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A New Geometric Approach to Lie Systems and Physical Applications

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Abstract. The characterization of systems of differential equations admitting a superposition function allowing us to write the general solution in terms of any fundamental set of particular solutions is discussed. These systems are shown to be related with equations on a Lie group and with some connections in fiber bundles. We develop two methods for dealing with such systems: the generalized Wei–Norman method and the reduction method, which are very useful when particular solutions of the original problem are known. The theory is illustrated with some applications in both classical and quantum mechanics.

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1. Introduction

Time evolution of many physical systems is described by nonautonomous systems of differential equations

$$\frac{dx^i(t)}{dt} = X^i(t, x), \quad i = 1, \ldots, n, \tag{1}$$

for instance, Hamilton equations or Lagrange equations when transformed to the first-order case by doubling the number of degrees of freedom.

The Theorem of existence and uniqueness of solution for such systems establishes that the initial conditions $x(0)$ determine the future evolution. It is also well-known that for the simpler case of a homogeneous linear system the general solution can be written as a linear combination of n independent particular solutions, $x_{(1)}, \ldots, x_{(n)}$,

$$x = F(x_{(1)}, \ldots, x_{(n)}, k_1, \ldots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}, \tag{2}$$

and for each set of initial conditions, the coefficients can be determined. For an inhomogeneous linear system, the general solution can be written as an affine function of $(n + 1)$-independent particular solutions:

$$x = F(x_{(1)}, \ldots, x_{(n+1)}, k_1, \ldots, k_n) \\ = x_{(1)} + k_1(x_{(2)} - x_{(1)}) + \cdots + k_n(x_{(n+1)} - x_{(1)}). \quad (3)$$

Under a nonlinear change of coordinates, both systems become nonlinear. However, the fact that the general solution is expressible in terms of a set of particular solutions is maintained, but the superposition function is no longer linear or affine.

The very existence of such examples of systems of differential equations admitting a superposition function suggests to us an analysis of such systems. We are lead in this way to the problem of characterizing the systems of differential equations for which a superposition function, allowing us to express the general solution in terms of m particular solutions, does exist. The solution of this problem is due to Lie [1]. Our aim here is to review the theory developed by Lie from a modern geometric viewpoint and to present different applications both in mathematics and physics.

The paper is organized as follows: Section 2 present the main Theorem due to Lie and some simple examples are given in Section 3. In Section 4, after the introduction of some notation concerning the ingredients of Lie group theory, a particular case in which the systems are defined in a Lie group G is analyzed, and we show how to relate them with a particular type of equation in a group. We also show that Lie systems in homogeneous spaces are naturally associated with these systems in Lie groups. The theory is illustrated with a pair of examples which point out the universal character of the equation in the group. Section 5 is devoted to presenting a generalization to the general case of a method proposed by Wei and Norman for linear systems and an example of the affine group in one dimension is used to illustrate the theory. The relation of the problem at hand with the theory of connections is studied in Section 6: it is shown that Lie systems define horizontal curves with respect to a connection. The reduction method developed in Section 7 corresponds to considering the action of a group of automorphisms of the principal bundle on the set of connections, transforming, in this way, the given problem into a simpler one. Some examples and references to different applications of this reduction method are also given. The applications of the general theory to different problems in both Classical and Quantum Mechanics are indicated in Section 8 with an especial emphasis on time evolution of time-dependent Hamiltonian systems. The example of the time-dependent linear potential model has been explicitly developed in Section 9 in both the classical and quantum cases.

2. Lie Theorem

The characterization of nonautonomous systems (1) having the property that a general solution can be written as a function of m independent particular solutions and some constants determining each specific solution is due to Lie. The statement of the theorem, which can be found in the book edited and revised by Scheffers [1], is as follows:

THEOREM 1. *Given a nonautonomous system of n first-order differential equations like* (1)*, a necessary and sufficient condition for the existence of a function $F\colon \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that the general solution is*

$$x = F(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n),$$

with $\{x_{(a)} \mid a = 1, \ldots, m\}$ being any set of particular solutions of the system and $k_1, \ldots, k_n$, n arbitrary constants, is that the system can be written as

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = Z_1(t)\xi^{1i}(x) + \cdots + Z_r(t)\xi^{ri}(x), \quad i = 1, \ldots, n, \tag{4}$$

where $Z_1, \ldots, Z_r$, are r functions depending only on t and $\xi^{\alpha i}$, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$Y^{(\alpha)} \equiv \sum_{i=1}^{n} \xi^{\alpha i}(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, r, \tag{5}$$

close on a real finite-dimensional Lie algebra, i.e. the vector fields $Y^{(\alpha)}$ are linearly independent and there exist r^3 real numbers, $f^{\alpha\beta}{}_{\gamma}$, such that

$$[Y^{(\alpha)}, Y^{(\beta)}] = \sum_{\gamma=1}^{r} f^{\alpha\beta}{}_{\gamma} Y^{(\gamma)}. \tag{6}$$

The number r satisfies $r \leqslant m\,n$. For a geometric proof, see [2].

From the geometric viewpoint, the system of first-order differential equations (1) provides the integral curves of the t-dependent vector field on an n-dimensional manifold M

$$X = \sum_{i=1}^{n} X^i(x, t)\frac{\partial}{\partial x^i},$$

in the same way as happens for autonomous systems and true vector fields, and the t-dependent vector fields satisfying the hypothesis of the theorem are those which can be written as a t-dependent linear combination of vector fields

$$X(x, t) = \sum_{\alpha=1}^{r} Z_\alpha(t)\, Y^{(\alpha)}(x),$$

with vector fields $Y^{(\alpha)}$ closing on a finite-dimensional real Lie algebra. They will be called Lie (or even Lie–Scheffers) systems. Many of its applications in physics and mathematics have been developed by Winternitz and coworkers [3–10].

3. Some Examples

In the Introduction, we have mentioned two types of systems of differential equations whose general solution can be written as described by Theorem 1: homogeneous linear systems like

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j, \quad i = 1, \ldots, n, \tag{7}$$

for which $m = n$ and the (linear) superposition function is given by (2), and the inhomogeneous ones

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t), \quad i = 1, \ldots, n, \tag{8}$$

for which $m = n + 1$ and the (affine) superposition function is (3).

In the first case, the linear system can be considered as the one giving the integral curves of the t-dependent vector field

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, x^j\, \frac{\partial}{\partial x^i}, \tag{9}$$

which is a linear combination with time-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, X_{ij}, \tag{10}$$

of the n^2 vector fields

$$X_{ij} = x^j\, \frac{\partial}{\partial x^i}, \quad i, j = 1, \ldots, n. \tag{11}$$

Notice that

$$[X_{ij}, X_{kl}] = \left[x^j \frac{\partial}{\partial x^i}, x^l \frac{\partial}{\partial x^k}\right] = \delta^{il}\, x^j \frac{\partial}{\partial x^k} - \delta^{kj}\, x^l \frac{\partial}{\partial x^i},$$

i.e.

$$[X_{ij}, X_{kl}] = \delta^{il}\, X_{kj} - \delta^{kj}\, X_{il}, \tag{12}$$

which means that the vector fields $\{X_{ij}\}$, with $i, j = 1, \ldots, n$, appearing in the case of a homogeneous system, close on a n^2-dimensional Lie algebra isomorphic to the $\mathfrak{gl}(n, \mathbb{R})$ algebra. It suffices to compare these commutation relations with those of the $\mathfrak{gl}(n, \mathbb{R})$ algebra. The latter is generated by the matrices E_{ij} with elements $(E_{ij})_{kl} = \delta_{ik}\, \delta_{jl}$, which satisfy

$$[E_{ij}, E_{kl}] = \delta_{jk}\, E_{il} - \delta_{il}\, E_{kj}.$$

Therefore, in this homogeneous linear case, $r = n^2$ and $m = n$, hence the inequality $r \leqslant m\,n$ is actually an equality.

For the case of the inhomogeneous system (8), the time-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \right) \frac{\partial}{\partial x^i}, \tag{13}$$

which is a linear combination with t-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, X_{ij} + \sum_{i=1}^{n} B^i(t)\, X_i, \tag{14}$$

of the n^2 vector fields (11) and the n vector fields

$$X_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n. \tag{15}$$

Now, these last vector fields commute among themselves

$$[X_i, X_k] = 0, \quad \forall\, i, k = 1, \ldots, n,$$

and

$$[X_{ij}, X_k] = -\delta_{kj}\, X_i, \quad \forall\, i, j, k = 1, \ldots, n.$$

Therefore, the Lie algebra generated by the vector fields $\{X_{ij}, X_k \mid i, j, k = 1, \ldots, n\}$ is isomorphic to the (n^2+n)-dimensional Lie algebra of the affine group. In this case, $r = n^2 + n$ and $m = n + 1$ and the equality $r = m\,n$ also follows.

Another remarkable example, with many applications in physics, is that of the Riccati equation, which corresponds to $n = 1$ [5, 11, 12]:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c_2(t)\, x^2(t) + c_1(t)\, x(t) + c_0(t). \tag{16}$$

In this case $r = 3$ and

$$\xi^1(x) = 1, \qquad \xi^2(x) = x, \qquad \xi^3(x) = x^2,$$

while

$$Z_1(t) = c_0(t), \qquad Z_2(t) = c_1(t), \qquad Z_3(t) = c_2(t).$$

Equation (16) determines the integral curves of the t-dependent vector field

$$X(x, t) = c_2(t)\, Y^{(3)} + c_1(t)\, Y^{(2)} + c_0(t)\, Y^{(1)},$$

where the vector fields $Y^{(1)}$, $Y^{(2)}$, and $Y^{(3)}$ in the decomposition are given by

$$Y^{(1)} = \frac{\partial}{\partial x}, \qquad Y^{(2)} = x\, \frac{\partial}{\partial x}, \qquad Y^{(3)} = x^2\, \frac{\partial}{\partial x}. \tag{17}$$

It is quite easy to check that they close on the following three-dimensional real Lie algebra,

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}, \qquad [Y^{(1)}, Y^{(3)}] = 2\,Y^{(2)}, \qquad [Y^{(2)}, Y^{(3)}] = Y^{(3)}, \quad (18)$$

i.e. the $\mathfrak{sl}(2, \mathbb{R})$ algebra.

It can be shown that, for the Riccati equation, $m = 3$ and, hence, as $r = 3$ the equality $r = mn$ holds. The superposition function comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k, \tag{19}$$

or, in other words [11],

$$x = \frac{x_1(x_3 - x_2) + k\,x_2(x_1 - x_3)}{(x_3 - x_2) + k\,(x_1 - x_3)}. \tag{20}$$

In particular, the solutions x_1, x_2 and x_3 are obtained for $k = 0, \infty$, and 1, respectively.

We show next an example of nonlinear superposition formula for a specific Lie system, in order to illustrate how complicated the explicit expressions can become. For the sake of brevity, we give only the result.

Consider the differential equation system

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= b_1(t) + b_2(t)\,x + b_3(t)\,(x^2 - y^2), \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= b_2(t)\,y + 2b_3(t)\,xy, \end{aligned} \tag{21}$$

which determines the integral curves of the t-dependent vector field

$$X(x, t) = b_1(t)\,Y^{(1)} + b_2(t)\,Y^{(2)} + b_3(t)\,Y^{(3)},$$

where now $Y^{(1)}$, $Y^{(2)}$, and $Y^{(3)}$ are given by

$$\begin{aligned} &Y^{(1)} = \frac{\partial}{\partial x}, \qquad Y^{(2)} = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}, \\ &Y^{(3)} = (x^2 - y^2)\,\frac{\partial}{\partial x} + 2xy\,\frac{\partial}{\partial y}. \end{aligned} \tag{22}$$

These vector fields satisfy the commutation relations (18), hence the previous system is a Lie system with associated Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, and with $r = 3$, $n = 2$. The number of particular solutions needed is $m = 3$. In fact, suppose we know a set of particular solutions $x_{(i)} = \{x_i,\ y_i\}$, $i = 1,\ 2,\ 3$ of the system (21). Then the general solution can be written as

$$\begin{aligned} x &= F_1(x_{(1)},\ x_{(2)},\ x_{(3)},\ k_1,\ k_2) = \frac{N_x}{D}, \\ y &= F_2(x_{(1)},\ x_{(2)},\ x_{(3)},\ k_1,\ k_2) = \frac{N_y}{D}, \end{aligned} \tag{23}$$

where

$$\begin{aligned}
N_x &= x_1\{(x_2-x_3)^2+(y_2-y_3)^2\} + \\
&\quad + k_1\{x_2^2\, x_3 + (x_3-x_2)x_1^2 + (y_1-y_2)^2 x_3 - \\
&\quad - x_2(x_3^2+(y_1-y_3)^2) - x_1((x_2-x_3)^2+(y_2-y_3)^2)\} + \\
&\quad + k_2\{x_3^2(y_2-y_1) + x_2^2\,(y_1-y_3) + \\
&\quad + (y_3-y_2)(x_1^2+(y_1-y_2)(y_1-y_3))\} + \\
&\quad + (k_1^2+k_2^2)x_2\{(x_1-x_3)^2+(y_1-y_3)^2\},
\end{aligned}$$

$$\begin{aligned}
N_y &= y_1\{(x_2-x_3)^2+(y_2-y_3)^2\} + \\
&\quad + k_1\{x_2^2(y_3-y_1) - x_3^2(y_1+y_2) + 2x_2(x_3y_1 - x_1y_3) + \\
&\quad + 2x_1x_3y_2 - (x_1^2+(y_1+y_2)(y_1-y_3))(y_2-y_3)\} + \\
&\quad + k_2\{x_1^2(x_2-x_3) + x_2^2\, x_3 + x_3(y_2^2-y_1^2) - \\
&\quad - x_2(x_3^2+y_3^2-y_1^2) + x_1(x_3^2-x_2^2+y_3^2-y_2^2)\} + \\
&\quad + (k_1^2+k_2^2)y_2\{(x_1-x_3)^2+(y_1-y_3)^2\},
\end{aligned}$$

$$\begin{aligned}
D &= (x_2-x_3)^2+(y_2-y_3)^2 - \\
&\quad - 2k_1\{(x_1-x_3)(x_2-x_3)+(y_1-y_3)(y_2-y_3)\} + \\
&\quad + 2k_2\{x_3(y_2-y_1)+x_2(y_1-y_3)+x_1(y_3-y_2)\} + \\
&\quad + (k_1^2+k_2^2)\{(x_1-x_3)^2+(y_1-y_3)^2\},
\end{aligned}$$

and k_1, k_2 are two arbitrary real constants determining each particular solution. For example, the particular solutions $\{x_1,\ y_1\}$, $\{x_2,\ y_2\}$ and $\{x_3,\ y_3\}$ can be obtained by taking $k_1 = k_2 = 0$, the limit $k_1 \to \infty$ (or $k_2 \to \infty$), and $k_1 = 1$, $k_2 = 0$, respectively.

In particular, if we look for solutions of the system (21) with $y = 0$, we recover, essentially, the Riccati equation (16); likewise, the superposition formula (23) reduces to (20) in such a particular case.

For a more complete information about the explicit construction and use of superposition formulas, see, e.g., [2–11] and references therein.

4. Lie–Scheffers Systems on Lie Groups

The most important example, which will be shown to give rise to many other related systems, occurs when M is a Lie group G and we consider vector fields X_α in G that are either left-invariant or right-invariant as corresponding either to the Lie algebra $\mathfrak{g}$ of G or to the opposite algebra [13, 14].

Let us choose a basis $\{a_1, \ldots, a_r\}$ for the tangent space T_eG at the neutral element $e \in G$, and denote $\{\vartheta_1, \ldots, \vartheta_r\}$ the corresponding dual basis of T_e^*G. In the following, X_α^R denotes the right-invariant vector field in G such that $X_\alpha^R(e) = a_\alpha$, i.e. $X_\alpha^R(g) = R_{g*e}(a_\alpha)$, and in an analogous way, X_α^L will denote the left-invariant vector field $X_\alpha^L(g) = L_{g*e}(a_\alpha)$. Similarly, θ_α^R and θ_α^L are the right- and left-invariant 1-forms in G determined by ϑ_α, i.e.

$$\theta_\alpha^R(g) = (R_{g^{-1}})_e^*(\vartheta_\alpha), \qquad \theta_\alpha^L(g) = (L_{g^{-1}})_e^*(\vartheta_\alpha).$$

If we consider the right-invariant Lie–Scheffers system on G

$$X(g,t) = -\sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g), \tag{24}$$

its integral curves will be determined by the system of differential equations

$$\dot g(t) = -\sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g(t)). \tag{25}$$

Applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides of this equation we obtain the equivalent equation

$$(R_{g(t)^{-1}})_{*g(t)}(\dot g(t)) = -\sum_{\alpha=1}^r b_\alpha(t) a_\alpha, \tag{26}$$

which we will write as well, with a slight abuse of notation, as

$$(\dot g\, g^{-1})(t) = -\sum_{\alpha=1}^r b_\alpha(t) a_\alpha, \tag{27}$$

although (26) reduces to (27) only when G is a matrix group. This equation is right-invariant, and so, out of a solution $\bar g(t)$ of (26) with initial condition $\bar g(0) = e$, the solution with initial conditions $g(0) = g_0$ is given by $\bar g(t) g_0$. This means that for the Lie–Scheffers system (24) on the Lie group G, $m = 1$.

Of course, given a homomorphism of Lie groups $F\colon G \to G'$, the right-invariant Lie–Scheffers system on G (24) produces a right-invariant Lie–Scheffers system on G',

$$X(g',t) = -\sum_{\alpha=1}^r b_\alpha(t)\, (F_* X)_\alpha^R(g'),$$

where $(F_* X)_\alpha^R$ is the right-invariant vector field on G' which is F-related with the vector field X_α^R.

Let us consider a left action of a Lie group G, with Lie algebra $\mathfrak{g}$, on a manifold M, $\Phi\colon G \times M \to M$, and denote $\Phi_g\colon M \to G$ and $\Phi_p\colon G \to G$, where $g \in G$, $p \in M$, the maps defined by $\Phi_g(p) = \Phi_p(g) = \Phi(g, p)$. The fundamental vector field X_a associated to the element a of $\mathfrak{g}$ is given by

$$(X_a f)(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\Phi(\exp(-ta), p))\Big|_{t=0}, \qquad f \in C^\infty(M),$$

where the minus sign has been introduced for $X\colon \mathfrak{g} \to \mathfrak{X}(M)$ to be a Lie algebra homomorphism, i.e. a $\mathbb{R}$-linear map such that $X_{[a,b]} = [X_a, X_b]$. Note that $X_a(p) = \Phi_{p*e}(-a)$. These vector fields are always complete. As an example, let us consider the left action action of G on itself by left translations, $\Phi(g, g') = g\,g'$. The fundamental vector fields X_a are right invariant because

$$(X_a)(g) = \Phi_{g*e}(-a) = R_{g*e}(-a) = -(X_a^R)(g),$$

where X_a^R is the right-invariant vector field in G determined by its value at the neutral element $(X_a^R)(e) = a$.

Given two actions Φ_1 and Φ_2 of a Lie group G on two differentiable manifolds M_1 and M_2, a map $F\colon M_1 \to M_2$ is said to be equivariant if $F \circ \Phi_{1g} = \Phi_{2g} \circ F$. The remarkable property is that when G is connected, the map $F\colon M_1 \to M_2$ is equivariant if and only if, for each $a \in T_eG$, the corresponding fundamental vector fields in M_1 and M_2 are F-related [2, 13, 14].

Now, let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. Then, G acts on M by $\lambda(g', gH) = (g'g)H$. Moreover, G can be seen as a principal bundle $(G, \tau, G/H)$ over G/H, where τ denotes the canonical projection. The important point is (see, e.g., [13]) that the map $\tau\colon G \to G/H$ is equivariant, with respect to the left action of G on itself by left translations and the action λ on G/H and, consequently, the fundamental vector fields corresponding to the two actions are τ-related. Therefore, the right-invariant vector fields X_α^R are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M, $\tau_{*g}X_\alpha^R(g) = -X_\alpha(gH)$. In this way we will have an associated Lie–Scheffers system on M:

$$X(x, t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x), \tag{28}$$

where $x = gH$, whose integral curve will be determined by

$$\dot{x} = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x). \tag{29}$$

Thus, the solution of (28) starting from x_0 will be $x(t) = \Phi(g(t), x_0)$, with $g(t)$ being the solution of (26) with $g(0) = e$. This is the main point: the knowledge of one particular solution of (26) allows us to obtain the general solution of (28).

The converse property is true in the following sense: Given a Lie–Scheffers system in a manifold M defined by complete vector fields and with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action given by integrating the vector fields. Then, the restriction to an orbit will provide a homogeneous space of the above type. The choice of a point x_0 in the homogeneous space allows us to identify the homogeneous space M with G/H, where H is the stability group of x_0. Different choices for x_0 will lead to conjugate subgroups [13].

For instance, the vector fields appearing in (17) close on a Lie algebra but the third one is not complete on $\mathbb{R}$. We can however consider the one-point compactification of $\mathbb{R}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and then the flows of vector fields in (17) are, respectively,

$$x \mapsto x + \epsilon, \qquad x \mapsto e^{\epsilon} x, \qquad x \mapsto \frac{x}{1 - x\,\epsilon},$$

and therefore they can be considered as the fundamental vector fields corresponding to the action of $\mathrm{SL}(2, \mathbb{R})$ on the completed real line $\overline{\mathbb{R}}$, given by [12]

$$\begin{aligned} &\Phi(A, x) = \frac{\alpha\, x + \beta}{\gamma\, x + \delta}, \quad \text{if } x \neq -\frac{\delta}{\gamma}, \\ &\Phi(A, \infty) = \frac{\alpha}{\gamma}, \qquad \Phi\,(A, -\delta/\gamma) = \infty, \\ &\quad \text{when} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \end{aligned} \tag{30}$$

The stability group of ∞ is the subgroup of matrices with $\gamma = 0$, which is isomorphic to the affine group $\mathcal{A}_1$ in one-dimensional space, while the stability group of 0 is made up by the matrices with $\beta = 0$, a group isomorphic to $\mathcal{A}_1$. Indeed

$$\begin{pmatrix} \delta & 0 \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The remarkable fact is that Equation (26) has a universal character. There will be many Lie–Scheffers systems associated with such an equation. It is enough to consider homogeneous spaces and the corresponding fundamental vector fields. In this way we will get a set of different systems corresponding to the same equation on the Lie group G. In particular, we can consider an action of G on a linear space given by a linear representation, and then the associated Lie systems are linear systems. Hence, we obtain a kind of linearization of the original problem [5]. Therefore, the theory can be useful in the study of both classical and quantum problems.

As an example we can consider both the Riccati equation

$$\dot{x} = b_0(t) + 2\, b_1(t)\, x + b_2(t)\, x^2,$$

and the linear system of first-order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1(t) & b_0(t) \\ -b_2(t) & -b_1(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

They are two different Lie systems for $G = \mathrm{SL}(2, \mathbb{R})$ corresponding to the same equation

$$\dot{g} g^{-1} = -b_0(t)\, M_0 - 2\, b_1(t)\, M_1 - b_2(t)\, M_2,$$

with the matrices

$$M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad M_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

being a basis of $\mathfrak{sl}(2, \mathbb{R})$. They satisfy the commutation relations

$$[M_0, M_1] = -M_0, \qquad [M_0, M_2] = -2\, M_1, \qquad [M_1, M_2] = -M_2.$$

As another illustrative example, we can consider the nonrelativistic dynamics of a spin 1/2 particle, when only the spinorial part is considered [14]. The dynamics of such a particle in a time-dependent magnetic field is described by the so-called Schrödinger–Pauli equation:

$$i\hbar \frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = H|\psi\rangle = -\mu\, \boldsymbol{B} \cdot \boldsymbol{S}|\psi\rangle,$$

with μ proportional to the Bohr magneton, $\boldsymbol{B} = (B^1, B^2, B^3)$ the t-dependent magnetic field, and $S_i = (\hbar/2)\,\sigma_i$. More explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} iB^3 & iB^1 + B^2 \\ iB^1 - B^2 & -iB^3 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{31}$$

The matrices $-i\sigma^1$, $-i\sigma^2$ and $-i\sigma^3$ generate the real Lie algebra of traceless skew-Hermitian 2×2 matrices, the Lie algebra of the group $\mathrm{SU}(2, \mathbb{C})$ and therefore of $\mathrm{SO}(3, \mathbb{R})$.

As a consequence of the theory that we have developed in this section, in order to find the general solution of the evolution equation (31), it suffices to determine the curve $R(t)$ in $\mathrm{SO}(3, \mathbb{R})$ starting from the identity map, $R(0) = I$ and such that

$$\dot{R}\, R^{-1} = B^3 M_3 + B^2 M_2 + B^1 M_1,$$

where

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Such a curve gives us the general solution for the dynamics

$$|\psi(t)\rangle = \bar{R}(t)|\psi(0)\rangle,$$

where $\bar{R}$ is an element in $\mathrm{SU}(2, \mathbb{C})$ corresponding to R.

5. The Wei–Norman Method

In order to directly solve Equation (26), we can use a method which is a generalization of the one proposed by Wei and Norman [15, 16] for finding the time evolution operator for a linear systems of type $\mathrm{d}U(t)/\mathrm{d}t = H(t)U(t)$, with $U(0) = I$, see also [11]. However, as will be mentioned in a later section, alternative methods exist for solving (26) by reducing the problem to a simpler one.

Both procedures are based on the following property ([13]): If $g(t)$, $g_1(t)$ and $g_2(t)$ are differentiable curves in G such that $g(t) = g_1(t)g_2(t)$, $\forall t \in \mathbb{R}$, then,

$$\begin{aligned} &R_{g(t)^{-1}*g(t)}(\dot g(t)) \\ &\quad = R_{g_1(t)^{-1}*g_1(t)}(\dot g_1(t)) + \mathrm{Ad}(g_1(t))\left\{R_{g_2(t)^{-1}*g_2(t)}(\dot g_2(t))\right\}. \end{aligned} \tag{32}$$

The generalization of this property to several factors is as follows. Let now $g(t)$ be a curve in G which is given by the product of other l curves

$$g(t) = g_1(t)g_2(t)\cdots g_l(t) = \prod_{i=1}^{l} g_i(t).$$

Then, denoting

$$h_s(t) = \prod_{i=s+1}^{l} g_i(t), \qquad \text{for } s \in \{1, \ldots, l-1\},$$

and applying (32) to $g(t) = g_1(t)\, h_1(t)$, we have

$$R_{g(t)^{-1}*g(t)}(\dot g(t)) = R_{g_1(t)^{-1}*g_1(t)}(\dot g_1(t)) + \mathrm{Ad}(g_1(t))\left\{R_{h_1(t)^{-1}*h_1(t)}(\dot h_1(t))\right\}.$$

Simply iterating, and using $\mathrm{Ad}(gg') = \mathrm{Ad}(g)\mathrm{Ad}(g')$ for all $g,\ g' \in G$, we obtain

$$\begin{aligned} &R_{g(t)^{-1}*g(t)}(\dot g(t)) \\ &\quad = R_{g_1(t)^{-1}*g_1(t)}(\dot g_1(t)) + \mathrm{Ad}(g_1(t))\left\{R_{g_2(t)^{-1}*g_2(t)}(\dot g_2(t))\right\} + \cdots \\ &\qquad + \mathrm{Ad}\left(\prod_{i=1}^{l-1} g_i(t)\right)\left\{R_{g_l(t)^{-1}*g_l(t)}(\dot g_l(t))\right\} \\ &= \sum_{i=1}^{l} \mathrm{Ad}\left(\prod_{j<i} g_j(t)\right)\left\{R_{g_i(t)^{-1}*g_i(t)}(\dot g_i(t))\right\} \\ &= \sum_{i=1}^{l}\left(\prod_{j<i} \mathrm{Ad}(g_j(t))\right)\left\{R_{g_i(t)^{-1}*g_i(t)}(\dot g_i(t))\right\}, \end{aligned} \tag{33}$$

where it has been taken $g_0(t) = e$ for all t.

The generalized Wei–Norman method consists on writing the solution $g(t)$ of (26) in terms of its second kind canonical coordinates w.r.t. a basis $\{a_1, \ldots, a_r\}$ of the Lie algebra $\mathfrak{g}$, for each value of t, i.e.

$$g(t) = \prod_{\alpha=1}^{r} \exp(-v_\alpha(t)a_\alpha) = \exp(-v_1(t)a_1)\cdots\exp(-v_r(t)a_r),$$

and transforming the differential equation (26) into a differential equation system for the $v_\alpha(t)$, with initial conditions $v_\alpha(0) = 0$ for all $\alpha = 1, \ldots, r$. The minus signs in the exponentials have been introduced for computational convenience. Then, we use the result (33), taking

$$l = r = \dim G \quad \text{and} \quad g_\alpha(t) = \exp(-v_\alpha(t)a_\alpha) \quad \text{for all } \alpha.$$

Now, since $R_{g_\alpha(t)^{-1}*g_\alpha(t)}(\dot g_\alpha(t)) = -\dot v_\alpha(t)a_\alpha$, we see that (33) reduces to

$$\begin{aligned} R_{g(t)^{-1}*g(t)}(\dot g(t)) &= -\sum_{\alpha=1}^{r} \dot v_\alpha \Bigg(\prod_{\beta<\alpha} \mathrm{Ad}(\exp(-v_\beta(t)a_\beta))\Bigg) a_\alpha \\ &= -\sum_{\alpha=1}^{r} \dot v_\alpha \Bigg(\prod_{\beta<\alpha} \exp(-v_\beta(t)\mathrm{ad}(a_\beta))\Bigg) a_\alpha, \end{aligned}$$

where it has been used the identity $\mathrm{Ad}(\exp(a)) = \exp(\mathrm{ad}(a))$, for all $a \in \mathfrak{g}$. Substituting in Equation (26), we obtain the fundamental expression of the Wei–Norman method

$$\sum_{\alpha=1}^{r} \dot v_\alpha \Bigg(\prod_{\beta<\alpha} \exp(-v_\beta(t)\mathrm{ad}(a_\beta))\Bigg) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha, \tag{34}$$

with $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$. The resulting differential equation system for the functions $v_\alpha(t)$ is integrable by quadratures if the Lie algebra is solvable [15, 16] and, in particular, for nilpotent Lie algebras.

As a simple but illustrative example, we can consider the affine group in one dimension, $\mathcal{A}_1$, i.e. the set of transformations of the real line

$$\bar x = \alpha_1 x + \alpha_0, \tag{35}$$

with $\alpha_1 \neq 0$ and α_0 being real numbers. The group composition law is

$$(\alpha_0', \alpha_1') * (\alpha_0, \alpha_1) = (\alpha_0' + \alpha_1' \alpha_0, \alpha_1' \alpha_1).$$

Denoting by (x_0, x_1) the coordinate system in $\mathcal{A}_1$ given by

$$x_0(\alpha_0, \alpha_1) = \alpha_0, \qquad x_1(\alpha_0, \alpha_1) = \alpha_1,$$

we see that a basis of right-invariant vector fields in $\mathcal{A}_1$ is given by

$$X_0^R = \frac{\partial}{\partial x_0}, \qquad X_1^R = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1},$$

while the corresponding basis of left-invariant vector fields is given by

$$X_0^L = x_1 \frac{\partial}{\partial x_0}, \qquad X_1^L = x_1 \frac{\partial}{\partial x_1}.$$

Therefore, the defining relations are $[a_0, a_1] = -a_0$. Then,

$$\mathrm{ad}(a_0)a_0 = 0, \qquad \mathrm{ad}(a_0)a_1 = -a_0,$$

and if $g = \exp(-u_0 a_0)\exp(-u_1 a_1)$, Equation (34) in this case becomes

$$\dot{u}_0\, a_0 + \dot{u}_1 (a_1 + u_0\, a_0) = b_0\, a_0 + b_1\, a_1,$$

so we obtain the system

$$\dot{u}_0 = b_0 - b_1\, u_0, \qquad \dot{u}_1 = b_1, \tag{36}$$

with the initial conditions $u_0(0) = u_1(0) = 0$. Note that the first equation is nothing but an inhomogeneous linear equation. The explicit solution can be obtained through two quadratures:

$$u_0(t) = \mathrm{e}^{-\int_0^t dt'\, b_1(t')} \int_0^t \mathrm{d}t'\, b_0(t')\, \mathrm{e}^{\int_0^{t'} \mathrm{d}t''\, b_1(t'')}, \qquad u_1(t) = \int_0^t \mathrm{d}t'\, b_1(t'),$$

In a similar way, if we consider instead $g = \exp(-v_1 a_1)\exp(-v_0 a_0)$, and we take into account that

$$\mathrm{ad}(a_1)a_0 = a_0, \qquad \mathrm{ad}(a_1)a_1 = 0,$$

then we will find

$$\dot{v}_1\, a_1 + \dot{v}_0 \exp(-v_1 \mathrm{ad}(a_1))a_0 = \dot{v}_1\, a_1 + \dot{v}_0\, \mathrm{e}^{-v_1}\, a_0 = b_0\, a_0 + b_1\, a_1,$$

yielding the system

$$\dot{v}_0 = b_0\, \mathrm{e}^{v_1}, \qquad \dot{v}_1 = b_1, \tag{37}$$

also with the initial conditions $v_0(0) = v_1(0) = 0$. The system (37), with such initial conditions, can be easily integrated by two quadratures:

$$v_0(t) = \int_0^t \mathrm{d}t'\, b_0(t')\, \mathrm{e}^{\int_0^{t'} \mathrm{d}t''\, b_1(t'')}, \qquad v_1(t) = \int_0^t \mathrm{d}t'\, b_1(t').$$

When we consider the previous action (35) on the real line, we get as fundamental vector fields

$$X_0 = -\frac{\partial}{\partial x} \quad \text{and} \quad X_1 = -x\frac{\partial}{\partial x},$$

thus the Lie system in $\mathbb{R}$ which corresponds to $\dot{g}\, g^{-1} = -b_0\, a_0 - b_1\, a_1$ is $\dot{x} = -b_0 - b_1\, x$. Our theory gives us the formula for the explicit general solution of such an inhomogeneous linear differential equation, by making use of

$$\begin{aligned} x(t) &= \Phi(g(t), x_0) = \Phi(\exp(-u_0(t)a_0), \Phi(\exp(-u_1(t)a_1), x_0)) \\ &= \mathrm{e}^{-u_1(t)} x_0 - u_0(t), \end{aligned}$$

namely

$$x(t) = \mathrm{e}^{-\int_0^t \mathrm{d}t'\, b_1(t')} \left\{ x_0 - \int_0^t \mathrm{d}t'\, b_0(t')\, \mathrm{e}^{\int_0^{t'} \mathrm{d}t''\, b_1(t'')} \right\}. \tag{38}$$

Likewise, the same solution can be obtained from the second factorization:

$$x(t) = \Phi(\exp(-v_1(t)a_1), \Phi(\exp(-v_0(t)a_0), x_0)) = \mathrm{e}^{-v_1(t)}(x_0 - v_0(t)), \tag{39}$$

which clearly gives the same result.

6. Connections and Lie Systems

If G is a connected Lie group, the set of curves

$$\gamma\colon \mathbb{R} \to G, \qquad t \mapsto g(t)$$

is also a group when the following composition law is considered: $\gamma_2 * \gamma_1\colon t \mapsto g_2(t)\, g_1(t)$.

Given a curve γ in G, $g(t)$, such that $g(0) = e$, then $\bar{g}(t) = g(t)\, g_0$ is another curve in G starting from g_0 (it is said to be right translated of γ by g_0) and similarly $\bar{\bar{g}}(t) = g_0\, g(t)$ is also a curve in G starting from g_0 (called left-translated from γ by g_0).

Now, given the curve γ, we have a vector field along γ given by the tangent vector $\dot{g}(t)$, and then, translating these tangent vectors to the neutral element by $R_{g^{-1}(t)*g(t)}$, we obtain a curve in T_eG like in (27). The curve $\bar{g}(t) = g(t)\, g_0$, right-translated by g_0 of γ, gives rise to the same equation.

But we can consider Equation (27) as an equation for the curve $g(t)$ in G determined by the curve $a(t) = -\sum_{\alpha=1}^r b_\alpha(t)\, a_\alpha$ in T_eG. This equation is right-invariant in the sense that if $g(t)$ is a solution such that $g(0) = e$, then, for each $g_0 \in G$, $\bar{g}(t) = g(t)\, g_0$ is a new solution, now such that $\bar{g}(0) = g_0$.

We remark that if G is a Lie group, $\pi_2\colon\ P = G \times \mathbb{R} \to \mathbb{R}$ defines a principal G-bundle. The right action of G on P is given by

$$\Psi((g', t), g) = \Psi_g((g', t)) = (g'\, g, t),$$

i.e. $\Psi_g = R_g \times \mathrm{id}_{\mathbb{R}}$.

Giving a connection in P is equivalent to giving a curve in G, for instance, one such that $g(0) = e$. It is also well known that each global section provides a different trivialization of the principal bundle P. The given curve furnishes a section for π_2, $\sigma(t) = (g(t), t)$, and a family of sections right-translated from such a section,

$$\{\sigma'(t) = \Psi(\sigma(t), g_0) \mid g_0 \in G\}.$$

The tangent vectors to such a family of sections span the horizontal spaces in each point. More specifically, horizontal and vertical spaces in a point of P are given by

$$VP_{(g_0,t)} = \langle (X_\alpha^R(g_0), 0) \rangle,$$

$$HP_{(g_0,t)} = \langle (R_{g_0*e}(\dot{g}(t)\, g^{-1}(t)), 1)\rangle.$$

Note that

$$\Psi_{g^{-1}(t)*g(t)}(\dot{g}(t), 1) = (R_{g^{-1}(t)*g(t)}\dot{g}(t), 1) = (\dot{g}(t)\, g^{-1}(t), 1).$$

The choice of the connection given by γ amounts to choose a basis of the tangent space at the point (g_0, t) as follows, $\{X_\alpha^R(g_0), \partial/\partial t + R_{g_0*e}(\dot{g}(t)\, g^{-1}(t))\}$, while the dual basis is made up by $\{\theta_\alpha^R(g_0) - \tau_\alpha(t)\, \mathrm{d}t, \mathrm{d}t\}$ where the coefficients $\tau_\alpha(t)$ are determined by

$$R_{g_0*e}(\dot{g}(t)\, g^{-1}(t)) = \sum_\beta \tau_\beta(t)\, X_\beta^R(g_0),$$

i.e.

$$\tau_\alpha(t) = \langle \theta_\alpha^R(g_0), R_{g_0*e}(\dot{g}(t)\, g^{-1}(t))\rangle.$$

Therefore, the vertical projector associated to the connection is

$$\begin{aligned} v_{(g_0,t)} &= \sum_\beta X_\beta^R(g_0) \otimes (\theta_\beta^R(g_0) - \tau_\beta(t)\, \mathrm{d}t) \\ &= \mathrm{id}_{T_{g_0}G} - \big(R_{g_0*e}(\dot{g}(t)\, g^{-1}(t))\big)\, \mathrm{d}t. \end{aligned}$$

It is also well known that when a left action $\Phi\colon\ G \times M \to M$ of G on M is considered, there exists an associated bundle E with base $\mathbb{R}$ and typical fiber M. The total space of such bundle is the set of orbits of the right action of G on $P \times M$,

$$(u, x)g = (\Psi(u, g), \Phi(g^{-1}, x)),$$

being the projection $\pi_E[u, x] = \pi_2(u)$, where $[u, x]$ denotes the equivalence class of $(u, x) \in P \times M$ and u is of the form (g', t). A connection in the principal bundle translates into a connection in the associated bundle E, and so the horizontal curves will then be $[\widetilde{\gamma}(t), x]$, where $\widetilde{\gamma}(t)$ is a horizontal curve in P. More explicitly, as the curves $\widetilde{\gamma}$ are of the form $\widetilde{\gamma}(t) = (g(t)g_0, t)$, we find that the horizontal curves in the associated bundle are

$$[(g(t)g_0, t), x_0] = [\Psi((e, t), g(t)g_0), x_0] = [(e, t), \Phi(g(t)g_0, x_0)]$$

and, consequently,

$$[(g(t)g_0, t), x_0] = [(e, t), \Phi(g(t), \Phi(g_0, x_0))].$$

Since the principal bundle is trivial, E is equivalent to a product. When Φ is transitive, $E = M \times \mathbb{R}$, where $[(e, t), x]$ corresponds to (x, t) and with this identification, the horizontal curve here considered corresponds to the integral curve starting from the point $\Phi(g_0, x_0)$ of the associated Lie system in M with respect to the action of G on M given by Φ.

Of course the simplest case is when a linear representation of G on the vector space V is considered, the associated bundle being then a vector bundle and the corresponding Lie system being a linear system. This means that a system as in (7) can be seen as defining the horizontal curves corresponding to a connection in an associated vector bundle. The fact that linear systems, as Schrödinger equations, could be thought of as defining horizontal curves of a connection were considered several years ago [17] and it has been suggested recently by looking at the transformation properties of the equation under certain gauge changes [18].

7. The Reduction Method

Given an equation on a Lie group,

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, a_\alpha \in T_eG, \tag{40}$$

with $g(0) = e \in G$, it may happen that the only nonvanishing coefficients are those corresponding to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the equation reduces to a simpler equation on a subgroup, involving less coordinate functions in the Wei–Norman method.

On the other hand, we know that Equation (40) can be seen as a connection in a principal bundle, and it is also well known that the group of automorphisms of the principal bundle acts on the set of connections. The automorphims of the bundle we are considering are given by curves $g'(t)$ in the group G, and so the group of curves in G defines an action on the set of connections and therefore on the Lie systems on the group. We can take advantage of such an action for transforming a given Lie system in another simpler one.

Now, let us choose a curve $g'(t)$ in the group G, corresponding to a given automorphism, and define the curve $\overline{g}(t)$ by $\overline{g}(t) = g'(t)g(t)$, where $g(t)$ is the previous solution of (26). The new curve in G, $\overline{g}(t)$, determines a new connection and therefore a new Lie system.

Indeed, from (32),

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}(g'(t))a_\alpha, \tag{41}$$

which is an equation similar to (26) but with different right-hand side. Therefore, the aim is to choose the curve $g'(t)$ appropriately, i.e. in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right-hand side of (41) lies in T_eH and, hence, $\overline{g}(t) \in H$ for all t.

Now, suppose we consider a transitive action $\Phi\colon\ G \times M \to M$ of G on a homogeneous space M, which can be identified with the set G/H of left-cosets, by choosing a fixed point x_0: H is then the stability subgroup of x_0. The horizontal curves starting from the point x_0 associated to both connections are related by

$$\begin{aligned}\overline{x}(t) &= \Phi(\overline{g}(t), x_0) = \Phi(g'(t)g(t), x_0)\\ &= \Phi(g'(t), \Phi(g(t), x_0)) = \Phi(g'(t), x(t)).\end{aligned}$$

Therefore, the action of the group of curves in G on the set of connections translates to the homogeneous space and gives an action on the corresponding set of associated Lie systems. More explicitly, if we consider the automorphism defined by $g'(t)$, the Lie system (29) transforms into a new one (see [13])

$$\dot{\bar{x}} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t) X_\alpha(\bar{x}), \tag{42}$$

in which

$$\bar{b} = \mathrm{Ad}(g'(t))b(t) + \dot{g}'\, g'^{-1}.$$

The important result proved in [13] is that the knowledge of a particular solution of the associated Lie system to (26) in G/H allows us to reduce the problem to one in the subgroup H. For any choice of the curve $g'(t)$ we can consider a curve $x'(t)$ defined in the homogeneous space G/H by $g'(t)$ as follows:

$$x'(t) = \tau(g'^{-1}(t)\overline{g}(t)) = g'^{-1}(t)H.$$

Then, if $g'(t)$ is chosen such that the curve $x'(t)$ is a solution of the associated system, then the automorphism defined by $g'(t)$ transforms the original problem into one in the subgroup H:

THEOREM 2. *Every integral curve of the time-dependent vector field on the group G, given by the right-hand side of* (25), *can be written in the form $g(t) = g_1(t)\, h(t)$, where $g_1(t)$ is a curve projecting onto a solution $x_1(t)$ of an equation of type* (29) *for the natural left action of G on the homogeneous space G/H, and $h(t)$ is a solution of a type* (26) *equation but for the subgroup H, given explicitly by*

$$(\dot{h}\, h^{-1})(t) = -\mathrm{Ad}(g_1^{-1}(t))\left(\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha + (\dot{g}_1\, g_1^{-1})(t)\right) \in T_eH.$$

As an example, we can consider once again the affine group in one dimension, $\mathcal{A}_1$, of the preceding section. We can choose first the Lie subgroup $H_0 = \{(a_0, 1) \mid a_0 \in \mathbb{R}\}$ and consider the corresponding one-dimensional homogeneous space $\mathcal{A}_1/H_0$. Its points can be characterized by $y = x_1$, with $x_1 \neq 0$. In this coordinate system for $\mathcal{A}_1/H_0$ the fundamental vector fields are $X_0 = 0$ and $X_1 = y\partial/\partial y$. The Lie system associated to $\dot{g}\, g^{-1} = -b_0\, a_0 - b_1\, a_1$ is $\dot{y} = b_1 y$ and, according to the result of the preceding theorem, once we know a solution of this last homogeneous linear equation, we can carry out the reduction procedure. More explicitly, when we know a solution of $\dot{y} = b_1 y$, the change of variable $x = y\,\zeta$ will simplify the equation $\dot{x} = b_0 + b_1\,x$ to one on the subgroup H_0, $\dot{\zeta} = b_0\,y^{-1}$.

If we consider instead the Lie subgroup $H_1 = \{(0, a_1) \mid a_1 \in \mathbb{R} - \{0\}\}$, then the elements of the one-dimensional homogeneous space $\mathcal{A}_1/H_1$ can be characterized by $z = x_0$. The expression of the fundamental vector fields in this coordinate system are $X_0 = \partial/\partial z$ and $X_1 = z\partial/\partial z$. Then, as soon as we know a solution of $\dot{z} = b_1 z + b_0$, namely a particular solution of the inhomogeneous equation, we can reduce the problem of finding the general solution of $\dot{x} = b_0 + b_1\,x$ to solving an equation on H_1, which is a homogeneous linear equation. This procedure corresponds to the change of variables $x = z + \zeta$, which leads to the reduced equation $\dot{\zeta} = b_1\,\zeta$.

Therefore the two methods usually found in textbooks for solving the inhomogeneous linear differential equation appear here as particular cases of a more general methodology for reduction of differential equation systems to simpler ones.

In this way, the last method can be generalized when one considers an inhomogeneous linear system, whose associated group is the corresponding affine group. Given a particular solution, the problem is reduced to another one on its stabilizer, i.e. the group $\mathrm{GL}(n, \mathbb{R})$ or, in other words, to a homogeneous linear system.

As another example, we consider the Riccati equation, which has been shown to be an example of Lie system, corresponding to the left action (30) of $\mathrm{SL}(2, \mathbb{R})$ on the (compactified) real line $\overline{\mathbb{R}}$ by homographies, see, e.g., [5, 12]. The action of the group of automorphisms of the principal bundle translates to the space $\overline{\mathbb{R}}$ into an action of the group of curves in $\mathrm{SL}(2, \mathbb{R})$ on the set of Riccati equations. This action was used in [12] for studying the integrability properties of the Riccati equation. The stabilizer of the point at the infinity is the affine group in one dimension $\mathcal{A}_1$. Therefore, if we know a particular solution, x_1, of the Riccati equation, the problem reduces to one on $\mathcal{A}_1$, i.e. an inhomogeneous linear equation, by means of the well-known change of variable $x = x_1 + z$. Had we chosen the origin $0 \in \overline{\mathbb{R}}$ as the initial point, the stabilizer (isomorphic to $\mathcal{A}_1$) would be generated by dilations and cotranslations. This corresponds to a new reduction of the Riccati equation by means of the change of variable

$$x' = \frac{x}{-\frac{x}{x_1} + 1} = \frac{x\,x_1}{x_1 - x},$$

which transforms $\dot{x} = c_0(t) + c_1(t)\,x + c_2(t)\,x^2$ into

$$\dot{x}' = \frac{\mathrm{d}x'}{\mathrm{d}t} = \left(\frac{2\,c_0(t)}{x_1} + c_1(t)\right) x' + c_0(t),$$

with associated group $\mathcal{A}_1$. For more details, see [12] and [13].

Now, suppose we know not only one but two different particular solutions, $x_{(1)}$ and $x_{(2)}$, of a Lie system in a homogeneous space. They will be determined by the choice of initial conditions which provide different presentations of the homogeneous space as G/H_1 and G/H_2, respectively, where H_i is the stability subgroup of $x_{(i)}(0)$. Using the result of Theorem 2, with $g_1(t)$ being a lifting to G of both curves $x_{(1)}(t)$ and $x_{(2)}(t)$, we will get an equation like the one in the theorem but where the right-hand side will be in the intersection $T_eH_1 \cap T_eH_2$, and therefore the Lie system is reduced to one on the subgroup $H_1 \cap H_2$. The example of the Riccati equation was explicitly considered in [12], where it was also shown that the knowledge of a third solution reduces the problem to a trivial equation $\dot{x} = 0$, and therefore giving rise in this way to the superposition function (20).

8. Some Applications in Classical and Quantum Mechanics

Nonautonomous linear systems and Riccati equations are examples of Lie systems that appear very often in physics. For instance, linear systems appear in the time evolution of time-dependent harmonic oscillators and the latter is a condition for the super-potential W in the factorization of a typical quantum Hamiltonian $H = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ as $H - \epsilon = (-\mathrm{d}/\mathrm{d}x + W)(\mathrm{d}/\mathrm{d}x + W)$, where ϵ is a constant (see, e.g., [19–21]), and it plays a relevant rôle in the search for the so-called Shape invariant potentials (see [14, 20, 22–24]). As we have pointed out in preceding sections, the Riccati equation may appear each time that the group $\mathrm{SL}(2, \mathbb{R})$ plays a rôle, and because of the isomorphism of the Lie algebras of $\mathrm{SL}(2, \mathbb{R})$ and the linear symplectic group in two dimensions, it will be useful in the linear approximation of symplectic transformations and the theory of aberrations in optics [25].

However, the Riccati equation is particularly important because it appears as a consequence of Lie reduction theory when taking into account that dilations are symmetries of linear second-order differential equations [11]. Actually, the homogeneous linear second-order differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}z}{\mathrm{d}x} + c(x)z = 0, \tag{43}$$

admits as an infinitesimal symmetry the vector field $X = z\,\partial/\partial z$ generating dilations in the variable z, which is defined for $z \neq 0$. According to Lie theory, we should change the coordinate z to a new one, $u = \varphi(z)$, such that $X = \partial/\partial u$. This change is determined by the equation $Xu = 1$, which leads to $u = \log|z|$, i.e. $|z| = \mathrm{e}^u$. In both cases of regions with $z > 0$ or $z < 0$, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = z\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + z\,\frac{\mathrm{d}^2 u}{\mathrm{d}x^2},$$

so Equation (43) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + c(x) = 0,$$

and the order can be lowered by introducing the new variable $w = \mathrm{d}u/\mathrm{d}x$. We arrive to the following Riccati equation for w:

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -w^2 - b(x)w - c(x). \tag{44}$$

Notice that $w = z^{-1}\mathrm{d}z/\mathrm{d}x$, and that this relation together with (44) is equivalent to the original second-order equation. In the particular case of the one-dimensional time-independent Schrödinger equation

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + (V(x) - \epsilon)\phi = 0,$$

the reduced Riccati equation for $W = \phi^{-1}\mathrm{d}\phi/\mathrm{d}x$ is

$$W' = -W^2 + (V(x) - \epsilon), \tag{45}$$

which is the equation that W must satisfy in the previously mentioned factorization of $H = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$.

Equations of type (45) are particular cases of Riccati equations. We have shown in a preceding section that it is possible to act with the group of curves in $\mathrm{SL}(2, \mathbb{R})$ on the set of Riccati equations. That means that a given Riccati equation can be transformed into other related equations (for more explicit details, see [12]). Therefore, when using curves in $\mathrm{SL}(2, \mathbb{R})$ preserving the form of a given equation like (45), we are transforming the spectral problem for a given Hamiltonian into that of another one. This method is carefully explained in [26], where explicit examples of the usefulness of the theory are given.

In the typical problem of classical mechanics, we are dealing with Hamiltonian vector fields in a symplectic manifold (M, Ω), and then we should consider the case in which the vector fields arising in the expression of the t-dependent vector field describing a Lie system, are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra. These vector fields correspond to a symplectic action of the group G on the symplectic manifold (M, Ω). The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -\mathrm{d}h_\alpha$, however, do not close on the same Lie algebra when the Poisson bracket is considered, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = 0,$$

and therefore they span a Lie algebra extension of the original one.

The situation in quantum mechanics is quite similar. It is well known that the separable complex Hilbert space of states $\mathcal{H}$ can be seen as a real manifold admitting a global chart [27]. The Abelian translation group allows us to identify the tangent space $T_\phi\mathcal{H}$ at any point $\phi \in \mathcal{H}$ with $\mathcal{H}$ itself, the isomorphism being obtained by associating with $\psi \in \mathcal{H}$ the vector $\dot{\psi} \in T_\phi\mathcal{H}$ given by

$$\dot{\psi} f(\phi) := \left(\frac{\mathrm{d}}{\mathrm{d}t} f(\phi + t\psi)\right)_{|t=0},$$

for any $f \in C^\infty(\mathcal{H})$.

The symplectic 2-form Ω is given by

$$\Omega_\phi(\dot\psi, \dot\psi') = 2\,\mathrm{Im}\langle\psi|\psi'\rangle,$$

with $\langle\cdot|\cdot\rangle$ denoting the Hilbert inner product on $\mathcal{H}$.

Through the identification of $\mathcal{H}$ with $T_\phi\mathcal{H}$ a continuous vector field is just a continuous map $A\colon \mathcal{H} \to \mathcal{H}$; therefore a linear operator A on $\mathcal{H}$ is a special kind of vector field.

Given a smooth function $a\colon \mathcal{H} \to \mathbb{R}$, its differential $\mathrm{d}a_\phi$ at $\phi \in \mathcal{H}$ is an element of the (real) dual $\mathcal{H}'$ given by

$$\langle \mathrm{d}a_\phi\,,\, \psi\rangle := \left(\frac{\mathrm{d}}{\mathrm{d}t} a(\phi + t\psi)\right)_{|t=0}.$$

Now, as it was pointed out in [27] the skew-Hermitian linear operators in $\mathcal{H}$ define Hamiltonian vector fields, the Hamiltonian function of $-i\,A$ for a self-adjoint operator A being $a(\phi) = \frac{1}{2}\langle\phi, A\phi\rangle$. The Schrödinger equation plays the rôle of Hamilton equations because it determines the integral curves of the vector field $-i\,H$.

Now, Lie system theory applies to the case in which a t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra. The remarkable point, however, is that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension. An example will be given in next section.

9. An Example: Classical and Quantum Time-Dependent Linear Potential

The linear potential model, with many applications in physics, has recently been studied by Guedes [28]. We can use this problem in order to illustrate the possible applications of the theory by means of a simple example. Let us consider the classical system described by a classical Hamiltonian

$$H_c = \frac{p^2}{2m} + f(t)\,x,$$

and the corresponding quantum Hamiltonian

$$H_q = \frac{P^2}{2m} + f(t)\,X,$$

describing, for instance when $f(t) = q\,E_0 + q\,E\,\cos\omega t$, the motion of a particle of electric charge q and mass m driven by a monochromatic electric field. E_0 is the strength of the constant confining electric field and E that of the time-dependent electric field that drives the system with a frequency $\omega/2\pi$. Instead of using the Lewis and Riesenfeld invariant method [29] as was done in [28], we will simultaneously study the classical and the quantum problem by reduction of both problems to similar equations and using the Wei–Norman method to solve such an equation. As it will be shown, the only difference is that the Lie algebra arising in the quantum problem is not the same as in the classical one, but a central extension.

The classical Hamilton equations of motion are

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = -f(t), \tag{46}$$

and therefore the motion is given by

$$x(t) = x_0 + \frac{p_0\, t}{m} - \frac{1}{m}\int_0^t \mathrm{d}t' \int_0^{t'} f(t'')\,\mathrm{d}t'', \tag{47}$$

$$p(t) = p_0 - \int_0^t f(t')\,\mathrm{d}t'. \tag{48}$$

The t-dependent vector field describing the time evolution is

$$X = \frac{p}{m}\frac{\partial}{\partial x} - f(t)\,\frac{\partial}{\partial p}.$$

This vector field can be written as a linear combination

$$X = \frac{1}{m}\,X_1 - f(t)\,X_2,$$

with

$$X_1 = p\,\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial p},$$

being two vector fields closing a three-dimensional Lie algebra with $X_3 = \partial/\partial x$, isomorphic to the Heisenberg algebra, namely,

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0. \tag{49}$$

The flow of these vector fields is given, respectively, by

$$\begin{aligned}
\phi_1(t,(x_0,p_0)) &= (x_0 + p_0\, t,\, p_0),\\
\phi_2(t,(x_0,p_0)) &= (x_0,\, p_0 + t),\\
\phi_3(t,(x_0,p_0)) &= (x_0 + t,\, p_0).
\end{aligned}$$

In other words, this corresponds to the action of the Lie group of upper triangular 3×3 matrices on $\mathbb{R}^2$,

$$\begin{pmatrix}\bar{x}\\ \bar{p}\\ 1\end{pmatrix} = \begin{pmatrix}1 & a_1 & a_3\\ 0 & 1 & a_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ p\\ 1\end{pmatrix}.$$

It is to be remarked that the three vector fields X_1, X_2 and X_3 are Hamiltonian vector fields with respect to the usual symplectic structure, $\Omega = \mathrm{d}x \wedge \mathrm{d}p$, the corresponding Hamiltonian functions h_i such that $i(X_i)\Omega = -\mathrm{d}h_i$ being

$$h_1 = -\frac{p^2}{2}, \qquad h_2 = x, \qquad h_3 = -p,$$

therefore

$$\{h_1, h_2\} = -h_3, \qquad \{h_1, h_3\} = 0, \qquad \{h_2, h_3\} = -1, \tag{50}$$

which close on a four-dimensional Lie algebra with $h_4 = 1$, that is, a central extension of that given by (49). Let $\{a_1,\ a_2,\ a_3\}$ be a basis of the Lie algebra with nonvanishing defining relations $[a_1, a_2] = -a_3$. Then, the corresponding equation in the group (27) becomes in this case

$$\dot g\, g^{-1} = -\frac{1}{m} a_1 + f(t)\, a_2.$$

Now, choosing the factorization

$$g = \exp(-u_3\, a_3)\ \exp(-u_2\, a_2)\ \exp(-u_1\, a_1)$$

and using the Wei–Norman formula (34) we will arrive to the system of differential equations

$$\dot u_1 = \frac{1}{m}, \qquad \dot u_2 = -f(t), \qquad \dot u_3 - \dot u_1\, u_2 = 0,$$

together with the initial conditions

$$u_1(0) = u_2(0) = u_3(0) = 0,$$

with solution

$$u_1 = \frac{t}{m}, \qquad u_2 = -\int_0^t f(t')\, \mathrm{d}t', \qquad u_3 = -\frac{1}{m} \int_0^t \mathrm{d}t' \int_0^{t'} f(t'')\, \mathrm{d}t''.$$

Therefore the motion will be given by

$$\begin{pmatrix} x \\ p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{m} & -\frac{1}{m} \int_0^t \mathrm{d}t' \int_0^{t'} f(t'')\, \mathrm{d}t'' \\ 0 & 1 & -\int_0^t f(t')\, \mathrm{d}t' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \\ 1 \end{pmatrix},$$

which reproduces (48). So, obviously we will recover the constant of motion given in [28], $I_1 = p(t) + \int_0^t f(t')\, \mathrm{d}t'$, together with the other one

$$I_2 = x(t) - \frac{1}{m} \left(p(t) + \int_0^t f(t')\, \mathrm{d}t' \right) t + \frac{1}{m} \int_0^t \mathrm{d}t' \int_0^{t'} f(t'')\, \mathrm{d}t''.$$

As far as the quantum problem is concerned, also studied in a very recent paper [30], notice that the quantum Hamiltonian H_q may be written as a sum

$$H_q = \frac{1}{m} H_1 - f(t)\, H_2,$$

with

$$H_1 = \frac{P^2}{2}, \qquad H_2 = -X,$$

and $-i\,H_1$ and $-i\,H_2$ close on a four-dimensional Lie algebra with $-i\,H_3 = -i\,P$, and $-i H_4 = i\,I$, isomorphic to that of (50), which is an extension of the Heisenberg Lie algebra (49),

$$[-i\,H_1, -i\,H_2] = -i\,H_3, \qquad [-i\,H_1, -i\,H_3] = 0,$$
$$[-i\,H_2, -i\,H_3] = -i\,H_4.$$

The Schrödinger equation given by the Hamiltonian H_q is like that of a Lie system. Note that this Hamiltonian is time–dependent and that such systems are seldom studied, because it is generally difficult to find the time evolution of such systems. However, this system is a Lie system and therefore we can find the time-evolution operator by applying the reduction of the problem to an equation on the Lie group and using the Wei–Norman method.

Let $\{a_1,\, a_2,\, a_3,\, a_4\}$ be a basis of the Lie algebra with nonvanishing defining relations $[a_1, a_2] = a_3$ and $[a_2, a_3] = a_4$. Equation (27) in the group to be considered is now

$$\dot g\, g^{-1} = -\frac{1}{m}\, a_1 + f(t)\, a_2.$$

Using the factorization

$$g = \exp(-u_4\, a_4) \exp(-u_3\, a_3)\, \exp(-u_2\, a_2)\, \exp(-u_1\, a_1)$$

the Wei–Norman method provides the following equations:

$$\dot u_1 = \frac{1}{m}, \qquad \dot u_2 = -f(t),$$
$$\dot u_3 + u_2\, \dot u_1 = 0, \qquad \dot u_4 + u_3\, \dot u_2 - \frac{1}{2}\, u_2^2\, \dot u_1 = 0,$$

and written in normal form

$$\dot u_1 = \frac{1}{m}, \qquad \dot u_2 = -f(t),$$
$$\dot u_3 = -\frac{1}{m}\, u_2, \qquad \dot u_4 = f(t)\, u_3 + \frac{1}{2m}\, u_2^2,$$

together with the initial conditions $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 0$, whose solution is

$$u_1(t) = \frac{t}{m}, \qquad u_2(t) = -\int_0^t f(t')\,\mathrm{d}t',$$
$$u_3(t) = \frac{1}{m}\int_0^t \mathrm{d}t' \int_0^{t'} f(t'')\,\mathrm{d}t'',$$

and

$$u_4(t) = \frac{1}{m}\int_0^t \mathrm{d}t' f(t') \int_0^{t'} \mathrm{d}t'' \int_0^{t''} f(t''')\,\mathrm{d}t''' + \frac{1}{2m}\int_0^t \mathrm{d}t' \left(\int_0^{t'} \mathrm{d}t'' f(t'')\right)^2.$$

These functions provide the explicit form of the time-evolution operator:

$$U(t,0) = \exp(-iu_4(t))\exp(iu_3(t)P)\exp(-iu_2(t)X)\exp(iu_1(t)P^2/2).$$

Notwithstanding, in order to find the expression of the wave-function in a simple way, it is advantageous to use the factorization

$$g = \exp(-v_4\,a_4)\exp(-v_2\,a_2)\ \exp(-v_3\,a_3)\ \exp(-v_1\,a_1).$$

In such a case, the Wei–Norman method gives the system

$$\dot v_1 = \frac{1}{m}, \qquad \dot v_2 = -f(t),$$
$$\dot v_3 = -\frac{1}{m}\,v_2, \qquad \dot v_4 = -\frac{1}{2m}\,v_2^2,$$

jointly with the initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$. The solution is

$$v_1(t) = \frac{t}{m}, \qquad v_2(t) = -\int_0^t \mathrm{d}t'\, f(t'), \tag{51}$$
$$v_3(t) = \frac{1}{m}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' f(t''), \tag{52}$$
$$v_4(t) = -\frac{1}{2m}\int_0^t \mathrm{d}t' \left(\int_0^{t'} \mathrm{d}t'' f(t'')\right)^2. \tag{53}$$

Then, applying the evolution operator onto the initial wave-function $\psi(p,0)$, which is assumed to be written in momentum representation, we have

$$\begin{aligned}
&\psi(p,t)\\
&\quad = \mathrm{U}(t,0)\psi(p,0)\\
&\quad = \exp(-iv_4(t))\exp(-iv_2(t)X)\exp(iv_3(t)P)\exp(iv_1(t)P^2/2)\psi(p,0)\\
&\quad = \exp(-iv_4(t))\exp(-iv_2(t)X)\mathrm{e}^{i(v_3(t)p+v_1(t)p^2/2)}\psi(p,0)\\
&\quad = \exp(-iv_4(t))\mathrm{e}^{i(v_3(t)(p+v_2(t))+v_1(t)(p+v_2(t))^2/2)}\psi(p+v_2(t),0),
\end{aligned}$$

where the functions $v_i(t)$ are given by (51), (52) and (53), respectively.

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