GENERALIZED SPACES AND THE INTEGRATION OF CERTAIN CLASSES OF DIFFERENTIAL EQUATIONS.

ELIE CARTAN

Abstract. English Translation of Elie Cartan's *Les espaces généralises et l'intégration de certaines classes d'équations différentielles.*

The problem of the geometrization of differential systems leads to certain remarkable classes of ordinary differential equations, that we call classes (C).

Definition 1. *A given class of differential equations of order n*

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right)$$

will be said to be a class (C) if there exists an infinite group (in the sense of Lie) G transforming equations of the class into equations of the class and such that the differential invariants with respect to G of an equation of the class be the first integrals of the equation.[1]

The study of differential invariants not requiring any integration, one sees that in general one can obtain n independent first integrals of a given equation of the class, except for the case in which this equation admits a continuous subgroup of G; in this case the integration of the equation requires the integration of a differential system of Lie the nature of which depends on the structure of the subgroup. In the examples which we give, the differential invariants are rational functions of the function F and its derivatives up to a certain order.[2]

I. K. Wünschmann,[3] studying the case in which the condition of contact of two infinitely near integral curves of a third order differential equation

$$y''' = F(x, y, y', y'') \tag{0.1}$$

is furnished by an equation of Monge of the second degree

$$\Phi(a^1, a^2, a^3, da^1, da^2, da^3) = 0 \tag{0.2}$$

[1]One can evidently also consider systems of differential equations. It can be shown that the systems of differential equations that have the characteristics of a system in involution of two equations in the partial derivatives of the second order of a function z of two variables x, y form a class (C) invariant for the group of contact transformations of the space (x, y, z) (see E. CARTAN, *Annales École Normale*, 27, 1920, p. 109-192). This is the first example of a differential system to which the method of integration of the present text has been applied.

[2]All of the functions that we consider will be analytic, although in reality it suffices for the reader to suppose that the derivatives exist up to a certain order.

[3]K. WÜNSCHMANN, *Ueber Berührungsbestimmungen bei Integralkurven von Differentialgleichungen (Inaug. Dissert.*, Leipzig, 1905, p.6-13).

between the parameters a^i and $a^i + da^i$ of two curves, has obtained for the function F the condition

$$\frac{d^2F_{y''}}{dx^2} - 2F_{y''}\frac{dF_{y''}}{dx} - 3\frac{dF_{y'}}{dx} + \frac{4}{9}(F_{y''})^3 + 3F_{y'}F_{y''} + 6F_y = 0. \tag{0.3}$$
$$\left(\frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + F\frac{\partial}{\partial y''}\right).$$

The equation (0.2) can then be written, *without knowing the integral curves of* (0.1),

$$\begin{aligned}(dy' - y''dx)^2 - 2(dy - y'dx)(dy'' - Fdx) + \tfrac{2}{3}F_{y''}(dy - y'dx)(dy' - y''dx)\\ + \left[-\frac{1}{3}\frac{dF_{y''}}{dx} + \frac{2}{9}(F_{y''})^2 + F_y\right](dy - y'dx)^2 = 0.\end{aligned} \tag{0.4}$$

The preceeding equations form a class (C). In effect each contact transformation of the plane (x, y) changes evidently each differential equation of the class into another equation of the same class; transforming the integral curves of the first equation into those of the second, it likewise transforms the equation of Monge (0.2) associated to the first equation of the class into that equation of Monge associated to the second of the class by means of a point transformation of the space (a^1, a^2, a^3). One can associate, as I showed,[4] to a quadratic equation of Monge a space $\mathcal{E}$ with normal conformal connection: the spaces $\mathcal{E}$ associated to two equations of the class considered equivalent with respect to G have therefore the same geometric structure. One consequence of the results obtained by Mr. Shiing-shen Chern in a recent Note ([5]) is that reciprocally two spaces $\mathcal{E}$ with the same conformal connection correspond to two differential equations equivalent with respect to G. The differential invariants of the equations of the class considered with respect to G are therefore identical to the differential invariants of the spaces $\mathcal{E}$ with respect to a group of point transformations; as these are rational entire functions of the components of the curvature tensor of $\mathcal{E}$ and its successive derivatives, they are functions of a^1, a^2, a^3 and it follows that they are first integrals of the differential equation associated to $\mathcal{E}$. The class of differential equations (0.1) satisfying the condition (0.3) is therefore a class (C).

One can show that to each quadratic Monge equation there corresponds a differential equation (0.1) (defined up to a contact transformation). One can obtain this in the following manner: to the equation (0.2) one can associate on the space (a^1, a^2, a^3) a first order partial differential equation for which the cone attached to each point of the space is of the second order. One determines a complete first integral of this equation with parameters x, y; the sought after differential equation expresses that the complete integral surface passes through a fixed point. In the corresponding space $\mathcal{E}$ the characteristics of the partial differential equation are none other than the curves which generalize the isotropic lines.

It is necessary to comment on how to obtain the components of the curvature of the space with conformal connection defined by the equation (0.4). This problem is analogous to the following, a bit simpler: *Knowing that a quadratic differential form* $\Phi \equiv g_{ij}dx^idx^j$ *in three variables* x^1, x^2, x^3 *can be expressed as a quadratic differential form in two variables* u^1 *and* u^2*, where* u^1 *and* u^2 *are conveniently chosen functions of* x^1, x^2, x^3 *, determine the Riemannian curvature of this second form.*

[4]E. CARTAN. *Ann. Soc. Polon. Math.*, 2, 1923, p. 171-221.

[5]*Comptes rendus*, 204, 1937, p. 1227.

The method of moving frames gives as follows the solution to this problem. One decomposes the form Φ, which is always possible, into a sum of two squares, $\omega_1^2 + \omega_2^2$, in which ω_1 and ω_2 are differential forms linear in dx^1, dx^2, dx^3. There exists a form $\omega_{12} = \omega_{21}$ satisfying

$$\omega_1' = [\omega_2\omega_{21}], \qquad \omega_2' = [\omega_1\omega_{12}];$$

this form is completely determined and its exterior derivation gives

$$\omega_{12}' = -K[\omega_1\omega_2],$$

in which K, a function of x^1, x^2, x^3, is the sought after Riemannian curvature. One knows in advance that K is a function of u^1, u^2, that is to say a first integral of the differential system

$$g_{ik}dx^k = 0 \qquad (i = 1, 2, 3),$$

of which u^1 and u^2 are first integrals. One calculates in the same way the differential parameters of K.

II.. The differential equations of the *second* order lead us to a second class (C). We consider first a differential equation

$$\frac{d^2v}{du^2} = \Phi(u, v, \frac{dv}{du}), \tag{0.5}$$

Φ being an entire polynomial of the third degree in $\frac{dv}{du}$. One knows that to this equation one can associate a space of two dimensions (u, v) having a normal projective connection.[6] We consider now the equation dual to (0.5). This is obtained by considering the general equation

$$\Psi(u, v, x, y) = 0 \tag{0.6}$$

of the integral curves, where x and y are the constants of integration, x and y being regarded as the variables of the equation, and u and v as its parameters; y regarded as a function of x satisfies then a second differential equation of the second order

$$y'' = F(x, y, y'), \tag{0.7}$$

said to be dual to the first.[7] It is apparent that this is equation is only defined up to a point transformation and that two equations (0.5) equivalent with respect to a group of point transformations of the plan (u, v) yield dual equations equivalent with respect to the group G of point transformations of the plane (x, y).

The class of equations (0.7) dual to the equations (0.5) is characterized by the relation

$$\frac{d^2F_{y''}}{dx^2} - 4\frac{dF_{yy'}}{dx} - F_{y'}\frac{dF_{y''}}{dx} + 4F_{y'}F_{yy'} - 3F_yF_{y''} + 6F_{y''} = 0. \tag{0.8}$$

To each equation of this class one can associate a space $\mathcal{E}$ of two dimensions (u, v) with a normal projective connection. The differential invariants of the equation (0.7) with respect to G are the differential invariants of $\mathcal{E}$ with respect to the group of coordinate changes;

---

[6]See for example E. CARTAN, *Leçons sur la théorie des espaces à connexion projective*, Paris, 1937, Chap. V, p. 242-257.

[7]See A. KOPPISCH, *Zur Invariantentheorie der gewöhnliche Differentialgleichungen zweiter Ordnung (Inaug. Dissert.*, Leipzig, 1905).

these are again first integrals of the equation (0.7). The condition (0.8) therefore defines a class (C). One can show that with the exception of the equations reducible to $y'' = 0$, all of the equations of this class can be integrated by differentiations and at most two quadratures.

Translator's Comments

This is a translation of [4]. Cartan's notation has been preserved.

References to the articles to which Cartan makes references are included in the bibliography below, as is the reference to the article itself. Note that Cartan gives incorrectly (as 1920 instead of 1910) the year of his own article which he cited in the first footnote. This error has been reproduced in the translation.

References

1. Elie Cartan, *Les systèmes de Pfaff à cinq variables et les équations dérivées partielles du second ordre*, Annales Scientifiques de l'Ecole Normale Supérieure **27** (1910), 109–192.
2. ______, *Les espaces à connexion conforme*, Annales de la Société Polonaise des Mathématiques **2** (1923), 171–221.
3. ______, *Lecons sur la thorie des espaces connexion projective. Redigees par P. Vincensini.*, (Cahiers scient. 17) Paris: Gauthier-Villars. VI, 308 p., 34 fig. , 1937.
4. ______, *Les espaces généralises et l'intégration de certaines classes d'équations différentielles.*, C. R. Acad. Sci., Paris **206** (1938), 1689–1693.
5. Shiing-shen Chern, *Sur la géométrie d'une équation différentielle du troisieme ordre*, C. R. Acad. Sci., Paris **204** (1937), 1227–1229.
6. Alfred Koppisch, *Zur Invariantentheorie der gewöhnlichen Differentialgleichung zweiter Ordnung*, Inaugural dissertation, Greifswald, 1905, Teubner, Leipzig.
7. Karl Wünschmann, *Über Berührungsbedingungen bei Integralkurven von Differentialgleichungen*, Inaugural dissertation, Greifswald, 1905, Teubner, Leipzig.

Translated by DANIEL J. F. FOX

Departamento de Matemáticas, Consejo Superior de Investigaciones Científicas, C/ Serrano 113bis, 28006 Madrid España

E-mail address: fox@imaff.cfmac.csic.es