Separation of variables on n-dimensional Riemannian manifolds. I. The n-sphere S_n and Euclidean n-space R_n

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The following problem is solved: What are all the "different" separable coordinate systems for the Laplace–Beltrami eigenvalue equation on the n-sphere S_n and Euclidean n-space R_n and how are they constructed? This is achieved through a combination of differential geometric and group theoretic methods. A graphical procedure for construction of these systems is developed that generalizes Vilenkin's construction of polyspherical coordinates. The significance of these results for exactly soluble dynamical systems on these manifolds is pointed out. The results are also of importance for the analysis of the special functions appearing in the separable solutions of the Laplace–Beltrami eigenvalue equation on these manifolds.

I. INTRODUCTION

In this paper we find all separable coordinate systems on the real n-sphere S_n and Euclidean n-space for the Hamilton–Jacobi equation

$$H = g^{ij} S_{x^i} S_{x^j} = E, \quad S_{x^i} = \frac{\partial S}{\partial x^i}, \quad i = 1,...,N, \tag{I}$$

and the Helmholtz equation

$$\Delta_n = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial \Psi}{\partial x^i} \right) = \lambda \Psi,$$
$$i,j = 1,...,n, \quad g = \det(g_{ij}). \tag{II}$$

There are several reasons why this is an important problem.

(1) The list of 11 coordinate systems in R_3 that provide a separation of variables for these equations are well known.[1] Their value in the solution of boundary value problems is unquestioned. More recently there has been an interest in separation of variables on the spheres S_2 and S_3 (see Refs. 2 and 3). In the case of S_3 the relationship with the hydrogen atom has been extensively studied.[4] More recently the importance of separable coordinate systems on S_n has been discussed[5] for dynamical symmetries in a spherical geometry. It is also of interest to study classical and quantum mechanics on S_n and R_n as a means for finding exactly soluble dynamical systems interacting under a suitable potential so as to admit solution via a separation of variables.

(2) On the mathematical side the solution of the problem we solve here gives the basic results necessary for a complete analysis of the special functions that are solutions of (II) via a separation of variables on S_m and R_n. In doing so all the separable solutions can be characterized in terms of symmetric second-order elements in the enveloping algebra of the corresponding symmetry group. This provides the basis for an all-embracing theory of such solutions and a systematic treatment of relations amongst these solutions. For an introduction to these methods we refer to Miller's book.[6] In solving this problem we also extend Vilenkin's work, which dealt with a restricted class of separable solutions.[7]
We should also note at this point the articles of Lucquaud,[8–10] which give a discussion of spherical harmonics on SO(n) via an elegant tensorial approach. For some of the crucial results concerning separation of variables we refer the reader to the papers of Levi-Civita,[11] Eisenhart,[12] and Benenti.[13] Referring now to equations (I) and (II) we should, of course, mention that these equations are expressed in an arbitrary coordinate system in terms of which the infinitesmal distance on the underlying manifold is

$$ds^2 = g_{ij}\, dx^i\, dx^j, \quad i,j = 1,...,n. \tag{1.1}$$

[Formulas (I), (II), and (1.1) use the summation convection on indices i, j.] Separation of variables for (I) is understood to mean that there is a coordinate system $\{x^i\}$ for which it is possible to find a solution $S = S(\mathbf{x};\lambda_1,...,\lambda_n)$ of (I) such that

$$S = \sum_{i=1}^{n} S_i(x^i;\lambda_1,...,\lambda_n) \tag{1.2}$$

and $\det(\partial^2 S/\partial x^i\, \partial\lambda_j)_{n\times n} \neq 0$, i.e., S is a complete integral.[14] This type of variable separation is *additive*.

Separation of variables for (II) is normally understood in the product sense,[15] i.e., the coordinates $\{x^i\}$ should be such that there is a soluton of (II) depending on n parameters $c_1,...,c_n$ of the form

$$\Psi = \prod_{i=1}^{n} \Psi_i(x^i;c_1,...,c_n). \tag{1.3}$$

In this article we determine all coordinate systems that provide additive separation for (I) and product separation for (II) for the following Riemannian manifolds: (i) the real n-sphere S_n and (ii) real Euclidean n-space R_n. We also describe a graphical procedure for constructing these coordinates, which includes Vilenkin's description of polyspherical coordinates as a special case.

We recall a few basic facts about variable separation. For a positive definite Riemannian space a separable coordinate system $\{x^i\}$ for (I) can always be chosen[16,17] such that

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the contravariant metric tensor is

$$(g^{ij}) = \left[\begin{array}{c|c} \delta^{ab} H_a^{-2} & 0 \\ \hline 0 & g^{\alpha\beta} \end{array}\right]. \tag{1.4}$$

The functions H_a^{-2} and $g^{\alpha\beta}$ have the form

$$H_a^{-2} = \frac{\Phi^{a1}}{\Phi}, \quad g^{\alpha\beta} = \sum_b A_b^{\alpha\beta}(x^b)\,\frac{\Phi^{b1}}{\Phi}, \tag{1.5}$$

where $\Phi = \det(\Phi_{ab}(x^a))$. The variables x^α are such that $\partial g^{ij}/\partial x^\alpha = 0$, for all i, j.

A typical separable solution takes the form

$$S = \sum_a S_a(x^a) + \sum_\alpha c_\alpha x^\alpha.$$

The choice of ignorable variables x^α is not unique; we would get a similar system if we defined new coordinates $x^{i'}$ by $x^{\alpha'} = a^\alpha_\beta x^\beta$, $x^{a'} = x^a$, where $\det(a^\alpha_\beta) \neq 0$. We say that two such coordinate systems are *equivalent* and will not distinguish between them.

The standard form (1.4) will be central to our arguments. If a coordinate system is separable for (II), it is automatically separable for (I). A separable system $\{x^i\}$ for (I) separates (II) if and only if

$$R_{ab} = 0, \quad a \neq b,$$

where R_{ij} is the Ricci tensor expressed in these coordinates. In particular, for orthogonal coordinates, (I) and (II) separate in the same systems.

A. The n-sphere S_n

This space is most readily realized in terms of $n+1$ real "standard" coordinates $(s_1,\ldots,s_{n+1}) \in R_{n+1}$, which satisfy

$$s_1^2 + \cdots + s_{n+1}^2 = 1. \tag{1.6}$$

The infinitesimal distance is given by

$$ds^2 = ds_1^2 + \cdots + ds_{n+1}^2. \tag{1.7}$$

The n-sphere admits the group $SO(n+1)$ of isometries. The algebra $so(n+1)$ is realized on the cotangent bundle of R_{n+1} by the Killing vectors

$$I_{ij} = s_i p_{s_j} - s_j p_{s_i}, \quad i \neq j. \tag{1.8}$$

We recall in the normal correspondence, $\partial S/\partial x^i = p_i$ and L is a Killing vector if L is linear in the p_i's and $\{H, L\} = 0$ where $\{\ ,\ \}$ is the Poisson bracket. It is then seen that the ignorable coordinates x^α of a given separable coordinate system are such that p_α is a Killing vector. The Lie algebra $so(n+1)$ also can be realized by means of linear differential operators, with the identification $p_{s_i} \to \partial/\partial s_i$. The symmetry operator $\hat{I}_{ij} = s_i(\partial/\partial s_j - s_j \partial/\partial s_i)$ satisfies $[\Delta_n, \hat{I}_{ij}] = 0$, where $[\ ,\ ]$ is the commutator bracket and Δ_n is the operator (II) on S_n. We note that the two realizations of $so(n+1)$ directly relate to the $SO(n+1)$-invariant equations (I) and (II). For equation (I) the algebra is realized as the set of all Killing vectors L that are in involution with H, i.e., $\{H, L\} = 0$. For equation (II) the algebra is realized by all first-order linear differential operators $\mathscr{L}$ that commute with Δ_n. The n-sphere as a Riemannian manifold is a space of constant curvature -1 and is completely characterized by the Riemann curvature tensor conditions[18]

$$R_{hijk} = (g_{hk} g_{ij} - g_{hj} g_{ik}), \tag{1.9}$$

in any coordinate system.

B. Euclidean n-space R_n

Here, a point is given by n real (Cartesian) coordinates $(y_1,\ldots,y_n)$ and the infinitesimal distance is

$$ds^2 = dy_1^2 + \cdots + dy_n^2, \tag{1.10}$$

where R_n admits the isometry group $E(n) = T_n \circledS SO(n)$. This is the semidirect product of the n-dimensional Abelian group of translations T_n and $SO(n)$. On the cotangent bundle of R_n the Lie algebra $\mathscr{E}(n)$ has a realization by Killing vectors:

$$M_{ij} = y_i p_{y_j} - y_j p_{y_i}, \quad P_k = p_{y_k},$$
$$i, j, k = 1,\ldots,n, \quad i \neq j. \tag{1.11}$$

The corresponding realization in terms of symmetry operators can be obtained by the correspondence $p_j \to \partial/\partial y_j$. Euclidean n-space is characterized by the Riemann curvature tensor condition $R_{hijk} \equiv 0$ in any coordinate system.

We note that the study of variable separation will give a complete enumeration of the scope and extent of special function identities available in these spaces. In addition, exactly which special functions appear can be determined. The problem of separation of variables on S_n is also intimately related to the separation of variables problem on $CP(n)$,[19] n-dimensional complex projective space.

II. SEPARATION OF VARIABLES ON S_n

The following is a crucial result in the classification of separable coordinate systems on S_n.

Theorem: Let $\{x^i\}$ be a coordinate system on S_n for which the Hamilton–Jacobi equation admits a separation of variables. Then, by passing to an equivalent system of coordinates if necessary, we have $g^{ij} = \delta^{ij} H_i^{-2}$, i.e., separation of variables occurs only in orthogonal coordinates. Furthermore, in terms of the standard coordinates on the sphere $s_1,\ldots,s_{n+1}$, the ignorable variables can be chosen such that

$$p_{\alpha_1} = I_{12}, \quad p_{\alpha_2} = I_{34},\ldots, p_{\alpha_q} = I_{2q+1,2q+2}. \tag{2.1}$$

where the number of ignorable variables is q.

Proof: This is based on the general block-diagonal expression of the canonical form of the contravariant metric tensor for a separable coordinate system. It is well known[20] that any element of the symmetry algebra $so(n+1)$ of S_n is conjugate to an element of the form

$$L = I_{12} + b_2 I_{34} + \cdots + b_v I_{2v-1,2v}. \tag{2.2}$$

If this element corresponds to the ignorable variable α_1, i.e., $L = p_{\alpha_1}$, then by local Lie theory the standard coordinates on the n-sphere can be taken as

$$(s_1,\ldots,s_{n+1}) = (\rho_1 \cos(\alpha_1 + w_1), \rho_1 \sin(\alpha_1 + w_1), \rho_2 \cos(b_2\alpha_1 + w_2),$$
$$\rho_2 \sin(b_2\alpha_1 + w_2),\ldots,\rho_v \cos(b_v\alpha_1 + w_v), \rho_v \sin(b_v\alpha_1 + w_v), s_{2v+1},\ldots,s_{n+1}), \tag{2.3}$$

where $\rho_1^2 + \cdots + \rho_\nu^2 + s_{2\nu+1}^2 + \cdots + s_{n+1}^2 = 1$. The infinitesimal distance then has the form

$$ds^2 = d\rho_1^2 + \cdots + d\rho_\nu^2 + \rho_1^2(d\alpha_1 + dw_1)^2 + \cdots + \rho_\nu^2(b_\nu\, d\alpha_1 + dw_\nu)^2 + ds_{2\nu+1}^2 + \cdots + ds_{n+1}^2. \quad (2.4)$$

If there is only one ignorable variable then the coordinate system must be orthogonal and this is only possible if $b_2 = \cdots = b_\nu = 0$, i.e., $p_{\alpha_1} = I_{12}$. Indeed, the requirement that the contravariant metric have the form (1.4) (orthogonal in this case) is that

$$-dw_1 = \sum_{j=2}^{\nu} \frac{\rho_j^2}{\rho_1^2} b_j\, dw_j. \quad (2.5)$$

Since the differentials $d\rho_j$, dw_j $(j\geqslant 2)$, must be independent and the only condition on ρ_1^2 is

$$\sum_{i=1}^{\nu} \rho_i^2 + s_{2\nu+1}^2 + \cdots + s_{n+1}^2 = 1,$$

the condition $dw_1^2 = 0$ implies $b_j = 0, j = 2,\ldots,\nu$, and $dw_1 = 0$. We can then take the constant $w_1 = 0$ by suitably redefining α_1. The theorem is proven in this case.

Now suppose there are $q > 1$ ignorable variables. The Killing vectors p_{α_i}, $i = 1,\ldots,q$, must form an involutive set. It follows from the spectral theorem for commuting skew-adjoint matrices that for each i, p_{α_i} has a representation of the form

$$p_{\alpha_i} = b_1^i I_{12} + b_2^i I_{34} + \cdots + b_{\nu_i}^i I_{2\nu_i-1,2\nu_i}, \quad (2.6)$$

for $i = 2,\ldots,q$. In fact we can assume

$$p_{\alpha_i} = I_{2i-1,2i} + \sum_{l>q}^{N} b_l^i I_{2l-1,2l}, \quad i = 1,\ldots,q. \quad (2.7)$$

The projective coordinates on the sphere then have the form

$$(s_1,\ldots,s_{n+1}) = \Big(\rho_1 \cos(\alpha_1 + w_1), \rho_1 \sin(\alpha_1 + w_1),\ldots,\rho_q \cos(\alpha_q + w_q), \rho_q \sin(\alpha_q + w_q),$$
$$\rho_{q+1} \cos\Big(\sum_{i=1}^{q} b_{q+1}^i \alpha_i + w_{q+1}\Big),\ \rho_{q+1} \sin\Big(\sum_{i=1}^{q} b_{q+1}^i \alpha_i + w_{q+1}\Big),\ldots,$$
$$\rho_N \sin\Big(\sum_{i=1}^{q} b_N^i \alpha_i + w_N\Big), s_{2N+1},\ldots,s_{n+1}\Big). \quad (2.8)$$

We now make the crucial requirement that the ignorable variables α_i, $i = 1,\ldots,q$, are part of a separable coordinate system. If we compute the covariant metric, it should be in block-diagonal form with respect to the two classes of variables. Just as in the case $q = 1$, this is only possible if $b_l^i = 0$, $i = 1,\ldots,q$, $l = q+1,\ldots,N$ and $dw_i = 0$, $1<i<q$. We can therefore assume that $L_1 = I_{12}, L_2 = I_{34},\ldots,L_q = I_{2q-1,2q}$; the ignorable coordinates α_i then can always be chosen such that $w_i = 0$, $1<i<q$, and the system is orthogonal. Q.E.D.

This theorem enables us to bring to bear Eisenhart's[12] results on orthogonal systems of the Stäckel type. Our problem reduces to the enumeration of all orthogonal separable coordinate systems. We use an inductive procedure such that given all separable systems for S_j, $j < n$, we can give the rules for construction of all systems on S_n.

If $\{x^i\}$ is an orthogonal coordinate system with infinitesimal distance $ds^2 = \Sigma_{i=1}^n H_i^2 (dx^i)^2$, then the conditions that the space be of constant curvature -1, i.e., that we are dealing with S_n, are

$$\text{(i)}\quad R_{ijji} = -H_i^2 H_j^2, \quad i \neq j,$$
$$\text{(ii)}\quad R_{hiik} = 0, \quad i\neq h \neq k. \quad (2.9)$$

Eisenhart[12] showed that in order for orthogonal separation to occur on any n-dimensional Riemannian manifold the contravariant metric $g^{ij} = \delta^{ij} H_i^{-2}$ must be in Stäckel form and that the necessary and sufficient conditions for this are

$$\frac{\partial^2}{\partial x^j\, \partial x^k} \log H_i^2 - \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_i^2 + \frac{\partial}{\partial x_j} \log H_i^2 \frac{\partial}{\partial x_k} \log H_j^2 + \frac{\partial}{\partial x_k} \log H_i^2 \frac{\partial}{\partial x_j} \log H_k^2 = 0, \quad (2.10)$$

for $j \neq k$. He then went further to show that these conditions, together with the equations (2.9) (ii), are equivalent to the equations

$$\frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_i^2 - \frac{\partial}{\partial x^j} \log H_i^2 \frac{\partial}{\partial x^k} \log H_j^2 - \frac{\partial}{\partial x^k} \log H_i^2 \frac{\partial}{\partial x^j} \log H_k^2 = 0, \quad i,j,k \text{ pairwise distinct.} \quad (2.11)$$

It follows that the metric for a separable system can be written in the form

$$g_{ii} = H_i^2 = X_i \prod_{j\neq i} (\sigma_{ij} + \sigma_{ji}), \quad i = 1,\ldots,n, \quad (2.12)$$

where X_i, σ_{ij} are functions of x^i at most. The conditions (2.9) (i) are then equivalent to

$$\sigma_{ji}'\sigma_{ki}'(\sigma_{jk} + \sigma_{kj}) - \sigma_{ji}'\sigma_{kj}'(\sigma_{ki} + \sigma_{ik}) - \sigma_{ki}'\sigma_{jk}'(\sigma_{ij} + \sigma_{ji}) = 0, \quad i,j,k \text{ distinct,} \quad (2.13)$$

where $\sigma'_{kl} = (\partial_x k)\sigma_{kl}$, etc. We now study various possibilities for the functions σ_{ij}. If all the functions σ_{ij} are such that $\sigma'_{ij} \neq 0$ then Eisenhart has shown that the metric coefficients have the form

$$g_{ii} = H_i^2 = X_i \prod_{j \neq i} (\sigma_i - \sigma_j), \tag{2.14}$$

where $\sigma_i = \sigma_i(x^i)$ and $\sigma'_i \neq 0$. This metric will be the basic building block on which we can formulate our inductive construction. Without loss of generality we can redefine variables $\{x^i\}$ in such a way that $\sigma_i = x^i$, i.e.,

$$H_i^2 = X_i \prod_{j \neq i} (x^i - x^j). \tag{2.15}$$

The conditions (2.9) (i) then amount to

$$\left[\prod_{l \neq j} (x^j - x^l)\right]^{-1} \left\{\frac{-2}{(x^i - x^j)^2}\left(\frac{1}{X_j}\right) + \frac{-1}{(x^i - x^j)}\left(\frac{1}{X_j}\right)'\right\} + \left[\prod_{l \neq i} (x^i - x^l)\right]^{-1} \left\{\frac{-2}{(x^i - x^j)^2}\left(\frac{1}{X_i}\right) + \frac{-1}{(x^j - x^i)}\left(\frac{1}{X_i}\right)'\right\}$$
$$+ \sum_{l \neq i,j} \frac{1}{X_l (x^l - x^i)(x^l - x^j)\Pi_{k \neq l}(x^l - x^k)} = -4. \tag{2.16}$$

These equations have the solution

$$(1/X_i)^{(n+1)} + 4(n+1)! = 0, \quad i = 1,...,n. \tag{2.17}$$

i.e.,

$$\frac{1}{X_i} = -4(x^i)^{n+1} + \sum_{l=0}^{n} a_l (x^i)^{n-l} = f(x^i).$$

The function $f(x)$ can also be written

$$f(x) = -4 \prod_{i=1}^{n+1} (x - e_j). \tag{2.18}$$

There are two requirements to determine which metrics of this type occur on S_n: (1) the metric must be positive definite and (2) the variables x^i should vary in such a way that they correspond to a coordinate patch that is compact. There is a unique solution to these requirements: the x^i, e_i should satisfy

$$e_1 < x^1 < e_2 < \cdots < e_n < x^n < e_{n+1}. \tag{2.19}$$

These are elliptic coordinates on the n-sphere S_n. They can be related to the coordinates $\{s_j\}$ via

$$s_j^2 = \frac{\Pi_{i=1}^n (x^i - e_j)}{\Pi_{j \neq i}(e_i - e_j)}, \quad j = 1,...,n+1. \tag{2.20}$$

These systems are the basic building blocks for separable coordinate systems on real spheres. To complete the analysis of possible orthogonal separable systems we need to consider the case when some of the σ_{ij} functions are constants. If $\sigma_{ij} = a_{ij}$ (const), Eisenhart has shown that there are four possibilities:

(i) $\sigma_{ij} = a_{ij}, \quad \sigma_{ji} = a_{ji},$
$\sigma_{ik} = a_{ik}, \quad \sigma_{jk} = a_{jk};$

(ii) $\sigma_{ij} = a_{ij}, \quad \sigma_{ji} = a_{ji},$
$\sigma_{ik} = a_{ik}, \quad \sigma_{ki} = a_{ki};$

(iii) $\sigma_{ij} = a_{ij}, \quad \sigma_{ik} = a_{ik},$
$\sigma_{ji} = a_{ji}\sigma_j, \quad \sigma_{jk} = a_{jk}\sigma_j,$
$\sigma_{ki} = a_{ki}\sigma_k, \quad \sigma_{kj} = a_{kj}\sigma_k;$

(iv) $\sigma_{ij} = a_{ij}, \quad \sigma_{kj} = a_{kj},$
$\sigma_{ji} = a_{ji}\sigma_j, \quad \sigma_{jk} = a_{jk}\sigma_j,$

$$a_{ji}a_{kj} - a_{jk}a_{ij} = 0; \tag{2.21}$$

where σ_j is a function of x^j only and i,j,k are pairwise distinct. If we fix i and j, then, for k values corresponding to cases (i)–(iii), $\sigma_{ik} = a_{ik}$. To examine how the inductive process works let us take $\sigma_{1l} = a_{1l}$ for $l = k+1,...,n$ and $\sigma'_{1j} \neq 0$ for $j = 2,...,k$. Then we have

$$\sigma_{jl} = a_{jl}, \quad \sigma_{l1} = a_{l1}\sigma_l, \quad \sigma_{lj} = a_{lj}\sigma_l,$$
$$a_{l1}a_{jl} - a_{lj}a_{1l} = 0, \quad \text{for } l = k+1,...,n, \quad j = 2,...,k.$$

Assuming that $a_{lj} \neq 0$ for $l = k+1,...,n, j = 2,...,k,$ we find the metric coefficients have the form

$$H_i^2 = \left[X_i \prod_{j \neq i} (\sigma_{ij} + \sigma_{ji})\right]\left[\prod_{l=k+1}^{n} (a_{il} + a_{li}\sigma_l)\right],$$
$$i = 1,...,k, \tag{2.22}$$

$$H_l^2 = X_l \prod_{\substack{m \neq l \\ m > k+1}}^{1<j<k} (\sigma_{lm} + \sigma_{ml}), \quad l = k+1,...,n. \tag{2.23}$$

Let us assume that no further functions σ_{ij}, σ_{lm} are constants. Then using the results of Eisenhart we can take the metric coefficients as

$$H_i^2 = \left[X_i \prod_{j \neq i} (x^i - x^j)\right]\left(\prod_{l=k+1}^{n} \sigma_l\right),$$
$$H_l^2 = \left[X_l \prod_{m \neq l} (x^l - x^m)\right]. \tag{2.24}$$

The conditions $R_{kllk} = -H_k^2 H_l^2$ are equivalent to (2.16) and (2.17) with $i = k+1,...,n$ and $n \to k = n'$. Putting

$$H_i^2 = \left[X_i \prod_{j \neq i} (x^i - x^j)\right],$$

the conditions $R_{ijji} = H_i^2 H_j^2$ and $R_{illi} = -H_i^2 H_l^2$ are equivalent to

$$H_i^{-2}H_j^{-2}R_{ijji} + \left(\prod_{l=k+1}^{n} \sigma_l\right)$$
$$\times \left[\sum_{l'=k+1}^{n} \frac{1}{4H_{l'}^2}\left(\frac{\sigma'_l}{\sigma_l}\right)^2 + 1\right] = 0, \tag{2.25}$$

$$2\frac{\sigma_l''}{\sigma_l}-\left(\frac{\sigma_l'}{\sigma_l}\right)^2-\left(\frac{\sigma_l'}{\sigma_l}\right)$$
$$\times\left[\frac{\partial}{\partial x^l}\log H_l^2+H_l^2\sum_{m\neq l}\frac{1}{H_m^2(x^l-x^m)}\right]=-4H_l^2, \tag{2.26}$$

where $\widetilde{R}_{ijji}$ is the Riemann curvature tensor for the Riemannian manifold with infinitesimal distance $ds^2=\Sigma_{i=1}^k\widetilde{H}_i^2(dx^i)^2$. These equations are satisfied if and only if

$$\frac{1}{X_l}=-4\prod_{m=1}^{n-k+1}(x^l-f_m),\quad l=k+1,\ldots,n, \tag{2.27}$$

and $\sigma_l=(x^l-f_{n-k+1})/(f_{l-k}-f_{n-k+1})$, where we take $f_1<f_2<\cdots<f_{n-k+1}$. The remaining condition then is $\widetilde{R}_{ijji}=-\widetilde{H}_i^2\widetilde{H}_j^2$ so that

$$\frac{1}{X_i}=-4\prod_{j=1}^{k+1}(x^i-e_j). \tag{2.28}$$

The coordinates on S_n can be taken as

$$(s_1,\ldots,s_{n+1})=(u_1v_1,\ldots,u_1v_{k+1},u_2,\ldots,u_{n-k+1}), \tag{2.29}$$

where

$$\sum_{i=1}^{k+1}v_i^2=1,\qquad\sum_{l=1}^{n-k+1}u_l^2=1,$$

and

$$v_j^2=-\frac{\Pi_{i=1}^k(x^i-f_j)}{\Pi_{j\neq i}(f_i-f_j)}, \tag{2.30}$$

$$u_m^2=-\frac{\Pi_{l=k+1}^n(x^l-e_m)}{\Pi_{n\neq m}(e_n-e_m)}. \tag{2.31}$$

The infinitesimal distance has the form

$$ds_0^2=ds_1^2\left[\frac{\Pi_{l=k+1}^n(x^l-f_{n-k+1})}{\Pi_{m\neq n-k+1}(f_m-f_{n-k+1})}\right]+ds_2^2, \tag{2.32}$$

where

$$ds_1^2=-\frac{1}{4}\sum_{i=1}^k\left[\frac{\Pi_{j\neq i}(x^i-x^j)}{\Pi_{j=1}^{k+1}(x^i-e_j)}\right](dx^i)^2, \tag{2.33}$$

$$ds_2^2=\frac{-1}{4}\sum_{l=k+1}^n\left[\frac{\Pi_{m\neq l}(x^l-x^j)}{\Pi_{m=1}^{n+k+1}(x^l-f_m)}\right](dx^l)^2,$$
$$m=k+1,\ldots,n,\quad j=1,\ldots,k. \tag{2.34}$$

The choice of embedding of the sphere S_k in the n-sphere S_n given by (2.29) is not, of course, unique. It is here we meet the second concept involved in regarding two choices of coordinates $(s_1,\ldots,s_{n+1})$ as giving "equivalent" coordinate systems. Clearly we could subject the coordinates $\{s_i\}$ to an arbitrary SO$(n+1)$ group action. The infinitesimal distance would remain unchanged in the process. We regard the new set of coordinates $(s_1',\ldots,s_{n+1}')$ as equivalent to the original set. This is just the mathematical formulation of the geometric identification of coordinate systems that differ only by an isometry. This aspect of equivalence is obviously group related. If the Riemannian manifold had no isometry group it would not be relevant.

Now suppose one of the constants $a_{lj}=0$ for some fixed l and j. Then from the relations

$$a_{l1}a_{jl}-a_{lj}a_{1l}=0, \tag{2.35}$$

we have $a_{l1}=0$ and consequently $a_{li}=0$, for $i=1,\ldots,k$. This implies that σ_l does not appear in H_i^2, $i=1,\ldots,k$.

Referring to the curvature equation $R_{illi}=-H_i^2H_l^2$ we see that it cannot be satisfied if $\sigma_{li}=a_{li}\sigma_l=0$ as this would imply $-4H_l^2=0$. Thus $a_{lj}\neq 0$ for each l,j. Recall here that we have assumed that none of the functions σ_{ij} $(i,j=1,\ldots,k,\ i\neq j)$, σ_{lm} $(l,m=k+1,\ldots,n,\ l\neq m)$ is a constant. Let us now push this process one step further: Let $\sigma_{k+1,s}=a_{k+1,s}$ for $s=p+1,\ldots,n$ and $\sigma_{k+1,s}'\neq 0$ for $s=k+1,\ldots,p$. Then applying the same arguments as previously, we see that the metric coefficients H_l^2, $l=k+1,\ldots,n$, can be brought to the form

$$H_l^2=X_l\left[\prod_{\substack{m\neq l\\ k+1<l<p}}(\sigma_{lm}+\sigma_{ml})\right]\left[\prod_{s=p+1}^n(a_{ls}+a_{sl}\sigma_s)\right], \tag{2.36}$$

$$H_t^2=X_t\left[\prod_{\substack{s\neq t\\ s>p+1}}(\sigma_{st}+\sigma_{ts})\right]. \tag{2.37}$$

Here the indices run over the ranges

$$i,j,\ldots=1,\ldots,k,\quad l,m,\ldots=k+1,\ldots,p, \tag{2.38}$$
$$s,t,u,\ldots=p+1,\ldots,n.$$

We follow this convention unless otherwise stated. If none of the remaining σ_{ab}'s are constants there are two cases to consider:

(i) $a_{ls}/a_{sl}=a_{is}/a_{si}$,

for $s=p+1,\ldots,n,\quad i=1,\ldots,k,\quad l=k+1,\ldots,p.$

Then the infinitesimal distance has the form

$$ds^2=\left(\prod_{t=p+1}^n\sigma_t\right)d\omega^2$$
$$+\sum_{t=p+1}^nX_t\left[\prod_{u\neq t}(\sigma_{ut}+\sigma_{tu})\right](dx^t)^2, \tag{2.39}$$

where

$$d\omega^2=\left(\prod_{l=k+1}^p\sigma_l\right)\sum_{i=1}^kX_i\left[\prod_{j\neq i}(\sigma_{ij}+\sigma_{ji})\right](dx^i)^2$$
$$+\sum_{l=k+1}^pX_l\left[\prod_{m\neq l}(\sigma_{lm}+\sigma_{ml})\right](dx^l)^2. \tag{2.40}$$

The form $d\omega^2$ corresponds to the choice of metric coefficients with $l=k+1,\ldots,p<n$. If we impose the conditions $R_{abba}=-H_a^2H_b^2$, then we see that for $a,b=1,\ldots,k,k+1,\ldots,p$ the conditions are identical with (2.16). Hence

$$\frac{1}{X_i}=-4\prod_{j=1}^{k+1}(x^i-e_j),\quad i=1,\ldots,k, \tag{2.41}$$

$$\frac{1}{X_l}=-4\prod_{m=1}^{p-k+1}(x^l-f_m),\quad l=k+1,\ldots,p, \tag{2.42}$$

and

$$\sigma_l=\frac{(x^l-f_{p-k+1})}{(f_{l-k}-f_{p-k+1})},\quad l=k+1,\ldots,p. \tag{2.43}$$

The remaining conditions

$$R_{tuut} = -H_t^2 H_u^2$$

and

$$R_{taat} = -H_a^2 H_t^2 \ (a=1,...,p)$$

also imply

$$\frac{1}{X_s} = -4 \prod_{t=1}^{n-p+1} (x^s - g_t), \quad s=p+1,...,n, \quad (2.44)$$

and

$$\sigma_t = \frac{(x^t - g_{n-p+1})}{(g_{t-p} - g_{n-p+1})}, \quad t=p+1,...,n.$$

These coordinates on S_n can then be constructed in a standard way:

$$(s_1,...,s_{n+1}) = (u_1 v_1 w_1,...,u_1 v_1 w_{k+1}, u_1 v_2,..., u_1 v_{p-k+1}, u_2,...,u_{n-p+1}), \quad (2.45)$$

where

$$\sum_{i=1}^{k+1} w_i^2 = 1, \quad \sum_{l=1}^{p-k+1} v_l^2 = 1, \quad \sum_{i=1}^{n-p+1} u_i^2 = 1,$$

and on each of the spheres defined by the u_i, v_j, and w_k coordinates, elliptic coordinates are chosen, i.e.,

$$v_j^2 = \frac{-\Pi_{q=1}^{k}(x^q - e_j)}{\Pi_{j\neq i}^{k+1}(e_i - e_j)}, \quad j,i = 1,...,k+1, \quad (2.46)$$

$$w_l^2 = \frac{-\Pi_{q=k+1}^{p}(x^q - f_l)}{\Pi_{m\neq l}(f_m - f_l)}, \quad m,l = 1,...,p-k+1, \quad (2.47)$$

$$u_t^2 = \frac{-\Pi_{q=p+1}^{n}(x^q - g_t)}{\Pi_{s\neq t}(g_s - g_t)}, \quad s,t = 1,...,n-p+1. \quad (2.48)$$

Now

(ii) $a_{ls}/a_{sl} \neq a_{is}/a_{si}$.

In this case $\sigma_l = a_l$, for $l = k+1,...,p$, as follows from Eisenhart's cases (2.21) (i)–(2.21) (iv). The infinitesimal distance has the form

$$ds^2 = \Big(\prod_{t=p+1}^{n} \sigma_t\Big) d\omega_1^2 + \Big(\prod_{t=p+1}^{n} (\sigma_t + \alpha)\Big) d\omega_2^2 + \sum_{t=p+1}^{n} X_t \Big[\prod_{\substack{u\neq t\\ u>p+1}} (\sigma_{ut} + \sigma_{tu})\Big] (dx^t)^2, \quad \alpha \neq 0, \quad (2.49)$$

where

$$d\omega_1^2 = \sum_{i=1}^{k} X_i \Big[\prod_{\substack{j\neq i\\ j<k}} (\sigma_{ij} + \sigma_{ji})\Big] (dx^i)^2, \quad (2.50)$$

$$d\omega_2^2 = \sum_{l=k+1}^{p} X_l \Big[\prod_{\substack{m\neq l\\ k+1<m<p}} (\sigma_{lm} + \sigma_{ml})\Big] (dx^l)^2. \quad (2.51)$$

The conditions that this metric correspond to S_n require that we have the same functions X_a as in the previous case and now

$$\sigma_t = \frac{(x^t - g_1)}{(g_{t-p+1} - g_1)}, \quad \sigma_t + \alpha = \frac{(x^t - g_2)}{(g_{t-p+2} - g_2)}. \quad (2.52)$$

Here we have adopted the convention

$$g_{n-p+1+l} = g_l, \quad \text{for } k+1<l<p. \quad (2.53)$$

Consequently the infinitesimal distance has the form

$$ds^2 = \Big[\frac{\Pi_{t=p+1}^{n}(x^t - g_1)}{\Pi_{u\neq t}(g_u - g_1)}\Big] d\omega_1^2 + \Big[\frac{\Pi_{t=p+1}^{n}(x^t - g_2)}{\Pi_{u\neq t}(g_u - g_2)}\Big] d\omega_2^2 - \frac{1}{4}\sum_{t=p+1}^{n}\Big[\frac{\Pi_{u\neq t, p+1<u<n}(x^t - x^u)}{\Pi_{u=p+1}^{n+1}(x^t - g_u)}\Big] (dx^t)^2. \quad (2.54)$$

A standard choice of coordinates on S_n for this infinitesimal distance can be taken as

$$(s_1,...,s_{n+1}) = (u_1 v_1,...,u_1 v_{k+1}, u_2 w_1,..., u_2 w_{p+1}, u_3,...,u_{n-p-k-1}), \quad (2.55)$$

with u_i, v_j, and w_k coordinates as in (2.45). This procedure can be iterated without difficulty to find all separable coordinate systems on S_n. If we do this we obtain an infinitesimal distance of the form

$$ds^2 = \sum_{I=1}^{p}\Big\{\sum_{i\in N_I} (H_i^I)^2 (dx^i)^2\Big\}\Big[\prod_{l\in N_{p+1}} (\sigma_l + \alpha_I)\Big] + \sum_{j\in N_{p+1}} (H_j^{p+1})^2 (dx^j)^2,$$

$$\alpha_I \neq \alpha_J, \quad \text{if } I \neq J. \quad (2.56)$$

Here $\{N_1,...,N_{p+1}\}$ is a partition of the integers $1,...,n$ into mutually exclusive sets N_I, i.e., $N_I \cap N_J = \varnothing$. It follows from Eisenhart's types (2.21) (i)–(2.21) (iv) that $(\partial_{x^j}) H_i^{(I)} = 0$ if $j \notin N_I$. The curvature conditions can now be written down. The conditions $R_{ijji} = -H_i^2 H_j^2 \ (i\neq j)$ are equivalent to the equations

$$R_{ijji}^{(p+1)} = -(H_i^{p+1})^2 (H_u^{p+1})^2, \quad i,j \in N_{p+1}, \quad (2.57)$$

$$(H_i^I)^{-2}(H_j^I)^{-2} R_{ijji}^{(I)} + \Big[\prod_{K\in N_{p+1}} (\sigma_k + \alpha_I)\Big] \times \Big[\frac{1}{4}\sum_{l\in N_{p+1}} (H_l^{(p+1)})^{-2} \frac{\sigma_l'^2}{(\sigma_l + \alpha_I)^2} + 1\Big] = 0,$$

$$i,j \in N_I, \quad (2.58)$$

$$2\frac{\sigma_l''}{(\sigma_l + \alpha_I)} - \Big(\frac{\sigma_l'}{\sigma_l + \alpha_I}\Big)^2 - \Big(\frac{\sigma_l'}{(\sigma_l + \alpha_I)}\Big)\Big[\frac{\partial}{\partial x^l}\log(H_l^{(p+1)})^2 + (H_l^{(p+1)})^2 \sum_{\substack{m\neq l\\ m\in N_{p+1}}} \frac{1}{(H_m^{(p+1)})^2 (x^l - x^m)}\Big]$$

$$= -4(H_l^{(p+1)})^2, \quad l \in N_{p+1}, \quad (2.59)$$

$$\frac{1}{4}\sum_{l\in N_{p+1}}\frac{1}{(H_l^{(p+1)})^2}\frac{\sigma_l'^2}{(\sigma_l+\alpha_I)(\sigma_l+\alpha_J)}=-1. \tag{2.60}$$

Here we have used the notation $R^{(I)}_{hijk}$ to refer to the curvature tensor of the Riemannian manifold with infinitesimal distance

$$d\omega_I^2=\sum_{i\in N_I}(H_i^{(I)})^2(dx^i)^2. \tag{2.61}$$

These equations have the solutions

$$\left[\prod_{k\in N_{p+1}}(\sigma_l+\alpha_I)\right]=\frac{\Pi_{l\in N_{p+1}}(x^l-e_I)}{\Pi_{m\in N_{p+1}}(e_m-e_I)}, \tag{2.62}$$

$$(H_l^{p+1})^2=\frac{-1}{4}\left[\frac{\Pi_{m\in N_{p+1}\,(m\neq l)}(x^m-x^l)}{\Pi_{n=1}^{n_{p+1}+1}(x^l-e_n)}\right],\quad l\in N_{p+1}, \tag{2.63}$$

$$R^{(I)}_{ijji}=-(H_i^{(I)})^2(H_j^{(I)})^2,$$
$$I=1,\ldots,p+1,\quad i,j\in N_I, \tag{2.64}$$

where $n_{p+1}=\dim N_{p+1}$. The infinitesimal distance can always be written in the form

$$ds^2=\sum_{I=1}^{p}d\omega_I^2\left[\frac{\Pi_{l=1}^{n_1}(x^l-e_I)}{\Pi_{m\neq I}(e_m-e_I)}\right]$$
$$-\frac{1}{4}\sum_{i=1}^{n_1}\left[\frac{\Pi_{j=1}(x^i-x^j)}{\Pi_{j=1}^{n_1+1}(x^i-e_j)}\right](dx^i)^2, \tag{2.65}$$

where each $d\omega_I^2$ is the infinitesimal distance of a S_{p_a}. The coordinates on each S_{p_a} are again separable. Clearly we must have the constraint $\Sigma_{I=1}^{p}p_I+n_1=n$. Using this infinitesimal distance we can construct all separable coordinate systems inductively. The basic building blocks of separable coordinate systems are the elliptic coordinates on spheres of various dimensions. We will prescribe a graphical procedure for obtaining admissible coordinate systems, essentially giving the admissible embeddings of spheres inside spheres, which are allowed so as to correspond to separable coordinates.

III. THE CONSTRUCTION OF SEPARABLE COORDINATE SYSTEMS ON S_n

As we have seen in the previous section the basic building blocks of separable coordinate systems on S_n are the p-sphere elliptic coordinates

$${}_ps_j^2=\frac{\Pi_{l=1}^{p}(x^l-e_j)}{\Pi_{j\neq i}(e_i-e_j)},\quad \sum_{j=1}^{p+1}{}_ps_j^2=1,$$
$$p=1,\ldots,n,\quad j=1,\ldots,p+1. \tag{3.1}$$

Two important examples of these coordinates are

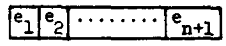
(i) $p=1$: $${}_1s_1^2=\frac{(x^1-e_1)}{(e_2-e_1)},\quad {}_1s_2^2=\frac{(x^1-e_2)}{(e_1-e_2)}, \tag{3.2}$$

where ${}_1s_1^2+{}_1s_2^2=1$, $e_1<x^1<e_2$; and

(ii) $p=2$: $${}_2s_1^2=\frac{(x^1-e_1)(x^2-e_1)}{(e_2-e_1)(e_3-e_1)},$$
$${}_2s_2^2=-\frac{(x^1-e_2)(x^2-e_2)}{(e_2-e_1)(e_3-e_1)},$$
$${}_2s_3^2=\frac{(x^1-e_3)(x^2-e_3)}{(e_3-e_1)(e_3-e_2)}, \tag{3.3}$$

where ${}_2s_1^2+{}_2s_2^2+{}_2s_3^2=1$, $e_1<x^1<e_2<x^2<e_3$.

We will develop a graphical calculus for calculating admissible coordinate systems. We represent elliptical coordinates on S_n by the "irreducible" block

 . (3.4)

Each separable coordinate system will be associated with a directed tree graph. Consider, for example, the sphere S_2. There are two possibilities.

(1) The first possibility is the irreducible block [e_1|e_2|e_3]. Most treatments of elliptic coordinates on S_2 correspond to the choice $e_1=0$, $e_2=1$, $e_3=a>1$. This is just a reflection of the fact that for Jacobi elliptic coordinates the variables x^i and e_i always can be subjected to the transformation

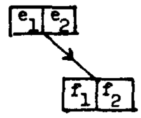
$$x^{i'}=ax^i+b,\quad e_j'=ae_j+b,$$
$$i=1,\ldots,n,\quad j=1,\ldots,n+1. \tag{3.5}$$

Thus we can always choose $e_1=0$ and $e_2=1$. [Note in particular that [e_1|e_2] can always be replaced by [0|1]. Putting $x^1=\cos^2\varphi$ we recover ${}_1s_1=\cos\varphi$, ${}_1s_2=\sin\varphi$ $(0<\varphi<2\pi)$.]

(ii) The second system is the usual choice of spherical coordinates

$$s_1=\sin\theta\cos\varphi,\quad s_2=\sin\theta\sin\varphi,\quad s_3=\cos\theta. \tag{3.6}$$

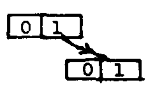
This system can be considered as the result of attaching a circle to a circle and is the prototype for the construction of more complicated systems. The graph

 (3.7)

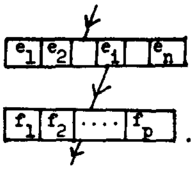
is taken to correspond to the choice of coordinates

$$s_1^2={}_1u_1^2=\frac{(x^1-e_1)}{(e_2-e_1)},$$
$$s_2^2=({}_1u_2^2)({}_1v_1^2)=\frac{(x^1-e_2)}{(e_1-e_2)}\frac{(x^2-f_1)}{(f_2-f_1)}, \tag{3.8}$$
$$s_3^2=({}_1u_2^2)({}_1v_2^2)=\frac{(x^1-e_2)}{(e_1-e_2)}\frac{(x^2-f_2)}{(f_1-f_2)},$$
$$e_1<x^1<e_2,\quad f_1<x^2<f_2.$$

Clearly, choosing angle variables on the S_1's, the choice of spherical coordinates corresponds to the graph

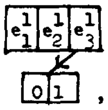
 . (3.9)

Only the square of origin of the arrow is of importance for a given arrow connecting two irreducible blocks, not the target square. The general branching law for an arrow connecting two irreducible blocks is readily given:

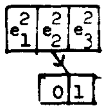


(3.10)

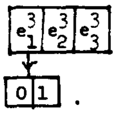
We should also note here that because of the availability of transformations of the type (3.5) some graphs that look different do in fact correspond to the same coordinate system. Indeed, consider graphs of type



(3.11a)



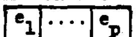
(3.11b)

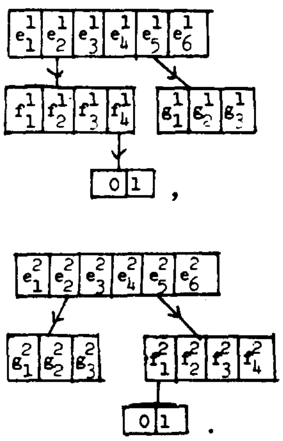


(3.11c)

These graphs correspond to Lamé[9] rotational coordinates on the sphere S_3. There are, however, only two distinct such coordinate systems. In fact, if the coordinates x^i and e_j^1 $(i=1,2, j=1,2,3)$ are subjected to the transformation

$$x^i \to -x^i = y^i;$$
$$e_1^1 \to -e_1^1 = e_3^3,\quad e_2^1 \to -e_2^1 = e_2^3, \tag{3.12}$$
$$e_3^1 \to -e_3^1 = e_1^3,$$

we see that the (3.11a) and (3.11c) correspond to the same type of coordinates. Graphs that are related in this way can be recognized by the feature that if the branch below a given irreducible block  is obtained from that of another graph by reflection about a vertical at the center of the corresponding $e_1^1 \cdots e_p^1$ block, then the two graphs are equivalent. (We are assuming, of course, that all other features of the graphs are identical.) Graphs that are essentially the same can be related by several transformations of the type (3.5) and the situation gets more complicated, e.g.,



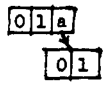
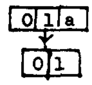
(3.13)

If the two irreducible blocks of S_n and S_p occur as indicated in (3.10), as part of some larger graph, this means that the elliptic coordinates ${}_nu_1,\ldots,{}_nu_{n+1}$ and ${}_pv_1,\ldots,{}_pv_{p+1}$ of these blocks must occur in the combinations

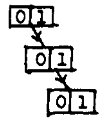
$$w_1 = {}_nu_1,\ldots,w_i = ({}_nu_i)({}_pv_1),\ldots,$$
$$w_{i+p+1} = ({}_nu_i)({}_pv_{p+1}),\quad w_{i+p+2} = {}_nu_{i+1},\ldots,$$
$$w_{p+n+1} = {}_nu_{n+1}. \tag{3.14}$$

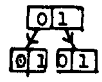
Arrows may emanate from different squares (e_i's) of the same block but cannot be directed at the same block. With these rules we may construct graphs corresponding to all separable coordinate systems on S_n. For $n=3$, we have the following possibilities[3,4]:

(1)  Jacobi elliptic coordinates, (3.15)

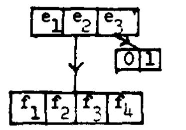
(2) (a)  (b)  Lamé rotational coordinates, (3.16)

(3)  Lamé subgroup reduction, (3.17)

(4)  spherical coordinates, (3.18)

(5)  cylindrical coordinates. (3.19)

The formation of more complicated graphs is now clear. Thus,



(3.20)

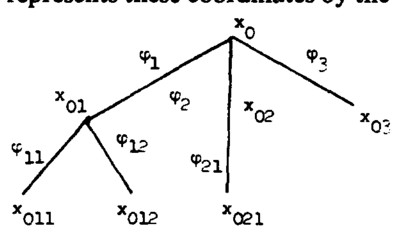
is a coordinate system on S_6 with coordinates

$$s_1^2 = ({}_2u_1)^2,\quad s_2^2 = ({}_2u_2)^2({}_3v_1)^2,$$
$$s_3^2 = ({}_2u_2)^2({}_3v_2)^2,\quad s_4^2 = ({}_2u_2)^2({}_3v_3)^2,$$
$$s_5^2 = ({}_2u_2)^2({}_3v_4)^2,\quad s_6^2 = ({}_2u_3)^2({}_1w_1)^2, \tag{3.21}$$
$$s_7^2 = ({}_2u_3)^2({}_1w_2)^2.$$

Vilenkin[7] has studied polyspherical coordinates on S_n and developed a graphical technique for constructing them. For example, he considers the coordinates on S_6:

$$x_0 = \cos\varphi_3\cos\varphi_2\cos\varphi_1,$$
$$x_{03} = \sin\varphi_3,$$
$$x_{02} = \cos\varphi_3\sin\varphi_2\cos\varphi_{21},$$
$$x_{01} = \cos\varphi_3\cos\varphi_2\sin\varphi_1\cos\varphi_{12}\cos\varphi_{11}, \tag{3.22}$$
$$x_{021} = \cos\varphi_3\sin\varphi_2\sin\varphi_{21},$$
$$x_{012} = \cos\varphi_3\cos\varphi_2\sin\varphi_1\sin\varphi_{12},$$
$$x_{011} = \cos\varphi_3\cos\varphi_2\sin\varphi_1\cos\varphi_{12}\sin\varphi_{11},$$

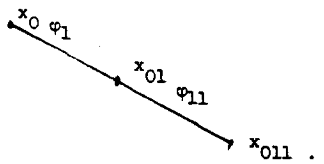
and represents these coordinates by the graph



For him, spherical coordinates on S_2,

$$x_0 = \cos \varphi_1,$$
$$x_{01} = \sin \varphi_1 \cos \varphi_{11}, \qquad (3.23)$$
$$x_{011} = \sin \varphi_1 \sin \varphi_{11},$$

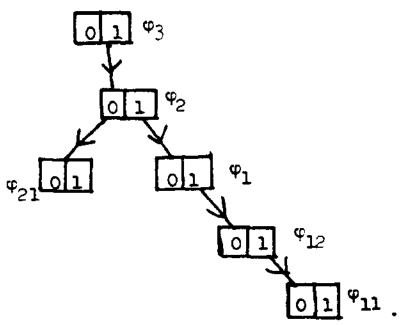
correspond to the graph



Vilenkin denotes coordinates of *rank r* by $x_{0i_1 \cdots i_r}$ and in the example of (3.22) arranges coordinates in the order

$$x_{011}, x_{012}, x_{021}, x_{01}, x_{02}, x_{03}, x_0, \qquad (3.24)$$

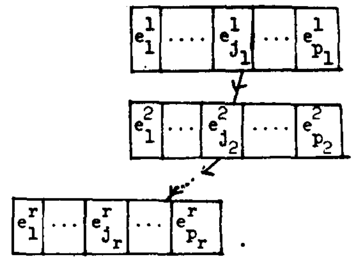
i.e., coordinates of higher rank precede those of lower rank while coordinates of equal rank are ordered lexicographically. Coordinates of the form $x_{0i_1 \cdots i_s i_{s+1} \cdots i_m}$ are called *subordinate* to the coordinate $x_{0i_1 \cdots i_s}$. Further, the coordinate $x_{0j_1 \cdots j_m}$ *essentially precedes* the coordinate $x_{0i_1 \cdots i_s}$ if $m > s$, and $j_k = i_k$ for $1 \leqslant k \leqslant s-1$ and $j_s < i_s$. The coordinate $x_{0i_1 \cdots i_s}$ *essentially follows* $x_{0j_1 \cdots j_m}$. To extract coordinates on S_n from this notation let $x_{0i_1 \cdots i_m}$ be a vertex of nonzero rank. A rotation $g(\varphi)$ by the angle $\varphi = \varphi_{i_1 \cdots i_m}$ in the $(x_{0i_1 \cdots i_{m-1}}, x_{0i_1 \cdots i_m})$ plane is then associated with this vertex. In this way Vilenkin constructs graphs representing the various possible polyspherical coordinates on S_n. In our notation his coordinate system (3.22) is represented by the graph



From these considerations we see that Vilenkin's polyspherical coordinates are the special case of separable coordinates on S_n consisting of those graphs that contain only the irreducible blocks of type [0|1].

IV. PROPERTIES OF SEPARABLE SYSTEMS IN S_n

Here we make more precise our graphic techniques through a prescription for writing down the standard coordinates s_i, $i = 1,\ldots,n+1$, on S_n in terms of the separable coordinates. A given standard coordinate coming from a given graph consists of a product of r factors, which we denote $x^{j_1 \cdots j_r}_{p_1 \cdots p_r} = ({}_{p_1}u_{j_1}) \cdots ({}_{p_r}u_{j_r})$. This is obtained by tracing the complete length of a branch of a given tree graph, i.e.,



We can then set up an ordering $<$ for the products $x^{j_1 \cdots j_r}_{P_1 \cdots P_r}$. We say that $x^{j_1 \cdots j_r}_{P_1 \cdots P_r} < x^{i_1 \cdots i_s}_{Q_1 \cdots Q_s}$ if $P_1 = Q_1, j_1 = i_1, \ldots, P_t = Q_t, j_t < i_t, P_{t+1} \neq Q_{t+1}, \ldots, j_s \neq i_s$. Then if we arrange the products in increasing order, say $x_1,\ldots,x_{n+1}$, we can identify this ordered n-tuple with $s_1,\ldots,s_{n+1}$. For the example (3.21) given above, the choice of coordinates corresponds to this ordering.

Having settled on a prescription for writing down the coordinates corresponding to a given coordinate system on S_n, we can now discuss the separation equations for both the Hamilton–Jacobi and Helmholtz equations. Let us first consider the coordinates corresponding to the irreducible block [e_1|e_2|······|e_{n+1}]. The Hamilton–Jacobi equation in these coordinates is

$$H = \sum_{i=1}^{n} \frac{1}{[\Pi_{j \neq i}(x^i - x^j)]} P_i^2 = E, \qquad (4.1)$$

where

$$P_i = \left[\prod_{j=1}^{n+1} (x^i - e_j)\right] \frac{\partial S}{\partial x^i}.$$

The separation equations are

$$\left[\prod_{j=1}^{n+1} (x^i - e_j)\right] (\partial_{x^i} S_i)^2 + \left[E(x^i)^{n-1} + \sum_{j=2}^{n} \lambda_j (x^i)^{n-j}\right] = 0. \qquad (4.2)$$

If we set $E = \lambda_1$, then the constants of the motion associated with the separation parameters $\lambda_1,\ldots,\lambda_n$ are

$$I_1^n = \sum_{i>j} I_{ij}^2 \quad \text{(second-order Casimir invariant)},$$
$$I_2^n = \sum_{i>j} S_1^{ij} I_{ij}^2, \qquad (4.3)$$
$$I_n^n = \sum_{i>j} S_n^{ij} I_{ij}^2,$$

where

$$S_l^{ij} = \frac{1}{l!} \sum_{i_1,\ldots,i_l \neq}{}' e_{i_1} \cdots e_{i_l},$$

and the summation extends over $i_1,\ldots,i_l \neq i,j$ and $i_l \neq i_m$ for $l \neq m$. For the associated Helmholtz equation the eigenvalues of Δ_n have the form $\sigma(\sigma + n - 1)$ and the Helmholtz equation becomes

$$\sum_{i=1}^{n} \frac{1}{[\Pi_{j \neq i}(x^i - x^j)]} \left\{\sqrt{\mathscr{P}_i} \frac{\partial}{\partial x^i}\left(\sqrt{\mathscr{P}_i} \frac{\partial \Psi}{\partial x^i}\right)\right\} = -\sigma(\sigma + n - 1)\Psi, \qquad (4.4)$$

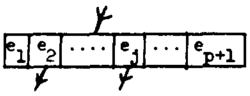
where

$$\mathscr{P}_i = \prod_{j=1}^{n} (x^i - e_j).$$

The separation equations are

$$\sqrt{\mathcal{P}_i}\,\frac{\partial}{\partial x^i}\left(\sqrt{\mathcal{P}_i}\,\frac{\partial \Psi_i}{\partial x^i}\right) + \left[\sigma(\sigma+n-1)(x^i)^n + \sum_{j=2}^{n-1}\tilde{\lambda}_j (x^i)^{n-j}\right]\Psi_i = 0. \tag{4.5}$$

The identification $\tilde{\lambda}_1 = \sigma(\sigma+n-1)$ enables us to further identify the symmetry operators whose eigenvalues are $\tilde{\lambda}_j$ with the expressions (4.3) where $I_{ij} \to \hat{I}_{ij}$ and $[\tilde{I}^n_j, \tilde{I}^m_k] = 0$. For an irreducible block appearing in an admissible graph the generalizations of these equations can be computed readily. Consider the block shown as part of a given graph:



Then define $d_i\,(i = 1,...,p+1)$ as follows: $d_i = 0$, if there is no arrow emanating downward from the block $\boxed{e_i}$; otherwise d_i is a parameter.

From the form of the metric we see the variables $x^1,...,x^p$ coming from this block satisfy an equation of the form

$$\sum_{i=1}^{p} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 + \sum_{i=1}^{p}\left[\frac{\Pi_{j\neq i}(e_i - e_j)}{\Pi_{j=1}^{p}(x^j - e_i)}\right] d_i = E_p. \tag{4.6}$$

Using the relation

$$\frac{1}{\Pi_{j=1}^{p}(x^j - e_k)} = \frac{1}{\Pi_{i>j}(x^i - x^j)}\left[\sum_{l=1}^{p}\frac{T_l}{(x^l - e_k)}\right], \tag{4.7}$$

where

$$T_l = (-1)^{l+1}\prod_{i>j}{}'\,(x^i - x^j),$$

with $i,j \neq l$, we see that the separation equations have the form

$$\left[\prod_{j=1}^{p}(x^i - e_j)\right]\left(\frac{dS_i}{dx^i}\right)^2 + \sum_{k=1}^{p+1}\frac{\Pi_{j\neq k}(e_k - e_j)d_k}{(x^i - e_k)} + \left[E_p (x^i)^{p-1} + \sum_{l=2}^{p}\lambda_l (x^i)^{p-l}\right] = 0. \tag{4.8}$$

For the corresponding Helmholtz equation the situation is somewhat more complicated. With each ${}_p u_j$ $(j = 1,...,p+1)$ we associate an index k_j, which is calculated as follows: If the irreducible block occurs as the rth step down from the trunk of the graph and if we write out the S_i in terms of our coordinates then k_j is the number of coordinates for which $x^{j_1 \cdots j \cdots j_q}_{p_1 \cdots p \cdots p_q}$ (rth column) occurs. The Helmholtz equation assumes the form

$$\sum_{i=1}^{p}\frac{1}{[\Pi_{j\neq i}(x^i - x^j)]}\left\{\sqrt{\frac{\mathcal{P}_i}{\mathcal{Q}_i}}\,\frac{d}{dx^i}\left(\sqrt{\mathcal{P}_i \mathcal{Q}_i}\,\frac{d\Psi}{dx^i}\right)\right\} + \sum_{i=1}^{p}\left[\frac{\Pi_{j\neq i}(e_i - e_j)}{\Pi_{j=1}^{p}(x^j - e_i)}\right] t_i \Psi = -\sigma(\sigma+p-1)\Psi, \tag{4.9}$$

where

$$\mathcal{P}_i = \prod_{j=1}^{p}(x^i - e_j), \quad \mathcal{Q}_i = \prod_{j=1}^{p}(x^i - e_j)^{k_j - 1},$$

$t_i = 0$ if $k_i = 1$ and $t_i = j_i(j_i + k_i - 1)$ if $k_i \neq 1$. The separation equations become

$$\sqrt{\frac{\mathcal{P}_i}{\mathcal{Q}_i}}\,\frac{\partial}{\partial x^i}\left(\sqrt{\mathcal{P}_i \mathcal{Q}_i}\,\frac{\partial \Psi_i}{\partial x^i}\right) + \left\{\sum_{k=1}^{p}\frac{\Pi_{j\neq k}(e_k - e_j)}{(x^i - e_k)} t_k + \left[\sigma(\sigma+p-1)(x^i)^{p-1} + \sum_{l=2}^{p}\tilde{\lambda}_l (x^i)^{p-l}\right]\right\}\Psi_i = 0. \tag{4.10}$$

If we take the coordinates (3.21) and choose

$$ {}_2u_j^2 = \frac{\Pi_{i=1}^{2}(x^i - e_j)}{\Pi_{j\neq i}(e_i - e_j)}, \quad j = 1,2,3, \quad i = 1,2,$$
$$ {}_3v_l^2 = \frac{\Pi_{i=3}^{5}(x^i - f_l)}{\Pi_{m\neq l}(f_m - f_l)}, \quad l = 1,2,3,4, \quad l = 3,4,5, \tag{4.11}$$
$$ {}_1w_s^2 = \frac{(x^6 - g_s)}{(g_t - g_s)}, \quad t,s = 1,2, \quad t \neq s,$$

then the separation equations for the Hamilton–Jacobi equation are

$$\text{(i)} \quad \left[\prod_{j=1}^{3}(x^i - e_j)\right]\left(\frac{dS_i}{dx^i}\right)^2 + \frac{(e_2 - e_3)(e_2 - e_1)}{(x^i - e_2)} d_2 + \frac{(e_3 - e_2)(e_3 - e_1)}{(x^i - e_3)} d_3 + Ex^i + \lambda_1 = 0, \quad i = 1,2;$$

$$\text{(ii)} \quad \left[\prod_{m=1}^{4}(x^l - f_m)\right]\left(\frac{dS_l}{dx^l}\right)^2 + d_2 (x^l)^2 + \lambda_2 x^l + \lambda_3 = 0, \quad l = 3,4,5; \tag{4.12}$$

$$\text{(iii)} \quad \left[\prod_{s=1}^{2}(x^6 - g_s)\right]\left(\frac{dS_6}{ds^6}\right)^2 + d_3 = 0;$$

and for the Helmholtz equation the corresponding separation equations are

$$\text{(i)}\quad \sqrt{\frac{\Pi_{j=1}^{3}(x^i-e_j)}{(x^i-e_2)^3(x^i-e_3)}}\frac{d}{dx^i}\left(\sqrt{\prod_{j=1}^{3}(x^i-e_j)(x^i-e_2)^3(x^i-e_3)}\frac{d\Psi_i}{dx^i}\right)$$
$$+\left[\frac{(e_3-e_2)(e_3-e_1)}{(x^i-e_3)}j_1(j_1+2)+\frac{(e_3-e_2)(e_3-e_1)}{(x^i-e_3)}j_2^2+j(j+5)x^i+\lambda_1\right]\Psi_i=0,\quad i=1,2;$$
$$\text{(ii)}\quad \sqrt{\prod_{m=1}^{4}(x^l-f_m)}\frac{d}{dx^l}\left(\sqrt{\prod_{m=1}^{4}(x^l-f_m)}\frac{d\Psi_l}{dx^l}\right)+[l(l+2)(x^l)^2+\lambda_2x^l+\lambda_3]\Psi_l=0,\quad l=3,4,5; \tag{4.13}$$
$$\text{(iii)}\quad \sqrt{\prod_{s=1}^{2}(x^6-g_s)}\frac{d}{dx^6}\left(\sqrt{\prod_{s=1}^{2}(x^6-g_s)}\frac{d\Psi_6}{dx^6}\right)+j_2^2\,\psi_6=0;$$
$$ds^2=ds_1^2+\cdots+ds_Q^2.$$

Once we are given the coordinates and have computed the associated separation equations for (I) and (II) we can also compute the Killing tensors corresponding to the separation constants: In (4.8) we put $\lambda_1=E_p$. Given ${}_pu_j$, two coordinates s_i,s_k are said to be *connected* if they both contain ${}_pu_j$. The corresponding Killing tensors are then calculated from the formulas (4.3) with I_{ij}^2 replaced by $\Sigma_{r>s}I_{rs}^2$, where the sum extends over all indices r connected to i and s connected to j. The Killing tensors correspond to I_1^m-type operators of the next irreducible block of dimension m connected further up the branch in question. For example, consider the coordinates (3.21). The corresponding Killing tensors are

$$L_1=\sum_{i>j}I_{ij}^2,$$
$$L_2=e_1\left(\sum_{i=2}^{5}I_{6i}^2+I_{7i}^2\right)+e_2\,(I_{16}^2+I_{17}^2)+e_3\left(\sum_{j=2}^{5}I_{ij}^2\right),$$
$$L_3=\sum_{k>l}{}'\,I_{kl}^2,\quad k,l=2,3,4,5,$$
$$L_4=(f_1+f_2)I_{45}^2+(f_1+f_3)I_{35}^2+(f_1+f_4)I_{34}^2+(f_2+f_3)I_{25}^2+(f_2+f_4)I_{24}^2+(f_3+f_4)I_{23}^2, \tag{4.14}$$
$$L_5=f_1f_2I_{45}^2+f_1f_3I_{35}^2+f_1f_4I_{34}^2+f_2f_3I_{25}^2+f_2f_4I_{24}^2+f_3f_4I_{23}^2,$$
$$L_6=I_{67}^2.$$

For the Hamilton–Jacobi equation these tensors have the constant values

$$L_1\sim E_1,\quad L_2\sim\lambda_1,\quad L_3\sim d_2,$$
$$L_4\sim\lambda_2,\quad L_5\sim\lambda_3,\quad L_6\sim d_3,$$

and for the Helmholtz equation with $I_{ij}\to\hat I_{ij}$ the resulting operators L_i $(i=1,\ldots,6)$ have the eigenvalues $\tilde L_1\sim j(j+5)$, $\tilde L_2\sim\lambda_1$, $\tilde L_3\sim l(l+2)$, $\tilde L_4\sim\lambda_2$, $\tilde L_5\sim\lambda_3$, $\tilde L_6\sim j_2^2$.

V. SEPARATION OF VARIABLES ON R_n

As was the case for S_n all separable coordinate systems in R_n can be chosen to be orthogonal.

Theorem: Let $\{x^i\}$ be a coordinate system on R_n for which the Hamilton–Jacobi equation admits separation of variables and let q be the number of ignorable variables. Then it is always possible to choose an equivalent coordinate system $\{x^i\}$ such that $g^{ij}=\delta^{ij}H_i^{-2}$, i.e., the coordinates are orthogonal. Furthermore, the ignorable variables $\alpha_1,\ldots,\alpha_q$ can always be taken such that

$$p_{\alpha_1}=I_{12},\ldots,p_{\alpha_p}=I_{2p-1,2p}\,p_{\alpha_{p+1}}=P_{2p+1},\ldots,p_{\alpha_q}=P_{p+q}.$$

Proof: We use methods similar to those for S_n. Any element of the algebra $\mathscr{E}(n)$ is conjugate to one of the two forms

(1) $L=I_{12}+b_2I_{34}+\cdots b_\nu I_{2\nu-1,2\nu}+\beta P_{2\nu+1}$,

where $\beta=0$ if $n=2\nu$; and

(2) $L'=P_n$.

Let $\{x^i\}$ be a separable system with $q=1$. It follows from the block-diagonal form that this system must be orthogonal. Furthermore, without loss of generality we can assume that $p_{\alpha_1}=L$ or $p_{\alpha_1}=L'$. It is evident that the second case can occur and is in accordance with the statement of the theorem. For the first case we can always choose the ignorable variable α_1 so that it is related to the Cartesian coordinates $(y_1,\ldots,y_n)$ by

$$(y_1,\ldots,y_n)=(\rho_1\cos(\alpha_1+w_1),\rho_1\sin(\alpha_1+w_1),\ldots,\rho_\nu\cos(b_\nu\alpha_1+w_\nu),\rho_\nu\sin(b_\nu\alpha_1+w_\nu),\rho_{\nu+1}+\beta\alpha_1,y_{2\nu+2},\ldots,y_n). \tag{5.1}$$

The infinitesimal metric then has the form

$$ds^2=d\rho_1^2+\cdots+d\rho_\nu^2+\rho_1^2(d\alpha_1+dw_1)^2+\cdots+\rho_\nu^2(b_\nu\,d\alpha_1+dw_\nu)^2+(d\rho_{\nu+1}+\beta\,d\alpha_1)^2+dy_{2\nu+2}^2+\cdots+dy_n^2. \tag{5.2}$$

If there is only one ignorable variable the coordinate system must be orthogonal and consequently

$$\rho_1^2\,dw_1+\sum_{j=2}^{\nu}b_j\rho_j^2\,dw_j+\beta\,d\rho_{\nu+1}=0. \tag{5.3}$$

This is possible only if $b_2=\cdots=b_\nu=\beta=0$ and $dw_1=0$. (By redefining α_1 we then can take $w_1=0$.) Therefore if we have only one ignorable variable then $p_\alpha=I_{12}$ or P_n.

Now suppose we have q Killing vectors p_{α_i}, $i=1,\ldots,q$. Then they must be of the form

$$L_1 = I_{12} + \sum_{l>p}^{s} b_l^1 I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^1 P_m,$$
$$L_2 = I_{34} + \sum_{l>p}^{s} b_l^2 I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^2 P_m,$$
$$\vdots$$
$$L_p = I_{2p-1,2p} + \sum_{l>p}^{s} b_l^p I_{2l-1,2l} + \sum_{m=2s+1}^{n} \gamma_m^p P_m,$$
$$L_{p+1} = \sum_{m=2s+1}^{n} \gamma_m^{p+1} P_m, \tag{5.4}$$
$$\vdots$$
$$L_q = \sum_{m=2s+1}^{n} \gamma_m^q P_m.$$

The condition $\{L_i, L_l\} = 0$ implies

$$b_k^i \gamma_{2k-1}^l = b_k^i \gamma_{2k}^l = 0, \tag{5.5}$$

for $i = 1,...,p$, $l = 1,...,q$, $k = p+1,...,s$. We are assuming that there is always one b_k^i nonzero for each k and some i. Then $\gamma_{2k-1}^l = \gamma_{2k}^l = 0$ for $k = p+1,...,s$ and $l = 1,...,q$. The Cartesian coordinates are

$$(y_1,...,y_n) = \Big(\rho_1 \cos(\alpha_1 + w_1), \rho_1 \sin(\alpha_1 + w_1),..., \rho_p \cos(\alpha_p + w_p), \rho_p \sin(\alpha_p + w_p),$$
$$\rho_{p+1} \cos\Big(\sum_{l=1}^{p} b_{p+1}^l \alpha_l + w_{p+1}\Big),..., \rho_s \sin\Big(\sum_{l=1}^{p} b_s^l \alpha_l + w_s\Big), \sum_{l=1}^{q} \gamma_{2s+1}^l \alpha_l + w_{2s+1},..., \sum_{l=1}^{q} \gamma_n^l \alpha_l + w_n\Big). \tag{5.6}$$

This set of candidate ignorable variables can take the necessary block-diagonal form only if $dw_i = 0$, $b_k^i = 0$, for $i = 1,...,p$ and $k = p+1,...,s$. Also $dw_l = 0$, for $l = 2s+1,...,n$. We can thus assume that $w_1 = \cdots = w_p = 0$, $w_{2s+1} = \cdots = w_n = 0$. This implies $\gamma_m^i = 0$, for $i = 1,...,q$, $m = 2s+q-p+1,...,n$, and we can also assume $\gamma_m^i = 0$, for $i = 1,...,p$ and $m = 2s+1,...,2s+q-p+1$. Consequently we can take

$$L_1 = I_{12},...,L_p = I_{2p-1,2p},$$
$$L_{p+1} = P_{2s+1},...,L_q = P_{2s+q-p}, \tag{5.7}$$

and there are no nonzero elements $g^{\alpha_i \alpha_j}$, $1 \leqslant i < j \leqslant q$, in the metric. By a suitable $E(n)$ motion we can always choose $s = p$. All separable coordinates in R_n must be orthogonal. Q.E.D.

To find all possible separable coordinate systems on R_n we proceed in analogy with what we have done for S_n. If we choose orthogonal coordinates in which none of the σ_{ij} are constant functions, then

$$H_i^2 = X_i \Big[\prod_{j \neq i} (x^i - x^j)\Big] \quad (i = 1,...,n), \tag{5.8}$$

where, as usual,

$$ds^2 = \sum_{i=1}^{n} H_i^2 (dx^i)^2.$$

The conditions $R_{ijji} = 0$ are equivalent to (2.16) in which the right-hand side is zero. These conditions have the solution

$$(1/X_i)^{(n+1)} = 0 \quad (i = 1,...,n)$$

and

$$\frac{1}{X_i} = \sum_{l=0}^{n} a_l (x^i)^{n-l} = g(x^i).$$

Again we look for choices of $g(x)$ that are compatible with a positive definite metric. There are only two possibilities:

$$\text{(i)} \quad g(x) = \prod_{i=1}^{n} (x - e_i) \quad \text{(elliptic coordinates)},$$
$$e_1 < x^1 < e_2 < \cdots < x^{n-1} < e_n < x^n;$$
$$\text{(ii)} \quad g(x) = \prod_{i=1}^{n-1} (x - e_i) \quad \text{(parabolic coordinates)}, \tag{5.9}$$
$$x^1 < e_1 < x^2 < e_2 < \cdots < x^{n-1} < e_{n-1} < x^n.$$

These metrics give coordinates in n dimensions that are the analog of elliptic and parabolic coordinates, familiar in Euclidean spaces of dimension $n = 2,3$. To these systems we may associate Cartesian coordinates by

$$\text{(i)} \quad y_j^2 = c^2 \frac{\Pi_{i=1}^n (x^i - e_j)}{\Pi_{i \neq j} (e_i - e_j)}, \quad j = 1,...,n, \quad c \in \mathbb{R};$$
$$\text{(ii)} \quad y_1 = (c/2)(x^1 + \cdots + x^n + e_1 + \cdots + e_{n-1}), \tag{5.10}$$
$$y_j^2 = -c^2 \frac{\Pi_{i=1}^n (x^i - e_{j-1})}{\Pi_{i \neq j-1} (e_i - e_{j-1})}, \quad j = 2,...,n.$$

These two systems are fundamental for generating all separable systems on R_n. As an example of the relevance of these systems we consider the case when some of the σ_{ij} functions are constants. We first treat, as we did for S_n, the case in which the metric coefficients have the form (2.22) and (2.23). Then, as we have shown, these coefficients reduce to

$$H_i^2=\left[X_i\prod_{j\neq i}(x^i-x^j)\right]\left(\prod_{h=k+1}^{n}\sigma_h\right),$$
$$H_l^2=\left[X_l\prod_{m\neq l}(x^l-x^m)\right]. \qquad (5.11)$$

The conditions $R_{kllk}=0$ imply that the quadratic form

$$ds'^2=\sum_l H_l^2(dx^l)^2$$

is that of a flat space. The remaining nonzero conditions are

$$\widetilde{H}_i^{-2}\widetilde{H}_j^{-2}\widetilde{R}_{ijji}+\left(\prod_{l=k+1}^{n}\sigma_l\right)\left[\sum_{m=k+1}^{n}\frac{1}{4H_m^2}\left(\frac{\sigma_m'}{\sigma_m}\right)^2\right]=0, \qquad (5.12)$$

$$2\frac{\sigma_l''}{\sigma_l}-\left(\frac{\sigma_l'}{\sigma_l}\right)^2-\left(\frac{\sigma_l'}{\sigma_l}\right)\times\left[\frac{\partial}{\partial x^l}\log H_l^2+H_l^2\sum_{m\neq l}\frac{1}{H_m^2(x^l-x^m)}\right]=0, \qquad (5.13)$$

with $\widetilde{R}_{ijji}$ as in (2.25). These equations are satisfied provided $\widetilde{R}_{ijji}=-\widetilde{H}_i^2\widetilde{H}_j^2$ and the function $\Sigma=(\Pi_{l=k+1}^{n}\sigma_l)$ is given as follows:

$$\frac{1}{X_l}=\prod_{m=k+1}^{N}(x^m-e_l),\quad l=k+1,\dots,n, \qquad (5.14)$$

$$\Sigma=\frac{\Pi_{l=k+1}^{N}(x^l-e_m)}{\Sigma_{l\neq m}(e_l-e_m)},\quad \text{for some } m \text{ fixed}, \qquad (5.15)$$

where $N=n,n-1$. The functions $1/X_i$ are given by

$$\frac{1}{X_i}=-4\prod_{j=1}^{k+1}(x^i-e_j'). \qquad (5.16)$$

The systems are related to Cartesian coordinates on R_n according to

$$(y_1,\dots,y_n)=(w_1s_1,\dots,w_1s_{k+1},w_2,\dots,w_{n-k}), \qquad (5.17)$$

where

$$\sum_{i=1}^{k+1}s_i^2=1\quad\text{and}\quad s_j^2=\frac{\Pi_{i=1}^{k}(x^i-e_j')}{\Pi_{j\neq i}(e_i'-e_j')},$$

(i) $$w_l^2=\frac{\Pi_{m=1}^{n-k}(x^m-e_l)}{\Pi_{m\neq l}(e_m-e_l)},\quad l=1,\dots,n-k;$$

(ii) $$w_l^2=\frac{\Pi_{m=1}^{n-k}(x^m-e_l)}{\Pi_{m\neq l}(e_m-e_l)},\quad l=1,\dots,n-k-1, \qquad (5.18)$$

$$w_{n-k}=\frac{1}{2}\left(\sum_{m=1}^{n-k}x^m+e_1+\cdots+e_{n-k}\right).$$

There exists an additional possibility that could be discounted for S_n: $\sigma_l=a_l$, $l=k+1,\dots,n$. This corresponds to the case in which the infinitesimal distance can be written

$$ds^2=ds_1^2+ds_2^2, \qquad (5.19)$$

where ds_1^2 is the infinitesimal distance for elliptic or parabolic coordinates in R_k and ds_2^2 is a similar infinitesimal distance on R_{n-k}. We can mimic the procedure adopted for S_n. The only essential difference is that the infinitesimal distance can be expressed, in general, as a sum of distances that can be identified with Euclidean subspaces. This reflects the fact that if $\{y^i\}$, $i=1,\dots,n_1$, and $\{z^j\}$, $j=1,\dots,n_2$, are separable coordinate systems in Euclidean spaces R_{n_1} and R_{n_2} with respective infinitesimal distances ds_1^2, ds_2^2, then the coordinates $\{y^i,z^j\}$, $i=1,\dots,n_1$, $j=1,\dots,n_2$, can be regarded as a separable coordinate system on $R_{n_1+n_2}$ with corresponding infinitesimal distance $ds^2=ds_1^2+ds_2^2$. This is, of course, not the case for S_n. This property of Euclidean space coordinates naturally extends to separable coordinate systems $\{x_p^i\}$, $i=1,\dots,n_p$, $p=1,\dots,Q$, on R_p in such a way that

$$ds^2=ds_1^2+\cdots+ds_Q^2.$$

In general the infinitesimal distance can be written as a sum of basic forms

$$ds^2=\sum_{I=1}^{Q}ds_I^2, \qquad (5.20)$$

where

$$ds_I^2=\sum_{i=1}^{n_I}\left[\frac{\Pi_{l=1}^{N_I}(x^l-e_i^I)}{\Pi_{j\neq i}(e_j^I-e_i^I)}\right]d\omega_i^2+d\sigma_I^2. \qquad (5.21)$$

Here the $d\sigma_I^2$ is the infinitesimal distance corresponding to elliptic or parabolic coordinates for a flat space of dimension N_I. Also $n_I\leqslant N_I$ for elliptic coordinates with a strict inequality for parabolic coordinates.

The $d\omega_i^2$ is the infinitesimal distance of some separable coordinate system on the sphere S_{p_I} and $n=\Sigma_{I=1}^{Q}(N_I+p_I)$. To establish a graphic procedure for construction of separable coordinates we need only analyze one of the basic forms ds_I^2. We should also mention here that if $N_I=1$, then the basic form is written

$$ds_I^2=w^2\,d\omega^2+dw^2. \qquad (5.21')$$

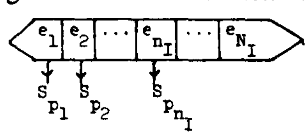
A basic form could in fact correspond to elliptic or parabolic coordinates on R_{N_I} and no $d\omega_i^2$ terms. We associate this with $n_I=0$ in (5.21).

For our construction we need only invent graphic representations for elliptic and parabolic coordinates in R_n, the analog of the irreducible blocks on S_n. We adopt the following notation:

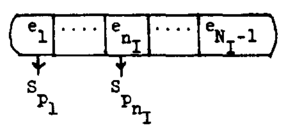
(1) elliptic coordinates , $n\geqslant 1$,

(2) parabolic coordinates , $n\geqslant 2$.

It is clear that only elliptic coordinates exist in one dimension. The graphical representation of a basic form corresponding to the infinitesimal distance ds_I^2 has the appearance

(i) 

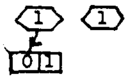
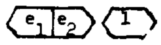
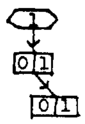
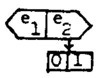
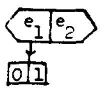
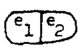
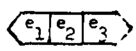
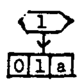
,

(ii) 

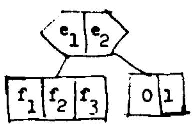
.

Attached to each leg descending from the top block is the appropriate graph of the coordinate system on the S_{p_i} giving

TABLE I. Separable systems for R_3.

1)		Cartesian coordinates
2)		cylindrical coordinates
3)		elliptic cylindrical
4)		parabolic cylindrical
5)		spherical coordinates
6)		prolate spheroidal
7)		oblate spheroidal
8)		parabolic coordinates
9)		paraboloidal coordinates
10)		ellipsoidal coordinates
11)		conical coordinates

rise to the form $d\omega_i^2$. The general graph corresponding to a separable system then can be constructed as a sum of disconnected graphs for basic forms. We first illustrate this technique for the separable systems of R_3 (see Ref. 21) (see Table I). As an additional nonstandard example, consider the graph

 ,

which defines a coordinate system in R_5. The coordinates can be chosen as

$$y_i^2 = c^2\left[\frac{(x^1-e_1)(x^2-e_1)}{(e_2-e_1)}\right]({}_2u_i)^2, \quad i=1,2,3,$$
$$y_4^2 = c^2\left[\frac{(x^1-e_2)(x^2-e_2)}{(e_1-e_2)}\right]\cos^2 x^5, \tag{5.22}$$
$$y_5^2 = c^2\left[\frac{(x^1-e_2)(x^2-e_2)}{(e_1-e_2)}\right]\sin^2 x^5,$$

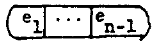
where

$$({}_2u_i)^2 = \frac{(x^3-f_k)(x^4-f_i)}{(f_j-f_i)(f_k-f_i)}, \quad i,j,k \text{ distinct.}$$

We can set up a natural ordering for separable systems in R_n. For a given basic form we can suppose the natural ordering of the e_i's in the leading irreducible block on the ordering of the S_{p_i} branches and then write down coordinates in a standard way.

The ordering of the disconnected parts of the graph is presumed already given. There are equivalences (relating graphs of various coordinate systems) that we have already discussed for the n-sphere and, of course, there is an additional equivalence corresponding to the permutation of disconnected parts of a given graph. The separation equations can be readily computed also. For the elliptic and parabolic coordinate blocks

(1)  ,

(2)  ,

the Hamilton–Jacobi equation has the form

$$H = \sum_{i=1}^{n} \frac{1}{[\Pi_{j\neq i}(x^i - x^j)]} P_i^2 = E, \tag{5.23}$$

where

$$P_i = \left[\prod_{j=1}^{N_k}(x^i - e_j)\right]\frac{\partial S}{\partial x^i},$$

with $N_1 = n$ (elliptic coordinates) and $N_2 = n-1$ (parabolic coordinates). The separation equations are

$$\left[\prod_{j=1}^{N_k}(x^i-e_j)\right](\partial_{x^i}S_i)^2 + \left[E(x^i)^{n-1} + \sum_{j=2}^{n}\lambda_j(x^i)^{n-j}\right] = 0. \tag{5.24}$$

If we identify $E = \lambda_1$, the constants of the motion associated with the separation parameters $\lambda_1,...,\lambda_n$ are

$$_1I_1^n = P_1^2 + \cdots + P_n^2,$$
$$_1I_2^n = \sum_{i>j} I_{ij}^2 + c^2\sum_{i=1}^{n} S_1^i P_i^2, \tag{5.25a}$$
$$\vdots$$
$$_1I_n^n = \sum_{i>j} S_{n-2}^{ij}\, I_{ij}^2 + c^2\sum_{i=1}^{n} S_{n-1}^i P_i^2,$$

where

$$S_l^i = \frac{1}{l!}\sum_{i_1,...,i_l\neq} e_{i_1}\cdots e_{i_l}$$

and the sum is over $i_1,...,i_l \neq i$; and

$$ {}_2I_1^n = P_1^2 + \cdots + P_n^2, $$

$$ {}_2I_2^n = c\sum_{k=2}^{n}\{I_{1k},P_k\} + c^2S_1P_1^2 + \sum_{j=2}^{n} c^2S_1^jP_j^2, $$

$$ {}_2I_3^n = \sum_{k=2}^{n} cS_1^k\{I_{1k},P_k\} + \sum_{i>j>2} I_{ij}^2 + c^2S_2P_1^2 + c^2\sum_{j=2}^{n} S_2^jP_j^2, \qquad (5.25b) $$

$$ {}_2I_4^n = \sum_{k=2}^{n} cS_2^k\{I_{1k},P_k\} + \sum_{i>j>2} S_1^{ij}I_{ij}^2 + c^2S_3P_1^2 + c^2\sum_{j=2}^{n} S_3^jP_j^2, $$

$$ \vdots $$

$$ {}_2I_n^n = \sum_{k=2}^{n} cS_{n-2}^k\{I_{1k},P_k\} + \sum_{i>j>2} S_{n-3}^{ij}I_{ij}^2 + c^2S_{n-1}P_1^2, $$

where the S_n^j have the same significance as for elliptic coordinates and

$$ S_l = \frac{1}{l!}\sum_{i_1,\ldots,i_l\neq} e_{i_1}\cdots e_{i_l}. $$

For the corresponding Helmholtz equation the eigenvalues of Δ_n are $-k^2$ (k real) and the Helmholtz equation reads

$$ \sum_{i=1}^{n}\frac{1}{[\Pi_{j\neq i}(x^i-x^j)]}\left\{\sqrt{\mathscr{P}_i}\frac{\partial}{\partial x^i}\left(\sqrt{\mathscr{P}_i}\frac{\partial\Psi}{\partial x^i}\right)\right\} = -k^2\Psi, \qquad (5.26) $$

where $\mathscr{P}_i = \Pi_{j=1}^{N_k}(x^i-e_j)$. The separation equations are

$$ \sqrt{\mathscr{P}_i}\frac{\partial}{\partial x^i}\left(\sqrt{\mathscr{P}_i}\frac{\partial\Psi_i}{\partial x^i}\right) + \left[k^2(x^i)^n + \sum_{j=2}^{n-1}\lambda_j(x^i)^{n-j}\right]\Psi_i = 0. \qquad (5.27) $$

For a basic form such as ds_I^2 the separation equations for the Hamilton–Jacobi equation have the form

$$ \left[\prod_{j=1}^{N_{I_k}}(x^i-e_j)\right]\left(\frac{\partial S_i}{\partial x^i}\right)^2 + \sum_{l=1}^{n_I}\frac{\Pi_{j\neq k}(e_l-e_j)}{(x^i-e_l)}k_l + \left[E_I(x^i)^{N_{I_k}-1} + \sum_{l=2}^{N_{I_k}}\lambda_l(x^i)^{N_{I_k}-l}\right] = 0, \qquad (5.28) $$

where k_l is the constant value of the Hamiltonian on the sphere whose infinitesimal distance is $d\omega_l^2$. For the Helmholtz equation the corresponding contribution of this basic form is the equation

$$ \sum_{i=1}^{N_{I_k}}\frac{1}{[\Pi_{j\neq i}(x^i-x^j)]}\left[\sqrt{\frac{\mathscr{P}_i}{\mathscr{Q}_i}}\frac{\partial}{\partial x^i}\left(\sqrt{\mathscr{P}_i\mathscr{Q}_i}\frac{\partial\Psi}{\partial x^i}\right) + \sum_{i=1}^{n_I}\left[\frac{\Pi_{j\neq i}(e_i-e_j)}{\Pi_{j=1}^{n_{I_k}}(x^j-e_i)}\right]j_i(j_i+p_i-1)\Psi = -k_I^2\Psi, \right. \qquad (5.29) $$

where

$$ \mathscr{P}_i = \prod_{k=1}^{N_{I_k}}(x^i-e_k), \quad \mathscr{Q}_i = \prod_{k=1}^{N_{I_k}}(x^i-e_k)^{d_k-1}, $$

$$ d_k = \begin{cases} p_k+1, & \text{if } k=1,\ldots,n_I,\\ 1, & \text{if } k=n_I+1,\ldots,N_{I_k}. \end{cases} $$

The separation equations are

$$ \sqrt{\frac{\mathscr{P}_i}{\mathscr{Q}_i}}\frac{d}{dx^i}\left(\sqrt{\mathscr{P}_i\mathscr{Q}_i}\frac{d\Psi_i}{dx^i}\right) + \left[\sum_{k=1}^{n_I}\left[\frac{\Pi_{j\neq k}(e_k-e_j)}{(x^i-e_k)}\right]j_i(j_i+p_i-1) + k_I^2(x^i)^{N_{I_k}-1} + \sum_{l=2}^{N_{I_k}}\lambda_l(x^i)^{N_{I_k}-l}\right]\Psi_i = 0. \qquad (5.30) $$

In the example on R_5 the separation equations for the Hamilton–Jacobi equation are

$$ \left[\prod_{j=1}^{2}(x^i-e_j)\right]\left(\frac{dS_i}{dx^i}\right)^2 + \frac{(e_2-e_1)}{(x^i-e_2)}k_1 + \frac{(e_1-e_2)}{(x^i-e_1)}k_2 + k^2x^i + \lambda_1 = 0, \quad i=1,2, \qquad (5.31) $$

and for the Helmholtz equation they are

$$ \sqrt{\frac{\Pi_{j=1}^2(x^i-e_j)}{(x^i-e_1)^2(x^i-e_2)}}\frac{d}{dx^i} \times\left[\prod_{j=1}^{2}(x^i-e_j)(x^i-e_1)^2(x^i-e_2)\frac{d\Psi_i}{dx^i}\right] + \left[\frac{(e_2-e_1)}{(x^i-e_2)}j_1^2 + \frac{(e_1-e_2)}{(x^i-e_1)}j_2(j_2+1) + k^2x^i + \tilde{\lambda}_1\right]\Psi_i = 0. \qquad (5.32) $$

For the elliptic case the only new prescription required is that P_i^2 be replaced by $\Sigma_r P_r^2$, where the sum extends over all induces r connected to i. Similar comments apply to expressions of the form $\{I_{kl},P_l\}$.

For our example the operators that describe separation are

$$ \begin{aligned} L_1 &= I_{12}^2 + I_{23}^2 + I_{13}^2,\\ L_2 &= f_1I_{23}^2 + f_2I_{13}^2 + f_3I_{12}^2,\\ L_3 &= I_{45}^2,\\ L_4 &= \sum_{i=1}^{5}P_i^2,\\ L_6 &= \sum_{i=1}^{3}(I_{i4}^2 + I_{i5}^2) + c^2[e_2(P_1^2+P_2^2+P_3^2) + e_1(P_4^2+P_5^2)]. \end{aligned} \qquad (5.33) $$

The operator L_6 corresponds to the separation constant λ_1.

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