ON THE PRINCIPLES OF HAMILTON AND CARTAN

BY J. W. CAMPBELL

1. *Introduction.* A holonomic dynamical system can be characterized by a Hamilton stationary action integral or by a Cartan integral invariant. A. E. Taylor* has extended Cartan's principle to the case of non-holonomic systems. It is the purpose of this paper to obtain by a different method both Taylor's extension of Cartan's principle and the corresponding extension of Hamilton's principle. The latter is an extension in a sense not hitherto obtained.

2. *Extensions to Non-holonomic Systems.* As we shall make the extensions by transforming non-holonomic systems into equivalent holonomic systems, we shall first state the principles for holonomic systems in the form which is most suitable for our purposes.

Suppose we have a dynamical system defined by the equations

$$(1) \qquad \dot{q}_r = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \qquad (r = 1, \cdots, k).$$

Then in the qpt space these equations determine a $(2k)$-parameter family of trajectories.

To make use of the notation of Cartan,† we shall denote the parameters of the system by $\alpha, \beta_1, \cdots, \beta_{2k-1}$, where the parameters are such that for the β's constant the trajectories given by $\alpha_0 \leqq \alpha \leqq \alpha_1$ form a tube of trajectories.

Also a parameter u is introduced such that $dt = \rho du$, where ρ is a function of u, α, and the β's, arbitrary except that it is always of the same sign.

When the β's and u are fixed and α varies from α_0 to α_1, then a locus of corresponding points on the tube determined by the β's is obtained. The locus is arbitrary owing to the presence of ρ,

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* A. E. Taylor, *On integral invariants of non-holonomic dynamical systems*, this Bulletin, vol. 40 (1934), pp. 735–742.

† E. Cartan, *Leçons sur les Invariants Intégraux*, 1922.

and the trajectories given by α_0 and α_1 are one and the same trajectory. Such a locus we shall call a *Cartan locus*.

Now we have

$$\sum_1^k \left\{ \left(dq_r - \frac{\partial H}{\partial p_r} dt \right) \delta p_r + \left(- dp_r - \frac{\partial H}{\partial q_r} dt \right) \delta q_r + \left(dH - \frac{\partial H}{\partial t} dt \right) \delta t \right\} \tag{2}$$

$$= \sum_1^k (\delta p_r dq_r - dp_r \delta q_r) - (\delta H dt - dH \delta t).$$

Moreover, the integral of the second member of (2) over a Cartan locus can be written as the negative of

$$d \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t, \tag{3}$$

and its integral over an arc of a trajectory becomes

$$\delta \int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt, \tag{4}$$

if in the second case we take variation in the qpt space with fixed end points; for in both cases the total integral arising from integration by parts vanishes. It follows that the vanishing of (3) or (4) identically in δq_r, δp_r, and δt implies equations (1), and that equations (1) imply the vanishing of (3) and (4). That is, *a holonomic system*

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \qquad (r = 1, \cdots, k),$$

is characterized either by the Cartan integral invariant

$$\int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t,$$

or by the Hamilton extremal integral

$$\int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt.$$

Consider now a non-holonomic dynamical system defined by the equations

$$(5)\qquad \dot q_r = \frac{\partial H}{\partial p_r},\qquad \dot p_r = -\frac{\partial H}{\partial q_r} + \sum_1^m a_{rs}\lambda_s,\qquad (r = 1, \cdots, k),$$

subject to the non-integrable relations

$$(6)\qquad \sum_1^k a_{rs}\dot q_r + a_s = 0,\qquad (s = 1, \cdots, m).$$

Let

$$(7)\qquad \int_{t_0}^{t}\sum_1^m a_{rs}\lambda_s dt = f_r,\qquad \int_{t_0}^{t}\sum_1^m a_s\lambda_s dt = f,$$

where the integrals are taken along a trajectory starting from its intersection with the hyperplane $t=t_0$. Then $(f_1, \cdots, f_k, f)$ is a vector-function, and by (5) and (6) we may write

$$(8)\qquad \begin{aligned} dq_r - \frac{\partial H}{\partial p_r}dt &= 0, \\ dp_r + \frac{\partial H}{\partial q_r}dt - df_r &= 0, \qquad (r = 1, \cdots\ k), \\ dH - \frac{\partial H}{\partial t}dt + df &= 0. \end{aligned}$$

Let us now make the transformation

$$(9)\qquad p_r = p_r' + f_r,\qquad q_r = q_r',\qquad H = H' - f.$$

Then

$$\frac{\partial H}{\partial p_r} = \frac{\partial H'}{\partial p_r'},\qquad \frac{\partial H}{\partial q_r} = \frac{\partial H'}{\partial q_r'}.$$

Moreover, since $H' = H(q_r', p_r' + f_r, t) + f$, we have

$$(11)\qquad \begin{aligned} \frac{\partial H'}{\partial t} &= \frac{\partial H}{\partial t} + \sum_1^k \frac{\partial H}{\partial p_r}\frac{df_r}{dt} + \frac{df}{dt} \\ &= \frac{\partial H}{\partial t} + \sum_1^k \dot q_r \sum_1^m a_{rs}\lambda_s + \sum_1^m a_s\lambda_s = \frac{\partial H}{\partial t}, \end{aligned}$$

by (6). Hence equations (8) may be written in the form

$$(12)\qquad \dot{q}_r' = \frac{\partial H'}{\partial p_r'}, \qquad \dot{p}_r' = -\frac{\partial H'}{\partial q_r'}, \qquad (r = 1, \cdots, k).$$

Conversely, if (12) hold, where q_r', p_r', H' are defined by (9), then (8) hold; and with the f_r and f defined by (7), equations (8) imply (5) and (6), for the λ's are functions of t which are not identically zero. Therefore we may state the following theorem.

THEOREM 1. *Every non-holonomic system given by* (5) *and* (6) *is by the transformations* (7) *and* (9) *equivalent to the holonomic system given by* (12).

By the known theory for holonomic systems, the system given by (12) is characterized either by the Cartan integral invariant

$$(13)\qquad \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r'\delta q_r' - H'\delta t,$$

or by the Hamilton extremal integral

$$(14)\qquad \int_{u_0}^{u_1} \sum_1^k p_r' dq_r' - H'dt.$$

Then since equations (12) are equivalent to (5) and (6), by applying (7) and (9) to (13) and (14), we may state the following result.

THEOREM 2. *A non-holonomic system which is defined by* (5) *and* (6) *is characterized either by the Cartan integral invariant*

$$(15)\qquad \int_{\alpha_0}^{\alpha_1} \left\{ \sum_1^k p_r\delta q_r - H\delta t - \sum_1^k \left(\int_{t_0}^t \sum_1^m \lambda_s a_{rs} dt\right)\delta q_r - \left(\int_{t_0}^t \sum_1^m \lambda_s a_s dt\right)\delta t\right\},$$

or by the Hamilton extremal integral

$$(16)\qquad \int_{u_0}^{u_1} \left\{ \sum_1^k p_r dq_r - H dt - \sum_1^k \left(\int_{t_0}^t \sum_1^m \lambda_s a_{rs} dt\right) dq_r - \left(\int_{t_0}^t \sum_1^m \lambda_s a_s dt\right) dt\right\}.$$

The first part of Theorem 2 is the same as Taylor's result, and the second part is the corresponding extension of Hamilton's principle.

3. *Remarks*. (a) In the expressions (15) and (16), the order of integration is not reversible.

(b) Comparison may be made with a previous extension of Hamilton's principle. It has long been known that for non-holonomic systems the action integral

$$\int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt$$

is an extremal with respect to varied paths which are related to the trajectories by virtual displacements which are consistent with the non-holonomic constraints; that is, when the virtual displacements satisfy the relations

$$\sum_1^k a_{rs}\delta q_r + a_s \delta t = 0, \qquad (s = 1, \cdots, m).$$

When so obtained, however, the varied paths are themselves not in general kinematically possible paths, and in summing up his discussion on this point Whittaker* has written as follows:

"Hamilton's principle applies to every dynamical system whether holonomic or not. In every case the varied path considered is to be derived from the actual orbit by displacements which do not violate the kinematical equations representing the constraints; but it is only for holonomic systems that the varied motion is a possible motion, so that if we compare the actual motion with adjacent motions which obey the kinematical equations of constraint, Hamilton's principle is true only for holonomic systems."

Thus this extension is in a restricted sense, and implies nothing about the class of all motions which are real in the sense of being possible. On the other hand, (16) is an extremal with respect to all paths, whether kinematically possible or not.

It is obvious that all the results obtained for non-holonomic systems apply equally well to systems with superfluous coordinates.

THE UNIVERSITY OF ALBERTA

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* E. T. Whittaker, *Analytical Dynamics*, 1927, p. 250.