Integrability of Riccati equation from a group theoretical viewpoint

José F. Cariñena and Arturo Ramos

Departamento de Física Teórica. Facultad de Ciencias.
Universidad de Zaragoza, E-50009, Zaragoza, Spain.

Abstract

In this paper we develop some group theoretical methods which are shown to be very useful for a better understanding of the properties of the Riccati equation and we discuss some of its integrability conditions from a group theoretical perspective. The nonlinear superposition principle also arises in a simple way.

1 Introduction

The Riccati equation is very often used in many different fields of theoretical physics (see e.g. [1] and references therein) and this is particularly true during the last years, because this differential equation has recently been shown to be related with the factorization method (see e.g. [2]) and its importance has been increasing since Witten's introduction of supersymmetric Quantum Mechanics. On the other side, the first order differential equation obtained by reduction from linear second order differential equations, according to Lie theory, when taking into account the invariance under the one-parameter group of dilations, is a Riccati equation. Moreover, from the mathematical viewpoint it is esentially the only differential equation, with one dependent variable, admitting a non–linear superposition principle, and this fact shows that there is a symmetry group for the theory which can be used for a better understanding of the properties of such equation.

The main point is that the general solution of a Riccati equation cannot be expressed by means of quadratures but in some particular cases. This motivated the study of different cases in which such an explicit solution is possible. So, Kamke's book (1959) presents some integrability conditions which have also been completed recently by Strelchenya (1991).

The criterion established by Strelchenya was based on the use of some transformations by t–dependent matrices with values in $SL(2,\mathbb{R})$. We aim in this paper to develop the ideas pointed out by Strelchenya, but our interpretation will be different. In particular, it will be shown that his new criterion reduces to a very well known fact: the general solution can be explicitly written by means of two quadratures if one particular solution of the differential equation is known.

The paper is organized as follows: In Section 2 we recall some well known properties of Riccati equation and we have summarized its general properties of integrability which are known to date. The geometric interpretation of the general Riccati equations as a time–dependent vector field in the one–point compactification of the real line is given in Section 3. The particular case of Riccati equations with constant coefficients that is carried out in Section 4 from this geometric approach is based on the action of the group $SL(2,\mathbb{R})$ on the set of such Riccati equations, as a way of introducing in a particular case what we will do for the general case in Section 5 by means of the group $\mathcal{G}$ of maps of $\mathbb{R}$ in $SL(2,\mathbb{R})$. The transformation formulae for the coefficients of the Riccati equation involve time derivatives of the coefficients, but we will show that these additional terms define a cocycle and therefore it is possible to define an affine action of such a group $\mathcal{G}$ on the set of Riccati equations. The action so obtained furnishes a method for studying the reduction when one or two solutions of Riccati equation are known and provides a method of introducing some integrability criteria. Finally, in the last section the non–linear superposition principle is revisited from this new group theoretical perspective.

2 Integrability criteria for Riccati equation

We recall that the Riccati equation is a non–linear first order differential equation

$$\frac{dx(t)}{dt} = a_0(t) + a_1(t)\, x(t) + a_2(t)\, x^2(t) \tag{1}$$

and that there is no way of writing the general solution, in the most general case, by using some quadratures. However, there are some particular cases for which one can write the general solution by such an expression. Of course the simplest case is when $a_2 = 0$, i.e., when the equation is linear: then, two quadratures allow us to find the general solution, given explicitly by

$$x(t) = \exp\left\{\int_0^t a_1(s)\, ds\right\} \times \left\{x_0 + \int_0^t a_0(t') \exp\{-\int_0^{t'} a_1(s)\, ds\}\, dt'\right\} .$$

It is also remarkable that under the change of variable

$$w = -\frac{1}{x} \tag{2}$$

the Riccati equation (1) becomes a new Riccati equation

$$\frac{dw(t)}{dt} = a_0(t)\, w^2(t) - a_1(t)\, w(t) + a_2(t) . \tag{3}$$

This shows that if in the original equation $a_0 = 0$ (Bernoulli with $n = 2$), then the mentioned change of variable transforms the given equation into a homogeneous linear one, and therefore the general solution can be written by means of two quadratures.

Two other integrability conditions can be found in [3]:

a) The coefficients satisfy $a_0 + a_1 + a_2 = 0$.

b) There exist constants c_1 and c_2 such that $c_1^2\, a_2 + c_1\, c_2\, a_1 + c_2^2\, a_0 = 0$.

In a recent paper Strelchenya (1991) claimed to give a *new* integrability criterion:

c) There exist functions $\alpha(t)$ and $\beta(t)$ such that

$$a_2 + a_1 + a_0 = \frac{d}{dt} \log\frac{\alpha}{\beta} - \frac{\alpha - \beta}{\alpha\, \beta}(\alpha\, a_2 - \beta\, a_0) , \tag{4}$$

which can also be rewritten as

$$\alpha^2\, a_2 + \alpha\,\beta\, a_1 + \beta^2\, a_0 = \alpha\beta\, \frac{d}{dt}\log\frac{\alpha}{\beta}\ . \tag{5}$$

All these preceding conditions, including what Strelchenya called a new integrability condition, are nothing but three particular cases of a well known result (see e.g. [4]): if one particular solution x_1 of (1) is known, then the change of variable

$$x = x_1 + x' \tag{6}$$

leads to a new Riccati equation for which the new coefficient a_0 vanishes:

$$\frac{dx'}{dt} = (2\, x_1\, a_2 + a_1)\, x' + a_2\, x'^2, \tag{7}$$

that, as indicated above, can be reduced to a linear equation with the change $x' = -1/u$. Consequently, when one particular solution is known, the general solution can be found by means of two quadratures: is given by $x = x_1 - (1/u)$ with

$$\begin{aligned} u(t) \;&=\; \exp\Big\{ -\int_0^t [2\, x_1(s)\, a_2(s) + a_1(s)]\, ds \Big\} \\ &\quad \times \Big\{ u_0 + \int_0^t a_2(t') \exp\{ \int_0^{t'} [2\, x_1(s)\, a_2(s) + a_1(s)]\, ds\}\, dt' \Big\}\ . \end{aligned} \tag{8}$$

The criteria a) and b) correspond to the fact that either the constant function $x = 1$, in case a), or $x = c_1/c_2$, in case b), are solutions of the given Riccati equation [5]. What is not so obvious is that, actually, the condition given in c) is equivalent to say that the function $x = \alpha/\beta$ is a solution of (1).

Moreover, it is also known [4] that when not only one but two particular solutions of (1) are known, $x_1(t)$ and $x_2(t)$, the general solution can be found by means of only one quadrature. In fact, the change of variable

$$\bar{x} = \frac{x - x_1}{x - x_2} \tag{9}$$

transforms the original equation into a homogeneous linear differential equation in the new variable $\bar{x}$,

$$\frac{d\bar{x}}{dt} = a_2(t)\, (x_1(t) - x_2(t))\, \bar{x}\ ,$$

which has general solution

$$\bar{x}(t) = \bar{x}(t=0)\, e^{\int_0^t a_2(s)\,(x_1(s)-x_2(s))\,ds} \ .$$

Alternatively, we can consider the change

$$x'' = (x_1 - x_2)\,\frac{x - x_1}{x - x_2} \ , \tag{10}$$

and the original Riccati equation (1) becomes

$$\frac{dx''}{dt} = (2\,x_1(t)\,a_2(t) + a_1(t))\,x'' \ ,$$

and therefore the general solution can be immediately found:

$$x'' = x''(t=0)\, e^{\int_0^t (2\,x_1(s)\,a_2(s)+a_1(s))\,ds} \ .$$

We will comment the relation between both changes of variables and find another possible one later on.

Even more interesting is the following property: once three particular solutions, $x_1(t), x_2(t), x_3(t)$, are known, the general solution can be written, without making use of any quadrature, in the following way:

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k \ , \tag{11}$$

where k is an arbitrary constant characterizing each particular solution.

Notice that the theorem for uniqueness of solutions of differential equations shows that the difference between two solutions of the Riccati equation (1) has a constant sign and therefore the difference between two different solutions never vanishes, and the previous quotients are always well defined.

This is a non–linear superposition principle: given three particular fundamental solutions there exists a superposition function $\Phi(x_1, x_2, x_3, k)$ such that the general solution is expressed as $x = \Phi(x_1, x_2, x_3, k)$. In this particular case the superposition function is given by

$$x = \frac{k\,x_1(x_3 - x_2) + x_2(x_1 - x_3)}{k\,(x_3 - x_2) + (x_1 - x_3)} \ . \tag{12}$$

This superposition principle and the generalization given by the so called Lie–Scheffers theorem [6], which has had a revival after several interesting papers by Winternitz and coworkers (see e.g. [7]), have been studied in [1] from a group theoretical perspective. We aim here to investigate other interesting properties by using appropriate group theoretical methods and the relationship of these properties with the usual integrability criteria.

3 Geometric Interpretation of Riccati equation

From the geometric viewpoint the Riccati equation can be considered as a differential equation determining the integral curves of the time–dependent vector field

$$\Gamma = (a_0(t) + a_1(t)x + a_2(t)x^2)\frac{\partial}{\partial x} \ . \tag{13}$$

The simplest case is when all the coefficients $a_i(t)$ are constant, because then Γ, given by (13), is a true vector field. Otherwise, Γ is a vector field along the projection $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by $\pi(x, t) = x$ (see e.g. [8] where it is shown that these vector fields along π also admits integral curves).

The important point is that Γ is a linear combination with time–dependent coefficients of the three following vector fields

$$L_0 = \frac{\partial}{\partial x}\,, \quad L_1 = x\,\frac{\partial}{\partial x}\,, \quad L_2 = x^2\,\frac{\partial}{\partial x}\,, \tag{14}$$

that one can check that close on a three-dimensional real Lie algebra, with defining relations

$$[L_0, L_1] = L_0\,, \quad [L_0, L_2] = 2L_1\,, \quad [L_1, L_2] = L_2\,, \tag{15}$$

and consequently this Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, which is made up by traceless 2×2 matrices. A basis is given by

$$M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,,\ M_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\ M_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\,. \tag{16}$$

Let us remark that the matrices (16) have the same commuting relations as the corresponding L's but with opposite sign, because the identification between the L's and the M's is an antihomomorphism of Lie algebras.

Notice also that the commutation relations (15) show that L_0 and L_1 generate a two–dimensional Lie subalgebra isomorphic to the Lie algebra of the affine group of transformations in one dimension, and the same holds for L_1 and L_2.

The one–parameter subgroups of local transformations of $\mathbb{R}$ generated by L_0, L_1 and L_2 are

$$x \mapsto x+\epsilon\,, \quad x \mapsto e^{\epsilon}x\,, \quad x \mapsto \frac{x}{1-x\epsilon}\ .$$

Notice that L_2 is not a complete vector field on $\mathbb{R}$. However we can do the one-point compactification of $\mathbb{R}$ and then L_0, L_1 and L_2 can be considered as the fundamental vector fields corresponding to the action of $SL(2,\mathbb{R})$ on the completed real line $\overline{\mathbb{R}}$, given by

$$\begin{aligned} \Phi(A,x) &= \frac{\alpha x+\beta}{\gamma x+\delta}, \quad \text{if } x \neq -\frac{\delta}{\gamma}, \\ \Phi(A,\infty) &= \frac{\alpha}{\gamma}\ , \quad \Phi(A,-\frac{\delta}{\gamma}) = \infty, \\ \text{when} \quad A &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R}). \end{aligned} \tag{17}$$

This suggests us that the group $SL(2,\mathbb{R})$ should play a prominent role in the study of Riccati equation, as it will be shown shortly.

4 The Riccati equation with constant coefficients

Let us first consider the simplest case of Riccati equations with constant coefficients

$$\dot{x} = a_0 + a_1\,x + a_2\,x^2\ . \tag{18}$$

As indicated above, the set of such equations is a $\mathbb{R}$–linear space that can be identified with the set of fundamental vector fields corresponding to the above mentioned action Φ of $SL(2,\mathbb{R})$ on the extended real line $\overline{\mathbb{R}}$. It is very easy to check that under the change of variable

$$x' = \frac{\alpha x+\beta}{\gamma x+\delta}\ , \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R}),\ \text{if } x \neq -\frac{\delta}{\gamma}\ ,$$

the original Riccati equation (18) becomes a new Riccati equation

$$\dot{x}' = a_2'\, x'^{\,2} + a_1'\, x' + a_0'\ , \tag{19}$$

where the relation amongst the old and new coefficients is given by

$$\begin{array}{rcl} a_2' &=& \delta^2\, a_2 - \delta\gamma\, a_1 + \gamma^2\, a_0 \\ a_1' &=& -2\,\beta\delta\, a_2 + (\alpha\delta + \beta\gamma)\, a_1 - 2\,\alpha\gamma\, a_0 \\ a_0' &=& \beta^2\, a_2 - \alpha\beta\, a_1 + \alpha^2\, a_0 \end{array} \quad . \tag{20}$$

This can also be written in a matrix form

$$\begin{pmatrix} a_2' \\ a_1' \\ a_0' \end{pmatrix} = \begin{pmatrix} \delta^2 & -\delta\gamma & \gamma^2 \\ -2\,\beta\delta & \alpha\delta + \beta\gamma & -2\,\alpha\gamma \\ \beta^2 & -\alpha\beta & \alpha^2 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} \quad . \tag{21}$$

In this way we define a linear action of the group $SL(2,\mathbb{R})$ on the set of Riccati equations with constant coefficients, i.e., on the set of fundamental vector fields and therefore on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. The action so obtained is nothing but the adjoint representation. Moreover, using the Killing–Cartan form on $\mathfrak{sl}(2,\mathbb{R})$ we can establish a one-to-one correspondence of it with $\mathfrak{sl}(2,\mathbb{R})^*$ and the action turns out to be the coadjoint action. The orbits of the action are then easily found: they are symplectic manifolds characterized by the values of the Casimir function corresponding to the natural Poisson structure defined on $\mathfrak{sl}(2,\mathbb{R})^*$, which is $a_1^2 - 4\,a_0 a_2$.

But the knowledge of the orbits is very important because in order to solve the given equation we can analyze whether there exists another easily solvable Riccati differential equation in the same orbit or not.

For instance, we would like to find some constant coefficients $\alpha, \beta, \gamma, \delta$ with $\alpha\gamma - \beta\delta = 1$ and such that either

$$a_0' = \beta^2\, a_2 - \alpha\beta\, a_1 + \alpha^2\, a_0 = 0\ , \tag{22}$$

or

$$a_2' = \delta^2\, a_2 - \delta\gamma\, a_1 + \gamma^2\, a_0 = 0\ . \tag{23}$$

In this case the corresponding transformation would reduce the original equation to a simpler one.

Notice however that if such coefficients do exist, then $x = -\beta/\alpha = k_1$, or $x = -\delta/\gamma = k_1$, respectively, is a constant solution of the original Riccati equation. Remember that the condition for the existence of at least one constant solution is $a_1^2 - 4\,a_2 a_0 \geq 0$. Conversely, if $x = k_1$ is a constant solution then the matrix

$$\begin{pmatrix} 1 & -k_1 \\ 0 & 1 \end{pmatrix} \tag{24}$$

which corresponds to the change $x' = x - k_1$ will transform the original equation (18) into a new one with $a_0' = 0$. That is,

$$\dot{x}' = a_2'\, x'^{\,2} + a_1'\, x' \ , \tag{25}$$

where $a_1' = 2\,k_1 a_2 + a_1$ and $a_2' = a_2$.

Respectively, a similar assertion follows for the matrix

$$\begin{pmatrix} 1 & 0 \\ -k_1^{-1} & 1 \end{pmatrix} \tag{26}$$

which corresponds to the change

$$\tilde{x}' = \frac{k_1\, x}{k_1 - x} \tag{27}$$

which transforms the equation (18) into one with $\tilde{a}_2' = 0$.

Let us assume now that there exists another constant solution k_2, for which a necessary and sufficient condition is $a_1^2 - 4\,a_0 a_2 > 0$. If we do first the change given by (24) then $k_2 - k_1$ will be a solution of (25) and a new change given by the matrix

$$\begin{pmatrix} 1 & 0 \\ (k_1 - k_2)^{-1} & 1 \end{pmatrix} \ ,$$

i.e.,

$$x'' = \frac{(k_1 - k_2)x'}{x' + k_1 - k_2} \ , \tag{28}$$

will lead to a new Riccati equation with $a_2'' = a_0'' = 0$, namely

$$\dot{x}'' = a_1''\, x'' \ , \tag{29}$$

with $a_1'' = a_1' = 2\,k_1\,a_2 + a_1$. If we take into account that if k_1, k_2 are constant solutions of (18) the following relations will be satisfied

$$a_1 = -a_2\,(k_1 + k_2) \ , \qquad a_0 = a_2\,k_1 k_2 \ , \tag{30}$$

and then $a''_1 = 2\,k_1\,a_2 + a_1 = a_2\,(k_1 - k_2)$. So, the equation (29) can be integrated by means of a quadrature:

$$x''(t) = x''(0)e^{a_2(k_1-k_2)t}.$$

Notice that the final equation (29) is linear, so any constant multiple of the transformation (28) also does the work.

When $a_0 \neq 0$ and two constant solutions k_1, k_2 of (18) are known we can also follow an alternative way because the expressions for a'_2 and a'_0 in terms of the old coefficients are similar, with the interchange of β by δ and α by γ, as can be seen in (20). Therefore, the knowledge of a constant solution $k_1 \neq 0$ can be used to put first $\tilde{a}'_2 = 0$ by means of the transformation (27) associated to the matrix (26) obtaining the linear equation

$$\dot{\tilde{x}}' = \tilde{a}'_1\,\tilde{x}' + \tilde{a}'_0 \tag{31}$$

where

$$\tilde{a}'_0 = a_0, \quad \tilde{a}'_1 = a_1 + 2k_1^{-1}a_0\ .$$

Then, as another constant solution k_2 of (18) is known, the constant function $k_1\,k_2\,(k_1 - k_2)^{-1}$ is a solution of (31). As a consequence, we can consider the transformation corresponding to the matrix

$$\begin{pmatrix} 1 & -k_1k_2(k_1-k_2)^{-1} \\ 0 & 1 \end{pmatrix}\ , \tag{32}$$

i.e.,

$$\tilde{x}'' = \tilde{x}' - \frac{k_1\,k_2}{k_1 - k_2} \tag{33}$$

which transforms (31) into the homogeneous linear equation

$$\dot{\tilde{x}}'' = \tilde{a}''_1\,\tilde{x}''\ , \tag{34}$$

with $\tilde{a}''_1 = a_2\,(k_2 - k_1)$.

5 The general Riccati equation with coefficients depending on t

In the general case in which the coefficients can depend on t, each Riccati equation can be considered as a curve in $\mathbb{R}^3$, or, in other words, as an element of $E = \mathrm{Map}(\mathbb{R}, \mathbb{R}^3)$.

The point now is that we can transform every function in $\mathbb{R}$, $x(t)$, under an element of the group of smooth $SL(2,\mathbb{R})$–valued curves $\mathrm{Map}(\mathbb{R},\, SL(2,\mathbb{R}))$, which from now on will be denoted $\mathcal{G}$, as follows:

$$\Theta(A, x(t)) = \frac{\alpha(t)x(t)+\beta(t)}{\gamma(t)x(t)+\delta(t)}\ , \quad \text{if}\ \ x(t) \neq -\frac{\delta(t)}{\gamma(t)}\ , \tag{35}$$

$$\Theta(A, \infty) = \frac{\alpha(t)}{\gamma(t)}\ , \qquad \Theta(A, -\frac{\delta(t)}{\gamma(t)}) = \infty\ , \tag{36}$$

$$\text{when}\ A = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \in \mathrm{Map}(\mathbb{R},\, SL(2,\mathbb{R}))\ , \tag{37}$$

and it is now easy to check that the Riccati equation (1) transforms under these changes into a new Riccati equation with coefficients given by

$$a'_2 = \delta^2\, a_2 - \delta\gamma\, a_1 + \gamma^2\, a_0 + \gamma\dot\delta - \delta\dot\gamma\ , \tag{38}$$

$$a'_1 = -2\,\beta\delta\, a_2 + (\alpha\delta+\beta\gamma)\, a_1 - 2\,\alpha\gamma\, a_0 + \delta\dot\alpha - \alpha\dot\delta + \beta\dot\gamma - \gamma\dot\beta\ , \tag{39}$$

$$a'_0 = \beta^2\, a_2 - \alpha\beta\, a_1 + \alpha^2\, a_0 + \alpha\dot\beta - \beta\dot\alpha\ . \tag{40}$$

Some particular instances of transformations of this type are those given by (2), (6), (9) and (10).

We can use this expression for defining an affine action of the group $\mathcal{G}$ on the set of general Riccati equations. The relation amongst new and old coefficients can be written in a matrix form

$$\begin{pmatrix} a'_2 \\ a'_1 \\ a'_0 \end{pmatrix} = \begin{pmatrix} \delta^2 & -\delta\gamma & \gamma^2 \\ -2\,\beta\delta & \alpha\delta+\beta\gamma & -2\,\alpha\gamma \\ \beta^2 & -\alpha\beta & \alpha^2 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} + \begin{pmatrix} \gamma\dot\delta - \delta\dot\gamma \\ \delta\dot\alpha - \alpha\dot\delta + \beta\dot\gamma - \gamma\dot\beta \\ \alpha\dot\beta - \beta\dot\alpha \end{pmatrix}\ . \tag{41}$$

In the first term of the right hand side we see the adjoint action. As far as the second term is concerned, we can check that it is a 1-cocycle for the adjoint action because if A is given by (37), then

$$\dot A = \begin{pmatrix} \dot\alpha & \dot\beta \\ \dot\gamma & \dot\delta \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\ ,$$

and therefore

$$\theta(A) = \dot{A}A^{-1} = \begin{pmatrix} \delta\dot\alpha - \gamma\dot\beta & \alpha\dot\beta - \beta\dot\alpha \\ \delta\dot\gamma - \gamma\dot\delta & \alpha\dot\delta - \beta\dot\gamma \end{pmatrix} ,$$

is a zero trace matrix because of the condition $\alpha\delta - \beta\gamma = 1$. Then, we can make use of the natural identification of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the one of zero trace matrices, taking into account our election of the basis (16). We arrive to the following value for the image of $\theta(A)$ under such identification, which with a slight abuse of notation we also denote by $\theta(A)$:

$$\theta(A) = \begin{pmatrix} \gamma\dot\delta - \delta\dot\gamma \\ \delta\dot\alpha - \alpha\dot\delta + \beta\dot\gamma - \gamma\dot\beta \\ \alpha\dot\beta - \beta\dot\alpha \end{pmatrix} ,$$

that is, the second term of the right side of (41). It is quite simple to check that the cocycle condition holds, because

$$\begin{aligned} \theta(A_2A_1) &= (A_2A_1)^{\cdot}(A_2A_1)^{-1} = (\dot A_2A_1 + A_2\dot A_1)A_1^{-1}A_2^{-1} \\ &= \dot A_2A_2^{-1} + (A_2\dot A_1)A_1^{-1}A_2^{-1} , \end{aligned} \qquad (42)$$

or in a different way,

$$\theta(A_2A_1) = \theta(A_2) + A_2\theta(A_1)A_2^{-1} ,$$

which is the 1–cocycle condition for the adjoint action. Consequently, see e.g [9], the expression (41) defines an affine action of $\mathcal{G}$ on the set of general Riccati equations. In other terms, to transform the coefficients of a general Riccati equation by means of two successive transformations of type (41) which are associated respectively with two elements A_1, A_2 of $\mathcal{G}$, gives exactly the same result as doing only *one* transformation of type (41) with associated element $A_2\,A_1$ of $\mathcal{G}$.

In similar way to what happens in the constant coefficients case, we can take advantage of some particular transformation to reduce a given equation to a simpler one. So, (39) shows that if we choose $\beta = \gamma = 0$ and $\delta = \alpha^{-1}$, i.e.,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} , \qquad (43)$$

then $a'_1 = 0$ if and only if the function α is such that

$$a_1 = -2\,\frac{\dot\alpha}{\alpha}\ ,$$

equation which has the particular solution

$$\alpha = \exp\left[-\frac{1}{2}\int a_1(t)dt\right]\ ,$$

i.e., the change is $x' = e^{-\phi}x$ with $\phi = \int a_1(t)\,dt$, and then $a'_2 = a_2 e^{\phi}$ and $a'_0 = a_0 e^{-\phi}$, which is the property **3-1-3.a.i** of [5]. In fact, under the transformation (43)

$$a'_2 = \alpha^{-2}a_2\ ,\quad a'_1 = a_1 + 2\,\frac{\dot\alpha}{\alpha}\ ,\quad a'_0 = \alpha^2 a_0\ , \tag{44}$$

and therefore with the above choice for α we see that $a'_1 = 0$.

If we use instead $\alpha = \delta = 1$, $\gamma = 0$, the function β can be chosen in such a way that $a'_1 = 0$ if and only if

$$\beta = \frac{a_1}{2a_2}\ ,$$

and then

$$a'_0 = a_0 + \dot\beta - \frac{a_1^2}{4a_2}\ ,\ a'_2 = a_2\ ,$$

which is the property **3-1-3.a.ii** of [5].

As another instance, the original equation (1) would be reduced to one with $a'_0 = 0$ if and only if there exist functions $\alpha(t)$ and $\beta(t)$ such that

$$\beta^2 a_2 - \alpha\beta a_1 + \alpha^2 a_0 + \alpha\dot\beta - \beta\dot\alpha = 0\ .$$

This was considered in [10], although written in the slightly modified way (4), as a criterion for the integrability of Riccati equation.

We should remark as an important fact that when dividing the preceding expression by α^2 we find that $x_1 = -\beta/\alpha$ is a solution of the original Riccati equation, and conversely, if a particular solution is known, x_1, then the element of $\mathcal{G}$

$$\begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix} \tag{45}$$

with associated change

$$x' = x - x_1 \ , \tag{46}$$

will transform the equation (1) into a new one with $a'_0 = 0$, $a'_2 = a_2$ and $a'_1 = 2\,x_1\,a_2 + a_1$, i.e., equation (7), which can be easily integrated by two quadratures. Consequently, the criterion given in [10] is nothing but the well known fact that once a particular solution is known, the original Riccati equation can be reduced to a Bernoulli one and therefore the general solution can be easily found. However, in our opinion the previous remark gives a very appropriate group theoretical explanation of the convenience of the change of variables given by (6).

As a particular case of the previous, one can interpret the property **3-1-3.b.iii** of [5], which assumes that a special solution x_1 of (1) is known such that the quantity

$$X(t) = 2\,x_1(t)\,a_2(t) + a_1(t) \tag{47}$$

is determined by the equation

$$\dot{x}_1 = -a_2\,x_1^2 + X(t)\,x_1 + a_0, \tag{48}$$

and then considers three possible values of $X(t)$. Replacing (47) in (48) we simply recover the fact that x_1 is a solution of (1). The quantity $X(t)$ is nothing but the coefficient a'_1 obtained with the transformation given by the matrix (45). Selecting $X(t)$ to be 0, $-\dot{a}_2/a_2$ and $a_1 - 2\sqrt{a_0\,a_2}$, respectively, means to consider special types of the Riccati equation with one *known* particular solution which can be expressed in terms of the coefficients of the equation, $a_i(t)$, and their time derivatives, $\dot{a}_i(t)$, and whose general solution involves an immediate quadrature, as it can be seen in (8). The special case which is said in [10] to be absent in [3] is simply expressed in these terms by saying that $X(t) = a_0 - a_2 + \dot{a}_0/a_0 - \dot{a}_2/a_2$. We would like to remark that the properties **3-1-3.a.i**, **3-1-3.a.ii** and **3-1-3.b.iii** of [5] can also be found in [3].

As indicated in the constant case, we can also follow a similar path by first reducing the original equation (1) to a new one with $\tilde{a}'_2 = 0$. Then, we should look for functions $\gamma(t)$ and $\delta(t)$ such that

$$\tilde{a}'_2 = \delta^2 a_2 - \delta\gamma a_1 + \gamma^2 a_0 + \gamma\dot{\delta} - \delta\dot{\gamma} = 0 \ .$$

This equation is similar to the one satisfied by α and β in order to obtain $a'_0 = 0$ with the replacement of β by δ and α by γ, and therefore we should consider the transformation given by the element of $\mathcal{G}$

$$\begin{pmatrix} 1 & 0 \\ -x_1^{-1} & 1 \end{pmatrix} , \tag{49}$$

that is,

$$\tilde{x}' = \frac{x_1\, x}{x_1 - x} \tag{50}$$

in order to obtain a new Riccati with $\tilde{a}'_2 = 0$. More explicitly, the new coefficients are

$$\tilde{a}'_2 = 0\,, \quad \tilde{a}'_1 = \frac{2\,a_0}{x_1} + a_1\,, \quad \tilde{a}'_0 = a_0 \ , \tag{51}$$

i.e., the original Riccati equation (1) becomes

$$\frac{d\tilde{x}'}{dt} = \left(\frac{2\,a_0}{x_1} + a_1\right)\tilde{x}' + a_0 \ . \tag{52}$$

Therefore, the transformation (50) will *directly* produce a linear equation (52). Such a change seems to be absent in the literature as far as we know.

Let us suppose now that another solution x_2 of (1) is also known. If we make the change (46) the difference $x_2 - x_1$ will be a solution of the resulting equation (7) and therefore, after using the change given by (45), the element of $\mathcal{G}$

$$\begin{pmatrix} 1 & 0 \\ (x_1 - x_2)^{-1} & 1 \end{pmatrix} \tag{53}$$

will transform the Riccati equation (7) into a new one with $a''_2 = a''_0 = 0$ and $a''_1 = a'_1 = 2\,x_1\,a_2 + a_1$, namely,

$$\frac{dx''}{dt} = (2\,x_1\,a_2 + a_1)\,x'' \ , \tag{54}$$

which can be integrated with just one quadrature. This fact can also be considered as a very appropriate group theoretical explanation of the introduction of the change of variable (10).

In fact, we can check directly that if we use the transformation with $\alpha = 1$, $\beta = 0$, $\delta = 1$ and $\gamma = (x_1 - x_2)^{-1}$ over the coefficients of (7), then we find that still $a''_0 = 0$ and

$$a''_2 = a_2 - (x_1 - x_2)^{-1} a'_1 + (x_1 - x_2)^{-2}(\dot{x}_1 - \dot{x}_2) \ ,$$

and as x_1 and x_2 are solutions of (1), taking the difference we see that

$$\dot{x}_1 - \dot{x}_2 = a_1(x_1 - x_2) + a_2(x_1^2 - x_2^2) \ , \tag{55}$$

from which we obtain

$$\begin{aligned} a_2'' &= (x_1 - x_2)^{-2}\{a_2(x_1 - x_2)^2 + (x_2 - x_1)(a_1 + 2x_1 a_2) \\ &\quad + a_1(x_1 - x_2) + a_2(x_1^2 - x_2^2)\} = 0 \ . \end{aligned}$$

The composition of both transformations (45) and (53) leads to the element of $\mathcal{G}$

$$\begin{pmatrix} 1 & -x_1 \\ (x_1 - x_2)^{-1} & -x_2(x_1 - x_2)^{-1} \end{pmatrix} \tag{56}$$

and therefore to the transformation (10).

Now we can compare the transformations (9) and (10). The first one corresponds to the element of $\mathcal{G}$ (we assume that $x_1(t) > x_2(t)$, for all t)

$$\frac{1}{\sqrt{x_1 - x_2}} \begin{pmatrix} 1 & -x_1 \\ 1 & -x_2 \end{pmatrix} \ , \tag{57}$$

and therefore both matrices (56) and (57) are obtained one from the other by multiplication by an element of type (43) with $\alpha = (x_1 - x_2)^{-1/2}$, and then (44) relates the coefficients a_1'' and $\bar{a}_1$ arising after one or the other transformation when taking into account (55):

$$\bar{a}_1 = a_1'' - a_1 - a_2(x_1 + x_2) = a_2(x_1 - x_2) \ ,$$

as expected.

By the other hand, If we use first the change of variable given by (50), the function

$$\tilde{x}_2' = \frac{x_2\, x_1}{x_1 - x_2}$$

will be a solution of (52). Then, a new transformation given by the element of $\mathcal{G}$

$$\begin{pmatrix} 1 & -\frac{x_2\, x_1}{x_1 - x_2} \\ 0 & 1 \end{pmatrix} \tag{58}$$

will lead to a new equation in which $\tilde{a}_0'' = 0$. More explicitly,

$$\tilde{a}_2'' = 0 \ , \quad \tilde{a}_1'' = \tilde{a}_1' = \frac{2\, a_0}{x_1} + a_1 \ , \quad \tilde{a}_0'' = 0 \ . \tag{59}$$

The composition of the two transformations is

$$\begin{pmatrix} 1 & -\frac{x_2\,x_1}{x_1-x_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{x_1} & 1 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{x_1-x_2} & -\frac{x_2\,x_1}{x_1-x_2} \\ -\frac{1}{x_1} & 1 \end{pmatrix} , \tag{60}$$

which corresponds to the change of variable

$$\tilde{x}'' = \frac{x_1^2}{(x_2-x_1)} \frac{(x-x_2)}{(x-x_1)} , \tag{61}$$

leading to the homogeneous linear equation

$$\dot{\tilde{x}}'' = \Big(\frac{2\,a_0}{x_1} + a_1\Big)\tilde{x}'' , \tag{62}$$

which can be integrated by means of just one quadrature.

6 The non–linear superposition principle

In a recent paper [1] the non–linear superposition principle for the Riccati equation was considered by using the Wei–Norman method of integrating differential equations on a Lie group, the superposition principle for Riccati equation arising in a very natural way.

The most important fact is that we can obtain the general solution even without solving the differential equation in the group. This is the case when we know a fundamental set of solutions that can be used to determine the explicit time–dependence of the canonical coordinates: this is the reason for the existence of such a superposition principle (for more details, see [1]).

This superposition principle can also be understood from a group theoretical viewpoint. Let us now suppose that we know three particular solutions x_1, x_2, x_3 of (1) and we can assume that $x_1 > x_2 > x_3$ for any value of the parameter t. Following the method described in the previous section we can use the two first solutions for reducing the Riccati equation to the simpler form of a linear equation, either to

$$\dot{x}'' = (2\,x_1\,a_2 + a_1)\,x , \tag{63}$$

or

$$\dot{\tilde{x}}'' = \Big(\frac{2\,a_0}{x_1} + a_1\Big)\tilde{x}'' . \tag{64}$$

The set of solutions of such differential equations is an one-dimensional linear space, so it suffices to know a particular solution to find the general solution. As we know that

$$x_3'' = (x_1 - x_2)\,\frac{x_3 - x_1}{x_3 - x_2} \tag{65}$$

is then a solution of equation (63), and

$$\tilde{x}_3'' = \frac{x_1^2}{(x_2 - x_1)}\frac{(x_3 - x_2)}{(x_3 - x_1)} \tag{66}$$

is a solution of (64), we can take advantage of an appropriate diagonal element of $\mathcal{G}$ of the form

$$\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} ,$$

with z being one of the two mentioned solutions in order to reduce the equations either to $\dot{x}''' = 0$ or $\dot{\tilde{x}}''' = 0$, respectively. These last equations have the general solutions

$$x''' = k \ ,$$

or

$$\tilde{x}''' = k \ ,$$

which show the superposition formula (11).

More explicitly, for the first case (63) the product transformation will be given by

$$\begin{pmatrix} \sqrt{\dfrac{(x_2 - x_3)}{(x_1 - x_3)(x_1 - x_2)}} & -x_1\sqrt{\dfrac{(x_2 - x_3)}{(x_1 - x_3)(x_1 - x_2)}} \\ \sqrt{\dfrac{(x_1 - x_3)}{(x_2 - x_3)(x_1 - x_2)}} & -x_2\sqrt{\dfrac{(x_1 - x_3)}{(x_2 - x_3)(x_1 - x_2)}} \end{pmatrix} , \tag{67}$$

or written in a different way

$$\frac{-1}{\sqrt{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}}\begin{pmatrix} x_2 - x_3 & -x_1(x_2 - x_3) \\ x_1 - x_3 & -x_2(x_1 - x_3) \end{pmatrix} . \tag{68}$$

The transformation defined by this element of $\mathcal{G}$ is

$$x''' = \frac{(x - x_1)(x_2 - x_3)}{(x - x_2)(x_1 - x_3)} \tag{69}$$

and therefore we arrive in this way to the superposition principle giving the general solution of the Riccati equation (1) in terms of three particular solutions and the value of a constant k characterizing each particular solution:

$$\frac{(x - x_1)(x_2 - x_3)}{(x - x_2)(x_1 - x_3)} = k \ . \tag{70}$$

The other case (64) can be treated in a similar way, leading also to the no linear superposition principle of the Riccati equation.

7 Conclusions and outlook

We have carried out the application of group theoretical methods in order to understand several well known properties of Riccati equation that seems to have been unrelated until now. This work has been motivated by the recent applications in Supersymmetric Quantum Mechanics and the factorization problems, on the physical side, and from the mathematical viewpoint by a very interesting property of such equation: it admits a non–linear superposition principle. This property was shown by Lie to be related with the particular form of the Riccati equation [6], because the time–dependent vector field of the general Riccati equation can be written as a linear combination with time–dependent coefficients of vector fields closing on a $\mathfrak{sl}(2, \mathbb{R})$ algebra.

The use of such group theoretical methods has allowed us to get a better understanding of the reduction of the problem when some particular solutions are known, and we have discovered a previously, as far as we know, unknown alternative way for such reduction, that is, the one given by the changes (49) and (58). In particular, we have obtained explicitly the non–linear superposition principle of the Riccati equation. As a by-product, we have made clear the meaning of the integrability conditions pointed out by Strelchenya in [10] as well as other previously known criteria.

The question is now open to the possible generalization of these properties in the case of other differential equation systems admitting a non–linear superposition principle, for which similar techniques should be useful.

For such a generalization, it might be useful the observation that for some particular cases, the studied Bernoulli and linear ones, for which the general solution is easily found, the time–dependent vector field associated to the equation takes values in a Lie subalgebra isomorphic to the Lie algebra of the affine group of transformations in one dimension. Those Riccati equations that are in the same orbit under the action of $\mathcal{G}$ as one of these particular cases can be reduced to them and therefore they are integrable.

The elements of $\mathcal{G}$ which take the Riccati equation to their reduced form provided some particular solutions are known, are constructed *with* such solutions, giving an explanation to the previously known properties of the Riccati equation involving changes of variable which seemed to be rather miraculous.

We hope that the application of these techniques will also be enlightening for the above mentioned problems of factorization of a quantum Hamiltonian as a product $A^\dagger A$ of first order differential operators and the corresponding problem of related Hamiltonian operators [2].

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