Superposition rules, Lie theorem, and partial differential equations

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Abstract

A rigorous geometric proof of the Lie's Theorem on nonlinear superposition rules for solutions of non-autonomous ordinary differential equations is given filling in all the gaps present in the existing literature. The proof is based on an alternative but equivalent definition of a superposition rule: it is considered as a foliation with some suitable properties. The problem of uniqueness of the superposition function is solved, the key point being the codimension of the foliation constructed from the given Lie algebra of vector fields. Finally, as a more convincing argument supporting the use of this alternative definition of superposition rule, it is shown that this definition allows an immediate generalization of Lie's Theorem for the case of systems of partial differential equations.

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1 Introduction

The integration of systems of differential equations admitting infinitesimal symmetries was the main concern of Lie in developing what is nowadays called the theory of Lie algebras and Lie groups. In particular, in a remarkable work [15] he was able to prove an important theorem connecting Lie algebras and nonlinear superposition rules for solutions of some non-autonomous systems of nonlinear ordinary differential equations. These systems can be considered as a generalization of linear systems but the superposition rule is no longer a linear function. Our aim in this paper is to study once again, from a geometric viewpoint, the theory of systems of differential equations admitting a (maybe nonlinear) superposition rule, allowing us to express the general solution of the system by means of a superposition function in terms of a (fundamental) set of particular solutions, with the hope of establishing clearly the necessary and sufficient conditions for a system to admit such a superposition rule. This time, however, we include also partial differential equations into our considerations.

Even if the hypotheses of Lie's theorem were not accurately stated from the today level of rigor, the resulting systems characterized by means of an associated Lie algebra appear very often in physical problems and in many cases the problem is related with another one on the corresponding Lie group. This provides us with both methods of reduction to simpler problems on one side, and another method, introduced by Wei and Norman which involves some algebraic manipulations based on Lie groups and Lie algebra theories, on the other.

As far as we know, there is no rigorous proof of the if part in Lie's theorem and the attempts known to us to get a rigorous geometric proof share the same pseudo-argument [2, 17]. In this paper we prove that actually the existence of a superposition rule for the solutions of a given non-autonomous system implies that it has the explicit form which is usually accepted. The proof is based on an alternative but equivalent definition of superposition rule: we consider it as a foliation with some appropriate properties explicitly formulated later on. An auxiliary lemma is necessary to overcome the weak point in previous derivations of the theorem [2, 17].

On the other hand, the converse part is not given in its full generality and almost nothing is said about the uniqueness of the superposition function for these Lie systems. Only in [10] an example given in [15], for which there are two different superposition functions, is pointed out. Our approach provides us with an answer to this important question, the key point being the codimension of the foliation constructed from the given Lie algebra of vector fields. Moreover, this codimension is very relevant when the action of the Lie algebra of vector fields on the initial manifold is not transitive.

Finally, as a more convincing argument supporting the use of this alternative definition of superposition rule, it will be shown that this definition allows an immediate generalization of Lie's Theorem for the case of systems of partial differential equations.

The organization of the paper is as follows. Next section discuss the concept of superposition function and gives a geometric characterization of such superposition in terms of a foliation. Section 3 is devoted to a complete proof of the statement of Lie's theorem by establishing a lemma which allows us to overcome the weak point of other previous derivations. The number of solutions in a fundamental set is discussed in Section 4 and in Section 5 the problem of uniqueness of the superposition rule is studied. Lie systems on Lie groups and homogeneous spaces are considered in Section 6 as most important examples. Finally, a generalization of the Lie's Theorem for the case of systems of first-order partial differential equations is given in Section 7. An outlook with future applications is given in the last section of the paper.

2 Superposition rules for ordinary differential equations

By a *superposition rule* (or a *superposition principle*) for a given system of ordinary differential equations

$$\frac{dx^i}{dt} = Y^i(t,x)\,, \qquad i = 1,\ldots,n\,, \tag{1}$$

one usually understands, after [15], a superposition function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ given by

$$x = \Phi(x_{(1)},\ldots,x_{(m)};k_1,\ldots,k_n)\ , \tag{2}$$

such that the general solution can be written, at least for sufficiently small t, as

$$x(t) = \Phi(x_{(1)}(t),\ldots,x_{(m)}(t);k_1,\ldots,k_n)\ , \tag{3}$$

with $\{x_{(a)}(t) \mid a = 1,\ldots,m\}$ being a fundamental set of particular solutions of the system (1) and $k = (k_1,\ldots,k_n)$ being a set of n arbitrary constants associated with each particular solution.

The standard example is the system of linear differential equations

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\,x^j\ , \quad i = 1,\ldots,n\,, \tag{4}$$

which admits the superposition function, with $m = n$,

$$x = \Phi(x_{(1)},\ldots,x_{(n)};k_1,\ldots,k_n) = \sum_{i=1}^{n} k_i\,x_{(i)}\ .$$

Of course, we can obtain every solution by superposing $x_{(1)}(t),\ldots,x_{(m)}(t)$, for certain $k_1,\ldots,k_n$, only if the functions $x_{(1)}(t),\ldots,x_{(m)}(t)$, are appropriately independent, i.e. if they form a fundamental set of solutions. In the above example it means that if the matrix

$$X(t) = (x^i_{(j)}(t))^i_j \qquad i,j = 1,\ldots,n\,, \tag{5}$$

is invertible for small t.

The order in which the particular solutions are chosen is irrelevant and therefore the superposition function should be such that a permutation of two arguments only amounts to a change of the parameters k. Note also that it is assumed that the superposition function Φ does not depend explicitly on the independent variable t, and this fact has strong consequences (see later the Lie theorem).

From a geometric perspective, systems of differential equations as (4) appear as those determining the integral curves of a t-dependent vector field in $\mathbb{R}^n$,

$$Y(t,x) = \sum_{i=1}^{n} Y^i(t,x)\,\frac{\partial}{\partial x^i}\,,$$

the generalization to the case of a n-dimensional manifold being immediate. Note that then in any point $x \in N$, the t-dependent vector field Y in N, determines not only one vector but a linear subspace, spanned by the set of vectors $\{Y(t,x) \mid t \in \mathbb{R}\}$, of the corresponding tangent space. Actually, under a time re-parametrization the vectors are rescaled and, when changing the value of t, different vectors are obtained. In this way it defines a 'generalized' distribution for which the dimension of the linear subspace can change from one point to another. We will see later on that the case we are interested in is such that the distribution defined by the t-dependent vector field is involutive.

In order to look for superposition rules we need a more geometric picture. Let us first observe that, as a consequence of the Implicit Function Theorem, the function $\Phi(x_{(1)},\ldots,x_{(m)};\,\cdot\,):\mathbb{R}^n \to \mathbb{R}^n$ can be, at least locally around generic points, inverted, so we can write

$$k = \Psi(x_{(0)},\ldots,x_{(m)}) \tag{6}$$

for a certain function $\Psi : \mathbb{R}^{n(m+1)} \longrightarrow \mathbb{R}^n$. Hereafter in order to handle a short notation we start writing $x_{(0)}$ instead of x. The foliation defined by the function Ψ is now invariant under permutations of the $(m+1)$ variables.

With some abuse of terminology we will also call the function Ψ a superposition function. The relation between Φ and Ψ is given by:

$$k = \Psi(\Phi(x_{(1)},\ldots,x_{(m)};k_1,\ldots,k_n),x_{(1)},\ldots,x_{(m)})\,. \tag{7}$$

For instance, for the system (4) we have

$$\Psi(x_{(0)}(t),\ldots,x_{(n)}(t)) = X^{-1}(t)x_{(0)}(t),$$

where $X(t)$ is the matrix given in (5). This example indicates the obvious fact that, in general, the superposition function Ψ is defined on an open dense subset of $\mathbb{R}^{n(m+1)}$ rather than the whole $\mathbb{R}^{n(m+1)}$.

The fundamental property of the superposition function Ψ is that as

$$k = \Psi(x_{(0)}(t),x_{(1)}(t),\ldots,x_{(m)}(t))\,, \tag{8}$$

the function $\Psi(x_{(0)},\ldots,x_{(m)})$ is constant on any $(m+1)$-tuple of solutions of the system (1). This property is true for any choice of $(m+1)$ solutions and this means that the foliation is invariant under the permutation of the $(m+1)$ arguments of the function Ψ.

After differentiation of relation (8) with respect to t, as the functions $x_i(t)$ are solutions of (1), we get

$$D\Psi(Y(t,x_{(0)}),\ldots,Y(t,x_{(m)}))=0\,, \tag{9}$$

i.e.

$$\sum_{i=1}^{n}\sum_{a=0}^{m}\frac{\partial\Psi}{\partial x^i_{(a)}}Y^i(t,x_{(a)})=0\,,$$

and therefore the 'diagonal prolongations' $\widetilde{Y}(t,x_{(0)},\ldots,x_{(m)})$ of the t-dependent vector field $Y(t,x)$, given by

$$\widetilde{Y}(t,x_{(0)},\ldots,x_{(m)})=\sum_{a=0}^{m}Y_a(t,x_{(a)})\,,\qquad t\in\mathbb{R}\,,$$

where

$$Y_a(t,x_{(a)})=\sum_{i=1}^{n}Y^i(t,x_{(a)})\,\frac{\partial}{\partial x^i_{(a)}} \tag{10}$$

are t-dependent vector fields on $\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ ($(m+1)$ factors) which are tangent to the level sets of Ψ as displayed by (9).

The level sets of Ψ corresponding to regular values define a n-codimensional foliation $\mathcal{F}$ on an open dense subset $U\subset\mathbb{R}^n\times\cdots\times\mathbb{R}^n$ ($(m+1)$ factors) and the family $\{\widetilde{Y}(t),\,t\in\mathbb{R}\}$ of vector fields in $\mathbb{R}^{n(m+1)}$ consists of vector fields tangent to the leaves of this foliation.

This foliation has another important property. Since on the level set $\mathcal{F}_k$ corresponding to $k=(k_1,\ldots,k_n)\in\mathbb{R}^n$ and given $(x_{(1)},\ldots,x_{(m)})\in\mathbb{R}^{nm}$, there is a unique point $(x_{(0)},x_{(1)},\ldots,x_{(m)})\in\mathcal{F}_k$, namely, $(\Phi(x_{(1)},\ldots,x_{(m)};k),x_{(1)},\ldots,x_{(m)})\in\mathcal{F}_k$ (cf. (7)), then the projection onto the last m factors

$$\mathrm{pr}:(x_{(0)},x_{(1)},\ldots,x_{(m)})\in\mathbb{R}^{n(m+1)}\mapsto(x_{(1)},\ldots,x_{(m)})\in\mathbb{R}^{nm}$$

induces diffeomorphisms on the leaves $\mathcal{F}_k$ of $\mathcal{F}$.

This can also be viewed as the fact that the foliation $\mathcal{F}$ corresponds to a connection Δ in the bundle $\mathrm{pr}:\mathbb{R}^n\times\cdots\times\mathbb{R}^n=\mathbb{R}^{n(m+1)}\to\mathbb{R}^n\times\cdots\times\mathbb{R}^n=\mathbb{R}^{nm}$ with trivial curvature. The restriction of the projection pr to a leaf gives a one-to-one map. In this way there is a linear map among vector fields in $\mathbb{R}^{nm}$ and (horizontal) vector fields tangent to a leaf.

Note that the knowledge of this connection (foliation) gives us the superposition principle without referring to the function Ψ (which we can change by composing it, for instance, with a diffeomorphism of $\mathbb{R}^n$): if we fix the point $x_{(0)}(0)$, i.e. we choose a $k=(k_1,\ldots,k_n)$, and m solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$, then $x_{(0)}(t)$ is the unique point in $\mathbb{R}^n$ such that $(x_{(0)}(t),x_{(1)}(t),\ldots,x_{(m)}(t))$ belongs to the same leaf of $\mathcal{F}$ as $(x_{(0)}(0),x_{(1)}(0),\ldots,x_{(m)}(0))$. Thus, it is only $\mathcal{F}$ that really matters when the superposition rule is concerned.

On the other hand, if we have a connection ∇ in the bundle $\mathrm{pr}:\mathbb{R}^{n(m+1)}\to\mathbb{R}^{nm}$ with a trivial curvature, i.e. we have a horizontal distribution ∇ in $T\mathbb{R}^{n(m+1)}$ that is involutive, which therefore can be integrated to give a foliation in $\mathbb{R}^{n(m+1)}$ such that the vector fields $\widetilde{Y}(t)$ belong to ∇ (equivalently, are tangent to $\mathcal{F}$, i.e. are horizontal), then the procedure described above determines a superposition rule for the system (1).

Indeed, let $k \in \mathbb{R}^n$ enumerate smoothly the leaves $\mathcal{F}_k$ of $\mathcal{F}$ (e.g. by a choice of a small cross-section of $\mathcal{F}$), then $\Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k)$ defined as the unique point $x_{(0)}(t) \in \mathbb{R}^n$ such that $(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \in \mathcal{F}_k$, is a superposition rule for the system (1) of ordinary differential equations. To see this, let us observe that the inverse is $\Psi(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) = k$, which is equivalent to $(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \in \mathcal{F}_k$. If we fix k and take solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ of (1), then $x_{(0)}(t)$ defined by the condition $\Psi(x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) = k$ satisfies (1). Indeed, let $x'_{(0)}(t)$ be the solution of (1) with initial value $x'_{(0)}(0) = x_{(0)}(0)$. Since the t-dependent vector fields $\widetilde{Y}(t)$ are tangent to $\mathcal{F}$, the curve $(x'_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t))$ lies entirely on a leaf of $\mathcal{F}$, so on $\mathcal{F}_k$. But the point of one leaf is entirely determined by its projection pr, so $x'_{(0)}(t) = x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution. Thus we have proved the following geometric characterization of superposition rules:

Proposition 1 *Giving a superposition rule (2) for a system of differential equations (1) is equivalent to giving a zero curvature connection in the bundle* $\mathrm{pr} : \mathbb{R}^{(m+1)n} \to \mathbb{R}^{nm}$ *for which the diagonal prolongations* $\widetilde{Y}(t)$ *of the* t*-dependent vector fields* $Y(t)$, $t \in \mathbb{R}$, *defining the system (1) are horizontal.*

Note that we can also consider arbitrary manifolds N instead of $\mathbb{R}^n$. The superposition functions are then given by maps $\Phi : N^{m+1} \to N$ or by appropriate foliations in N^{m+1}, i.e. zero curvature connections in the bundle $\mathrm{pr} : N^{m+1} \to N^m$.

Remark 1 Frankly speaking, the connection is defined only generically, usually over an open-dense subset. But this is a general problem with superpositions, which hold only 'generically'. *In the sequel all objects and constructions will be 'generic' in this sense.*

Example 1 Consider the (generalized) foliation $\mathcal{F}$ of codimension one generated by the vector field $x\,\partial/\partial x + y\,\partial/\partial y$ on $\mathbb{R}^2 - \{(0,0)\}$. It defines a zero curvature connection for the bundle $\mathrm{pr} : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R} - \{0\}$, $\mathrm{pr}(x, y) = y \in \mathbb{R}$. The leaves of $\mathcal{F}$ are of the form $(e^t x, e^t y)$, $t \in \mathbb{R}$, $y \neq 0$. In particular, the function $\Psi(x, y) = x/y$ is constant on the leaves. The diagonal prolongation of the t-dependent vector fields $a(t)\,x\,\partial/\partial x$ are of the form $a(t)\,(x\,\partial/\partial x + y\,\partial/\partial y)$ and are tangent to $\mathcal{F}$. This gives us the superposition rule for the linear differential equation $\dot{x} = a(t)\,x$ as follows: If $x_{(1)}(t)$ is a solution, then $(x_{(0)}(t), x_{(1)}(t))$ belongs to one leaf, e.g with $x/y = k$, if and only if $x_{(0)}(t) = k\,x_{(1)}(t)$. This is the standard superposition rule for this equation:

$$\Phi(x_{(1)}(t); k) = k\,x_{(1)}(t)\,.$$

Remark 2 It is clear from the above proposition that when the diagonal prolongations $\widetilde{Y}(t)$ generate a foliation of dimension smaller than the dimension of $\mathcal{F}$, we can change $\mathcal{F}$ (i.e. we can change the connection ∇) respecting the required properties. This means that the corresponding system (1) admits many different superposition rules even if we regard compositions of Φ with diffeomorphisms of $\mathbb{R}^n$ as equivalence relations.

3 Lie theorem for ODE's systems admitting superposition rules

In this section we shall give a proof of the classical Lie's theorem on ODE's admitting a superposition rule (cf. [15, 2, 3, 4]).

First of all, in this proof we shall fill in all gaps which appear in the known literature. Second, we shall use the alternative characterization of superposition rules given in Proposition 1 and last, but not least, the proof, as it will be presented, can be immediately extended to the case of partial differential equations.

Remark first that all considerations will be 'generic' and local. Let us recall that a foliation $\mathcal{F}$ of an open dense subset of $\widetilde{N} = N^{m+1} = N \times \cdots \times N$ (($m+1$) factors) defines a superposition rule for the system of differential equations (1), if and only if $\mathcal{F}$ is of codimension n, the projection $\mathrm{pr} : \widetilde{N} \to N \times \cdots \times N = N^m$ (only m factors) onto the last m arguments maps leaves of $\mathcal{F}$ diffeomorphically and, furthermore, the generalized foliation $\mathcal{F}_0$ generated by the family $\{\widetilde{Y}(t) \mid t \in \mathbb{R}\}$ of diagonal prolongations of $Y(t)$ is contained in $\mathcal{F}$.

We shall work only with the regular part of $\mathcal{F}_0$. Such regular part is spanned by $\{\widetilde{Y}(t) \mid t \in \mathbb{R}\}$, i.e. in any case, it is spanned by diagonal prolongations of some vector fields on N as $[\widetilde{Y}(t), \widetilde{Y}(t')] = \widetilde{[Y(t), Y(t')]}$, etc. Let $\widetilde{X}_1, \dots, \widetilde{X}_r$ be diagonal prolongations spanning locally the regular part of $\mathcal{F}_0$ of dimension r, and therefore $\mathrm{pr}_*(\widetilde{X}_1), \dots, \mathrm{pr}_*(\widetilde{X}_r)$ are assumed to be linearly independent at a generic point. We clearly have $r \leq mn$.

Since $\widetilde{X}_1, \dots, \widetilde{X}_r$ span an r-dimensional foliation, then

$$[\widetilde{X}_\alpha, \widetilde{X}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\, \widetilde{X}_\gamma\,,$$

for some r^3 functions $c_{\alpha\beta}{}^{\gamma}$ defined on $\widetilde{N}$. Note also that $[\widetilde{X}_\alpha, \widetilde{X}_\beta]$ are diagonal prolongations as brackets of diagonal prolongations and the projections $\mathrm{pr}_*(\widetilde{X}_1), \dots, \mathrm{pr}_*(\widetilde{X}_r)$, are assumed to be functionally independent. Then we shall use the following lemma.

Lemma 1 *Let* $\widetilde{X}_\alpha = \sum_{a=0}^{m} X_{\alpha(a)}$*, for* $\alpha = 1, \dots r$*, and with* $r \leq mn$*, be diagonal prolongations to* $\widetilde{N}$ *of vector fields* X_α *on* N*, and such that at each point* p *of* N^m *the vectors that are the projections of their values,* $\mathrm{pr}_*(\widetilde{X}_\alpha)(p) = \sum_{a=1}^{m} X_{\alpha(a)}(p)$*, are linearly independent. Then,* $\sum_{\alpha=1}^{r} b_\alpha\, \widetilde{X}_\alpha$*, with* $b_\alpha \in C^\infty(\widetilde{N})$*, is again a diagonal prolongation if and only if the coefficients* b_α *are constant.*

Proof.- Let us write in local coordinates

$$X_\alpha = \sum_{i=1}^{n} A^i_\alpha(x) \frac{\partial}{\partial x^i}\,,$$

which implies that

$$\widetilde{X}_\alpha = \sum_{i=1}^{n}\sum_{a=0}^{m} A^i_\alpha(x_{(a)})\frac{\partial}{\partial x^i_{(a)}}\,.$$

Then,

$$\sum_{\alpha=1}^{r} b_\alpha(x_{(0)},\ldots,x_{(m)})\widetilde{X}_\alpha = \sum_{\alpha=1}^{r}\sum_{i=1}^{n}\sum_{a=0}^{m} b_\alpha(x_{(0)},\ldots,x_{(m)})A^i_\alpha(x_{(a)})\,\frac{\partial}{\partial x^i_{(a)}}\,,$$

which is a diagonal prolongation if and only if there are functions $B^i_a(x)$, for $a = 0,\ldots,m$, and $i = 1,\ldots,n$, such that for each pair of indexes i and a,

$$\sum_{\alpha=1}^{r} b_\alpha(x_{(0)},\ldots,x_{(m)})\,A^i_\alpha(x_{(a)}) = B^i_a(x_{(a)})\;,\quad a = 0,\ldots,m\;\; i = 1,\ldots,n.$$

In particular, the r functions $b_\alpha(x_{(0)},\ldots,x_{(m)})$ solve the following subsystem of linear equations in the unknown u_α, for $\alpha = 1,\ldots,r$:

$$\sum_{\alpha=1}^{r} u_\alpha\,A^i_\alpha(x_{(a)}) = B^i_a(x_{(a)})\,,\qquad \text{with } i = 1,\ldots,n\,,\; a = 1,\ldots,m\,. \tag{11}$$

But the rank of the matrix $(A^i_\alpha(x_{(a)}))^{i,a}_\alpha$ is $r \le mn$ as the projections $\mathrm{pr}_*(\widetilde{X}_1),\ldots,\mathrm{pr}_*(\widetilde{X}_r)$ are assumed to be linearly independent. Thus the solutions $u_1,\ldots,u_r$ of (11) are unique and are completely determined by the matrix $A^i_\alpha(x_{(a)})^{i,a}_\alpha$ and the vector $B^i_a(x_{(a)})^{i,a}$, with $a = 1,\ldots,m$, so they do not depend on $x_{(0)}$. But since the diagonal prolongations are invariant with respect to the symmetry group S_{m+1} acting on $\widetilde{N} = N^{m+1}$ in an obvious way, the functions $b_\alpha(x_{(0)},\ldots,x_{(m)})$ do not depend also on the other variables $x_{(1)},\ldots,x_{(m)}$.

Remark 3 Let us note that the assumption on the projections is crucial for the above lemma and actually without such assumption the result of this fundamental lemma is simply wrong as the following example shows. Consider the following two vector fields in $\mathbb{R}^2$ which are prolongations of vector fields in $\mathbb{R}$:

$$\widetilde{X}_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,,\qquad \widetilde{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$$

and the functions

$$b_1(x,y) = x\,y\,,\qquad b_2(x,y) = -(x+y)$$

for which

$$b_1(x,y)\widetilde{X}_1(x,y) + b_2(x,y)\,\widetilde{X}_2(x,y) = -\left(x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)$$

is also a prolongation. However the coefficients b_1 and b_2 are not constant. This is the standard gap in the proofs of Lie's theorem we found in the literature. One usually claims that a functional combination of diagonal prolongations is a diagonal prolongation only if the coefficients are constant without assuming that the corresponding projections are linearly independent.

Now, we can now prove the above mentioned Lie theorem using the previous results.

Theorem 1 *The system (1) on a differentiable manifold N admits a superposition rule if and only if the t-dependent vector field $Y(t,x)$ can be locally written in the form*

$$Y(t,x) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x)$$

where the t-dependent vector fields X_α, $\alpha = 1,\ldots,r$, close on a finite-dimensional real Lie algebra, i.e. there exist r^3 real numbers $c_{\alpha\beta}{}^\gamma$ such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma\, X_\gamma\,, \qquad \forall \alpha,\beta = 1,\ldots,r\,.$$

Proof.- Suppose that the system admits a superposition rule and let $\mathcal{F}$ be the foliation corresponding to the superposition function. We know already that the generators $\{\widetilde{X}_\alpha \mid \alpha = 1,\ldots,r\}$ of the regular part of $\mathcal{F}_0 \subset \mathcal{F}$ close on a Lie algebra

$$[\widetilde{X}_\alpha, \widetilde{X}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma \widetilde{X}_\gamma, \tag{12}$$

where the coefficients $c_{\alpha\beta}{}^\gamma$ are constant, so also

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma\, X_\gamma\,.$$

Since every $\widetilde{Y}(t)$ is tangent to $\mathcal{F}_0$ there are functions $b_t^\alpha(x_{(0)},\ldots,x_{(m)})$ such that

$$\widetilde{Y}(t) = \sum_{\alpha=1}^{r} b_t^\alpha\, \widetilde{X}_\alpha\,.$$

But $\widetilde{Y}(t)$ is a diagonal prolongation, so, using the fundamental lemma once more, we get that the $b_t^\alpha = b^\alpha(t)$ are independent on $x_{(0)},\ldots,x_{(m)}$. Hence

$$\widetilde{Y}(t) = \sum_{\alpha=1}^{r} b^\alpha(t)\, \widetilde{X}_\alpha \tag{13}$$

and also

$$Y(t) = \sum_{\alpha=1}^{r} b^\alpha(t)\, X_\alpha\,. \tag{14}$$

To prove the converse property, assume that the t-dependent vector field $Y(t,x)$ can be written as in (13) and define $\widetilde{Y}(t)$ by (14). We can additionally assume that the vector fields X_α are linearly independent over $\mathbb{R}$. Thus they define an r-dimensional Lie algebra with structure constants $c_{\alpha\beta}{}^\gamma$ (and the corresponding simply connected Lie group if the vector fields are complete).

Since only non-trivial functional dependence of $X_1, \dots, X_r$ is possible, there is a number $m \leq r$ such that their diagonal prolongations to $N^m = N \times \cdots \times N$ (m times) are generically linearly independent at each point. The distribution spanned by the diagonal prolongations $\widetilde{X}_1, \dots, \widetilde{X}_r$ to $\widetilde{N} = N^{m+1}$ is clearly involutive, so it defines an r-dimensional foliation $\mathcal{F}_0$ of $\widetilde{N}$. Moreover, the leaves of this foliation project onto the product of the last m factors diffeomorphically and they are at least n-codimensional. Now, it is obvious that we can extend this foliation to an n-codimensional foliation $\mathcal{F}$ with the latter property, and this foliation, according to proposition 1, defines a superposition rule. Here, of course, extending of a foliation means that the leaves of the smaller are submanifolds of the leaves of the extension.

Remark 4 There is another way to look at the superposition the way we proposed. We can consider two projections: the first one, $\mathrm{pr} : \widetilde{N} = N^{n(m+1)} \to N^{nm}$ is on the last m factors and the second one, $\mathrm{pr}_1 : \widetilde{N} \to N$ the projection on the first factor. These projections are clearly transversal and the first one is a fibration if the foliation $\mathcal{F}$ is conserved, i.e. every curve in N^m has a unique lifting in a fixed leaf of $\mathcal{F}$. Then, the superposition associated with this lift is just pr_1 of the lift of particular solutions.

Remark 5 We hope that it is clear to the reader that in our picture of superposition rules there is no real need to take all the manifolds N equal in the product where the superposition foliation lives. We can consider as well $\widetilde{N} = N_0 \times \cdots N_m$ with analogous projection and fibration property. This means that we can get a solution of a system on N_0 out of solutions of some systems on N_a, $a = 1, \dots, m$. This is a geometric picture for the Darboux (Bäcklund) transformations. We will, however, discuss this problem in a separate paper.

4 Determination of the number m of solutions of a fundamental set

Our proof of the Lie's Theorem contains an information about the number m of solutions involved in the superposition rule. For a Lie system defined by a t-dependent vector field of the form

$$Y(t, x) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x)\,,$$

with generic $b_\alpha(t)$ the number m turned out to be the minimal k such that the diagonal prolongations of $X_1, \dots, X_r$ to N^k are linearly independent at (generically) each point: the only real numbers solution of the linear system

$$\sum_{\alpha=1}^{r} c_\alpha\, X_\alpha(x_{(a)}) = 0\,, \qquad a = 1, \dots, k$$

at a generic point $(x_{(1)}, \dots, x_{(k)})$ is the trivial solution $c_\alpha = 0$, $\alpha = 1, \dots, m$, for $k = m$ and there are nontrivial solutions for $k < m$.

For instance, for the Riccati equation

$$\dot{x} = b_0(t) + b_1(t)\,x + b_2(t)x^2\,,$$

in which the vector fields generating the foliation $\mathcal{F}_0$ are prolongations of the vector fields

$$X_0 = \frac{\partial}{\partial x}\,, \qquad X_1 = x\frac{\partial}{\partial x}\,, \qquad X_2 = x^2\frac{\partial}{\partial x}\,, \tag{15}$$

representing an $sl(2,\mathbb{R})$-action, we see that the system

$$c_0 + c_1x_1 + c_2x_1^2 = 0\,, \qquad c_0 + c_1x_2 + c_2x_2^2 = 0$$

has a nontrivial solution but the one given by

$$c_0 + c_1x_1 + c_2x_1^2 = 0\,, \qquad c_0 + c_1x_2 + c_2x_2^2 = 0\,, \qquad c_0 + c_1x_3 + c_2x_3^2 = 0\,,$$

does not admit non-trivial solutions because the determinant of the coefficients is invertible when the three points x_1, x_2 and x_3 are different. This implies that $m = 3$ in the superposition rule for the Riccati equation.

5 Nonuniqueness of the superposition rule

In some cases the foliation $\mathcal{F}_0$ spanned by the prolongations of t-dependent vector fields defining the dynamics is already n-codimensional and we get a unique (minimal) superposition rule (e.g. Riccati equation). In the cases with $\operatorname{codim}\mathcal{F}_0 > n$, we have some ambiguity in choosing the superposition rule as we can extend $\mathcal{F}_0$ to an n-codimensional foliation in different ways.

Example 2 Consider the action of the Abelian group $\mathbb{R}$ acting on $N = \mathbb{R}^2$ by horizontal translations., i.e. $X_1 = \partial/\partial x$. This action on $\mathbb{R}^2$ is free and we have $m = 1$, so $\widetilde{N} = \mathbb{R}^2 \times \mathbb{R}^2$ and $\mathcal{F}_0$ is spanned by $\partial/\partial x_{(0)} + \partial/\partial x_{(1)}$, where the coordinates in $\widetilde{N}$ are denoted $(x_{(0)}, y_{(0)}, x_{(1)}, y_{(1)})$. We can extend such foliation to a 2-dimensional foliation $\mathcal{F}$ of $\widetilde{N}$ with the required property with respect to the projection $\mathrm{pr}(x_{(0)}, y_{(0)}, x_{(1)}, y_{(1)}) = (x_{(1)}, y_{(1)})$ in different ways. For instance, we can take $\mathcal{F}$ to be given by the level sets of the mapping

$$F(x_{(0)}, y_{(0)}, x_{(1)}, y_{(1)}) = (x_{(0)} - x_{(1)}, f(y_{(0)}, y_{(1)}))$$

with f being an arbitrary function such that $\partial f/\partial y_{(0)} \neq 0$. Then, every solution $(x_{(1)}(t), y_{(1)}(t) = y_{(1)}(0))$ of the system of differential equations

$$\dot{x} = a(t)\,, \qquad \dot{y} = 0\,,$$

gives a new solution $(x_{(0)}(t), y_{(0)}(t) = y_{(0)}(0))$ associated with the level set of (k_1, k_2) by

$$(x_{(0)}(t) = x_{(1)}(t) + k_1, y_{(0)}(t) = y_{(0)}(0))\,.$$

where $y_{(0)}(0)$ is the unique point in $\mathbb{R}$ satisfying

$$f(y_{(0)}(0), y_{(1)}(0)) = k_2 \,.$$

In the case $f(y_{(0)}, y_{(1)}) = y_{(0)} - y_{(1)}$ we recover the 'standard' superposition rule:

$$\Phi(x_{(1)}, y_{(1)}; k_1, k_2) = (x_{(1)} + k_1, y_{(1)} + k_2) \,.$$

Example 3 A very simple example is given by the separable first-order differential equation

$$\dot{x} = a(t)\, f(x) \,,$$

with a and f being arbitrary smooth functions, and where f is assumed to be of a constant sign (otherwise we can restrict ourselves to a neighbourhood of a point in which f does not vanish). In this case $N = \mathbb{R}$ and we can consider the vector field in $\mathbb{R}$

$$X(x) = f(x)\, \frac{\partial}{\partial x} \,.$$

As the function f does not vanish, we have $m = 1$ and the diagonal prolongation

$$\widetilde{X}(x_{(0)}, x_{(1)}) = f(x_{(0)})\, \frac{\partial}{\partial x_{(0)}} + f(x_{(1)})\, \frac{\partial}{\partial x_{(1)}}$$

generates a one-dimensional foliation in $\mathbb{R}^2$ whose leaves are the level sets of a function $\Psi(x_{(0)}, x_{(1)})$ such that

$$f(x_{(0)})\, \frac{\partial \Psi}{\partial x_{(0)}} + f(x_{(1)})\, \frac{\partial \Psi}{\partial x_{(1)}} = 0 \,,$$

which gives rise to the following characteristic system

$$\frac{dx_{(0)}}{f(x_{(0)})} = \frac{dx_{(1)}}{f(x_{(1)})} \,.$$

Therefore, if the function $\phi(y)$ is defined by

$$\phi(y) = \int_0^y \frac{d\zeta}{f(\zeta)} \,,$$

then we find that the leaves are characterized by a constant k in such a way that

$$\phi(x_{(0)}) - \phi(x_{(1)}) = k \,.$$

The function ϕ is a monotone function, because $\phi'(x) = f(x)$ and $f(x)$ has constant sign. Therefore, there exists an inverse function which allows to write the superposition rule as

$$x = \phi^{-1}\left(k + \phi(x_{(1)})\right) \,.$$

For instance, if $f(x) = 1/x^2$, we find that $\phi(x) = -1/x = \phi^{-1}(x)$, an we obtain the following superposition rule.

$$x = \frac{x_{(1)}}{1 - k\, x_{(1)}} \,.$$

Example 4 It has been pointed out in [11] the following example of the original Lie's work:

$$\left\{\begin{array}{rcl}\dfrac{dx}{dt} &=& a_{12}(t)\,y+b_1(t)\\ \dfrac{dy}{dt} &=& -a_{12}(t)\,x+b_2(t)\end{array}\right.$$

In principle, it is a particular example of an inhomogeneous linear system and we expect to have an affine superposition rule involving three different solutions:

$$x=\Phi_1(x_{(1)},x_{(2)},x_{(3)})=x_{(1)}+k_1(x_{(2)}-x_{(1)})+k_2(x_{(3)}-x_{(1)})\,.$$

However, we can obtain a superposition rule which is not linear but involves only two solutions. It is due to the fact that here is not the affine group in two dimensions which is play the relevant rôle, but the Euclidean group. The foliation corresponding to this Lie system is generated by the prolongations of the vector fields

$$X_1=\frac{\partial}{\partial x}\,,\qquad X_2=\frac{\partial}{\partial y}\,,\qquad X_3=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\,.$$

First of all, m is different from 1, because there exist nontrivial coefficients λ_1, λ_2 and λ_3 such that $\lambda_1\,X_1(x_{(1)})+\lambda_2\,X_2(x_{(1)})+\lambda_3\,X_3(x_{(1)})=0$, at a given point $x_{(1)}$, for instance, $\lambda_1=-y_{(1)}$, $\lambda_1=x_{(1)}$, $\lambda_3=1$. However, the only coefficients λ_1, λ_2 and λ_3 such that

$$\lambda_1\,X_1(x_{(1)})+\lambda_2\,X_2(x_{(1)})+\lambda_3\,X_3(x_{(1)})=0\,,\qquad \lambda_1\,X_1(x_{(2)})+\lambda_2\,X_2(x_{(2)})+\lambda_3\,X_3(x_{(2)})=0\,,$$

with $x_{(1)}\neq x_{(2)}$ are $\lambda_1=\lambda_2=\lambda_3=0$ and therefore $m=2$.

The function Ψ defining the superposition rule satisfies $\widetilde{X}_1\Psi=\widetilde{X}_2\Psi=\widetilde{X}_3\Psi=0$ with

$$\widetilde{X}_1=\frac{\partial}{\partial x}+\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\,,\qquad \widetilde{X}_2=\frac{\partial}{\partial y}+\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\,,$$

and

$$\widetilde{X}_3=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+y_1\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial y2}\,.$$

The two first conditions imply that Ψ must be of the form

$$\Psi(x_0,y_0,x_1,y_1,x_2,y_2)=\psi(x_0-x_1,x_0-x_2,y_0-y_1,y_0-y_2)\,,$$

what suggests the change of variables

$$u_1=x_0-x_1,\quad u_2=x_0-x_2,\quad u_3=x_0,\quad v_1=y_0-y_1,\quad v_2=y_0-y_2,\quad v_3=y_0\,,$$

and then the third condition $\widetilde{X}_3\Psi=0$ is written

$$v_1\frac{\partial}{\partial u_1}+v_2\frac{\partial}{\partial u_2}-u_1\frac{\partial}{\partial v_1}-u_2\frac{\partial}{\partial v_2}\,,$$

for which the characteristic system is

$$\frac{du_1}{v_1}=\frac{du_2}{v_2}=\frac{dv_1}{-u_1}=\frac{dv_2}{-u_2}$$

from where we find the first integrals

$$u_1^2+v_1^2=(x_0-x_1)^2+(y_0-y_1)^2=C_1\,,\qquad u_2^2+v_2^2=(x_0-x_2)^2+(y_0-y_2)^2=C_2$$

which determine the superposition foliation and provide us with a superposition rule for the given system involving only two particular solutions (i.e. with $m=2$).

6 Lie systems in Lie groups and homogeneous spaces

Let us consider now the particular case $m = 1$, i.e. when a single solution is enough to obtain any other solution. Let us assume additionally that $\mathcal{F} = \mathcal{F}_0$, i.e. that the superposition rule is unique as a foliation. This means that $r = n$ and the vector fields $X_1, \ldots, X_n$ generically span ßTN. Assume for simplicity that they span ßTN globally and are complete vector fields. Since these vector fields close on an n-dimensional Lie algebra $\mathfrak{g}$, we conclude that there is a transitive action on N of the simple-connected n-dimensional Lie group G associated with $\mathfrak{g}$, so that $N = G/H$ with a discrete subgroup H of G and the foliation $\mathcal{F} = \mathcal{F}_0$ is generated by the fundamental vector fields of the G-action. If H is trivial and we consider the standard action of G on itself by left translations L_g, the vector fields X_i are just right-invariant vector fields on G. As a superposition function corresponding to $\mathcal{F} = \mathcal{F}_0$ we can choose the group multiplication $\Phi : G \times G \to G$, $\Phi(g_{(1)}, k) = g_{(1)}k$. In this case $\Psi(g_{(0)}, g_{(1)}) = g_{(1)}^{-1} g_{(0)}$ is left invariant $\Psi(g' g_{(0)}, g' g_{(1)}) = \Psi(g_{(0)}, g_{(1)})$.

Conversely, given a Lie system defined by a t-dependent vector field of the form

$$Y(t, g) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X^{\mathsf{R}}_\alpha(g)\,,$$

where X^{R}_α is a basis of right-invariant vector fields, then the projectability condition is satisfied and there is a uniquely defined superposition rule. Note however that if the vector fields X^{R}_α generate a smaller Lie subalgebra, the superposition rule is not unique.

Let $\{a_1, \ldots, a_n\}$ be a basis in T_eG. This linear space can be identified with the Lie algebra $\mathfrak{g}$ of G, the set of left invariant vector fields on G: for each $a \in T_eG$ let X^{L}_a denote the corresponding left-invariant vector field in G, given by $X^{\mathsf{L}}_a(g) = L_{g*e}a$. Similarly, X^{R}_a denotes the right-invariant vector field in G given by $X^{\mathsf{R}}_a(g) = R_{g*e}a$. A curve in T_eG,

$$a(t) = \sum_{\alpha=1}^{n} b_\alpha(t)\, a_\alpha\ ,$$

gives rise to a t-dependent vector field on G

$$X^{\mathsf{R}}(t, g) = X^{\mathsf{R}}_{a(t)}(g) = \sum_{\alpha=1}^{n} b_\alpha(t)\, X^{\mathsf{R}}_\alpha(g)\ , \tag{16}$$

where in the right hand side X^{R}_α is a shorthand notation for $X^{\mathsf{R}}_{a_\alpha}$. The associated system of differential equations determining the integral curves of such a t-dependent vector field reads

$$\dot{g}(t) = \sum_{\alpha=1}^{n} b_\alpha(t)\, X^{\mathsf{R}}_\alpha(g(t))\ , \tag{17}$$

and applying $R_{g^{-1}(x)*g(x)}$ to both sides of (17) we find the equation

$$R_{g^{-1}(t)*g(t)}\dot{g}(t) = \sum_{\alpha=1}^{n} b_\alpha(t) a_\alpha = a(t)\ , \tag{18}$$

that with some abuse of notation we will write

$$(\dot{g}\, g^{-1})(t) = a(t)\ . \tag{19}$$

The solution of this equation starting from the neutral element can be solved by making use of a generalization of the method developed by Wei and Norman [18, 19] (see e.g. [5] and [3]) for solving an analogous linear problem. A simpler situation is when $a(t)$ takes values not in the full Lie algebra $\mathfrak{g} = T_eG$ but in a subalgebra.

Now, let H be an arbitrary closed subgroup of G and consider the homogeneous space $N = G/H$. Then, G can be seen as a principal bundle $\tau : G \to G/H$. Moreover, it is also known that the right-invariant vector fields X^{R}_α are τ-projectable and the τ-related vector fields in N are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on N, $\tau_{*g}X^{\mathsf{R}}_\alpha(g) = -X_\alpha(gH)$. In this way we can associate with the Lie system on the group X given by (16) a Lie system on N:

$$\bar{X}(t,x) = -\sum_{\alpha=1}^{n} b_\alpha(t)\, X_\alpha(x)\ . \tag{20}$$

Therefore, the integral curve of (20) starting from x_0 are given by $x(t) = \Phi(g(t), x_0)$, with $g(t)$ being the solution of (18) with $g(0) = e$.

Let us note that even if the original t-dependent vector field on N is projectable to N_0 by means of a submersion $\pi : N \to N_0$, the superposition rule cannot be projected in general and the number m of solutions appearing in the superposition rule changes. For instance, in the case of Riccati equation a fundamental set is made of three solutions, while for the linear realization of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ only two solutions are needed, and for $SL(2,\mathbb{R})$ acting on itself only one solution is sufficient.

In the particular case of a Lie system $\dot{g}\, g^{-1} = a$ in a Lie group G, it was recently shown [4] that the knowledge of a particular solution of the corresponding system in a homogeneous space for G reduces the problem to one on the isotopy group of a point in the homogeneous space. So, if $x(t)$ is the particular solution of the associated Lie system in a homogeneous space starting from x_0, then we can choose a curve $\bar{g}(t)$ such that $\Phi(\bar{g}(t), x_0) = x(t)$ and there should exist a curve $h(t) \in G_{x_0}$ such that $g(t) = \bar{g}(t)\, h(t)$. Such curve $h(t)$ is a solution of the Lie equation in G_{x_0}, $\dot{h}\, h^{-1} = \mathrm{Ad}\, \bar{g}^{-1}(a + \dot{\bar{g}}\bar{g}^{-1})$. Therefore, finding a solution of such equation in the subgroup G_{x_0}, we can recover the solution $g(t)$ of the Lie system in G as $g(t) = \bar{g}(t)\, h(t)$. Using a new solution starting from a new point, the problem is further reduced and therefore with a number of known solutions we can directly write the general solution.

Another relevant case is when there exists an equivariant map $F : N_1 \to N_2$ between two homogeneous spaces of a Lie group G. In this case the corresponding fundamental fields are F-related and then the image under F of an integral curve of a Lie system in M_1 is an integral curve of the corresponding system in M_2. A very simple example is the following: the function $F : \mathbb{R}^2 - \{(0,0)\} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ given by

$$F(x_1, x_2) = \begin{cases} \dfrac{x_1}{x_2} & \text{if} \quad x_2 \neq 0 \\ \infty & \text{if} \quad x_2 = 0 \end{cases}$$

is equivariant with respect to the linear action of the Lie group $SL(2,\mathbb{R})$ on $\mathbb{R}^2-\{(0,0)\}$, and its action on the completed real line $\overline{\mathbb{R}}$, given by

$$\begin{aligned}\Phi(A,x) &= \frac{\alpha\,x+\beta}{\gamma\,x+\delta}, \quad \text{if } x\neq -\frac{\delta}{\gamma},\\ \Phi(A,\infty) &= \frac{\alpha}{\gamma}\,, \qquad \Phi(A,-\delta/\gamma)=\infty,\end{aligned}$$

when A is the matrix given by

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\mathbb{R})\ . \tag{21}$$

Consequently, given an integral curve of the system of differential equations

$$\left\{\begin{aligned}\dot x_1 &= \frac{1}{2}\,b_2\,x_1+b_1\,x_2\,,\\ \dot x_2 &= -b_3\,x_1-\frac{1}{2}b_2\,x_2\,,\end{aligned}\right.$$

the curve $x(t)=x_1(t)/x_2(t)$ is an integral curve of the Riccati equation

$$\dot x = b_1+b_2\,x+b_3\,x^2\ .$$

This is precisely the method by which Riccati arrived to this last equation.

7 Partial superposition rules

Finding new solutions from know ones is a very usual method in the theory of both ordinary and partial differential equations and this procedure has many applications in physics. In order to deal in a geometric way with this problem we introduce next a concept generalizing those of nonlinear superposition introduced so far and the one in [12] as well as that of connecting function used in [13] (see also [14] for some examples).

A *partial superposition rule* of rank s of m solutions for the system of ordinary differential equations (1) is given by a function $\Phi:\mathbb{R}^{n\,m+s}\to\mathbb{R}^n$,

$$x=\Phi(x_{(1)},\ldots,x_{(m)};k_1,\ldots,k_s)\ , \tag{22}$$

such that if $\{x_{(a)}(t)\mid a=1,\ldots,m\}$ is a set of m particular solutions of the system (1), then, at least for sufficiently small t,

$$x(t)=\Phi(x_{(1)}(t),\ldots,x_{(m)}(t);k_1,\ldots,k_s)\ , \tag{23}$$

is also a solution of the system (1), where $k=(k_1,\ldots,k_s)$ is a set of s arbitrary constants.

Note that this new concept reduces to the previously considered superposition rule for $s=n$ and coincides with a s-parameter family of connecting functions in the sense of [13].

Given such a partial superposition function, there is a non-uniquely defined function $\Psi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ such that

$$\Psi^i(x_{(0)}, x_{(1)}, \ldots, x_{(m)}) = \begin{cases} k_i & \text{if} \quad i \leq s \\ 0 & \text{if} \quad i > s \end{cases}$$

The last $(n-s)$ equations are restrictions defining a submanifold M of codimension $(n-s)$ of $\mathbb{R}^{n(m+1)}$, i.e. of dimension $m\,n+s$, and the other equations define a foliation of codimension s in M. Now, following the same procedure as in Section 2 we will arrive to a distribution in $\mathbb{R}^{n(m+1)}$ spanned by the t-dependent vector fields Y_a defined by (10) which provide us with a distribution in M, because they are tangent to M, whose integral leaves are $n\,m$-dimensional and each leaf is fixed by the choice of s constants $k_1, \ldots, k_s$.

Moreover, the restriction $\mathrm{pr}_{|M}$ of pr on the submanifold M defines a subbundle of $\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n\,m}$ and establishes diffeomorphisms among the different leaves and allows us to identify among them the leaves of the foliation defined by the prolongation of the given non-autonomous system.

Conversely, if M is a submanifold of $\mathbb{R}^{n(m+1)}$ of codimension $(n-s)$ which defines a subbundle of $\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n\,m}$ such that the distribution defined in $\mathbb{R}^{n(m+1)}$ by the prolongation of vector fields is also a distribution in M, i.e. such vectors are tangent to M, and the restriction $\mathrm{pr}_{|M}$ provides us with diffeomorphisms among the different leaves allowing us to identify among them the leaves of the foliation defined by the prolongation of the given non-autonomous system. Such diffeomorphisms can be used to define a superposition rule of m solutions involving s constants,

If for instance we consider the linear system

$$\frac{dx^1}{dt} = a_{11}(t)\,x^1 + a_{12}(t)\,x^2 \tag{24}$$

$$\frac{dx^2}{dt} = a_{21}(t)\,x^1 + a_{22}(t)\,x^2 \tag{25}$$

then it admits a superposition function of rank one and involving one particular solution, $F(x_{(1)}; k) = k\; x_{(1)}$, which determines the three-dimensional subbundle M of $\mathrm{pr} : \mathbb{R}^4 \to \mathbb{R}^2$ defined by the restriction to the subset given by the relation

$$x^1\; x^2_{(1)} - x^2\; x^1_{(1)} = 0\ ,$$

which is endowed with a foliation: each leaf is characterized by a real number k and is defined on the set of points $(x^1, x^2, x^1_{(1)}, x^2_{(1)})$ such that $x^1\; x^2_{(1)} - x^2\; x^1_{(1)} = 0$.

However we have also a superposition function of rank one but involving two constants:

$$F(x_{(1)}, x_{(2)}; k) = x_{(1)} + k\; x_{(2)}\ ,$$

The subbundle now will be defined by

$$x^1_{(2)}(x^2 - x^2_{(1)}) - x^2_{(2)}(x^1 - x^1_{(1)}) = 0\ .$$

8 Superposition rules for PDE's

Consider now the system of first-order PDE's of the form:

$$\frac{\partial x^i}{\partial t^a} = Y^i_a(t, x)\,, \qquad x \in \mathbb{R}^n,\ t = (t^1, \ldots, t^s) \in \mathbb{R}^s\,, \tag{26}$$

whose solutions are maps $x(t) : \mathbb{R}^s \to \mathbb{R}^n$.

A particular case of (26) when $s = 1$ is (1). The main difference of (26) with respect to (1) is that for $s > 1$ we have no, in general, existence of a solution with a given initial value $x(0) \in \mathbb{R}^n$. For a better understanding of this problem, let us put (26) in a more general and geometric framework.

For a manifold N of dimension n consider the trivial fibre bundle

$$P^s_N = \mathbb{R}^s \times N \to \mathbb{R}^s\,.$$

A connection $\bar{Y}$ in this bundle is a horizontal distribution in TP^s_N. i.e. an s-dimensional distribution transversal to the fibres. It is determined by horizontal lifts of the coordinate vector fields $\partial/\partial t^a$ in $\mathbb{R}^s$ which read

$$\bar{Y}_a = \frac{\partial}{\partial t^a} + Y_a(t, x)$$

with

$$Y_a(t, x) = Y^i_a(t, x)\frac{\partial}{\partial x^i}\,.$$

Thus, the solutions of (26) can be identified with integral submanifolds of the distribution $\bar{Y}$,

$$(t, Y(t))\,, \qquad t \in \mathbb{R}^s\,.$$

It is now clear that there is an (obviously unique) solution of (26) for every initial data if and only if the distribution $\bar{Y}$ is integrable, i.e. the connection has a trivial curvature. This means that

$$[\bar{Y}_a, \bar{Y}_b] = \sum_{c=1}^{r} f_{ab}{}^c\, \bar{Y}_c$$

for some functions $f_{ab}{}^c$ in P^s_N. But the commutators $[\bar{Y}_a, \bar{Y}_b]$ are clearly vertical while $\bar{Y}_c$ are linearly independent horizontal vector fields, so $f_{ab}{}^c = 0$ which yields the integrability condition in the form of the system of equations $[\bar{Y}_a, \bar{Y}_b] = 0$, i.e., in local coordinates,

$$\frac{\partial Y^i_b}{\partial t^a}(t, x) - \frac{\partial Y^i_a}{\partial t^b}(t, x) + \sum_{j=1}^{n}\left(Y^j_a(t, x)\frac{\partial Y^i_b}{\partial x^j}(t, x) - Y^j_b(t, x)\frac{\partial Y^i_a}{\partial x^j}(t, x)\right) = 0\,. \tag{27}$$

Let us assume now that we work with a system of first-order PDE's of the form (26) and satisfying the integrability conditions (27). Then we are sure that, for a given initial value, there is a unique solution of (26). Now, we can think about superposition rules for such solutions. It is, however, completely obvious that the concepts of superposition rules we have developed can be applied with no real changes to our case of PDE's. In the formula (2) we should now think that t is not a real parameter but $t \in \mathbb{R}^s$. The only difference when passing to the foliation induced by the superposition function Ψ is that we differentiate (6) not with respect to the simple parameter t but with respect to all t^a. Therefore, the proposition 1 takes the form:

Proposition 2 *Giving a superposition rule for the system (26) satisfying the integrability condition (27) is equivalent with giving a connection in the bundle* $\mathrm{pr} : N^{(m+1)} \to N^m$ *with a zero curvature and for which the diagonal prolongations* $\widetilde{Y}_a(t)$ *of all the vector fields* $Y_a(t)$, $t \in \mathbb{R}^s$, $a = 1, \ldots, s$, *are horizontal.*

Also the proof of Lie theorem remains unchanged. Therefore we get the following analog of the Lie theorem for PDE's:

Theorem 2 *The system (26) of PDE's defined on a manifold N and satisfying the integrability condition (27) admits a superposition rule if and only if the vector fields $Y_a(t,x)$ on N depending on the parameter $t \in \mathbb{R}^s$, can be written locally in the form*

$$Y_a(t,x) = \sum_{\alpha=1}^{r} u_a^\alpha(t) X_\alpha(x)\,, \qquad a = 1, \ldots s\,, \tag{28}$$

where the vector fields X_α close on a finite-dimensional real Lie algebra, i.e. there exist r^3 real constants ${c_{\alpha\beta}}^\gamma$ such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} {c_{\alpha\beta}}^\gamma X_\gamma\,.$$

Let us observe that the integrability condition for $Y_a(t,x)$ of the form (28) can be written as

$$\sum_{\alpha,\beta,\gamma=1}^{r} \left[(u_b^\gamma)'(t) - (u_a^\gamma)'(t) + u_a^\alpha(t) u_b^\beta(t) c_{\alpha\beta}^\gamma\right] X_\gamma = 0.$$

Example 5 Consider the following system of partial differential equations on $\mathbb{R}^2$ associated with the $sl(2,\mathbb{R})$-action on $\mathbb{R}$ represented by vector fields (15):

$$\begin{aligned} u_x &= a(x,y)u^2 + b(x,y)u + c(x,y)\,, \\ u_y &= d(x,y)u^2 + e(x,y)u + f(x,y)\,. \end{aligned}$$

This equation can be written in the form of a 'total differential equation'

$$(a(x,y)u^2 + b(x,y)u + c(x,y))\mathrm{d}x + (d(x,y)u^2 + e(x,y)u + f(x,y))\mathrm{d}y = \mathrm{d}u\,.$$

The integrability condition just says that the one-form

$$\omega = (a(x,y)u^2 + b(x,y)u + c(x,y))\mathrm{d}x + (d(x,y)u^2 + e(x,y)u + f(x,y))\mathrm{d}y$$

is closed for arbitrary function $u = u(x,y)$. If this is the case, then there is a unique solution with the initial condition $u(x_0, y_0) = u_0$ and there is a superposition rule giving a general solution as a function of three independent solutions exactly as in the case of Riccati equation:

$$\frac{(u-u_{(1)})(u_{(2)}-u_{(3)})}{(u-u_{(2)})(u_{(1)}-u_{(3)})}=k\,,$$

or

$$u=\frac{(u_{(1)}-u_{(3)})u_{(2)}k+u_{(1)}(u_{(3)}-u_{(2)})}{(u_{(1)}-u_{(3)})k+(u_{(3)}-u_{(2)})}\,.$$

9 Concluding remarks

In this paper we have identified and solved a gap present in previous proofs of the necessary and sufficient conditions for the existence of a superposition rule for ordinary differential equations. In doing this we provided the superposition rule with a much better geometrical interpretation which allows us to exhibit many interesting properties. For instance, it is now clear by inspection that the vector field as given in the Theorem 1 may be multiplied by a function of time, i.e. we may perform a re-parametrization in time, and still get an equation admitting a superposition principle. In this way we find that if the superposition rule holds true for a vector field field, it is also true for all re-parametrized vector fields. In particular, it would allow us to write a superposition rule also for vector fields which may be reduced to autonomous ones via re-parametrization. The new geometrical interpretation also paves the way to a proper treatment of the superposition rule for partial differential equations. We hope to be able to extend the treatment to field theory and perhaps be able to get interacting field theories out of free ones very much as it happens for ordinary differential equations. Indeed it is known that Riccati equation obtains from a linear system. In previous papers it has been shown how to cast completely integrable systems in a generalized version of Lie-Scheffers systems, as most completely integrable systems do arise as reduction of simple systems we hope to be able to show in general that systems which allow for a superposition rule may be derived from 'simple ones', both for ordinary and partial differential equations.

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