

The non-linear superposition principle and the Wei–Norman method

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ABSTRACT

Group theoretical methods are used to study some properties of the Riccati equation, which is the only differential equation admitting a nonlinear superposition principle. The Wei–Norman method is applied for obtaining the associated differential equation in the group $SL(2, \mathbb{R})$. The superposition principle for first order differential equation systems and Lie-Scheffers theorem are also analysed from this group theoretical perspective. Finally, the theory is applied in the solution of second order differential equations like time independent Schrödinger equation.

1 Introduction

Nonlinear phenomena are having everyday more and more importance in almost all branches of science, and in particular in Physics. One of the most important nonlinear differential equation is the Riccati equation (see e.g. [1] and references therein). This differential equation has recently been shown to be related with the factorization method (see e.g.[2, 3, 4, 5]). The recent interest of this equation is steadily increasing since Witten’s introduction of supersymmetric Quantum Mechanics [6, 7].

Two important features of this Riccati type differential equation are:

i) Even if we cannot find the general solution by means of a finite number of quadratures over elementary functions of the coefficients $a_i(t)$ defining the differential equation,

$$\frac{dx(t)}{dt} = a_0(t) + a_1(t)x(t) + a_2(t)x^2(t), \quad (1.1)$$

once a particular solution $x_1(t)$ is known, the change of variable $x = x_1 + z$ leads to a new differential equation of the Bernoulli (for $n = 2$) type:

$$\frac{dz}{dt} = (2a_2x_1 + a_1)z + a_2z^2. \quad (1.2)$$

This is a particular instance of (1.1) for which $a_0 = 0$. Notice that under the change of variable $w = 1/x$ the Riccati equation (1.1) becomes

$$\frac{dw(t)}{dt} + a_0(t) w^2(t) + a_1(t) w(t) + a_2(t) = 0. \quad (1.3)$$

In particular, the Bernoulli equation (1.2) can be reduced to a linear one just by introducing the new variable $u = 1/z$. In this way, if we know a particular solution the general solution can be found through two quadratures.

ii) When three particular solutions, $x_1(t), x_2(t), x_3(t)$, of the differential equation (1.1) are known, the general solution can be written with no quadrature at all:

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k, \quad (1.4)$$

where k is an arbitrary constant determining each particular solution.

In this sense we can say that there exists a nonlinear superposition principle for the Riccati equation, because the general solution can be expressed as a function $x = \Phi(x_1, x_2, x_3, k)$ of three particular solutions and one arbitrary constant k .

Our aim is to explain these facts from a group theoretic viewpoint and present some new ideas both about the Riccati equation itself and on the nonlinear superposition principle, for which the Riccati equation is the simplest case.

The organization of this paper is as follows. In Section 2 we review the problem of reducing a second order linear differential equation to a nonlinear first order Riccati equation, what means that the original linear superposition principle for the second order equation should be replaced by a nonlinear superposition principle. We also remark that this fact is due to a relation of the Riccati equation with the $SL(2, \mathbb{R})$ group to be explicated later. In Section 3 we explain a method developed by Wei and Norman [8] for determining the solution of a differential equation in a Lie group and we apply the method for the study of the Riccati equation, finding in this way the explicit form of the superposition principle as a consequence of some group theoretical computations. The superposition principle for first order differential equation systems and Lie–Scheffers theorem are studied in Section 4. It is shown that for an important class of such systems the problem of finding the general solution is reduced to the simpler problem of finding one particular solution of another system on a Lie group G , and moreover, even without solving directly this new system the solution of the original system can be easily found as soon as we know a fundamental set of solutions of it. Once again the simplest case is Riccati equation and the superposition principle can be found by determining the first integral of a system. Finally in Section 5 we give as an example the application of the Wei–Norman method in the solution of second order differential equations taking as a prototype the Schrödinger equation for the harmonic oscillator.

2 The nonlinear superposition principle

There is a well known method of relating a linear second order differential equation with a Riccati equation. Actually, given the linear second order differential equation

$$\frac{d^2 u}{dt^2} + b(t) \frac{du}{dt} + c(t) u = 0, \quad (2.1)$$

the property of linearity means that the vector field

$$X = u \frac{\partial}{\partial u}$$

generates a one-parameter Lie group of point symmetries of the equation (2.1)

$$\bar{t}(\epsilon) = t, \quad \bar{u}(\epsilon) = e^\epsilon u.$$

Changing coordinate u to a new one $v = \varphi(u)$ such that the vector field $X = u\partial/\partial u$ becomes a translation generator (Straightening-out Theorem), i.e. $X = \partial/\partial v$, and therefore determined by the equation $Xv = 1$, leads to $v = \log u$, i.e. $u = e^v$, and then,

$$\frac{du}{dt} = e^v \frac{dv}{dt} = u \frac{dv}{dt}. \quad (2.2)$$

When written in terms of this new coordinate the equation (2.1) becomes

$$\frac{d^2v}{dt^2} + b(t) \frac{dv}{dt} + \left(\frac{dv}{dt} \right)^2 + c(t) = 0.$$

The unknown function v does not appear in the preceding equation and therefore a lowering of the order is obtained when introducing the change of variable $x = \frac{dv}{dt}$, and then we will get a Riccati equation

$$\frac{dx}{dt} = -c - bx - x^2, \quad (2.3)$$

as it was pointed out in [9].

Therefore, the linear superposition principle for solutions of (2.1) translates in a nonlinear superposition principle for those of this Riccati equation, as it will be shown later.

Notice that (2.2) shows that

$$x = \frac{1}{u} \frac{du}{dt}. \quad (2.4)$$

The second order differential equation (2.1) is equivalent to the set of (2.3) and (2.4). We should also remark that two solutions u_1, u_2 of (2.1) project on the same solution of (2.3) if and only if there exists a nonzero real number $\lambda \in \mathbb{R}$ such that $u_2(0) = \lambda u_1(0)$ and $u_2'(0) = \lambda u_1'(0)$.

From the geometric viewpoint, Riccati equation can be interpreted as the one determining the integral curves of a time-dependent vector field

$$\Gamma = (a_0(t) + a_1(t)x + a_2(t)x^2) \frac{\partial}{\partial x}.$$

Let us remark that this vector field can be written as a linear combination with time dependent coefficients of the vector fields

$$L_0 = \frac{\partial}{\partial x}, \quad L_1 = x \frac{\partial}{\partial x}, \quad L_2 = x^2 \frac{\partial}{\partial x}, \quad (2.5)$$

which generate a three dimensional real Lie algebra with defining relations

$$[L_0, L_1] = L_0, \quad [L_0, L_2] = 2L_1 \quad [L_1, L_2] = L_2, \quad (2.6)$$

and therefore isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In fact, it is an easy matter to check that the (local) one-parameter transformation Lie groups of \mathbb{R} generated by L_0 , L_1 and L_2 are

$$x \mapsto x + \epsilon, \quad x \mapsto e^\epsilon x, \quad x \mapsto \frac{x}{1 - x\epsilon},$$

i.e., they are fundamental vector fields corresponding to the action of $SL(2, \mathbb{R})$ on the real line \mathbb{R} extended with a point at the infinity, $\bar{\mathbb{R}}$, given by

$$\Phi(A, x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ if } x \neq -\frac{\delta}{\gamma},$$

and

$$\Phi(A, \infty) = \frac{\alpha}{\gamma}, \quad \Phi(A, -\frac{\delta}{\gamma}) = \infty.$$

3 The Wei–Norman method

Let us consider a differential equation system

$$\frac{dx^i(t)}{dt} = X^i(x, t), \quad i = 1, \dots, n, \quad (3.1)$$

which can be seen as the differential equation system whose solutions are the integral curves of the time-dependent vector field

$$X = X^i(x, t) \frac{\partial}{\partial x^i}. \quad (3.2)$$

The theorem for existence and uniqueness of solutions of the preceding differential equation tells us that, for small enough t , there exists a map Φ_t applying the initial condition $x(0) = (x^i(0))$ into the corresponding value $x^i(t)$. Correspondingly, functions f defined in a neighborhood of $x(0)$ transform as

$$[U(t)f](x) = f(\Phi_t^{-1}(x)), \quad (3.3)$$

and taking derivatives with respect to t we obtain

$$\begin{aligned} \left[\frac{dU(t)}{dt} f \right] (x) &= \frac{d}{dt} \left((f \circ \Phi_t^{-1})(x) \right) = \frac{dx^i}{dt} \frac{\partial f}{\partial x^i} (\Phi_t^{-1}(x)) \\ &= X(f) (\Phi_t^{-1}(x)) = [U(t)(Xf)](x). \end{aligned} \quad (3.4)$$

This relation is valid for any differentiable function f , and therefore

$$\frac{dU(t)}{dt} = U(t)X.$$

We recall [8] that given such a differential equation for operators for X being a linear combination of vector fields in \mathbb{R}^n ,

$$X = \sum_{k=1}^m a_k(t) L_k,$$

namely,

$$\frac{dU(t)}{dt} = \sum_{k=1}^m a_k(t)U(t)L_k, \quad (3.5)$$

where the L_k span a finite dimensional real Lie algebra, it is possible to write the general solution in the form

$$U(t) = \prod_{i=1}^m \exp(g_i(t)L_i), \quad (3.6)$$

where the functions $g_i(t)$ are given by the solution of the system obtained from the relation

$$\sum_{i=1}^m a_i(t)L_i = \sum_{i=1}^m \dot{g}_i(t) \left[\prod_{j=i+1}^m \exp(-g_j(t)\text{ad}L_j) \right] L_i, \quad (3.7)$$

and the initial condition $g_i(0) = 0$, $i = 1, \dots, m$.

This method proposed by Wei–Norman (see also [10]) can be used in the case of the Riccati equation and the generalization for other differential equation system involving several degrees of freedom is immediate. In fact, given the differential equation (1.1) there will be an evolution operator $U(t)$ which takes values in $SL(2, \mathbb{R})$ and satisfies the differential equation

$$\frac{dU(t)}{dt} = U(t)[a_0(t)L_0 + a_1(t)L_1 + a_2(t)L_2], \quad (3.8)$$

together with the initial condition $U(0) = I$, and where the vector fields L_k , for $k = 0, 1, 2$, are fundamental vector fields associated to the left action of $SL(2, \mathbb{R})$ on the extended real line $\bar{\mathbb{R}}$ that are explicitly given in (2.5) and generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

According to the Wei–Norman method [8], there will be functions $u_0(t), u_1(t), u_2(t)$ such that $u_0(0) = u_1(0) = u_2(0) = 0$ and

$$U(t) = \exp(u_1L_1) \exp(u_2L_2) \exp(u_0L_0). \quad (3.9)$$

Then, when using (3.7), we obtain

$$\begin{aligned} a_0(t)L_0 + a_1(t)L_1 + a_2(t)L_2 &= \dot{u}_1(t) \exp(-u_0(t)\text{ad}L_0) \exp(-u_2(t)\text{ad}L_2)L_1 \\ &\quad + \dot{u}_2(t) \exp(-u_0(t)\text{ad}L_0)L_2 + \dot{u}_0(t)L_0 \\ &= \dot{u}_1(L_1 - u_0L_0 + u_2L_2 - 2u_0u_2L_1 + u_0^2u_2L_0) \\ &\quad + \dot{u}_2(L_2 - 2u_0L_1 + u_0^2L_0) + \dot{u}_0L_0, \end{aligned}$$

and therefore the system of differential equations for the functions u_i

$$\begin{aligned} a_0(t) &= -u_0\dot{u}_1 + u_0^2u_2\dot{u}_1 + u_0^2\dot{u}_2 + \dot{u}_0 \\ a_1(t) &= \dot{u}_1 - 2u_0u_2\dot{u}_1 - 2u_0\dot{u}_2 \\ a_2(t) &= u_2\dot{u}_1 + \dot{u}_2, \end{aligned} \quad (3.10)$$

that can be written in normal form,

$$\begin{aligned} \dot{u}_0(t) &= a_0(t) + a_1(t)u_0(t) + a_2(t)u_0^2(t) \\ \dot{u}_1(t) &= a_1(t) + 2a_2(t)u_0(t) \\ \dot{u}_2(t) &= a_2(t) - a_1(t)u_2(t) - 2a_2(t)u_0(t)u_2(t). \end{aligned} \quad (3.11)$$

We remark that the first equation for u_0 is nothing but the original Riccati equation and therefore it seems that there is no advantage at all. However when looking in the other two equations, we see that provided that the appropriate solution for the first equation, the one determined by $u_0(0) = 0$ has been found, the solution for the second one is almost immediate and when introducing these values in the third equation this one becomes a first order differential equation and the solution reduces to a quadrature. In this sense we have reduced the problem of finding the general solution of (1.1) to the one of finding the particular solution such that $u_0(0) = 0$. This is quite similar to the property i) we mentioned in the Introduction.

Once the solution of (3.11) determined by $u_0(0) = u_1(0) = u_2(0) = 0$ has been found, the general solution of the differential equation will be written

$$x(t) = (U(t)x)(0) = [\exp(u_1 L_1) \exp(u_2 L_2) \exp(u_0 L_0)x]|_{t=0}, \quad (3.12)$$

and therefore

$$x(t) = \frac{e^{u_1} x_0}{1 - u_2 e^{u_1} x_0} + u_0, \quad (3.13)$$

where $x_0 = x(0)$.

Let us now fix three independent initial conditions. A possible set is the one given by $x_1(0) \rightarrow \infty$, $x_2(0) = 0$ y $x_3(0) = 1$, (actually any other three different initial conditions will be transformed to this one under an appropriate transformation of the group $SL(2, \mathbb{R})$).

Having in mind the form (3.13) of the general solution, we see that the above mentioned initial conditions determine the particular solutions

$$\begin{aligned} x_1(t) &= -\frac{1}{u_2(t)} + u_0(t) \\ x_2(t) &= u_0(t) \\ x_3(t) &= \frac{e^{u_1(t)}}{1 - u_2(t)e^{u_1(t)}} + u_0(t), \end{aligned} \quad (3.14)$$

and then the functions u_0, u_1, u_2 are determined as

$$u_0(t) = x_2(t) \quad (3.15)$$

$$u_1(t) = \log \left[\frac{(x_3(t) - x_2(t))(x_2(t) - x_1(t))}{(x_3(t) - x_1(t))} \right] \quad (3.16)$$

$$u_2(t) = -\frac{1}{x_2(t) - x_1(t)}, \quad (3.17)$$

and therefore, when putting these values in (3.13), we find that for the general solution

$$x(t) = \frac{x_0 x_1(t)(x_3(t) - x_2(t)) + x_2(t)(x_1(t) - x_3(t))}{x_0(x_3(t) - x_2(t)) + (x_1(t) - x_3(t))}, \quad (3.18)$$

which is the well known superposition principle for Riccati equation (1.1) which can also be written as

$$\frac{(x - x_2)(x_3 - x_1)}{(x - x_1)(x_3 - x_2)} = x_0. \quad (3.19)$$

The factorization (3.9) for the evolution operator is not the only possible one, but there are other five alternative factorizations. We next present the results for all possible reorderings:

II) When we consider the factorization

$$U(t) = \exp(g_0 L_0) \exp(g_1 L_1) \exp(g_2 L_2),$$

the associated system turns out to be

$$\begin{aligned}\dot{g}_0 &= a_0 e^{-g_1} \\ \dot{g}_1 &= a_1 - 2a_0 g_2 \\ \dot{g}_2 &= a_2 - a_1 g_2 + a_0 g_2^2,\end{aligned}\tag{3.20}$$

with $g_0(0) = g_1(0) = g_2(0) = 0$, and then the solution is

$$x(t) = \frac{e^{g_1}(x_0 + g_0)}{1 - g_2 e^{g_1}(x_0 + g_0)},\tag{3.21}$$

the three particular solutions x_1, x_2, x_3 being expressed as

$$\begin{aligned}x_1(t) &= -\frac{1}{g_2} \\ x_2(t) &= \frac{g_0 e^{g_1}}{1 - g_0 g_2 e^{g_1}} \\ x_3(t) &= \frac{e^{g_1}(1 + g_0)}{1 - g_2 e^{g_1}(1 + g_0)},\end{aligned}\tag{3.22}$$

with the inverse relation

$$\begin{aligned}g_0 &= \frac{x_2(x_1 - x_3)}{x_1(x_3 - x_2)} \\ g_1 &= \log \left[\frac{x_1^2(x_3 - x_2)}{(x_1 - x_2)(x_1 - x_3)} \right] \\ g_2 &= -\frac{1}{x_1}.\end{aligned}\tag{3.23}$$

We should remark that the third equation in (3.20) is a new Riccati equation and that once the solution for this new Riccati equation is found, we substitute its value in the second one and integrate without any difficulty, and when this value for g_1 is put in the first equation we can integrate it. Therefore the new result we have found here is the following: if one knows a solution of a related Riccati equation, given by the third one in (3.20) equation, satisfying $g_2(0) = 0$, the general solution of (1.1) can be found in a straightforward way.

III) The third possibility corresponds to the factorization

$$U(t) = \exp(h_2 L_2) \exp(h_1 L_1) \exp(h_0 L_0),\tag{3.24}$$

the associated system being

$$\dot{h}_0 = a_0 + a_1 h_0 + a_2 h_0^2\tag{3.25}$$

$$\dot{h}_1 = a_1 + 2a_2 h_0\tag{3.26}$$

$$\dot{h}_2 = a_2 e^{h_1},\tag{3.27}$$

and the general solution is then

$$x(t) = \frac{e^{h_1} x_0}{1 - h_2 x_0} + h_0. \quad (3.28)$$

The three particular solutions x_1, x_2, x_3 will be written as

$$\begin{aligned} x_1(t) &= -\frac{e^{h_1}}{h_2} + h_0 \\ x_2(t) &= h_0 \\ x_3(t) &= \frac{e^{h_1}}{1 - h_2} + h_0, \end{aligned} \quad (3.29)$$

the inverse relations being

$$\begin{aligned} h_0 &= x_2 \\ h_1 &= \log \left[\frac{(x_3 - x_2)(x_2 - x_1)}{(x_3 - x_1)} \right] \\ h_2 &= \frac{x_2 - x_3}{x_1 - x_3}. \end{aligned} \quad (3.30)$$

Notice that in this approach the first equation in (3.27) is the original equation (1.1).

IV) The fourth reordering leads to the factorization

$$U(t) = \exp(f_1 L_1) \exp(f_0 L_0) \exp(f_2 L_2), \quad (3.31)$$

associated system

$$\begin{aligned} \dot{f}_0 &= a_0 + a_1 f_0 - 2a_0 f_0 f_2 \\ \dot{f}_1 &= a_1 - 2a_0 f_2 \\ \dot{f}_2 &= a_2 - a_1 f_2 + a_0 f_2^2, \end{aligned} \quad (3.32)$$

and general solution

$$x(t) = \frac{e^{f_1} x_0 + f_0}{1 - f_2(e^{f_1} x_0 + f_0)}. \quad (3.33)$$

The three particular solutions are now

$$\begin{aligned} x_1(t) &= -\frac{1}{f_2} \\ x_2(t) &= \frac{f_0}{1 - f_2 f_0} \\ x_3(t) &= \frac{e^{f_1} + f_0}{1 - f_2(e^{f_1} + f_0)}, \end{aligned} \quad (3.34)$$

and the inverse relation

$$\begin{aligned} f_0 &= \frac{x_1 x_2}{x_1 - x_2} \\ f_1 &= \log \left[\frac{x_1^2 (x_3 - x_2)}{(x_1 - x_2)(x_1 - x_3)} \right] \end{aligned}$$

$$f_2 = -\frac{1}{x_1}. \quad (3.35)$$

We remark that now the third differential equation for f_2 in (3.32) is the same as in the case II) and it provides a new method of finding the general solution of (1.1) once the particular solution satisfying $f_2(0) = 0$ of the associated Riccati equation is found.

V) The fifth possibility is

$$U(t) = \exp(v_0 L_0) \exp(v_2 L_2) \exp(v_1 L_1), \quad (3.36)$$

with associated system

$$\begin{aligned} \dot{v}_0 &= a_0 e^{-v_1} \\ \dot{v}_1 &= a_1 - 2a_0 v_2 e^{-v_1} \\ \dot{v}_2 &= a_2 e^{v_1} - a_0 v_2^2 e^{-v_1}, \end{aligned} \quad (3.37)$$

and solution

$$x(t) = \frac{e^{v_1}(x_0 + v_0)}{1 - v_2(x_0 + v_0)}; \quad (3.38)$$

the expressions for the three particular solutions x_1, x_2, x_3 are

$$\begin{aligned} x_1(t) &= -\frac{e^{v_1}}{v_2} \\ x_2(t) &= \frac{e^{v_1} v_0}{1 - v_2 v_0} \\ x_3(t) &= \frac{e^{v_1}(1 + v_0)}{1 - v_2(1 + v_0)}, \end{aligned} \quad (3.39)$$

with the inverse relation

$$\begin{aligned} v_0 &= \frac{(x_3 - x_1)x_2}{(x_2 - x_3)x_1} \\ v_1 &= \log \left[\frac{x_1^2(x_2 - x_3)}{(x_2 - x_1)(x_1 - x_3)} \right] \\ v_2 &= -\frac{(x_2 - x_3)x_1}{(x_2 - x_1)(x_3 - x_1)}. \end{aligned} \quad (3.40)$$

VI) The last possibility is the factorization

$$U(t) = \exp(w_2 L_2) \exp(w_0 L_0) \exp(w_1 L_1), \quad (3.41)$$

and then the associated system is

$$\begin{aligned} \dot{w}_0 &= a_0 e^{-w_1} - a_2 w_0^2 e^{w_1} \\ \dot{w}_1 &= a_1 + 2a_2 w_0 e^{w_1} \\ \dot{w}_2 &= a_2 e^{w_1}, \end{aligned} \quad (3.42)$$

the general solution

$$x(t) = \frac{e^{w_1} x_0}{1 - w_2 x_0} + w_0 e^{w_1}, \quad (3.43)$$

and the expressions for x_1, x_2, x_3 ,

$$\begin{aligned}x_1(t) &= -\frac{e^{w_1}}{w_2} + w_0 e^{w_1} \\x_2(t) &= w_0 e^{w_1} \\x_3(t) &= \frac{e^{w_1}}{1 - w_2} + w_0 e^{w_1},\end{aligned}\tag{3.44}$$

with inverse relation

$$\begin{aligned}w_0 &= \frac{(x_3 - x_1)x_2}{(x_2 - x_1)(x_3 - x_2)} \\w_1 &= \log \left[\frac{(x_2 - x_1)(x_3 - x_2)}{(x_3 - x_1)} \right] \\w_2 &= \frac{(x_3 - x_2)}{(x_3 - x_1)}.\end{aligned}\tag{3.45}$$

Finally, we remark that in these two last approaches there is no uncoupled differential equation of the Riccati type whose solution allows us to find the solution of the two other remaining equations in the system, and therefore the general solution of the original Riccati equation, anymore. However, if in the third equation in (3.37) we define $\phi(t) = \dot{v}_1$ we will get the Riccati equation

$$\dot{\phi} = \frac{1}{2}\phi^2 + q(t)\phi + p(t)\tag{3.46}$$

with

$$q(t) = \frac{\dot{a}_2}{a_0}, \quad p(t) = \dot{a}_1 - 2a_0 a_2 - \frac{a_1 \dot{a}_2}{a_0} + \frac{a_1^2}{2}.\tag{3.47}$$

In a similar way, in the sixth case, taking derivatives in the second equation in (3.42) and after some manipulations, the equation for $\varphi = \dot{w}_1$ becomes a Riccati equation

$$\dot{\varphi} = s(t) + r(t)\varphi + \frac{1}{2}\varphi^2\tag{3.48}$$

where

$$r(t) = \frac{\dot{a}_2}{a_2}, \quad s(t) = \dot{a}_1 - \frac{a_1 \dot{a}_2}{a_2} + 2a_0 a_2 - \frac{1}{2}a_1^2.\tag{3.49}$$

Let us summarize the results we have got. We have reduced the problem of finding the general solution of the Riccati equation to the one of determining a curve in $SL(2, \mathbb{R})$ which is defined through its second class canonical coordinates, and this leads to a differential equation system. Once the curve in $SL(2, \mathbb{R})$ is known we are able to find the general solution of the Riccati equation. However, even if we are not able to find the solution of the corresponding differential equation system for the second class coordinates in the group, we know the form (3.12) of the general solution of the original Riccati equation. Even more, given a set of (three in the Riccati case) fundamental particular solutions we can determine the function giving us the superposition principle (3.18).

We have seen that the general solution of Riccati equation is given by

$$x(t) = \frac{x_0 x_1 (x_3 - x_2) + x_2 (x_1 - x_3)}{x_0 (x_3 - x_2) + (x_1 - x_3)},\tag{3.50}$$

where x_0 is a constant depending on the initial conditions. We aim now to show how it is possible to reconstruct the original differential equation once the superposition formula is given. In the case of Riccati equation the superposition formula (3.50) is equivalent to

$$x_0 = \frac{a - c x}{d x - b}$$

and it is easy to check that taking derivatives in the preceding relation we find

$$(\dot{a} - \dot{c}x - c\dot{x})(dx - b) - (a - cx)(d\dot{x} + d\dot{x} - \dot{b}) = 0$$

from where we find the following expression for \dot{x}

$$\dot{x} = \frac{(\dot{c}d - c\dot{d})}{(bc - ad)}x^2 + \frac{(-\dot{a}d + a\dot{d} + \dot{b}c - b\dot{c})}{(bc - ad)}x + \frac{(\dot{a}b - a\dot{b})}{(bc - ad)},$$

that is a Riccati equation.

On the other side, if we assume that there is a superposition formula for Riccati equation and we try to determine the function ϕ giving that formula, $x = \phi(x_1, x_2, x_3, k)$, we will have

$$\dot{x} = \dot{\phi} = \frac{\partial\phi}{\partial x_1}\dot{x}_1 + \frac{\partial\phi}{\partial x_2}\dot{x}_2 + \frac{\partial\phi}{\partial x_3}\dot{x}_3 = a_0 + a_1\phi + a_2\phi^2$$

from where the following system of partial differential equations is found

$$\begin{aligned} \frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} + \frac{\partial\phi}{\partial x_3} &= 1 \\ x_1 \frac{\partial\phi}{\partial x_1} + x_2 \frac{\partial\phi}{\partial x_2} + x_3 \frac{\partial\phi}{\partial x_3} &= \phi \\ x_1^2 \frac{\partial\phi}{\partial x_1} + x_2^2 \frac{\partial\phi}{\partial x_2} + x_3^2 \frac{\partial\phi}{\partial x_3} &= \phi^2. \end{aligned} \tag{3.51}$$

Now a long computation leads to the following expression for ϕ :

$$x(t) = \frac{kx_1(x_3 - x_2) + x_2(x_1 - x_3)}{k(x_3 - x_2) + (x_1 - x_3)}. \tag{3.52}$$

4 The superposition principle for first order differential equation systems and Lie-Scheffers theorem

We are now interested in studying the existence of a superposition principle for first order differential equation systems generalizing the one obtained for the Riccati equation. More explicitly, given a system

$$\frac{dx^i}{dt} = X^i(x, t), \quad i = 1, \dots, n, \tag{4.1}$$

we ask whether there exists a set $\{x^{(1)}, \dots, x^{(m)}\}$ of fundamental solutions and a function $\Phi : \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^n$ such that the general solution of (4.1) can be expressed as

$$x = \Phi(x^{(1)}, \dots, x^{(m)}; a_1, \dots, a_n),$$

where a_1, \dots, a_n are constants related with the initial conditions.

Before studying the general case, we remark that we know that, at least in the case where the system is the autonomous linear system

$$\frac{dx}{dt} = Ax, \quad x \in \mathbb{R}^n,$$

with A being a constant matrix, it defines a flow ϕ_t which can be considered as a curve on the general linear group $GL(\mathbb{R}, n)$ given by $\phi_t(x) = e^{At}x$. The flow satisfies

$$\frac{d\phi_t}{dt} = \phi_t \circ A$$

and it can be determined from a fundamental set of solutions, i.e., when the $n \times n$ matrix X whose columns are the vectors defining the solutions $\{x_1(t), \dots, x_n(t)\}$,

$$X(t) = (x_1(t), \dots, x_n(t)),$$

is an invertible matrix, then the equation

$$X(t) = e^{tA}X(0)$$

shows that the evolution operator e^{tA} is determined as $e^{tA} = X(t)X(0)^{-1}$, i.e., the fundamental system of solutions allows us to find the flow of our first order differential equation system. In other words, we have a positive answer to our previous question with $m = n$ and Φ being the linear map $\Phi(x^{(1)}, \dots, x^{(n)}; a_1, \dots, a_n) = a_1 x^{(1)} + \dots + a_n x^{(n)}$.

From here it is clear that with any linear differential equation system on \mathbb{R}^n we can associate an equation on $GL(n, \mathbb{R})$ by setting

$$\dot{g} = gA,$$

or

$$g^{-1} \frac{dg}{dt} = A.$$

The matrix A is an element of $\mathfrak{gl}(n, \mathbb{R})$.

Moreover, this way of finding the resolvent can also be used for time-dependent systems

$$\frac{dx}{dt} = A(t)x.$$

In this case if we have a fundamental set of solutions, denoted by $X(t) = (x_1(t), \dots, x_n(t))$, from

$$X(t) = R(t, 0)X(0)$$

we get $R(t, 0) = X(t)X(0)^{-1}$.

We could equally well associate a time-dependent equation on $GL(n, \mathbb{R})$ by setting

$$g^{-1} \frac{dg}{dt} = a^i(t)A_i,$$

with A_i being elements of the natural basis of $\mathfrak{gl}(n, \mathbb{R})$.

The point we want to remark is that if the evolution preserves some structure, then

$$A(t) = \phi_t^{-1} \circ \frac{d\phi_t}{dt}$$

lies in certain subalgebra of $\mathfrak{gl}(\mathbb{R}, n)$. The flow now defines a one-parameter family of transformations, and conversely, given a one-parameter family of transformations it will determine a vector field in \mathbb{R}^n , the corresponding fundamental vector field.

The critical fact is that the general solution is determined by a linear operator from a set of fundamental solutions, and this is a linear superposition principle. It is very natural to ask what happens when the vector field is nonlinear. The answer is that, at least in some cases, to be explicated shortly, there is a kind of non-linear superposition principle, as it was proved by Lie ([11]). This nonlinear superposition principle is simply a generalization of the previous construction to those cases where the action of the group is not linear and \mathbb{R}^n is replaced for a manifold M .

In the general case of the system (4.1), the Theorem for existence and uniqueness of solutions of such systems tells us that there will be, for each small enough t , a local diffeomorphism of \mathbb{R}^n which establishes the correspondence among the initial values and the corresponding ones for the explicit value of the parameter t . In other words, the evolution is described by a curve g_t in the group of diffeomorphisms of \mathbb{R}^n . We have seen that when we consider a linear autonomous system this curve lies in the group $GL(n, \mathbb{R})$ and is just the exponential of the matrix A giving the system, $g(t) = \exp tA$. In the linear time-dependent case, we also have a curve g_t in $GL(n, \mathbb{R})$ but it is not the exponential anymore: the only thing we can say is that

$$A(t) = \phi_t^{-1} \circ \frac{d\phi_t}{dt}$$

takes values in the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$. Actually, the solution is obtained by the Dyson, time-ordered, exponential.

The point is that for other types of vector fields the curve described by g_t belongs to other Lie subgroups of the group of diffeomorphisms and these are just the cases for which the idea of the nonlinear superposition principle is generalizable.

The result established by Lie and Scheffers [11] is that the general evolution defined by (4.1) can be expressed in terms of m fundamental solutions if there are r vector fields Y_1, \dots, Y_r , such that the vector field X ,

$$X = X^i(x, t) \frac{\partial}{\partial x^i}$$

can be expressed as a linear combination

$$X = a_1(t)Y_1 + \dots + a_r(t)Y_r \tag{4.2}$$

and furthermore the vector fields

$$Y_\alpha = \xi_\alpha^i(x) \frac{\partial}{\partial x^i}$$

close a finite dimensional real (or complex) Lie algebra, with dimension r , i.e., there exist r^3 real numbers $c_{\alpha\beta}^\gamma$ such that

$$[Y_\alpha, Y_\beta] = c_{\alpha\beta}^\gamma Y_\gamma. \tag{4.3}$$

Moreover, in this case $mn \geq r$. When $\xi_\alpha^i(x) = a_{\alpha j}^i x^j + b_\alpha^i$, with $a_{\alpha j}^i$ and b_α^i arbitrary constants, the system is linear.

Let us consider an effective action of a Lie group G of dimension r on n -dimensional differentiable manifold M , $\Phi : G \times M \rightarrow M$, and $\Phi_g : M \rightarrow M$ and $\Phi_x : G \rightarrow M$ denote the maps $\Phi_g(x) = \Phi_x(g) = \Phi(g, x)$, for $g \in G$, $x \in M$. Choosing an initial point $x(0)$, every curve $g : I \rightarrow G$ in the group determine a curve in the manifold M by

$$x(t) = \Phi(g(t), x(0)) = \Phi_{g(t)}(x(0)) = \Phi_{x(0)}(g(t)),$$

and taking derivatives with respect to t we see that the tangent vectors to the curves $g(t)$ and $x(t)$, respectively, are related by

$$\dot{x}(t) = \Phi_{x(0)*g(t)}\dot{g}(t).$$

Let us remark that $\dot{g}(t) \in T_{g(t)}G$ and $\dot{x}(t) \in T_{x(t)}M$.

We can express $\dot{x}(t)$ in terms of $x(t)$: we recall that if $x_2 = \Phi_g(x_1)$, then

$$\Phi_{x_2} = \Phi_{x_1} \circ R_g,$$

where R_g denotes right translation in the Lie group G , because for any $g' \in G$,

$$\Phi_{x_2}(g') = \Phi(g', x_2) = \Phi(g', \Phi(g, x_1)) = \Phi(g'g, x_1) = (\Phi_{x_1} \circ R_g)(g').$$

Now, using the chain rule for computing the differentials we find that

$$\Phi_{x_2*} = \Phi_{x_1*} \circ R_{g*},$$

and then,

$$\Phi_{x_1*} = \Phi_{x_2*} \circ R_{g^{-1}*}.$$

We can use this relation for $g = g(t)$ and $x_1 = x(0)$ and then we find the following expression for $\dot{x}(t)$:

$$\dot{x}(t) = \Phi_{x(t)*e}(R_{g^{-1}(t)*g(t)}\dot{g}(t)).$$

Since $R_{g^{-1}(t)}$ is the right translation leading $g(t)$ to the neutral element $e \in G$ and $\dot{g}(t) \in T_{g(t)}G$, then $R_{g^{-1}(t)*g(t)}\dot{g}(t) \in T_eG$ and we know that T_eG may be identified with the Lie algebra of G , \mathfrak{g} . Moreover, for linear Lie groups, i.e., subgroups of $GL(n, \mathbb{R})$, right translation reduces to right multiplication by the corresponding matrix, and hence $R_{g^{-1}(t)*g(t)}\dot{g}(t)$ is just the product $\dot{g}(t) g^{-1}(t)$.

Let $\{e_\alpha\}_{\alpha=1}^r$ be a basis of the corresponding Lie algebra \mathfrak{g} with defining relations

$$[e_\alpha, e_\beta] = c_{\alpha\beta}{}^\gamma e_\gamma,$$

and denote X_α the corresponding fundamental vector fields defined by

$$(X_\alpha f)(m) = \frac{d}{dt} [f(\exp(-te_\alpha)m)]|_{t=0},$$

for any differentiable function f . We recall that in this case

$$[X_\alpha, X_\beta] = c_{\alpha\beta}{}^\gamma X_\gamma.$$

Now, if the time-dependent vector field defining the system (4.1) is of the form

$$X(t) = a_\alpha(t)X_\alpha$$

we associate with it the following differential equation on the Lie group itself

$$g^{-1}(t) \frac{dg}{dt} = a_\alpha(t) e_\alpha.$$

In this way the given system is replaced by a higher dimensional system of first order linear equations. Or in other words, we have replaced the original system of differential equations with a new system on the group G , but the important point is that it is enough to find a particular solution, the one starting from the neutral element, for obtaining the general solution of the system(4.1): the solution starting from x_0 is given by $\Phi(g(t), x_0)$.

Moreover, even if we do not know the solution $g(t)$ of the new system, it is possible to find it from the knowledge of a convenient set of particular solutions of (4.1) such that (4.2) with the additional condition (4.3). More explicitly, given a curve $x_1(t)$ that is a particular solution of the given system, there are, in principle, different possible choices for the curve $g(t)$ such that $x_1(t) = \Phi(g(t), x_1(0))$, because the stability group of the point $x_1(0)$ may be nontrivial. If we choose a different particular solution, $x_2(t)$, then the ambiguity reduces to the group intersection of the isotopy groups of $x_1(0)$ and $x_2(0)$. We can, if necessary iterate the procedure until we arrive to a set of m particular solutions x_1, \dots, x_m allowing us the determination of the curve $g(t)$. Of course as we have r unknown functions, the second class canonical coordinates, and we have mn conditions, it should be $mn \geq r$.

More explicitly, a set of x_1, \dots, x_m of solutions is said to be a fundamental system of solutions, if

$$\begin{aligned} x_1(t) &= \Phi(g(t), x_1(0)) \\ \dots &= \dots\dots\dots \\ x_m(t) &= \Phi(g(t), x_m(0)) \end{aligned} \tag{4.4}$$

is a minimal set allowing us to solve for $g(t)$ via the implicit function Theorem. If this can be done we get

$$g(t) = F(x_1(t), \dots, x_m(t); x_1(0), \dots, x_m(0)),$$

and then any other solution can be written as

$$x(t) - \Phi(F(x_1(t), \dots, x_m(t); x_1(0), \dots, x_m(0)), x(0)) = 0.$$

Therefore the left hand side of this relation defines a constant of the motion.

Starting with the action $\Phi : G \times M \rightarrow M$ we should find the minimal integer number m such that the isotopy group of the action of G on the product $M^m = M \times \dots \times M$ (m times), extended from Φ by $\Phi^m(g, x_1, \dots, x_m) = (\Phi(g, x_1), \dots, \Phi(g, x_m))$, reduces to the neutral element for a point such that any two coordinates are different.

We recall that the fundamental vector field corresponding to an element of \mathfrak{g} generating a one-parameter Lie subgroup contained in the isotopy group of a point, vanishes in such a point, and conversely. Therefore, when expressed in terms of fundamental vector fields that means that the extensions to M^m of the fundamental vector fields X_α do not vanish in a point whose coordinates are different. The general solution then is found by adding a new component and looking for constants of motion.

The procedure is next illustrated with an example for the simplest case $n = 1$. According to Lie's Theorem we should look for a finite dimensional real Lie algebra of differential operators

$$X_\alpha = f_\alpha(x) \frac{\partial}{\partial x}.$$

It can be shown that the only finite dimensional Lie algebra that can be found from vector fields in one real variable are $\mathfrak{sl}(2, \mathbb{R})$ and its subalgebras. The uniquely defined (up to a change of variables) realization of $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$X_0 = x \frac{\partial}{\partial x}, \quad X_- = \frac{\partial}{\partial x}, \quad X_+ = x^2 \frac{\partial}{\partial x}.$$

These vector fields close the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra

$$[X_0, X_-] = -X_-, \quad [X_0, X_+] = X_+, \quad [X_-, X_+] = 2X_0.$$

Therefore, it suffices to consider the case in which the time-dependent vector field X defining the equation can be written as a linear combination $X = a_1 X_0 + a_2 X_+ + a_3 X_-$ with $a_0 = a_0(t)$, $a_1 = a_1(t)$ and $a_2 = a_2(t)$ real functions. For simplicity we will consider the case when a_1 , a_2 and a_3 are real numbers and then we obtain the differential equation

$$\frac{dx}{dt} = X(x, t) = a_0 + a_1 x + a_2 x^2,$$

which is nothing but the well known Riccati equation.

First we note that the determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

is generically different from zero, while the system $a + bx_1 + cx_1^2 = 0$, $a + bx_2 + cx_2^2 = 0$ always has a solution. Consequently, $m = 3$ in this case.

Now, for obtaining the general solution we should define the vector fields

$$\begin{aligned} V_0 &= x \frac{\partial}{\partial x} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}, \\ V_- &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \\ V_+ &= x^2 \frac{\partial}{\partial x} + x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} + x_3^2 \frac{\partial}{\partial x_3} \end{aligned} \tag{4.5}$$

and look for a solution of the system

$$V_0 f = V_+ f = V_- f = 0.$$

This system of partial differential equations is integrable, because the vector fields V_0 , V_+ and V_- close a Lie algebra, and therefore they define an integrable distribution.

The last equation $V_- f = 0$ tell us that the function f depends only on the differences $u_1 = x - x_1$, $u_2 = x_1 - x_2$ and $u_3 = x_2 - x_3$, because the characteristic system is

$$\frac{dx}{1} = \frac{dx_1}{1} = \frac{dx_2}{1} = \frac{dx_3}{1},$$

and it has as first integrals the differences $u_1 = x - x_1$, $u_2 = x_1 - x_2$ and $u_3 = x_2 - x_3$. Now, if $f(x, x_1, x_2, x_3) = \varphi(u_1, u_2, u_3)$, the condition $V_0 f = 0$ is written

$$u_1 \frac{\partial \varphi}{\partial u_1} + u_2 \frac{\partial \varphi}{\partial u_2} + u_3 \frac{\partial \varphi}{\partial u_3} = 0,$$

i.e., the function φ should be homogeneous of degree zero, and therefore it only can depend on the quotients $v_1 = u_1/u_2$ and $v_2 = u_3/u_2$, $\varphi(u_1, u_2, u_3) = \phi(v_1, v_2)$. Finally, the condition $V_+ f = 0$ can be written in these coordinates, after a long computation, as

$$v_1(v_1 + 1) \frac{\partial \phi}{\partial v_1} - v_2(v_2 + 1) \frac{\partial \phi}{\partial v_2} = 0.$$

The corresponding characteristic system is

$$\frac{dv_1}{v_1(v_1 + 1)} = -\frac{dv_2}{v_2(v_2 + 1)}$$

and taking into account that

$$\int \frac{d\xi}{\xi(\xi + 1)} = \log \frac{\xi}{\xi + 1},$$

we obtain that the constant of motion f should be a function of

$$\zeta = \frac{v_1}{v_1 + 1} \frac{v_2}{v_2 + 1},$$

and therefore

$$\frac{(x - x_1)(x_2 - x_3)}{(x - x_2)(x_1 - x_3)} = c$$

provides the non-linear evolution principle giving $x(t)$ as a function of three independent solutions

$$x = \frac{(x_1 - x_3)x_2 c + x_1(x_3 - x_2)}{(x_1 - x_3)c + (x_3 - x_2)}.$$

Let us remark that for $n = 1$ there is only one nonlinear differential equation family satisfying Lie–Scheffers theorem: the Riccati equation. Of course, proper subalgebras of $\mathfrak{sl}(2, \mathbb{R})$ lead to the linear inhomogeneous equation when $a_2 = 0$ or to a linear homogeneous equation when $a_0 = a_2 = 0$. However, for $n = 2$ in addition to $SL(3, \mathbb{R})$, $O(3, 1)$ and $O(2, 2)$, we can realize families of Lie algebras with arbitrary large Abelian ideals.

5 Application in the solution of second order differential equations

Algebraic methods have very often been used in the search for eigenvalues of operators and the corresponding eigenvector. The particular case of the harmonic oscillator is the prototype and it is based on creation and annihilation operators, and therefore it is related with the Heisenberg group. The possibility of relating linear second order differential equations with a Riccati equation, as indicated above, and the related group $SL(2, \mathbb{R})$ has not been exploited till now, as far as we know.

In this section we will explore the use of Wei–Norman method based on the $SL(2, \mathbb{R})$ group for studying the spectral problem of the second order differential operator determined for the Hamiltonian of the Harmonic oscillator

$$H = \frac{P^2}{2M} + \frac{k}{2}X^2, \quad (5.1)$$

where k is a constant.

We will use the following notation:

$$\omega = \sqrt{k/M}, \quad (5.2)$$

$$\alpha = \sqrt{M\omega/\hbar}, \quad (5.3)$$

$$\xi = \alpha x, \quad (5.4)$$

$$\lambda = 2E/\hbar\omega, \quad (5.5)$$

and then the Hamiltonian can be written

$$H = \frac{\hbar\omega}{2} \left[-\frac{d^2}{d\xi^2} + \xi^2 \right]. \quad (5.6)$$

The eigenvectors $\psi(\xi)$ of the preceding Hamiltonian operator corresponding to the eigenvalues $\lambda \frac{\hbar\omega}{2}$ are the normalizable solutions of the differential equation

$$-\frac{d^2\psi}{d\xi^2} + \xi^2\psi = \lambda\psi. \quad (5.7)$$

We proved in Section 2 that if ψ is a solution of (5.7), then the function $z = \frac{1}{\psi} \frac{d\psi}{d\xi}$ will be a solution of the following Riccati equation

$$\frac{dz}{d\xi} = -z^2 + (\xi^2 - \lambda). \quad (5.8)$$

As it was stated in Section 3, such equation admits a nonlinear superposition principle based on the $SL(2, \mathbb{R})$ group, and therefore the general solution can be found by means of an appropriate factorization

$$z(\xi) = \exp(g_2 L_2) \exp(g_1 L_1) \exp(g_0 L_0)(z)|_{\xi=0}. \quad (5.9)$$

The functions g_0, g_1, g_2 , are to be determined from the first order differential equation system

$$\begin{aligned} \dot{g}_0 &= \xi^2 - \lambda - g_0^2 \\ \dot{g}_1 &= -2g_0 \\ \dot{g}_2 &= -e^{g_1}, \end{aligned} \quad (5.10)$$

together with the initial conditions $g_0(0) = g_1(0) = g_2(0) = 0$.

Let us first remark that the Riccati equation

$$\frac{dz}{d\xi} + z^2 - \xi^2 + \lambda = 0,$$

under the change of variables given by

$$z = 2\xi v - \xi, \quad y = \xi^2, \quad (5.11)$$

becomes a new Riccati equation,

$$\frac{dv}{dy} + v^2 + v \left(\frac{1}{2y} - 1 \right) - \frac{1 - \lambda}{4y} = 0, \quad (5.12)$$

On the other side, the Riccati equation associated, according to the method described in Section 2, with the linear second order confluent hypergeometric differential equation

$$yW'' + (b - y)W' - aW = 0, \quad (5.13)$$

where a and b are constants and W' and W'' are the first and second derivative, respectively, of the function $W(y)$, is

$$\frac{dv}{dy} + v^2 + v \left(\frac{b}{y} - 1 \right) - \frac{a}{y} = 0. \quad (5.14)$$

with

$$v = \frac{W'}{W}.$$

A simple comparison between (5.12) and (5.14) shows that both coincide when

$$a = \frac{1 - \lambda}{4} \quad b = \frac{1}{2}. \quad (5.15)$$

It is well known that the general solution of (5.13) is given by

$$W(y) = AM \left(\frac{1 - \lambda}{4}, \frac{1}{2}, y \right) + By^{\frac{1}{2}} M \left(\frac{3 - \lambda}{4}, \frac{3}{2}, y \right)$$

with A and B arbitrary constants, and $M(a, b, y)$ is such that for large values of the variable y ,

$$M(a, b, y) = \frac{\Gamma(b)}{\Gamma(b - a)} e^{i\epsilon\pi a} y^{-a} g(a, a - b + 1, -y) \quad (5.16)$$

$$+ \frac{\Gamma(b)}{\Gamma(a)} e^y y^{a-b} g(1 - a, b - a, y), \quad (5.17)$$

where $\epsilon = 1$ if $-\pi/2 < \text{Arg } y < 3\pi/2$ and $\epsilon = -1$ when $-3\pi/2 < \text{Arg } y \leq -\pi/2$ and g denotes the function

$$g(a, b, y) = \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1) b(b+1) \cdots (b+n-1)}{n! y^n} \quad (5.18)$$

Therefore, the function $W(y)$ behaves for large values of y as $e^y = e^{\xi^2}$ unless that

$$\Gamma \left(\frac{1}{4} - \frac{\lambda}{4} \right) = \infty \text{ and } B = 0 \quad (5.19)$$

$$\Gamma \left(\frac{3}{4} - \frac{\lambda}{4} \right) = \infty \text{ and } A = 0, \quad (5.20)$$

Now, we recall that Gamma function is such that $\Gamma(0) = -\infty$ and takes finite values when $-z$ is not a positive integer number, while $\Gamma(-m) = \infty$ for $0 < m \in \mathbb{Z}$. Therefore if we want to have a solution of (5.13) with values a and b satisfying (5.15) being square integrable, it is necessary that

$$\frac{1}{4} - \frac{\lambda}{4} = -m, \text{ and then } \lambda = 2n + 1 \text{ with } n = 2m = 2, 4, 6, \dots \quad (5.21)$$

$$\frac{3}{4} - \frac{\lambda}{4} = -m, \text{ and then } \lambda = 2n + 1 \text{ with } n = 2m + 1 = 3, 5, 7, \dots, \quad (5.22)$$

or,

$$\frac{1}{4} - \frac{\lambda}{4} = 0 \text{ and then } \lambda = 2n + 1 \text{ with } n = 0 \quad (5.23)$$

$$\frac{3}{4} - \frac{\lambda}{4} = 0 \text{ and then } \lambda = 2n + 1 \text{ with } n = 1. \quad (5.24)$$

Consequently, a necessary condition for (5.13) with values a and b satisfying (5.15) to be square integrable is

$$\lambda = 2n + 1,$$

where $n = 0, 1, 2, 3, \dots$

Therefore, we will restrict ourselves to the case $\lambda = 2n + 1$. We will show that the solution for g_0 can be obtained by induction on the index n .

We first consider the case in which n is an even number. The first equation reduces for $n = 0$ to

$$\dot{g}_0 = \xi^2 - 1 - g_0^2, \quad (5.25)$$

and then it is an easy matter to check that the solution we are looking for is $g_0 = -\xi$.

In the case $n = 2$ the equation becomes

$$\dot{g}_0 = \xi^2 - 5 - g_0^2, \quad (5.26)$$

and again it is easy to check that then the solution is

$$g_0 = \frac{8\xi}{4\xi^2 - 2} - \xi. \quad (5.27)$$

In a similar way, when studying the cases $n = 4, 6, \dots$ we will see that for an even number n the solution satisfying $g_0(0) = 0$ is given by

$$g_0 = \frac{H'_n(\xi)}{H_n(\xi)} - \xi, \quad (5.28)$$

with H_n being the Hermite polynomial of order n and H'_n means the derivative of H_n with respect to ξ . In fact, using several properties of Hermite polynomials, we see that

$$\frac{H''_n}{H_n} - \frac{(H'_n)^2}{H_n^2} - 1 = \xi^2 - \lambda - \left(\frac{H'_n}{H_n} - \xi \right)^2, \quad (5.29)$$

$$H''_n H_n - (H'_n)^2 - H_n^2 = (\xi^2 - \lambda) H_n^2 - (H'_n - H_n \xi)^2, \quad (5.30)$$

and taking into account that $H'_n = 2nH_{n-1}$ and the corresponding relation for the derivatives $H''_n = 4n(n-1)H_{n-2}$, we will get

$$4n(n-1)H_{n-2}H_n - H_n^2 = -\lambda H_n^2 + 2\xi H_n H'_n, \quad (5.31)$$

$$4n(n-1)H_{n-2} - H_n = -\lambda H_n + 2\xi H'_n. \quad (5.32)$$

Now, the recurrence relation $H_n = 2\xi H_{n-1} - 2(n-1)H_{n-2}$, leads to

$$2(n-1)H_{n-2}(2n+1-\lambda) = 2\xi H_{n-1}(2n+1-\lambda), \quad (5.33)$$

and the right hand side vanishes because of $\lambda = 2n+1$.

In summary, we have checked that for any even number n the solution such that $g_0(0) = 0$ is given by

$$g_0 = \frac{H'_n}{H_n} - \xi. \quad (5.34)$$

When introducing this value for g_0 in the second equation (5.11) for g_1 we obtain the new equation

$$\dot{g}_1 = -2g_0 = -2\left(\frac{H'_n}{H_n} - \xi\right), \quad (5.35)$$

which can be easily integrated

$$g_1 = -2 \int \left(\frac{H'_n}{H_n} - \xi\right) d\xi = -2 \log H_n + \xi^2 + C_1 = -\log(H_n^2 + e^{\xi^2}) + C_1, \quad (5.36)$$

and as $g_1(0) = 0$, C_1 takes the value $C_1 = \log(H_n^2(0)) = \log k_1$. From the relation $H_n = 2\xi H_{n-1} - 2(n-1)H_{n-2}$ we see that $H_n(0) = -2(n-1)H_{n-2}(0)$ and iterating this reasoning we obtain the chain of relations

$$H_{n-2}(0) = -2(n-3)H_{n-4}(0) \quad (5.37)$$

$$H_{n-4}(0) = -2(n-5)H_{n-6}(0) \quad (5.38)$$

$$\dots = \dots \quad (5.39)$$

$$H_2(0) = -2(n-n+1)H_0(0) = -2 \cdot 1 \quad (5.40)$$

and therefore,

$$\begin{aligned} H_n(0) &= (-2(n-1))(-2(n-3))(-2(n-5))\dots(-2 \cdot 1) \\ &= (-2)^{n/2} (n-3)(n-5)\dots 1, \end{aligned} \quad (5.41)$$

and then,

$$\begin{aligned} k_1 = H_n^2(0) &= \left[2^{n/2}(n-1)(n-3)(n-5)\dots 1\right]^2 \\ &= \left[2^{n/2} \frac{n(n-1)(n-2)(n-3)\dots 1}{n(n-2)(n-4)\dots 2}\right]^2 \\ &= \left[\frac{n!}{\frac{n}{2} \left(\frac{n}{2}-1\right) \left(\frac{n}{2}-2\right) \dots 1}\right]^2 = \left[\frac{n!}{\left(\frac{n}{2}\right)!}\right]^2. \end{aligned} \quad (5.42)$$

So, the function g_1 is given by the expression

$$g_1 = -\log \left[k_1 H_n^2 e^{-\xi^2} \right]. \quad (5.43)$$

Finally, the function g_2 is to be determined from the differential equation

$$\dot{g}_2 = -\frac{1}{k_1 H_n^2 e^{-\xi^2}}, \quad (5.44)$$

whose solution is

$$g_2 = -\int \frac{1}{k_1 H_n^2 e^{-\xi^2}} d\xi + k_2 \quad (5.45)$$

where k_2 is to be chosen such that $g_2(0) = 0$, i.e.,

$$k_2 = \left[\int \frac{1}{k_1 H_n^2 e^{-\xi^2}} \right]_{(\xi=0)}. \quad (5.46)$$

Putting together all previous results we get for the general solution $z(\xi)$ the following expression:

$$\begin{aligned} z(\xi) &= \frac{[e^{g_1} z]_{(\xi=0)}}{1 - [g_2 z]_{(\xi=0)}} + g_0 \\ &= \frac{\left[\frac{1}{k_1 H_n^2 e^{-\xi^2}} z \right]_{(\xi=0)}}{1 + \left[\int \frac{1}{k_1 H_n^2 e^{-\xi^2}} d\xi - k_2 \right] z}_{(\xi=0)} + \frac{H'_n}{H_n} - \xi \\ &= \left[\frac{\frac{1}{k_1 H_n^2 e^{-\xi^2}} z}{1 + \left[\int \frac{1}{k_1 H_n^2 e^{-\xi^2}} d\xi - k_2 \right] z}_{(\xi=0)} \right] + \frac{H'_n}{H_n} - \xi. \end{aligned} \quad (5.47)$$

We recall that the wave function ψ was given by

$$\psi = e^{\int z(\xi) d\xi}, \quad (5.48)$$

and from

$$\int z(\xi) d\xi = \log H_n - \frac{\xi^2}{2}, \quad (5.49)$$

we obtain

$$\psi = e^{\log H_n - \frac{\xi^2}{2}} = H_n e^{-\frac{\xi^2}{2}}. \quad (5.50)$$

Notice that when ψ has a well defined parity, the quotient $z = \frac{1}{\psi} \frac{d\psi}{d\xi}$ is an odd function and then the limit when $\xi \rightarrow 0$ of z should be zero. However, when ψ is a continuous odd function, then $\lim_{\xi \rightarrow 0} \psi = 0$ and then the quotient $\frac{1}{\psi} \frac{d\psi}{d\xi}$ cannot be finite. This was the reason for leaving aside the case $\lambda = 2n + 1$ for an odd number n . In this last case it is convenient first to introduce the change of variable $z = 1/v$, and then the equation (5.8) becomes

$$\frac{dv}{d\xi} = 1 - (\xi^2 - \lambda) v^2, \quad (5.51)$$

and the corresponding system of differential equations

$$\begin{aligned}\dot{g}_0 &= 1 - (\xi^2 - \lambda) g_0^2 \\ \dot{g}_1 &= -2 (\xi^2 - \lambda) g_0 \\ \dot{g}_2 &= -(\xi^2 - \lambda) e^{g_1}.\end{aligned}\tag{5.52}$$

with initial conditions $g_0(0) = g_1(0) = g_2(0) = 0$.

It is now easy to check that the solution for g_0 is

$$g_0 = \frac{H_n}{H'_n - \xi H_n},\tag{5.53}$$

that g_1 is given by

$$g_1 = \xi^2 + \log \left[\frac{k_1}{H'_n - \xi H_n} \right]^2\tag{5.54}$$

and that g_2 is

$$g_2 = - \int \frac{(\xi^2 - \lambda) k_1}{(H'_n - \xi H_n)^2 e^{-\xi^2}} d\xi + k_2,\tag{5.55}$$

where

$$k_1 = \frac{2n!}{\left(\frac{n-1}{2}\right)!},\tag{5.56}$$

$$k_2 = \left[\int \frac{(\xi^2 - \lambda) k_1}{(H'_n - \xi H_n)^2 e^{-\xi^2}} d\xi \right]_{(\xi=0)}\tag{5.57}$$

Then, under the change $z = 1/v$ the general solution

$$v(\xi) = \frac{(e^{g_1} v)_{(\xi=0)}}{1 - (g_2 v)_{(\xi=0)}} + g_0,\tag{5.58}$$

becomes after some computations the expression given by (5.47).

6 Conclusions

The analysis developed in this paper of the reduction process leading to the Riccati equation starting from a second order linear differential equation just by a simple application of the general Lie theory of symmetry of differential equations, now a well established subject (see e.g. [12]), allows us a better understanding of the so called factorization method, which is a method that have been used for finding new solvable potentials once one of such problem is known and that motivated the study of supersymmetric quantum mechanics. The Riccati equation was chosen here as the simplest example of first order differential equation systems admitting a nonlinear superposition principle and the deep reasons for the existence of such principle have been clarified in this paper. We hope that this will allow a geometric interpretation of the converse part of Lie-Scheffers theorem. This nonlinear superposition principle is very important and has played a very important role because it provides an explicit expression for the solution of nonlinear equations admitting such superposition principle. For a review see e.g. the review lecture given by Winternitz [13]. We have also developed many examples of the use of Wei-Norman method of computation of the solution of differential equations in a Lie group and we have applied it as an academical example for reobtaining the eigenvalues and eigenvectors of the harmonic oscillator problem.

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