ arbitrary constants. Then

$$Z = \nabla\phi = (\alpha + \gamma x_2)\partial_{x_1} + (\beta + \gamma x_1)\partial_{x_2}$$

is divergence free and $Y = X - Z$ is curl free.

8.4 Exercises

1. Show that for any vector field $X \in \mathcal{X}(M)$ we have

$$X_{i;j} - X_{j;i} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}.$$

2. Show that for any vector field X on a Riemannian manifold M,

$$2X_{i;j} = (L_X g)_{ij} + (curl\, X)_{ij}.$$

3. A vector field X is called *geodesic* if $\nabla_X X = 0$. Show that if X is a Killing vector field provided by a potential, then X is geodesic. (*Hint: Use* $(\nabla_X X)_a = X^a X_{a;b}$ *and Exercise 1.*)

4. (i) Show that

$$(L_X g)_{ij} = X_{i;j} + X_{j;i}.$$

(ii) Taking the trace on both sides, show $Trace\,(L_X g) = 2\,div\,X$.
(iii) Show that any Killing vector field has zero divergence.

5. Consider the unit vector field $N(x) = \sum_{i=1}^{n} \frac{x_i}{|x|} \partial_{x_i}$ on $\mathbb{R}^n \backslash \{0\}$. Show that

$$div\,N(x) = \frac{n-1}{|x|}.$$

6. Let $N = fV$ be a vector field, with $f = 1/(x^2 + y^2)$ and consider the vector fields $V = (x, y, -\sqrt{(x^2 + y^2)(1 + x^2 + y^2)})$.
(i) Show $f div V = 2f$.
(ii) Show $V(f) = -2f$.
(iii) Use the formula $div(fV) = f div V + V(f)$ to show that $div N = 0$.

7. Consider $f = \big((1 + x^2 + y^2)(x^2 + y^2)\big)^{-1/2}$ and the vector field on $\mathbb{R}^3$ given by $X = y\partial_x - x\partial_y + (x^2 + y^2)\partial_z$. Show the following:
(i) $div X = 0$.
(ii) $X(f) = 0$.
(iii) Using $div(fX) = f div X + X(f)$ prove that $div(fX) = 0$.

8. Show that the function $f(x, y) = a_0 y + a_1 xy + a_2 x$ is a solution for equation (8.2.17) if and only if $a_1 = 0$.

9. Show that:
(i) Ellipsoids, paraboloids and hyperboloids are not minimal surfaces in $\mathbb{R}^3$.
(ii) Consider $f(x, y) = \sum_{i,j=0}^{N} a_{ij} x^i y^j$. The function $f(x, y)$ is a solution for the equation (8.2.17) if and only if $N = 1$.
(iii) The only minimal surfaces given as $(x, y) \to (x, y, f(x, y))$ are planes.

10. Let (M, g) be a hypersurface in $\mathbb{E}^{n+1} = (\mathbb{R}^{n+1}), \delta_{ij}$ and let S denote the Weingarten map. Show that

$$Ric(X, Y) = g(SX, Y) \cdot Trace\,S - g(SX, SY), \quad \forall X, Y \in \mathcal{X}(M).$$

9

Radially Symmetric Spaces

9.1 Existence and uniqueness of geodesics

Consider the Hamiltonian on the Riemannian manifold (M, g),

$$H(x, p) = \frac{1}{2}|p|_g^2 = \frac{1}{2}g^{ij}p_i p_j, \tag{9.1.1}$$

and let $\nabla H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial p}\right)$. With this notation, the Hamilton system can be written as only one equation

$$\dot{y} = J\nabla H(y) \tag{9.1.2}$$

where $y = (x, p)$ and $J^2 = -I_{2n}$. Using the Hamiltonian equation $p = \dot{x}$ (see Chapter 6), the initial condition becomes

$$y_0 = (x_0, p_0) = (x_0, v),$$

where x_0 is the initial point and v is the initial velocity.

Denote $f(y) = J\nabla H(y)$. The existence and uniqueness problem for geodesics with initial condition $y_0 = (x_0, v)$ becomes:

Under what conditions does the Cauchy problem

$$\dot{y} = f(y), \quad y(0) = y_0, \tag{9.1.3}$$

have solutions, and when is the solution unique?

There are a few theorems that handle this problem. They are based on the regularity of the function f. In the present case this is reduced to the smoothness of the Riemannian metric (g_{ij}).

Existence of geodesics

In the following "$|\ \ |$" denotes any norm on $\mathbb{R}^m$. The following result is a particular case of Peano's existence theorem and the proof can be found in Hartman [20]:

Theorem 9.1. *Denote* $\mathbf{B}(y_0, b) = [y_0 - b, y_0 + b] \subset \mathbb{R}^m$. *Assume the function* $f(y)$ *is continuous on* $\mathbf{B}(y_0, b)$ *with the bound* $|f(y)| \leq M$. *Then there is at least a solution* $y = y(t)$ *for the system (9.1.3) on* $[t_0, t_0 + b/M]$.

When $f(y) = J\nabla H(y)$ the function f is continuous if and only if $\dfrac{\partial H}{\partial x} = \dfrac{1}{2}\dfrac{\partial g^{ij}}{\partial x} p_i p_j$ is continuous. This means that the metric g^{ij} is differentiable with continuous derivatives (*i.e.*, continuous Christoffel symbols). We arrive at the following result:

Proposition 9.2 *Consider* $x_0 \in M$ *such that* $g^{ij} \in \mathcal{C}^1(\mathbf{B}(x_0, b))$. *Given* $v \in T_{x_0}M$, *there is* $a > 0$ *and at least one geodesic* $\phi : [t_0, t_0 + a] \to (M, g)$ *with* $\phi(t_0) = x_0$ *and* $\dot{\phi}(t_0) = v$.

Example 9.1.1 *(Hartman) Consider the Riemannian metric*

$$(g_{ij}) = \begin{pmatrix} 1 + y^{4/3} & 0 \\ 0 & 1 + y^{4/3} \end{pmatrix}$$

on $\mathbb{R}^2$. *The functions* $\dfrac{\partial g_{ii}}{\partial y}$ *are continuous,* $i = 1, 2$. *Then there are at least three geodesics emanating at* $x_0 = (0, 0)$ *with the same initial velocity* $v = (1, 0)$.

By the above theorem we have at least a geodesic. We shall find three distinct geodesics. The Lagrangian and the Hamiltonian are

$$L = \frac{1}{2}(1 + y^{4/3})(\dot{x}^2 + \dot{y}^2),$$

$$H = \frac{1}{2}\frac{1}{1 + y^{4/3}}(p_1^2 + p_2).$$

As H does not depend on x, $\dot{p}_1 = -\dfrac{\partial H}{\partial x} = 0 \Longrightarrow p_1 = k$ constant. On the other hand $p_2 = \dfrac{\partial L}{\partial \dot{y}} = (1 + y^{4/3})\dot{y}$ and using the fact that H is preserved along the solutions ($\partial H/dt = 0$), we write $H = \frac{1}{2}C^2$. This yields

$$k^2 + (1 + y^{4/3})^2\dot{y}^2 = C^2(1 + y^{4/3}).$$

Solving for $\dot{y}$,

$$\frac{dy}{dt} = \pm\frac{\sqrt{C^2(1 + y^{4/3}) - k^2}}{1 + y^{4/3}}. \tag{9.1.4}$$

The equilibrium solution verifies $C^2 y^{4/3} = k^2 - C^2$. Choosing $C = k = 1$, we get $y(t) = 0$. From one of the Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p_1} = \frac{p_1}{1 + y^{4/3}} = k = 1.$$

We obtain the geodesic $\phi(t) = (t, 0)$ with $\phi(0) = (0, 0)$ and $\dot{\phi} = (1, 0)$.

To find more geodesics we apply the separation in the equation (9.1.4) with $C = k = 1$,

$$\frac{dy}{dt} = \pm\frac{y^{2/3}}{1 + y^{4/3}}.$$

Integrating

$$\int y^{-2/3}\, dy + \int y^{2/3}\, dy = \pm t + C_1.$$

Using $y(0) = 0$, the constant of integration vanishes

$$5y^{1/3} + y^{5/3} = \pm\frac{5}{3}t. \tag{9.1.5}$$

This gives two distinct solutions for the equation (9.1.4) written implicitly. We can find $\dot{y}(0)$ by implicit differentiation

$$y^{-2/3}\dot{y} + y^{2/3}\dot{y} = \pm 1$$

and hence

$$\dot{y}(0) = \frac{\pm y^{2/3}(0)}{1 + y^{4/3}(0)} = 0.$$

The x-component is given by

$$\dot{x} = \frac{p_1}{1 + y^{4/3}} = \frac{1}{1 + y^{4/3}}.$$

Then the initial velocity is $\dot{x}(0) = 1$. Hence we have obtained three geodesics which start at $x_0 = (0, 0)$ with the initial velocity $v = (1, 0)$:

$$\phi(t) = (t, 0),$$

$$\psi_\pm(t) = \big(x_\pm(t), y_\pm(t)\big),$$

where

$$x_\pm(t) = \int_0^t \frac{ds}{1 + y_\pm^{4/3}(s)},$$

and $y_\pm$ are the solutions of the equation (9.1.5). As the function $y \to 5y^{1/3} + y^{5/3}$ is symmetric about the origin, the solutions $y_-(t)$ and $y_+(t)$ will be symmetric too. Hence the geodesics ψ_- and ψ_+ start tangent to the x-axis and point towards opposite semiplanes.

Uniqueness

The following result is known in the theory of ordinary differential equations as the Picard–Lindeleöf theorem. It holds in more restrictive conditions than the ones stated below (see Hartman [20], chapter ii). It is a useful tool in investigating the uniqueness of solutions.

Theorem 9.3. *Denote* $\mathbf{B}(y_0, b) = [y_0 - b, y_0 + b] \subset \mathbb{R}^m$. *Assume the function* $f(y)$ *is* $\mathcal{C}^1\big(\mathbf{B}(y_0, b)\big)$ *with the bound* $|f(y)| \le M$. *Then the system (9.1.3) has a unique solution* $y = y(t)$ *on* $[t_0, t_0 + b/M]$.

The function $f(y) = J\nabla H(y)$ is $\mathcal{C}^1$ iff $\dfrac{\partial g^{ij}}{\partial x_r}$ is $\mathcal{C}^1$, or $\dfrac{\partial \Gamma^k_{ij}}{\partial x_r}$ is continuous, *i.e.*, the Riemannian tensor $R^l_{ijk} = \dfrac{\partial}{\partial x^i}\Gamma^l_{jk} - \dfrac{\partial}{\partial x^j}\Gamma^l_{ik} + \sum_{r=1}^{m}\big(\Gamma^l_{ir}\Gamma^r_{jk} - \Gamma^l_{jr}\Gamma^r_{ik}\big)$ is continuous. Then Theorem 9.3 yields the following result:

Proposition 9.4 *Consider* $x_0 \in M$ *such that* g^{ij} *has a continuous Riemannian tensor* R^i_{jk} *in a neighborhood* $\mathbf{B}(x_0, b)$ *of* x_0. *Given* $v \in T_{x_0}M$, *there is* $a > 0$ *and only one geodesic* $\phi : [t_0, t_0 + a] \to (M, g)$ *with* $\phi(t_0) = x_0$ *and* $\dot\phi(t_0) = v$.

Example 9.1.2 *Consider the Riemannian metric*

$$(g_{ij}) = \begin{pmatrix} 1 + y^{2/3} & 0 \\ 0 & 1 + y^{2/3} \end{pmatrix}$$

on $\mathbb{R}^2$. *There are at least two geodesics starting at* $(0, 0)$ *with initial velocity* $(1, 0)$.

This example is very similar to Example 9.1.1, but the functions $\dfrac{\partial g_{ii}}{\partial y}$ are not continuous at $y = 0$. In this case we should be able to find explicit formulas for the geodesics. Using the Hamiltonian $H = \dfrac{1}{2}\dfrac{1}{1 + y^{2/3}}(p_1^2 + p_2^2)$ and the Lagrangian $L = \dfrac{1}{2}(1 + y^{2/3})(\dot x^2 + \dot y^2)$ we see in a similar way that $p_1 = k$, constant and $p_2 = (1 + y^{2/3})\dot y$. The conservation of energy yields

$$k^2 + (1 + y^{2/3})^2 \dot y^2 = C^2(1 + y^{2/3}),$$

which becomes for $C = k = 1$,

$$\frac{dy}{dt} = \pm\frac{y^{1/3}}{1 + y^{2/3}}. \tag{9.1.6}$$

The equilibrium solution is $y = 0$. The corresponding x-component is $x(t) = t$. The first geodesic is $\phi(t) = (t, 0)$. Separating and integrating in (9.1.6) yields

$$\frac{3}{2}y^{2/3} + \frac{3}{4}y^{4/3} = \pm t. \tag{9.1.7}$$

Implicit differentiation yields

$$\dot y(0) = \frac{\pm y^{1/3}(0)}{1 + y^{2/3}(0)} = 0.$$

Denoting $u = y^{2/3}$ in (9.1.7) and choosing the positive sign for t,

$$u^2 + 2u - \frac{4}{3}t = 0$$

with the positive solution $u = \left(1 + \frac{4}{3}t\right)^{1/2} - 1$. Hence

$$y(t) = \left(\left(1 + \frac{4}{3}t\right)^{1/2} - 1\right)^{3/2}. \tag{9.1.8}$$

The x-component is $\dot{x}(t) = \dfrac{1}{1 + y^{2/3}(t)} = \dfrac{1}{(1 + \frac{4}{3}t)^{1/2}}$ and hence $\dot{x}(0) = 1$. Integrating

$$x(t) = \int_0^t \frac{ds}{\left(1 + \frac{4}{3}s\right)^{1/2}} = \frac{3}{2}\left(\left(1 + \frac{4}{3}t\right)^{1/2} - 1\right). \tag{9.1.9}$$

The second geodesic which starts at $(0, 0)$ with the initial velocity $(1, 0)$ is $\psi(t) = \big(x(t), y(t)\big)$, with $x(t)$ and $y(t)$ given by relations (9.1.9) and (9.1.8).

9.2 Geodesic spheres

If in Picard–Lindeleóf Theorem 9.3 we denote $a = b/M$, then a depends on the initial condition y_0.

Lemma 9.5 *One may choose $a = b/M$ as a continuous function of y_0.*

Proof. We shall show $\forall \epsilon > 0$, $\exists \delta = \delta_\epsilon > 0$ such that

$$|y_0 - y_0'| < \delta \Longrightarrow |a(y_0) - a(y_0')| < \epsilon.$$

Consider an interior tangent sphere $\mathbf{B}(y_0', b') \subset \mathbf{B}(y_0, b)$. Then the distance between the centers is the difference of radii $|y_0 - y_0'| = |b - b'|$. Let M' be an upper bound for $|f|$ on $\mathbf{B}(y_0', b')$. As we have $M \geq \sup_{y \in \mathbf{B}(y_0, b)} |f(y)| \geq \sup_{y \in \mathbf{B}(y_0', b')} |f(y)|$, we may choose $M' = M$. Take $\delta = \epsilon M$ and consider $|y_0 - y_0'| < \delta$. Then

$$|a(y_0) - a(y_0')| = \left|\frac{b}{M} - \frac{b'}{M'}\right| = \frac{|b - b'|}{M} = \frac{|y_0 - y_0'|}{M} < \frac{\delta}{M} = \frac{\epsilon M}{M} = \epsilon.$$

■

Proposition 9.6 *Consider in Proposition 9.4 only velocities $|v| = 1$. Then one may choose $a > 0$, uniformly with respect to v.*

Proof. Choose $y_0 = (x_0, v)$ in Lemma 9.5 with x_0 fixed. Hence y_0 belongs to the compact set $y_0 \in \{(x_0, v); v \in T_{x_0}M, |v| = 1\}$. On this set the continuous function $a(y_0)$ will reach a minimum $a_0 > 0$, which depends only on x_0 and it is independent of v. ■

We shall denote the minimum given by the above proposition by $a(x_0) = a_0$. For any $0 < t < a(x_0)$ consider all the geodesics emanating at the point x_0 with unit initial speed. If the geodesic is parametrized by arc length, the velocity will be unitary along the geodesic.

Definition 9.7 *The geodesic sphere centered at x_0 with radius t is defined by*

$$\mathbb{S}(x_0, t) = \{\gamma(t); \gamma : [0, a(x_0)) \to M, \gamma(0) = x_0, \gamma \textit{ unit speed geodesic}\},$$

with $0 < t < a(x_0)$.

As the geodesics are locally length minimizing curves, the Riemannian distance is measured along the geodesics and it is equal to the arc length parameter t,

$$d(x_0, \gamma(t)) = \text{length}(\gamma) = t.$$

Hence the geodesic sphere can be written as

$$\mathbb{S}(x_0, t) = \{x \in M; d(x_0, x) = t\}.$$

Consider the vector field, locally about x_0, given by

$$X_{\gamma(t)} = \dot{\gamma}(t), \qquad t \in [0, a(x_0)).$$

X is called a *geodesic* vector field.

Proposition 9.8 *If X is a geodesic vector field, $curl\ X = 0$.*

Proof. If X is geodesic vector field, it is provided by a gradient $X_x = \nabla S(x)$, where S is the action associated with the geodesics. By Proposition 8.2, $curl\ X = curl\ \nabla S = 0$. ∎

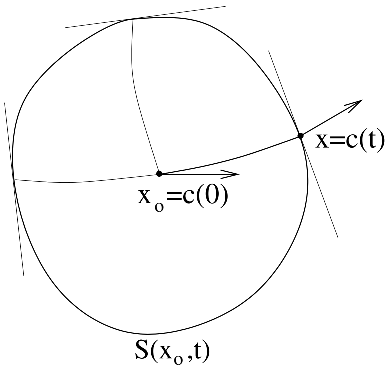


Figure 9.1: The geodesic sphere $\mathcal{S}(x_0, t)$.

Lemma 9.9 *(Gauss) Any geodesic emanating from a point x_0 meets the geodesic sphere $\mathbb{S}(x_0, t)$ perpendicularly.*

Proof. Using the formula for action $S(x,t) = \dfrac{d(x_0,x)^2}{2t}$, a computation shows

$$\dot{\gamma}(t) = X_{\gamma(t)} = \nabla S(\gamma(t),t) = \frac{d(x_0,\gamma(t))}{t}\nabla d(x_0,\gamma(t)).$$

Assuming arc length parametrization, $d(x_0,\gamma(t)) = t$. Hence

$$X_{\gamma(t)} = \nabla d(x_0,\gamma(t)).$$

Let $\mathbb{S}(x_0,t) = d^{-1}(t)$, where d denotes the distance. This yields an $X_{\gamma(t)}$ unit normal vector field to the geodesic sphere. ∎

The following result contains a formula for the mean scalar curvature of geodesic spheres.

Proposition 9.10 *Let $x \in \mathbb{S}(x_0,t)$ be a point on the geodesic sphere of radius t. Then the mean scalar curvature*

$$\alpha(x) = \frac{\Delta d(x_0,x)}{n-1}\bigg|_{|x|=t}. \tag{9.2.10}$$

Proof. From Gauss's lemma, the geodesic flow is perpendicular to the geodesic sphere. If it is parametrized by arc length, it is unitary. Hence the unit normal vector field is $N_x = X_x = \nabla d(x_0,x)$ and using Theorem 8.16 yields

$$\alpha = -\frac{div\, N}{n-1} = -\frac{div\, \nabla d(x_0,x)}{n-1} = \frac{\Delta d(x_0,x)}{n-1}.$$

∎

Definition 9.11 *Let Σ be a compact hypersurface in $\mathbb{R}^n$. Then the total mean scalar curvature of Σ is*

$$\alpha_T = \int_\Sigma \alpha(x)\, d\sigma_x. \tag{9.2.11}$$

Consider the compact manifold $M = f(\mathbb{S}^n)$, where $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is an isometric immersion. The manifolds M and $\mathbb{S}^n$ have the same intrinsic structure but different second fundamental forms with respect to $\mathbb{R}^{n+1}$. Denote by $\mathcal{N}$ and $\mathcal{S}$ the North and the South poles of $\mathbb{S}^n$. Let $x_0 = f(\mathcal{N})$, $x_1 = f(\mathcal{S})$ be the images of the poles through the isometry f. Consider geodesic spheres $\mathbb{S}(x_0,t)$ on M centered at x_0 of radius $t \in [0, 2\pi]$. The divergence and Fubini's theorem yield

$$0 = \int_M \alpha(x)\, dv_x = \int_0^{2\pi}\left(\int_{S(x_0,\tau)} \alpha(x)\, d\sigma_x\right) dt.$$

We arrive at:

Proposition 9.12 *There is $t \in (0, 2\pi)$ such that the total scalar mean curvature of $\mathbb{S}(x_0,t)$ vanishes, $\alpha_T = 0$.*

Definition 9.13 *A Riemannian manifold* (M, g) *is called radially symmetric if for any* $x_0 \in M$*, the geodesic sphere* $\mathcal{S}(x_0, t)$ *centered at* x_0 *with radius* t *has constant scalar mean curvature.*

For a radially symmetric Riemannian manifold the scalar mean curvature of the geodesic sphere $\mathcal{S}(x_0, t)$ depends only on the radius t, which is the distance from the center x_0.

From Gauss's Lemma 9.9, the unit normal vector field to the geodesic sphere $\mathcal{S}(x_0, t)$ is the vector field

$$N(x) = \dot{c}(t),$$

where $c : [0, t] \to M$ is the unit speed geodesic which joins $x_0 = c(0)$ and $x = c(t)$, $t < a(x_0)$. For any $x \in \mathbb{S}(x_0, t)$, we may choose the geodesic for which $x = c(t)$. A computation provides the following sequence of identities for the scalar mean curvature of the geodesic sphere:

$$\begin{aligned}
\alpha(x) &= -\frac{1}{n-1} div\, N(x) \\
&= -\frac{1}{n-1} div\, \dot{c}(t) \\
&= -\frac{1}{n-1} div\, \nabla S(c(t)) \\
&= \frac{1}{n-1} \Delta S(c(t)),
\end{aligned}$$

where $S(c(t))$ denotes the action between x_0 and $c(t)$. Hence we arrived at the following result.

Proposition 9.14 *Let* (M, g) *be a Riemannian manifold. The following are equivalent:*

1) (M, g) *is a radially symmetric space,*

2) $div\, \dot{c}(t)$ *depends only on* t*,*

3) $\Delta S(c(t))$ *depends only on* t*.*

Example 9.2.1 *The Euclidean space* $(\mathbb{R}^n, \delta_{ij})$ *is radially symmetric. In this case the geodesics are lines through* x_0 *given by*

$$c(s) = \Big(x_0^1 + \frac{s}{t} x^1, \dots, x_0^n + \frac{s}{t} x^n\Big),$$

with $c(t) = x$*. The velocity vector is*

$$\dot{c}(s) = \frac{1}{t}(x^1, \dots, x^n) = \frac{1}{t} x.$$

Because the geodesic is unit speed, $t = |x|$*. For any* $0 < s \le t$*, we have*

$$div\, \dot{c}(t) = \frac{1}{t} div \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} = \frac{n}{t},$$

i.e., depends on t *only.*

Lemma 9.15 *Let S be the action between x_0 and x within time t. Let $d = d(x_0, x)$ denote the Riemannian distance. Then*

$$\Delta S = \frac{1}{t}(d\Delta d - 1).$$

Proof.

$$\begin{aligned} \Delta S &= \Delta\Big(\frac{d^2}{2t}\Big) = \frac{1}{2t}\Delta d^2 \\ &= \frac{1}{2t}(2d\Delta d - 2|\nabla d|^2) \\ &= \frac{1}{t}(d\Delta d - 1), \end{aligned}$$

where we used the eiconal equation $|\nabla d|^2 = 1$. ∎

Example 9.2.2 *On the circle $\mathbb{S}^1$ the Laplacian is $\Delta_{\mathbb{S}^1} = -\frac{d^2}{ds^2}$ and the distance is $d = s$, where s denotes the arc length. Then $\Delta_{\mathbb{S}^1}(d) = 0$, and hence Lemma 9.15 yields $\Delta_{\mathbb{S}^1}(S) = -\frac{1}{t}$, i.e., it depends only on t. Hence $\mathbb{S}^1$ is a radially symmetric space.*

The volume function about a point x_0

Let (M, g) be a Riemannian manifold with the volume element $dv = \sqrt{|g_{ij}|}\, dx_1 \wedge \cdots \wedge dx_n$. If L denotes the Lie derivative, we have shown in Proposition 2.7 that for any vector field $X \in \mathcal{X}(M)$, we have $L_X dv = -(div X)\, dv$. If X is the vector field along a geodesic flow defined by the geodesics emanating at the point x_0, *i.e.*,

$$X_{c(t)} = \dot c(t) = c_*\Big(\frac{d}{ds}\Big),$$

with $c(0) = x_0$, then

$$\begin{aligned} L_{\dot c}\, dv &= -(div\, \dot c)\, dv \\ &= (n-1)\alpha(c(t)) \\ &= \Delta S(c(t)), \end{aligned}$$

with α the scalar mean curvature of the geodesic sphere centered at x_0.

Inspired by the above formula, we shall define the following volume function associated with a geodesic flow on (M, g) emanating from a point x_0.

Definition 9.16 *A function $v(\tau)$ is called a volume function along a geodesic flow parametrized by τ if it verifies the initial value problem*

$$\frac{dv(\tau)}{d\tau} = \frac{1}{2}\Delta S(x_0, x, \tau)v(\tau),$$
$$\lim_{\tau \to 0} \tau^{n/2} v(\tau) = 1$$

where $c(0) = x_0$ *and* $c(\tau) = x$, *with* $c(s)$ *geodesic.* $S(x_0, x, \tau)$ *stands for the classical action between* x_0 *and* x *within time* τ, *i.e.,* $S(x_0, x, \tau) = \dfrac{d^2(x_0, x)}{2\tau}$.

Example 9.2.3 *The volume function on* $\mathbb{R}^n$ *about any point* x_0.
From Example 9.2.1 we have $\Delta S = -div\, \dot{c} = -\dfrac{n}{\tau}$. *The volume function about any point* x_0 *satisfies the equation*

$$\frac{dv}{d\tau} = -\frac{n}{2\tau}v.$$

Separating and integrating between $v(\tau_0) = v_0$ *and* $v = v(\tau)$, *yields*

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{n}{2}\int_{\tau_0}^{\tau} \frac{d\tau}{\tau} \Longleftrightarrow \ln\frac{v}{v_0} = \ln\Big(\frac{\tau_0}{\tau}\Big)^{n/2},$$

and hence $v(\tau) = v_0 \tau_0^{n/2} \dfrac{1}{\tau^{n/2}}$. *The boundary condition* $\lim_{\tau \searrow 0} \tau^{n/2} v(\tau) = 1$ *yields*

$$v(\tau) = \frac{1}{\tau^{n/2}}.$$

The volume function will play an important role in finding heat kernels on radially symmetric spaces. In this case, there is a function $h(\tau) = \frac{1}{2}\Delta S(x_0, x, \tau)$ and the volume function will be

$$v(\tau) = v(\tau_0)e^{\int_{\tau_0}^{\tau} h(u)\, du}.$$

We shall construct the heat kernel on radially symmetric spaces. The method yields exact solutions.

9.3 A radially non-symmetric space

We shall show that the sphere $\mathbb{S}^2$ with the induced metric from $\mathbb{R}^3$ is not a radially symmetric space. Consider the spherical coordinates defined on $\mathbb{S}^2$ without the North and South poles

$$h(\phi, \psi) = (\cos\phi\cos\psi, \sin\phi\cos\psi, \sin\psi), \quad 0 \le \phi \le 2\pi, -\frac{\pi}{2} < \psi < \frac{\pi}{2}.$$

The tangent vector fields

$$\partial_\phi = -\sin\phi\cos\psi\, \partial_{x_1} + \cos\phi\cos\psi\, \partial_{x_2},$$
$$\partial_\psi = -\cos\phi\sin\psi\, \partial_{x_1} - \sin\psi\cos\psi\, \partial_{x_2} + \cos\psi\, \partial_{x_3}$$

define the coefficients of a Riemannian metric

$$g_{\phi\phi} = \langle \partial_\phi, \partial_\phi \rangle = \cos^2 \psi, \quad g_{\phi\psi} = g_{\psi\phi} = \langle \partial_\phi, \partial_\psi \rangle = 0, \quad g_{\psi\psi} = \langle \partial_\psi, \partial_\psi \rangle = 1,$$

with the inverse metric

$$g^{\phi\phi} = \frac{1}{\cos^2 \psi}, \quad g^{\phi\psi} = g^{\psi\phi} = 0, \quad g^{\psi\psi} = 1.$$

Hence the Laplace–Beltrami operator on $\mathbb{S}^2$ is

$$\Delta_{\mathbb{S}^2} = -\frac{1}{\cos^2 \psi} \partial_\phi^2 - \partial_\psi^2 + \tan \psi \, \partial_\psi. \tag{9.3.12}$$

Let $M(\cos\phi \cos\psi, \sin\phi \cos\psi, \sin\psi)$ be a point on the sphere, see Figure 9.2. We shall compute the Riemannian distance $d = d(M, A)$ between the points M and $A(1, 0, 0)$. At the point A we also have $\phi = \psi = 0$. The distance d is the arc length between M and A of a great circle. As the sphere has unit radius, then $d = \theta$, where $\theta = m(\widehat{MOA})$, see Figure 9.2.

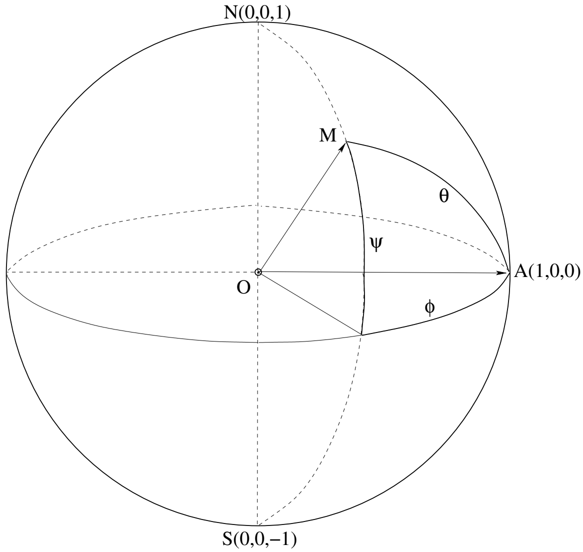


Figure 9.2: The sphere $\mathbb{S}^2$ and the point $M(\cos\phi \cos\psi, \sin\phi \cos\psi, \sin\psi)$.

From $\cos\theta = \langle \overrightarrow{OM}, \overrightarrow{OA} \rangle = \cos\phi \, \cos\psi$ we obtain

$$d(M, A) = \theta = \arccos(\cos\phi \, \cos\psi).$$

In the following we shall compute Δd. In order to do this we need to compute the following derivatives:

$$\partial_\psi \theta = \frac{\cos\phi\,\sin\psi}{\sqrt{1-\cos^2\phi\,\cos^2\psi}}, \qquad \partial^2_\psi \theta = \frac{\cos\phi\,\cos\psi\,\sin^2\phi}{(1-\cos^2\phi\,\cos^2\psi)^{3/2}},$$
$$\partial^2_\phi = \frac{\cos\phi\,\cos\psi\,\sin^2\psi}{(1-\cos^2\phi\,\cos^2\psi)^{3/2}}.$$

Then

$$\begin{aligned}
\Delta\theta &= -\frac{1}{\cos^2\psi}\,\frac{\cos\phi\,\cos\psi\,\sin^2\psi}{(1-\cos^2\phi\,\cos^2\psi)^{3/2}} \\
&\quad -\frac{\cos\phi\,\cos\psi\,\sin^2\phi}{(1-\cos^2\phi\,\cos^2\psi)^{3/2}} \\
&\quad +\frac{\sin\psi}{\cos\psi}\,\frac{\cos\phi\,\sin\psi\,(1-\cos^2\phi\,\cos^2\psi)}{(1-\cos^2\phi\,\cos^2\psi)^{3/2}} \iff \\
(1-\cos^2\phi\,\cos^2\psi)^{3/2}\Delta\theta &= -\cos\phi\,\cos\psi\Big(\sin^2\phi + \frac{\sin^2\psi}{\cos\psi}\cos^2\phi\,\cos\psi\Big) \\
&= -\cos\phi\,\cos\psi\Big(\sin^2\phi + \sin^2\psi\,\cos^2\phi\Big) \\
&= -\cos\phi\,\cos\psi\Big(1-\cos^2\phi + \sin^2\psi\,\cos^2\phi\Big) \\
&= -\cos\phi\,\cos\psi\Big(1-\cos^2\phi\,(1-\sin^2\psi)\Big) \\
&= -\cos\phi\,\cos\psi\Big(1-\cos^2\phi\,\cos^2\psi\Big) \\
\iff \Delta\theta &= -\frac{\cos\phi\,\cos\psi}{\sqrt{1-\cos^2\phi\,\cos^2\psi}} = -\frac{\cos\theta}{\sqrt{1-\cos^2\theta}} \\
&= -\frac{\cos\theta}{\sin\theta} = -\cot\theta.
\end{aligned}$$

We have arrived at the following result.

Proposition 9.17 *Consider the sphere $\mathbb{S}^2$ with the induced Riemannian metric from $\mathbb{R}^3$. Let A be a point on the sphere $\mathbb{S}^2$. Let d denote the distance on $\mathbb{S}^2$ measured from the point A. Then*

$$\Delta d + \cot\,d = 0.$$

Now Lemma 9.15 yields

$$\Delta S = \frac{1}{t}(d\Delta d - 1) = -\frac{1}{t}(d\cot\,d + 1),$$

which does not depend only on time t. Hence $\mathbb{S}^2$ is not a radially symmetric space.

9.4 The Heisenberg group

9.4.1 The left invariant metric

The 3-dimensional Heisenberg group $\mathbf{H}_1$ may be realized as $\mathbb{R}^3 = \mathbb{R}^2_x \times \mathbb{R}_t$ endowed with the group law

$$(x,t)\circ_H (x',t') = (x+x', t+t'+2x_2x_1' - 2x_1x_2'). \tag{9.4.13}$$

The vector fields

$$X_1 = \partial_{x_1} + 2x_2\partial_t, \quad X_2 = \partial_{x_2} - 2x_1\partial_t, \quad T = \partial_t \tag{9.4.14}$$

are left invariant with respect to the group law (9.4.13) and generate the Lie algebra of $\mathbf{H}_1$. The elliptic operator

$$\Delta_{Cas} := \frac{1}{2}\big(X_1^2 + X_2^2 + T^2\big)$$

is called a Casimir operator. We shall construct a left invariant Riemannian metric h on $\mathbf{H}_1$ in which the vector fields (9.4.14) are orthonormal. For more about Lie groups theory, see [1].

Proposition 9.18 *Consider the Riemannian space* $(\mathbb{R}^3, h)$*, where the metric coefficients are given by*

$$h_{ij} = \begin{pmatrix} 1+4x_2^2 & -4x_1x_2 & -2x_2 \\ -4x_1x_2 & 1+4x_1^2 & 2x_1 \\ -2x_2 & 2x_1 & 1 \end{pmatrix}. \tag{9.4.15}$$

Then $h(X_i, X_j) = \delta_{ij}$, $h(X_j, T) = 0$, $i, j = 1, 2, 3$.

Proof. It is a direct verification.

$$\begin{aligned} h(X_1, T) &= h_{13}X_1^1T^3 + h_{23}X_1^2T^3 + h_{33}X_1^3T^3 \\ &= (-2x_2) + 0 + 2x_2 = 0, \\ h(X_2, T) &= h_{13}X_2^1T^3 + h_{23}X_2^2T^3 + h_{33}X_2^3T^3 \\ &= 0 + 2x_1 + (-2x_1) = 0, \\ h(X_1, X_2) &= h_{12}X_1^1X_2^2 + h_{13}X_1^1X_2^3 + h_{32}X_1^3X_2^2 + h_{33}X_1^3X_2^3 \\ &= -4x_1x_2 + (-2x_2)(-2x_1) + (2x_1)(2x_2) + (2x_2)(-2x_1) = 0. \end{aligned}$$

■

The Lagrangian is defined as the kinetic energy associated with the Riemannian metric h,

$$L(x,t,\dot{x},\dot{t}) = \frac{1}{2}\sum_{i,j=1}^{3} h_{ij}\dot{x}_i\dot{x}_j.$$

Proposition 9.19 *The Lagrangian is given by*

$$L(x,t,\dot{x},\dot{t}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{t}^2) + 2(x_1\dot{x}_2 - x_2\dot{x}_1)(\dot{t} + x_1\dot{x}_2 - x_2\dot{x}_1). \tag{9.4.16}$$

Proof. A straightforward computation yields

$$\begin{aligned}\sum h_{ij}\dot{x}_i\dot{x}_j &= (1+4x_2^2)\dot{x}_1^2 + (1+4x_1^2)\dot{x}_2^2 + \dot{t}^2 - 8x_1x_2\dot{x}_1\dot{x}_2 - 4x_2\dot{x}_1\dot{t} + 4x_1\dot{x}_2\dot{t}\\ &= (\dot{x}_1^2+\dot{x}_2^2+\dot{t}^2) + 4[(x_2\dot{x}_1)^2 + (x_1\dot{x}_2)^2 - 2x_2\dot{x}_1x_1\dot{x}_2 - x_2\dot{x}_1\dot{t} + x_1\dot{x}_2\dot{t}]\\ &= (\dot{x}_1^2+\dot{x}_2^2+\dot{t}^2) + 4(x_1\dot{x}_2 - x_2\dot{x}_1 + \dot{t})(x_1\dot{x}_2 - x_2\dot{x}_1).\end{aligned}$$

■

In polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$ the Lagrangian becomes

$$\begin{aligned}L &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{t}^2) + 2r^2\dot{\phi}(\dot{t} + r^2\dot{\phi})\\ &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{t}^2) + 2\dot{t}r^2\dot{\phi} + 2r^4\dot{\phi}^2.\end{aligned}$$

9.4.1.1 The Euler–Lagrange system

The momenta are

$$\begin{aligned}\theta &= \frac{\partial L}{\partial \dot{t}} = \dot{t} + 2r^2\dot{\phi},\\ \eta &= \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} + 2\dot{t}r^2 + 4r^4\dot{\phi},\\ \rho &= \frac{\partial L}{\partial \dot{r}} = \dot{r}.\end{aligned}$$

As the Lagrangian L does not depend on t and ϕ, the Euler–Lagrange equations yield

$$\theta = \text{constant}, \qquad \eta = \text{constant}.$$

The momentum η can be written in terms of θ as

$$\eta = r^2(\dot{\phi} + 2\dot{t} + 4r^2\dot{\phi})s = r^2(\dot{\phi} + 2\theta).$$

The Euler–Lagrange equation $\dot{\rho} = \dfrac{\partial L}{\partial r}$ becomes

$$\begin{aligned}\ddot{r} &= r\dot{\phi}^2 + 4\dot{t}r\dot{\phi} + 8r^3\dot{\phi}^2\\ &= r\dot{\phi}^2 + 4r\dot{\phi}(\dot{t} + 2r^2\dot{\phi})\\ &= r\dot{\phi}^2 + 4r\dot{\phi}\theta\\ &= r\dot{\phi}(\dot{\phi} + 4\theta).\end{aligned}$$

Hence the Euler–Lagrange system is

$$\begin{cases}\ddot{r} & = r\dot{\phi}(\dot{\phi} + 4\theta),\\ r^2(\dot{\phi} + 2\theta) & = \eta,\\ \dot{t} + 2r^2\dot{\phi} & = \theta\\ \theta & = \text{constant},\\ \eta & = \text{constant}.\end{cases} \tag{9.4.17}$$

It suffices to study only the geodesics from the origin, because of the Heisenberg translation. In this case $r(0) = 0$ and hence $\eta = 0$. It follows that $\dot\phi = -2\theta$ and the system (9.4.17) becomes

$$\begin{cases} \ddot r &= -4\theta^2 r, \\ \dot\phi &= -2\theta, \\ \dot t &= \theta - 2r^2\dot\phi = \theta(1+4r^2), \\ \theta &= \text{constant} \end{cases} \tag{9.4.18}$$

with the boundary conditions

$$r(0) = 0, \quad \phi(0) = \phi_0, \qquad t(0) = t_0 = 0, \tag{9.4.19}$$

$$r(\tau) = \mathbf{r}, \quad \phi(\tau) = \Phi, \qquad t(\tau) = \mathbf{t}. \tag{9.4.20}$$

We shall show in the following that the system (9.4.18) has solutions if and only if some compatibility of the above boundary conditions holds. The solutions are

$$r(s) = \frac{\sin(2\theta s)}{\sin(2\theta\tau)}\mathbf{r}, \tag{9.4.21}$$

$$\phi(s) = -2\theta s + \phi_0. \tag{9.4.22}$$

The boundary condition $\phi(\tau) = \Phi$ yields

$$\theta = \frac{1}{2\tau}(\phi_0 - \Phi). \tag{9.4.23}$$

Integrating in (9.4.21) yields

$$\begin{aligned} t(s) &= \theta\int_0^s (1+4r^2(u))\,du = \theta\Big(s + 4\int_0^s r^2\Big) \\ &= \theta\Big(s + \frac{4\mathbf{r}^2}{\sin^2(2\theta\tau)}\int_0^s \sin^2(2\theta u)\,du\Big) \\ &= \theta\Big[s + \frac{4\mathbf{r}^2}{2\theta\sin^2(2\theta\tau)}\Big(\frac{1}{2}(2\theta s) - \frac{1}{4}\sin(4\theta s)\Big)\Big] \\ &= \theta s + \frac{\mathbf{r}^2}{\sin^2(2\theta\tau)}\Big(2\theta s - \frac{1}{2}\sin(4\theta s)\Big). \end{aligned} \tag{9.4.24}$$

The boundary condition $t(\tau) = \mathbf{t}$ yields

$$\begin{aligned} \mathbf{t} &= \theta\tau + \frac{\mathbf{r}^2}{\sin^2(2\theta\tau)}\Big(2\theta\tau - \sin(2\theta\tau)\cos(2\theta\tau)\Big) \\ &= \theta\tau + \mathbf{r}^2\Big(\frac{2\theta\tau}{\sin^2(2\theta\tau)} - \cot(2\theta\tau)\Big). \end{aligned} \tag{9.4.25}$$

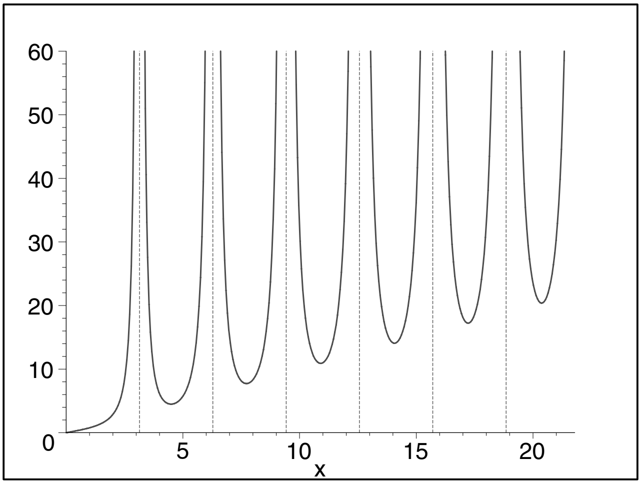


Figure 9.3: The graph of $\mu(x)$.

Let

$$\mu(x) = \frac{x}{\sin^2 x} - \cot x. \tag{9.4.26}$$

The graph of μ for $x > 0$ is sketched in Figure 9.3. It suffices to study only the case $\theta > 0$. The case $\theta < 0$ can be obtained from the previous one changing $t \to -t$ and $\phi \to -\phi$. This follows from the relation $\theta = \dot{t} + 2r^2\dot{\phi}$. Then (9.4.25) becomes

$$\mathbf{t} = \theta\tau + \mathbf{r}^2\mu(2\theta\tau). \tag{9.4.27}$$

In order to understand the exact number of geodesics, which join the origin with any given point, we need the following lemma, see Beals, Gaveau and Greiner [37].

Lemma 9.20 *μ is a monotone increasing diffeomorphism of the interval $(-\pi, \pi)$ onto $\mathbb{R}$. On each interval $(m\pi, (m+1)\pi)$, $m = 1, 2, \ldots$, μ has a unique critical point x_m. On this interval μ decreases strictly from $+\infty$ to $\mu(x_m)$ and then increases strictly from $\mu(x_m)$ to $+\infty$. Moreover*

$$\mu(x_m) + \pi < \mu(x_{m+1}), \quad m = 1, 2, \ldots, \tag{9.4.28}$$

$$0 < \Big(m + \frac{1}{2}\Big)\pi - x_m < \frac{1}{m\pi}. \tag{9.4.29}$$

Proof. As μ is an odd function, it suffices to show that it is a monotone increasing diffeomorphism of the interval $(0, \pi)$ onto $(0, +\infty)$. We note that $\sin x - x\cos x$ vanishes at $x = 0$ and it is increasing in $(0, \pi)$. Then

$$\frac{1}{2}\mu'(x) = \frac{\sin x - x\cos x}{\sin^3 x} = \begin{cases} = 1/3, & x = 0, \\ > 1/3, & x \in (0, \pi). \end{cases}$$

The first identity holds as an application of the l'Hospital rule:

$$\lim_{x\to 0}\frac{\sin x - x\cos x}{\sin^3 x} = \lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x}$$
$$= \frac{1}{3}\lim_{x\to 0}\frac{x}{\sin x} = \frac{1}{3}.$$

The second inequality holds because

$$\frac{1}{2}\mu''(x) = \frac{x + 2x\cos^2 x - 3\cos x\sin x}{\sin^4 x} > 0.$$

The numerator vanishes at $x = 0$, and its derivative is

$$4\sin x(\sin x - x\cos x) > 0, \quad x \in (0, \pi).$$

Therefore μ is a diffeomorphism of the interval $(0, \pi)$ onto $(0, \infty)$. In the interval $(m\pi, (m+1)\pi)$ μ approaches $+\infty$ at the endpoints. In order to find the critical points, we set

$$\frac{1}{2}\mu'(x) = \frac{\sin x - x\cos x}{\sin^3 x} = \frac{1 - x\cot x}{\sin^4 x} = 0.$$

Hence the critical point x_m is the solution of the equation $x = \tan x$ on the interval $(m\pi, (m+1)\pi)$. Note that

$$\mu(x+\pi) = \frac{x+\pi}{\sin^2(x+\pi)} - \cot(x+\pi)$$
$$= \frac{x}{\sin^2(x+\pi)} - \cot(x+\pi) + \frac{\pi}{\sin^2 x}$$
$$= \mu(x) + \frac{\pi}{\sin^2 x},$$

so the successive minimum values increase by more than π. From Figure 9.4 we have

$$m\pi < x_m < m\pi + \frac{\pi}{2} = (m + \frac{1}{2})\pi. \tag{9.4.30}$$

Using $x_m = \tan x_m$ yields

$$\cot x_m = \frac{1}{x_m} < \frac{1}{m\pi}. \tag{9.4.31}$$

Let $f(x) = \cot x$. As $f'(x) = -\frac{1}{\sin^2 x} < -1$, there is a ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x - y) < -(x - y).$$

Hence $x - y < f(y) - f(x)$. Choosing $x = m\pi + \frac{\pi}{2}$, $y = x_m$ and using

$$f(m\pi + \frac{\pi}{2}) = \frac{\cos(m\pi + \frac{\pi}{2})}{\sin(m\pi + \frac{\pi}{2})} = 0,$$

and (9.4.31) yields

$$0 < (m + \frac{1}{2})\pi - x_m < \cot x_m < \frac{1}{m\pi}.$$

■

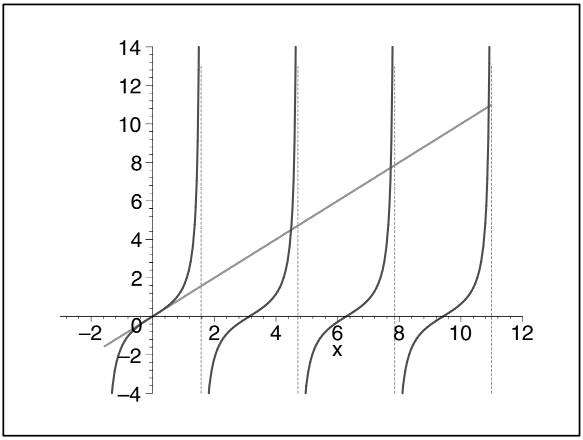


Figure 9.4: Critical points of μ are solutions of $\tan x = x$.

The number of geodesics that join the origin with an arbitrary given point is given in the following theorems.

Theorem 9.21. *(i) Given a point* $P(\mathbf{x}, \mathbf{t})$*,* $\mathbf{r} = |\mathbf{x}| \neq 0$*,* $\mathbf{t} > 0$*, there are finitely many geodesics between the origin and* P*. Let* $0 < \zeta_1 < \cdots < \zeta_N$ *be the solutions of*

$$\mathbf{t} - \frac{1}{2}\zeta = \mathbf{r}^2 \mu(\zeta). \tag{9.4.32}$$

Then, with $\theta_m = \dfrac{\zeta_m}{2\tau}$*, the geodesic equations are*

$$\begin{aligned} r_m(s) &= \frac{\sin(2\theta_m s)}{\sin(2\theta_m \tau)} \mathbf{r}, \\ \phi_m(s) &= -2\theta_m s + \phi_0, \\ t_m(s) &= \theta_m s + \frac{\mathbf{r}^2}{\sin^2(2\theta_m \tau)} \Big(2\theta_m \tau - \frac{1}{2} \sin(4\theta_m s) \Big), \quad m = 1, 2, \ldots N. \end{aligned}$$

(ii) The compatibility condition for the boundary conditions is

$$\zeta_m = \phi_0 - \Phi, \quad m = 1, 2, \ldots, N. \tag{9.4.33}$$

Given the point $P(\mathbf{x}, \mathbf{t})$*, let* $\Phi = \arctan(|\mathbf{x}|)$ *be the final argument. Then the initial arguments of the geodesics joining the origin and* P *are*

$$\phi_{0,m} = \zeta_m - \Phi, \quad m = 1, 2, \ldots, N. \tag{9.4.34}$$

Proof. (i) It is obvious that equation (9.4.32) has finitely many solutions, see Figure 9.3. For each solution of (9.4.27), substitute θ in the equations (9.4.21), (9.4.22) and (9.4.24).
(ii) It follows from (i) and condition (9.4.23). See Figure 9.5. ∎

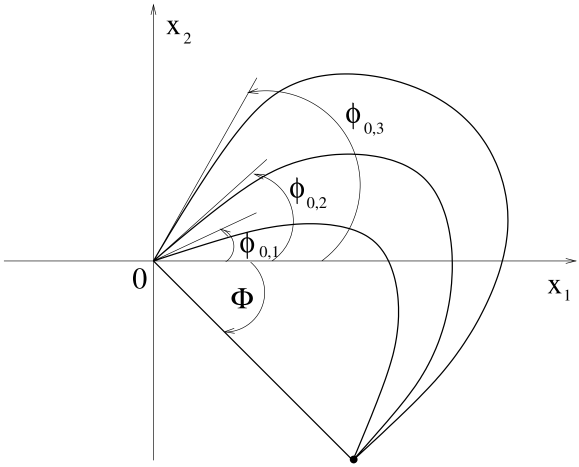


Figure 9.5: The projections of the geodesics on an x-plane start with different arguments.

Remark 9.22 *A similar theorem works for the case* $\mathbf{t} < 0$.

It is well known that locally, there is only one geodesic joining the origin and the point P. The size and the shape of the neighborhood is given by the following result.

Theorem 9.23. *Given a point* $P(\mathbf{x}, \mathbf{t})$*, with* $|\mathbf{t}| < \big(\frac{1}{2} + |\mathbf{x}|^2\big)\pi$ *and* $|\mathbf{x}| \neq 0$*, there is a unique geodesic joining the origin and the point* P.

Proof. We shall discuss the following cases: $0 < \mathbf{t} < \big(\frac{1}{2} + |\mathbf{x}|^2\big)\pi$, $\mathbf{t} = 0$ and $-\big(\frac{1}{2} + |\mathbf{x}|^2\big)\pi < \mathbf{t} < 0$. The third case can be treated in a similar way as the first case.
Case $0 < \mathbf{t} < \big(\frac{1}{2} + |\mathbf{x}|^2\big)\pi$.
We shall show that equation (9.4.32) has only one solution $\zeta > 0$. Consider the function $\varphi(\zeta) = \dfrac{1}{|\mathbf{x}|^2}(\mathbf{t} - \dfrac{1}{2}\zeta)$. We shall show that the solutions of the equation $\varphi(\zeta) = \mu(\zeta)$ are only in the interval $(0, \pi)$. It suffices to show that

$$\varphi(\zeta) < \mu(\zeta), \qquad \text{for} \quad \pi < \zeta. \tag{9.4.35}$$

Let $x_1 \in (\pi, 2\pi)$ be the first critical point of μ. Using Lemma 9.20, the monotonicity of μ and convexity of μ yields

$$\varphi(\zeta) < \varphi(\pi) = \frac{1}{|\mathbf{x}|^2}(\mathbf{t} - \frac{1}{2}\pi) < \pi < \mu(x_1) = \min_{\pi<\zeta} \mu(\zeta),$$

which yields (9.4.35). Then there are no solutions on $(\pi, +\infty)$. As φ is decreasing and μ is increasing on $(0, \pi)$, there is only one solution for the equation $\varphi(\zeta) = \mu(\zeta)$, see Figure 9.6

Case $\mathbf{t} = 0$.
If $\mathbf{t} = 0$, then $-\frac{1}{2}\zeta = |\mathbf{x}|\mu(\zeta)$ yields only the solution $\zeta = 0$. Then $\theta = \frac{\zeta}{2\tau} = 0$. Theorem 9.21 yields $\phi_0 = \Phi$, $t(s) = 0$. $r(s)$ satisfies $\ddot{r} = 0$, with solution $r(s)|\mathbf{x}|s$. There is a unique solution, which is a straight line from the origin to P, in the x-plane. ■

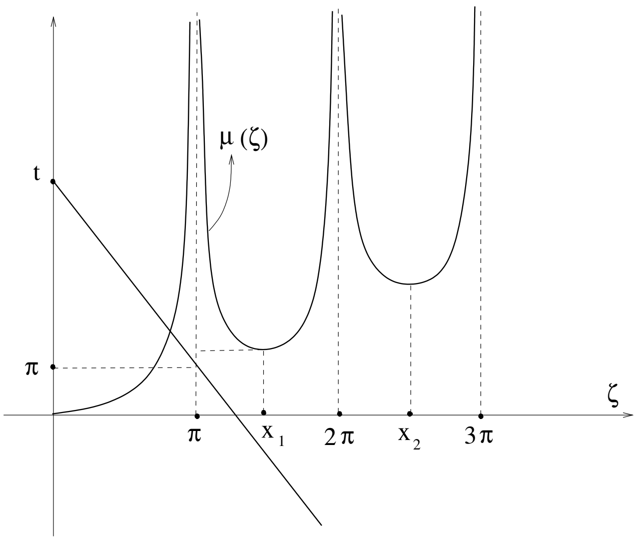


Figure 9.6: The case when $\phi(\zeta) = \mu(\zeta)$ has a unique solution.

Corollary 9.24 *Given a point* $P(\mathbf{x}, 0)$, $|\mathbf{x}| \neq 0$, *there is a unique geodesic between the origin and* P. *The geodesic is given by the equations* $r(s) = |\mathbf{x}|s$, $\phi(s) = \Phi$, *and* $t(s) = 0$, *i.e., it is a straight segment in the* x*-plane.*

In Theorems 9.21 and 9.23 we assumed $|\mathbf{x}| \neq 0$. In the following we shall cover the case when $|x| = 0$.

Theorem 9.25. *Given a point* $P(0, \mathbf{t})$ *on the* t*-axis, there is a unique geodesic between the origin and the point* P.

Proof. If $|\mathbf{x}| = \mathbf{r} = 0$, from (9.4.21) we get $r(s) = 0$. Using (9.4.24) yields $t(s) = \theta s$, with $\theta = \mathbf{t}/\tau$. The geodesic is along the t-axis. ■

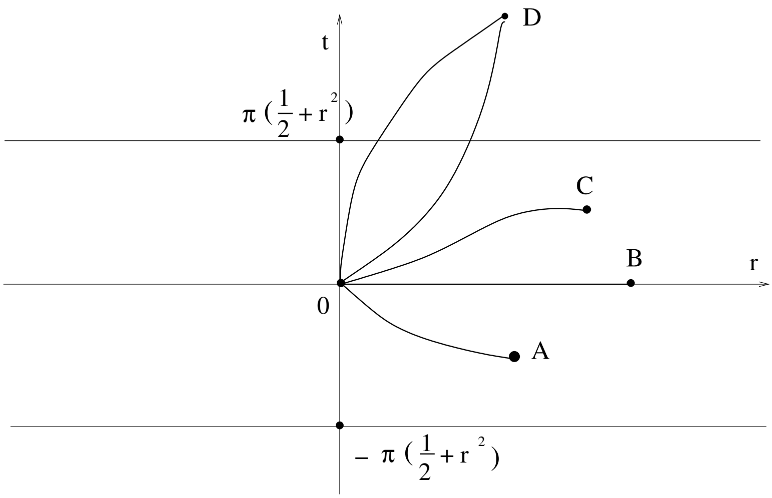


Figure 9.7: There is a unique geodesic in the strip $|\mathbf{t}| < \pi(\frac{1}{2} + \mathbf{r}^2)$ between O and (x, t).

Remark 9.26 *Theorem* 9.23 *works also in the case* $|\mathbf{x}| = 0$.

9.4.2 The classical action

In a strip like in Theorem 9.23 the geodesic is unique. Let θ denote the unique solution. The Lagrangian along the geodesic is

$$\begin{aligned}
L &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{t}^2) + 2(r^2\dot{\phi} + \dot{t})r^2\dot{\phi} \\
&= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{t}^2) + 2(\theta - r^2\dot{\phi})r^2\dot{\phi} \\
&= \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{t}^2) + 2\theta r^2\dot{\phi} - 2r^4\dot{\phi}^2 \\
&= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \frac{1}{2}(\theta - 2r^2\dot{\phi})^2 + 2\theta r^2\dot{\phi} - 2r^4\dot{\phi}^2 \\
&= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\, 4\theta^2 + \frac{1}{2}\theta^2 \\
&= \frac{1}{2}\dot{r}^2 + \frac{1}{2}\theta^2(1 + 4r^2) \\
&= \frac{1}{2}\dot{r}^2 + \frac{1}{2}\theta\dot{t}.
\end{aligned}$$

The classical action is obtained by integrating the Lagrangian along the geodesic

$$\begin{aligned}
S(\tau) = S(\mathbf{x}, \mathbf{y}, \tau) &= \int_0^\tau L\, ds = \int_0^\tau \Big(\frac{1}{2}\dot{r}^2 + \frac{1}{2}\theta\dot{t}\Big)\, ds \\
&= \frac{1}{2}\int_0^\tau \dot{r}^2(s)\, ds + \frac{1}{2}\theta\big(t(\tau) - t(0)\big).
\end{aligned} \tag{9.4.36}$$

Integrating the first term yields

$$\begin{aligned}\int_0^\tau \dot{r}(s)^2\,ds &= \frac{4\theta^2\mathbf{r}^2}{\sin^2(2\theta\tau)}\int_0^\tau \cos^2(2\theta s)\,ds = \frac{2\theta\mathbf{r}^2}{\sin^2(2\theta\tau)}\int_0^{2\theta\tau}\cos^2 v\,dv\\ &= \frac{2\theta\mathbf{r}^2}{\sin^2(2\theta\tau)}\Big[\theta\tau + \frac{1}{4}\sin(4\theta\tau)\Big]\\ &= \frac{\theta\mathbf{r}^2}{\sin^2(2\theta\tau)}\Big(2\theta\tau + \sin(2\theta\tau)\cos(2\theta\tau)\Big)\\ &= \theta\mathbf{r}^2\Big[\frac{2\theta\tau}{\sin^2(2\theta\tau)} + \cot(2\theta\tau)\Big] = \theta\mathbf{r}^2\widetilde{\mu}(2\theta\tau), \qquad (9.4.37)\end{aligned}$$

where

$$\widetilde{\mu}(x) = \frac{x}{\sin^2 x} + \cot x \qquad (9.4.38)$$

Proposition 9.27 *The classical action starting at the origin is*

$$\begin{aligned}S(\mathbf{x},\mathbf{t},\tau) &= \theta^2|\mathbf{x}|^2\Big(\frac{1}{2} + \frac{2|\mathbf{x}|^2}{\sin^2(2\theta\tau)}\Big)\\ &= \theta\mathbf{t} - \frac{1}{2}\theta^2\tau + \theta|\mathbf{x}|^2\cot(2\theta\tau).\end{aligned}$$

Proof. Using $t(0) = 0$, substituting (9.4.37) in equation (9.4.36) yields

$$\begin{aligned}S(\mathbf{x},\mathbf{t},\tau) &= \frac{1}{2}\theta\mathbf{r}^2\widetilde{\mu}(2\theta\tau) + \frac{1}{2}\theta\mathbf{t}\\ &= \frac{1}{2}\theta\mathbf{r}^2\widetilde{\mu}(2\theta\tau) + \frac{1}{2}\theta\big(\theta\tau + \mathbf{r}^2\mu(2\theta\tau)\big)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}\theta\mathbf{r}^2\widetilde{\mu}(2\theta\tau) + \frac{1}{2}\theta^2\tau + \frac{1}{2}\theta\mathbf{r}^2\mu(2\theta\tau)\\ &= \frac{1}{2}\theta\mathbf{r}^2\big[\widetilde{\mu}(2\theta\tau) + \mu(2\theta\tau)\big] + \frac{1}{2}\theta^2\tau\\ &= \theta\mathbf{r}^2\frac{2\theta\tau}{\sin^2(2\theta\tau)} + \frac{1}{2}\theta^2\tau\\ &= \theta^2\mathbf{r}\Big(\frac{1}{2} + \frac{2\mathbf{r}^2}{\sin^2(2\theta\tau)}\Big).\end{aligned}$$

For the second identity, using $\mu(x) = \widetilde{\mu}(x) + 2\cot(x)$, we have

$$\begin{aligned}S(\mathbf{x},\mathbf{t},\tau) &= \frac{1}{2}\theta\mathbf{r}^2\Big(\mu(2\theta\tau) + 2\cot(2\theta\tau)\Big) + \frac{1}{2}\theta\mathbf{t}\\ &= \frac{1}{2}\theta\mathbf{r}^2\mu(2\theta\tau) + \theta\mathbf{r}^2\cot(2\theta\tau) + \frac{1}{2}\theta\mathbf{t}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\theta(\mathbf{t} - \theta\tau) + \theta\mathbf{r}^2\cot(2\theta\tau) + \frac{1}{2}\theta\mathbf{t} \\
&= \frac{1}{2}\theta\mathbf{t} - \frac{1}{2}\theta^2\tau + \theta\mathbf{r}^2\cot(2\theta\tau) + \frac{1}{2}\theta\mathbf{t} \\
&= \theta\mathbf{t} - \frac{1}{2}\theta^2\tau + \theta\mathbf{r}^2\cot(2\theta\tau).
\end{aligned}
$$

Replacing $\mathbf{r}$ by $|\mathbf{x}|$ we obtain the desired formulas. ∎

9.4.3 The complex action

The space $(\mathbb{R}^3, h)$ with h given by (9.4.15) is *not a radially symmetric space*. The reason is the fact that the momentum $\theta = \theta(\mathbf{x}, \mathbf{t}, \tau)$, which appears in the classical action given by Proposition 9.27, is a solution of the equation $\mathbf{t} = \theta\tau + |\mathbf{x}|^2\mu(2\theta\tau)$, and hence depends on the boundary conditions $\mathbf{t}$ and $\mathbf{x}$ in a complicated manner. Therefore we do not expect $\Delta_{Cas}S(\mathbf{x}, \mathbf{t}, \tau)$ to be a function that depends just on τ.

However, we can fix the situation. In the next chapter, when computing the heat kernels, we need an action function, which satisfies the Hamilton–Jacobi equation. We define the *complex action* for our problem to be the function obtained by substituting $\theta = -i$ in the classical action. Let $S_{\mathbb{C}}$ denote the complex action. Using the properties $\sin(-ix) = -i\sinh(x)$ and $\cos(-ix) = i\cosh(x)$ yields

$$
S_{\mathbb{C}} = -i\mathbf{t} + \frac{1}{2}\tau + (\mathbf{x}_1^2 + \mathbf{x}_2^2)\coth(2\tau). \tag{9.4.39}
$$

Proposition 9.28 *The complex action* (9.4.39) *satisfies the Hamilton–Jacobi equation*

$$
\frac{\partial S_{\mathbb{C}}}{\partial\tau} + \frac{1}{2}\Big(X_1 S_{\mathbb{C}}\Big)^2 + \frac{1}{2}\Big(X_2 S_{\mathbb{C}}\Big)^2 + \frac{1}{2}\Big(T S_{\mathbb{C}}\Big)^2 = 0. \tag{9.4.40}
$$

Proof. A computation provides

$$
\begin{aligned}
\partial_t S_{\mathbb{C}} &= -i, \\
\partial_{x_1} S_{\mathbb{C}} &= 2\mathbf{x}_1\coth(2\tau), \\
\partial_{x_2} S_{\mathbb{C}} &= 2\mathbf{x}_2\coth(2\tau).
\end{aligned}
$$

$$
\begin{aligned}
2H(\nabla S_{\mathbb{C}}) :&= \big(X_1 S_{\mathbb{C}}\big)^2 + \big(X_2 S_{\mathbb{C}}\big)^2 + \big(T S_{\mathbb{C}}\big)^2 \\
&= \big(\partial_{x_1} S_{\mathbb{C}} + 2\mathbf{x}_2\partial_t S_{\mathbb{C}}\big)^2 + \big(\partial_{x_2} S_{\mathbb{C}} - 2\mathbf{x}_1\partial_t S_{\mathbb{C}}\big)^2 + \big(\partial_t S_{\mathbb{C}}\big)^2 \\
&= \big(2\mathbf{x}_1\coth(2\tau) - 2i\mathbf{x}_2\big)^2 + \big(2\mathbf{x}_2\coth(2\tau) + 2i\mathbf{x}_1\big)^2 + (-i)^2 \\
&= 4\mathbf{x}_1^2\coth^2(2\tau) - 4\mathbf{x}_2^2 - 8i\mathbf{x}_1\mathbf{x}_2\coth(2\tau) \\
&\quad +4\mathbf{x}_2^2\coth^2(2\tau) - 4\mathbf{x}_1^2 + 8i\mathbf{x}_1\mathbf{x}_2\coth(2\tau) - 1 \\
&= 4|\mathbf{x}|^2\coth^2(2\tau) - 4|\mathbf{x}|^2 - 1.
\end{aligned} \tag{9.4.41}
$$

On the other hand

$$\frac{\partial S_{\mathbb{C}}}{\partial \tau} = \frac{1}{2} + |\mathbf{x}|^2 \frac{\partial}{\partial \tau}\big[\coth(2\tau)\big] = \frac{1}{2} - \frac{2|\mathbf{x}|^2}{\sinh^2(2\tau)}. \tag{9.4.42}$$

Adding (9.4.41) and (9.4.42) yields

$$\begin{aligned} \frac{\partial S_{\mathbb{C}}}{\partial \tau} + H(\nabla S_{\mathbb{C}}) &= \frac{1}{2} - \frac{2|\mathbf{x}|^2}{\sinh^2(2\tau)} + \frac{1}{2}\Big(4|\mathbf{x}|^2\coth^2(2\tau) - 4|\mathbf{x}|^2 - 1\Big) \\ &= -\frac{2|\mathbf{x}|^2}{\sinh^2(2\tau)} + 2|\mathbf{x}|^2\Big(\coth^2(2\tau) - 1\Big) \\ &= -\frac{2|\mathbf{x}|^2}{\sinh^2(2\tau)} + 2|\mathbf{x}|^2\frac{1}{\sinh^2(2\tau)} = 0. \end{aligned}$$

■

Now, we can easily check that $\Delta_{Cas} S_{\mathbb{C}}$ depends only on τ.

Proposition 9.29 *We have* $\Delta_{Cas} S_{\mathbb{C}} = 2\coth(2\tau)$.

Proof. Obviously $T^2 S_{\mathbb{C}} = 0$. We have

$$X_1^2 S_{\mathbb{C}} = X_1\big(2\mathbf{x}_1\coth(2\tau) - 2\mathbf{x}_2\big) = 2\coth(2\tau).$$

Similarly, $X_2^2 S_{\mathbb{C}} = 2\coth(2\tau)$, and hence

$$\Delta_{Cas} S_{\mathbb{C}} = \frac{1}{2}X_1^2 S_{\mathbb{C}} + \frac{1}{2}X_2^2 S_{\mathbb{C}} + \frac{1}{2}T^2 S_{\mathbb{C}} = 2\coth(2\tau).$$

■

9.4.4 The volume function at the origin

The volume function equation

$$\frac{dv(\tau)}{d\tau} + \big(\Delta_{Cas} S_{\mathbb{C}}\big)v(\tau) = 0$$

becomes

$$\frac{dv(\tau)}{d\tau} = -2\coth(2\tau)v(\tau).$$

Separating

$$\frac{dv}{v} = -2\coth(2\tau),$$

and integrating

$$\ln|v(\tau)| = -\ln|\sinh(2\tau)| + C_0.$$

Hence

$$v(\tau) = \frac{2}{\sinh(2\tau)} \tag{9.4.43}$$

is the solution with $\lim_{\tau\to 0}\tau v(\tau) = 1$. Formula (9.4.43) will be useful when we compute the heat kernel for the Casimir operator in Chapter 10.

9.5 Exercises

1. Denote by γ_v the geodesic emanating at the point x_0 with initial velocity v. Show

$$\gamma_v(\lambda t) = \gamma_{\lambda v}(t),$$

for any λ such that $t, \lambda t \in [0, a(x_0))$.

2. The mean curvature vector field to the geodesic sphere $S(x_0, t)$ is given by

$$H_x = \frac{\Delta d(x_0, x)}{n-1} \nabla d(x_0, x)\Big|_{|x|=t}.$$

3. If $\nabla_X X = 0$, then $curl\ X = 0$.

4. Given a point $x_0 \in M$, there is a compact neighborhood U of x_0 and $a > 0$ such that $\forall x \in U$ and $\forall v \in T_{x_0}M$, $|v| = 1$, there is only one geodesic $\gamma : [0, a) \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.

5. Compute the exponential map on the Heisenberg group with respect to the metric h.

6. Let $x \in \mathbb{R}^n$ and $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. Show the following:
(i) $\Delta|x| = -\dfrac{n-1}{|x|}$.

(ii) $\Delta S = -\dfrac{n}{t}$, and use Lemma 9.15 to deduce that $\mathbb{R}^n$ with the standard metric is a radially symmetric space.

(iii) $\Delta^2(|x|) = 0$, for $x \neq 0$.

7. Show that there are no compact Riemannian manifolds M, without boundary, such that

$$d\Delta d = k, \quad k \neq 1.$$

Hint:

$$\begin{aligned} 0 &= -\int_M div(\nabla d^2)\, dv = \int_M \Delta(d^2)\, dv \\ &= 2\int_M \underbrace{d\Delta d}_{=k}\, dv - 2\int_M \underbrace{|\nabla d|^2}_{=1}\, dv \\ &= 2k vol(M) - 2vol(M) = 2(k-1)vol(M), \end{aligned}$$

which is a contradiction.

10

Fundamental Solutions for Heat Operators with Potentials

10.1 The heat operator on Riemannian manifolds

Let (M, g) be a Riemannian manifold and let $C^{1,2}(M)$ be the space of functions $f : (0, \infty) \times M \to \mathbb{R}$, which are continuous on $[0, \infty) \times M$, C^1-differentiable in the first variable, and C^2-differentiable in the second variable. Let the Laplacian be $\Delta = -div\nabla$.

Definition 10.1 *The operator* $P = \dfrac{\partial}{\partial t} + \Delta$ *defined on the space* $\mathcal{C}^{1,2}(M)$ *is called the heat operator on* (M, g).

In order to invert the heat operator, one needs to study the fundamental solution.

Definition 10.2 *A fundamental solution* K *for the heat operator* $P = \dfrac{\partial}{\partial t} + \Delta_y$ *is a function* $K : M \times M \times (0, \infty) \to \mathbb{R}$ *with the following properties:*

$i)$ $K \in \mathcal{C}(M \times M \times (0, \infty))$, $\mathcal{C}^2$ *in the* 1^{st} *variable, and* $\mathcal{C}^1$*in the* 2^{nd}*variable,*
$ii)$ $\left(\frac{\partial}{\partial t} + \Delta_y\right)K(\,.\,, y, t) = 0, \quad \forall t > 0,$

$iii)$ $\lim_{t\searrow 0} K(x, \cdot\,, t) = \delta_x, \quad \forall x \in M,$
where δ_x *is the Dirac distribution centered at* x *and the limit* $iii)$ *is considered in the distribution sense, i.e.,*

$$\lim_{t\searrow 0} \int_M K(x, y, t)\phi(x)\,dv(x) = \phi(y), \quad \forall\phi \in C_0(M),\ \forall x \in M,$$

where $C_0(M)$ *denotes the set of smooth functions with compact support, and* $dv(x) = \sqrt{|g_{ij}(x)|}dx_1 \wedge \cdots \wedge dx_n$.

10.1.1 The case of compact manifolds

Let (M, g) be a compact Riemannian manifold. We define the inner product

$$\Big(f, g\Big)_0 = \int_M fg\,dv, \qquad \forall f, g \in \mathcal{F}(M).$$

Let $\|f\|_{L^2} = \Big(f, f\Big)_0^{1/2}$. The space $L^2(M)$ is obtained from $\mathcal{F}(M) = \{f : M \to \mathbb{R};\ f \in \mathcal{C}^\infty\}$ by completeness with respect to the norm $\|\cdot\|_{L^2}$.

The real numbers λ for which there is a nonzero smooth function f such that $\Delta f = \lambda f$ are called eigenvalues. f is an eigenfunction of λ. Let $V_\lambda(M, g) = \{f : M \to \mathbb{R};\ \Delta f = \lambda f\}$ be the vectorial space of the eigenfunctions together with the zero function. The number $m_\lambda = \dim V_\lambda(M, g)$ is called the multiplicity of λ.
In the following we shall find the fundamental solution of P in the case of a compact manifold. The spectral theory of the Laplace operator is a consequence of the Riesz–Schauder theory. Hence the following spectral theorem holds for the Laplace operator on Riemannian manifolds:

Theorem 10.3. *(i) The eigenvalues are nonnegative and form a countable infinite set*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

with $\lambda_k \to +\infty$, as $k \to +\infty$ and the series $\sum_{k\geq 1} \frac{1}{\lambda_k^2}$ converges.
(ii) Each eigenvalue λ_k has finite multiplicity m_k. The eigenspaces $V_{\lambda_k}(M, g)$ and $V_{\lambda_j}(M, g)$, $k \neq j$ are orthogonal with respect to the inner product $(\ ,\ )_0$.
(iii) From the system of eigenfunctions, using the Gram–Schmidt procedure, one may obtain a complete orthonormal system $\{f_{kj};\ k \in \mathbb{N}, j = 1, \ldots, m_k\}$ of eigenfunctions, such that

$$h = \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} a_{kj} f_{kj}, \quad \forall h \in L^2(M),$$

with $a_{kj} = (h, f_{kj})_0$. In particular, the Parseval identity holds

$$\|h\|_0^2 = \sum_{k=0}^{\infty}\sum_{j=1}^{m_k} (h, f_{kj})_0^2.$$

The following result provides a formula for the fundamental solution on a compact Riemannian manifold.

Proposition 10.4 *Let $\{f_i;\ i \in \mathbb{N}\}$ be a complete orthonormal system of eigenfunctions for the Laplace operator on the compact Riemannian manifold (M, g), such that*

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots.$$

Then the fundamental solution is given by

$$K(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} f_i(x) f_i(y). \tag{10.1.1}$$

Proof. Since the system $\{f_i; i \in \mathbb{N}\}$ is an orthonormal basis of the Hilbert space $L^2(M)$, we assume the existence of a fundamental solution for fixed x and t. Thus,

$$K(x, \cdot, t) = \sum_{i=0}^{\infty} \rho_i(x, t) f_i,$$

where

$$\rho_i(x, t) = \int_M K(x, y, t) f_i(y)\, dv(y).$$

Differentiating with respect to t yields

$$\begin{aligned}\frac{\partial \rho_i}{\partial t} &= \int_M \frac{\partial K}{\partial t}(x, y, t) f_i(y)\, dv(y) = \langle \frac{\partial K}{\partial t}, f_i \rangle \\ &= -\langle \Delta_y K, f_i \rangle = -\langle K, \Delta_y f_i \rangle = -\lambda_i \langle K, f_i \rangle = -\lambda_i \rho_i.\end{aligned}$$

Hence $\dfrac{\partial \rho_i}{\partial t} = -\lambda_i \rho_i$, where $\rho_i(x, t) = c_i(x) e^{-\lambda_i t}$. The function c_i satisfies

$$\begin{aligned}\lim_{t \searrow 0} \rho_i(x, t) &= \lim_{t \searrow 0} \int_M K(x, y, t) f_i(y)\, dv(y) \\ &= \int_M \delta_x(y) f_i(y)\, dv(y) = f_i(x).\end{aligned}$$

On the other side

$$\lim_{t \searrow 0} \rho_i(x, t) = c_i(x),$$

and hence $c_i(x) = f_i(x)$. Therefore equation (10.1.1) is proved. ∎

The above proof assumes the existence of a fundamental solution for the heat operator. This result is proved in [28]. The series $\sum_{i=0}^{\infty} \rho_i(x, t) f_i(y)$ is pointwise convergent on $(0, \infty) \times M \times M$ and its sum is $K(x, y, t)$. For the proof the reader may consult [28].

One may be interested in solving the initial value problem for the heat operator: Given a continuous function $g \in C^0(M)$, find a function $f \in \mathcal{C}^{1,2}(M)$ such that

i) $(\dfrac{\partial}{\partial t} + \Delta) f = 0,$

ii) $\lim_{t \searrow 0} f(x, t) = g(x), \quad \forall x \in M.$

Proposition 10.5 *The solution for the above $i) - ii)$ initial value problem is given by the formula*

$$f(x, t) = \int_M K(x, y, t) g(y)\, dv(y), \tag{10.1.2}$$

where K is given by (10.1.1).

Proof. A straightforward computation provides

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial}{\partial t} \int_M \sum_{i=0}^{\infty} e^{-\lambda_i t} f_i(x) f_i(y) g(y)\, dv(y)$$
$$= -\int_M \sum_{i=0}^{\infty} \lambda_i e^{-\lambda_i t} f_i(x) f_i(y) g(y)\, dv(y).$$
$$\Delta_x f(x,t) = \Delta_x \int_M \sum_{i=0}^{\infty} e^{-\lambda_i t} f_i(x) f_i(y) g(y)\, dv(y)$$
$$= \int_M \sum_{i=0}^{\infty} e^{-\lambda_i t} \Delta_x f_i(x) f_i(y) g(y)\, dv(y)$$
$$= \int_M \sum_{i=0}^{\infty} \lambda_i e^{-\lambda_i t} f_i(x) f_i(y) g(y)\, dv(y).$$

Hence

$$(\frac{\partial}{\partial_t} + \Delta) f = 0.$$

We still need to show that

$$\lim_{t \searrow 0} f(x,t) = g(x).$$

Using definition 10.2 $iii)$ yields

$$\lim_{t \searrow 0} f(x,t) = \lim_{t \searrow 0} \int_M K(x,y,t) g(y)\, dv(y) = \int_M \lim_{t \searrow 0} K(x,y,t) g(y)\, dv(y)$$
$$= \int_M \delta_x(y) g(y)\, dv(y) = \langle \delta_x, g \rangle = g(x).$$

■

10.2 Heat kernel on radially symmetric spaces

We have seen that $\mathbb{R}^n$ with the standard metric is a radially symmetric space, *i.e.*, the scalar mean curvature of the geodesic sphere depends only on its radius. It is known that the fundamental solution in this case is given by

$$K(x,y,t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0. \tag{10.2.3}$$

This is a product between the volume function $v(t) = t^{-n/2}$ and an exponential with the exponent

$$-\frac{|x-y|^2}{4t} = -\frac{1}{2} S,$$

where S is the classical action between the points x and y within time t.

The goal of this section is to prove a similar formula for radially symmetric spaces. We shall use the following result.

Lemma 10.6 *For any smooth function φ on a Riemannian manifold (M, g) we have*

$$\Delta e^{\varphi} = e^{\varphi}(\Delta\varphi - |\nabla\varphi|^2). \tag{10.2.4}$$

Proof. First we shall show that

$$\nabla e^{\varphi} = e^{\varphi}\nabla\varphi. \tag{10.2.5}$$

This comes from the definition of the gradient. For any vector field X,

$$\begin{aligned} g(\nabla e^{\varphi}, X) = X(e^{\varphi}) &= \sum X^i \partial_{x_i} e^{\varphi} = e^{\varphi} X(\varphi) \\ &= e^{\varphi} g(\nabla\varphi, X) = g(e^{\varphi}\nabla\varphi, X), \end{aligned}$$

and hence (10.2.5). Using the formula

$$div(fX) = f div\, X + g(\nabla f, X), \quad \forall X \in \mathcal{X}(M)$$

we have

$$\begin{aligned} -\Delta e^{\varphi} &= div(\nabla e^{\varphi}) = div(e^{\varphi}\nabla\varphi) \\ &= e^{\varphi}(div\nabla\varphi) + g(\nabla e^{\varphi}, \nabla\varphi) \\ &= -e^{\varphi}\Delta\varphi + e^{\varphi} g(\nabla\varphi, \nabla\varphi) \\ &= -e^{\varphi}(\Delta\varphi - |\nabla\varphi|^2). \end{aligned}$$

■

Let $d = d(x_0, x)$ be the Riemannian distance between the points x_0 and $x \in M$. Let

$$f = \frac{1}{2} d^2(x_0, x). \tag{10.2.6}$$

It was proved in section 7.3 (see Corollary 7.16) that $|\nabla d^2|^2 = 4d^2$. Hence the function f satisfies the eiconal equation

$$|\nabla f|^2 = 2f. \tag{10.2.7}$$

The classical action starting at x_0 is

$$S = S(x_0, x, t) = \frac{d^2(x_0, x)}{2t} = \frac{f}{t}.$$

Then

$$|\nabla S|^2 = \left|\nabla\left(\frac{f}{t}\right)\right|^2 = \frac{1}{t^2}|\nabla f|^2 = \frac{2f}{t^2} = \frac{2S}{t} = 2E,$$

where $E = \dfrac{d^2(x_0, x)}{2t^2}$ is the energy.

Inspired by the formula (10.2.3), we shall look for a fundamental solution of the form

$$K(x_0, x, t) = V(t)e^{kS}, \tag{10.2.8}$$

where $k \in \mathbb{R}$ is a constant, $V(t)$ is a differentiable function, and S is the above action. Differentiating and using the Hamilton–Jacobi equation $\dfrac{\partial}{\partial t} S = -E$, we have

$$\begin{aligned}\frac{\partial}{\partial t} K &= V'(t)e^{kS} + kV(t)e^{kS}\frac{\partial}{\partial t} S \\ &= e^{kS}\Big(V'(t) - kEV(t)\Big).\end{aligned}$$

Lemma 10.6 yields

$$\begin{aligned}\Delta\Big(V(t)e^{kS}\Big) &= e^{kS}V(t)\Big(k\Delta S - k^2|\nabla S|^2\Big) \\ &= e^{kS}V(t)\Big(k\Delta S - 2k^2 E\Big).\end{aligned}$$

Hence

$$\begin{aligned}(\frac{\partial}{\partial t} + \Delta)\Big(V(t)e^{kS}\Big) &= e^{kS}V(t)\Big(\frac{V'(t)}{V(t)} - kE\Big) + e^{kS}V(t)\Big(k\Delta S - 2k^2 E\Big) \\ &= e^{kS}V(t)\Big(\frac{V'(t)}{V(t)} + k\Delta S - kE(2k+1)\Big).\end{aligned}$$

Choose $k = -\dfrac{1}{2}$ and let $V(t)$ satisfy the equation $\dfrac{V'(t)}{V(t)} + k\Delta S = 0$, *i.e.*,

$$V'(t) = \frac{1}{2}\Delta S\, V(t). \tag{10.2.9}$$

As the manifold (M, g) is radially symmetric, ΔS is a function of t only, *i.e.*, there is a function $h(t) = \dfrac{1}{2}\Delta S = \dfrac{n-1}{2}\alpha(t)$, where $\alpha(t) = \alpha(c(t))$ is the mean scalar curvature of the geodesic sphere centered at x_0 with radius t. The solution is given by

$$V(t) = V(t_0)e^{\int_{t_0}^{t} h(u)\,du}.$$

Theorem 10.7. *Let (M, g) be a radially symmetric space about the point $x_0 \in M$. Then the fundamental solution for the heat operator is given by*

$$K(x_0, x, t) = CV(t)e^{-\frac{1}{2}S} = CV(t)e^{-\frac{d^2(x_0,x)}{4t}},$$

where $V(t)$ is the solution of (10.2.9) with the condition $\lim_{t\searrow 0} t^{n/2}V(t) = 1$ and

$$1/C = 2^n \int_0^\infty e^{-y^2}\omega(x_0, y)\,dy,$$

with ω defined by (10.2.10).

Proof. We still need to prove iii) of Definition 10.2, *i.e.*, for any ϕ compact supported function,

$$\lim_{t \searrow 0} \int_M K(x_0, x, t)\phi(x)\, dv(x) = \phi(x_0).$$

Substitute $y = \dfrac{d(x_0, x)}{2\sqrt{t}}$ and let $x \in d^{-1}(2\sqrt{t}y) = \mathbb{S}(x_0, 2\sqrt{t}y)$, a geodesic sphere centered at x_0. As ϕ is compact supported, let $\mathrm{D} = \mathrm{supp}(\phi)$. Then let $\delta = \max_{x \in \mathrm{D}} d(x_0, x)$ and $y \in [0, \delta/(2\sqrt{t})]$. Let $\omega(x_0, y)$ be defined by

$$vol\mathbb{S}(x_0, 2\sqrt{t}y) \sim (2\sqrt{t})^n \omega(x_0, y), \quad \text{as } t \searrow 0. \tag{10.2.10}$$

$$\begin{aligned}
\lim_{t \searrow 0} \int_M K(x_0, x, t)\phi(x)\, dv(x) &= C \lim_{t \searrow 0} V(t) \int_M e^{-\frac{d^2(x_0, x)}{4t}} \phi(x)\, dv(x) \\
&= C \lim_{t \searrow 0} V(t) \int_0^{\delta/(2\sqrt{t})} \int_{\mathbb{S}(x_0, 2\sqrt{t}y)} e^{-y^2} \phi(x)\, d\sigma_x\, dy \\
&= C \lim_{t \searrow 0} V(t) \int_0^{\delta/(2\sqrt{t})} e^{-y^2} \phi(x_t) vol\mathbb{S}(x_0, 2\sqrt{t}y)\, dy \\
&= C \lim_{t \searrow 0} V(t)\phi(x_t) \int_0^{\delta/(2\sqrt{t})} e^{-y^2} (2\sqrt{t})^n \omega(x_0, y)\, dy \\
&= C \lim_{t \searrow 0} 2^n t^{n/2} V(t)\phi(x_t) \int_0^{\infty} e^{-y^2} \omega(x_0, y)\, dy \\
&= \phi(x_0) = \delta_x(\phi),
\end{aligned}$$

where we have applied Fubini's theorem and the mean value theorem for integrals to obtain $x_t \in \mathbb{S}(x_0, 2\sqrt{t}y)$. ■

We shall extend this formula to spaces which are not radially symmetric but can be reduced to them. In those cases we shall compute the volume function $V(t)$ explicitly.

10.3 Heat kernel for the Casimir operator

We have defined the Casimir operator in Chapter 9 as an elliptic operator given by a sum of squares of vector fields

$$\Delta_{Cas} = \frac{1}{2}\Big(X_1^2 + X_2^2 + T^2\Big),$$

where X_1, X_2 and T are given by (9.4.14) and are left invariant vector fields with respect to the Heisenberg group law (9.4.13).

Theorem 10.8. *There is a constant* **c** *such that the fundamental solution for the operator* $\partial_\tau - \Delta_{Cas}$ *is*

$$K\big(y,\sigma,x,t,\tau\big) = K\big(0,0,(y,\sigma)^{-1}\circ_H(x,t),\tau\big), \tag{10.3.11}$$

where " $\circ_H$ " stands for the Heisenberg group law, and

$$K\big(0,0,x,t,\tau\big) = \frac{2\mathbf{c}}{\sinh(2\tau)} e^{-\frac{1}{2}\big(-it+\frac{\tau}{2}|x|^2\coth(2\tau)\big)},$$

and $x=(x_1,x_2)$, $y=(y_1,y_2)$.

Proof. The complex action from the origin and the volume function at the origin had been computed in Chapter 9, see equations (9.4.39) and (9.4.43). Theorem 10.7 yields a fundamental solution at the origin

$$\begin{aligned} K\big(0,0,x,t,\tau\big) &= v(\tau)e^{-\frac{1}{2}S_{\mathbb{C}}} \\ &= \frac{2\mathbf{c}}{\sinh(2\tau)} e^{-\frac{1}{2}\big(-it+\frac{\tau}{2}|x|^2\coth(2\tau)\big)}. \end{aligned}$$

We have that $K\big(0,0,x,t,\tau\big)$ is the kernel relative to the origin. It follows from the left invariance of Δ_{Cas} that the full heat kernel is obtained by left translations. The Heisenberg convolution provides formula (10.3.11). See Exercise 5. ■

10.4 Heat kernel for operators with potential

In the next few sections we shall compute the action and volume functions explicitly and provide closed form solutions for heat operators with potential. The first few sections will deal with the heat kernel of a Hermite operator.

10.4.1 The kernel of $\partial_t - \partial_x^2 \pm b^2x^2$

We start with the operator

$$\mathrm{L} = \frac{d^2}{dx^2} - a^2x^2,$$

where $a \in \mathbb{R}_+$ is a nonnegative real parameter. We associate the Hamiltonian function as half of the principal symbol

$$H(\xi,x) = \frac{1}{2}(\xi^2 - a^2x^2). \tag{10.4.12}$$

The Hamiltonian system is

$$\begin{cases} \dot{x} = H_\xi = \xi, \\ \dot{\xi} = -H_x = a^2x. \end{cases}$$

As we are interested in finding the geodesic between the points $x_0, x \in \mathbb{R}$, $x(s)$ will satisfy the boundary problem

$$\begin{cases} \ddot{x} = a^2 x, \\ x(0) = x_0, \ x(t) = x. \end{cases}$$

The conservation of energy law is

$$\frac{1}{2}\dot{x}^2(s) - \frac{1}{2}a^2x^2(s) = E,$$

where E is the energy constant. This can be used to obtain an ODE for the solution $x(s)$,

$$\frac{dx}{ds} = \sqrt{2E + a^2x^2} \Longrightarrow \frac{dx}{\sqrt{2E + a^2x^2}} = ds.$$

Integrating between $s = 0$ and $s = t$, with $x(0) = x_0$ and $x(t) = x$, yields

$$\int_{x_0}^{x} \frac{du}{\sqrt{2E + a^2u^2}} = t \Longleftrightarrow \int_{v_0}^{v} \frac{dv}{\sqrt{1 + v^2}} = at,$$

with $v = \dfrac{ax}{\sqrt{2E}}$ and $v_0 = \dfrac{ax_0}{\sqrt{2E}}$. Integrating yields

$$\begin{aligned}
\sinh^{-1}(v) - \sinh^{-1}(v_0) &= at \\
\Longleftrightarrow \sinh^{-1}(v) &= \sinh^{-1}(v_0) + at \\
\Longleftrightarrow v &= \sinh\Big(\sinh^{-1}(v_0) + at\Big) \\
\Longleftrightarrow v &= v_0 \cosh(at) + \cosh(\sinh^{-1}(v_0)) \sinh(at) \\
\Longleftrightarrow v &= v_0 \cosh(at) + \sqrt{1 + v_0^2}\sinh(at) \\
\Longleftrightarrow \frac{ax}{\sqrt{2E}} &= \frac{ax_0}{\sqrt{2E}}\cosh(at) + \sqrt{1 + \frac{a^2x_0^2}{2E}}\sinh(at) \\
\Longleftrightarrow ax &= ax_0 \cosh(at) + \sqrt{2E + a^2x_0^2}\sinh(at) \\
\Longleftrightarrow \frac{a(x - x_0\cosh(at))}{\sinh(at)} &= \sqrt{2E + a^2x_0^2}.
\end{aligned}$$

Solving for E yields

$$\begin{aligned}
2E &= \frac{a^2(x - x_0\cosh(at))^2}{\sinh(at)^2} - a^2x_0^2 \\
&= \frac{a^2\Big(x^2 - 2xx_0\cosh(at) + x_0^2\cosh(at)^2 - x_0^2\sinh(at)^2\Big)}{\sinh(at)^2} \\
&= \frac{a^2\Big(x^2 + x_0^2 - 2xx_0\cosh(at)\Big)}{\sinh(at)^2}.
\end{aligned}$$

Proposition 10.9 *The energy along a geodesic derived from the Hamiltonian (10.4.12) between the points* x_0 *and* x *is*

$$E = \frac{a^2\Big(x^2 + x_0^2 - 2xx_0\cosh(at)\Big)}{2\sinh(at)^2}. \tag{10.4.13}$$

Making $x_0 = 0$, we obtain the following result.

Corollary 10.10 *The energy along a geodesic derived from the Hamiltonian (10.4.12) joining the origin and* x *is given by*

$$E = \frac{a^2x^2}{2\sinh(at)^2}. \tag{10.4.14}$$

We note that if we take the limit $a \to 0$ in (10.4.13), we obtain the Euclidian energy

$$\begin{aligned}\lim_{a\to 0} E &= \lim_{a\to 0}\frac{a^2t^2}{\sinh(at)^2}\frac{\Big(x^2 + x_0^2 - 2xx_0\cosh(at)\Big)}{2t^2}\\ &= \frac{(x-x_0)^2}{2t^2}.\end{aligned}$$

The action

Let $S = S(x_0, x, t)$ be the action with initial point x_0 and final point x, within time t. The action satisfies the Hamilton–Jacobi equation

$$\partial_t S + H(\nabla S) = 0.$$

We note that

$$H = \frac{1}{2}(\xi^2 - a^2x^2) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}a^2x^2 = E,$$

and hence $\partial_t S = -E$. Using (10.4.13) yields

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\frac{a^2\Big(x^2 + x_0^2 - 2xx_0\cosh(at)\Big)}{2\sinh(at)^2}\\ &= \frac{a}{2}(x^2 + x_0^2)\frac{\partial}{\partial t}\coth(at) - axx_0\frac{\partial}{\partial t}\frac{1}{\sinh(at)}\\ &= \frac{\partial}{\partial t}\Big[\frac{a}{2}(x^2 + x_0^2)\coth(at) - \frac{axx_0}{\sinh(at)}\Big].\end{aligned}$$

Hence we have arrived at the action

$$\begin{aligned}S(x_0, x, t) &= \frac{a}{2}\Big[(x^2 + x_0^2)\coth(at) - \frac{2xx_0}{\sinh(at)}\Big]\\ &= \frac{a}{2}\frac{1}{\sinh(at)}\Big[(x^2 + x_0^2)\cosh(at) - 2xx_0\Big].\end{aligned} \tag{10.4.15}$$

We also note that

$$\lim_{a \to 0} S = \frac{(x - x_0)^2}{2t},$$

which is the Euclidian action.

Lemma 10.11 *We have*

$$1)\ \ (\partial_x S)^2 = a^2 x^2 + 2E,$$
$$2)\qquad \partial_x^2 S = a \coth(at).$$

Proof. 1) Differentiating in (10.4.15) yields

$$\partial_x S = \frac{a}{\sinh(at)} \Big(x \cosh(at) - x_0 \Big), \tag{10.4.16}$$

$$\begin{aligned}
(\partial_x S)^2 &= \frac{a^2 \Big(x^2 \cosh^2(at) + x_0^2 - 2 x x_0 \cosh(at) \Big)}{\sinh^2(at)} \\
&= \frac{a^2 \Big(x^2 + x^2 \sinh^2(at) + x_0^2 - 2 x x_0 \cosh(at) \Big)}{\sinh^2(at)} \\
&= a^2 x^2 + \frac{a^2 (x^2 + x_0^2 - 2 x x_0 \cosh(at))}{\sinh^2(at)} \\
&= a^2 x^2 + 2E.
\end{aligned}$$

2) Differentiating in (10.4.16) yields

$$\partial_x^2 S = \frac{a}{\sinh(at)} \cosh(at) = a \coth(at).$$

■

We shall look for a fundamental solution of the type

$$K(x_0, x, t) = V(t) e^{k S(x_0, x, t)}, \tag{10.4.17}$$

where $V(t)$ will satisfy a volume function equation and k is a real constant. Lemma 10.11 provides

$$\begin{aligned}
\partial_t K &= V'(t) e^{kS} + V(t) k e^{kS} \partial_t S \\
&= e^{kS} \Big(V'(t) - k V(t) E \Big),
\end{aligned}$$

$$\begin{aligned}
\partial_x e^{kS} &= k e^{kS} \partial_x S, \\
\partial_x^2 e^{kS} &= k^2 e^{kS} (\partial_x S)^2 + k e^{kS} \partial_x^2 S \\
&= k e^{kS} \big[k (\partial_x g)^2 + \partial_x^2 S \big] \\
&= k e^{kS} \big[k (a^2 x^2 + 2E) + a \coth(at) \big].
\end{aligned}$$

We shall find the heat kernel using a multiplier method. Let

$$P = \partial_t - \partial_x^2 + \alpha a^2 x^2, \tag{10.4.18}$$

where α is a real multiplier, which will be determined such that $PK(x_0, x, t) = 0$ for any $t > 0$.

$$\begin{aligned} PK(x_0, x, t) &= e^{kS}\Big(V'(t) - kEV(t)\Big) \\ &\quad -ke^{kS}\Big(k(a^2x^2 + 2E) + a\coth(at)\Big)V(t) \\ &\quad +\alpha a^2 x^2 e^{kS} V(t) \\ &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} - kE - k^2(a^2x^2 + 2E) - ka\coth(at) + \alpha a^2 x^2\Big] \\ &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} - kE - k^2a^2x^2 - 2k^2E + \alpha a^2x^2 - ka\coth(at)\Big] \\ &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} - kE(2k+1) + (\alpha - k^2)a^2x^2 - ka\coth(at)\Big]. \end{aligned}$$

In order to eliminate the middle two terms in the brackets, we choose $k = -\frac{1}{2}$ and $\alpha = \frac{1}{4}$. Let $b = \frac{a}{2} > 0$. Then the operator (10.4.18) becomes

$$P = \partial_t - \partial_x^2 + b^2 x^2 \tag{10.4.19}$$

and

$$PK(x_0, x, t) = K(x_0, x, t)\Big(\frac{V'(t)}{V(t)} + b\coth(2bt)\Big).$$

We shall choose $V(t)$ such that

$$\frac{V'(t)}{V(t)} = -b\coth(2bt), \quad t > 0.$$

Integrating yields

$$\ln V(t) = -\frac{1}{2}\ln\Big(\sinh(2bt)\Big) \Longrightarrow V(t) = \frac{C}{\sqrt{\sinh(2bt)}}.$$

Using the action (10.4.15), the fundamental solution formula (10.4.17) becomes

$$\begin{aligned} K(x_0, x, t) &= \frac{C}{\sqrt{\sinh(2bt)}} e^{-\frac{2b}{4}\frac{1}{\sinh(2bt)}[(x^2 + x_0^2)\cosh(2bt) - 2xx_0]} \\ &= \frac{C}{\sqrt{2bt}}\sqrt{\frac{2bt}{\sinh(2bt)}}\, e^{-\frac{1}{4t}\cdot\frac{2bt}{\sinh(2bt)}[(x^2 + x_0^2)\cosh(2bt) - 2xx_0]}. \end{aligned}$$

We shall find the constant C by investigating the limit case $b \to 0$, when the operator (10.4.19) becomes the usual one-dimensional heat operator $\partial_t - \partial_x^2$. As $\dfrac{2bt}{\sinh(2bt)} \to 1$, the above fundamental solution becomes

$$K(x_0, x, t) \sim \frac{C}{\sqrt{2bt}} e^{\frac{1}{4t}(x-x_0)^2}, \quad b \to 0.$$

By comparison with the fundamental solution for the usual heat operator, which is

$$\frac{1}{\sqrt{4\pi t}} e^{\frac{1}{4t}(x-x_0)^2},$$

we find $C = \sqrt{\dfrac{b}{2\pi}}$. We arrive at the following result.

Theorem 10.12. *Let $b \geq 0$. The fundamental solution for the operator $P = \partial_t - \partial_x^2 + b^2x^2$ is*

$$K(x_0, x, t) = \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{2bt}{\sinh(2bt)}}\, e^{-\frac{1}{4t}\frac{2bt}{\sinh(2bt)}[(x^2+x_0^2)\cosh(2bt) - 2xx_0]}, \quad t > 0.$$

The computations are similar in the case when $b = -i\beta$. Using $\cosh(i\beta t) = \cos(\beta t)$ and $\sinh(2i\beta t) = i\sin(2\beta t)$, we obtain a dual theorem.

Theorem 10.13. *Let $\beta \geq 0$. The fundamental solution for the operator $P = \partial_t - \partial_x^2 - \beta^2x^2$ is*

$$K(x_0, x, t) = \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{2\beta t}{\sin(2\beta t)}}\, e^{-\frac{1}{4t}\frac{2\beta t}{\sin(2\beta t)}[(x^2+x_0^2)\cos(2\beta t) - 2xx_0]}, \quad t > 0.$$

10.4.2 The kernel of $\partial_t - \sum \partial_{x_i}^2 \pm a^2|x|^2$

Consider the operator

$$\Delta_n - a^2|x|^2 = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 - a^2(x_1^2 + \cdots + x_n^2), \quad a \geq 0.$$

The associated Hamiltonian is

$$H = \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2) - \frac{1}{2}a^2(x_1^2 + \cdots + x_n^2),$$

with the Hamiltonian system

$$\begin{cases} \dot{x}_j = H_{\xi_j} = \xi_j, \\ \dot{\xi}_j = -H_{x_j} = a^2 x_j, \quad j = 1, \ldots, n. \end{cases}$$

The geodesic $x(s)$ starting at $x_0 = (x_1^0, \ldots, x_n^0)$ and having the final point $x = (x_1, \ldots, x_n)$ satisfies the equations

$$\begin{cases} \ddot{x}_j = a^2 x_j, \\ x_j(0) = x_j^0, \\ x_j(t) = x_j, \quad j = 1 \ldots n. \end{cases}$$

As in the one-dimensional case, we have the law of conservation of energy

$$\dot{x}_j^2(s) - a^2 x_j^2(s) = 2E_j, \quad j = 1, \ldots, n$$

where E_j is the energy constant for the j-th component. The total energy, which is the Hamiltonian, is given by

$$H = \sum_{j=1}^{n} (\frac{1}{2}\dot{x}_j^2 - \frac{1}{2}a^2 x_j^2) = E_1 + \cdots + E_n = E(\text{constant}).$$

Proposition 10.9 yields

$$E_j = \frac{a^2[x_j^2 + (x_j^0)^2 - 2x_j x_j^0 \cosh(at)]}{2\sinh^2(at)},$$

and hence

$$H = E = \sum_{j=1}^{n} E_j = \frac{a^2[|x|^2 + |x_0|^2 - 2\langle x, x_0\rangle \cosh(at)]}{2\sinh^2(at)},$$

where $|x|^2 = \sum_{j=1}^{n} x_j^2$ and $\langle x, x_0\rangle = \sum_{j=1}^{n} x_j x_j^0$.

The action

The action between x_0 and x in time t satisfies the equation $\dfrac{\partial}{\partial t} S = -E$ or

$$\begin{aligned} \frac{\partial}{\partial t} S &= -\frac{a^2[|x|^2 + |x_0|^2 - 2\langle x, x_0\rangle \cosh(at)]}{2\sinh^2(at)} \\ &= \frac{\partial}{\partial t}\left[\frac{a}{2}(|x|^2 + |x_0|^2)\coth(at) - \frac{a\langle x, x_0\rangle}{\sinh(at)}\right]. \end{aligned}$$

Hence we shall choose

$$S = \frac{a}{2}\frac{1}{\sinh(at)}\left[(|x|^2 + |x_0|^2)\cosh(at) - 2\langle x, x_0\rangle\right]. \tag{10.4.20}$$

Let

$$S_j = \frac{a}{2}\frac{1}{\sinh(at)}\Big[(x_j^2 + (x_j^0)^2)\cosh(at) - 2x_j\, x_j^0\Big]. \qquad (10.4.21)$$

Then $S = S_1 + \cdots + S_n$ and $\partial_{x_j} S = \partial_{x_j} S_j$. Then Lemma 10.11 yields

$$\begin{aligned}\sum_{j=1}^{n}(\partial_{x_j} S)^2 = \sum_{j=1}^{n}(\partial_{x_j} S_j)^2 &= \sum_{j=1}^{n}(a^2 x_j^2 + 2E_j)\\ &= a^2|x|^2 + 2E,\end{aligned}$$

$$\sum_{j=1}^{n}\partial_{x_j}^2 S = \sum_{j=1}^{n}\partial_{x_j}^2 S_j = na\coth(at).$$

We shall look for a kernel of the form

$$K(x_0, x, t) = V(t)e^{kS(x_0,x,t)}, \qquad k \in \mathbb{R}. \qquad (10.4.22)$$

A computation similar to the one-dimensional case yields

$$\frac{\partial}{\partial t}K = e^{kS}\Big(V'(t) - kEV(t)\Big),$$

and

$$\partial_{x_j}^2 e^{kS} = e^{kS}k\Big[k(\partial_{x_j} S)^2 + \partial_{x_j}^2 S\Big]$$

and hence

$$\Delta_n e^{kS} = ke^{kS}\Big[k(a^2|x|^2 + 2E) + n\,a\coth(at)\Big].$$

In order to find the kernel for the heat operator we employ the multiplier method again. We shall consider the parabolic operator

$$P_n = \partial_t - \Delta_n + \alpha a^2|x|^2,$$

where α is a multiplier subject to being found later. Then

$$\begin{aligned}P_n K &= e^{kS}\Big[V'(t) - kEV(t)\Big]\\ &\quad -ke^{kS}\Big[k(a^2|x|^2 + 2E) + n\,a\coth(at)\Big]V(t)\\ &\quad +\alpha a^2|x|^2 V(t)e^{kS}\\ &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} - kE(1+2k) + (\alpha - k^2)a^2|x|^2 - k\,n\,a\coth(at)\Big]\\ &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} + \frac{na}{2}\coth(at)\Big],\end{aligned}$$

where we choose $k = -\dfrac{1}{2}$ and $\alpha = \dfrac{1}{4}$. Let $b = \dfrac{a}{2} \geq 0$ and choose $V(t)$ satisfying

$$\frac{V'(t)}{V(t)} = -nb\coth(2bt), \quad t > 0.$$

Integrating yields $V(t) = \dfrac{C}{\sinh^{n/2}(2bt)}$. Hence the fundamental solution for the operator $P_n = \partial_t - \Delta_n + b^2|x|^2$ expressed in the form (10.4.22) is

$$\begin{aligned} K(x_0, x, t) &= \frac{C}{\sinh^{n/2}(2bt)} e^{-\frac{2b}{4}\frac{1}{\sinh(2bt)}\left((|x|^2+|x_0|^2)\cosh(2bt)-2\langle x, x_0\rangle\right)} \\ &= \frac{C}{(2bt)^{n/2}} \frac{(2bt)^{n/2}}{\sinh^{n/2}(2bt)} e^{-\frac{1}{4t}\frac{2bt}{\sinh(2bt)}\left((|x|^2+|x_0|^2)\cosh(2bt)-2\langle x, x_0\rangle\right)}. \end{aligned}$$

When $b \to 0$ we should obtain the kernel of the heat operator $\partial_t - \Delta_n$, which is

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{1}{4t}|x-x_0|^2}, \quad t > 0.$$

By comparison, we obtain the value

$$C = \frac{b^{n/2}}{(2\pi)^{n/2}}.$$

Theorem 10.14. *Let $b \geq 0$ and $\Delta_n = \sum_{j=1}^n \partial_{x_j}^2$. The fundamental solution for the operator $P_n = \partial_t - \Delta_n + b^2|x|^2$ is*

$$\begin{aligned} &K(x_0, x, t) \\ &= \frac{1}{(4\pi t)^{n/2}} \left(\frac{2bt}{\sinh(2bt)}\right)^{n/2} e^{-\frac{1}{4t}\frac{2bt}{\sinh(2bt)}[(|x|^2+|x_0|^2)\cosh(2bt)-2\langle x, x_0\rangle]} \end{aligned}$$

for $t > 0$.

In a similar way as in the one-dimensional case, choosing $b = -i\beta$, yields the following result.

Theorem 10.15. *Let $\beta \geq 0$ and $\Delta_n = \sum_{j=1}^n \partial_{x_j}^2$. The fundamental solution for the operator $P = \partial_t - \Delta_n - \beta^2|x|^2$ is*

$$\begin{aligned} &K(x_0, x, t) \\ &= \frac{1}{(4\pi t)^{n/2}} \left(\frac{2\beta t}{\sin(2\beta t)}\right)^{n/2} e^{-\frac{1}{4t}\frac{2\beta t}{\sin(2\beta t)}[(|x|^2+|x_0|^2)\cos(2\beta t)-2\langle x, x_0\rangle]} \end{aligned}$$

for $t > 0$.

10.4.3 Fourier transform method

The Hermite operator has been studied by mathematicians and physicists for a few generations (see *e.g.*, [5], [18]). The Fourier transform method used in this section follows the idea of Chang and Tie, see [8]. In the following we derive the fundamental solution and the heat kernel of the Hermite operator

$$H_\alpha = \alpha + \sum_{j=1}^{n} \left(\lambda_j^2 x_j^2 - \frac{\partial^2}{\partial x_j^2} \right)$$

in $\mathbb{R}^n$, *i.e.,* we are looking for a distribution $K_\alpha(\mathbf{x}, \mathbf{y})$ such that

$$\left[\alpha + \sum_{j=1}^{n} \left(\lambda_j^2 x_j^2 - \frac{\partial^2}{\partial x_j^2} \right) \right] K_\alpha(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \tag{10.4.23}$$

We first compute the fundamental solution with singularity at the origin when

$$\alpha \notin \Lambda = \left\{ -\sum_{j=1}^{n} (2k_j + 1)\lambda_j; \quad \mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n \right\}.$$

We also construct the relative fundamental solution for the operator H_{α_0} while $\alpha_0 \in \Lambda$, *i.e.,*

$$I = K_{\alpha_0} H_{\alpha_0} + J_{\alpha_0}.$$

Here J_{α_0} is a projection operator. Since the operator H_α is not left invariant under the Euclidean group action, we have to compute the fundamental solution with singularity at any point $\mathbf{y}$. Another reason for dividing these into two cases is to use a different method to sum up the infinite series involved.

10.4.3.1 Fundamental solution with singularity at the origin

In this section, we shall find $K_\alpha(\mathbf{x}) = K(\mathbf{x}, 0)$, *i.e.,* the fundamental solution with singularity at the origin first. Taking the Fourier transform

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\xi} f(\mathbf{x}) d\mathbf{x}$$

to the Hermite operator and applying the formulae

$$\mathcal{F}\left(\frac{\partial f}{\partial x_j} \right) = i\xi_j \mathcal{F}(f)(\xi) \qquad \text{and} \qquad \mathcal{F}(x_j f(\mathbf{x})) = i\frac{\partial}{\partial \xi_j} (\mathcal{F}(f))(\xi),$$

then when $\mathbf{y} = 0$, equation (10.4.23) becomes

$$(\alpha + |\xi|^2 - \sum_{j=1}^{n} \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}) \widehat{K}_\alpha(\xi) = 1.$$

First note that the Hermite function $\psi_k(x)$ is defined by its usual generating function formula:

$$\sum_{k=0}^{\infty} \frac{\psi_k(x)}{k!} t^k = e^{2tx - t^2 - \frac{1}{2}x^2}.$$

Here $\psi_k(x)$ is the eigenfunction of $(x^2 - \frac{d^2}{dx^2})$ with eigenvalue $2k+1$, *i.e.,*

$$\left(x^2 - \frac{d^2}{dx^2}\right)\psi_k(x) = (2k+1)\psi_k(x). \tag{10.4.24}$$

Besides the generating function formula, $\psi_k(x)$ has another representation

$$\psi_k(x) = e^{\frac{1}{2}x^2}\left(-\frac{d}{dx}\right)^k (e^{-x^2}) = H_k(x)e^{-\frac{1}{2}x^2}, \quad k \in \mathbf{Z}^+, \tag{10.4.25}$$

where $H_k(x)$ is the Hermite polynomial of degree k. The system $\{\psi_k(x)\}_{k=0}^{\infty}$ is complete in $L^2(\mathbb{R})$ and satisfies the orthogonal condition

$$< \psi_k, \psi_\ell >= \int_{-\infty}^{\infty} \psi_k(x)\psi_\ell(x)dx = 2^k\sqrt{\pi}k!\delta_{k\ell} \qquad \text{with} \quad \delta_{k\ell} = \begin{cases} 1 & \ell = k, \\ 0 & \ell \neq k. \end{cases} \tag{10.4.26}$$

Going back to the differential operator $\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}$, we introduce the new variable $\eta_j = \frac{\xi_j}{\sqrt{\lambda_j}}$, then

$$\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2} = \lambda_j \left(\eta_j^2 - \frac{\partial^2}{\partial \eta_j^2}\right).$$

Equation (10.4.24) yields

$$\left(\eta_j^2 - \frac{\partial^2}{\partial \eta_j^2}\right)\psi_k(\eta_j) = (2k+1)\psi_k(\eta_j).$$

This implies

$$\left(\frac{\alpha}{n} + \xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)\psi_k(\frac{\xi_j}{\sqrt{\lambda_j}}) = \left[\frac{\alpha}{n} + \lambda_j(2k+1)\right]\psi_k(\frac{\xi_j}{\sqrt{\lambda_j}}), \tag{10.4.27}$$

i.e., $\psi_k(\frac{\xi_j}{\sqrt{\lambda_j}})$ is the eigenfunction of $\frac{\alpha}{n} + \xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}$ with eigenvalue $\frac{\alpha}{n} + \lambda_j(2k+1)$. Next, for $\mathbf{k} = (k_1, \ldots, k_n)$ we define the n-tuple Hermite function

$$\Psi_{\mathbf{k}}(\xi) = \prod_{j=1}^{n} \psi_{k_j}(\xi_j/\sqrt{\lambda_j})$$

and let

$$\widehat{K}_\alpha(\xi) = \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \Psi_{\mathbf{k}}(\xi), \qquad \text{where} \quad |\mathbf{k}| = k_1 + \cdots + k_n.$$

Then we apply the operator $\left(\alpha + |\xi|^2 - \sum_{j=1}^{n} \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)$ to $\widehat{K}_\alpha(\xi)$ and obtain:

$$\left(\alpha + |\xi|^2 - \sum_{j=1}^{n} \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right) \widehat{K}_\alpha(\xi) = \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \left[\alpha + \sum_{j=1}^{n} \lambda_j (2k_j + 1)\right] \Psi_{\mathbf{k}}(\xi).$$

We will use the orthogonality property (10.4.26) to find $c_{\mathbf{k}}$.

$$\sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \left[\alpha + \sum_{j=1}^{n} \lambda_j (2k_j + 1)\right] \Psi_{\mathbf{k}}(\xi) = 1$$

implies

$$\left[\alpha + \sum_{j=1}^{n} \lambda_j (2k_j + 1)\right] c_{\mathbf{k}} < \Psi_{\mathbf{k}}, \Psi_{\mathbf{k}} >=< 1, \Psi_{\mathbf{k}} > .$$

Here $< \Psi_{\mathbf{k}}, \Psi_{\mathbf{m}} >$ is the usual inner product in $L^2(\mathbb{R})$. Since

$$< \Psi_{\mathbf{k}}, \Psi_{\mathbf{k}} >= \prod_{j=1}^{n} \sqrt{\lambda_j \pi}\, 2^{k_j} k_j!, \quad < 1, \Psi_{2\mathbf{k}+1} >= 0 \quad \text{and}$$

$$< 1, \Psi_{2\mathbf{k}} >= \prod_{j=1}^{n} \sqrt{2\lambda_j \pi}\, \frac{(2k_j)!}{k_j!}$$

we have $c_{2\mathbf{k}+1} = 0$ for $\mathbf{k} \in (\mathbb{Z}_+)^n$ and

$$\begin{aligned} c_{2\mathbf{k}} &= \frac{< 1, \Psi_{2\mathbf{k}} >}{\left[\alpha + \sum_{j=1}^{n} \lambda_j (4k_j + 1)\right] < \Psi_{2\mathbf{k}}, \Psi_{2\mathbf{k}} >} \\ &= \frac{1}{\left[\alpha + \sum_{j=1}^{n} \lambda_j (4k_j + 1)\right]} \cdot \frac{\prod_{j=1}^{n} \sqrt{2\lambda_j \pi}\, \frac{(2k_j)!}{k_j!}}{\prod_{j=1}^{n} \sqrt{\lambda_j \pi}\, 2^{2k_j} (2k_j)!} \\ &= \frac{2^{\frac{n}{2}}}{[\alpha + \sum_{j=1}^{n} \lambda_j (4k_j + 1)]} \cdot \frac{1}{\prod_{j=1}^{n} 2^{2k_j} k_j!}. \end{aligned}$$

Hence

$$\widehat{K}_\alpha(\xi) = \sum_{|\mathbf{k}|=0}^{\infty} c_{2\mathbf{k}} \Psi_{2\mathbf{k}} = \sum_{|\mathbf{k}|=0}^{\infty} \frac{2^{\frac{n}{2}}}{[\alpha + \sum_{j=1}^{n} \lambda_j (4k_j + 1)]} \prod_{j=1}^{n} \frac{\psi_{2k_j}(\frac{\xi_j}{\sqrt{\lambda_j}})}{2^{2k_j} k_j!}.$$

From the above discussion, it is easy to see that H_α is not invertible when

$$\alpha \in \Lambda = \Big\{ -\sum_{j=1}^{n}(2k_j+1)\lambda_j; \quad \mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n \Big\}.$$

We call Λ the exceptional set of H_α. Next we apply

$$\frac{1}{A} = \int_0^\infty e^{-As} ds \qquad \text{for} \qquad A = \alpha + \sum_{j=1}^{n} \lambda_j(4k_j+1)$$

and obtain

$$\begin{aligned}
\widehat{K}_\alpha(\xi) &= 2^{\frac{n}{2}} \sum_{|\mathbf{k}|=0}^{\infty} \int_0^\infty \prod_{j=1}^{n} \frac{\psi_{2k_j}(\frac{\xi_j}{\sqrt{\lambda_j}})}{2^{2k_j} k_j!} e^{-(4k_j+1)\lambda_j s} e^{-\alpha s} ds \\
&= \int_0^\infty 2^{\frac{n}{2}} \prod_{j=1}^{n} e^{-\lambda_j s} \sum_{k_j=0}^{\infty} \frac{\psi_{2k_j}(\eta_j)}{2^{2k_j} k_j!} e^{-4k_j\lambda_j s} e^{-\alpha s} ds \\
&= \int_0^\infty 2^{\frac{n}{2}} \prod_{j=1}^{n} e^{-\lambda_j s} g_j(\eta_j, s) e^{-\alpha s} ds
\end{aligned}$$

with $g_j(\eta_j, s) = \sum_{k_j=0}^{\infty} \frac{\psi_{2k_j}(\eta_j)}{2^{2k_j} k_j!} e^{-4k_j\lambda_j s}$. To sum up with respect to k_j in $g_j(\eta_j, s)$, we apply the relationship between the Hermite function and Laguerre polynomial (see p. 252 in [47]):

$$\psi_{2k}(x) = e^{-\frac{x^2}{2}} (-1)^k 2^{2k} k! L_k^{(-\frac{1}{2})}(x^2) \qquad \Leftrightarrow \qquad \frac{\psi_{2k}(x)}{2^{2k}k!} = e^{-\frac{x^2}{2}} (-1)^k L_{k_j}^{(-\frac{1}{2})}(x^2).$$

Therefore,

$$g_j(x, s) = \sum_{k_j=0}^{\infty} (-1)^{k_j} e^{-\frac{x^2}{2}} L_{k_j}^{(-\frac{1}{2})}(x^2) e^{-4k_j\lambda_j s} \tag{10.4.28}$$

$$= e^{-\frac{x^2}{2}} \sum_{k_j=0}^{\infty} L_{k_j}^{(-\frac{1}{2})}(x^2)(-e^{-4\lambda_j s})^{k_j}. \tag{10.4.29}$$

The Laguerre polynomials are defined by the generating formula (see *e.g.,* [6]):

$$\sum_{k=0}^{\infty} L_k^{(\beta)}(w) z^k = \frac{1}{(1-z)^{\beta+1}} \exp\left\{ \frac{wz}{z-1} \right\}.$$

Now we may apply the generating formula of the Laguerre polynomials to sum up the series (10.4.28) and find $g_j(x, s)$.

$$\begin{aligned} g_j(x,s) &= \frac{e^{-\frac{x^2}{2}}}{(1+e^{-4\lambda_j s})^{\frac{1}{2}}} \exp\left\{\frac{x^2 e^{-4\lambda_j s}}{e^{-4\lambda_j s}+1}\right\} \\ &= \frac{1}{(1+e^{-4\lambda_j s})^{\frac{1}{2}}} \exp\left\{-\frac{x^2}{2}\left[1-\frac{2e^{-4\lambda_j s}}{1+e^{-4\lambda_j s}}\right]\right\} \\ &= \frac{1}{(1+e^{-4\lambda_j s})^{\frac{1}{2}}} \exp\left\{-\frac{x^2}{2}\cdot\frac{1-e^{-4\lambda_j s}}{1+e^{-4\lambda_j s}}\right\}. \end{aligned}$$

Hence,

$$\widehat{K}_\alpha(\xi) = \int_0^\infty 2^{\frac{n}{2}} \left[\prod_{j=1}^n \frac{e^{-\lambda_j s}}{(1+e^{-4\lambda_j s})^{\frac{1}{2}}}\right] \exp\left\{-\sum_{j=1}^n \frac{|\xi_j|^2}{2\lambda_j}\cdot\frac{1-e^{-4\lambda_j s}}{1+e^{-4\lambda_j s}}\right\} e^{-\alpha s} ds.$$

We may rewrite the above formula in terms of hyperbolic functions

$$\widehat{K}_\alpha(\xi) = \int_0^\infty \left\{\prod_{j=1}^n \cosh(2\lambda_j s)]^{-\frac{1}{2}}\right\} \exp\left\{-\sum_{j=1}^n \frac{|\xi_j|^2}{2\lambda_j} \tanh(2\lambda_j s)\right\} e^{-\alpha s} ds. \tag{10.4.30}$$

Let

$$G(\xi,s) = e^{-\alpha s} \prod_{j=1}^n [\cosh(2\lambda_j s)]^{-\frac{1}{2}} \exp\left\{-\sum_{j=1}^n \frac{|\xi_j|^2}{2\lambda_j} \tanh(2\lambda_j s)\right\} \tag{10.4.31}$$

be the integrand of the above integral. We can prove directly that

$$\left[\alpha + \sum_{j=1}^n \left(\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)\right] \widehat{K}_\alpha(\xi) = 1$$

by showing that the function $G(\xi, s)$ satisfies the heat equation

$$\frac{\partial G}{\partial s} + \left[\alpha + \sum_{j=1}^n \left(\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)\right] G(\xi,s) = 0 \quad \text{and} \quad \lim_{s\to 0^+} G(\xi,s) = 1. \tag{10.4.32}$$

Then the fundamental theorem of calculus yields

$$\begin{aligned} \left[\alpha + \sum_{j=1}^n \left(\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)\right] \widehat{K}_\alpha(\xi) &= \int_0^\infty \left[\alpha + \sum_{j=1}^n \left(\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)\right] G(\xi,s) ds \\ &= \int_0^\infty \left(-\frac{\partial G}{\partial s}\right) ds = G(0) = 1. \end{aligned}$$

The fact that $G(\xi, s)$ satisfies the heat equation (10.4.32) can be proved directly by simple differentiation. Since

$$\frac{\partial G}{\partial \xi_j} = (-\frac{\xi_j}{\lambda_j}\tanh(2\lambda_j s))G,$$
$$\frac{\partial^2 G}{\partial \xi_j^2} = \left[\frac{\xi_j^2}{\lambda_j^2}(\tanh(2\lambda_j s))^2 - \frac{\tanh(2\lambda_j s)}{\lambda_j}\right]G$$

one has

$$\begin{aligned}
&\sum_{j=1}^n \left(\frac{\alpha}{n} + \xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right) G(\xi, s) \\
&= G(\xi, s)\sum_{j=1}^n \left[\frac{\alpha}{n} - \xi_j^2(\tanh(2\lambda_j s))^2 + \lambda_j \tanh(2\lambda_j s) + \xi_j^2\right] \\
&= G(\xi, s)\sum_{j=1}^n \left[\frac{\alpha}{n} + \xi_j^2(1 - (\tanh(2\lambda_j s))^2) + \lambda_j \tanh(2\lambda_j s)\right] \\
&= G(\xi, s)\sum_{j=1}^n \left[\frac{\alpha}{n} + \frac{\xi_j^2}{(\cosh(2\lambda_j s))^2} + \lambda_j \tanh(2\lambda_j s)\right].
\end{aligned}$$

Next the product rule of differentiation yields

$$\begin{aligned}
\frac{\partial G}{\partial s} &= -\alpha G(\xi, s) - G(\xi, s)\sum_{j=1}^n \lambda_j (\cosh(2\lambda_j s))^{-1}\sinh(2\lambda_j s) \\
&\quad - G(\xi, s)\sum_{j=1}^n \frac{\xi_j^2}{2\lambda_j}\cdot\frac{2\lambda_j}{(\cosh(2\lambda_j s))^2} \\
&= -G(\xi, s)\left[\alpha + \sum_{j=1}^n \left(\frac{\xi_j^2}{(\cosh(2\lambda_j s))^2} + \lambda_j \tanh(2\lambda_j s)\right)\right] \\
&= -\sum_{j=1}^n \left(\frac{\alpha}{n} + \xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right) G(\xi, s).
\end{aligned}$$

Therefore

$$\frac{\partial G}{\partial s} + \sum_{j=1}^n \left(\frac{\alpha}{n} + \xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right) G(\xi, s) = 0.$$

This shows $G(\xi, s)$ is the heat kernel of the Hermite operator $\alpha + \sum_{j=1}^n \left(\xi_j^2 - \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2}\right)$ with $G(\xi, 0) = 1$. Finally, let us compute the fundamental solution $K_\alpha(\mathbf{x})$ by taking the inverse Fourier transform with respect to ξ.

$$
\begin{aligned}
&K_\alpha(\mathbf{x})\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\xi}\hat{K}(\xi)d\xi\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\xi}\left\{\int_0^\infty \prod_{j=1}^n [\cosh(2\lambda_j s)]^{-\frac{1}{2}} \exp\left\{-\sum_{j=1}^n \frac{\xi_j^2}{2\lambda_j}\tanh(2\lambda_j s)\right\} e^{-\alpha s} ds\right\} d\xi\\
&= \frac{1}{(2\pi)^n}\int_0^\infty \prod_{j=1}^n [\cosh(2\lambda_j s)]^{-\frac{1}{2}} \left\{\prod_{j=1}^n \int_{-\infty}^\infty e^{ix_j\xi_j} e^{-\frac{\xi_j^2}{2\lambda_j}\tanh(2\lambda_j s)} d\xi_j\right\} e^{-\alpha s} ds.
\end{aligned}
$$

First, we need to compute $\int_{-\infty}^\infty e^{ix_j\xi_j} e^{-\frac{\tanh(2\lambda_j s)}{2\lambda_j}\xi_j^2} d\xi_j$. Using the formula

$$
\int_{-\infty}^\infty e^{ixw-\frac{w^2}{2a}} dw = \sqrt{2\pi a} e^{-\frac{a}{2}x^2}
$$

with $a = \dfrac{\lambda_j}{\tanh(2\lambda_j s)}$, we obtain

$$
\int_{-\infty}^\infty e^{ix_j\xi_j - \left(\frac{\tanh(2\lambda_j s)}{2\lambda_j}\right)\xi_j^2} d\xi_j = \sqrt{\frac{2\pi\lambda_j}{\tanh(2\lambda_j s)}} e^{-\frac{\lambda_j x_j^2}{2\tanh(2\lambda_j s)}}.
$$

This implies that

$$
K_\alpha(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \left[\prod_{j=1}^n \frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \exp\left\{-\sum_{j=1}^n \frac{\lambda_j x_j^2}{2\tanh(2\lambda_j s)}\right\} e^{-\alpha s} ds.
$$

We summarize the computation and formulate as a theorem:

Theorem 10.16. *For* $\alpha \notin \Lambda = \left\{-\sum_{j=1}^n \lambda_j(2k_j+1),\ \mathbf{k} = (k_1, \ldots, k_n) \in (\mathbf{Z}_+)^n\right\}$, *the fundamental solution* $K_\alpha(\mathbf{x})$ *of the Hermite operator* $H_\alpha K_\alpha(\mathbf{x}) = \delta(\mathbf{x})$ *is*

$$
K_\alpha(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \left[\prod_{j=1}^n \frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \exp\left\{-\sum_{j=1}^n \frac{\lambda_j x_j^2}{2\tanh(2\lambda_j s)}\right\} e^{-\alpha s} ds. \tag{10.4.33}
$$

The associated heat kernel is given by

$$
P_s(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left[\prod_{j=1}^n \frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \exp\left\{-\sum_{j=1}^n \frac{\lambda_j x_j^2}{2\tanh(2\lambda_j s)}\right\}
$$

i.e., $P_s(\mathbf{x})$ *satisfies the heat equation*

$$
\frac{\partial P_s}{\partial s} + \alpha P_s + \sum_{j=1}^n \left(\lambda_j^2 x_j^2 - \frac{\partial^2}{\partial x_j^2}\right) P_s(\mathbf{x}) = 0 \quad \text{with} \quad \lim_{s\to 0}\int_{\mathbb{R}^n} P_s(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = f(0).
$$

10.4.3.2 Isotropic case: $\lambda_j = \lambda$ for all j

We now consider the special case of $\lambda_j = \lambda$ for all $j = 1, \cdots, n$. Then the fundamental solution reduces to

$$K_\alpha(\mathbf{x}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_0^\infty e^{-\alpha s} \left[\sinh(2\lambda s)\right]^{-\frac{n}{2}} \exp\left\{-\frac{\lambda|\mathbf{x}|^2}{2}\coth(2\lambda s)\right\} ds$$

by introducing a new variable $u = \coth(2\lambda s)$. We have

$$e^{-\alpha s} = \left(\frac{u-1}{u+1}\right)^{\frac{\alpha}{4\lambda}}, \; du = -2\lambda(\sinh(2\lambda s))^{-2} ds \text{ and}$$

$$(\sinh(2\lambda s))^{-1} = \sqrt{(\coth(2\lambda s))^2 - 1} = \sqrt{u^2-1}.$$

Hence,

$$K_\alpha(\mathbf{x}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{2\lambda} \int_1^\infty (u-1)^{\frac{n}{4}-1+\frac{\alpha}{4\lambda}} (u+1)^{\frac{n}{4}-1-\frac{\alpha}{4\lambda}} e^{-\frac{\lambda}{2}|\mathbf{x}|^2 u} du. \quad (10.4.34)$$

Introducing the new integral variable $u = 2v + 1$, we reduce equation (10.4.32) to the form:

$$K_\alpha(\mathbf{x}) = \frac{\lambda^{\frac{n}{2}-1}}{4\pi^{\frac{n}{2}}} e^{-\frac{\lambda}{2}|\mathbf{x}|^2} \int_0^\infty v^{\frac{n}{4}-1+\frac{\alpha}{4\lambda}} (v+1)^{\frac{n}{4}-1-\frac{\alpha}{4\lambda}} e^{-\lambda|\mathbf{x}|^2 v} dv.$$

Then the integral can be reduced to the Whittaker function. Let

$$\mu - \chi - \frac{1}{2} = \frac{n}{4} - 1 + \frac{\alpha}{4\lambda} \quad \text{and} \quad \mu + \chi - \frac{1}{2} = \frac{n}{4} - 1 - \frac{\alpha}{4\lambda},$$

then we have $\mu = \frac{n}{4} - \frac{1}{2}$ and $\chi = -\frac{\alpha}{4\lambda}$ and can write the above as the Whittaker function $W_{\chi,\mu}(\lambda|\mathbf{x}|^2)$. We omit the detail and just give the final formula:

$$K_\alpha(\mathbf{x}) = \frac{\lambda^{\frac{n}{4}-1}\Gamma(\frac{n}{4}+\frac{\alpha}{4\lambda})}{4\pi^{\frac{n}{2}}|\mathbf{x}|^{\frac{n}{2}} W_{-\frac{\alpha}{4\lambda},\frac{n}{2}-\frac{1}{2}}(\lambda|\mathbf{x}|^2)}. \quad (10.4.35)$$

We can write $K_\alpha(\mathbf{x})$ as a modified Bessel function when $\alpha = 0$ by applying the following integral formula (see p. 250 in [47]):

$$\int_1^\infty (x^2-1)^{\gamma-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu}\right)^{\gamma-\frac{1}{2}} \Gamma(\gamma)\mathcal{K}_{\gamma-\frac{1}{2}}(\mu),$$

where $\mathcal{K}_\nu(z)$ is the modified Bessel function :

$$\mathcal{K}_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt.$$

Therefore, with $\mu = \frac{\lambda}{2}|\mathbf{x}|^2$, $\gamma = \frac{n}{4}$, we have

$$K_0(\mathbf{x}) = \frac{\Gamma\left(\frac{n}{4}\right)\lambda^{\frac{n}{4}-\frac{1}{2}}}{\pi^{\frac{n+1}{2}}} \cdot \frac{\mathcal{K}_{\frac{n}{4}-\frac{1}{2}}(\frac{\lambda}{2}|\mathbf{x}|^2)}{|\mathbf{x}|^{\frac{n}{2}-1}}. \tag{10.4.36}$$

In the case of $\frac{n}{4} - \frac{1}{2} = m + \frac{1}{2} \Leftrightarrow n = 4(m+1)$, we have the explicit formula for the modified Bessel function

$$\mathcal{K}_{m+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{\ell=0}^{m}\frac{(m+\ell)!}{\ell!(m-\ell)!}(2z)^{-\ell}.$$

Hence when $n = 4(m+1)$, we can find a closed form of $K_0(\mathbf{x})$:

$$K_0(\mathbf{x}) = \frac{\Gamma(m+1)}{\pi^{2(m+1)}}e^{-\frac{\lambda}{2}|\mathbf{x}|^2}\sum_{\ell=0}^{m}\frac{(m+\ell)!}{\ell!(m-\ell)!}\frac{\lambda^{m-\ell}}{|\mathbf{x}|^{2(m+\ell)+2}}.$$

The formal argument is therefore complete. We now need to justify the integral (10.4.28) and calculations in (10.4.31). In view of the hyperbolic cosine term in (10.4.30), we know that

$$|G(\xi, s)| \le 2^{\frac{n}{2}}\exp\left(-\sum_{j=1}^{n}\lambda_j\xi^2 s\right)$$

for $s \ge 0$. Therefore, the integral (10.4.28) converges rapidly. It also justifies the interchange of integrals in (10.4.31).

10.4.3.3 Partial inverse and projection to the kernel

We now consider the behavior of H_α near a singular value α, *i.e.,* $\alpha \in \Lambda$. Since we emphasize the dependence on the value of α, we see H_α and K_α as functions of α and denote $K_\alpha = K(\alpha)$ and $H_\alpha = H(\alpha)$. From (10.4.27) it follows that $K(\alpha) = H(\alpha)^{-1}$ has a simple pole at each point of Λ. Let $\alpha_0 \in \Lambda$. We can expand $K(\alpha)$ at α_0,

$$K(\alpha) = \frac{J(\alpha_0)}{\alpha - \alpha_0} + K(\alpha_0) + O(|\alpha - \alpha_0|).$$

For α sufficiently near α_0, $\alpha \ne \alpha_0$, $H(\alpha)K(\alpha) = K(\alpha)H(\alpha) = I$, this implies

$$I = \frac{J(\alpha_0)H(\alpha)}{\alpha - \alpha_0} + K(\alpha_0)H(\alpha) + O(|\alpha - \alpha_0|).$$

Since $H(\alpha) = \alpha + \sum_{j=1}^{n}\left(\lambda_j^2 x_j^2 - \frac{\partial^2}{\partial x_j^2}\right)H(\alpha_0) + (\alpha - \alpha_0)$, we have

$$I = \lim_{\alpha \to \alpha_0} \frac{J(\alpha_0)H(\alpha_0)}{\alpha - \alpha_0} + J(\alpha_0) + K(\alpha_0)H(\alpha_0).$$

Interchanging $K(\alpha)$ and $H(\alpha)$ in the above, we have

$$I = \lim_{\alpha \to \alpha_0} \frac{H(\alpha_0)J(\alpha_0)}{\alpha - \alpha_0} + J(\alpha_0) + H(\alpha_0)K(\alpha_0).$$

This yields

$$H(\alpha_0)J(\alpha_0) = J(\alpha_0)H(\alpha_0) = 0$$

and

$$I = K(\alpha_0)H(\alpha_0) + J(\alpha_0).$$

Apply $H(\alpha_0)$ to the above and we have

$$H(\alpha_0) = H(\alpha_0)K(\alpha_0)H(\alpha_0).$$

Therefore, $[H(\alpha_0)K(\alpha_0)]^2 = H(\alpha_0)K(\alpha_0)$ and $[J(\alpha_0)]^2 = J(\alpha_0)$. This yields that $H(\alpha_0)K(\alpha_0)$ and $J(\alpha_0)$ are complementary projections on L^2. The operator $K(\alpha_0)$ and $J(\alpha_0)$ can be computed from the integrals

$$K(\alpha_0) = \frac{1}{2\pi i}\int_\Gamma \frac{K(\alpha)}{\alpha - \alpha_0} d\alpha \quad \text{and} \quad J(\alpha_0) = \frac{1}{2\pi i}\int_\Gamma K(\alpha) d\alpha.$$

Here Γ represents a sufficiently small circle about α_0.

The first singular value is $\alpha_0 = -\sum_{j=1}^{n} \lambda_j$ with $k_j = 0$ for $j = 1, 2, \cdots, n$. We will calculate $J(\alpha_0)$ and $K(\alpha_0)$ explicitly. The residues of $\widehat{K}_\alpha$ at this pole are

$$\sigma\,(J(\alpha_0)) = 2^{\frac{n}{2}} \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n} \frac{\xi_j^2}{\lambda_j} \right\}.$$

Here $\sigma\,(J(\alpha_0))$ is the symbol of the projection $J(\alpha_0)$. The kernel is the inverse Fourier transform of the symbol

$$J(-\sum_{j=1}^{n} \lambda_j) = \frac{1}{2^n} \prod_{j=1}^{n} \left[\sqrt{\frac{\lambda_j}{\pi}} e^{-\frac{\lambda_j}{2} x_j^2} \right].$$

$\widehat{K}_\alpha$ can be written as

$$\widehat{K}_\alpha(\xi) = \int_0^\infty e^{-\alpha s} G_0(\xi, s) ds,$$

where

$$G_0(\xi, s) = \prod_{j=1}^{n} [\cosh(2\lambda_j s)]^{-\frac{1}{2}} \exp\left\{ -\sum_{j=1}^{n} \frac{\xi_j^2}{2\lambda_j} \tanh(2\lambda_j s) \right\}.$$

Integration by parts gives

$$\widehat{K}_\alpha(\xi) = \frac{1}{\alpha + \sum_{j=1}^n \lambda_j}\left[1 + \int_0^\infty e^{-\alpha s}\left(\frac{\partial}{\partial s} + \sum_{j=1}^n \lambda_j\right) G_0(\xi, s)ds\right]. \quad (10.4.37)$$

This implies that $\widehat{K}(\alpha)$ has a pole at $\alpha = -\sum_{j=1}^n \lambda_j$. Thus $\widehat{K}(\alpha_0)$ is the term of order zero in the expansion of (10.4.37) at $\alpha = -\sum_{j=1}^n \lambda_j$:

$$\widehat{K}(\alpha_0) = -\int_0^\infty s\frac{\partial}{\partial s}\left[e^{s\sum_{j=1}^n \lambda_j} G_0(\xi, s)\right]ds.$$

Taking the inverse Fourier transform, one can find the corresponding kernels. The computation is almost identical to those of the computation of $K_\alpha(\mathbf{x})$, so we omit the details here and list the final formula only:

$$\begin{aligned} &K(\alpha_0) \\ &= -\left(\prod_{j=1}^n \sqrt{\frac{\lambda_j}{2\pi}}\right)\int_0^\infty s\frac{d}{ds}\left[\left(\prod_{j=1}^n \frac{e^{2\lambda_j s}}{\sinh 2\lambda_j s}\right)^{\frac{1}{2}}\right. \\ &\qquad \left.\times \exp\left\{-\frac{1}{2}\sum_{j=1}^n \lambda_j x_j^2 \coth(2\lambda_j s)\right\}\right]ds. \end{aligned}$$

10.4.3.4 Fundamental solution with singularity at an arbitrary point y

Let us start with the operator H_0, *i.e.,* $\alpha = 0$. We want to derive the following kernel $K(\mathbf{x}, \mathbf{y})$ which satisfies

$$(-\Delta + \sum_{j=1}^n \lambda_j^2 x_j^2)K(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}),$$

and is the case of $\alpha = 0$ in (10.4.23). Taking the Fourier transform with respect to the $\mathbf{x}$-variable, we have

$$(|\xi|^2 - \sum_{j=1}^n \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2})\hat{K}(\xi, \mathbf{y}) = \int_{\mathbb{R}^n} \delta(\mathbf{x} - \mathbf{y})e^{-i\mathbf{x}\cdot\xi}d\mathbf{x} = e^{-i\mathbf{y}\cdot\xi}.$$

As before we let

$$\hat{K}(\xi, \mathbf{y}) = \sum_{|\mathbf{k}|=0}^\infty c_{\mathbf{k}}(\mathbf{y})\Psi_{\mathbf{k}}(\xi) \quad \text{with} \quad \Psi_{\mathbf{k}}(\xi) = \prod_{j=1}^n \psi_{k_j}(\frac{\xi_j}{\sqrt{\lambda_j}}).$$

Then

$$(|\xi|^2 - \sum_{j=1}^{n} \lambda_j^2 \frac{\partial^2}{\partial \xi_j^2})\hat{K}(\xi, \mathbf{y}) = \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}}(\mathbf{y}) \left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right] \Psi_{\mathbf{k}}(\xi).$$

Hence, we need to solve $\sum_{|\mathbf{k}|=0}^{\infty} \left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right] c_{\mathbf{k}}(\mathbf{y}) \Psi_{\mathbf{k}}(\xi) = e^{-i\mathbf{y}\cdot\xi}$ to find $c_{\mathbf{k}}(\mathbf{y})$. The orthogonality of the Hermite function yields

$$\left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right] c_{\mathbf{k}}(\mathbf{y}) < \Psi_{\mathbf{k}}(\xi), \Psi_{\mathbf{k}}(\xi) >=< e^{-i\mathbf{y}\cdot\xi}, \Psi_{\mathbf{k}}(\xi) > .$$

We first have to find

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\xi} \Psi_{\mathbf{k}}(\xi) d\xi &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-iy_j\xi_j} \psi_{k_j}(\frac{\xi_j}{\sqrt{\lambda_j}}) d\xi_j \\ &= \prod_{j=1}^{n} \sqrt{\lambda_j} \int_{-\infty}^{\infty} e^{-i\sqrt{\lambda_j} y_j \eta_j} \psi_{k_j}(\eta_j) d\eta_j . \end{aligned}$$

Applying the formula

$$\int_{-\infty}^{\infty} e^{-iy\xi} \psi_\ell(\xi) d\xi = \sqrt{2\pi} (-i)^\ell \psi_\ell(y) \tag{10.4.38}$$

and $< \Psi_{\mathbf{k}}(\xi), \Psi_{\mathbf{k}}(\xi) >= \pi^{\frac{n}{2}} \prod_{j=1}^{n} \sqrt{\lambda_j} 2^{k_j} k_j!$, one has

$$\left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right] c_{\mathbf{k}}(\mathbf{y}) \pi^{\frac{n}{2}} \left(\prod_{j=1}^{n} \sqrt{\lambda_j} 2^{k_j} k_j! \right) = (2\pi)^{\frac{n}{2}} \prod_{j=1}^{n} (-i)^{k_j} \sqrt{\lambda_j} \psi_{k_j}(\sqrt{\lambda_j} y_j).$$

It follows that

$$c_{\mathbf{k}}(\mathbf{y}) = \frac{2^{\frac{n}{2}}}{\left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right]} \prod_{j=1}^{n} \frac{(-i)^{k_j}}{2^{k_j} k_j!} \psi_{k_j}(\sqrt{\lambda_j} y_j).$$

Hence we have

$$\hat{K}(\xi, \mathbf{y}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{2^{\frac{n}{2}}}{\left[\sum_{j=1}^{n} (2k_j + 1)\lambda_j \right]} \prod_{j=1}^{n} \frac{(-i)^{k_j}}{2^{k_j} k_j!} \psi_{k_j}(\sqrt{\lambda_j} y_j) \psi_{k_j}(\frac{\xi_j}{\sqrt{\lambda_j}}).$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
&K(\mathbf{x},\mathbf{y})\\
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\xi}\hat{K}(\xi,\mathbf{y})d\xi\\
&=\frac{1}{(2\pi)^n}\sum_{|\mathbf{k}|=0}^{\infty}\frac{2^{\frac{n}{2}}}{\left[\sum_{j=1}^n(2k_j+1)\lambda_j\right]}\prod_{j=1}^n\frac{(-i)^{k_j}}{2^{k_j}k_j!}\psi_{k_j}(\sqrt{\lambda_j}y_j)\\
&\quad\times\int_{-\infty}^{\infty}e^{ix_j\cdot\xi_j}\psi_{k_j}(\frac{\xi_j}{\sqrt{\lambda_j}})d\xi_j.
\end{aligned}$$

Applying the identity (10.4.37) again, we have

$$\begin{aligned}
K(\mathbf{x},\mathbf{y})&=\frac{1}{(2\pi)^n}\sum_{|\mathbf{k}|=0}^{\infty}\frac{2^n\pi^{\frac{n}{2}}}{\left[\sum_{j=1}^n(2k_j+1)\lambda_j\right]}\\
&\quad\times\prod_{j=1}^n\frac{(-i)^{k_j}i^{k_j}\sqrt{\lambda_j}}{2^{k_j}k_j!}\psi_{k_j}(\sqrt{\lambda_j}y_j)\psi_{k_j}(\sqrt{\lambda_j}x_j)\\
&=\frac{1}{\pi^{\frac{n}{2}}}\sum_{|\mathbf{k}|=0}^{\infty}\frac{1}{\left[\sum_{j=1}^n(2k_j+1)\lambda_j\right]}\prod_{j=1}^n\frac{\sqrt{\lambda_j}}{2^{k_j}k_j!}\psi_{k_j}(\sqrt{\lambda_j}y_j)\psi_{k_j}(\sqrt{\lambda_j}x_j).
\end{aligned}$$

Here we have used the identity $\psi_k(-x)=(-1)^k\psi_k(x)$. Now we apply the formula $\frac{1}{A}=\int_0^{\infty}e^{-As}ds$ with $A=\sum_{j=1}^n(2k_j+1)\lambda_j$ again and obtain

$$\begin{aligned}
K(\mathbf{x},\mathbf{y})&=\frac{1}{\pi^{\frac{n}{2}}}\sum_{|\mathbf{k}|=0}^{\infty}\left(\int_0^{\infty}\prod_{j=1}^n\frac{\sqrt{\lambda_j}e^{-2k_j\lambda_j s-\lambda_j s}}{2^{k_j}k_j!}\psi_{k_j}(\sqrt{\lambda_j}y_j)\psi_{k_j}(\sqrt{\lambda_j}x_j)ds\right)\\
&=\frac{1}{\pi^{\frac{n}{2}}}\int_0^{\infty}\prod_{j=1}^n\sqrt{\lambda_j}e^{-\lambda_j s}\left(\sum_{k_j=0}^{\infty}\frac{e^{-2k_j\lambda_j s}}{2^{k_j}k_j!}\psi_{k_j}(\sqrt{\lambda_j}y_j)\psi_{k_j}(\sqrt{\lambda_j}x_j)\right)ds.
\end{aligned}$$

We next sum up the infinite series on the right hand side by applying the formula:

$$\sum_{k=0}^{\infty}\frac{H_k(x)H_k(y)}{k!}z^k=(1-4z^2)^{-\frac{1}{2}}\exp\left\{y^2-\frac{(y-2zx)^2}{1-4z^2}\right\}$$

where $H_k(x)$ is the Hermite polynomial (see page 280 in [17]). Denote

$$g(x,y,s)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}e^{-2s}\right)^k}{k!}\psi_k(x)\psi_k(y)$$

where $\psi_k(x) = e^{-\frac{x^2}{2}} H_k(x)$. Then we have

$$
\begin{aligned}
& g(x, y, s) \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}e^{-2s}\right)^k}{k!} \psi_k(x)\psi_k(y) = e^{-\frac{x^2}{2}-\frac{y^2}{2}}(1-e^{-4s})^{-\frac{1}{2}} \exp\left\{y^2 - \frac{(y - xe^{-2s})^2}{1-e^{-4s}}\right\} \\
&= (1-e^{-4s})^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{2} + \frac{y^2}{2} - \frac{y^2 - 2e^{-2s}xy + e^{-4s}x^2}{1-e^{-4s}}\right\} \\
&= (1-e^{-4s})^{-\frac{1}{2}} \exp\left\{(1-e^{-4s})^{-1}\left[-\frac{(x^2+y^2)}{2} - \frac{(x^2+y^2)}{2}e^{-4s} + 2e^{-2s}xy\right]\right\} \\
&= (1-e^{-4s})^{-\frac{1}{2}} \exp\left\{-\left(\frac{x^2}{2} + \frac{y^2}{2}\right)\frac{1+e^{-4s}}{1-e^{-4s}} + \frac{2e^{-2s}}{1-e^{-4s}}xy\right\} \\
&= (1-e^{-4s})^{-\frac{1}{2}} \exp\left\{-\frac{x^2+y^2}{2}\coth(2s) + \frac{xy}{\sinh(2s)}\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& K(\mathbf{x}, \mathbf{y}) \\
&= \frac{1}{\pi^{\frac{n}{2}}} \int_0^{\infty} \prod_{j=1}^{n} \frac{\sqrt{\lambda_j} e^{-\lambda_j s}}{(1-e^{-4\lambda_j s})^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\lambda_j(x_j^2 + y_j^2)\coth(2\lambda_j s) + \frac{\lambda_j x_j y_j}{\sinh(2\lambda_j s)}\right\} ds \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^{\infty} \left[\prod_{j=1}^{n} \frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \\
&\quad \times \exp\left\{-\sum_{j=1}^{n} \frac{\lambda_j(x_j^2+y_j^2)\cosh(2\lambda_j s) - 2\lambda_j x_j y_j}{2\sinh(2\lambda_j s)}\right\} ds.
\end{aligned}
$$

The heat kernel is

$$
\begin{aligned}
P_s(\mathbf{x}, \mathbf{y}) = & \frac{1}{(2\pi)^{\frac{n}{2}}} \left[\prod_{j=1}^{n} \frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \\
& \times \exp\left\{-\sum_{j=1}^{n} \frac{\lambda_j(x_j^2+y_j^2)\cosh(2\lambda_j s) - 2\lambda_j x_j y_j}{2\sinh(2\lambda_j s)}\right\}.
\end{aligned}
\tag{10.4.39}
$$

Using the formula $\cosh(2s) = 1 + 2\sinh^2 s$ and $\sinh(2s) = 2\sinh s \cosh s$, we can rewrite the heat kernel as

$$P_s(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left[\prod_{j=1}^{n}\frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \times \exp\left\{-\sum_{j=1}^{n}\left[\frac{\lambda_j(x_j-y_j)^2}{2\sinh(2\lambda_j s)}+\frac{\lambda_j(x_j^2+y_j^2)}{2}\tanh(\lambda_j s)\right]\right\}.$$

We summarize the computation with the following theorem.

Theorem 10.17. *The kernel*

$$P_s(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\left[\prod_{j=1}^{n}\frac{\lambda_j}{\sinh(2\lambda_j s)}\right]^{\frac{1}{2}} \times \exp\left\{-\sum_{j=1}^{n}\left[\frac{\lambda_j(x_j-y_j)^2}{2\sinh(2\lambda_j s)}+\frac{\lambda_j(x_j^2+y_j^2)}{2}\tanh(\lambda_j s)\right]\right\}$$

satisfies the associated heat equation

$$\frac{\partial P_s}{\partial s}-\left[\Delta-\sum_{j=1}^{n}\lambda_j^2x_j^2\right]P_s(\mathbf{x},\mathbf{y})=0 \quad \textit{and} \quad \lim_{s\to 0^+}P_s(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y}),$$

with the initial condition

$$\lim_{s\to 0^+}\int_{\mathbb{R}^n}P_s(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y}=f(\mathbf{x}).$$

Now we may use a similar method as before to obtain the following corollary.

Corollary 10.18 *For* $\alpha\notin\Lambda=\{-\sum_{j=1}^{n}\lambda_j(2k_j+1),\ \mathbf{k}=(k_1,\ldots,k_n)\in(\mathbf{Z}_+)^n\}$, *the Hermite operator* $H_\alpha=\alpha-\Delta+\sum_{j=1}^{n}\lambda_j^2x_j^2$ *has the fundamental solution*

$$K_\alpha(\mathbf{x},\mathbf{y})=\int_0^\infty e^{-\alpha s}P_s(\mathbf{x},\mathbf{y})ds$$

where $P_s(\mathbf{x},\mathbf{y})$ *is defined in Theorem 10.17*

10.5 Heat kernel on radially symmetric spaces with potential

We shall investigate the fundamental solution for the operator

$$P = \partial_t + \Delta - U(x),$$

where $\Delta = -div\nabla$ and $U : M \to \mathbb{R}$ is a potential function defined on the radially symmetric space (M, g). The associated Hamiltonian is half of the principal symbol of $-\Delta + U(x)$,

$$H(p, x) = \frac{1}{2}|p|_g^2 + \frac{1}{2}U(x).$$

As H does not depend explicitly on the time parameter t, then $H = E$, where E is the constant of the total energy along the solutions of the Hamiltonian system. The action S will satisfy the Hamilton–Jacobi equation

$$\begin{aligned}\frac{\partial}{\partial t}S &= -H(\nabla S)\\ &= -\frac{1}{2}|\nabla S|^2 - \frac{1}{2}U(x)\\ &= -E.\end{aligned}$$

We also note the useful relation

$$|\nabla S|^2 = 2E - U(x).$$

For the zero potential $U(x) = 0$ the action $S = d^2(x_0, x)/(2t)$. For general potentials $U(x)$ the action S is not easy to compute. This shall be seen in the next section. The action S is a function of the endpoints x_0, x and time t.

In this section we shall perform a formal computation for the heat kernel. As before, we shall look for a fundamental solution of the form

$$K = K(x_0, x, t) = V(t)e^{kS}, \quad t > 0.$$

By straightforward computation

$$\begin{aligned}\partial_t K &= e^{kS}\Big(V'(t) + kV(t)\partial_t S\Big)\\ &= e^{kS}V(t)\Big(\frac{V'(t)}{V(t)} - kE\Big),\\ \Delta K &= V(t)e^{kS}\Big(k\Delta S - k^2|\nabla S|^2\Big)\\ &= V(t)e^{kS}\Big(k\Delta S - k^2(2E - U(x))\Big)\\ &= V(t)e^{kS}\Big(k\Delta S - 2k^2E + k^2U(x)\Big).\end{aligned}$$

Following the idea from the previous sections, we shall consider the following operator with multiplier λ,

$$P_\lambda = \partial_t + \Delta + \lambda U(x).$$

We shall find λ and k such that

$$P_\lambda(K(x_0, x, t)) = 0, \quad t > 0.$$

A straightforward computation provides

$$\begin{aligned} P_\lambda(V(t)e^{kS}) &= e^{kS}V(t)\Big[\frac{V'(t)}{V(t)} - kE\Big] \\ &\quad + e^{kS}V(t)\Big(k\Delta S - 2k^2E + k^2U(x)\Big) \\ &\quad + \lambda\, U(x)e^{kS}V(t) \\ &= e^{kS}V(t)\Big(\frac{V'(t)}{V(t)} + k\Delta S - kE(2k+1) + (k^2+\lambda)U(x)\Big). \end{aligned}$$

We choose $k = -\frac{1}{2}$, $\lambda = -\frac{1}{4}$ and let $V(t)$ satisfy the volume equation

$$\frac{V'(t)}{V(t)} = -\frac{1}{2}\Delta S.$$

This shows that $P_\lambda(V(t)e^{kS}) = 0$, for $t > 0$. The volume function $V(t)$ is determined up to a multiplicative constant C. The condition

$$\lim_{t\searrow 0} V(t)\int_M e^{-\frac{1}{2}S(x_0,x,t)}\phi(x)\,dv(x) = \phi(x_0), \quad \forall\phi \in C_0^\infty(M)$$

fixes the constant C.

We would expect to have the following result for the fundamental solution:

Theorem 10.19. *Let (M, g) be a radially symmetric space. The fundamental solution for the operator*

$$P = \partial_t + \Delta - \frac{1}{4}U(x)$$

is given by

$$K(x_0, x, t) = V(t)e^{-\frac{1}{2}S(x_0,x,t)},$$

where $V(t)$ is the above volume function and S is the action associated with the Hamiltonian $H(p, x) = \frac{1}{2}|p|_g^2 + \frac{1}{2}U(x)$.

The above theorem provides a general formula for the heat kernel. For each potential $U(x)$ one needs to find the action S and the volume function V. As will be shown in the next section, this cannot be done explicitly for all potentials. However, for some potentials U (like the quartic one) there are more than one energy, which makes the problem more difficult.

10.6 The case of the quartic potential

The case of quartic potential is much different than the case of the quadratic potential. The kernel of the operator

$$P = \partial_t - \partial_x^2 - \frac{1}{4}a^4x^4,$$

with $a \geq 0$, is expected to be of the form

$$K(x_0, x, t) = V(t)e^{-\frac{1}{2}S(x_0,x,t)},$$

where S is the action between x_0 and x in time t, associated with the Hamiltonian $H(\xi, x) = \frac{1}{2}\xi^2 + \frac{1}{2}a^4x^4$. The volume function $V(t)$ depends on S, which depends on the energy E,

$$\partial_t S = -E.$$

If for given x_0, x and t we are able to find the energy E, then the problem is solved. The Hamiltonian system is

$$\begin{cases} \dot{x} = H_\xi = \xi, \\ \dot{\xi} = -H_x = -2a^4x^3, \end{cases}$$

and hence $x(s)$ satisfies the boundary value problem

$$\begin{cases} \ddot{x} = -2a^4x^3, \\ x(0) = x_0, \\ x(t) = x. \end{cases} \tag{10.6.40}$$

The conservation of energy yields

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}a^4x^4 = E,$$

with E the constant of energy. Writing

$$\dot{x} = \pm\sqrt{2E - a^4x^4},$$

separating and integrating between $x_0 = x(0)$ and $x = x(t)$, yields

$$\int_{x_0}^{x} \frac{du}{\sqrt{2E - a^4u^4}} = \pm t.$$

With the substitution $v = au/(2E)^{1/4}$ the above integral becomes

$$\int_{w_0}^{w} \frac{dv}{\sqrt{1 - v^4}} = \pm a(2E)^{1/4}\, t, \tag{10.6.41}$$

where $w_0 = ax_0/(2E)^{1/4}$ and $w = ax/(2E)^{1/4}$. The integral can be written in terms of the elliptic function cn, see [23],

$$\begin{aligned} \int_{w_0}^{w} \frac{dv}{\sqrt{1 - v^4}} &= \int_{w_0}^{1} \frac{dv}{\sqrt{1 - v^4}} - \int_{w}^{1} \frac{dv}{\sqrt{1 - v^4}} \\ &= \frac{1}{\sqrt{2}}\Big[\mathrm{cn}^{-1}(w_0, \frac{1}{\sqrt{2}}) - \mathrm{cn}^{-1}(w, \frac{1}{\sqrt{2}})\Big]. \end{aligned}$$

Hence (10.6.41) yields

$$\mathrm{cn}^{-1}(w_0) - \mathrm{cn}^{-1}(w) = \pm 2^{3/4} a\, E^{1/4} t. \tag{10.6.42}$$

Let $u = \mathrm{cn}^{-1}(w_0)$ and $v = \mathrm{cn}^{-1}(w)$. Then $\mathrm{sn}\, u = \sqrt{1 - w_0^2}$, $\mathrm{sn}\, v = \sqrt{1 - w^2}$,

$$\mathrm{dn}^2\, u = k'^2 + k^2 \mathrm{cn}^2\, u = \frac{1}{2}(1 + \mathrm{cn}^2 u) = \frac{1}{2}(1 + w_0^2),$$

and in a similar way $\mathrm{dn}^2\, v = \dfrac{1}{2}(1 + w^2)$. We have used $k = k' = \sqrt{2}/2$. Applying cn, which is an even function, to (10.6.42) yields

$$\begin{aligned}
\mathrm{cn}(2^{3/4} a\, E^{1/4} t) = \mathrm{cn}(u - v) &= \frac{\mathrm{cn} u\, \mathrm{cn} v + \mathrm{sn} u\, \mathrm{sn} v\, \mathrm{dn} u\, \mathrm{dn} v}{1 - k^2 \mathrm{sn}^2 u\, \mathrm{sn}^2 v} \\
&= \frac{w_0 w + \sqrt{1 - w_0^2}\sqrt{1 - w^2}\, \frac{1}{\sqrt{2}}\sqrt{1 + w_0^2}\, \frac{1}{\sqrt{2}}\sqrt{1 + w^2}}{1 - \frac{1}{2}(1 - w_0^2)(1 - w^2)} \\
&= \frac{2 w_0 w + \sqrt{(1 - w_0^2)(1 - w^4)}}{2 - (1 - w_0^2)(1 - w^2)} \\
&= \frac{\dfrac{2a^2 x x_0}{\sqrt{2E}} + \dfrac{\sqrt{(2E - a^4 x_0^4)(2E - a^4 x^4)}}{2E}}{2 - \dfrac{(\sqrt{2E} - a^2 x_0^2)(\sqrt{2E} - a^2 x^2)}{2E}} \\
&= \frac{2a^2 \sqrt{2E} x_0 x + \sqrt{(2E - a^4 x_0^4)(2E - a^4 x^4)}}{4E - (\sqrt{2E} - a^2 x_0^2)(\sqrt{2E} - a^2 x^2)}.
\end{aligned}$$

Let

$$\Phi_{x_0,x}(E) = \frac{2a^2 \sqrt{2E} x_0 x + \sqrt{(2E - a^4 x_0^4)(2E - a^4 x^4)}}{4E - (\sqrt{2E} - a^2 x_0^2)(\sqrt{2E} - a^2 x^2)}. \tag{10.6.43}$$

Lemma 10.20 *We have:*

(i)
$$\Phi_{x_0,x}(E) < 1, \quad \forall E \geq \frac{a^4}{2} \min(|x_0|, |x|),$$

(ii)
$$\lim_{E \to \infty} \Phi_{x_0,x}(E) = 1.$$

Proof. (i) The inequality between the geometric and arithmetic means yields

$$2a^2\sqrt{2E}x_0x + \sqrt{(2E - a^4x_0^4)(2E - a^4x^4)} \le 2a^2\sqrt{2E}x_0x + 2E - \frac{a^4}{2}(x_0^4 + x^4).$$

In order to show $\Phi_{x_0,x}(E) < 1$ it suffices to show that

$$\begin{aligned} & 2a^2\sqrt{2E}x_0x + 2E - \frac{a^4}{2}(x_0^4 + x^4) \le 4E - (\sqrt{2E} - a^2x_0^2)(\sqrt{2E} - a^2x^2) \\ \Longleftrightarrow\ & 2a^2\sqrt{2E}x_0x + 2E - \frac{a^4}{2}(x_0^4 + x^4) \le 2E + a^4x_0^2x^2 + a^2\sqrt{2E}(x_0^2 + x^2) \\ \Longleftrightarrow\ & 4\sqrt{2E}x_0x - a^2(x_0^4 + x^4) \le 2a^2x_0^2x^2 + 2\sqrt{2E}(x_0^2 + x^2), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & 0 \le a^2(x_0^4 + 2x_0^2x^2 + x^4) + 2\sqrt{2E}(x_0^2 - 2x_0x + x^2) \\ \Longleftrightarrow\ & 0 \le a^2(x_0^2 + x^2)^2 + 2\sqrt{2E}(x - x_0)^2, \end{aligned}$$

which is always true.
(ii) We have

$$\lim_{E\to\infty} \Phi_{x_0,x}(E) = \lim_{E\to\infty} \frac{\dfrac{2x_0x}{\sqrt{2E}} + \sqrt{\Big(1 - \dfrac{x_0^4}{2E}\Big)\Big(1 - \dfrac{x^4}{2E}\Big)}}{2 - \Big(1 - \dfrac{x_0^2}{\sqrt{2E}}\Big)\Big(1 - \dfrac{x^2}{\sqrt{2E}}\Big)} = 1.$$

∎

Theorem 10.21. (i) *Given* x_0, x, t *and* $a \ge 0$*, there is an infinite sequence of energies*

$$0 < E_1 < E_2 < \cdots < E_n < \cdots < +\infty$$

parametrized by the solutions $\theta = E^{1/4}$ *of the equation*

$$\Phi_{x_0,x}(\theta^4) = cn(2^{3/4}a\theta t), \tag{10.6.44}$$

(ii)
$$E_n \sim 2\Big(\frac{nK}{2at}\Big)^4, \text{ as } n \to \infty$$

where $K = K(\sqrt{2}/2) = \dfrac{\Gamma(1/4)^2}{4\sqrt{\pi}} \approx 1.854$*. Hence the asymptotics of the energy depend only on* t *and do not depend on the end points* x_0 *and* x*.*

Proof. (i) As from the above lemma $\Phi_{x_0,x}(\theta^4) \nearrow 1$ and $\text{cn}(2^{3/4}a\theta t)$ oscillates between -1 and 1 with the period $T = \dfrac{2K}{2^{3/4}at} = \dfrac{2^{1/4}K}{at}$, the equation (10.6.44) will have infinitely countable solutions $\theta_n = E_n^{1/4}$, see Figure 10.1.

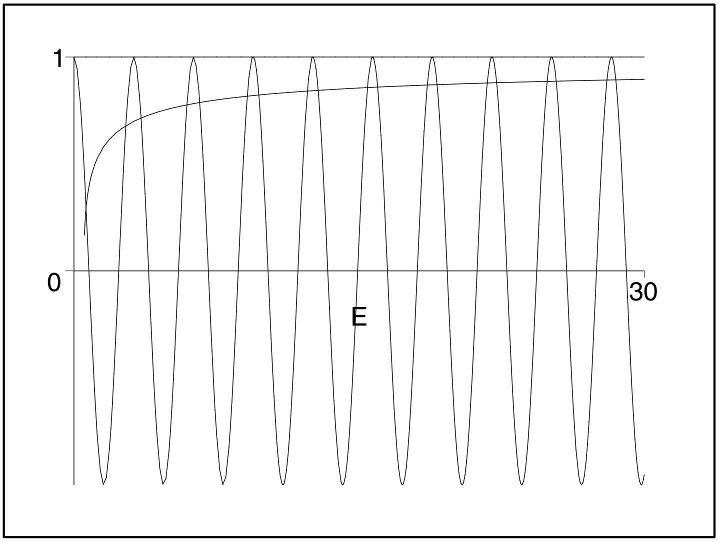


Figure 10.1: The energies $E_n, n = 1, 2, 3 \dots$

(ii) For θ large, the solutions of the equation (10.6.44) are approximated by the solutions of the equation $\mathrm{cn}(2^{3/4}a\theta t) = 1$, which are $\theta = \dfrac{2mK}{2^{3/4}at}$, $m = 1, 2, 3 \dots$.
Hence $(E_{2m})^{1/4} \sim \dfrac{2mK}{2^{3/4}at}$ or $E_n \sim \dfrac{n^4K^4}{2^3a^4t^4} = 2\Big(\dfrac{nK}{2at}\Big)^4$. ∎

In the case of a quartic potential there are infinitely many solutions with the end points x_0 and x joined in time t. Their energies form an increasing unbounded sequence E_n. The solution $x_n(s)$ of the Hamiltonian system associated with the energy E_n is given implicitly by

$$\mathrm{cn}(2^{3/4}aE_n^{1/4}s) = \frac{2a^2\sqrt{2E_n}x_0x_n(s) + \sqrt{(2E_n - a^4x_0^4)(2E_n - a^4x_n^4(s))}}{4E_n - (\sqrt{2E_n} - a^2x_0^2)(\sqrt{2E_n} - a^2x_n^2(s))}.$$

This is quite different behavior than the quadratic potential case, where there is only one energy and one solution between two given points. This behavior makes the quartic potential heat operator $P = \partial_t - \partial_x^2 - \dfrac{1}{4}a^4x^4$ difficult to invert.

The fundamental solution

Given any two points x_0 and x and a time $t > 0$, there is a sequence of energies $E_n = E_n(x_0, x, t)$ provided by Theorem 10.21. For each energy we associate an action $S_n = S_n(x_0, x, t)$, which satisfies the Hamilton–Jacobi equation

$$\partial_t S_n = -E_n(x_0, x, t).$$

Using Theorem 10.21 (ii), the asymptotics of S_n do not depend on the end points

$$S_n \sim \frac{2}{3}\Big(\frac{nK}{2a}\Big)^4 \frac{1}{t^3}, \quad \text{as} \ \ n \to \infty.$$

For each action S_n we associate a volume function V_n. If $\partial_x^2 S_n$ does not depend on x, then $V_n = V_n(t)$ is a solution for the equation $V_n'(t) + \frac{1}{2}(\Delta S_n) V_n(t) = 0$. But if $\partial_x^2 S_n$ depends on both x and t, then $V_n = V_n(t, x)$ will satisfy a more general equation, which will be introduced in the next section. Formally, the fundamental solution will be of the form

$$K(x_0, x, t) = \sum_{n=1}^{\infty} C_n V_n(t, x) e^{-\frac{1}{2} S_n}. \tag{10.6.45}$$

The constants C_n should be chosen such that

$$\sum_{n=1}^{\infty} C_n \lim_{t \searrow 0} V_n(t) \int_{\mathbb{R}} e^{-\frac{1}{2} S_n} \phi(x)\, dx = \phi(x_0),$$

for any compact supported function ϕ.

10.7 The kernel of the operator $\partial_t - \partial_x^2 - U(x)$

In the case of the quadratic potential $U(x) = a^2 x^2$ there is a unique solution joining two given points x_0 and x and in this case the action is unique. This is no longer true in the case of the quartic potential when $U(x) = a^2 x^4$. In this case the fundamental solution is a sum over all paths joining the end points x_0 and x in time t. A similar non-uniqueness behavior is expected for potentials $U(x) = a^2 x^m$, $m \geq 4$.

We shall study the case of a general potential function $U(x)$. Consider the operator $L = \partial_x^2 + U(x)$ with the principal symbol as a Hamiltonian

$$H(\xi, x) = \frac{1}{2}\xi^2 + \frac{1}{2} U(x). \tag{10.7.46}$$

Hamilton's equations are

$$\dot{x} = H_\xi = \xi,$$
$$\dot{\xi} = -H_x = -\frac{1}{2} U'(x), \quad \text{and hence} \quad \ddot{x} = \dot{\xi} = -\frac{1}{2} U'(x).$$

Given two points x_0 and x, we are interested in solving the system

$$\begin{cases} \ddot{x} = -\frac{1}{2} U'(x), \\ x(0) = x_0, \\ x(t) = x. \end{cases} \tag{10.7.47}$$

Since the Hamiltonian (10.7.46) does not depend explicitly on the variable t, it will be preserved along the solutions of (10.7.47), and

$$H = \frac{\dot{x}^2}{2} + \frac{1}{2}U(x) = E, \tag{10.7.48}$$

where $E = E(x_0, x, t)$ is the constant of energy. Hence $x(s)$ verifies the integral equation

$$\int_{x_0}^{x(s)} \frac{dw}{\sqrt{2E - U(w)}} = \pm s,$$

where the positive (negative) sign is taken in the right-hand side if $x > x_0 (x < x_0)$. The energy $E = E(x_0, x, t)$ satisfies the equation

$$\int_{x_0}^{x} \frac{dw}{\sqrt{2E - U(w)}} = \pm t,$$

with the same sign convention. The action S verifies $\partial_t S = -E(x_0, x, t)$. As along the solutions $\xi = S_x$, then $\dot{x} = \xi$ yields $\dot{x} = S_x$ and hence (10.7.48) becomes

$$(S_x)^2 = 2E - U(x). \tag{10.7.49}$$

We shall look for a fundamental solution of the type $K = V(t, x)e^{kS}$. A computation provides

$$\begin{aligned}
\partial_t K &= K\Big(\frac{V'}{V} - kE\Big),\\
\partial_x K &= V_x e^{kS} + V e^{kS} k S_x\\
&= K\Big(\frac{V_x}{V} + kS_x\Big),\\
\partial_x^2 K &= V_{xx} e^{kS} + V_x e^{kS} k S_x + kK_x S_x + kK S_{xx}\\
&= V_{xx} e^{kS} + kV_x e^{kS} S_x + kS_x\Big(V_x e^{kS} + kK S_x\Big) + kK S_{xx}\\
&= V_{xx} e^{kS} + 2kV_x e^{kS} S_x + k^2 K (S_x)^2 + kK S_{xx}\\
&= K\Big(\frac{V_{xx}}{V} + 2k\frac{V_x}{V} S_x + k^2 (S_x)^2 + kS_{xx}\Big).
\end{aligned}$$

Let $P = \partial_t - \partial_x^2 + \lambda U(x)$, where λ is a real multiplier. We shall find λ and k such that $PK = 0$. We have

$$\begin{aligned}
PK &= K\Big(\frac{V'}{V} - kE\Big),\\
&\quad -K\Big(\frac{V_{xx}}{V} + 2k\frac{V_x}{V} S_x + k^2 (S_x)^2 + kS_{xx}\Big)\\
&\quad +K\lambda U(x)\\
&= K\Big(\frac{V'}{V} - kE - \frac{V_{xx}}{V} - 2kS_x\frac{V_x}{V} - k^2(S_x)^2 - kS_{xx} + \lambda U(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= K\Big(\frac{V' - V_{xx} - 2kS_x V_x}{V} - kS_{xx} - kE + \lambda U(x) - k^2 \cdot \underbrace{(S_x)^2}_{=2E-U(x)}\Big) \\
&= K\Big(\frac{V' - V_{xx} - 2kS_x V_x}{V} - kS_{xx} - kE\,\underbrace{(2k+1)}_{=0} + \underbrace{(\lambda + k^2)}_{=0}\,U(x)\Big) \\
&= 0,
\end{aligned}
$$

where we choose $k = -\frac{1}{2}$ and $\lambda = -\frac{1}{4}$. Let $V(t, x)$ satisfy the generalized volume equation

$$V' - V_{xx} + S_x V_x = -\frac{1}{2} S_{xx} V, \tag{10.7.50}$$

where $V' = \partial_t V$ and $V_x = \partial_x V$. Using that $\dot{x} = S_x$, we have

$$\frac{d}{dt} V(t, x) = \partial_t V + \dot{x}\partial_x V = \partial_t V + S_x \partial_x V,$$

and the equation (10.7.50) becomes

$$\frac{d}{dt} V(t, x) = V_{xx}(t, x) - \frac{1}{2} S_{xx} V(t, x). \tag{10.7.51}$$

In the case when S_{xx} depends on t only, it makes sense to look for a function $V = V(t)$, which satisfies $V' = \frac{1}{2} S_{xx} V$.

Summing up the corresponding products for all the solutions that join x_0 and x we arrive at the following formula for the fundamental solution.

Theorem 10.22. *Let $x_n(s)$ be all solutions of the boundary value problem (10.7.47). Let S_n be the action and V_n be the generalized volume function associated with the solution $x_n(s)$ satisfying (10.7.51). Then the kernel of the operator*

$$P = \partial_t - \partial_x^2 - \frac{1}{4} U(x)$$

is given by the formula

$$K(x_0, x, t) = \sum_{n=1}^{\infty} C_n V_n(t, x) e^{-\frac{1}{2} S_n}, \quad t > 0,$$

where the relation

$$\lim_{t \searrow 0} \int_{\mathbb{R}} \sum_n C_n V_n(t, x) e^{-\frac{1}{2} S_n(x_0, x, t)} \phi(x)\, dv(x) = \phi(x_0), \quad \forall \phi \in C_0^\infty(\mathbb{R}) \tag{10.7.52}$$

fixes the constant C_n.

For any potential $U(x)$ we need to find the action S and the volume function V. This cannot be done explicitly all the time. It can be done explicitly for quadratic potentials of the form $U(x) = ax^2 + bx + c$, but it cannot be done for polynomial potentials of degree greater than 3. Formally, in the latter case the kernel is a sum over all the paths joining the points x and x_0.

10.7.1 The linear potential

Consider $U(x) = -ax$. In this case the solution $x(s)$ between x_0 and x is unique. The associated energy $E = E(x_0, x, t)$ is defined by the integral

$$\int_{x_0}^{x} \frac{dw}{\sqrt{2E + aw}} = \pm t \Longleftrightarrow \sqrt{2E + ax} = \sqrt{2E + ax_0} \pm \frac{a}{2} t,$$

$$a(x - x_0) = \frac{a^2}{4} t^2 \pm at\sqrt{2E + ax_0} \Longleftrightarrow \left(a(x - x_0) - \frac{a^2}{4} t^2\right)^2 = a^2 t^2 (2E + ax_0),$$

$$2E + ax_0 = \frac{\left(a(x - x_0) - \frac{a^2}{4} t^2\right)^2}{a^2 t^2} \Longleftrightarrow 2E + ax_0 = \frac{(x - x_0)^2}{t^2} - \frac{a}{2}(x - x_0) + \frac{a^2}{16} t^2,$$

$$E = \frac{(x - x_0)^2}{2t^2} - \frac{a}{4}(x - x_0) + \left(\frac{a}{4}\right)^2 t^2.$$

The action S satisfies

$$\begin{aligned} \partial_t S &= -E \\ &= -\frac{(x - x_0)^2}{2t^2} + \frac{a}{4}(x - x_0) - \left(\frac{a}{4}\right)^2 t^2, \end{aligned}$$

with the solution

$$S = \frac{(x - x_0)^2}{2t} + b(x + x_0)t - \frac{b^2}{12} t^3,$$

where $b = \dfrac{a}{4}$. As $S_{xx} = \dfrac{1}{t}$, the volume function satisfies (10.7.51), which becomes $V' = \dfrac{1}{2t} V$ and hence $V(t) = \dfrac{C}{\sqrt{t}}$.

Theorem 10.23. *Let $b \in R$. The kernel of the operator $P = \partial_t - \partial_x^2 + bx$ is given by*

$$K(x, x_0, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x - x_0)^2}{4t} - \frac{b}{2}(x + x_0)t + \frac{b^2}{12} t^3}, \quad t > 0.$$

Proof. Applying Theorem 10.22, the kernel will be $K = Ve^{-\frac{1}{2}S}$. Making $b \to 0$, the operator P tends to the usual heat equation. Comparing this with its fundamental solution yields $C = \dfrac{1}{\sqrt{4\pi}}$. ∎

10.8 Propagators for Schrödinger's equation in the one-dimensional case

A quantum particle situated in a potential $U(x)$ is characterized by a wave function, which satisfies Schrödinger's equation

$$ih\partial_t\Psi + \frac{1}{2}h^2\partial_x^2\Psi = U(x)\Psi, \tag{10.8.53}$$

where $h > 0$ is the Planck constant. Given the initial value of the wave function $\Psi_0(x) = \Psi(x, t_0)$, the solution of (10.8.53) at any instance of time $t > t_0$ is given by

$$\Psi(x,t) = \int K(x,t;x_0,t_0)\Psi_0(x_0)\,dx_0,$$

where $K(x,t;x_0,t_0)$ is the fundamental solution of the Schrödinger's operator $L = ih\partial_t + \frac{1}{2}h^2\partial_x^2 - U(x)$. In Quantum Mechanics $K(x,t;x_0,t_0)$ is also referred to as a propagator. The previous section is very useful to provide propagators for different expressions of the potential function $U(x)$. There are only a few cases when we can compute explicit formulas for the propagators. These kernels are computed in Quantum Mechanics using path integrals formalism, see [41]. Here we use the geometric method provided by the previous sections.

10.8.1 Free quantum particle

In this case the potential energy $U(x) = 0$. The propagator in this case is obtained from the heat kernel. It is known that the heat operator $\partial_t - \partial_x^2$ has the fundamental solution $K(x,x_0,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{1}{4t}(x-x_0)^2}$. Consider the substitution

$$\mathbf{t} = iht, \qquad \mathbf{x} = \frac{ih}{\sqrt{2}}x. \tag{10.8.54}$$

Then the heat equation becomes a Schrödinger operator

$$\partial_t - \partial_x^2 = ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2,$$

and the fundamental solution becomes a propagator

$$\begin{aligned}
K(x,x_0,t) &= \frac{1}{\sqrt{4\pi t}}e^{-\frac{1}{4t}(x-x_0)^2}\\
&= \sqrt{\frac{ih}{4\pi\mathbf{t}}}\,e^{\frac{ih}{2\mathbf{t}}(\mathbf{x}-\mathbf{x}_0)^2}\\
&= \mathbf{K}(\mathbf{x},\mathbf{x}_0,\mathbf{t},0).
\end{aligned}$$

Making a time translation $0 \to \mathbf{t}_0$ yields the following result.

Theorem 10.24. *The propagator for a one-dimensional free quantum particle is given by*

$$\mathbf{K}(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0) = \sqrt{\frac{ih}{4\pi(\mathbf{t}-\mathbf{t}_0)}}\, e^{\dfrac{ih}{2(\mathbf{t}-\mathbf{t}_0)}(\mathbf{x}-\mathbf{x}_0)^2}, \qquad \mathbf{t} > \mathbf{t}_0.$$

10.8.2 Quantum particle in a linear potential

The substitution (10.8.54) yields

$$\begin{aligned}\partial_t - \partial_x^2 + bx &= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - \frac{ib\sqrt{2}}{h}\mathbf{x} \\ &= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - a\mathbf{x},\end{aligned}$$

where

$$a = i\alpha = \frac{ib\sqrt{2}}{h}.$$

Using Theorem 10.23, the same substitution yields

$$\begin{aligned}K(x, x_0, t) &= \frac{1}{\sqrt{4\pi t}} e^{-\dfrac{(x-x_0)^2}{4t} - \dfrac{b}{2}(x+x_0)t + \dfrac{b^2}{12}t^3} \\ &= \sqrt{\frac{ih}{4\pi\mathbf{t}}}\, e^{\dfrac{ih}{2\mathbf{t}}(\mathbf{x}-\mathbf{x}_0)^2}\, e^{-\dfrac{\alpha}{2i}(\mathbf{x}+\mathbf{x}_0)\dfrac{\mathbf{t}}{ih} + \left(\dfrac{\alpha h}{\sqrt{2}}\right)^2 \dfrac{1}{12}\dfrac{\mathbf{t}^3}{(ih)^3}} \\ &= \sqrt{\frac{ih}{4\pi\mathbf{t}}}\, e^{\dfrac{ih}{2\mathbf{t}}(\mathbf{x}-\mathbf{x}_0)^2}\, e^{\dfrac{\alpha}{2h}(\mathbf{x}+\mathbf{x}_0)\mathbf{t} + \dfrac{\alpha^2}{24}\dfrac{\mathbf{t}^3}{(-i)h}} \\ &= \sqrt{\frac{ih}{4\pi\mathbf{t}}}\, e^{\dfrac{ih}{2\mathbf{t}}(\mathbf{x}-\mathbf{x}_0)^2}\, e^{-\dfrac{i}{2h}[a(\mathbf{x}+\mathbf{x}_0)\mathbf{t} + \dfrac{a^2}{12}\mathbf{t}^3]}.\end{aligned}$$

Replacing $\mathbf{t}$ by $\mathbf{t} - t_0$ yields the formula for the propagator for a quantum mechanical particle in the presence of a homogeneous force due to a linear potential $U(x) = ax$.

Theorem 10.25. *The propagator for the Schrödinger operator*

$$ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - a\mathbf{x}$$

is given by

$$\begin{aligned}&\mathbf{K}(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0) \\ &= \sqrt{\frac{ih}{4\pi(\mathbf{t}-\mathbf{t}_0)}}\, e^{\dfrac{ih}{2(\mathbf{t}-\mathbf{t}_0)}(\mathbf{x}-\mathbf{x}_0)^2 - \dfrac{i}{2h}[a(\mathbf{x}+\mathbf{x}_0)(\mathbf{t}-\mathbf{t}_0) + \dfrac{a^2}{12}(\mathbf{t}-\mathbf{t}_0)^3]},\end{aligned}$$

with $\mathbf{t} > \mathbf{t}_0$.

10.8.3 Linear harmonic quantum oscillator

This is the case of a quantum particle in a quadratic potential $U(x) = \frac{1}{2}\alpha^2\mathbf{x}^2, \alpha \in \mathbb{R}$. Let a and b be such that

$$\frac{1}{2}\alpha^2 = a^2 = 2\frac{b^2}{h^2}.$$

The substitution (10.8.54) yields the Schrödinger operator

$$\begin{aligned}
\partial_t - \partial_x^2 + b^2x^2 &= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 + b^2\frac{-2}{h^2}\mathbf{x}^2 \\
&= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - 2\frac{b^2}{h^2}\mathbf{x}^2 \\
&= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - \frac{\alpha^2}{2}\mathbf{x}^2.
\end{aligned}$$

With substitution (10.8.54), the fundamental solution given by Theorem 10.12 becomes

$$\begin{aligned}
&K(x_0, x, t) \\
&= \frac{1}{\sqrt{4\pi t}}\sqrt{\frac{2bt}{\sinh(2bt)}}\, e^{-\frac{1}{4t}\frac{2bt}{\sinh(2bt)}[(x^2+x_0^2)\cosh(2bt)-2xx_0]} \\
&= \sqrt{\frac{ih}{4\pi\mathbf{t}}}\sqrt{\frac{-i\sqrt{2}a\mathbf{t}}{\sinh(-i\sqrt{2}a\mathbf{t})}}\, e^{-\frac{ih}{4}\frac{\alpha}{\sin(\alpha\mathbf{t})}\frac{4}{h^2}\left[-\frac{1}{2}(\mathbf{x}^2+\mathbf{x}_0^2)\cos(\alpha\mathbf{t})+\mathbf{x}\mathbf{x}_0\right]} \\
&\qquad = \sqrt{\frac{ih}{4\pi\mathbf{t}}}\sqrt{\frac{\alpha\mathbf{t}}{\sin(\alpha\mathbf{t})}}\, e^{\frac{-i\alpha}{h\sin(\alpha\mathbf{t})}\left[-\frac{1}{2}(\mathbf{x}^2+\mathbf{x}_0^2)\cos(\alpha\mathbf{t})+\mathbf{x}\mathbf{x}_0\right]} \\
&\qquad = \sqrt{\frac{ih}{4\pi\mathbf{t}}}\sqrt{\frac{\alpha\mathbf{t}}{\sin(\alpha\mathbf{t})}}\, e^{\frac{i\alpha}{h}\left[\frac{1}{2}(\mathbf{x}^2+\mathbf{x}_0^2)\cot(\alpha\mathbf{t})-\frac{\mathbf{x}\mathbf{x}_0}{\sin(\alpha\mathbf{t})}\right]}.
\end{aligned}$$

Replacing $\mathbf{t}$ by $\mathbf{t} - \mathbf{t}_0$ yields the formula for the propagator for a quantum harmonic oscillator.

Theorem 10.26. *The propagator for the Schrödinger's operator with quadratic potential*

$$ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}}^2 - \frac{1}{2}\alpha^2\mathbf{x}^2$$

is given by

$$\mathbf{K}(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0) = \sqrt{\frac{ih}{4\pi\mathbf{t}}}\sqrt{\frac{\alpha\mathbf{t}}{\sin(\alpha(\mathbf{t}-\mathbf{t}_0))}}\, e^{\frac{i\alpha}{h}\left[\frac{1}{2}(\mathbf{x}^2+\mathbf{x}_0^2)\cot(\alpha\mathbf{t})-\frac{\mathbf{x}\mathbf{x}_0}{\sin(\alpha(\mathbf{t}-\mathbf{t}_0))}\right]},$$

with $\mathbf{t} > \mathbf{t}_0$.

10.9 Propagators for Schrödinger's equation in the n-dimensional case

Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$. The n-dimensional Schrödinger equation with potential energy $U(\mathbf{x})$ is

$$ih\partial_{\mathbf{t}}\Psi + \frac{1}{2}h^2\big(\partial_{\mathbf{x}_1} + \cdots + \partial_{\mathbf{x}_n}\big)\Psi = U(\mathbf{x})\Psi. \tag{10.9.55}$$

Let $\Psi_0(\mathbf{x}) = \Psi(\mathbf{x}, \mathbf{t}_0)$ be the initial value of the wave function. Then the solution of (10.9.55) is

$$\Psi(x, \mathbf{t}) = \int K(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0)\Psi_0(\mathbf{x}_0)\, d\mathbf{x}_0, \qquad \mathbf{t} > \mathbf{t}_0,$$

where $K(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0)$ is the propagator. The potential $U(x) = 0$ yields the propagator for an n-dimensional free particle

$$K(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0) = \left(\frac{ih}{4\pi(\mathbf{t} - \mathbf{t}_0)}\right)^{n/2} e^{\dfrac{ih}{2(\mathbf{t} - \mathbf{t}_0)}|\mathbf{x} - \mathbf{x}_0|^2}.$$

The potential $U(x) = \frac{1}{2}\alpha^2|\mathbf{x}|^2 = \frac{1}{2}\alpha^2(\mathbf{x}_1^2 + \cdots + \mathbf{x}_n^2)$ yields the propagator for an n-dimensional linear harmonic oscillator

$$K(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0) = \left[\frac{ih}{4\pi\mathbf{T}} \cdot \frac{\alpha\mathbf{T}}{\sin\big(\alpha\mathbf{T}\big)}\right]^{n/2} e^{\dfrac{i\alpha}{h}\left[\dfrac{1}{2}\left(|\mathbf{x}|^2 + |\mathbf{x}_0|^2\right)\cot(\alpha\mathbf{T}) - \dfrac{\langle \mathbf{x}, \mathbf{x}_0\rangle}{\sin(\alpha\mathbf{T})}\right]},$$

where $\mathbf{T} = \mathbf{t} - \mathbf{t}_0 > 0$, and $\langle \mathbf{x}, \mathbf{x}_0\rangle = \mathbf{x}_1\mathbf{x}_{01} + \cdots + \mathbf{x}_n\mathbf{x}_{0n}$.

The following result deals with the potential energy

$$U(x) = \langle Mx, x\rangle = \sum_{j=1}^{n} \alpha_j^2 x_j^2,$$

where

$$M = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}$$

is a real matrix.

Theorem 10.27. *The propagator for the Schrödinger operator*

$$ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\partial_{\mathbf{x}} - \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2 \mathbf{x}_j^2 \tag{10.9.56}$$

is

$$K(\mathbf{x}, \mathbf{x}_0, \mathbf{t}, \mathbf{t}_0)$$
$$= \left(\frac{ih}{4\pi}\right)^{n/2} \prod_{j=1}^{n} \left(\frac{\alpha_j}{\sin(\alpha_j \mathbf{T})}\right)^{1/2} e^{\frac{i}{2h}\sum_{j=1}^{n}\left[\alpha_j(\mathbf{x}_j^2+\mathbf{y}_j^2)\cot(\alpha_j\mathbf{T}) - \frac{2\mathbf{x}_j\mathbf{y}_j}{\sin(\alpha_j\mathbf{T})}\right]},$$

where $\mathbf{y} = \mathbf{x}_0$ *and* $\mathbf{T} = \mathbf{t} - \mathbf{t}_0 > 0$.

Proof. Let $\alpha_j = \frac{2}{h}\lambda_j$, $j = 1, ..., n$. With substitution (10.8.54) we have

$$\begin{aligned} \partial_x - \Delta_x + \sum \lambda_j^2 x_j^2 &= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\Delta_{\mathbf{x}} - \frac{2}{h^2}\sum \lambda_j^2 \mathbf{x}_j^2 \\ &= ih\partial_{\mathbf{t}} + \frac{1}{2}h^2\Delta_{\mathbf{x}} - \frac{1}{2}\sum \alpha_j^2 \mathbf{x}_j^2, \end{aligned}$$

which is the operator (10.9.56). The fundamental solution of the operator $\partial_x - \Delta_x + \sum \lambda_j^2 x_j^2$ is given by Theorem 10.17. Using (10.8.54) the fundamental solution given by formula (10.4.39) becomes

$$K(x, y, t, 0)$$
$$= \frac{1}{(4\pi t)^{n/2}} \prod_{j=1}^{n} \left[\frac{2\lambda_j t}{\sinh(2\lambda_j t)}\right]^{1/2} e^{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j\left[(x_j^2+y_j^2)\coth(2\lambda_j t) - \frac{2x_j y_j}{\sinh(2\lambda_j t)}\right]}$$
$$= \frac{1}{(4\pi t)^{n/2}} \prod_{j=1}^{n} \left[\frac{h\alpha_j t}{\sinh(h\alpha_j t)}\right]^{1/2} e^{-\frac{1}{2}\sum_{j=1}^{n}\frac{h\alpha_j}{2}\left[(x_j^2+y_j^2)\coth(h\alpha_j t) - \frac{2x_j y_j}{\sinh(h\alpha_j t)}\right]}$$
$$= \left(\frac{ih}{4\pi}\right)^{n/2} \prod_{j=1}^{n} \left[\frac{\alpha_j}{\sin(\alpha_j \mathbf{t})}\right]^{1/2} e^{-\frac{i}{2h}\sum_{j=1}^{n}\alpha_j\left[-(\mathbf{x}_j^2+\mathbf{y}_j^2)\cot(\alpha_j\mathbf{t}) + \frac{2\mathbf{x}_j\mathbf{y}_j}{\sin(\alpha_j\mathbf{t})}\right]}.$$

Replacing $\mathbf{t}$ by $\mathbf{T} = \mathbf{t} - \mathbf{t}_0$ yields the desired relation. ∎

10.10 The operator $P = \partial_t - \partial_x^2 - U(x)\partial_x$

We shall study the fundamental solution function for the operator

$$P = \partial_t - \partial_x^2 - U(x)\partial_x,$$

where $U(x)$ is a potential function. We shall study different potentials U (linear, quadratic, square root, exponential). A last section will deal with the physical significance of this operator.

10.10.1 The linear potential

Consider the operator

$$L = \frac{d^2}{dx^2} + 2ax\frac{d}{dx}, \quad a \in \mathbb{R}$$

with the associated Hamiltonian function

$$H(\xi, x) = \frac{1}{2}(\xi^2 + 2ax\xi).$$

The Hamiltonian system yields

$$\dot{x} = H_\xi = \xi + ax \Longrightarrow \xi = \dot{\xi} - ax,$$
$$\dot{\xi} = -H_x = -a\xi = -a(\dot{x} - ax) = -a\dot{x} + a^2x,$$

and hence

$$\ddot{x} = \dot{\xi} + a\dot{x} = -a\dot{x} + a^2x + a\dot{x} = a^2x.$$

Then $x(s)$ satisfies the boundary problem

$$\begin{cases} \ddot{x} = a^2x, \\ x(0) = x_0, \\ x(t) = x. \end{cases}$$

The above boundary problem has a unique solution. The associated energy is the same as in Proposition 10.9

$$E = \frac{a^2\Big(x^2 + x_0^2 - 2xx_0\cosh(at)\Big)}{2\sinh(at)^2}. \tag{10.10.57}$$

The corresponding action is the same as (10.4.15)

$$S(x_0, x, t) = \frac{a}{2}\frac{1}{\sinh(at)}\Big[(x^2 + x_0^2)\cosh(at) - 2xx_0\Big].$$

From the conservation of energy

$$H(\nabla_x S) = E,$$

we obtain

$$(\partial_x S)^2 + 2ax\,\partial_x S = 2E \Longrightarrow 2ax(\partial_x S) = 2E - (\partial_x S)^2. \tag{10.10.58}$$

We shall look again for a fundamental solution of the type

$$K = K(x_0, x, t) = V(t)e^{kS(x_0,x,t)},$$

with k constant. A straightforward computation yields

$$\begin{aligned}
\partial_t K &= K\Big(\frac{V'}{V} - kE\Big)\partial_x K = kK\partial_x S,\\
\partial_x^2 K &= k\,\partial_x K\,\partial_x S + kK\partial_x^2 S\\
&= k^2 K(\partial_x S)^2 + kK\partial_x^2 S\\
&= K\Big(k^2(\partial_x S)^2 + k\partial_x^2 S\Big).
\end{aligned}$$

Consider the operator

$$P = \partial_t - \partial_x^2 - 2\alpha a x\partial_x,$$

where α is a multiplier determined by the relation $PK = 0$. A computation provides

$$\begin{aligned}
PK &= K\Big(\frac{V'}{V} - kE\Big) - K\Big(k^2(\partial_x S)^2 + k\partial_x^2 S\Big) - 2\alpha akxK\partial_x S\\
&= K\left(\frac{V'}{V} - kE - k^2(\partial_x S)^2 - k\partial_x^2 S - 2\alpha akx\partial_x S\right)\\
&= K\left(\frac{V'}{V} - kE - k^2(\partial_x S)^2 - k\partial_x^2 S - \alpha k(2E - (\partial_x S)^2)\right)\\
&= K\left(\frac{V'}{V} - kE(1+2\alpha) - k\partial_x^2 S + k(\alpha - k)(\partial_x S)^2\right),
\end{aligned}$$

where we have used relation (10.10.58). Choosing $\alpha = k = -1/2$ yields

$$PK = K\left(\frac{V'}{V} + \frac{1}{2}\partial_x^2 S\right).$$

Using $\partial_x^2 S = a\coth(at)$, we let V satisfy

$$\frac{V'(t)}{V(t)} + \frac{a}{2}\coth(at) = 0,$$

with the solution

$$V(t) = \frac{C}{\sqrt{\sinh(at)}}, \quad C \in \mathbb{R}.$$

Hence the operator $P = \partial_t - \partial_x^2 + ax\partial_x$ has the kernel

$$\begin{aligned}
K(x_0, x, t) &= V(t)e^{\frac{1}{2}S}\\
&= \frac{C}{\sqrt{\sinh(at)}}\, e^{-\frac{1}{4t}\frac{at}{\sinh(at)}\Big((x^2 + x_0^2)\cosh(at) - 2xx_0\Big)}.
\end{aligned}$$

When $a \to 0$, the operator becomes the usual heat operator $\partial_t - \partial_x^2$, with the fundamental solution $\frac{1}{\sqrt{4\pi t}}e^{-\frac{1}{4}(x-x_0)^2}$. By comparison we obtain $C = \sqrt{\frac{a}{\pi}}$.

Theorem 10.28. *Let $a \in \mathbb{R}$. The fundamental solution for the operator $P = \partial_t - \partial_x^2 + ax\partial_x$ is*

$$K(x_0, x, t) = \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{at}{\sinh(at)}}\, e^{-\frac{1}{4t}\frac{at}{\sinh(at)}[(x^2 + x_0^2)\cosh(at) - 2xx_0]}, \quad t > 0.$$

The computations are similar in the case when a is replaced by $-ia$. Using $\cosh(iat) = \cos(at)$ and $\sinh(iat) = i\sin(at)$, we obtain a dual theorem.

Theorem 10.29. *Let $a \in \mathbb{R}$. The fundamental solution for the operator $P = \partial_t - \partial_x^2 + iax\partial_x$ is*

$$K(x_0, x, t) = \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{at}{\sin(at)}}\, e^{-\frac{1}{4t}\frac{at}{\sin(at)}[(x^2 + x_0^2)\cos(at) - 2xx_0]}, \quad t > 0.$$

10.10.2 The quadratic potential

The operator considered in this section is $P = \partial_t - L$, with $L = \partial_x^2 + 2ia^2x^2\partial_x$. This corresponds to a quartic harmonic oscillator in Quantum Mechanics. The Hamiltonian associated with the operator L is

$$H(\xi, x) = \frac{1}{2}\xi^2 + ia^2x^2\xi.$$

From the Hamiltonian system we have

$$\begin{aligned}
\dot{x} &= H_\xi = \xi + ia^2x \Longrightarrow \xi = \dot{x} - ia^2x^2,\\
\dot{\xi} &= -H_x = -2ia^2x\xi = -2ia^2x(\dot{x} - ia^2x^2)\\
&= -2ia^2x\dot{x} - 2a^4x^3,\\
\ddot{x} &= \dot{\xi} + 2ia^2x\dot{x}\\
&= -2ia^2x\dot{x} - 2a^4x^3 + 2ia^2x\dot{x}\\
&= -2a^4x^3.
\end{aligned}$$

Then $x(s)$ will satisfy the boundary value problem (10.6.40)

$$\begin{cases}
\ddot{x} = -2a^4x^3,\\
x(0) = x_0,\\
x(t) = x.
\end{cases}$$

This problem has infinitely many solutions $x_n(s)$, even for $|x - x_0|$ small. They correspond to an increasing unbounded sequence of energies $(E_n)_n$ given by the Theorem 10.21. The actions S_n cannot be found explicitly. This explains the difficulty of the problem. We shall find the kernel in the case of a general potential $U(x)$ in the next section.

10.10.3 The kernel of $\partial_t - \partial_x^2 - U(x)\partial_x$

Consider the operator $L = \partial_x^2 + U(x)\partial_x$ with the associated Hamiltonian

$$H(\xi, x) = \frac{1}{2}\xi^2 + \frac{1}{2}U(x)\xi.$$

The Hamiltonian system yields

$$\begin{aligned}
\dot{x} &= H_\xi = \xi + \frac{1}{2}U(x) \Longrightarrow \xi = \dot{x} - \frac{1}{2}U(x),\\
\dot{\xi} &= -H_x = -\frac{1}{2}U'(x)\xi,\\
\ddot{x} &= \dot{\xi} + \frac{1}{2}U'(x)\dot{x} = -\frac{1}{2}U'(x)\xi + \frac{1}{2}U'(x)\dot{x}\\
&= -\frac{1}{2}U'(x)(\dot{x} - \frac{1}{2}U(x)) + \frac{1}{2}U'(x)\dot{x}\\
&= \frac{1}{4}U(x)U'(x) = \frac{1}{8}\frac{d}{dx}U^2(x).
\end{aligned}$$

We are interested in the solutions of the boundary value problem

$$\begin{cases} \ddot{x} = \dfrac{1}{8}\dfrac{d}{dx}U^2(x),\\ x(0) = x_0,\\ x(t) = x. \end{cases} \tag{10.10.59}$$

The conservation law is

$$\frac{1}{2}\dot{x}^2 - \frac{1}{8}U^2(x) = E, \tag{10.10.60}$$

where E is the constant of energy along the solution $x(s)$ which joins the end points x_0 and x. The solution $x(s)$ can be obtained by integration

$$\int_{x_0}^{x(s)} \frac{dw}{\sqrt{2E + \frac{1}{4}U^2(w)}} = \pm s,$$

where the energy $E = E(x_0, x)$ satisfies the equation

$$\int_{x_0}^{x} \frac{dw}{\sqrt{2E + \frac{1}{4}U^2(w)}} = \pm t. \tag{10.10.61}$$

The equation (10.10.61) has always at least a solution $E > 0$. It might have even infinitely many solutions E_n. There is an action associated with each energy E such that

$$H(\nabla_x S) = E \Longrightarrow (S_x)^2 + U(x)S_x = 2E,$$

and

$$\partial_t S = -E,$$

where $\nabla_x S = S_x = \partial_x S$. For each solution $x(s)$ we shall consider the product $K = V(t,x)e^{kS}$. Let $\lambda \in \mathbb{R}$ be a multiplier and consider the operator

$$P_\lambda = \partial_t - \partial_x^2 - \lambda U(x)\partial_x.$$

A straightforward computation yields

$$\begin{aligned}
P_\lambda(K) &= K\left(\frac{V'}{V} - kE\right) - K\left(\frac{V_{xx}}{V} + 2k\frac{V_x}{V}S_x + k^2(S_x)^2 + kS_{xx}\right) \\
&\quad -\lambda U(x)K\left(\frac{V_x}{V} + kS_x\right) \\
&= K\left(\frac{V'}{V} - kE - \frac{V_{xx}}{V} - 2k\frac{V_x}{V}S_x - k^2(S_x)^2 - kS_{xx} - \lambda U(x)\frac{V_x}{V} - \lambda kU(x)S_x\right) \\
&= K\left(\frac{1}{V}\Big(V' - V_{xx} - 2kV_xS_x - \lambda U(x)V_x\Big) - kE - k^2(S_x)^2 - kS_{xx} - \lambda kU(x)S_x\right) \\
&= K\left(\frac{1}{V}\Big(V' - V_{xx} - 2kV_xS_x - \lambda U(x)V_x\Big)\right. \\
&\quad \left. -kE - k^2(2E - S_xU(x)) - kS_{xx} - \lambda kU(x)S_x\right) \\
&= K\left(\frac{1}{V}\Big(V' - V_{xx} - 2kV_xS_x - \lambda U(x)V_x\Big) - kS_{xx} - kE\underbrace{(1+2k)}_{=0}\right. \\
&\quad \left. +k\underbrace{(k-\lambda)}_{=0}U(x)S_x\right) \\
&= 0,
\end{aligned}$$

where we choose $\lambda = k = -\dfrac{1}{2}$ and let $V(t,x)$ satisfy the generalized volume function equation

$$V' - V_{xx} + [S_x + \frac{1}{2}U(x)]V_x + \frac{1}{2}S_{xx}V = 0. \tag{10.10.62}$$

A well-known result of Classical Mechanics states that $\xi = S_x$ along the solutions of the Hamiltonian system. The first equation of the Hamiltonian system yields $\dot{x} = \xi + \frac{1}{2}U(x) = S_x + \frac{1}{2}U(x)$, and hence the generalized volume function equation becomes

$$V' - V_{xx} + \dot{x}\,V_x + \frac{1}{2}S_{xx}V = 0. \tag{10.10.63}$$

Using

$$\frac{d}{dt}V(t,x(t)) = \partial_t V + \dot{x}\partial_x V = V' + \dot{x}V_x$$

yields the following form for equation (10.10.62),

$$\frac{d}{dt}V(t, x(t)) - \partial_x^2 V(t, x) = -\frac{1}{2}S_{xx}V(t, x). \tag{10.10.64}$$

In the case when S_{xx} depends only on t, it makes sense to look for a function V which does not depend on x. Equation (10.10.64) in this case becomes

$$V'(t) = -\frac{1}{2}S_{xx}V(t).$$

This happens just in a few particular cases.

Theorem 10.30. *Let $x_n(s)$ be all solutions of the boundary value problem (10.10.59). Let S_n be the action and V_n be the generalized volume function associated with the solution $x_n(s)$. Then the kernel of the operator*

$$P = \partial_t - \partial_x^2 + \frac{1}{2}U(x)\partial_x$$

is given by the formula

$$K(x_0, x, t) = \sum_n C_n V_n(t, x) e^{-\frac{1}{2}S_n(x_0, x, t)}$$

where $V_n(t, x)$ satisfies (10.10.64) and the constants C_n satisfy an analogue of equation (10.7.52).

There are only a few cases when the boundary value problem (10.10.59) can be solved and we are able to find explicit formulas for the action S. The linear potential is one of them. In the next section we shall present other particular cases, which have unique solutions.

10.10.4 The square root potential

Let $U(x) = 2\sqrt{2x}$. Then the equation (10.10.61) becomes

$$\int_{x_0}^{x} \frac{dw}{\sqrt{E + w}} = \pm\sqrt{2}t.$$

If $x > x_0$ we choose the $+$ sign and if $x < x_0$ we shall choose the $-$ sign in the right hand side. The sign does not affect the solution E. Integrating yields

$$2\sqrt{E + w}\Big|_{x_0}^{x} = \pm\sqrt{2}t \Longleftrightarrow \sqrt{E + x} - \sqrt{E + x_0} = \pm\frac{t}{\sqrt{2}},$$

$$\sqrt{E + x} = \sqrt{E + x_0} \pm \frac{t}{\sqrt{2}} \Longrightarrow E + x = \left(\sqrt{E + x_0} \pm \frac{t}{\sqrt{2}}\right)^2$$

$$\Longleftrightarrow E + x = E + x_0 \pm 2\sqrt{E + x_0}\frac{t}{\sqrt{2}} + \frac{t^2}{2}$$

$$\Longleftrightarrow x - x_0 - \frac{t^2}{2} = \pm 2\sqrt{E + x_0}\frac{t}{\sqrt{2}}$$

$$\Longrightarrow \Big(x - x_0 - \frac{t^2}{2}\Big)^2 = 2t^2(E + x_0)$$

$$\begin{aligned}\Longleftrightarrow E &= \frac{\Big(x - x_0 - \frac{t^2}{2}\Big)^2}{2t^2} - x_0 \\ &= \frac{(x - x_0)^2}{2t^2} + \frac{t^2}{8} - \frac{x - x_0}{2} - x_0 \\ &= \frac{(x - x_0)^2}{2t^2} + \frac{t^2}{8} - \frac{x + x_0}{2}.\end{aligned}$$

Theorem 10.31. *Given $x \neq x_0$, there is a unique solution of the boundary value problem (10.10.59) with the potential $U(x) = 2\sqrt{2x}$. The solution is a parabola given by*

$$x(s) = \begin{cases} \dfrac{s^2}{2} + \sqrt{2}s\sqrt{E + x_0} + x_0 & \text{if } x > x_0, \\ \\ \dfrac{s^2}{2} - \sqrt{2}s\sqrt{E + x_0} + x_0 & \text{if } x < x_0, \end{cases} \tag{10.10.65}$$

where the energy $E = \dfrac{(x - x_0)^2}{2t^2} + \dfrac{t^2}{8} - \dfrac{x + x_0}{2}$ is the same for both cases.

Proof. We solve the following integral for $x(s)$,

$$\int_{x_0}^{x(s)} \frac{dw}{\sqrt{E + w}} = \pm\sqrt{2}s \Longrightarrow \sqrt{x(s) + E} = \pm\frac{s}{\sqrt{2}} + \sqrt{x_0 + E}.$$

Taking the square we obtain (10.10.65). ∎

In the following we shall find the action S, which satisfies the Hamilton–Jacobi equation

$$\partial_t S = -E = \frac{(x - x_0)^2}{2t^2} + \frac{t^2}{8} - \frac{x + x_0}{2}$$

$$\Longrightarrow S(x, x_0, t) = \frac{(x - x_0)^2}{2t} + \frac{x + x_0}{2}t - \frac{t^3}{24}.$$

An obvious computation shows that $S_{xx} = \dfrac{1}{t}$ does not depend on x. Then the volume function V depends only on t and satisfies

$$V'(t) = -\frac{1}{2t}V(t),$$

which can be easily integrated to obtain

$$V(t) = \frac{C}{\sqrt{t}}.$$

Theorem 10.32. *The kernel of the operator*

$$P = \partial_t - \partial_x^2 + \sqrt{2x}\,\partial_x$$

is given by

$$K(x, x_0, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\left(\frac{(x-x_0)^2}{2t} + \frac{x+x_0}{2}t - \frac{t^3}{24}\right)}. \tag{10.10.66}$$

Proof. From Theorem 10.31 there is a unique solution $x(s)$ and hence the sum in the Theorem 10.30 yields a fundamental solution $K = V(t)e^{-\frac{1}{2}S}$. Equation (10.7.52) yields $C = \dfrac{1}{\sqrt{2\pi}}$. ■

10.10.5 The constant potential case $U(x) = a$, with $a \in \mathbb{R}$

In this case the boundary value problem (10.10.59) becomes

$$\begin{cases} \ddot{x} = 0, \\ x(0) = x_0 \\ x(t) = x. \end{cases}$$

The solution is unique and it is given by

$$x(s) = (x - x_0)\frac{s}{t} + x_0, \qquad 0 \le s \le t.$$

The energy given by (10.10.60) is

$$\begin{aligned} E &= \frac{1}{2}\dot{x}^2 - \frac{1}{8}U^2(x) \\ &= \frac{(x-x_0)^2}{2t^2} - \frac{1}{8}a^2. \end{aligned}$$

The action S satisfies

$$\begin{aligned} \partial_t S \;&=\; -E = -\frac{1}{t^2}\frac{(x-x_0)^2}{2} - \frac{1}{8}a^2 \\ &\Longrightarrow S = \frac{(x-x_0)^2}{2t} - \frac{1}{8}a^2 t. \end{aligned}$$

It is easy to show that $S_{xx} = \dfrac{1}{t}$. Hence the volume function $V(t)$ will satisfy the equation $V'(t) = -\dfrac{1}{2t}V(t)$ with the solution $V(t) = \dfrac{C}{\sqrt{t}}$, $t > 0$. There is only one term in the sum provided by Theorem 10.30. The kernel will be

$$K(x, x_0, t) = V(t)e^{-\frac{1}{2}S} = \frac{C}{\sqrt{t}}\, e^{-\frac{(x-x_0)^2}{4t} + \frac{1}{16}a^2 t}.$$

Making $a \longrightarrow 0$, we get $C = \dfrac{1}{\sqrt{4\pi}}$ by comparison with the kernel of the usual heat equation. Making $b = \dfrac{a}{2}$ yields the following theorem.

Theorem 10.33. *Let $b \in \mathbb{R}$.*
(i) The kernel of the operator

$$P = \partial_t - \partial_x^2 + b\partial_x$$

is

$$K(x, x_0, t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-x_0)^2}{4t} + \frac{b^2}{4}t}, \qquad t > 0.$$

(ii) The kernel of the operator

$$P = \partial_t - \partial_x^2 + ib\partial_x$$

is

$$K(x, x_0, t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x-x_0)^2}{4t} - \frac{b^2}{4}t}, \qquad t > 0.$$

10.10.6 The exponential potential

In this section we shall deal with the kernel of

$$P = \partial_t - \partial_x^2 + \sqrt{2}e^{x/2}\partial_t.$$

The potential in this case is $U(x) = 2\sqrt{2}e^{x/2}$ and the integral (10.10.61) becomes

$$\int_{x_0}^{x} \frac{dw}{\sqrt{E + e^w}} = \pm\sqrt{2}t, \qquad t \geq 0. \tag{10.10.67}$$

We choose a positive (negative) sign in the right-hand side if $x > x_0$ ($x < x_0$). Integrating yields

$$-\frac{2}{\sqrt{E}} \tanh^{-1} \sqrt{1 + \frac{e^w}{E}}\,\Bigg|_{x_0}^{x} = \pm\sqrt{2}t$$

$$\Longleftrightarrow \tanh^{-1}\sqrt{1+\frac{e^x}{E}} - \tanh^{-1}\sqrt{1+\frac{e^{x_0}}{E}} = \mp t\sqrt{\frac{E}{2}}.$$

Using $\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$ yields

$$\ln\frac{1+\sqrt{1+\frac{e^x}{E}}}{1-\sqrt{1+\frac{e^x}{E}}} - \ln\frac{1+\sqrt{1+\frac{e^{x_0}}{E}}}{1-\sqrt{1+\frac{e^{x_0}}{E}}} = \mp\sqrt{2E}\,t$$

$$\Longleftrightarrow \frac{1+\sqrt{1+\frac{e^x}{E}}}{1-\sqrt{1+\frac{e^x}{E}}}\cdot\frac{1-\sqrt{1+\frac{e^{x_0}}{E}}}{1+\sqrt{1+\frac{e^{x_0}}{E}}} = e^{\mp\sqrt{2E}\,t}.$$

Using $\frac{1+\sqrt{z}}{1-\sqrt{z}} = \frac{1+z+2\sqrt{z}}{1-z}$, the above relation becomes

$$\frac{2+\frac{e^x}{E}+2\sqrt{1+\frac{e^x}{E}}}{-\frac{e^x}{E}}\cdot\frac{-\frac{e^{x_0}}{E}}{2+\frac{e^{x_0}}{E}+2\sqrt{1+\frac{e^{x_0}}{E}}} = e^{\mp\sqrt{2E}\,t}$$

$$\Longleftrightarrow e^{x_0-x}\cdot\frac{2E+e^x+2\sqrt{E}\sqrt{E+e^x}}{2E+e^{x_0}+2\sqrt{E}\sqrt{E+e^{x_0}}} = e^{\mp\sqrt{2E}\,t}.$$

Let $\lambda = \sqrt{2E}$. Then λ satisfies the equation

$$\underbrace{e^{x_0-x}\cdot\frac{\lambda^2+e^x+\sqrt{2}\lambda\sqrt{\frac{1}{2}\lambda^2+e^x}}{\lambda^2+e^{x_0}+\sqrt{2}\lambda\sqrt{\frac{1}{2}\lambda^2+e^{x_0}}}}_{f(\lambda)} = e^{\mp\lambda\,t}.$$

Let $f(\lambda)$ be the left-hand side of the above relation. We have

$$f(0) = e^{x_0-x}e^{x-x_0} = 1,$$

$$\lim_{\lambda\to\infty} f(\lambda) = e^{x_0-x}\begin{cases} >1 & \text{if } x_0 > x,\\ <1 & \text{if } x_0 < x.\end{cases}$$

Case $x_0 > x$: The equation becomes $f(\lambda) = e^{\lambda t}$. The linear approximations around $\lambda = 0$ are

$$e^{\lambda t} = 1 + t\lambda + \mathcal{O}(\lambda^2),$$
$$f(\lambda) = 1 + \sqrt{2}\Big(\frac{1}{e^{x/2}} - \frac{1}{e^{x_0/2}}\Big)\lambda + \mathcal{O}(\lambda^2).$$

For any $0 < t < \sqrt{2}\Big(\frac{1}{e^{x/2}} - \frac{1}{e^{x_0/2}}\Big)$ there is an $\epsilon > 0$ such that

$$f(\lambda) > e^{\lambda t}, \qquad \text{for } 0 < \lambda < \epsilon.$$

We also have

$$f(\lambda) < e^{x_0 - x} < e^{\lambda t}, \qquad \text{for } \lambda > \frac{x_0 - x}{t} > 0.$$

Hence there is a solution $\lambda \in \Big(\epsilon, \frac{x_0 - x}{t}\Big)$, see Figure 10.2.

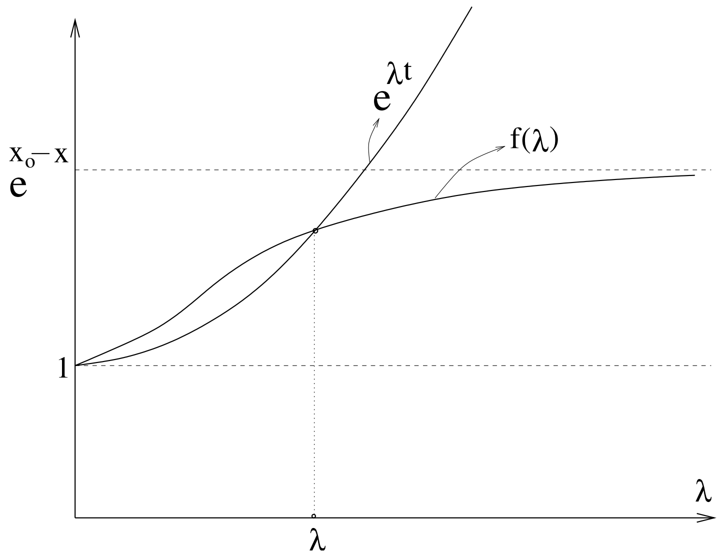


Figure 10.2: The functions $f(\lambda)$ and $e^{\lambda t}$ in the case $x_0 > x$.

Case $x_0 < x$: The equation becomes $f(\lambda) = e^{-\lambda t}$. The linear approximations around $\lambda = 0$ are

$$e^{-\lambda t} = 1 - t\lambda + \mathcal{O}(\lambda^2),$$
$$f(\lambda) = 1 - \sqrt{2}\Big(\frac{1}{e^{x_0/2}} - \frac{1}{e^{x/2}}\Big)\lambda + \mathcal{O}(\lambda^2).$$

A similar analysis yields that t can be chosen small enough such that the graph of the function $f(\lambda)$ is below the graph of $e^{-\lambda t}$ for small positive values of λ. For large values of λ the exponential has an asymptote at $y = 0$, while $f(\lambda)$ has an asymptote at $y = e^{x_0 - x} < 1$. Hence there is a solution $\lambda = \sqrt{2E(x_0, x, t)}$ only for small values of $t > 0$. See Figure 10.3.

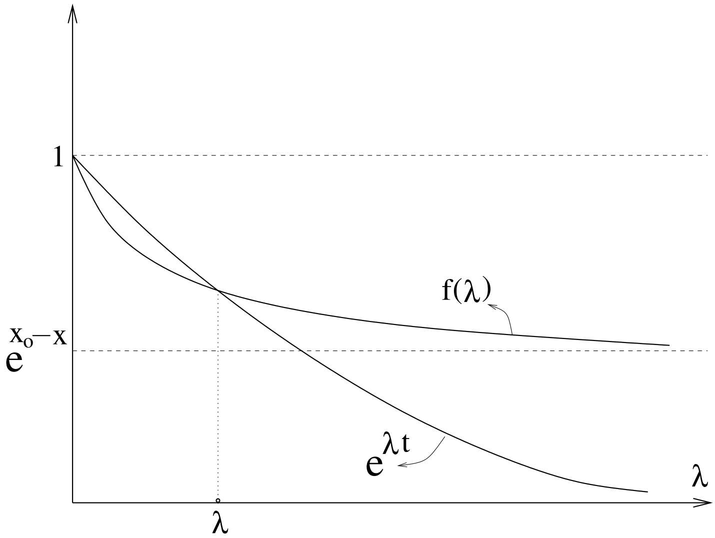


Figure 10.3: The functions $f(\lambda)$ and $e^{\lambda t}$ in the case $x_0 < x$.

A fundamental solution is provided by Theorem 10.30. In this case there is only one term in the sum

$$K = V(t,x)e^{-\frac{1}{2}S},$$

with $\partial_t S = -E = -\frac{1}{2}\lambda^2$. The function $V(t,x)$ satisfies

$$\partial_t V - V_{xx} + \sqrt{\lambda^2 + 2e^x}\, V_x + \frac{1}{2} S_{xx} V = 0,$$

where we used that $\dot{x} = \sqrt{2E + 2e^x} = \sqrt{\lambda^2 + 2e^x}$. The function $\lambda = \lambda(x_0, x, t)$ depends on x and t. This makes the above equation almost impossible to solve.

10.10.7 Physical interpretation

One way to look at the equation

$$u_t - U(x)u_x = \epsilon u_{xx} \tag{10.10.68}$$

is to think of it as the parabolic regularization of the transport equation

$$u_t - U(x)u_x = 0. \tag{10.10.69}$$

For equation (10.10.69), one can define the characteristic $x = x(t)$ by $\dfrac{dx(t)}{dt} = -U(x(t))$. Then equation (10.10.69) is $\dfrac{d(u(x(t),t)}{dt} = 0$.

Another way to look at it is to consider the viscous conservation laws

$$w_t + f(w)_x = \epsilon w_{xx}. \tag{10.10.70}$$

The corresponding hyperbolic conservation law is

$$w_t + f(w)_x = 0, \tag{10.10.71}$$

where $f(w)$ is called a flux function. In many physical situations w is a vector. For example, the famous Euler's equation of compressible fluids. In Euler's equation the vector $w = (\rho, v, E)$. Here ρ is the density, v is the velocity, and E is the total energy (kinetic energy and internal energy). In this case, equation (10.10.71) denotes the conservation of mass, momentum, and energy. In equation (10.10.71), some important physical effects such as viscosity and heat-conductivity are ignored, because in general they are small. The more physically realistic equation is (10.10.70), which takes account of those physical effects.

One may consider the linearized form of an equation around a specific solution. For example, let W be a specific solution of (10.10.70). Let u be the small perturbation, *i.e.*, $u = w - W$. So u satisfies the equation

$$u_t + (f(w) - f(W))_x = \epsilon u_{xx}. \tag{10.10.72}$$

Write $f(w) - f(W) = f'(W)u + Q(u, W)$. Then $Q(u, W)$ is a high order term of u. So equation (10.10.720 can be written as

$$u_t + (f'(W)u)_x = \epsilon u_{xx} - Q_x. \tag{10.10.73}$$

The corresponding linearized equation is

$$u_t + (f'(W)u)_x = \epsilon u_{xx}. \tag{10.10.74}$$

In order to understand the behavior of solutions of (10.10.73), it is very important to understand the Green function of linearized equation (10.10.74).

When W is a travelling wave solution of (10.10.70) of the form $W(\frac{x - st}{\epsilon})$, there is an extensive study of the Green function of (10.10.74). See the references [27], [26], [25], [46], [48], [7], [32], [9].

In the particular case when $\epsilon = 1$, and the flux is $f(w) = -U(x)w(x,t)$, the equation (10.10.71) becomes

$$w_t - \big(U(x)w\big)_x = w_{xx}. \tag{10.10.75}$$

If one sets

$$u(x,t) = \int_{-\infty}^{x} w(y,t)\,dy,$$

then integrating the equation (10.10.75) yields

$$u_t - U(x)u_x = u_{xx},$$

i.e., $Pu = 0$, with $P = \partial_t - \partial_x^2 - U(x)\partial_x$.

10.11 Exercises

1. Prove (ii) of Theorem 10.3 (see [29], p. 50).

2. Show that the fundamental solution for the heat equation on $\mathbb{R}^n$ has the following properties:
(i) $K(x, y, t) = K(y, x, t) \geq 0$,
(ii) $\int_{\mathbb{R}^n} K(x, y, t)\, dy = 1$,
(iii) $\int_{\mathbb{R}^n} K(x, z, t)K(z, y, s)\, dz = K(x, y, t + s)$,
(iv) $\lim\limits_{t \searrow 0} \displaystyle\int_{\mathbb{R}^n} K(x, y, t)\phi(y)\, dy = \phi(x)$, for any ϕ compact supported smooth function.

3. Using $e^{\varphi} = \sum_n \frac{\varphi^n}{n!}$ and a formula for $\Delta\varphi^n$, prove formula (10.2.4).

4. (i) Let $(E_j)_{j\geq 1}$ be the energies provided by Theorem 10.21. Given $x_0 = x(0)$ and $x = x(t)$, show that the solution $x(s)$ of the Hamiltonian system is given implicitly by

$$\mathrm{cn}(2^{3/4} a E_j^{1/4} s) = \frac{2a^2\sqrt{2E_j}x_0 x(s) + \sqrt{(2E_j - a^4x_0^4)(2E_j - a^4x^4(s))}}{4E_j - (\sqrt{2E_j} - a^2x_0^2)(\sqrt{2E_j} - a^2x^2(s))}.$$

(ii) Assume $x_0 = 0$ and find an explicit formula for $x(s)$ in terms of the energies $(E_j)_{j\geq 1}$.

5. Let K be given as in Theorem 10.8. Show that
(i) $\lim\limits_{\tau \searrow 0} K(\,\cdot\,, x, t, \tau) = \delta_{(x,t)}$,
(ii) $\lim\limits_{\tau \searrow 0} K(\,\cdot\,, (y, \sigma)^{-1} \circ_H (x, t), \tau) = \delta_{(x-y)}\delta_{(t-\sigma)}$.

6. Let M be a compact Riemannian manifold and let $\varphi : (M, g) \to \mathbb{R}^m$ be an isometric immersion. If there are $p, q \geq 1$ integers such that

$$\varphi = \phi_0 + \sum_{j=p}^{q} \varphi_j,$$

with $\Delta\varphi_j = \lambda_j\varphi_j$, and $\lambda_j \in \mathbb{R}$ is the j-th eigenvalue, then (M, g) is called a submanifold of $\mathbb{R}^m$ of finite type.

a) Show that φ_0 is the center of mass of (M, g), *i.e.*,

$$\varphi_0 = \frac{1}{\mathrm{vol}(M)} \int_M \varphi\, dv.$$

b) Show that $\int_M \varphi_j \varphi_k \, dv = 0$ for $j \neq k$.

c) If M is a 1-dimensional submanifold of $\mathbb{R}^2$ (a curve), then show that M is a piece of a line or arc of a circle.

d) If M is a closed plane curve in $\mathbb{R}^2$, then its type is finite if and only if M is a circle.

e) Show that the Euclidean sphere $\mathbb{S}^n(r)$ is a submanifold on $\mathbb{R}^{n+1}$ of finite type. Show that if j is the inclusion, then $\Delta j = \frac{n}{r^2} j$. What is the type?

7. (Getzler) Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a positive definite matrix. Show that the heat kernel of the harmonic oscillator $-\Delta + \langle Ax, x\rangle$ is

$$K(x, y, t, 0) = \frac{1}{(4\pi t)^{n/2}} \sqrt{\det C}\, e^{-\frac{1}{4}\Big(\langle Bx, x\rangle + \langle By, y\rangle - 2\langle Cx, y\rangle\Big)},$$

with

$$B = \frac{2\sqrt{A}t}{\tanh 2\sqrt{A}t}, \qquad C = \frac{2\sqrt{A}t}{\sinh 2\sqrt{A}t}, \qquad t > 0.$$

8. (Hörmander) Let $\Omega \in M_{n\times m}(R)$ be a skew symmetric matrix and denote $i = \sqrt{-1}$. Using the technique presented in this chapter show that the heat kernel of the operator

$$L = -\sum_{j=1}^{n} \Big(-\partial_{x_j} - i \sum_{k=1}^{n} \Omega_{jk} x^k \Big)^2$$

is

$$K(x, y, t, 0) = \frac{1}{(4\pi t)^{n/2}} \sqrt{\det A}\, e^{-\frac{1}{4t}\Big(\langle B(x-y), x-y\rangle + 4it\langle \Omega x, y\rangle\Big)},$$

where

$$A = \frac{2|\Omega|t}{\sinh 2|\Omega|t}, \quad B = \frac{2|\Omega|t}{\tanh 2|\Omega|t}, \quad |\Omega| := \sqrt{-\Omega^2}.$$

(see Hörmander [22], p. 158).

11

Fundamental Solutions for Elliptic Operators

11.1 Fundamental solutions for Laplace operators

In this chapter we shall find a formula for the fundamental solution of the Laplace operator on radially symmetric spaces. We recall the formulas for the action and energy along a geodesic which joins the points x_0 and x within time τ. The action is given by $S = \dfrac{d(x_0, x)^2}{2\tau}$ and satisfies the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial \tau} + H(\nabla S) = 0,$$

where the Hamiltonian $H(\nabla S) = E$ is constant along the geodesic and equal to the energy. Hence

$$E = -\frac{\partial S}{\partial \tau} = \frac{d(x_0, x)^2}{2\tau^2}.$$

We note that the quotient $\dfrac{E}{S} = \dfrac{1}{\tau}$ is independent of the end points x_0 and x.

11.2 The transport operator

Definition 11.1 *The transport operator is defined as $T : \mathcal{F}(\mathbb{R} \times M) \to \mathcal{F}(\mathbb{R} \times M)$,*

$$T = \frac{\partial}{\partial \tau} + \nabla S, \tag{11.2.1}$$

where ∇ stands for the gradient and S is the action along a geodesic $c : [0, \tau] \to M$ with endpoints $x_0 = c(0)$, $x = c(\tau)$.

This means that if $f \in \mathcal{F}(\mathbb{R} \times M)$, then

$$Tf = \frac{\partial f}{\partial \tau} + \nabla S(f) = \frac{\partial f}{\partial \tau} + g\big(\nabla S, \nabla f\big).$$

The following result shows that T is the derivation with respect to the parameter τ.

Theorem 11.2. *Let $v : \mathbb{R} \times M \to \mathbb{R}$ be a smooth function. Then*

$$T v = \frac{d}{d\tau} v\big(\tau, c(\tau)\big). \tag{11.2.2}$$

Proof. The chain rule yields

$$\begin{aligned} \frac{d}{d\tau} v\big(\tau, c(\tau)\big) &= \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial x_i} \dot{c}^i(\tau) \\ &= \frac{\partial v}{\partial \tau} + g_{ki}\, g^{kj} \frac{\partial v}{\partial x_j} \dot{c}^i(\tau) \\ &= \frac{\partial v}{\partial \tau} + g_{ki} (\nabla v)^k \dot{c}^i(\tau) \\ &= \frac{\partial v}{\partial \tau} + g\big(\nabla v, \dot{c}(\tau)\big). \end{aligned}$$

Using the relation $\dot{c} = \nabla S$ yields

$$g\big(\nabla v, \dot{c}(\tau)\big) = g\big(\nabla v, \nabla S\big) = \nabla S(v),$$

by the definition of the gradient. Hence

$$\frac{d}{d\tau} v\big(\tau, c(\tau)\big) = \frac{\partial v}{\partial \tau} + \nabla S(v) = T(v).$$

■

11.3 Properties of the transport operator

Proposition 11.3 *The operator T acts as a derivation*

(i) $T(u + v) = T(u) + T(v)$,
(ii) $T(u\, v) = u\, T(v) + v\, T(u), \quad \forall u, v \in \mathcal{F}(\mathbb{R} \times M)$.

Proof. As T is the sum of two derivations,

$$\begin{aligned} T(u\, v) &= \frac{\partial}{\partial \tau}(u\, v) + \nabla S(u\, v) \\ &= u \frac{\partial}{\partial \tau} v + v \frac{\partial}{\partial \tau} u + u\, \nabla S(v) + v\, \nabla S(u) \\ &= u\, T v + v\, T u. \end{aligned}$$

■

The following proposition deals with the eigenfunctions of the transport operator. Let S be the action function.

Proposition 11.4 *We have*

(i) $TS = \dfrac{1}{\tau} S,$

(ii) $T\Big(\dfrac{1}{S}\Big) = -\dfrac{1}{\tau}\dfrac{1}{S},$

(iii) *In general, for any* $n \in \mathbb{Z}$ *we have* $TS^n = \dfrac{n}{\tau} S^n.$

Proof. (i) Using the Hamilton–Jacobi equation,

$$\begin{aligned} TS &= \frac{\partial S}{\partial \tau} + g\big(\nabla S, \nabla S\big) = \frac{\partial S}{\partial \tau} + |\nabla S|^2 \\ &= \underbrace{\frac{\partial S}{\partial \tau} + \frac{1}{2}|\nabla S|^2}_{=0} + \frac{1}{2}|\nabla S|^2 = \frac{1}{2}|\nabla S|^2 \\ &= \frac{1}{2}|\dot{c}|^2 = E = \frac{d(x_0, x)^2}{2\tau^2} = \frac{S}{\tau}. \end{aligned}$$

(ii) Applying T to $1 = S\dfrac{1}{S}$ and using that T acts as a derivation yields

$$\begin{aligned} 0 = T\Big(S\frac{1}{S}\Big) &= S\,T\Big(\frac{1}{S}\Big) + \frac{1}{S}\,TS \\ &= S\,T\Big(\frac{1}{S}\Big) + \frac{1}{S}\frac{1}{\tau}S = S\,T\Big(\frac{1}{S}\Big) + \frac{1}{\tau}. \end{aligned}$$

Hence

$$S\,T\Big(\frac{1}{S}\Big) = -\frac{1}{\tau} \Longrightarrow T\Big(\frac{1}{S}\Big) = -\frac{1}{\tau}\frac{1}{S}.$$

(iii) Using (i), (ii) and the definition of T, we have

$$\begin{aligned} TS^n &= \frac{\partial S^n}{\partial \tau} + g\big(\nabla S, \nabla S^n\big) \\ &= nS^{n-1}\bigg(\frac{\partial S}{\partial \tau} + g\big(\nabla S, \nabla S\big)\bigg) = nS^{n-1}T(S) \\ &= nS^{n-1}\,\frac{S}{\tau} = \frac{n}{\tau}S^n. \end{aligned}$$

∎

Remark 11.5 *The set* $\big(S^{\pm n}\big)_{n\geq 1}$ *are eigenfunctions for the operator* T *with the corresponding eigenvalues* $\Big(\dfrac{\pm n}{\tau}\Big)_{n\geq 1}$.

11.4 The homogeneous transport equation

Consider the homogeneous equation $Tv = 0$. We shall look for a solution as a linear combination of powers of S,

$$v = \sum_{n\geq 1} a_n(\tau)S^n + \sum_{n\geq 1} b_n(\tau)\frac{1}{S^n}.$$

Using the properties of T yields

$$\begin{aligned} Tv &= \sum_{n\geq 1}[a_n(\tau)TS^n + S^n a_n'(\tau) + b_n(\tau)T(S^{-n}) + b_n'(\tau)S^{-n}] \\ &= \sum_{n\geq 1}[a_n(\tau)\frac{n}{\tau}S^n + S^n a_n'(\tau) + b_n(\tau)\Big(\frac{-n}{\tau}\Big)S^{-n} + b_n'(\tau)S^{-n}] \\ &= \sum_{n\geq 1}[a_n(\tau)\frac{n}{\tau} + a_n'(\tau)]S^n + \sum_{n\geq 1}[b_n'(\tau) - \frac{n}{\tau}b_n(\tau)]\frac{1}{S^n}. \end{aligned}$$

In order to have $Tv = 0$, it suffices to choose the coefficients $a_n(\tau)$, $b_n(\tau)$ such that the following ODEs are satisfied:

$$\begin{aligned} a_n'(\tau) &= -\frac{n}{\tau}\, a_n(\tau), \\ b_n'(\tau) &= \frac{n}{\tau}\, b_n(\tau). \end{aligned}$$

Integrating yields the solutions

$$\begin{aligned} a_n(\tau) &= C_n\tau^{-n}, \\ b_n(\tau) &= \widetilde{C}_n\tau^n, \end{aligned}$$

with $C_n, \widetilde{C_n} \in \mathbb{R}$ constants. Hence

$$\begin{aligned} v(\tau, x) &= \sum_{n\geq 1}[C_n\tau^{-n}S^n + \widetilde{C}_n\tau^n S^{-n}] \\ &= \sum_{n\geq 1}\Big[C_n\Big(\frac{S}{\tau}\Big)^n + \widetilde{C}_n\Big(\frac{\tau}{S}\Big)^n\Big] \\ &= \sum_{n\geq 1}[C_n E^n + \widetilde{C}_n E^{-n}], \end{aligned}$$

where E is the energy along the geodesics between x_0 and x within time τ. Hence $v = f(E)$, where f is a function, which has a Laurent series expansion at $E = 0$. As a consequence, we have the following result.

Proposition 11.6 (i) *T is E-homogeneous, i.e.,*

$$T(Ew) = ET(w), \quad \forall w \in \mathcal{F}(M).$$

(ii) *In general, T is $f(E)$-homogeneous where f is a function which has Laurent expansion around zero.*

Proof. (i) As T is a derivation,

$$T(Ew) = ET(w) + wT(E) = ET(w).$$

Replacing E by $f(E)$ yields (ii). ∎

11.5 The non-homogeneous transport equation

Consider the non-homogeneous equation $Tv = h$, where h has an expansion of the form

$$h(\tau, x) = \sum_{n\geq 1} [\alpha_n(\tau)S^n + \beta_n(\tau)\frac{1}{S^n}].$$

Looking for a solution of the form

$$v = \sum_{n\geq 1} a_n(\tau)S^n + \sum_{n\geq 1} b_n(\tau)\frac{1}{S^n}$$

yields

$$\sum_{n\geq 1}[a_n(\tau)\frac{n}{\tau} + a_n'(\tau)]S^n + \sum_{n\geq 1}[b_n'(\tau) - \frac{n}{\tau}b_n(\tau)]\frac{1}{S^n} = \sum_{n\geq 1}[\alpha_n(\tau)S^n + \beta_n(\tau)\frac{1}{S^n}].$$

It suffices to choose the coefficients $a_n(\tau)$ and $b_n(\tau)$ such that the following linear ODEs are satisfied:

$$a_n'(\tau) + \frac{n}{\tau}\, a_n(\tau) = \alpha_n(\tau),$$
$$b_n'(\tau) - \frac{n}{\tau}\, b_n(\tau) = \beta_n(\tau).$$

The integrand factors of the above equations are $\mu(\tau) = \tau^{\pm n}$. Integrating, we obtain the solutions

$$a_n(\tau) = \tau^{-n} \int \tau^n \alpha_n(\tau)\, d\tau,$$
$$b_n(\tau) = \tau^{n} \int \tau^{-n} \beta_n(\tau)\, d\tau.$$

Substituting back in the expression of v yields

$$
\begin{aligned}
v &= \sum_{n\ge 1} a_n(\tau) S^n + \sum_{n\ge 1} b_n(\tau)\frac{1}{S^n} \\
&= \sum_{n\ge 1} S^n \tau^{-n} \int \tau^n \alpha_n(\tau)\, d\tau + \sum_{n\ge 1} \frac{1}{S^n}\tau^n \int \tau^{-n}\beta_n(\tau)\, d\tau \\
&= \sum_{n\ge 1} E^n \int \tau^n \alpha_n(\tau)\, d\tau + \sum_{n\ge 1} \frac{1}{E^n} \int \tau^{-n}\beta_n(\tau)\, d\tau.
\end{aligned}
$$

In the case when $\alpha_n(\tau) = \beta_n(\tau) = 0$ the integrals in the above formula are replaced by constants C_n and $\widetilde{C}_n$.

11.6 Fundamental solution

The following lemmas will be useful in our study. They hold true on any Riemannian space (M, g).

Lemma 11.7 *Let S be the action. Then for any $\alpha \in \mathbb{R}$, we have*

$$
\Delta S^\alpha = \alpha S^{\alpha-1}\Delta S + 2\alpha \frac{\partial}{\partial\tau}\Big(S^{\alpha-1}\Big). \tag{11.6.3}
$$

Proof. Lemma 2.27 yields

$$
\begin{aligned}
\Delta S^\alpha &= -\alpha S^{\alpha-2}\Big(- S\Delta S + (\alpha-1)|\nabla S|^2\Big) \\
&= \alpha S^{\alpha-1}\Delta S - \alpha(\alpha-1)S^{\alpha-2}|\nabla S|^2.
\end{aligned} \tag{11.6.4}
$$

From the Hamilton–Jacobi equation we have

$$
-\frac{1}{2}|\nabla S|^2 = \frac{\partial S}{\partial \tau}. \tag{11.6.5}
$$

Multiplying (11.6.5) by $2\alpha(\alpha-1)S^{\alpha-2}$ yields

$$
\begin{aligned}
\alpha(\alpha-1)S^{\alpha-2}|\nabla S|^2 &= 2\alpha(\alpha-1)S^{\alpha-2}\frac{\partial S}{\partial\tau} \\
&= 2\alpha\frac{\partial}{\partial\tau}\Big(S^{\alpha-1}\Big).
\end{aligned}
$$

Substituting in (11.6.4) yields (11.6.3). ■

Lemma 11.8 *Let S be the action. Then for any $\alpha \in \mathbb{R}$ and $v \in \mathcal{F}(\mathbb{R}\times M)$, we have*

$$
\Delta(vS^\alpha) = S^\alpha \Delta v - 2\alpha S^{\alpha-1}\Big(Tv - \frac{1}{2}(\Delta S)v\Big) - 2\alpha\frac{\partial}{\partial\tau}\Big(vS^{\alpha-1}\Big), \tag{11.6.6}
$$

where T is the transport operator.

Proof. Lemma 2.24 yields

$$\Delta(vu) = u\Delta v + v\Delta u - 2g\big(\nabla v, \nabla u\big).$$

Substituting $u = S^\alpha$, and using Lemma 11.7 yields

$$\begin{aligned}
\Delta(vS^\alpha) &= S^\alpha \Delta v + v\Delta S^\alpha - 2g\big(\nabla v, \nabla S^\alpha\big)\\
&= S^\alpha \Delta v + \alpha S^{\alpha-1} v\Delta S + 2\alpha v \frac{\partial}{\partial \tau}\Big(S^{\alpha-1}\Big) - 2g\big(\nabla v, \alpha S^{\alpha-1}\nabla S\big)\\
&= S^\alpha \Delta v + \alpha S^{\alpha-1} v\Delta S + 2\alpha \frac{\partial}{\partial \tau}(vS^{\alpha-1})\\
&\quad -2\alpha S^{\alpha-1}\frac{\partial v}{\partial \tau} - 2\alpha S^{\alpha-1} g\big(\nabla v, \nabla S\big)\\
&= S^\alpha \Delta v + 2\alpha S^{\alpha-1}\Big[\frac{1}{2}(\Delta S)v - \frac{\partial v}{\partial \tau} - g(\nabla v, \nabla S)\Big] + 2\alpha\frac{\partial}{\partial \tau}\big(vS^{\alpha-1}\big)\\
&= S^\alpha \Delta v + 2\alpha S^{\alpha-1}\Big[\frac{1}{2}(\Delta S)v - Tv\Big] + 2\alpha\frac{\partial}{\partial \tau}\big(vS^{\alpha-1}\big)\\
&= S^\alpha \Delta v - 2\alpha S^{\alpha-1}\Big[Tv - \frac{1}{2}(\Delta S)v\Big] + 2\alpha\frac{\partial}{\partial \tau}\big(vS^{\alpha-1}\big).
\end{aligned}$$

■

In the following we shall assume that the space (M, g) is radially symmetric, *i.e.*, $h(\tau) = \Delta S(\tau)$ depends only on the parameter τ. Consider the function

$$F = \frac{Ew}{S^q},$$

where S and E denote the action and the energy, while w is a function with properties specified later. The following computations take place for $x \neq x_0$. Applying Lemma 11.8 with $v = Ew$ and $\alpha = q$ yields

$$\begin{aligned}
\Delta F &= \frac{1}{S^q}\Delta(Ew) + \frac{2q}{S^{q+1}}\Big[T(Ew) - \frac{1}{2}(\Delta S)(Ew)\Big] + 2q\frac{\partial}{\partial \tau}\Big(\frac{Ew}{S^{q+1}}\Big)\\
&= \frac{1}{S^q}\Delta(Ew) + \frac{2qE}{S^{q+1}}\Big[T(w) - \frac{1}{2}(\Delta S)(w)\Big] + 2q\frac{\partial}{\partial \tau}\Big(\frac{Ew}{S^{q+1}}\Big),
\end{aligned}$$

where we used that T is E-homogeneous. Assuming that any geodesic is infinitely extendible, we may integrate in τ between $-\infty$ and $+\infty$,

$$\int_{-\infty}^{\infty} \Delta F\, d\tau = \int_{-\infty}^{\infty} \frac{1}{S^q}\Delta(Ew) + \frac{2qE}{S^{q+1}}[Tw - h(\tau)w] + 2q\frac{Ew}{S^{q+1}}\bigg|_{\tau=-\infty}^{\tau=+\infty}. \quad (11.6.7)$$

Comparing with the fundamental singularity computed in section 7.4.2, we shall choose q such that $\frac{1}{S^q} \sim \frac{1}{d^{n-2}}$. Since $S = \frac{d^2}{2\tau^2}$, it follows that $2q = n - 2$, *i.e.*, $q = \frac{n}{2} - 1$.

We shall assume that the function w satisfies the following three conditions:

(i) $\Delta(Ew) = 0$;

(ii) $Tw = h(\tau)w$;

(iii) $\frac{Ew}{S^{n/2}}$ vanishes at $\tau = \pm\infty$.

Then

$$K(x_0, x) = \int_{-\infty}^{\infty} \frac{Ew}{S^{\frac{n}{2}-1}}\, d\tau$$

is a fundamental solution, because

$$\Delta K(x_0, x) = \int_{-\infty}^{\infty} \Delta\Big(\frac{Ew}{S^{\frac{n}{2}-1}}\Big)\, d\tau = \int_{-\infty}^{\infty} \Delta F\, d\tau = 0, \quad \text{for} \quad x \neq x_0.$$

In the following we shall find a function $w = w(\tau, x)$ satisfying properties $(i)-(iii)$.

We start solving equation (ii) and employ an expansion for w in Laurent series in the argument S,

$$w = \sum_{n\geq 0} \Big[\alpha_n(\tau)S^n + \beta_n(\tau)\frac{1}{S^n}\Big].$$

Using the properties of the transport operator T yields

$$\begin{aligned} Tw &= \sum_{n\geq 0} \Big[\alpha_n'(\tau)S^n + \alpha_n(\tau)T(S^n) + \beta_n'(\tau)\frac{1}{S^n} + \beta_n(\tau)T\Big(\frac{1}{S^n}\Big)\Big] \\ &= \sum_{n\geq 0} \Big[\alpha_n'(\tau)S^n + \frac{n}{\tau}\alpha_n(\tau)S^n + \beta_n'(\tau)\frac{1}{S^n} - \frac{n}{\tau}\beta(\tau)\frac{1}{S^n}\Big] \\ &= \sum_{n\geq 0} \Big[\alpha_n'(\tau) + \frac{n}{\tau}\alpha_n(\tau)\Big]S^n + \sum_{n\geq 0} \Big[\beta_n'(\tau) - \frac{n}{\tau}\beta_n(\tau)\Big]\frac{1}{S^n}. \end{aligned}$$

Comparing with

$$h(\tau)w = \sum_{n\geq 0} h(\tau)\alpha_n(\tau)S^n + \sum_{n\geq 0} h(\tau)\beta_n(\tau)\frac{1}{S^n}$$

yields

$$\begin{aligned} \alpha_n'(\tau) - \Big(h(\tau) - \frac{n}{\tau}\Big)\alpha_n(\tau) &= 0, \\ \beta_n'(\tau) - \Big(h(\tau) + \frac{n}{\tau}\Big)\beta_n(\tau) &= 0, \end{aligned}$$

which are linear ODEs with the integrand factors $\mu = \tau^{\pm n} e^{-\int h(\tau)\, d\tau}$. Integrating, we obtain the solutions

$$\alpha_n(\tau) = \frac{C_{1,n}}{\tau^n} e^{\int h(\tau)\,d\tau},$$
$$\beta_n(\tau) = C_{2,n}\tau^n e^{\int h(\tau)\,d\tau},$$

with $C_{1,n}, C_{2,n} \in \mathbb{R}$ constants. Hence

$$\begin{aligned} w &= \sum_{n\geq 0}\Big[\alpha_n(\tau)S^n + \beta_n(\tau)\frac{1}{S^n}\Big] \\ &= e^{\int h(\tau)\,d\tau}\sum_{n\geq 0}\Big[C_{1,n}\frac{S^n}{\tau^n} + C_{2,n}\frac{\tau^n}{S^n}\Big] \\ &= e^{\int h(\tau)\,d\tau}\sum_{n\geq 0}\Big[C_{1,n}E^n + C_{2,n}\frac{1}{E^n}\Big], \end{aligned}$$

where we used $S = \tau E$. The function $v(\tau) = e^{\int h(\tau)\,d\tau}$ was introduced and studied in Chapter 9, where it was called volume function. Then w is the product between the volume function and a Laurent series in $E = \dfrac{d^2(x_0, x)}{2\tau^2}$. This solves the equation (ii).

We need to choose the constants $C_{1,n}$ and $C_{2,n}$ in the expression of w such that (i) holds. We have

$$Ew = v(\tau)\sum_{n\geq 0}\Big[C_{1,n}E^{n+1} + C_{2,n}\frac{1}{E^{n-1}}\Big].$$

We make Ew dependent on only τ by choosing

$$C_{1,n} = 0,\ n \geq 0, \qquad C_{2,1} \neq 0, \quad C_{2,n} = 0, \ \text{ for } n \neq 1.$$

Hence $Ew = C_{2,1}v(\tau) = C_{2,1}e^{\int h(\tau)\,d\tau}$ is a volume function and hence (i) holds.

We still have to check condition (iii). We have

$$\begin{aligned} \frac{Ew}{S^{n/2}} &= \frac{C_{2,1}\, e^{\int_0^\tau h(u)\,du}}{\left(\frac{d^2}{2\tau}\right)^{n/2}} \\ &= \frac{C_{2,1}(2\tau)^{n/2}\, e^{\int_0^\tau h(u)\,du}}{d^n}. \end{aligned}$$

Hence, we need to employ the following condition on the volume function,

$$\lim_{\tau\to\pm\infty} \tau^{n/2} e^{\int_0^\tau h(u)\,du} = 0. \tag{11.6.8}$$

In the case $h(\tau) = \Delta S < -k^2 < 0$ the condition (11.6.8) holds. Geometrically, the condition $h(\tau) < 0$ corresponds to converging geodesics on the manifold (M, g). We have arrived at the following result.

Theorem 11.9. *Let (M, g) be a radially symmetric space and $S = \dfrac{d^2(x_0, x)}{2\tau}$ be the action and $v(\tau) = Ce^{\int \Delta S\, d\tau}$ be the volume function. If (11.6.8) is satisfied, then the fundamental solution is*

$$K(x_0, x) = \int_{-\infty}^{+\infty} \frac{v(\tau)}{S^{n/2}}\, d\tau. \tag{11.6.9}$$

Corollary 11.10 *Let (M, g) be a radially symmetric space with curvature greater than a positive constant. Then the fundamental solution is given by (11.6.9)*

Proof. On a Riemannian space with positive curvature the geodesics have negative convergence $h(\tau) < -k^2 < 0$ and hence (11.6.8) holds. ∎

11.7 The parametrix

The idea of looking for a parametrix as an expansion of powers of the action S goes back to Hadamard (see [19]). We shall construct a sequence of functions $v_1, v_2, \dots$ depending on τ such that

$$K = \int_{-\infty}^{+\infty} \Big(\frac{v_1}{S} + \frac{v_2}{S^2} + \frac{v_3}{S^3} + \cdots \Big)\, d\tau \tag{11.7.10}$$

is a fundamental solution for the Laplacian Δ on the Riemannian manifold (M, g). In this section, the space (M, g) is not assumed radially symmetric, *i.e.*, ΔS is allowed to be a function of both S and τ. Let

$$F = \frac{v_1}{S} + \frac{v_2}{S^2} + \frac{v_3}{S^3} + \cdots$$

and then

$$\Delta F = \Delta\Big(\frac{v_1}{S}\Big) + \Delta\Big(\frac{v_2}{S^2}\Big) + \Delta\Big(\frac{v_3}{S^3}\Big) + \cdots$$

Lemma 11.8 yields

$$\begin{aligned}
\Delta\Big(\frac{v_1}{S}\Big) &= \frac{1}{S}\Delta v_1 + \frac{2}{S^2}\Big(T v_1 - \frac{1}{2}(\Delta S)v_1\Big) + \frac{\partial}{\partial \tau}\Big(\frac{2v_1}{S^2}\Big),\\
\Delta\Big(\frac{v_2}{S^2}\Big) &= \frac{1}{S^2}\Delta v_2 + \frac{2\cdot 2}{S^3}\Big(T v_2 - \frac{1}{2}(\Delta S)v_2\Big) + \frac{\partial}{\partial \tau}\Big(2\frac{2v_2}{S^3}\Big),\\
\Delta\Big(\frac{v_3}{S^3}\Big) &= \frac{1}{S^3}\Delta v_3 + \frac{2\cdot 3}{S^4}\Big(T v_3 - \frac{1}{2}(\Delta S)v_3\Big) + \frac{\partial}{\partial \tau}\Big(2\frac{3v_3}{S^4}\Big),\\
\Delta\Big(\frac{v_4}{S^4}\Big) &= \frac{1}{S^4}\Delta v_4 + \frac{2\cdot 4}{S^5}\Big(T v_4 - \frac{1}{2}(\Delta S)v_4\Big) + \frac{\partial}{\partial \tau}\Big(2\frac{4v_4}{S^5}\Big),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Therefore

$$\Delta F = \frac{1}{S}\Delta v_1 + \frac{1}{S^2}\Big[\Delta v_2 + 2Tv_1 - (\Delta S)v_1\Big] + \frac{1}{S^3}\Big[\Delta v_3 + 2\cdot 2Tv_2 - 2(\Delta S)v_2\Big] + \frac{1}{S^4}\Big[\Delta v_4 + 2\cdot 3Tv_3 - 3(\Delta S)v_3\Big] + \dots + \frac{\partial}{\partial \tau}\Big(\frac{2v_1}{S^2} + 2\frac{2v_2}{S^3} + 2\frac{3v_3}{S^4} + 2\frac{4v_4}{S^5} + \dots\Big).$$

Assume that $2\frac{kv_k}{S^{k+1}}$ vanishes at $\tau = \pm\infty$. Integrating yields

$$\int_{-\infty}^{+\infty} \Delta F\, d\tau = \int_{-\infty}^{+\infty} \Big[\frac{1}{S}\Delta v_1 + \frac{1}{S^2}\Big[\Delta v_2 + 2Tv_1 - (\Delta S)v_1\Big] + \frac{1}{S^3}\Big[\Delta v_3 + 2\cdot 2Tv_2 - 2(\Delta S)v_2\Big] + \frac{1}{S^4}\Big[\Delta v_4 + 2\cdot 3Tv_3 - 3(\Delta S)v_3\Big] + \dots\Big] d\tau = 0,$$

providing $v_1, v_2, v_3, \dots$ satisfies the set of equations

$$(\Sigma)\begin{cases} -\Delta v_1 &= 0,\\ -\Delta v_2 &= 2Tv_1 - (\Delta S)v_1,\\ -\Delta v_3 &= 2\big(2Tv_2 - (\Delta S)v_2\big),\\ -\Delta v_4 &= 3\big(2Tv_3 - (\Delta S)v_3\big),\\ \dots\dots & \dots\dots\dots\dots\dots\dots\\ -\Delta v_{k+1} &= k\big(2Tv_k - (\Delta S)v_k\big),\\ \dots\dots & \dots\dots\dots\dots\dots\dots \end{cases}$$

Theorem 11.11. *Let $v_1, v_2, v_3, \dots$ be functions satisfying the system of equations (Σ) such that $\frac{v_k}{S^{k+1}}$ vanishes at $\tau = \pm\infty$, for all $k \geq 1$. Then the fundamental solution has the expansion*

$$K(x_0, x) = \int_{-\infty}^{+\infty} \Big(\frac{v_1}{S} + \frac{v_2}{S^2} + \frac{v_3}{S^3} + \cdots\Big) d\tau, \quad \forall x \neq x_0, \tag{11.7.11}$$

with the action $S = d^2(x_0, x)/(2\tau)$.

Proof. A formal interchange of Δ and $\int_{-\infty}^{+\infty}$ yields

$$\Delta K(x_0, x) = \Delta \int_{-\infty}^{+\infty} F\, d\tau = \int_{-\infty}^{+\infty} \Delta F\, d\tau = 0,$$

by the choice of v_k's. ∎

11.8 Solving the system (Σ)

We shall solve the system (Σ) in the case when (M, g) is a compact manifold without boundary *i.e.*, $\partial M = \emptyset$. In order to do this we shall use Hopf's lemma and the following result.

Lemma 11.12 *Let S be the action and T be the transport operator. For any $n \geq 0$ we have*

$$TS^n = \frac{n}{2} S^{n-1} |\nabla S|^2. \tag{11.8.12}$$

Proof. The definition of the transport operator and the Hamilton–Jacobi equation yields

$$\begin{aligned}
TS^n &= \frac{\partial}{\partial \tau} S^n + g\Big(\nabla S^n, \nabla S\Big) \\
&= nS^{n-1} \frac{\partial S}{\partial \tau} + nS^{n-1} g\Big(\nabla S, \nabla S\Big) \\
&= nS^{n-1} \left(\frac{\partial S}{\partial \tau} + |\nabla S|^2 \right) \\
&= nS^{n-1} \Bigg(\underbrace{\frac{\partial S}{\partial \tau} + \frac{1}{2} |\nabla S|^2}_{=0} + \frac{1}{2} |\nabla S|^2 \Bigg) \\
&= \frac{n}{2} S^{n-1} |\nabla S|^2.
\end{aligned}$$

■

Applying Hopf's lemma, the first equation of (Σ) yields $v_1 = c_1$, constant. Then the second equation of (Σ) becomes

$$-\Delta v_2 = -\Delta(c_1 S) \Longleftrightarrow -\Delta(v_2 - c_1 S) = 0.$$

Hopf's lemma yields $v_2 = c_1 S + c_2$, with c_2 constant. From Lemma 11.12,

$$T v_2 = c_1 T S + \underbrace{T c_2}_{=0} = c_1 \frac{1}{2} |\nabla S|^2$$

and hence the third equation of (Σ) becomes

$$\begin{aligned}
-\Delta v_3 &= 2\Big(c_1 |\nabla S|^2 - (\Delta S)(c_1 S + c_2)\Big) \\
&= c_1 \Big(2|\nabla S|^2 - 2S \Delta S\Big) - \Delta S(2c_2) \\
&= -c_1 \Delta S^2 - 2c_2 \Delta S \\
&= -\Delta(c_1 S^2 + 2c_2 S).
\end{aligned}$$

Hence

$$-\Delta(v_3 - c_1 S^2 - 2c_2 S) = 0 \Longrightarrow v_3 = c_1 S^2 + 2c_2 S + c_3,$$

where c_3 is a constant. Using Lemma 11.12 yields

$$\begin{aligned} T v_3 &= c_1 T S^2 + 2c_2 T S + T c_3 \\ &= c_1 S|\nabla S|^2 + c_2 |\nabla S|^2. \end{aligned}$$

The right side of the fourth equation of (Σ) becomes

$$\begin{aligned} &3\Big(2T v_3 - (\Delta S) v_3\Big) = 3\Big(2c_1 S|\nabla S|^2 + 2c_2|\nabla S|^2 - (\Delta S)(c_1 S^2 + 2c_2 S + c_3)\Big) \\ &= -c_1(3S^2\Delta S - 3\cdot 2S|\Delta S|^2) - 3c_2(2S\Delta S - 2|\nabla S|^2) - 3c_3\Delta S \\ &= -c_1\Delta S^3 - 3c_2\Delta S^2 - 3c_3\Delta S \\ &= -\Delta\big(c_1 S^3 + 3c_2 S^2 + 3c_3 S\big). \end{aligned}$$

Then the fourth equation of (Σ) becomes

$$-\Delta v_4 = -\Delta(c_1 S^3 + 3c_2 S^2 + 3c_3 S)$$

and Hopf's lemma yields

$$v_4 = c_1 S^3 + 3c_2 S^2 + 3c_3 S + c_4,$$

where c_4 is a constant. Lemma 11.12 yields

$$\begin{aligned} T v_4 &= c_1 T S^3 + 3c_2 T S^2 + 3c_3 T S + T c_4 \\ &= c_1 \frac{3}{2} S^2|\nabla S|^2 + 3c_2 \frac{2}{2} S|\nabla S|^2 + 3c_3 \frac{1}{2}|\nabla S|^2. \end{aligned}$$

Therefore

$$2T v_4 = 3c_1 S^2|\nabla S|^2 + 6c_2 S|\nabla S|^2 + 3c_3|\nabla S|^2. \tag{11.8.13}$$

We also have

$$(\Delta S) v_4 = c_1 S^3\Delta S + 3c_2 S^2\Delta S + 3c_3 S\Delta S + c_4\Delta S. \tag{11.8.14}$$

Subtracting (11.8.13) and (11.8.14) yields

$$\begin{aligned} 2T v_4 - (\Delta S) v_4 &= c_1(-S^3\Delta S + 3S^2|\nabla S|^2) \\ &\quad +3c_2(-S^2\Delta S + 2|\nabla S|^2) \\ &\quad +3c_3(-S\Delta S + |\nabla S|^2) - c_4\Delta S \\ &= -\frac{1}{4} c_1\Delta S^4 - c_2\Delta S^3 - \frac{3c_3}{2}\Delta S^2 - c_4\Delta S. \end{aligned}$$

The fifth equation of (Σ) becomes

$$-\Delta v_5 = -\Delta(c_1 S^4 + 4c_2 S^3 + 6c_3 S^2 + 4c_4 S)$$

with the solution

$$v_5 = c_1 S^4 + 4c_2 S^3 + 6c_3 S^2 + 4c_4 S + c_5, \qquad c_5 \in \mathbb{R}.$$

Inductively, we obtain the following result.

Proposition 11.13 *There is a sequence of constants* $c_1, c_2, c_3, \dots$ *such that for any* $n \geq 1$ *we have*

$$v_1 = c_1,$$
$$v_{n+1} = \sum_{k=0}^{n} \binom{k}{n} c_{k+1} S^{n-k}. \tag{11.8.15}$$

11.9 Exercises

1. Let $u : M \to \mathbb{R}$ be a smooth function preserved along a geodesic flow with respect to the Riemannian metric g. Show that
(i) $g(\nabla u, \nabla S) = 0$;
(ii) $Tu = 0$;
(iii) Eu and u/E satisfy the equation $Tu = 0$.

2. Let S be the action and E be the energy.
(i) If T is the transport operator, show that $TS = E$.
(ii) Show that $T^n S = 0$, for $n \geq 2$, where $T^1 = T$ and $T^{n+1} = T(T^n)$.

3. Consider the equation $Tv = d^2(x_0, x)$.
(i) Show that $v_p = \dfrac{\tau}{3} d^2(x_0, x)$ is a particular solution.
(ii) Find the general solution of the above equation.

4. Consider the equation $Tv = \dfrac{1}{d^2(x_0, x)}$.
(i) Show that $v_p = -\dfrac{\tau}{d^2(x_0, x)}$ is a particular solution.
(ii) Find the general solution.

5. Consider the radially symmetric space $(\mathbb{R}^n, \delta_{ij})$.
(i) Find the function $h(\tau)$ and the volume function $v(\tau)$ in this case.
(ii) Is the condition (11.6.8) satisfied?
(iii) Can formula (11.6.9) be used to find a fundamental solution of the Laplacian on $\mathbb{R}^n$? Why?

6. Let S be the action and T be the transport operator. Show that $TS^2 = |\nabla S|^2$.

7. What formula (11.7.11) becomes when v_n are given by the formula (11.8.15)?

12

Mechanical Curves

In this chapter we shall describe mechanical curves from the Lagrangian and Hamiltonian point of view. In this way, many geometric properties of these curves will be derived from the variational formalism.

A mechanical curve is a curve described by a particle on which acts an exterior force. For instance the circle, cycloid, hypocycloid, astroid, etc are models of particle trajectories under some exterior forces. A particle on which acts a central force of constant magnitude describes a circle. A point on a circle which rolls on a line, without slipping, describes a cycloid. A point on a circle tangent interior to another circle, which rotates without slipping in the interior of the large circle, describes a hypocycloid.

12.1 The areal velocity

Suppose an object moves in the plane from the point A to point B along a continuous arc $\widehat{AB}$. Let $\mathcal{A}$ be the area swept by the vectorial radius $\overrightarrow{OX}$ with $X \in \widehat{AB}$. We shall consider positive the orientation given by the clock-wise rotation of $\overrightarrow{OX}$. An elementary calculus formula in polar coordinates yields

$$\mathcal{A} = \frac{1}{2}\int_0^{\phi} r^2\, d\phi, \tag{12.1.1}$$

where $r = r(\phi)$ is the length of the vectorial radius and the argument angle $\phi = \angle AOX$.

Written in differential form, we have $d\mathcal{A} = \frac{1}{2}r^2\, d\phi$. Let t be the time parameter. Then

$$\frac{d\mathcal{A}}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt} = \frac{1}{2}r^2\dot{\phi}. \tag{12.1.2}$$

The derivative $d\mathcal{A}/dt$ is called *areal velocity*.

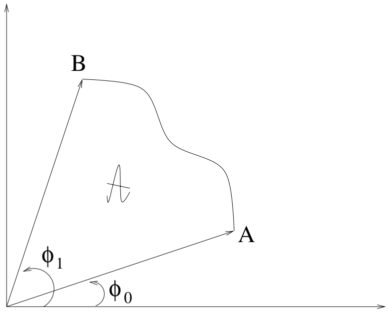


Figure 12.1: The area swept by the vectorial radius between two points.

12.1.0.1 Areal velocity as an angular momentum

Using polar coordinates $x = r\ \cos\phi,\ y = r\ \sin\phi$, the areal velocity becomes

$$\begin{aligned}\frac{d\mathcal{A}}{dt} &= \frac{1}{2}r^2\dot{\phi} = \frac{1}{2}\big(r\cos\phi\, r\cos\phi\,\dot{\phi} - r\sin\phi\,(-r\cos\phi)\,\dot{\phi}\big)\\ &= \frac{1}{2}(x\dot{y} - y\dot{x}).\end{aligned}$$

The expression $x\dot{y} - y\dot{x} = \langle (x, y), (\dot{x}, -\dot{y})\rangle$ is called *angular momentum.*

If $x\dot{y} - y\dot{x}$ is constant, the particle moves such that equal areas are described in equal amounts of time *i.e.*, the vectorial radius sweeps out equal areas in equal time. This happens for instance, in the case of a particle in uniform motion on a circle or in the case of a planet in the revolution motion around the sun (Kepler's second law).

12.2 The circular motion

Consider a particle in the (x, y)-plane which is described by the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - y\dot{x}) \tag{12.2.3}$$

i.e., the particle moves on a trajectory which is an extremizer for the action

$$S = \int L(x, y, \dot{x}, \dot{y}). \tag{12.2.4}$$

The Lagrangian L is the sum of the kinetic energy and the angular momentum.

Theorem 12.1. *The Euler–Lagrange system associated with the Lagrangian* (12.2.3) *is*

$$\begin{cases}\ddot{x} - 2\dot{y} &= 0,\\ \ddot{y} + 2\dot{x} &= 0.\end{cases} \tag{12.2.5}$$

The solutions of the system (12.2.5) *with the boundary conditions*

$$x(0) = \mathbf{x}_0, \qquad x(\tau) = \mathbf{x}, \qquad y(0) = \mathbf{y}_0, \qquad y(\tau) = \mathbf{y}, \tag{12.2.6}$$

with $0 < \tau < \pi$*, are*

$$x(s) = \pm C \sin s \ \sin(s + \alpha_0) + \mathbf{x}_0,$$
$$y(s) = \pm C \sin s \ \cos(s + \alpha_0) + \mathbf{y}_0,$$

with $C = \sqrt{\frac{E}{2}}$ *and the energy* E *given by*

$$E = \frac{(\mathbf{x} - \mathbf{x}_0)^2 + (\mathbf{y} - \mathbf{y}_0)^2}{2\sin^2 \tau}. \tag{12.2.7}$$

Proof. Using

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x} - \dot{y}, \qquad \frac{\partial L}{\partial x} = \dot{y},$$
$$\frac{\partial L}{\partial \dot{y}} = \dot{y} + x, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \ddot{y} + \dot{x}, \qquad \frac{\partial L}{\partial \dot{y}} = -\dot{x},$$

it is easy to see that the Euler–Lagrange system becomes (12.2.5).

Multiplying the first equation of (12.2.5) by $\dot{x}$ and the second by $\dot{y}$ and adding yields $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$, therefore

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 0 \Longrightarrow \dot{x}^2 + \dot{y}^2 = C^2 \tag{12.2.8}$$

where C is a constant along the trajectory. Let $E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ denote the first integral of energy. Using (12.2.8) there is a smooth function $\alpha = \alpha(s)$ such that

$$\dot{x}(s) = \pm C \sin\alpha(s) \Longrightarrow \ddot{x}(s) = \pm C \cos\alpha(s)\,\dot{\alpha}(s),$$
$$\dot{y}(s) = \pm C \cos\alpha(s) \Longrightarrow \ddot{y}(s) = \mp C \sin\alpha(s)\,\dot{\alpha}(s).$$

Substituting back in the system (12.2.5) yields

$$\pm \cos^2\alpha(s)\,\dot{\alpha}(s) = \pm 2\cos^2\alpha(s),$$
$$\mp \sin^2\alpha(s)\,\dot{\alpha}(s) = \mp 2\sin^2\alpha(s).$$

Subtracting we get

$$\dot{\alpha}(s) = 2 \Longrightarrow \alpha(s) = 2s + \alpha_0, \tag{12.2.9}$$

with α_0 constant. Hence

$$\begin{aligned}\dot{x}(s) &= \pm C\sin(2s+\alpha_0) \Longrightarrow x(s) = \pm C\int_0^s \sin(2u+\alpha_0)\,du + \mathbf{x}_0\\ &= \pm C\frac{1}{2}(-\cos w)\Big|_{\alpha_0}^{2s+\alpha_0} + \mathbf{x}_0 = \pm\frac{C}{2}\Big(\cos\alpha_0 - \cos(2s+\alpha_0)\Big) + \mathbf{x}_0\\ &= \pm C\sin s\sin(s+\alpha_0) + \mathbf{x}_0,\end{aligned}$$

where we used

$$\begin{aligned}\cos\alpha_0 - \cos(2s+\alpha_0) &= -2\sin\frac{\alpha_0+2s+\alpha_0}{2}\sin\frac{\alpha_0-2s-\alpha_0}{2}\\ &= 2\sin(s+\alpha_0)\sin s.\end{aligned}$$

Substituting $\alpha(s)$ in the formula for $\dot{y}(s)$ yields

$$\begin{aligned}\dot{y}(s) &= \pm C\cos\alpha(s) = \pm C\cos(2s+\alpha_0)\\ \Longrightarrow y(s) &= \pm C\int_0^s \cos(2u+\alpha_0)\,du + \mathbf{y}_0\\ &= \pm\frac{1}{2}C\sin w\Big|_{\alpha_0}^{2s+\alpha_0} + \mathbf{y}_0\\ &= \pm\frac{1}{2}C\Big(\sin(2s+\alpha_0) - \sin\alpha_0\Big) + \mathbf{y}_0\\ &= \pm C\sin s\cos(s+\alpha_0) + \mathbf{y}_0,\end{aligned}$$

where we used

$$\begin{aligned}\sin(2s+\alpha_0) - \sin\alpha_0 &= 2\sin\frac{2s+\alpha_0-\alpha_0}{2}\cos\frac{2s+\alpha_0+\alpha_0}{2}\\ &= 2\sin s\cos(s+\alpha_0).\end{aligned}$$

Hence we have arrived at

$$x(s) = \pm C\sin s\ \sin(s+\alpha_0) + \mathbf{x}_0, \tag{12.2.10}$$

$$y(s) = \pm C\sin s\ \cos(s+\alpha_0) + \mathbf{y}_0, \tag{12.2.11}$$

where α_0 is a constant. We shall show that energy $E = C^2/2$ does not depend on α_0. Making $s = \tau$ in (12.2.10), (12.2.11) yields

$$\mathbf{x} = \pm C\sin\tau\ \sin(\tau+\alpha_0) + \mathbf{x}_0, \tag{12.2.12}$$

$$\mathbf{y} = \pm C\sin\tau\ \cos(\tau+\alpha_0) + \mathbf{y}_0, \tag{12.2.13}$$

hence it follows that

$$\Big(\frac{\mathbf{x}-\mathbf{x}_0}{\sin\tau}\Big)^2 = C^2\sin^2(\tau+\alpha_0), \tag{12.2.14}$$

$$\Big(\frac{\mathbf{y}-\mathbf{y}_0}{\sin\tau}\Big)^2 = C^2\cos^2(\tau+\alpha_0). \tag{12.2.15}$$

Adding yields

$$2E = C^2 = \frac{(\mathbf{x} - \mathbf{x}_0)^2 + (\mathbf{y} - \mathbf{y}_0)^2}{\sin^2 \tau},$$

which is (12.2.7). ∎

Remark 12.2 *Let* $d = \sqrt{(\mathbf{x} - \mathbf{x}_0)^2 + (\mathbf{y} - \mathbf{y}_0)^2}$ *denote the Euclidean distance between* $(\mathbf{x}_0, \mathbf{y}_0)$ *and* $(\mathbf{x}, \mathbf{y})$. *We note the fact that the energy* $E = \dfrac{d^2}{2\sin^2 \tau}$ *is not Euclidean. Replacing* $\sin^2 \tau$ *by* τ^2 *we obtain the Euclidean energy.*
Let δ *be the Riemannian distance in which the solutions of the Euler–Lagrange equations become geodesics. Then* $E = \dfrac{\delta^2}{2\tau^2}$. *Then* $\delta^2 = \left(\dfrac{\tau}{\sin \tau}\right)^2 d^2$, *and hence* d *and* δ *are homothetic.*

The action

The action $S = S(\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}, \mathbf{y}, \tau)$ satisfies the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial \tau} = -E = -\frac{d^2}{2\sin^2 \tau} = \frac{d^2}{2} \frac{\partial}{\partial \tau}(\cot \tau)$$
$$\Longrightarrow 0 = \frac{\partial}{\partial \tau}\Big(S - \frac{d^2}{2} \cot \tau\Big) \Longrightarrow S = S_0 + \frac{d^2}{2} \cot \tau. \qquad (12.2.16)$$

Proposition 12.3 *The Hamiltonian associated with the Lagrangian* (12.2.3) *is*

$$H(x, y, p_1, p_2) = \frac{1}{2}\big(p_1 + y\big)^2 + \frac{1}{2}\big(p_2 - x\big)^2. \qquad (12.2.17)$$

Proof. The Hamiltonian system for the Hamiltonian (12.2.17) yields

$$\dot{x} = H_{p_1} = p_1 + y \Longrightarrow p_1 = \dot{x} - y,$$
$$\dot{y} = H_{p_2} = p_2 - x \Longrightarrow p_2 = \dot{y} + x.$$

Using the Legendre transform we have

$$\begin{aligned} L &= p_1\dot{x} + p_2\dot{y} - H \\ &= (\dot{x} - y)\dot{x} + (\dot{y} + x)\dot{y} - \frac{1}{2}\big(p_1 + y\big)^2 - \frac{1}{2}\big(p_2 - x\big)^2 \\ &= (\dot{x} - y)\dot{x} + (\dot{y} + x)\dot{y} - \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2 \\ &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x\dot{y} - y\dot{x}. \end{aligned}$$

∎

We note that the Hamiltonian (12.2.17) is the principal symbol of the operator

$$
\begin{aligned}
P &= \frac{1}{2}\big(\partial_x + y\big)^2 + \frac{1}{2}\big(\partial_y - x\big)^2 \\
&= \frac{1}{2}(\partial_x^2 + \partial_y^2) + y\partial_x - x\partial_y + \frac{1}{2}(x^2 + y^2),
\end{aligned}
$$

which describes the circular motion.

12.3 The astroid

The trajectory of a point P on the unit circle which rolls without slipping in the interior of a circle of radius 4 is a hypocycloid with four cuspidal points. This curve is called *astroid*. The equation of the astroid is

$$x^{2/3} + y^{2/3} = 1. \tag{12.3.18}$$

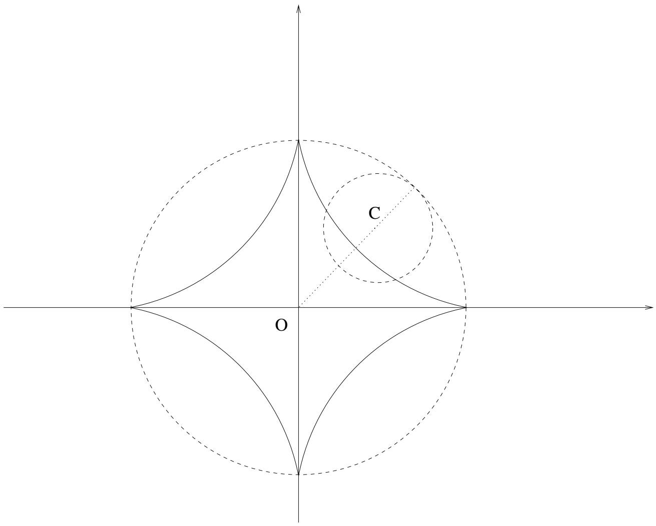


Figure 12.2: The astroid.

If P starts at the cuspidal point $(4, 0)$ and s denotes the angle argument of the center C, we have

$$x(s) = \cos^3 s, \qquad y(s) = \sin^3 s, \tag{12.3.19}$$

which are equivalent with

$$x(s) = 3\cos s + \cos\, 3s, \qquad y(s) = 3\sin s - \sin\, 3s. \tag{12.3.20}$$

A simple computation shows that (12.3.20) is the solution of the system

$$\begin{cases} \ddot{x} - 2\dot{y} + 3x = 0, \\ \ddot{y} + 2\dot{x} + 3y = 0, \end{cases} \tag{12.3.21}$$

with initial conditions

$$x(0) = 4, \quad \dot{x}(0) = \dot{y}(0) = y(0) = 0. \tag{12.3.22}$$

Standard ODE techniques show that the solution (12.3.20) is unique.

Proposition 12.4 *The system* (12.3.21) *is the Euler–Lagrange system associated with the Lagrangian*

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x\dot{y} - \dot{x}y - \frac{3}{2}(x^2 + y^2). \tag{12.3.23}$$

Proof. We have

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x} - \dot{y}, \quad \frac{\partial L}{\partial x} = \dot{y} - 3x,$$

$$\frac{\partial L}{\partial \dot{y}} = \dot{y} + x, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \ddot{y} + \dot{x}, \quad \frac{\partial L}{\partial y} = -\dot{x} - 3y.$$

Then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}$$

yields the system (12.3.21). ■

12.3.0.2 Noether's Theorem

The Lagrangian (12.3.23) is invariant under rotations centered at the origin. The vector field associated with this rotation at the point (x, y) is $(-y, x)$. Noether's theorem yields a first integral of motion given by

$$\begin{aligned} I &= \left\langle \left(\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{y}}\right), (-y, x) \right\rangle = (\dot{x} - y)(-y) + (\dot{y} + x)x \\ &= -\dot{x}y + y^2 + \dot{y}x + x^2 = x^2 + y^2 + x\dot{y} - \dot{x}y \\ &= r^2 + 2\frac{d\mathcal{A}}{ds}, \end{aligned}$$

where $x = r\cos\phi$, $y = r\sin\phi$. We have arrived at the following result.

Proposition 12.5 *For any solution of the system* (12.3.21) *there is a constant C such that*

(i)
$$r^2 + 2\frac{d\mathcal{A}}{ds} = C,$$

(ii)
$$\phi(s) = C\int_0^s \frac{du}{r^2(u)} - s$$

along the solution.

Proof. (i) It clearly follows from the fact that the first integral is constant along the solutions.
(ii) We have

$$\begin{aligned} C &= r^2 + x\dot{y} - \dot{x}y \\ &= r^2 + r^2\dot{\phi} \\ &= r^2(1+\dot{\phi}) \\ \Longrightarrow \frac{d\phi}{ds} &= \frac{C}{r^2} - 1. \end{aligned}$$

Integrating yields the desired result. ■

We can get the same result if we write the Euler–Lagrange system in polar coordinates. See Exercise 2.

As the astroid is a solution of the system (12.3.21), the above proposition applies to it. In this case the constant C is obtained by taking the value at $s = 0$,

$$\begin{aligned} C &= x^2(0) + y^2(0) + x(0)\dot{y}(0) - \dot{x}(0)y(0) \\ &= 16. \end{aligned}$$

Proposition 12.6 *The Hamiltonian associated with the Lagrangian* (12.3.23) *is*

$$H(p_1, p_2, x, y) = \frac{1}{2}[(p_1 + y)^2 + (p_2 - x)^2] + \frac{3}{2}(x^2 + y^2). \tag{12.3.24}$$

Proof. The momenta are $p_1 = \dfrac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad p_2 = \dfrac{\partial L}{\partial \dot{y}} = \dot{y} + x$, and then

$$\dot{x} = p_1 + y, \quad \dot{y} = p_2 - x. \tag{12.3.25}$$

Using (12.3.25), the Legendre transform yields

$$\begin{aligned} H(p_1, p_2, x, y) &= p_1\dot{x} + p_2\dot{y} - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - x\dot{y} + \dot{x}y + \frac{3}{2}(x^2 + y^2) \\ &= p_1(p_1 + y) + p_2(p_2 - x) - \frac{1}{2}\Big((p_1 + y)^2 + (p_2 - x)^2\Big) \\ &\quad -x(p_2 - x) + (p_1 + y)y + \frac{3}{2}(x^2 + y^2) \\ &= \frac{1}{2}(p_1^2 + p_2^2) + p_1 y - p_2 x + 2(x^2 + y^2) \\ &= \frac{1}{2}\Big[(p_1 + y)^2 + (p_2 - x)^2\Big] + \frac{3}{2}(x^2 + y^2). \end{aligned}$$

■

12.3.0.3 The first integral of energy

As
$$\frac{\partial H}{\partial t} = 0 \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{dH}{dt},$$
it follows that H is preserved along the trajectory. The value of H along the trajectory is called *the total energy*. In $x, y, \dot{x}, \dot{y}$ coordinates the energy takes the form

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{3}{2}(x^2 + y^2). \tag{12.3.26}$$

Note that E does not depend on the angular momentum as the Lagrangian does. It depends only on the magnitude of the velocity and the distance to the origin.

12.3.0.4 Physical interpretation

The speed of a particle described by a solution of the Euler–Lagrange system is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

If $r = \sqrt{x^2 + y^2}$ denotes the distance from the origin to the point (x, y), formula (12.3.26) yields
$$v^2 = 2E - 3r^2.$$

In the case of the astroid with the initial conditions (12.3.26) we have $E = 24$. Thus $v = \sqrt{3(16 - r^2)}$ with $r \in [0, 4]$. The speed on the astroid is zero iff $r = 4$, which occurs only at the cuspidal points.

12.4 The cycloid

Consider a particle described by a Lagrangian, which is the sum of the kinetic, angular momentum and potential energy in the x-direction

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x\dot{y} - \dot{x}y) + x. \tag{12.4.27}$$

The Euler–Lagrange system of equations associated with the Lagrangian (12.4.27) is

$$\begin{cases} \ddot{x} - \dot{y} = 1, \\ \ddot{y} + \dot{x} = 0. \end{cases} \tag{12.4.28}$$

If we consider the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 0$$

the solution will be the cycloid

$$x(y) = 1 - \cos t, \qquad y(t) = \sin t - t. \tag{12.4.29}$$

From the mechanical point of view, the cycloid is the trajectory of a point fixed on a circle which rolls without slipping on the real axis.

12.4.0.5 Solving the Euler–Lagrange system (12.4.28)

Set

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad v = \dot{u} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{J}^{-1} = -\mathcal{J}, \quad \mathcal{J}\mathbf{e}_1 = -\mathbf{e}_2, \quad \mathcal{J}\mathbf{e}_2 = \mathbf{e}_1.$$

The system (12.4.28) can be written as

$$\dot{v} - \mathcal{J}v = \mathbf{e}_1. \tag{12.4.30}$$

Multiplying by $e^{\mathcal{J}s}$ yields

$$\frac{d}{ds}(e^{-\mathcal{J}s}v) = e^{-\mathcal{J}s}\mathbf{e}_1.$$

Integrating we obtain

$$\begin{aligned} e^{-\mathcal{J}s}v &= -\mathcal{J}^{-1}e^{-\mathcal{J}s}\mathbf{e}_1 + C_0 = e^{-\mathcal{J}s}\mathcal{J}\mathbf{e}_1 + C_0 \\ &= -e^{-\mathcal{J}s}\mathbf{e}_2 + C_0. \end{aligned}$$

Multiplying by $e^{\mathcal{J}s}$ yields

$$\begin{aligned} \dot{u}(s) &= e^{\mathcal{J}s}C_0 - \mathbf{e}_2 \\ \Longrightarrow u(s) &= \mathcal{J}^{-1}e^{\mathcal{J}s}C_0 - \mathbf{e}_2 s + C_1 \\ &= -\mathcal{J}e^{\mathcal{J}s}C_0 - \mathbf{e}_2 s + C_1. \end{aligned} \tag{12.4.31}$$

The integration constants C_0 and C_1 depend on the boundary conditions: $u(0) = \mathbf{u}_0$, $u(\tau) = \mathbf{u}_1$, where $\tau > 0$. Let $A = e^{\mathcal{J}\tau}$. Making $s = 0$ and $s = \tau$ in the relation (12.4.31), yields

$$\begin{aligned} \mathbf{u}_0 &= -\mathcal{J}C_0 + C_1, \\ \mathbf{u}_1 &= -\mathcal{J}AC_0 - \mathbf{e}_2\tau + C_1. \end{aligned}$$

Subtracting, we eliminate C_1,

$$\begin{aligned} \mathbf{u}_0 - \mathbf{u}_1 &= -\mathcal{J}C_0 + \mathcal{J}AC_0 + \mathbf{e}_2\tau \\ &= -\mathcal{J}(I - A)C_0 + \mathbf{e}_2\tau \\ \Longrightarrow C_0 &= (I - A)^{-1}[J(\mathbf{u}_0 - \mathbf{u}_1) - \mathbf{e}_1\tau]. \end{aligned} \tag{12.4.32}$$

The elimination of C_0 gives us

$$\begin{aligned} \mathbf{u}_1 - A\mathbf{u}_0 &= (I - A)C_1 - \mathbf{e}_2\tau \\ \Longrightarrow C_1 &= (I - A)^{-1}[\mathbf{u}_1 - A\mathbf{u}_0 + \mathbf{e}_2\tau]. \end{aligned} \tag{12.4.33}$$

Substituting (12.4.32) and (12.4.33) back in (12.4.31) yields

$$\begin{aligned} u(s) &= -\mathcal{J}e^{\mathcal{J}s}C_0 - \mathbf{e}_2 s + C_1 \\ &= -\mathcal{J}e^{\mathcal{J}s}(I - A)^{-1}[\mathcal{J}(\mathbf{u}_0 - \mathbf{u}_1) - \mathbf{e}_1\tau] - \mathbf{e}_2 s \\ &\quad +(I - A)^{-1}[\mathbf{u}_1 - A\mathbf{u}_0 + \mathbf{e}_2\tau] \\ &= (I - A)^{-1}[-\mathcal{J}e^{\mathcal{J}s}\mathcal{J}(\mathbf{u}_0 - \mathbf{u}_1) + \mathcal{J}e^{\mathcal{J}s}\mathbf{e}_1\tau + \mathbf{u}_1 - A\mathbf{u}_0 + \mathbf{e}_2\tau] - \mathbf{e}_2\tau \\ &= (I - A)^{-1}[e^{\mathcal{J}s}(\mathbf{u}_0 - \mathbf{u}_1) - e^{\mathcal{J}s}\mathbf{e}_2\tau + \mathbf{u}_1 - A\mathbf{u}_0 + \mathbf{e}_2\tau] - \mathbf{e}_2 s \\ &= (I - A)^{-1}[(e^{\mathcal{J}s} - A)\mathbf{u}_0 + (I - e^{\mathcal{J}s})(\mathbf{u}_1 + \mathbf{e}_2\tau)] - \mathbf{e}_2 s. \end{aligned} \tag{12.4.34}$$

Proposition 12.7 *The solution of the Euler–Lagrange system (12.4.28) with the boundary conditions*

$$x(0) = x_0, \qquad x(\tau) = x_1, \qquad y(0) = y_0, \qquad y(\tau) = y_1$$

is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}(I - \cot\frac{\tau}{2}\mathcal{J})\left[(e^{\mathcal{J}s} - e^{\mathcal{J}\tau})\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + (I - e^{\mathcal{J}s})\begin{pmatrix} x_1 \\ y_1 + \tau \end{pmatrix}\right],$$

where

$$e^{\mathcal{J}s} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are rotations of angle s and $\pi/2$, respectively.

Proof. It follows from formula (12.4.34) and Exercise 3. ∎

Proposition 12.8 *The Hamiltonian associated with the Lagrangian* (12.4.27) *is*

$$H(p_1, p_2, x, y) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(p_1 y - x p_2) + \frac{1}{8}(x^2 + y^2) + x. \tag{12.4.35}$$

Proof. Using

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{x}} = \dot{x} - \frac{1}{2}y, \quad \dot{x} = p_1 + \frac{1}{2}y, \\ p_2 &= \frac{\partial L}{\partial \dot{y}} = \dot{y} - \frac{1}{2}x, \quad \dot{y} = p_2 - \frac{1}{2}x, \end{aligned}$$

the Legendre transform yields the Hamiltonian

$$
\begin{aligned}
H &= p_1\dot{x} + p_2\dot{y} - L \\
&= p_1(p_1 + \frac{1}{2}y) + p_2(p_2 - \frac{1}{2}x) - \frac{1}{2}\Big[(p_1 + \frac{1}{2}y)^2 + (p_2 - \frac{1}{2}x)^2\Big] \\
&\quad -\frac{1}{2}\Big[x(p_2 - \frac{1}{2}x) - y(p_1 + \frac{1}{2}y)\Big] + x \\
&= p_1^2 + \frac{1}{2}p_1 y + p_2^2 - \frac{1}{2}p_2 x - \frac{1}{2}\Big[p_1^2 + p_1 y + \frac{1}{4}y^2 + p_2^2 - p_2 x + \frac{1}{4}x^2\Big] \\
&\quad -\frac{1}{2}(xp_2 - \frac{1}{2}x^2 - p_1 y - \frac{1}{2}y^2) + x \\
&= (p_1^2 + p_2^2) - \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}p_1 y - \frac{1}{2}p_2 x + \frac{1}{2}p_2 x - \frac{1}{2}p_1 y \\
&\quad -\frac{1}{2}(\frac{1}{4}y^2 + \frac{1}{4}x^2) - \frac{1}{2}xp_2 + \frac{1}{4}x^2 + \frac{1}{2}p_1 y + \frac{1}{4}y^2 + x \\
&= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(p_1 y - xp_2) + \frac{1}{8}(x^2 + y^2) + x.
\end{aligned}
$$

■

12.4.0.6 The total energy

As the Hamiltonian does not depend explicitly on the parameter s, H will be constant along the solutions of the Euler–Lagrange equations. Let E be the constant value of H along the solution. Using x, y, $\dot{x}$, $\dot{y}$ coordinates yields

$$
\begin{aligned}
E &= \frac{1}{2}\Big[(\dot{x} - \frac{1}{2}y)^2 + (\dot{y} + \frac{1}{2}x)^2\Big] + \frac{1}{2}\Big[(\dot{x} - \frac{1}{2}y)y - x(\dot{y} + \frac{1}{2}x)\Big] + \frac{1}{8}(x^2 + y^2) + x \\
&= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 - \dot{x}y + \dot{y}x + \frac{1}{4}x^2 + \frac{1}{4}y^2) + \frac{1}{2}(\dot{x}y - \frac{1}{2}y^2 - x\dot{y} - \frac{1}{2}x^2) \\
&\quad + \frac{1}{8}(x^2 + y^2) + x = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x.
\end{aligned}
$$

In particular, as the cycloid is a solution of the Euler–Lagrange equations, it has the energy

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + x. \tag{12.4.36}$$

Using the initial data for the cycloid $x(0) = 0, \dot{x}(0) = \dot{y}(0) = 0$, it follows that $E = 0$. Hence $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) = -x$ along the cycloid, or

$$v = \sqrt{2|x|} \tag{12.4.37}$$

where v is the speed.

12.4.0.7 Galileo's law

A unit mass particle in a gravitational potential with acceleration $g = 1$, situated at a level h above the ground, has the potential energy $U = h$. When the particle is free falling, from the conservation of energy, the initial potential energy is equal to the final kinetic energy *i.e.*, $h = \frac{1}{2}v^2$. The formula for the speed $v = \sqrt{2h}$ is called Galileo's law. Comparing with (12.4.37) yields an important characteristic of the motion on a cycloid:

Two punctiform, unit-mass bodies are released in free gravitational fall, from the same height h, the first on a cycloid and the second vertically. Then at each level the speeds are the same and they will reach the ground with the same speed, $v = \sqrt{2h}$.

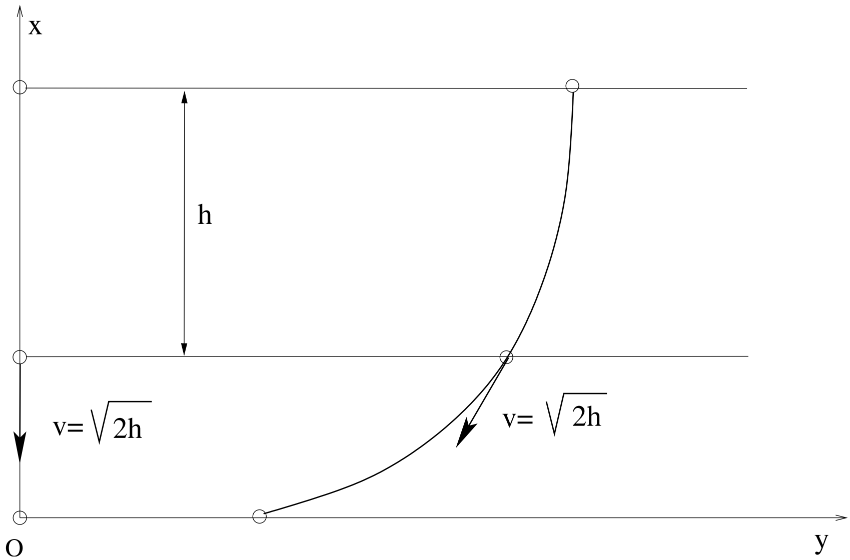


Figure 12.3: The speed at the same level $x = h$ is the same for both unit-mass bodies in free gravitational falling.

12.5 Curves that minimize a potential

Given two points A and B, we are interested in finding a curve in the (x, y)-plane, that joins A and B, and minimizes a given potential $U(y)$ along the trajectory. This means the particle moves such as to minimize the action

$$\int U(y)\, ds, \tag{12.5.38}$$

where $ds = \sqrt{dx^2 + dy^2}$ is the arc element along the curve. Using $ds = \sqrt{1 + y'^2}dx$, the action becomes $\int L(y, y')\, dy$, with the Lagrangian

$$L(y, y') = U(y)\sqrt{1 + y'^2}. \tag{12.5.39}$$

The extremizers of the above action will satisfy the Euler–Lagrange equation, which are provided in the next result.

Theorem 12.9. *Let $U(y) > 0$ be a differentiable potential function for $y > 0$. The Euler–Lagrange equation for the Lagrangian* (12.5.39) *is*

$$y'' = \frac{U'(y)}{U(y)}(1 + y'^2). \tag{12.5.40}$$

The solution $y = y(x)$ satisfies the integral equation

$$\int_{y_0}^{y(x)} \frac{dw}{\sqrt{k^2 U^2(w) - 1}} = x - x_0, \tag{12.5.41}$$

where $y(x_0) = y_0$ and k is a constant. The solutions of the equation (12.5.41) *are the Riemannian geodesics with respect to the metric $d\sigma^2 = U(y)(dx^2 + dy^2)$.*

Proof. We have

$$\frac{\partial L}{\partial y'} = \frac{U(y)y'}{\sqrt{1 + y'^2}}, \qquad \frac{\partial L}{\partial y} = U'(y)\sqrt{1 + y'^2},$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial y'}\Big) = \Big(\frac{U(y)y'}{\sqrt{1 + y'^2}}\Big)' = \frac{1}{1 + y'^2}\Big[\Big(U(y)y'\Big)'\sqrt{1 + y'^2} - U(y)y'(\sqrt{1} + y'^2)'\Big].$$

Then the Euler–Lagrange equation $\dfrac{d}{dt}\dfrac{\partial L}{\partial y'} = \dfrac{\partial}{\partial y}$ becomes

$$\begin{aligned}
\Big(U(y)y'\Big)'(1 + y'^2) - U(y)y'^2y'' &= U'(y)(1 + y'^2)^2 \\
\Longleftrightarrow (1 + y'^2)\Big(U'(y)y'^2 + U(y)y'' - U'(y) - U'(y)y'^2\Big) &= U(y)y'^2y'' \\
\Longleftrightarrow U(y)y'' &= U'(y)(1 + y'^2) \\
\Longleftrightarrow y'' &= \frac{U'(y)}{U(y)}(1 + y'^2).
\end{aligned}$$

In order to solve the equation, let $y' = p$. Then $y'' = \dfrac{dp}{dt} = \dfrac{dp}{dy}y' = \dfrac{dp}{dy}p$. The equation becomes $\dfrac{dp}{dy}p = \dfrac{U'(y)}{U(y)}(1 + p^2)$. Separating the variables yields $\dfrac{p}{1 + p^2}\,dp = \dfrac{U'(y)}{U(y)}\,dy$. Integrating, we obtain

$$\begin{aligned}
\frac{1}{2}\ln(1 + p^2) &= \ln U(y) + C \\
\Longleftrightarrow \sqrt{1 + p^2} &= k\,U(y) \\
\Longleftrightarrow p^2 &= k^2U^2(y) - 1 \\
\Longleftrightarrow y' &= \pm\sqrt{k^2U^2(y) - 1} \\
\Longleftrightarrow \frac{dy}{\sqrt{k^2U^2(y) - 1}} &= \pm dx.
\end{aligned}$$

Integrating yields (12.5.41). ■

In the following we shall consider a few cases in which the integration can be performed explicitly.

12.5.0.8 The gravitational potential

In particular, if $U(y) = y$, the Euler–Lagrange equation is

$$yy'' = 1 + y'^2 \tag{12.5.42}$$

with the solution $y(x)$ satisfying

$$\begin{aligned}
\int \frac{dy}{\sqrt{k^2y^2 - 1}} = x + \mathcal{C} \Longleftrightarrow \frac{1}{k}\int \frac{dy}{\sqrt{y^2 - (1/k)^2}} = x + \mathcal{C},\\
\cosh^{-1}(ky) = kx + \mathcal{C} \Longleftrightarrow ky(x) = \cosh(kx + \mathcal{C}),\\
y(x) \quad = \quad \frac{1}{k}\cosh(kx + \mathcal{C}).
\end{aligned} \tag{12.5.43}$$

This is called the *catenary* curve. The catenary is the shape of the curve that joins two given points and has minimum gravitational potential energy.

12.5.0.9 Minimal surfaces

If we consider the potential $U(y) = 2\pi y$, the action to be minimized is

$$2\pi \int y\sqrt{1 + y'^2}\, dx. \tag{12.5.44}$$

This is the area of the surface generated by revolving the curve $y = y(x)$ about the x-axis. The action (12.5.44) is minimized by the catenary curve. The revolution surface generated by the catenary is a minimal surface called a *catenoid*, See chapter 8, Figure 8.1.

The minimum surface property has an interesting physical significance. If two thin circular rings, initially in contact, are placed in a soap film surface, then the surface has the minimum area property, and it has the shape of a catenoid.

12.5.0.10 The brachistochrone curve

Another important case of physical interest is when the potential is $U(y) = \dfrac{1}{\sqrt{y}}$. The equation becomes

$$1 + y'^2 + 2yy'' = 0.$$

Multiplying by y' yields an exact equation

$$\frac{d}{dx}\left(y + yy'^2\right) = 0.$$

There is a constant $C \neq 0$ such that $y(1 + y'^2) = C$. Solving for y' yields

$$y' = \pm\sqrt{\frac{C}{y} - 1}. \qquad (12.5.45)$$

Introduce a new variable θ by the relation

$$y = C \sin^2 \theta. \qquad (12.5.46)$$

Then (12.5.45) becomes

$$\frac{2C \sin\theta \, d\theta}{dx} = \pm\frac{1}{\sin\theta}.$$

Separating yields

$$2C \sin^2 \theta \, d\theta = \pm dx. \qquad (12.5.47)$$

Substituting $t = 2\theta$, formula (12.5.47) can be written as

$$C \sin^2\left(\frac{t}{2}\right) dt = \pm dx$$

$$\Longleftrightarrow \frac{C}{2}(1 - \cos t)\, dt = \pm dx.$$

Integrating yields

$$x(t) = \pm\frac{C}{2}(t - \sin t) + x_0.$$

From (12.5.46) we obtain

$$y = C \sin^2 \theta = C \sin^2\left(\frac{t}{2}\right) = \frac{C}{2}(1 - \cos t).$$

Hence, if $C \neq 0$, the solution is a cycloid which starts at the point $(x_0, 0)$,

$$x(t) = \pm\frac{C}{2}(t - \sin t) + x_0, \qquad y(t) = \frac{C}{2}(1 - \cos t).$$

It is known that along the cycloid the speed is given by Galileo's law $v = \sqrt{2y}$. Thus the action

$$\int \frac{1}{\sqrt{y}}\, ds = \sqrt{2}\int \frac{ds}{v}$$

gives the time for a free falling particle necessary to move from one point to another under gravitational influence. This time-minimizing curve was discovered in 1696 by John Bernoulli, who called the curve a *brahistocrone* curve.

12.5.0.11 Coloumb potential

The potential $U(y) = \dfrac{1}{y}$ provides an important case related to hyperbolic geometry. The curves will extremize the action

$$\int U(y)\,ds = \int \frac{1}{y}\,ds = \int \frac{\sqrt{dx^2 + dy^2}}{y} \tag{12.5.48}$$

$$= \int \frac{\sqrt{1 + y'^2}}{y}\,dx = \int d\sigma, \tag{12.5.49}$$

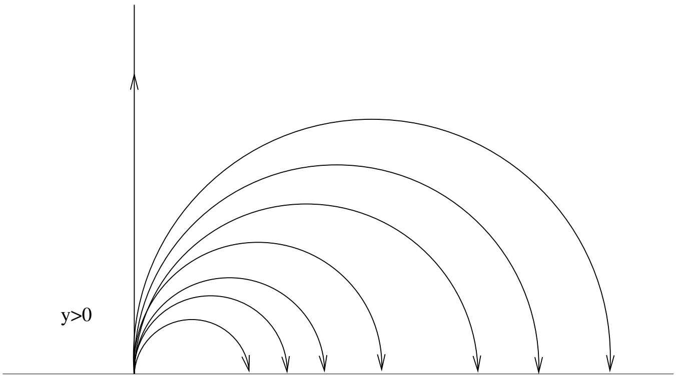


Figure 12.4: The geodesics in Poincaré's upper half-plane.

where

$$d\sigma^2 = \frac{dx^2 + dy^2}{y^2} \tag{12.5.50}$$

is the Riemannian metric of Poincaré's upper half-plane, see Chapter 6. The solutions of the Euler–Lagrange system will be geodesics in the above metric, and hence they will be arcs of circle and lines perpendicular on the $\{y = 0\}$ line.

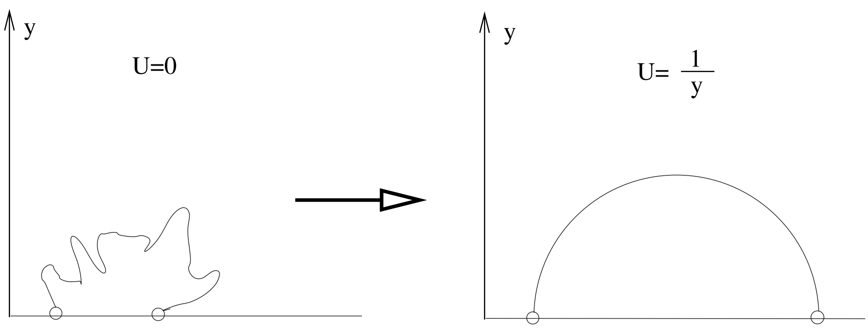


Figure 12.5: The uncharged thread in potential $U = 0$ and the charged thread in the potential $U = 1/y$.

12.5.0.12 Physical interpretation

Suppose a horizontal rod is crossed by an electrical current at a very high voltage. Around the rod there is a Coulomb potential $U(y) = 1/y$, where y is the distance to the rod. Suppose now that a thread with mobile ends is attached to the rod. When the thread gets charged, repelling electrical forces act between the thread and the rod. The equilibrium shape of the thread will be an arc of a circle normal to the rod, *i.e.*, a geodesic in the Poincaré space.

12.5.1 Hamiltonian approach

The problem may be approached also from the Hamiltonian point of view.

Proposition 12.10 *Let* $U(y) > 0$. *The Hamiltonian associated with the Lagrangian* $L(q, \dot{q}) = U(q)\sqrt{1 + \dot{q}^2}$ *is*

$$H(q, p) = -\sqrt{U(q)^2 - p^2}. \tag{12.5.51}$$

Proof. The momentum is $p = \dfrac{\partial L}{\partial \dot{q}} = U(q)\dfrac{\dot{q}}{\sqrt{1 + \dot{q}^2}}$. Solving for $\dot{q}$ yields

$$\dot{q}^2 = \frac{p^2}{U(q)^2 - p^2} \Longrightarrow \sqrt{1 + \dot{q}^2} = \frac{U(q)}{\sqrt{U(q)^2 - p^2}}.$$

The Hamiltonian is

$$\begin{aligned} H &= p\dot{q} - L(q, \dot{q}) = U(q)\frac{\dot{q}^2}{\sqrt{1 + \dot{q}^2}} - U(q)\sqrt{1 + \dot{q}^2} \\ &= U(q)\Big(\frac{\dot{q}^2}{\sqrt{1 + \dot{q}^2}} - \frac{1 + \dot{q}^2}{\sqrt{1 + \dot{q}^2}}\Big) = \frac{-U(q)}{\sqrt{1 + \dot{q}^2}} \\ &= \frac{-U(q)\sqrt{U(q)^2 - p^2}}{U(q)} = -\sqrt{U(q)^2 - p^2}. \end{aligned}$$

■

12.5.2 Hamiltonian system

The Hamilton system of equations becomes

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \dfrac{p}{\sqrt{U(q)^2 - p^2}}, \\ \dot{p} = -\dfrac{\partial H}{\partial q} = \dfrac{U(q)\,U'(q)}{\sqrt{U(q)^2 - p^2}}. \end{cases} \tag{12.5.52}$$

Dividing the equations yields

$$\frac{\dot q}{\dot p} = \frac{p}{U(q)U'(q)} \quad \text{or} \quad p\dot p = U(q)\, u'(q)\, \dot q,$$

which may be written as

$$\frac{d}{dt}\Big(\frac{1}{2}p^2(t)\Big) = \frac{d}{dt}\Big(\frac{1}{2}U^2(q(t)\Big).$$

Therefore $U(q)^2 - p^2$ is a first integral of motion. Hence the Hamiltonian $H = -\sqrt{U(q)^2 - p^2}$ will be constant along the solutions and the Hamiltonian system (12.5.52) becomes

$$\begin{cases} \dot q = -\frac{p}{H}, \\ \dot p = -\frac{U(q)\, U'(q)}{H}. \end{cases} \tag{12.5.53}$$

Differentiating the first equation and using the second one yields a second order equation in q,

$$\ddot q = -\frac{\dot p}{H} = \frac{U(q)U'(q)}{H^2} = -\frac{1}{2H^2}\frac{d}{dq}\big(- U(q)^2\big).$$

Let $V(q) = -U^2(q)$ denote the potential energy. Then q verifies

$$\ddot q = \frac{-1}{2H^2}\frac{dV(q)}{dq}, \tag{12.5.54}$$

which is a pendulum equation with potential energy $V(q)$, with the energy constant H. For instance, in the case of $U(q) = \frac{1}{\sqrt{q}}$, it follows that the cycloid may be interpreted as a pendulum in a Coulomb potential $V(q) = -\frac{1}{q}$.

12.6 Exercises

1. Prove that the system of equations

$$\begin{cases} \ddot x - 2\dot y = 0, \\ \ddot y + 2\dot x = 0, \end{cases}$$

with the initial conditions

$$x(0) = 0, \qquad y(0) = 1, \qquad \dot x(0) = 2, \qquad \dot y(0) = 0$$

has the solution $\big(x(t), y(t)\big) = \big(\sin 2t, \cos 2t\big)$, which is a circle.

2. Show that in polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, we have

(i)
$$x\dot{y} - \dot{x}y = r^2\dot{\phi}.$$

(ii)
$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2.$$

(iii) The Lagrangian (12.3.23) becomes

$$L(r, \dot{r}, \dot{\phi}) = \frac{1}{2}\dot{r}^2 + r^2\Big(\frac{1}{2}\dot{\phi}^2 + \dot{\phi} - \frac{3}{2}\Big).$$

(iv) Write the Euler–Lagrange equations and show there is constant C such that $r^2(1 + \dot{\phi}) = C$.

3. Let $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(i) Show that $e^{\mathcal{J}s} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$ and $\big(e^{\mathcal{J}s}\big)^n = \begin{pmatrix} \cos(ns) & \sin(ns) \\ -\sin(ns) & \cos(ns) \end{pmatrix}$.

(ii) Show that

$$(I - e^{\mathcal{J}\tau})^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\cot\frac{\tau}{2} \\ \cot\frac{\tau}{2} & 1 \end{pmatrix} = \frac{1}{2}(I - \cot\frac{\tau}{2}\mathcal{J}).$$

Hint: Use the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

4. Consider the metric $d\sigma^2 = U(y)(dx^2 + dy^2)$ on $\mathbb{R}^2$. Find a formula for the Laplace operator in this metric.

Bibliography

1. J. F. Adams. *Lectures on Lie groups*. Benjamin, New York, and University of Chicago Press, Chicago, IL, 1969.
2. V. I. Arnold. *Ordinary Differential Equations*. MIT Press, Cambridge, MA, London, 1973.
3. V. I. Arnold. *Mathematical Methods of Classical Mechanics*. GTM 60, Springer, Berlin, 1989.
4. L. Auslander and R. E. MacKenzie. *Introduction to Differentiable Manifolds*. Dover Publications, Inc., New York, 1977.
5. R. Beals. A note on fundamental solutions. *Comm. PDE*, **24**:1,2, (1999).
6. D.C. Chang, C. Berenstein, and T. Tie. *Laguerre Calculus and Its Applications on the Heisenberg Group*. AMS/IP Series in Advanced Mathematics #**22**, International Press, Cambridge, MA, 2001.
7. K. Zumbrun and C. Mascia. Pointwise Green function bounds for shock profiles of systems with real viscosity. *Arch. Ration. Mech. Anal.*, **169**, (2003).
8. D.C. Chang and J. Tie. A note on Hermite and subelliptic operators. *Acta Math. Sinica*, (2004).
9. K. Zumbrun and D. Hoff. Pointwise Green's function bounds for multidimensional scalar viscous shock fronts. *J. Diff. Eqs.*, **183**, (2002).
10. M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Englewoods Cliffs, NJ, 1976.
11. M. P. do Carmo. *Riemannian Geometry*. Birkhäuser, Cambridge, MA, 1992.
12. M. P. do Carmo. *Differential Forms and Applications*. Universitext, Springer-Verlag, Berlin, 1994.
13. J. Eells and J.H. Sampson. Harmonic Mappings of Riemannian Manifolds. *Amer. J. Math.*, 86, (1964).
14. J. Eells and L. Lemaire. Another report on harmonic maps. *Bull. London Math. Soc.*, **20**, (1978).
15. J. Eells and L. Lemaire. A report on harmonic maps. *Bull. London Math. Soc.*, **10**, (1978).
16. J. Eells and L. Lemaire. Selected topics in harmonic maps. *C.B.M.S. Regional Conf.*, Series 50 (Amer. Math. Soc., Providence R.I.), 1983.
17. R. Askey, G. Andrews, and R. Roy. *Special Functions*. Encyclopedia of Mathematics and its Applications #**71**, Cambridge University Press, Cambridge, UK, 1999.
18. J. Glimm and A. Jaffe. *Quantum physics: A functional integral point of view*, 2nd ed. Springer-Verlag, Berlin, New York, Heidelberg, 1987.
19. J. Hadamard. *Lectures on Cauchy's Problem*. Dover New York, 1952.

20. P. Hartman. *Ordinary Differential Equations*. Birkhäuser, Basel, 1982.
21. S. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-time*. Cambridge University Press, Cambridge, UK, 1973.
22. L. Hörmander. *Riemannian geometry*. Lectures given during the fall of 1990.
23. D. F. Lawden. *Elliptic Functions and Applications*, Applied Mathematical Sciences, Vol. 80, Springer, New York, 1989.
24. E.M. Lifschitz and L.D. Landau. *Course of Theoretical Physics, Vol I: Mechanics*. Pergamon Press, Oxford, 3rd corr. ed., 1994.
25. T.P. Liu. *Nonlinear waves for viscous conservation laws. Nonlinear evolutionary partial differential equations*, Volume 3. AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 1997. 35L65, 1993.
26. T.P. Liu. Pointwise convergence to shock waves for viscous conservation laws. *Comm. Pure Appl. Math.*, **50**, (1997).
27. T.P. Liu. *Hyperbolic and viscous conservation laws*. CBMS-NSF Regional Conference Series in Applied Mathematics, 72. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. x+73 pp. ISBN: 0-89871-436-2 (Reviewer: Ming Mei) 35L65 (35-01 76L05), 2000.
28. E. Mazet, M. Berger, and P. Gauduchon. Le spectre d'une variété riemannienne, *Lecture Notes in Math.*, Volume 194, Springer-Verlag, Berlin, 1971.
29. M. Puta and M. Craioveanu. *Introducere in Geometria Spectrala*. Editura Academiei RSR, Bucuresti, 1988.
30. M. Levi and G. Mann. The axial vector current in beta decay. *Nuovo Cimento*, no. 16, (1960).
31. S. Hildebrand and M. Giaquinta. *Calculus of Variations, I, II*, Volume 310. Springer, 1977.
32. K. Zumbrun and M. Oh. Stability of periodic solutions of conservation laws with viscosity. *Arch. Ration. Mech. Anal.*, **166**, (2003).
33. S. Ianus and O. Calin. A Note on Harmonic maps of semi-Riemannian manifold, Volume BSG Proc.,1. *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras*, Thessaloniki, 1995.
34. V. Mangione and O. Calin. Variational calculus on sub-Riemannian manifolds. *Balcan Journal of Geometry and Applications*, **8**, (2003).
35. B. O'Neill. *Semi-Riemannian Geometry*. Academic Press, 1983.
36. A. I. Pluzhnikov. Harmonic mappings of Riemannian surfaces and foliated manifolds; English translation. *Math. USSR. Sb.*, **41**, (1982).
37. P. Greiner, R. Beals, and B. Gaveau. Hamilton-Jacobi theory and the heat kernel on Heisenberg groups. *J. Math. Pure Appl.*, **79**, (2000).
38. A. Ratto. Harmonic maps of spheres and equivariant theory. Thesis, University of Warwick, 1987.
39. A. Sanini. Applicazioni tra varietá riemanniene con energia critica rispetto a deformazioni di metriche. *Rend. Math.*, **3**(7), 1983.
40. R. Schoen and S. T. Yau. On univalent harmonic maps between surfaces. *Invent. Math.*, **44**, (1978).
41. K. Schulten. *Notes on quantum mechanics*. Dept. of Physics and Beckman Inst., University of Illinois at Urbana-Champaign, 2000.
42. A. Sommerfeld. *Lectures on theoretical physics, Vol I: Mechanics*. Academic Press, NY, 1952.
43. M. Spivak. *Calculus on Manifolds*. Addison-Wesley, Reading, MA, 1965.
44. M. Spivak. *Differential Geometry*, Volume I-V. Publish or Perish Inc.
45. W. Thirring. *A Course in Mathematical Physics*, Volume I-II. Springer, Berlin, 1978.

46. Y. Zeng and T.P. Liu. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. *Mem. Amer. Math. Soc.*, **125**, (1997).
47. F. Oberhettinger, W. Magnus, and R.P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, Berlin, New York, Heidelberg, 1964.
48. S. H. Yu. Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws. *Arch. Ration. Mech. Anal.*, **146**, (1999).

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