Functional Integration: Action and Symmetries

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Dissertations often begin with a challenging statement. When completed,

† An expression of L. Rosenfeld.

a good dissertation is a wonderful document, understandable, carefully crafted, well referenced, presenting new results in a broad context.

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Symbols

$A := B$	A is defined by B
$A \overset{\int}{=} B$	both sides are equal only after they are integrated
$B \geqslant A$	B is inside the light cone of A
$d^\times l = dl/l$	multiplicative differential
$\partial^\times/\partial l = l\partial/\partial l$	multiplicative derivative

$\mathbb{R}^D, \mathbb{R}_D$	are dual of each other; $\mathbb{R}^D$ is a space of contra-variant vectors, $\mathbb{R}_D$ is a space of covariant vectors
$\mathbb{R}^{D\times D}$	space of D by D matrices
$\mathbb{X}, \mathbb{X}'$	$\mathbb{X}'$ is dual to $\mathbb{X}$
$\langle x', x \rangle$	dual product of $x \in \mathbb{X}$ and $x' \in \mathbb{X}'$
$(x\|y)$	scalar product of $x, y \in \mathbb{X}$, assuming a metric
$\left(\mathbb{M}^D, g\right)$	D-dimensional riemannian space with metric g
$T\mathbb{M}$	tangent bundle over $\mathbb{M}$
$T^*\mathbb{M}$	contangent bundle over $\mathbb{M}$
$\mathcal{L}_X$	Lie derivative in the X-direction

$U^{2D}(S)$	Space of critical points of the action functional S (Ch 4)
$\mathcal{P}_{\mu,\nu}\left(\mathbb{M}^D\right)$	Space of paths with values in $\mathbb{M}^D$, satisfying μ initial conditions and ν final conditions
$U_{\mu,\nu} := U^{2D}(S) \cap \mathcal{P}_{\mu,\nu}\left(\mathbb{M}^D\right)\}$	arena for WKB

$\hbar = h/2\pi$	Planck's constant
$[h] = ML^2T^{-1}$	physical (engineering) dimension of h
$\omega = 2\pi\nu$	ν frequency, ω pulsation
$t_B = -i\hbar\beta = -i\hbar k_B T$	(1.70)
$\tau = it$	(1.100)

Superanalysis (Ch 9)	
$\tilde{A}$	parity of $A \in \{0, 1\}$
$AB = (-1)^{\tilde{A}\tilde{B}}BA$	graded commutator or $[A, B]$ (9.5)
$A \wedge B = -(-1)^{\tilde{A}\tilde{B}}B \wedge A$	graded anticommutator $\{A, B\}$ (9.6)
$\xi^\mu\xi^\alpha = -\xi^\alpha\xi^\mu$	Grassmann generators (9.11)
$z = u + v$	supernumber, u even $\in \mathbb{C}_c$, v odd $\in \mathbb{C}_a$ (9.12)
$\mathbb{R}_c \subset \mathcal{C}_c$	real elements of $\mathcal{C}_c$ (9.16)

$\mathbb{R}_a \subset \mathbb{C}_a$	real elements of $\mathcal{C}_a$ (9.16)
$z = z_B + z_S$	supernumber z_B body, z_S soul (9.12)
$x^A = (x^a, \xi^\alpha) \in \mathbb{R}^{n\|\nu}$	superpoints

Conventions

Fourier transforms

$$(\mathcal{F}f)\left(x'\right) := \int_{\mathbb{R}^D} d^D x \;\exp\left(-2\pi i\langle x, x'\rangle\right) f(x) \qquad\qquad x \in \mathbb{R}^D, x' \in \mathbb{R}_D$$

For Grassmann variables

$$(\mathcal{F}f)(\kappa) := \int d\xi \;\exp\left(-2\pi i\kappa\xi\right) f(\xi)$$

In both cases

$$\begin{aligned} \langle\delta, f\rangle &= f(0) \quad \text{i.e.} \quad \delta(\xi) = c^{-1}\xi \\ \langle\mathcal{F}\delta, f\rangle &= f \qquad\;\; \text{i.e.} \quad c^2 = (2\pi i)^{-1} \end{aligned}$$

$$\int d\xi\;\xi = c\,, \qquad \text{here} \quad c^2 = (2\pi i)^{-1}$$

Formulary

(giving a context to symbols)

- Wiener integral

$$\mathbb{E}\left[\exp\left(-\int_{\tau_a}^{\tau_b} d\tau\ V(q(\tau))\right)\right] \qquad (1.1)$$

- Peierls bracket

$$(A, B) := D_A^- B - (-1)^{\tilde{A}\tilde{B}} D_A^- B \qquad (1.9)$$

- Schwinger variational principle

$$\delta\langle A|B\rangle = i\langle A|\delta S/\hbar|B\rangle \qquad (1.11)$$

- Quantum partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right) \qquad (1.71)$$

- Schrödinger equation

$$\left\{\begin{array}{lcl} i\partial_t\psi(x,t) & = & \left(-\frac{1}{2}\,\mu^2\delta_x + \hbar^{-1}V(x)\right)\psi(x,t) \\ \psi\left(x, t_a\right) & = & \phi(x) \end{array}\right. \qquad (1.77)$$

$$\mu^2 = \hbar/m$$

- Gaussian integral (2.29), (2.30)

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\ \exp\left(-2\pi i\langle x', x\rangle\right) := \exp\left(-s\pi W\left(x'\right)\right)$$

$$d\Gamma_{s,Q}x \stackrel{\int}{=} \mathcal{D}_{s,Q}x\ \exp\left(-\frac{\pi}{s}Q(x)\right)$$

$$Q(x) = \langle Dx, x\rangle\,, \qquad W\left(x'\right) = \langle x', Gx'\rangle \qquad (2.28)$$

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\ \langle x'_1, x\rangle \ldots \langle x'_{2n}, x\rangle = \left(\frac{s}{2\pi}\right)^n \sum{}' W\left(x'_{i_1}, x'_{i_2}\right)\ldots W\left(x'_{i_{2n-1}}, x'_{i_{2n}}\right)$$

sum without repetition

- linear maps

$$\langle\tilde{L}y', x\rangle = \langle y', Lx\rangle \qquad (2.58)$$

$$W_{\mathbb{Y}'} = W_{\mathbb{X}'} \circ \tilde{L}\,, \qquad Q_{\mathbb{X}} = Q_{\mathbb{Y}} \circ L \qquad \text{(Ch 3)}$$

- Scaling and coarse graining (section 2.5)

$$S_l u(x) = l^{[u]} u\left(\frac{u}{l}\right)$$
$$S_L[a,b[= \left[\frac{a}{l}, \frac{b}{l}\right[$$
$$P_l : S_{l/l_0} \cdot \mu_{[l_\infty, l[}* \qquad (2.83)$$

- Jacobi operator

$$S''(q) \cdot \xi\xi = \langle \mathcal{J}(q) \cdot \xi, \xi \rangle \qquad (5.7)$$

-

$$\langle b|\hat{O}|a\rangle = \int_{\mathcal{P}_{ab}} O(\gamma) \exp\left(iS(\gamma)/\hbar\right) \mu(\gamma) \mathcal{D}\gamma \qquad \text{(Ch 6)}$$

- Time ordered exponential

$$T \exp \int_{t_0}^{t} ds\ A(s) \qquad (6.38)$$

- Dynamical vector fields

$$dx(t,z) = X_{(A)}\left(x(t,z)\right) dz^A(t) + Y\left(x(t,z)\right) dt \qquad (7.14)$$

$$\Psi(t, x_0) := \int_{\mathcal{P}_0 \mathbb{R}^D} \mathcal{D}_{s,Q_0} z \cdot \exp\left(-\frac{\pi}{s} Q_0(z)\right)\ \phi\left(x_0 \cdot \sum(t,z)\right) \qquad (7.12)$$

$$Q_0(z) := \int_{\mathbb{T}} dt\ h_{AB} \dot{z}^A(t) \dot{z}^B(t) (7.8)$$

$$\begin{cases} \frac{\partial \Psi}{\partial t} = \frac{s}{4\pi} h^{AB} \mathcal{L}_{X_{(A)}} \mathcal{L}_{X_{(B)}} \Psi + \mathcal{L}_Y \Psi \\ \Psi\left(t_0, \mathbf{x}\right) = \phi(\mathbf{x}) \end{cases} \qquad (7.15)$$

- Homotopy

$$|K\left(b, t_b; a, t_a\right)| = \left|\sum_\alpha \chi(\alpha) K^\alpha\left(b, t_b; a, t_a\right)\right| \qquad \text{(Ch 8)}$$

- Koszul formula

$$\mathcal{L}_X \omega = \mathrm{Div}_\omega(X) \cdot \omega \qquad (11.1)$$

- Miscellaneous

$$\det \exp A = \exp \operatorname{tr} A \qquad (11.48)$$

$$d \log \det A = \operatorname{tr}\left(A^{-1} dA\right) \qquad (11.47)$$

$$\nabla_{g^{-1}} f := g^{ij} \partial f / \partial x^j \qquad \text{gradient} \qquad (11.73)$$

$$\left(\nabla_{g^{-1}} | V\right)_g = V^j{}_{,j} \qquad \text{divergence} \qquad (11.74)$$

$$(V | \nabla f) = -\left(\operatorname{div} V | f\right) \qquad \text{gradient/divergence} \qquad (11.79)$$

- Poisson processes

$$N(t) := \textstyle\sum_{k=1}^{\infty} \theta\left(t - T_k\right) \quad \text{counting process} \qquad (13.17)$$

- Density of energy states

$$\rho(E) = \sum_n \delta\left(E - E_n\right) , \qquad H\psi_n = E_n \psi_n$$

- Time ordering

$$T\left(\phi(x_j)\phi(x_i)\right) = \begin{cases} \phi(x_j)\phi(x_i) & \text{for} \quad j \geqslant i \\ \phi(x_i)\phi(x_j) & \text{for} \quad i \geqslant j \end{cases} \qquad (15.7)$$

- Wick (normal ordering)
  operator normal ordering

$$\left(a + a^\dagger\right)\left(a + a^\dagger\right) =: :\left(a + a^\dagger\right)^2 : +1 \qquad (D.1)$$

functional normal ordering

$$: F(\phi) :_G := \exp\left(-\frac{1}{2}\Delta_G\right) F(\phi) \qquad (D.4)$$

functional laplacian defined by the covariance G

$$\Delta_G = \int \mathrm{dvol}(x) \int \mathrm{dvol}(y)\ G(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)}$$

- The “Measure” (Ch 18)

$$\mu[\phi] \approx \left(\operatorname{sdet} G^+[\phi]\right) \qquad (18.3)$$

$${}_{i,}S_{,k}[\phi]\, G^{+kj}[\phi] = -{}_{i,}\delta^j , \qquad (18.4)$$

$$G^{+ij}[\phi] = 0 \text{ when } i \geqslant j \qquad (18.5)$$

$$\begin{aligned}\phi^i &= u^i_{\text{in}\,A} a^A_{\text{in}} + u^{I*}_{\text{in}\,A} a^{A*}_{\text{in}} \\ &= u^i_{\text{out}\,X} a^X_{\text{out}} + u^{i*}_{\text{out}\,A} a^{X*}_{\text{out}}\end{aligned} \tag{18.18}$$

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Part Six: Projects

Appendix

2

First Lesson: Gaussian Integrals

$$\int \mathcal{D}_{s,Q}x \, \exp\Big(-\frac{\pi}{s}Q(x)\Big) \exp(-2\pi i\langle x', x\rangle) = \exp(-\pi s W(x'))$$

Given the experience accumulated since Feynman's doctoral thesis, the time has come to extract simple and robust axioms for functional integration from the body of work done during the past sixty years, and to investigate approaches other than those dictated by an action functional.

Here, "simple and robust" means easy and reliable techniques for computing integrals by integration by parts, change of variable of integration, expansions, approximations, etc. ....

We begin with gaussian integrals in $\mathbb{R}$ and $\mathbb{R}^D$, defined in such a way that their definitions can be readily extended to gaussians in Banach spaces $\mathbb{X}$.

2.1 Gaussians in $\mathbb{R}$

A gaussian random variable and its concomitant the gaussian volume element are marvelous multifaceted tools. We summarize their properties in Appendix IC. In the following we focus on properties particularly relevant to functional integrals.

2.2 Gaussians in $\mathbb{R}^D$

Let

$$I_D(a) := \int_{\mathbb{R}^D} d^D x \;\; \exp\Big(-\frac{\pi}{a}\,|x|^2\Big) \quad \text{for} \quad a > 0, \tag{2.1}$$

with $d^D x := dx^1 \cdots dx^D$ and $|x|^2 = \sum_{j=1}^{D} \left(x^j\right)^2 = \delta_{ij} x^i x^j$. From elementary calculus, one gets

$$I_D(a) = a^{D/2}. \tag{2.2}$$

Therefore when $D = \infty$,

$$I_\infty(a) = \begin{cases} 0 & \text{if} \quad 0 < a < 1 \\ 1 & \text{if} \quad a = 1 \\ \infty & \text{if} \quad 1 < a \, , \end{cases} \tag{2.3}$$

which is clearly an unsatisfactory situation, but it can be corrected by introducing a volume element $D_a x$ scaled by the parameter a as follows:

$$D_a x := \frac{1}{a^{D/2}} dx^1 \cdots dx^D. \tag{2.4}$$

The volume element $D_a x$ can be characterized by the integral

$$\int_{\mathbb{R}^D} D_a x \ \exp\Bigl(-\frac{\pi}{a} |x|^2 - 2\pi i \left\langle x', x \right\rangle\Bigr) := \exp\Bigl(-a\pi \left|x'\right|^2\Bigr) \, , \tag{2.5}$$

where x' is in the dual $\mathbb{R}_D$ of $\mathbb{R}^D$. A point $x \in \mathbb{R}^D$ is a contravariant (or column vector). A point $x' \in \mathbb{R}_D$ is a covariant vector (or row vector).

The integral (2.5) suggests the following definition of a volume element $d\Gamma_a(x)$:

$$\int_{\mathbb{R}^D} d\Gamma_a(x) \ \exp(-2\pi i \left\langle x', x \right\rangle) := \exp\Bigl(-a\pi \left|x'\right|^2\Bigr) \, . \tag{2.6}$$

Here we can write

$$d\Gamma_a(x) = D_a x \ \exp\Bigl(-\frac{\pi}{a} |x|^2\Bigr) \, . \tag{2.7}$$

This equality is meaningless in infinite dimensions, however, the integral (2.5) and (2.6) remain meaningful. We introduce a different equality symbol:

$$\overset{\int}{=} \tag{2.8}$$

which is a qualified equality in integration theory; e.g. the expression

$$d\Gamma_a(x) \overset{\int}{=} D_a x \ \exp\Bigl(-\frac{\pi}{a}|x|^2\Bigr) \, . \tag{2.9}$$

indicates that both sides of the equality are defined by the same integral.

A linear map $A : \mathbb{R}^D \to \mathbb{R}^D$ given by

$$y = Ax, \quad \text{i.e.} \quad y^j = A^j{}_i \, x^i, \tag{2.10}$$

transforms the quadratic form $\delta_{ij}y^i y^j$ to a general positive quadratic form

$$Q(x) = \delta_{ij} A^i{}_k A^j{}_\ell x^k x^\ell =: Q_{k\ell} x^k x^\ell. \tag{2.11}$$

Consequently a linear change of variable in the integral (2.5) can be used for defining the gaussian volume element $d\Gamma_{a,Q}$ with respect to the quadratic form (2.11). We begin with the definition

$$\int_{\mathbb{R}^D} D_a y \ \exp\Big(-\frac{\pi}{a}\,|y|^2 - 2\pi i\,\langle y', y\rangle\Big) := \exp\Big(-a\pi\,|y'|^2\Big)\,. \tag{2.12}$$

Under the change of variable $y = Ax$, the volume element

$$D_a y = a^{-D/2} dy^1 \wedge \cdots \wedge dy^D$$

becomes

$$\begin{aligned} D_{a,Q} x &= a^{-D/2} |\det A| dx^1 \wedge \cdots \wedge dx^D \\ &= |\det Q/a|^{1/2}\, dx^1 \cdots dx^D. \end{aligned} \tag{2.13}$$

The change of variable

$$y'_j = x'_i B^i{}_j, \tag{2.14}$$

(shorthand $y' = x'B$) defined by transposition

$$\langle y', y\rangle = \langle x', x\rangle\,, \quad \text{i.e.} \quad y'_j y^j = x'_i x^i \tag{2.15}$$

implies

$$B^i{}_j A^j{}_k = \delta^i{}_k. \tag{2.16}$$

Equation (2.12) now reads

$$\int_{\mathbb{R}^D} D_{a,Q} x \ \exp\Big(-\frac{\pi}{a} Q(x) - 2\pi i\,\langle x', x\rangle\Big) := \exp(-a\pi W(x')), \tag{2.17}$$

where

$$\begin{aligned} W(x') &= \delta^{ij} x'_k x'_\ell B^k{}_i B^\ell{}_j, \\ &=: x'_k x'_\ell W^{k\ell}. \end{aligned} \tag{2.18}$$

The quadratic form $W(x') = x'_k x'_l W^{kl}$ on $\mathbb{R}_D$ can be said to be the inverse of $Q(x) = Q_{kl} x^k x^l$ on $\mathbb{R}^D$ since the matrices $\big(W^{kl}\big)$ and (Q_{kl}) are inverse to each other.

In conclusion, in $\mathbb{R}^D$, the gaussian volume element defined in (2.17) by the quadratic form aW is

$$\begin{aligned} d\Gamma_{a,Q}(x) &= D_{a,Q}x \ \exp\Big(-\frac{\pi}{a}Q(x)\Big) && (2.19) \\ &= dx^1 \dots dx^D \left(\det_{k,\ell} \frac{Q_{k\ell}}{a}\right)^{1/2} \exp\Big(-\frac{\pi}{a}Q(x)\Big). && (2.20) \end{aligned}$$

Remark: The volume element $D_a x$ has been chosen so as to be without physical dimension. In Feynman's dissertation, the volume element $\mathcal{D}x$ is the limit for $D = \infty$ of the discretized expression

$$\mathcal{D}x = \prod_i dx(t_i)\ A^{-1}(\delta t_i). \tag{2.21}$$

The normalization factor was determined by requiring that the wave function for a free particle of mass m moving in one dimension be continuous. It was found to be

$$A(\delta t_k) = (2\pi i\hbar\ \delta t_k/m)^{1/2}. \tag{2.22}$$

A general expression for the absolute value of the normalization factor was determined by requiring that the short time propagators be unitary[1]. For a system with action function $\mathcal{S}$, and paths taking their values in an n-dimensional configuration space,

$$|A(\delta t_k)| = \left|\det_{k,\ell} (2\pi\hbar)^{-1} \frac{\partial^2 \mathcal{S}(x^\mu(t_{k+1}), x^\nu(t_k))}{\partial x^\mu(t_{k+1})\partial x^\nu(t_k)}\right|^{1/2}. \tag{2.23}$$

The "intractable" product of the infinite number of normalization factors was found to be a Jacobian [1] later encountered by integrating out momenta from phase space path integrals. Equation (2.5) suggests equation (2.17) and equation (2.19) in which $D_{a,Q}(x)$ provides a volume element which can be generalized to infinite-dimensional spaces without working through an infinite product of short time propagators.

2.3 Gaussians on a Banach Space

In infinite dimensions, the reduction of a quadratic form to a sum of squares of linear forms (see formula (2.11)) is very often inconvenient, and shall be bypassed. Instead, we take formulae (2.17) and (2.19) as our stating point.

The set up

We denote by $\mathbb{X}$ a Banach space, which may consist of paths

$$x : \mathbb{T} \to \mathbb{R}^D \tag{2.24}$$

where $\mathbb{T} = [t_a, t_b]$ is a time interval, and $\mathbb{R}^D$ the configuration space in quantum mechanics. In quantum field theory, $\mathbb{X}$ may consist of fields, that is functions

$$\phi : \mathbb{M}^D \to \mathbb{C} \tag{2.25}$$

for scalar fields, or

$$\phi : \mathbb{M}^D \to \mathbb{C} \tag{2.26}$$

for spinor or tensor fields, where $\mathbb{M}^D$ is the physical space (or space-time). To specify $\mathbb{X}$, we take into account suitable smoothness and/or boundary conditions on x, or ϕ.

We denote by $\mathbb{X}'$ the dual of $\mathbb{X}$, that is the Banach space consisting of the continuous linear forms $x' : \mathbb{X} \to \mathbb{R}$, by $< x', x >$ we denote the value taken by x' in $\mathbb{X}'$ on the vector x in $\mathbb{X}$.

Our formulas require the existence of two quadratic forms $Q(x)$ for x in $\mathbb{X}$ and $W(x')$ for x' in $\mathbb{X}'$. By generalizing the fact that the matrices (Q_{ll}) in (2.11) and (W^{kl}) in (2.18) are inverse to each other, we require that the quadratic forms Q and W be inverse of each other in the following sense.

There exist two continuous linear maps

$$D : \mathbb{X} \to \mathbb{X}' , \qquad G : \mathbb{X}' \to \mathbb{X}$$

with the following properties:

- they are inverse of each other:

$$DG = 1 \text{ (on } \mathbb{X}') , \quad GD = 1 \text{ (on } \mathbb{X}) \tag{2.27}$$

- they are symmetrical:

$$\begin{aligned} < Dx, y > &= < Dy, x > \quad \text{for } x, y \text{ in } \mathbb{X}, \\ < x', Gy' > &= < y', Gx' > \quad \text{for } x', y' \text{ in } \mathbb{X}' ; \end{aligned}$$

- the quadratic forms are given by

$$Q(x) =< Dx, x > , \quad W(x') =< x', Gx' > \ . \tag{2.28}$$

We set also $W(x', y') :=< x', Gy' >$ for x', y' in $\mathbb{X}'$.

Definition

A gaussian volume element on the Banach space $\mathbb{X}$ is defined by its Fourier transform $\mathcal{F}\Gamma_{a,Q}$, namely

$$(\mathcal{F}\Gamma_{a,Q})(x') := \int_{\mathbb{X}} d\Gamma_{a,Q}(x)\ \exp(-2\pi i \langle x', x\rangle) = \exp(-a\pi W(x')) \quad (2.29)$$

for x' arbitrary in $\mathbb{X}'$. We also define formally a volume element $\mathcal{D}_{a,Q}x$ in $\mathbb{X}$ by†

$$d\Gamma_{a,Q}x \overset{\int}{=} D_{a,Q}x\ \exp\Big(-\frac{\pi}{a}Q(x)\Big) \quad (2.30)$$

So far we have been working with $d\Gamma_{a,Q}$ where a was a positive number. **As long as gaussians are defined by their Fourier transforms we can replace a by $s \in \{1, i\}$. Hence we rewrite (2.29) and (2.30):**

$$\boxed{\begin{aligned} &(\mathcal{F}\Gamma_{s,Q})(x') := \textstyle\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\ \exp(-2\pi i \langle x', x\rangle) = \exp(-s\pi W(x')) && (2.29)_s \\ &d\Gamma_{s,Q}x \overset{\int}{=} D_{s,Q}x\ \exp\big(-\tfrac{\pi}{s}Q(x)\big) && (2.30)_s \end{aligned}}$$

Important remark: Because of the presence of i in the exponent of the Feynman integral, it was (and occasionally still is) thought that the integral could not be made rigorous. The gaussian definition $(2.29)_s$ is rigorous for $Q(x) > 0$ when $s = 1$, and for $Q(x)$ real when $s = i$.

Physical interpretation

In our definitions, the case $s = 1$ (or $a > 0$) corresponds to problems in statistical mechanics, whereas the case $s = i$ corresponds to quantum physics via the occurrence of the phase factor $\exp \frac{i}{\hbar}\mathcal{S}(\psi)$.

The volume element definition corresponding to (2.29) and (2.30) can be written

$$\int \mathcal{D}\psi\ \exp\Big(\frac{i}{\hbar}S(\psi) - i\langle J, \psi\rangle\Big) = \exp\Big(\frac{i}{\hbar}W(J)\Big) = Z(J)\,, \quad (2.31)$$

where ψ is either a self-interacting field, or a collection of interacting fields. But the generating functional $Z(J)$ is difficult to ascertain *a priori* for the following reason. Let $\Gamma(\bar\psi)$ be the Legendre transform of $W(J)$. For given $\bar\psi$, $J(\bar\psi)$ is the solution of the equation $h\bar\psi = \delta W(J)/\delta J$ and

$$\Gamma(\bar\psi) := W\big(J(\bar\psi)\big) - \hbar\,\big\langle J(\bar\psi)\,, \bar\psi\big\rangle\,. \quad (2.32)$$

† We use the qualified equality symbol $\overset{\int}{=}$ for terms which are equal after integration.

Then $\Gamma(\bar{\psi})$ is the inverse of $W(J)$ in the same sense as Q and W are inverse of each other (2.27)and (2.28), but $\Gamma(\bar{\psi})$ is the *effective action* which is used for computing observables. If $S(\psi)$ is quadratic, the *bare action* $S(\psi)$ and the effective action $\Gamma(\psi)$ are identical, and the fields do not interact. In the case of interacting fields, the exact relation between the bare and effective actions is the main problem of quantum field theory. (see Chapters 15, 16, 17,18).

Examples

In this chapter we define a volume element on a space Φ of fields ϕ on $\mathbb{M}^D$ by the equation

$$\int_\Phi \mathcal{D}_{s,Q}\phi \cdot \exp\Big(-\frac{\pi}{s}Q(\phi)\Big) \; \exp(-2\pi i \, \langle J, \phi \rangle) := \exp(-\pi s W(J)) \; . \tag{2.33}$$

for given Q and W inverse of each other. For convenience, we will define instead the volume element $d\mu_G$ by†

$$\int_\Phi d\mu_G(\phi) \; \exp(-2\pi i \, \langle J, \phi \rangle) := \exp(-\pi s W(J)) \; . \tag{2.34}$$

As before W is defined by the covariance G

$$W(J) = \langle J, GJ \rangle \, ; \tag{2.35}$$

G is the inverse of the operator D defined by

$$Q(\phi) = \langle D\phi, \phi \rangle \; . \tag{2.36}$$

It is also the two-point function

$$\frac{s}{2\pi} G(x,y) = \int_\Phi d\mu_G(\phi) \, \phi(x) \, \phi(y) \, . \tag{2.37}$$

We shall construct covariances in quantum mechanics and quantum field theory in two simple examples.
In Quantum Mechanics:

Let $D = -\frac{d^2}{dt^2}$; its inverse on the space $\mathbb{X}_{ab}$ of paths with two fixed end

† Hence $d\mu_G$ is the same as $d\Gamma_{s,Q}$, but with the emphasis now placed on the covariance G.

points is‡

$$
\begin{aligned}
G(t,s) \quad &= \quad \theta(s-t)\,(t-t_a)\,(t_a-t_b)^{-1}\,(t_b-s) \\
&\quad\; -\theta(t-s)\,(t-t_b)\,(t_b-t_a)^{-1}\,(t_a-s)\,.
\end{aligned}
\tag{2.38}
$$

In Quantum Field Theory:

Let $D = -\Delta$ on $\mathbb{R}^D$; then

$$
G(x,y) = \frac{C_D}{|x-y|^{D-2}}, \tag{2.39}
$$

with a constant C_D equal to

$$
\Gamma\left(\frac{D}{2}-1\right)/4\pi^{D/2}. \tag{2.40}
$$

Notice that $G(t,s)$ is a continuous function. The function $G(x,y)$ is singular at the origin for euclidean fields and on the lightcone for minkowskian fields. However, we note that *the quantity of interest is not the covariance G, but the variance W*:

$$
W(J) = \langle J, GJ \rangle\,,
$$

which is singular only if J is a point-like source

$$
\langle J, \phi \rangle = c \cdot \phi(x_0)\ ,
$$

where c is a constant, and x_0 a fixed point.

2.4 Variances and covariances

The quadratic form W on $\mathbb{X}'$ that characterizes the Fourier transform $\mathcal{F}\Gamma_{s,Q}$ of the gaussian which in turn characterizes the gaussian $\Gamma_{s,Q}$ is known in probability theory as the variance. The kernel G in (2.27) is known as the covariance of the gaussian distribution. In quantum theory G is the propagator of the system. It is also the "two-point function" since (2.47) gives

$$
\int_{\mathbb{X}} d\Gamma_{s,Q}\left\langle x_1', x\right\rangle \left\langle x_2', x\right\rangle = \frac{s}{2\pi} W\left(x_1', x_2'\right). \tag{2.41}
$$

‡ We denote by $\theta(u)$ the Heaviside function

$$
\theta(u) = \begin{cases} 1 & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u < 0 \\ \text{undefined} & \text{for} \quad u = 0 \end{cases}.
$$

The exponential $\exp(-s\pi W(x'))$ is a generating functional which yields the moments (2.43) and (2.44) and the polarization (2.47). It has been used extensively by Schwinger who considers the term $\langle x', x\rangle$ $(2.29)_s$ as a source.

In this section, we work only with the variance W. In Chapter 3 we work with gaussian volume elements, i.e. with the quadratic form Q on $\mathbb{X}$. In other words we move from the algebraic theory of gaussians (Chapter 2) to their differential theory (Chapter 3), which is commonly used in physics.

Moments

The integral of polynomials with respect to a gaussian volume element follows readily from the definition (2.28) after replacing x' by $\frac{c}{2\pi i}x'$, i.e.

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\exp\bigl(-c\left\langle x', x\right\rangle\bigr) = \exp\bigl(c^2 sW\left(x'\right)/4\pi\bigr)\,. \tag{2.42}$$

Expanding both sides in powers of c, yields

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\left\langle x', x\right\rangle^{2n+1} = 0 \tag{2.43}$$

and

$$\begin{aligned}\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\left\langle x', x\right\rangle^{2n} &= \frac{2n!}{n!}\left(\frac{sW(x')}{4\pi}\right)^n \\ &= \frac{2n!}{2^n n!}\left(\frac{s}{2\pi}\right)^n W\left(x'\right)^n\,.\end{aligned} \tag{2.44}$$

Hint: $W(x')$ is an abbreviation of $W(x', x')$, therefore n-th order terms in expanding the right-hand side are equal to $2n$-th order terms of the left-hand side,

Polarization.1

The integral of a multilinear expression,

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\left\langle x'_1, x\right\rangle\cdots\left\langle x'_{2n}, x\right\rangle \tag{2.45}$$

can be readily be computed. Replacing x' in the definition $(2.28)_a$ by the linear combination $c_1x'_1 + \cdots + c_{2n}x'_{2n}$ and equating the $(c_1c_2\cdots c_{2n})$-terms in both sides of the equation yields

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\left\langle x_1',x\right\rangle\cdots\left\langle x_{2n}',x\right\rangle$$
$$= \frac{1}{2^n n!}\left(\frac{s}{2\pi}\right)^n \sum W\left(x_{i_1}',x_{i_2}'\right)\cdots W\left(x_{i_{2n-1}}',x_{i_{2n}}'\right), \qquad (2.46)$$

where the sum is performed over all possible distributions of the arguments. However each term is respected $2^n n!$ times in this sum since $W\left(x_{i_j}',x_{i_k}'\right) = W\left(x_{i_k}',x_{i_j}'\right)$ and since the product order is irrelevant. Finally†

$$\int_{\mathbb{X}} d\Gamma_{s,Q}(x)\left\langle x_1',x\right\rangle\cdots\left\langle x_{2n}',x\right\rangle$$
$$= \left(\frac{s}{2\pi}\right)^n {\sum}' W\left(x_{i_1}',x_{i_2}'\right)\cdots W\left(x_{i_{2n-1}}',x_{i_{2n}}'\right), \qquad (2.47)$$

where $\sum'$ is a sum without repetition of identical terms‡.

Example If $2n=4$, the sum consists of three terms which can be recorded by three diagrams as follows. Let $1,2,3,4$ designate x_1',x_2',x_3',x_4' respectively, and a line from i_1 to i_2 records $W\left(x_{i_1}',x_{i_2}'\right)$. Then the sum in (2.47) is recorded by the three diagrams

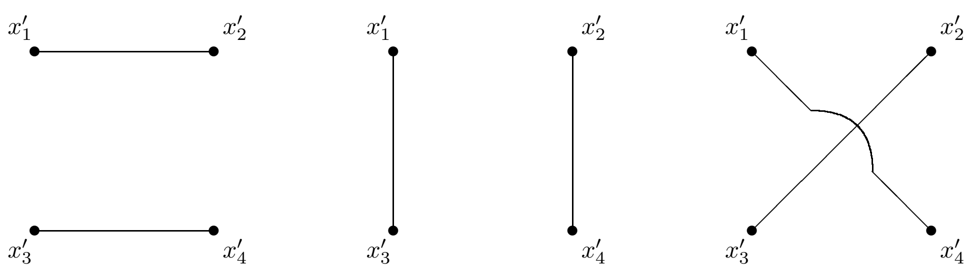


Fig. 2.1. Diagrams

† Corrected by Leila Javindpoor.
‡ For instance, we can assume the inequalities

$$i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}$$

$$i_1 < i_3 < i_5 < \ldots i_{2n-1}$$

in the summation.

Polarization.2

The following proof of the polarization formula (2.47) belongs also to several other chapters:

- Chapters 11 and 15 where the integration by parts (2.48) is justified
- Chapter 3 where we introduce (Section 3.3) the quadratic form Q on $\mathbb{X}$ inverse of the quadratic form W on $\mathbb{X}'$ in the sense of equations (2.26) and (2.27).

Given the qualified equality $(2.30)_s$

$$d\Gamma_{s,Q} \overset{\int}{=} \mathcal{D}_{s,Q}x \, \exp\Big(-\frac{\pi}{s}Q(x)\Big) \ ,$$

the gaussian defined in term of W by $(2.29)_s$ is then expressed in terms of Q by $(2.30)_s$.

We consider the case where $\mathbb{X}$ consists of paths $x = (x(t))_{t_a \le t \le t_b}$ in a one-dimensional space. Furthermore $D = D_t$ is a differential operator.

The basic integration by parts formula

$$\begin{aligned}
&\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \, \exp\Big(-\frac{\pi}{s}Q(x)\Big) \, \frac{\delta F(x)}{\delta x(t)} \\
&:= \quad -\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \, \exp\Big(-\frac{\pi}{s}Q(x)\Big) \, F(x) \, \frac{\delta}{\delta x(t)} \Big(-\frac{\pi}{s}Q(x)\Big)
\end{aligned} \tag{2.48}$$

yields the polarization formula (2.47) when

$$Q(x) = \int_{t_a}^{t_b} dr \, Dx(r) \cdot x\,(r)\,. \tag{2.49}$$

Indeed,

$$-\frac{\delta}{\delta x(t)} \frac{\pi}{s} Q(x) = -2\frac{\pi}{s} \int_{t_a}^{t_b} dr \, Dx(r) \, \delta(r-t) = -\frac{2\pi}{s} D_t x(t)\,. \tag{2.50}$$

When

$$F(x) = x(t_1) \ldots x(t_n) \ , \tag{2.51}$$

then

$$\frac{\delta F(x)}{\delta x(t)} = \sum_{i=1}^{n} \delta(t-t_i) \, x(t_1) \ldots \hat{x}(t_i) \ldots x(t_n) \ , \tag{2.52}$$

where a ˆ over a term means that the term is deleted. The n-point function with respect to the quadratic action $S = \frac{1}{2}Q$ is by definition

$$G_n(t_1, \ldots, t_n) := \left(\frac{2\pi}{s}\right)^{n/2} \int \mathcal{D}_{s,Q}(x) \exp\Bigl(-\frac{\pi}{s}Q(x)\Bigr) x(t_1) \ldots x(t_n) \,. \tag{2.53}$$

Therefore the left-hand side of the integration by parts formula (2.48) is

$$\begin{aligned} &\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \exp\left(-\frac{\pi}{s}Q(x)\right) \frac{\delta F(x)}{x(t)} \\ &\quad = \left(\frac{s}{2\pi}\right)^{(n-1)/2} \sum_{i=1}^{n} \delta(t - t_i)\, G_{n-1}\bigl(t_1, \ldots, \hat{t}_i, \ldots, t_n\bigr) \,. \end{aligned} \tag{2.54}$$

Given (2.50) the right-hand side of (2.48) is

$$\begin{aligned} &-\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \exp\Bigl(-\frac{\pi}{s}Q(x)\Bigr) x(t_1) \ldots x(t_n) \left(-\frac{2\pi}{s} D_t x(t)\right) \\ &\qquad = \frac{2\pi}{s} \left(\frac{s}{2\pi}\right)^{(n+1)/2} D_t G_{n+1}(t, t_1, \ldots, t_n) \,. \end{aligned} \tag{2.55}$$

The integration by parts formula (2.48) yields a recurrence formula for the n-point functions, G_n, namely

$$D_t G_{n+1}(t, t_1, \ldots, t_n) = \sum_{i=1}^{n} \delta(t - t_i)\, G_{n-1}\bigl(t_1, \ldots, \hat{t}_i, \ldots, t_n\bigr) \;; \tag{2.56}$$

equivalently (replace n by $n-1$)

$$D_{t_1} G_n(t_1, \ldots, t_n) = \sum_{i=2}^{n} \delta(t_1 - t_i)\, G_{n-2}\bigl(\hat{t}_1, \ldots, \hat{t}_i, \ldots, t_n\bigr) \,. \tag{2.57}$$

The solution is given by the rules:

- the 2-point function G_2 is a solution of the differential equation

$$D_{t_1} G_2\,(t_1, t_2) = \delta\,(t_1 - t_2) \;;$$

- the n-point function is 0 for m odd;
- for $m = 2n$ even, the m-point function is given by

$$G_{2n}\,(t_2, \ldots, t_{2n}) = \sum G_2\,(t_{i_1}, t_{i_2}) \ldots G_2\,\bigl(t_{i_{2n-1}}, t_{i_{2n}}\bigr)$$

with the same restrictions on the sum as in (2.47).

Linear Maps

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, possibly two copies of the same space. Let L be a linear continuous map $L : \mathbb{X} \to \mathbb{Y}$ by $x \mapsto y$ and $\tilde{L} : \mathbb{Y}' \to \mathbb{X}'$ by $y' \mapsto x'$ defined by

$$\left\langle \tilde{L}y', x \right\rangle = \langle y', Lx \rangle . \tag{2.58}$$

If L maps $\mathbb{X}$ onto $\mathbb{Y}$, then we can associate to a gaussian $\Gamma_{\mathbb{X}}$ on $\mathbb{X}$ another gaussian $\Gamma_{\mathbb{Y}}$ on $\mathbb{Y}$ such that the Fourier transforms $\mathcal{F}\Gamma_{\mathbb{X}}$, $\mathcal{F}\Gamma_{\mathbb{Y}}$ on $\mathbb{X}$ and $\mathbb{Y}$ respectively satisfy the equation

$$\mathcal{F}\Gamma_{\mathbb{Y}} = \mathcal{F}\Gamma_{\mathbb{X}} \circ \tilde{L}, \tag{2.59}$$

i.e. for the variances

$$W_{\mathbb{Y}'} = W_{\mathbb{X}'} \circ \tilde{L}. \tag{2.60}$$

If the map L is invertible, then we have similarly $Q_{\mathbb{X}} = Q_{\mathbb{Y}} \circ L$, but no simple relation exists between $Q_{\mathbb{X}}$ and $Q_{\mathbb{Y}}$ when L is not invertible. The following diagram will be used extensively

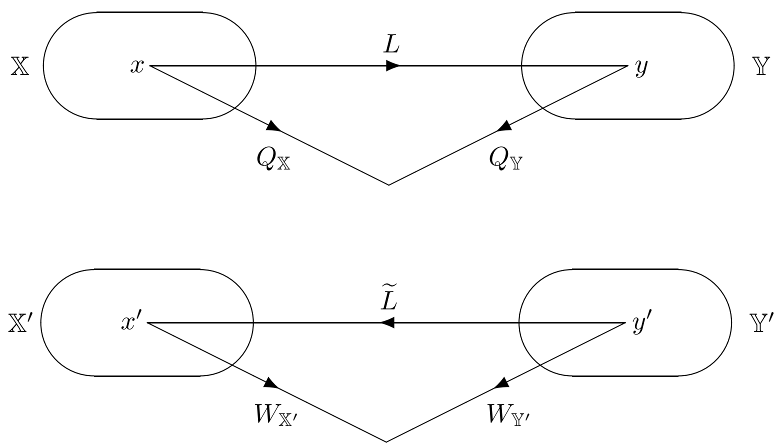


Fig. 2.2. Linear maps

2.5 Scaling and coarse-graining

In this section, we exploit the scaling properties of gaussian volume elements on spaces $\mathbf{\Phi}$ of fields ϕ on $\mathbb{M}^D$. These properties are valid for vector space $\mathbb{M}^D$ with either euclidean or minkowskian signature. These properties are applied to the $\lambda\phi^4$ system in Section 16.2.

The gaussian volume element μ_G is defined according to the conventions described in formulae (2.33) to (2.36). The covariance G is the two-point function (2.36). Objects defined by the covariance G include [See Wick Calculus in Appendix ID]

- convolution with volume element μ_G

$$(\mu_G * F)(\phi) := \int_{\mathbb{X}} d\mu_G(\psi)\, F(\phi + \psi)\,, \tag{2.61}$$

- which yields

$$\mu_G * F = \exp\left(\frac{1}{2}\Delta_G\right) F, \tag{2.62}$$

- where the functional Laplacian

$$\Delta_G := \frac{s}{2\pi} \int_{\mathbb{M}^D} d^D x \int_{\mathbb{M}^D} d^D y\, G(x,y)\, \frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)}, \quad s \in \{1, i\}, \tag{2.63}$$

- the Bargmann-Segal transform defined by

$$B_G := \mu_G * = \exp\left(\frac{1}{2}\Delta_G\right), \tag{2.64}$$

- and the Wick transform

$$:\quad :_G \quad := \exp\left(-\frac{1}{2}\Delta_G\right). \tag{2.65}$$

Scaling

The scaling properties of covariances can be used for investigating the transformation (or the invariance) of some quantum systems under a change of scale.

The definition of a gaussian volume element μ_G (2.34) in Quantum Field Theory reads

$$\int_\Phi d\mu_G(\phi)\ \exp\left(-2\pi i < J, \phi >\right) := \exp\left(-\pi i W(J)\right) \tag{2.66}$$

where ϕ is a field on spacetime (Minkowski or euclidean). The gaussian μ_G of covariance G can be decomposed into the convolution of any number of gaussians. For example, if

$$W = W_1 + W_2 \tag{2.67}$$

then

$$G = G_1 + G_2 \tag{2.68}$$

and

$$\mu_G = \mu_{G_1} + \mu_{G_2} \tag{2.69}$$

The convolution (2.69) can be defined by (2.70):

$$\begin{aligned}\int_{\mathbf{\Phi}} d\mu_G(\phi) \exp(-2\pi i \langle J, \phi \rangle) \\ = \int_{\mathbf{\Phi}} d\mu_{G_2}(\phi_2) \int_{\mathbf{\Phi}} d\mu_{G_1}(\phi_1) \exp(-2\pi i \langle J, \phi_1 + \phi_2 \rangle)\end{aligned} \tag{2.70}$$

where

$$\phi = \phi_1 + \phi_2 \,. \tag{2.71}$$

The additive property (2.68) makes it possible to express a covariance G as an integral over an independent scale variable.

Let $\lambda \in [0, \infty[$ be an independent scale variable†. A scale variable has no physical dimension

$$[\lambda] = 0 \tag{2.72}$$

The scaling operator S_λ acting on a function f of physical length dimension $[f]$ is by definition

$$S_\lambda \, f(x) := \lambda^{[f]} \, f\left(\frac{x}{\lambda}\right), \qquad x \in \mathbb{R}. \tag{2.73}$$

A physical dimension is often given in powers of mass, length, and time. Here we set $\hbar = 1$, $c = 1$, and the physical dimensions are physical *length* dimensions. We choose length dimension rather than the more conventional mass dimension because we define fields on coordinate space, not on momentum space. The subscript of the scaling operator has no dimension.

The scaling of an interval $[a, b[$ is given by

$$S_l \, [a, b[\; = \left\{ \frac{s}{l} \middle| s \in [a, b[\right\}, \quad \text{i.e.} \quad S_l \, [a, b[\; = \left[\frac{a}{l}, \frac{b}{l} \right[. \tag{2.74}$$

By definition the (dimensional) scaling of a functional F is

$$(S_l F)(\phi) = F(S_l \phi) \,. \tag{2.75}$$

† Brydges *et al* use $\lambda \in [1, \infty[$, and $\lambda^{-1} \in [0, 1[$.

We use multiplicative differentials which are scale invariant:

$$d^{\times}l = dl/l \tag{2.76}$$

$$\partial^{\times}/\partial l = l\,\partial/\partial l\,. \tag{2.77}$$

Scaled covariances

In order to control infrared problems at large distances, and ultraviolet divergences at short distances, of the following covariance

$$G(x,y) = c_D/|x-y|^{D-2}\,, \qquad x,y \in \mathbb{R}^D\,, \tag{2.78}$$

one introduces a scaled (truncated) covariance

$$G_{[l_0,l[}(x,y) := \int_{l_0}^{l} d^{\times}s\; S_{s/l_0}u(|x-y|) \tag{2.79}$$

where the length dimension of the various symbols are

$$[l] = 1, \quad [l_0] = 1, \quad [s] = 1, \quad [u] = [G] = 2-D\,. \tag{2.80}$$

In agreement with (2.37),

$$[G] = 2[\phi]\,. \tag{2.81}$$

The function u is chosen so that

$$\lim_{l_0=0,l=\infty} G_{[l_0,l[}(x,y) = G(x,y) \tag{2.82}$$

For $G(x,y)$ given by (2.78) the only requirement on u is

$$\int_0^{\infty} d^{\times}r \cdot r^{-[u]} = c_D\,. \tag{2.83}$$

In the Minkowski case, $|x-y|$ sums over the domain

$$[0,\infty[\, \cup\, i[0,\infty[\text{ in } \mathbb{C} \tag{2.84}$$

and we assume the homogeneity

$$u(ir) = i^{2-D}u(r)\,. \tag{2.85}$$

Example: $\Delta G = \mathbb{1}$ with $\Delta = \sum \partial^2/(\partial x^i)^2$.

$$c_D = \Gamma(D/2-1)/4\pi^{D/2} \tag{2.86}$$

The decomposition of the covariance G into scale dependent contributions (2.79) is also written

$$G = \sum_{j=-\infty}^{+\infty} G_{[2^j l_0, 2^{j+1} l_0[} \tag{2.87}$$

The contributions are self-similar in the following sense. According to (2.79)

$$G_{[a,b[}(\xi) = \int_{[a,b[} d^{\times} s \cdot S_{s/a} u(\xi) \ ; \tag{2.88}$$

given a scale parameter λ,

$$\begin{aligned} G_{[a,b[}(\xi) &= \int_{S_\lambda [a,b[} S_\lambda \left(d^{\times} s \cdot S_{s/a} u\,(\xi) \right) \\ &= \int_{a/\lambda}^{b/\lambda} d^{\times} s\, S_{\lambda s/a}\, l^{2[\phi]}\, u(\xi/\lambda) \ . \end{aligned}$$

Hence, by (2.88)

$$G_{[a,b[}(\xi) = \lambda^{[2\phi]}\, G_{[a/\lambda, b/\lambda[}(\xi/\lambda) \ . \tag{2.89}$$

Henceforth the suffix G in the objects defined by covariances such as μ_G, Δ_G, B_G, $: \quad :_G$ is replaced by the interval defining the scale dependent covariance.

Example: convolution

$$\mu_{[l_0 \infty[} * F = \mu_{[l_0, l[} * \left(\mu_{[l, \infty[} * F \right) \tag{2.90}$$

for any functional F of the fields. That is in terms of the Bargmann-Segal transform (2.64)

$$B_G = B_{G_1} B_{G_2} \tag{2.91}$$

where G, G_1, G_2 correspond respectively to the intervals $[l_0, \infty[$, $[l_0, l[$, $[l, \infty[$.

To the covariance decomposition (2.87) corresponds, according to (2.71) the field decomposition

$$\phi = \sum_{j=-\infty}^{+\infty} \phi_{[2^j l_0, 2^{j+1} l_0[} \ . \tag{2.92}$$

We also write

$$\phi(x) = \sum_{j=-\infty}^{+\infty} \phi_j(l_0, x) \ . \tag{2.93}$$

where the component fields $\phi_j(l_0, x)$ are stochastically independent.

Brydges coarse-graining operator P_l

D. Brydges, J. Dimock, and T.R. Hurd [2] introduced and developed the properties of a coarse-graining operator P_l which rescales the Bargmann-Segal transform so that all integrals are performed with a scale independent gaussian

$$P_l F := S_{l/l_0} B_{[l_0,l[} F := S_{l/l_0} \left(\mu_{[l_0,l[} * F \right) . \tag{2.94}$$

Here l_0 is a fixed length and l runs over $[l_0, \infty[$.

The six following properties of the coarse-graining operator are frequently used in Chapter 16 (Renormalization 2: Scaling):

i. P_l obeys a multiplicative semigroup property. Indeed

$$P_{l_2} P_{l_1} = P_{l_2 l_1 / l_0} , \tag{2.95}$$

whenever $l_1 \geq l_0$, $l_2 \geq l_0$.

Proof of multiplicative property (2.95)†

$$\begin{aligned} P_{l_2} P_{l_1} F &= S_{l_2/l_0} \left(\mu_{[l_0,l_2[} * \left(S_{l_1/l_0} \left(\mu_{[l_0,l_1[} * F \right) \right) \right) \\ &= S_{l_2/l_0} S_{l_1/l_0} \left(\mu_{l_0 l_1/l_0, l_2 l_1/l_0} * \left(\mu_{[l_0,l_1[} * F \right) \right) \\ &= S_{\frac{l_2 l_1}{l_0} \frac{1}{l_0}} \left(\mu_{\left[l_0, \frac{l_2 l_1}{l_0}\right[} * F \right) . \end{aligned}$$

□

ii. P_l does not define a group because convolution does not have an inverse. Information is lost by convolution. Physically, information is lost by integrating over some degrees of freedom.

iii. Wick ordered monomials defined by (2.65) and in Appendix ID are (pseudo) eigenfunctions of the coarse-graining operator.

$$\begin{aligned} & P_l \int_{\mathbb{M}^D} d^D x(x) \, : \phi^n(x) :_{[l_0,\infty[} \\ &= \left(\frac{l}{l_0} \right)^{n[\phi]+D} \int_{\mathbb{M}^D} d^D x(x) \, : \phi^n(x) :_{[l_0,\infty[} \end{aligned} \tag{2.96}$$

If the integral is over a finite volume, the volume is scaled down by S_{l/l_0}. Hence we use the expression "pseudo-eigenfunction" rather than "eigenfunction."

† In the proof we use the identity $S_l \left(\mu_{[a,b[} \right) = \mu_{[a/l, b/l[}$ which follows easily from (2.89).

Proof of eigenfunction equation (2.96)

$$\begin{aligned}
P_l \, :\phi^n(x):_{[l_0,\infty[} &= S_{l/l_0}\left(\mu_{[l_0,l[} * \left(\exp\left(-\frac{1}{2}\Delta_{[l_0,l[}\right)\phi^n(x)\right)\right)\\
&= S_{l/l_0}\left(\exp\left(\frac{1}{2}\Delta_{[l_0,l[} - \frac{1}{2}\Delta_{[l_0,\infty[}\right)\phi^n(x)\right)\\
&= S_{l/l_0}\left(\exp\left(-\frac{1}{2}\Delta_{[l,\infty[}\right)\phi^n(x)\right)\\
&= \exp\left(-\frac{1}{2}\Delta_{[l_0,\infty[}\right) S_{l/l_0}\phi^n(x)\\
&= \exp\left(-\frac{1}{2}\Delta_{[l_0,\infty[}\right)\left(\frac{l}{l_0}\right)^{n[\phi]}\phi^n\left(\frac{l_0}{l}x\right)\\
&= \left(\frac{l}{l_0}\right)^{n[\phi]} :\phi^n\left(\frac{l_0}{l}x\right):_{[l_0,\infty[} \qquad (2.97)
\end{aligned}$$

Note that P_l preserves the scale range. Integrating both sides of (2.86) over x gives, after a change of variable $\frac{l_0}{l}x \mapsto x'$, equation (2.85) and the scaling down of integration when the domain is finite. □

iv. The coarse-graining operator satisfies a parabolic evolution equation, valid for $l \geq l_0$ with initial condition $P_{l_0}F(\phi) = F(\phi)$

$$\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) P_l F(\phi) = 0\,, \qquad (2.98)$$

where

$$\dot{S} = \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} S_{l/l_0} \quad \text{and} \quad \dot{\Delta} = \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} \Delta_{[l_0,l[}. \qquad (2.99)$$

Explicitly

$$\begin{aligned}
\dot{\Delta}F(\phi) &= \int_{\mathbb{M}^D} d^D x \int_{\mathbb{M}^D} d^D y \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} G_{[l_0,l[}(|x-y|)\\
&\qquad\qquad \cdot \frac{\delta^2 F(\phi)}{\delta\phi(x)\,\delta\phi(y)}
\end{aligned}$$

with

$$\begin{aligned}
\left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} G_{[l_0,l[}(\xi) &= \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} \int_{l_0}^{l} d^\times s\, S_{s/l_0}\, u(\xi)\\
&= u(\xi)\,. \qquad (2.100)
\end{aligned}$$

Note that u is independent of the scale, and the final formula

$$\dot{\Delta} F(\phi) = \int_{\mathbb{M}^D} d^D x \int_{\mathbb{M}^D} d^D y \; u(|x-y|) \frac{\delta^2 F(\phi)}{\delta\phi(x)\,\delta\phi(y)} \tag{2.101}$$

Remark Frequently u is labeled $\dot{G}$, an abbreviation of (2.100) meaningful to the cognoscenti.

Proof of evolution equation (2.98) One computes $\frac{\partial^\times}{\partial l} P_l$ at $l = l_0$, then uses the semigroup property (2.95) to prove the validity of the evolution equation (2.98) for all l. Starting from the definition of P_l (2.94), one computes the convolution (2.61)

$$\big(\mu_{[l_0,l[} * F\big)(\phi) = \int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi)\, F(\phi+\psi)\,. \tag{2.102}$$

The functional Taylor expansion of $F(\phi+\psi)$ up to second order is sufficient for deriving (2.98)

$$\begin{aligned} &\big(\mu_{[l_0,l[} * F\big)(\phi) = \\ &\qquad \int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi) \left(F(\phi) + \frac{1}{2} F''(\phi)\cdot\psi\psi + \cdots \right) && (2.103)\\ &= \quad F(\phi) \int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi) + \frac{1}{2}\frac{s}{2\pi}\Delta_{[l_0,l[}\, F(\phi) && (2.104)\\ &\qquad \text{to second order only} \end{aligned}$$

where $\Delta_{[l_0,l[}$ is the functional laplacian (2.63)

$$\Delta_{[l_0,l[} = \int_{\mathbb{M}^D} d^D x \int_{\mathbb{M}^D} d^D y \, G_{[l_0,l[}(x,y) \, \frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)} \tag{2.105}$$

obtained by the ψ integration in (2.103) and the two-point function property (2.37)

$$\int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi)\ \psi(x)\ \psi(y) = \frac{s}{2\pi} G_{[l_0,l[}(x,y)\,. \tag{2.106}$$

Finally

$$\left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} S_{l/l_0}\big(\mu_{[l_0,l[} * F\big)(\phi) = \left(\dot{S} + \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) F(\phi)\,. \tag{2.107}$$

□

v. The generator H of the coarse-graining operator is defined by

$$H := \left.\frac{\partial^\times}{\partial l} P_l\right|_{l=l_0}, \tag{2.108}$$

equivalently

$$P_l =: \exp \frac{l}{l_0} H. \tag{2.109}$$

The evolution operator in (2.98) can therefore be written

$$\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta} = \frac{\partial^\times}{\partial l} - H. \tag{2.110}$$

The generator H operates on Wick monomials as follows (2.96)

$$\begin{aligned} H :\phi^n(x):_{[l_0,\infty[} &= \frac{\partial^\times}{\partial l} P_l :\phi^n(x): \Big|_{l=l_0} \\ &= \frac{\partial^\times}{\partial l}\left(\frac{l}{l_0}\right)^{n[\phi]} :\phi^n\left(\frac{l_0}{l}x\right):_{[l_0,\infty[}\Big|_{l=l_0}; \end{aligned}$$

hence

$$H :\phi^n(x):_{[l_0,\infty[} = n : [\phi]\phi^n(x) + \phi^{n-1}(x)E\phi(x):_{[l_0,\infty[} \tag{2.111}$$

where E is the Euler operator $\sum_{i=1}^D x^i \partial/\partial x^i$.

The second order operator H, consisting of scaling and convolution, operates on Wick monomials as a first order operator.

vi. Coarse-grained integrands in gaussian integrals
The following equation is used in Section 17.1 for deriving the scale evolution of the effective action.

$$\langle \mu_{[l_0,\infty[}, A\rangle = \langle \mu_{[l_0,\infty[}, P_l A\rangle \tag{2.112}$$

Proof of equation (2.112)

$$\begin{aligned} \langle \mu_{[l_0,\infty[}, A\rangle &= \langle \mu_{[l,\infty[}, \mu_{[l_0,l]} * A\rangle \\ &= \langle \mu_{[l_0,\infty[}, S_{l/l_0} \cdot \mu_{[l_0,l]} * A\rangle \\ &= \langle \mu_{[l_0,\infty[}, P_l A\rangle \end{aligned}$$

□

The important step in this proof is the second one,

$$\mu_{[l,\infty[} = \mu_{[l_0,\infty[} S_{l/l_0}\,. \tag{2.113}$$

We check on an example that $\mu_{[l_0,\infty[} S_{l/l_0} = \mu_{[l,\infty[}$.

$$\begin{aligned}
&\int d\mu_{[l_0,\infty[}(\phi)\, S_{l/l_0}\phi(x)\phi(y) \\
&\quad = \int d\mu_{[l_0,\infty[}(\phi) \left(\frac{l}{l_0}\right)^{2[\phi]} \phi\left(\frac{l_0}{l}x\right) \phi\left(\frac{l_0}{l}y\right) \\
&\quad = \frac{s}{2\pi}\left(\frac{l}{l_0}\right)^{2[\phi]} G_{[l_0,\infty[}\left(\frac{l_0}{l}|x-y|\right) \\
&\quad = \frac{s}{2\pi} G_{[l,\infty[}(|x-y|) \\
&\quad = \int d\mu_{[l,\infty[}(\phi)\, \phi(x)\phi(y)
\end{aligned}$$

where we have used (2.89), then (2.37).

From this example we learn the fundamental concepts involved in (2.102): the scaling operator S_{l/l_0} with $l/l_0 > 1$ shrinks the domain of the fields (first line), a change of range from $[l_0, \infty[$ to $[l, \infty[$ (third line) restores the original domain of the fields. These steps, shrinking, scaling, restoring, are at the heart of renormalization in condensed matter physics. The second and fourth lines, convenient for having an explicit presentation of the process relate the gaussian μ_G to its covariance G. The selfsimilarity of covariances in different ranges makes this renormalization process possible.

References

The first four sections summarize results scattered in too many publications to be listed here. The following publication contains many of these results and their references:
P. Cartier and C. DeWitt-Morette. “A new perspective on functional integration”, *J. Math. Phys.* **36**, 2237-2276 (1995).

[1] C. Morette. “On the definition and approximation of Feynman’s path integral”, *Phys. Rev.* **81**, 848-852 (1951).

[2] D.C. Brydges, J. Dimock and T.R. Hurd. “Estimates on Renormalization Group Transformations”, *Can. J. Math.* **50**, 756-793 (1998), and references therein. “A non-gaussian fixed point for ϕ^4 in $4-\epsilon$ dimensions”, *Commun. Math. Phys.* **198**, 111-156 (1998).

9

Grassmann analysis: basics

Parity

9.1 Introduction

Parity is ubiquitous, and Grassmann analysis is a tool well adapted for handling systematically parity and its implications in all branches of algebra, analysis, geometry and topology. Parity describes the behavior of a product under exchange of its two factors. The so-called Koszul's parity rule states: "*Whenever you interchange two factors of parity* 1*, you get a minus sign*". Formally the rule defines graded commutative products

$$AB = (-1)^{\tilde{A}\tilde{B}}\, BA\,, \tag{9.1}$$

where $\tilde{A} \in \{0, 1\}$ denotes the parity of A. Objects with parity zero are called *even*, and objects with parity one *odd*. The rule also defines graded anticommutative products. For instance,

$$A \wedge B = -(-1)^{\tilde{A}\tilde{B}}\, B \wedge A\,. \tag{9.2}$$

- A *graded* commutator $[A, B]$ can be either a commutator $[A, B]_- = AB - BA$, or an anticommutator $[A, B]_+ = AB + BA$.

- A *graded* anticommutative product $\{A, B\}$ can be either an anticommutator $\{A, B\}_+$, or a commutator $\{A, B\}_-$.

  Most often, the context makes it unnecessary to use the $+$ and $-$ signs.

There are no (anti)commutative rules for vectors and matrices. Parity is assigned to such objects in the following way.

- The parity of a vector is determined by its behavior under multiplication with a scalar z:

$$zX = (-1)^{\tilde{z}\tilde{X}}\, Xz\,. \tag{9.3}$$

- A matrix is even if it preserves the parity of graded vectors. A matrix is odd if it inverts the parity.

Vectors and matrices do not necessarily have well-defined parity, but they can always be decomposed into a sum of even and odd parts.

The usefulness of Grassmann analysis in physics became apparent in the works of F.A. Berezin [1], and M.S. Marinov [2]. We refer the reader to [3], [4], [5], [6], and [7] for references and recent developments. The next section summarizes the main formulae of Grassmann analysis.

As a rule of thumb, it is most often sufficient to insert the word "graded" in the corresponding ordinary situation. For example, an ordinary differential form is an antisymmetric covariant tensor. A Grassmann form is a graded antisymmetric covariant tensor: $\omega_{\ldots\alpha\beta\ldots} = -(-1)^{\tilde{\alpha}\tilde{\beta}}\,\omega_{\ldots\beta\alpha\ldots}$ where $\tilde{\alpha} \in \{0, 1\}$ is the grading of the index α. Therefore a Grassmann form is symmetric under the interchange of two Grassmann odd indices.

9.2 A compendium of Grassmann analysis

Contributed by Maria E. Bell†

This section is extracted from the Master's Thesis [8] of Maria E. Bell "Introduction to Supersymmetry." For convenience, we collect here formulae which are self-explanatory, as well as formulae whose meaning is given in the following sections.

Basic graded algebra

- $\tilde{A} :=$ parity of $A \in \{0, 1\}$
- Parity of a product:

$$\widetilde{AB} = \tilde{A} + \tilde{B} \bmod 2\,. \tag{9.4}$$

- Graded commutator:

$$[A, B] := AB - (-1)^{\tilde{A}\tilde{B}}\, BA \quad \text{or} \quad [A, B]_{\mp} = AB \mp BA\,. \tag{9.5}$$

† For an extended version see [8].

- Graded anticommutator:

$$\{A, B\} := AB + (-1)^{\tilde{A}\tilde{B}}\, BA \quad \text{or} \quad \{A, B\}_{\pm} = AB \pm BA \tag{9.6}$$

- Graded Leibnitz rule for a differential operator:

$$D(A \cdot B) = DA \cdot B + (-1)^{\tilde{A}\tilde{D}}\, (A \cdot DB) \tag{9.7}$$

(referred to as "anti-Leibnitz" when $\tilde{D} = 1$).
- Graded symmetry:
$A_{\dots}{}^{\alpha\beta\dots}$ has graded symmetry if

$$A^{\dots\alpha\beta\dots} = (-1)^{\tilde{\alpha}\tilde{\beta}}\, A^{\dots\beta\alpha\dots} \tag{9.8}$$

- Graded antisymmetry:
$A^{\dots\alpha\beta\dots}$ has graded antisymmetry if

$$A^{\dots\alpha\beta\dots} = -(-1)^{\tilde{\alpha}\tilde{\beta}}\, A^{\dots\beta\alpha\dots} \tag{9.9}$$

- Graded Lie derivative:

$$\mathcal{L}_X = [i_X, d]_+ \text{ for } \tilde{X} = 0 \quad \text{and} \quad \mathcal{L}_\Xi = [i_\Xi, d]_- \text{ for } \tilde{\Xi} = 1\,. \tag{9.10}$$

Basic Grassmann algebra

- Grassmann generators $\{\xi^\mu\} \in \Lambda_\nu, \Lambda_\infty, \Lambda$, algebra generated respectively by ν generators, an infinite or an unspecified number.

$$\xi^\mu \xi^\sigma = -\xi^\sigma \xi^\mu\,;\ \Lambda = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}\,. \tag{9.11}$$

- Supernumber (real) $z = u + v$, where u is even (has $\tilde{u} = 0$), and v is odd (has $\tilde{v} = 1$). Odd supernumbers anticommute among themselves; they are called a-numbers. Even supernumbers commute with everything; they are called c-numbers. The set $\mathbb{C}_c$ of all c-numbers is a commutative subalgebra of Λ. The set $\mathbb{C}_a$ of all a-numbers is not a subalgebra.

$$z = z_B + z_S\,,\ z_B \in \mathbb{R} \text{ is the body, } z_S \text{ is the soul.} \tag{9.12}$$

Similar definition for complex supernumber.
- Complex conjugation of a complex supernumber:

$$(zz')^* = z^* z'^*\,. \tag{9.13}$$

Complex conjugation is sometimes defined [4] by $(zz')^* = z'^* z^*$. We prefer the definition (9.13) for the following reason [5]:
Let a supernumber

$$\psi = c_0 + c_i\, \xi^i + \frac{1}{2!}\, c_{ij}\, \xi^i\, \xi^j + \dots \tag{9.14}$$

be called *real* if all its coefficients $c_{i_1 \dots i_p}$ are real numbers. Let

$$\psi = \rho + i\sigma$$

where both ρ and σ have real coefficients. Define complex conjugation by

$$(\rho + i\sigma)^* = \rho - i\sigma \,. \tag{9.15}$$

Then the generators $\{\xi^i\}$ are real, and the sum and product of two real supernumbers are real. Furthermore

$$\psi \text{ is real} \Leftrightarrow \psi^* = \psi \,. \tag{9.16}$$

If one uses the alternate definition of complex conjugation, one finds that the product of two real supernumbers is purely imaginary.

We denote by $\mathbb{R}_c$ the subset of all real elements of $\mathbb{C}_c$ and by $\mathbb{R}_a$ the subset of all real elements of $\mathbb{C}_a$.

- Superpoints. Real coordinates $x, y \in \mathbb{R}^n$, $x = (x^1, \dots, x^n)$. Superspace coordinates

$$(x^1, \dots, x^n, \xi^1, \dots, \xi^\nu) \in \mathbb{R}^{n|\nu} \,, \tag{9.17}$$

are also written in condensed notation $x^A = (x^a, \xi^\alpha)$

$$(u^1, \dots, u^n, v^1, \dots, v^\nu) \in \mathbb{R}^n_c \times \mathbb{R}^\nu_a \,. \tag{9.18}$$

- Supervector space, *i.e.* a graded module over the ring of supernumbers

$$\begin{aligned} X &= U + V, \text{ where } U \text{ is even, and } V \text{ is odd} \\ X &= e_{(A)}\ {}^{A}X \\ X^A &= (-1)^{\tilde{X}\tilde{A}}\ {}^{A}X \,. \end{aligned}$$

The even elements of the basis $(e_{(A)})_A$ are listed first. A supervector is even if each of its coordinates ${}^{A}X$ has the same parity as the corresponding basis element $e_{(A)}$. It is odd if the parity of each ${}^{A}X$ is opposite to the parity of $e_{(A)}$. Parity cannot be assigned in other cases.

- Graded Matrices. Four different uses of graded matrices:
Given $V = e_{(A)}\ {}^{A}V = \bar{e}_{(B)}\ {}^{B}\bar{V}$ with $A = (a, \alpha)$ and $e_{(A)} = \bar{e}_{(B)}\ {}^{B}M_A$ then ${}^{B}\bar{V} = {}^{B}M_A\ {}^{A}V$.
Given $\langle \omega, V \rangle = \omega_A\ {}^{A}V = \bar{\omega}_B\ {}^{B}\bar{V}$ where $\omega = \omega_A\ {}^{(A)}\theta = \bar{\omega}_B\ {}^{(B)}\bar{\theta}$ then $\langle \omega, V \rangle = \omega_A \langle {}^{(A)}\theta, e_{(B)} \rangle\ {}^{B}V$ implies $\langle {}^{(A)}\theta, e_{(B)} \rangle = {}^{A}\delta_B$, $\omega_A = \bar{\omega}_B\ {}^{B}M_A$, and ${}^{(B)}\bar{\theta} = {}^{B}M_A\ {}^{(A)}\theta$.
- Matrix parity:

$$\tilde{M} = 0, \text{ if forall } A \text{ and } B,\ \widetilde{{}^{B}M_A} + \widetilde{\text{column}}\, B + \widetilde{\text{row}}\, A = 0 \bmod 2 \,. \tag{9.19}$$

$$\tilde{M} = 1, \text{ if forall } A \text{ and } B, \ \widetilde{{}^{B}M_{A}} + \widetilde{\text{column}}\, B + \widetilde{\text{row}}\, A = 1 \bmod 2\,. \tag{9.20}$$

Parity cannot be assigned in other cases. Multiplication by an even matrix preserves the parity of the vector components, an odd matrix inverts the parity of the vector components.

- Supertranspose: Supertransposition, labeled "ST", is defined so that the basic rules apply

$$\left(M^{\mathrm{ST}}\right)^{\mathrm{ST}} = M$$

$$(MN)^{\mathrm{ST}} = (-1)^{\tilde{M}\tilde{N}}\, N^{\mathrm{ST}}\, M^{\mathrm{ST}}\,. \tag{9.21}$$

- Superhermitian conjugate:

$$M^{\mathrm{SH}} := (M^{\mathrm{ST}})^{*} = (M^{*})^{\mathrm{ST}} \tag{9.22}$$

$$(MN)^{\mathrm{SH}} = (-1)^{\tilde{M}\tilde{N}}\, N^{\mathrm{SH}}\, M^{\mathrm{SH}}\,. \tag{9.23}$$

- Graded operators on Hilbert spaces. Let $|\Omega\rangle$ be a simultaneous eigenstate of Z and Z' with eigenvalues z and z':

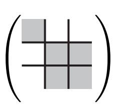
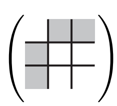
$$ZZ'\,|\Omega\rangle = zz'\,|\Omega\rangle \tag{9.24}$$

$$\langle\Omega|\, Z'^{\mathrm{SH}}\, Z^{\mathrm{SH}} = \langle\Omega|\, z'^{*} z^{*}\,. \tag{9.25}$$

- Supertrace:

$$\operatorname{Str} M = (-1)^{\tilde{A}}\ {}^{A}M_{A}\,. \tag{9.26}$$

Example: A matrix of order (p, q). Assume the p even rows and columns written first

$M_0 =$  $= \begin{pmatrix} A_0 & C_1 \\ D_1 & B_0 \end{pmatrix}$ $\qquad M_1 =$  $= \begin{pmatrix} A_1 & C_0 \\ D_0 & B_1 \end{pmatrix}$.

These are two matrices of order $(1, 2)$. The shaded areas indicate even elements. The matrix on the left is even; the matrix on the right is odd. Given the definitions above

$$\operatorname{Str} M_0 = \operatorname{tr} A_0 - \operatorname{tr} B_0\ ; \quad \operatorname{Str} M_1 = \operatorname{tr} A_1 - \operatorname{tr} B_1\,. \tag{9.27}$$

In general $M = M_0 + M_1$.

- Superdeterminant (a.k.a. Berezinian). It is defined so that it satisfies the basic properties

$$\operatorname{Ber} MN = \operatorname{Ber} M \operatorname{Ber} N \tag{9.28}$$

$$\delta \ln \operatorname{Ber} M = \operatorname{Str}(M^{-1}\delta M) \tag{9.29}$$

$$\operatorname{Ber} \exp M = \exp \operatorname{Str} M \tag{9.30}$$

$$\operatorname{Ber} \begin{pmatrix} A & C \\ D & B \end{pmatrix} := \det(A - CB^{-1}D)(\det B)^{-1} . \tag{9.31}$$

  The determinants on the right hand side are ordinary determinants defined only when the entries commute. It follows that the definition (9.31) applies only to the Berezinian of even matrices, i.e. A and B even, C and D odd.
- Parity assignments for differentials:

$$\tilde{d} = 1, \quad \widetilde{(dx)} = \tilde{d} + \tilde{x} = 1, \quad \widetilde{(d\xi)} = \tilde{d} + \tilde{\xi} = 0\,, \tag{9.32}$$

  where x is an ordinary variable, and ξ is a Grassmann variable,

$$\widetilde{(\partial/\partial x)} = \tilde{x} = 0, \quad \widetilde{(\partial/\partial \xi)} = \tilde{\xi} = 1 \tag{9.33}$$

$$\tilde{i} = 1, \quad \widetilde{i_X} = \tilde{i} + \tilde{X} = 1, \quad \widetilde{i_\Xi} = \tilde{i} + \tilde{\Xi} = 0\,. \tag{9.34}$$

  Parity of real p-forms: even for $p = 0$ mod 2, odd for $p = 1$ mod 2.
  Parity of Grassmann p-forms: always even.
  Graded exterior product: $\omega \wedge \eta = (-1)^{\tilde{\omega}\tilde{\eta}}\, \eta \wedge \omega$.

  Forms and densities will be introduced in Section 9.4. We first list definitions and properties of objects defined on ordinary manifolds $\mathbb{M}^D$ without metric tensors, then on riemannian manifolds $(\mathbb{M}^D, g)$.

Forms and densities of weight 1 on ordinary manifolds $\mathbb{M}^D$
(without metric tensors)

$(\mathcal{A}^\bullet, d)$ Ascending complex of forms $d : \mathcal{A}^p \to \mathcal{A}^{p+1}$
$(\mathcal{D}_\bullet, \nabla$ or $b)$ Descending complex of densities $\nabla : \mathcal{D}_p \to \mathcal{D}_{p-1}$
$\mathcal{D}_p \equiv \mathcal{D}^{-p}$ used for ascending complex in negative degrees.

- Operators on $\mathcal{A}^\bullet(\mathbb{M}^D)$:
  $M(f) : \mathcal{A}^p \to \mathcal{A}^p$, multiplication by a scalar function $f : \mathbb{M}^D \to \mathbb{R}$
  $e(f) : \mathcal{A}^p \to \mathcal{A}^{p+1}$ by $\omega \mapsto df \wedge \omega$
  $i(X) : \mathcal{A}^p \to \mathcal{A}^{p-1}$ by contraction with the vector field X

$\mathcal{L}_X \equiv \mathcal{L}(X) = i(X)d + di(X)$ maps $\mathcal{A}^p \to \mathcal{A}^p$ by the Lie derivative with respect to X.

- Operators on $\mathcal{D}_\bullet(\mathbb{M}^D)$:
  $M(f) : \mathcal{D}_p \to \mathcal{D}_p$, multiplication by scalar function $f : \mathbb{M}^D \to \mathbb{R}$
  $e(f) : \mathcal{D}_p \mapsto \mathcal{D}_{p-1}$ by $\mathcal{F} \to df \cdot \mathcal{F}$ (contraction with the form df)
  $i(X) : \mathcal{D}_p \to \mathcal{D}_{p+1}$ by multiplication and partial antisymmetrization
  $\mathcal{L}_X \equiv \mathcal{L}(X) = i(X)\nabla + \nabla i(X)$ maps $\mathcal{D}_p \to \mathcal{D}_p$ by the Lie derivative with respect to X.
- Forms and densities of weight 1 on a riemannian manifold $(\mathbb{M}^D, g)$:
  $C_g : \mathcal{A}^p \to \mathcal{D}_p$ (see equation (9.59))
  $* : \mathcal{A}^p \to \mathcal{A}^{D-p}$ such that $\mathcal{T}(\omega \mid \eta) = \omega \wedge *\eta$ (see equation (9.67)
  $\delta : \mathcal{A}^{p+1} \to \mathcal{A}^p$ is the metric transpose defined by

$$[d\omega \mid \eta] =: [\omega \mid \delta\eta] \text{ s.t. } [\omega \mid \eta] := \int \mathcal{T}(\omega \mid \eta)$$

  $\delta = C_g^{-1}\, b\, C_g$ (see equation (9.66))
  $\beta : \mathcal{D}_p \to \mathcal{D}_{p+1}$ is defined by $C_g\, d\, C_g^{-1}$.

We now list definitions and properties of objects defined on Grassmann variables.

Grassmann calculus on $\xi^\lambda \in \Lambda_\nu$, Λ_∞, Λ

$dd = 0$ remains true, therefore

$$\frac{\partial}{\partial \xi^\lambda} \frac{\partial}{\partial \xi^\mu} = -\frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \xi^\lambda} \tag{9.35}$$

$$d\xi^\lambda \wedge d\xi^\mu = d\xi^\mu \wedge d\xi^\lambda\,. \tag{9.36}$$

- Forms and densities of weight -1 on $\mathbb{R}^{0|\nu}$:
  Forms are graded *totally symmetric covariant* tensors. Densities are graded *totally symmetric contravariant* tensors of weight -1.
  $(\mathcal{A}^\bullet(\mathbb{R}^{0|\nu}), d)$ Ascending complex of forms not limited above
  $(\mathcal{D}_\bullet(\mathbb{R}^{0|\nu}), \nabla$ or $b)$ Descending complex of densities not limited above.

- Operators on $\mathcal{A}^\bullet(\mathbb{R}^{0|\nu})$:
  $M(\varphi) : \mathcal{A}^p(\mathbb{R}^{0|\nu}) \to \mathcal{A}^p(\mathbb{R}^{0|\nu})$ multiplication by a scalar function φ
  $e(\varphi) : \mathcal{A}^p(\mathbb{R}^{0|\nu}) \to \mathcal{A}^{p+1}(\mathbb{R}^{0|\nu})$ by $\omega \mapsto d\varphi \wedge \omega$
  $i(\Xi) : \mathcal{A}^p(\mathbb{R}^{0|\nu}) \to \mathcal{A}^{p-1}(\mathbb{R}^{0|\nu})$ by contraction with the vector field Ξ
  $\mathcal{L}_\Xi \equiv \mathcal{L}(\Xi) := i(\Xi)d - di(\Xi)$ maps $\mathcal{A}^p(\mathbb{R}^{0|\nu}) \to \mathcal{A}^p(\mathbb{R}^{0|\nu})$ by Lie derivative

with respect to Ξ.

- Operators on $\mathcal{D}_\bullet(\mathbb{R}^{0|\nu})$:
  $M(\varphi) : \mathcal{D}_p(\mathbb{R}^{0|\nu}) \to \mathcal{D}_p(\mathbb{R}^{0|\nu})$, multiplication by scalar function φ
  $e(\varphi) : \mathcal{D}_p(\mathbb{R}^{0|\nu}) \to \mathcal{D}_{p-1}(\mathbb{R}^{0|\nu})$ by $\mathcal{F} \mapsto d\varphi \cdot \mathcal{F}$ (contraction with the form $d\varphi$)
  $i(\Xi) : \mathcal{D}_p(\mathbb{R}^{0|\nu}) \to \mathcal{D}_{p+1}(\mathbb{R}^{0|\nu})$ by multiplication and partial symmetrization
  $\mathcal{L}_\Xi \equiv \mathcal{L}(\Xi) = i(\Xi)\nabla - \nabla i(\Xi)$ maps $\mathcal{D}_p(\mathbb{R}^{0|\nu}) \to \mathcal{D}_p(\mathbb{R}^{0|\nu})$ by Lie derivative with respect to Ξ.

In Section 10.2 we will construct a supersymmetric Fock space. The operators e and i defined above can be used for representing the following:
fermionic creation operators: $e(x^m)$
fermionic annihilation operators: $i(\partial/\partial x^m)$
bosonic creation operators: $e(\xi^\mu)$
bosonic annihilation operators: $i(\partial/\partial \xi^\mu)$.

We refer to [3]-[7] for the different definitions of graded manifolds, supermanifolds, supervarieties, superspace and sliced manifolds. Here we consider simply superfunctions F on $\mathbb{R}^{n|\nu}$: functions of n real variables $\{x^a\}$ and ν Grassmann variables $\{\xi^\alpha\}$

$$F(x,\xi) = \sum_{p=0}^{\nu} \frac{1}{p!} f_{\alpha_1 \ldots \alpha_p}(x)\, \xi^{\alpha_1} \ldots \xi^{\alpha_p} \tag{9.37}$$

where the functions $f_{\alpha_1 \ldots \alpha_p}$ are smooth functions on $\mathbb{R}^n$ that are antisymmetric in the indices $\alpha_1, \ldots, \alpha_p$.

9.3 Berezin integration†

A Berezin integral is a derivation

The fundamental requirement on a definite integral is expressed in terms of an integral operator I and a derivative operator D on a space of functions, and is

$$DI = ID = 0\,. \tag{9.38}$$

† See Project 19.4.6 "Berezin functional integrals. Roepstorff's formulation".

The requirement $DI = 0$ for functions of real variables $f : \mathbb{R}^D \to \mathbb{R}$ states the *definite integral* does not depend upon the variable of integration

$$\frac{d}{dx}\int f(x)\,dx = 0\,, \quad x \in \mathbb{R}\,. \tag{9.39}$$

The requirement $ID = 0$ on the space of functions that vanish on their domain boundaries states $\int df = 0$, or explicitly

$$\int \frac{d}{dx} f(x)\,dx = 0\,. \tag{9.40}$$

Equation (9.40) is the foundation of integration by parts

$$0 = \int d(f(x)\,g(x)) = \int df(x)\cdot g(x) + \int f(x)\cdot dg(x)\,, \tag{9.41}$$

and of Stokes' theorem for a form ω of compact support

$$\int_{\mathbb{M}} d\omega = \int_{\partial\mathbb{M}} \omega = 0, \tag{9.42}$$

since ω vanishes on. We shall use the requirement $ID = 0$ in Section 11.1 for imposing a condition on volume elements.

We now use the fundamental requirements on Berezin integrals defined on functions f of the Grassmann algebra Λ_ν. The condition $DI = 0$ states

$$\frac{\partial}{\partial\xi^\alpha} I(f) = 0 \qquad \text{for } \alpha \in \{1, \ldots, \nu\}\,. \tag{9.43}$$

Any operator on Λ_ν can be set in normal ordering†

$$\sum C^J_K \xi^K \frac{\partial}{\partial\xi^J} \tag{9.44}$$

where J and K are ordered multi-indices where $K = (\alpha_1 < \ldots < \alpha_q)$, $J = (\beta_1 < \ldots < \beta_p)$, $\xi^K = \xi^{\alpha_1}\ldots\xi^{\alpha_q}$, and $\partial/\partial\xi^J = \partial/\partial\xi^{\beta_1}\ldots\partial/\partial\xi^{\beta_p}$

Therefore the condition $DI = 0$ implies that I is a polynomial in $\partial/\partial\xi^i$,

$$I = Q\left(\frac{\partial}{\partial\xi^1}, \ldots, \frac{\partial}{\partial\xi^\nu}\right)\,. \tag{9.45}$$

† This ordering is also the operator normal ordering, in which the creation operator is followed by annihilation operator, since $e(\xi^\mu)$ and $i(\partial/\partial\xi^\mu)$ can be interpreted as creation and annihilation operators (see Section 10.2).

The condition $ID = 0$ states

$$Q\left(\frac{\partial}{\partial\xi^1}, \ldots, \frac{\partial}{\xi^\nu}\right)\frac{\partial}{\partial\xi^\mu} = 0 \quad \text{for every } \mu \in \{1, \ldots, \nu\}\,. \tag{9.46}$$

Equation (9.46) implies

$$I = C^\nu \frac{\partial}{\partial\xi^\nu} \cdots \frac{\partial}{\partial\xi^1} \quad \text{with } C \text{ a constant.} \tag{9.47}$$

A Berezin integral is a derivation. Nevertheless we write

$$I(f) = \int \delta\xi\, f(\xi)$$

where the symbol $\delta\xi$ is different from the differential form $d\xi$ satisfying (9.36). In a Berezian integral, one does not integrate a differential form. Recall ((9.36) and parity assignment (9.32)) that $d\xi$ is even

$$d\xi^\lambda \wedge d\xi^\mu = d\xi^\mu \wedge d\xi^\lambda$$

On the other hand, it follows from the definition of the Berezin integral (9.47) that

$$\int \delta\eta\delta\xi \cdot F(\xi,\eta) = c^2\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\xi}F(\xi,\eta)$$

Since the derivatives on the right hand side are odd,

$$\delta\xi\delta\eta = -\delta\eta\delta\xi\ ;$$

hence $\delta\xi$, like its counterpart dx in ordinary variables, is odd.

The constant is a normalization constant chosen for convenience in the given context. Notice that $C = \int \delta\xi \cdot \xi$. Typical choices include 1, $(2\pi i)^{1/2}$, $(2\pi i)^{-1/2}$. We choose $C = (2\pi i)^{-1/2}$ for the following reason.

Fourier transform and normalization constant

The constant C in (9.47) can be obtained from the Dirac δ-function defined by two conditions:

$$\begin{aligned} \langle\delta, f\rangle &= \int d\xi\,\delta(\xi)f(\xi) = f(0) \qquad (9.48)\\ \langle\mathcal{F}\delta, f\rangle &= f \quad \text{where } \mathcal{F}\delta \text{ is the Fourier transform of } \delta. \end{aligned}$$

The first condition implies

$$\delta(\xi) = C^{-1}\xi.$$

Define the Fourier transform $\tilde{f}$ of a function f by

$$\tilde{f}(\kappa) := \int d\xi\, f(\xi) \exp(-2\pi i \kappa \xi). \tag{9.49}$$

The inverse Fourier transform is

$$f(\xi) = \int d\delta\, \tilde{f}(\kappa)\ \exp(2\pi i \kappa \xi).$$

Then

$$f(\xi) = \int \delta p\, \delta(\xi - p) f(p)$$

provided

$$\delta(\xi - p) = \int \delta\kappa\ \exp\big(2\pi i \kappa(\xi - p)\big). \tag{9.50}$$

Hence, according to the definition of Fourier transforms given above

$$\mathcal{F}\delta = 1.$$

According to (9.50)

$$\delta(\xi) = 2\pi i \int \delta\kappa\, \kappa\xi = 2\pi i C\xi.$$

Together (9.49) and (9.50) imply

$$C^{-1}\xi = 2\pi i C\xi$$

and therefore

$$C^2 = (2\pi i)^{-1}. \tag{9.51}$$

Exercise: Use the Fourier transforms to show that

$$1 = \iint \delta\kappa\delta\xi\ \exp\left(-2\pi i \kappa\xi\right). \qquad \square$$

Conclusion

Let

$$f(\xi) = \sum_{\alpha_1 < \ldots \alpha_p} c_{\alpha_1 \ldots \alpha_p} \xi^{\alpha_1} \ldots \xi^{\alpha_p}, \quad \text{and} \quad d^\nu \xi = \delta\xi^\nu \ldots \delta\xi^1$$

then

$$\int \delta^\nu \xi\ f\left(\xi^1, \ldots, \xi^\nu\right) = (2\pi i)^{-\nu/2}\, c_{1 \ldots \nu}.$$

Remark: Berezin's integrals satisfy Fubini's theorem:

$$\iint \delta\eta\delta\xi\ f(\xi,\eta) = \int \delta\eta\ g(\eta)$$

where

$$g(\eta) = \int \delta\xi\ f(\xi,\eta)$$

Remark: The Fourier transform and its inverse are reciprocal in the following sense

$$\begin{aligned} f(\xi) &= \int \delta\kappa\ \tilde{f}(\kappa)\ \exp(2\pi i\kappa\xi) \\ &= \int \delta\kappa\ \tilde{f}(\kappa)\ \exp(-2\pi i\xi\kappa) \\ \tilde{f}(\kappa) &= \int \delta\xi\ f(\xi)\ \exp(-2\pi i\kappa\xi)\ . \end{aligned}$$

Change of variable of integration

Since integrating $f(\xi^1,\ldots,\xi^\nu)$ is equivalent to taking its derivatives with respect to $\xi^1,\ldots,\xi^\nu$, a change of variable of integration is most easily performed on the derivatives. Recall the induced transformations on the tangent and cotangent spaces given a change of coordinates f. Let $y = f(x)$ and $\theta = f(\zeta)$;

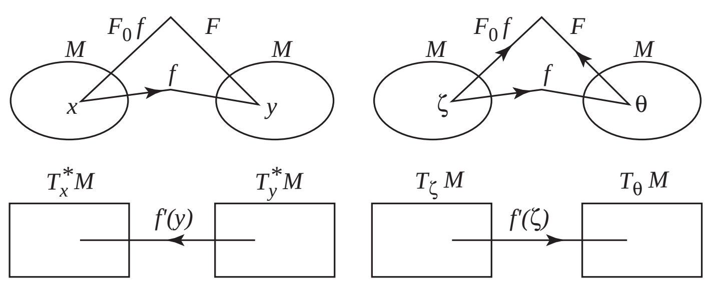


Fig. 9.1.

$$dy^1 \wedge \ldots \wedge dy^D = dx^1 \wedge \ldots \wedge dx^D \left(\det \frac{\partial y^i}{\partial x^j} \right) \tag{9.52}$$

and

$$\int dx^1 \wedge \ldots \wedge dx^D (F \circ f)(x) \left(\det \frac{\partial f^i}{\partial x^j} \right) = \int dy^1 \wedge \ldots \wedge dy^D F(y)\,. \tag{9.53}$$

For an integral over Grassmann variables, the antisymmetry leading to a determinant is the antisymmetry of the product $\partial_1 \ldots \partial_\nu$ where $\partial_\alpha = \partial / \partial \xi^\alpha$. Also

$$\left(\frac{\partial}{\partial \zeta^1} \cdots \frac{\partial}{\partial \zeta^\nu} \right) (F \circ f)(\zeta) = \left(\det \frac{\partial f^\lambda}{\partial \zeta^\mu} \right) \frac{\partial}{\partial \theta^1} \cdots \frac{\partial}{\partial \theta^\nu} F(\theta)\,. \tag{9.54}$$

The determinant is now on the right hand side; it will become an inverse determinant when brought to the left hand side as in (9.53).

Exercise: A quick check of (9.54). Let $\theta = f(\xi)$ be a linear map

$$\theta^\lambda = c^\lambda{}_\mu \xi^\mu, \quad \frac{\partial f^\lambda}{\partial \xi^\mu} = \frac{\partial \theta^\lambda}{\partial \xi^\mu} = c^\lambda{}_\mu, \quad \frac{\partial}{\partial \xi^\mu} = c^\lambda{}_\mu \frac{\partial}{\partial \theta^\lambda}\,.$$

$$\frac{\partial}{\partial \xi^\nu} \cdots \frac{\partial}{\partial \xi^1} = \det \left(\frac{\partial f^\lambda}{\partial \xi^\mu} \right) \frac{\partial}{\partial \theta^\nu} \cdots \frac{\partial}{\partial \theta^1} \qquad \square$$

Exercise: Change of variable of integration in the Fourier transform of the Dirac δ

$$\delta(\xi) = \int \delta\kappa \, \exp(2\pi i \kappa \xi) = 2\pi \int \delta\alpha \, \exp(i \alpha \xi)$$

where $\alpha = 2\pi\kappa$ and $\delta\alpha = \frac{1}{2\pi}\delta\kappa$.

9.4 Forms and densities

On an ordinary manifold $\mathbb{M}^D$, a volume form is an exterior differential form of degree D. It is called a "top form" because there are no forms of degree higher than D on $\mathbb{M}^D$; this is a consequence of the antisymmetry of forms. In Grassmann calculus, forms are symmetric. There are forms of arbitrary degrees on $\mathbb{R}^{0|\nu}$; therefore, there are *no* "top forms" on $\mathbb{R}^{0|\nu}$. We require another concept of volume element on $\mathbb{M}^D$ which can be generalized to $\mathbb{R}^{0|\nu}$.

In the 1930's [9], densities were used extensively in defining and computing integrals. Densities fell into disfavor, possibly because they do not form an algebra as forms do. On the other hand, complexes (ascending and descending) can be constructed with densities as well as with forms, in both ordinary and Grassmann variables.

A form (an exterior differential form) is a *totally antisymmetric covariant tensor*. A density (a linear tensor density) is a *totally antisymmetric contravariant* tensor-density of weight 1†.

Recall properties of forms and densities on ordinary D-dimensional manifolds $\mathbb{M}^D$ which can be established in the absence of a metric tensor. These properties can therefore be readily generalized to Grassmann calculus.

Ascending complex of forms on $\mathbb{M}^D$

Let $\mathcal{A}^p$ be the space of p-forms on $\mathbb{M}^D$, and let d be the exterior differentiation

$$d : \mathcal{A}^p \to \mathcal{A}^{p+1} . \tag{9.55}$$

Explicitly:

$$d\omega_{\alpha_1 \ldots \alpha_{p+1}} = \sum_{j=1}^{p+1} (-1)^{j-1} \partial_{\alpha_j} \omega_{\alpha_1 \ldots \alpha_{j-1} \alpha_{j+1} \ldots \alpha_p}$$

Since $dd = 0$, the graded algebra $\mathcal{A}^\bullet$ is an ascending complex with respect to the operator d

$$\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A}^D . \tag{9.56}$$

Descending complex of densities on $\mathbb{M}^D$

Let $\mathcal{D}_p$ be the space of p-densities on $\mathbb{M}^D$, and let ∇ be the divergence operator, also labeled b

$$\nabla : \mathcal{D}_p \to \mathcal{D}_{p-1} , \quad \nabla \equiv b . \tag{9.57}$$

Explicitly:

$$\nabla \mathcal{F}^{\alpha_1 \ldots \alpha_{p-1}} = \partial_\alpha \mathcal{F}^{\alpha \alpha_1 \ldots \alpha_{p-1}}$$

† See eqn. (9.75) for the precise meaning of weight 1.

(ordinary derivative, not covariant derivative).

Since $bb = 0$, $\mathcal{D}_\bullet$ (which is not a graded algebra) is a descending complex with respect to the divergence operator

$$\mathcal{D}_0 \xleftarrow{b} \mathcal{D}_1 \xleftarrow{b} \dots \xleftarrow{b} \mathcal{D}_D \,. \tag{9.58}$$

Metric-dependent and dimension-dependent transformations

The metric tensor g provides a correspondence C_g between a p-form and a p-density. Set

$$\mathcal{C}_g : \mathcal{A}^p \to \mathcal{D}_p \tag{9.59}$$

Example: The electromagnetic field F is a 2-form with components $F_{\alpha\beta}$ and

$$\mathcal{F}^{\alpha\beta} = \sqrt{\det g_{\mu\nu}} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}$$

are the components of 2-density $\mathcal{F}$. The metric g is used twice: 1) when raising indices, 2) when introducing weight 1 by multiplication with $\sqrt{\det g}$. This correspondence does not depend on the dimension D.

On an orientable manifold, the dimension D can be used for transforming a p-density into a $(D-p)$-form. Set

$$\lambda^D : \mathcal{D}_p \to \mathcal{A}^{D-p} \tag{9.60}$$

Example: Let $D = 4$ and $p = 1$, define

$$t_{\alpha\beta\gamma} := \epsilon^{1234}_{\alpha\beta\gamma\delta} \mathcal{F}^\delta$$

where the alternating symbol ϵ defines an orientation, $t_{\alpha\beta\gamma}$ are the components of a 3-form.

The star operator (Hodge-de Rham operator, see Ref. [10], p. 295) is the composition of the dimension-dependent transformation λ^D with the metric-dependent one $\mathcal{C}_g$. It transforms a p-form into a $D-p$-form.

$$\begin{tikzcd} \mathcal{A}^p \arrow[rr, "*"] \arrow[dr, "\mathcal{C}_g"'] & & \mathcal{A}^{D-p} \\ & \mathcal{D}_p & \arrow[ul, phantom] \end{tikzcd}$$

Let $\mathcal{T}$ be the volume element defined by

$$\mathcal{T}(\omega|\eta) = \omega \wedge *\eta \tag{9.61}$$

where the scalar product of two p-form ω and η is

$$(\omega|\eta) = \frac{1}{p!} g^{i_1 j_1} \dots g^{i_p j_p} \omega_{i_1 \dots i_p} \eta_{j_1 \dots j_p} \tag{9.62}$$

See (9.67) for the explicit expression.

We shall exploit the correspondence mentioned in the first paragraph

$$C_g : \mathcal{A}^p \to \mathcal{D}_p \tag{9.63}$$

for constructing a descending complex on $\mathcal{A}^\bullet$ with respect to the metric transpose δ of d (Ref. [10], p. 296)

$$\delta : \mathcal{A}^{p+1} \to \mathcal{A}^p \tag{9.64}$$

and an ascending complex on $\mathcal{D}_\bullet$

$$\beta : \mathcal{D}_p \to \mathcal{D}_{p+1} \tag{9.65}$$

where β is defined by the following diagram

$$\left.\begin{tikzcd}
\mathcal{A}^p \arrow[r, shift right, "d"'] \arrow[d, "C_g"'] & \mathcal{A}^{p+1} \arrow[l, shift right, "\delta"'] \arrow[d, "C_g"] \\
\mathcal{D}_p \arrow[r, shift left, "\beta"] & \mathcal{D}_{p+1} \arrow[l, shift left, "b"]
\end{tikzcd}\right\} \Longleftrightarrow \left\{ \begin{array}{lcl} \delta & = & C_g^{-1}\, b\, C_g \\ \beta & = & C_g\, d\, C_g^{-1} \end{array} \right. . \tag{9.66}$$

Example 1: Volume element on an oriented D-dimensional riemannian manifold. Set $\omega = \eta = 1$ in the definition (9.61), then

$$\mathcal{T} = *1 = \lambda^D \mathcal{C}_g 1$$

1 is a 0-form, $I := \mathcal{C}_g 1$ is a 0-density with component $\sqrt{\det g_{\mu\nu}}$. λ^D transforms the 0-density with component I into the D form

$$\mathcal{T} := dx^1 \wedge \dots \wedge dx^D \, \mathcal{I} = dx^1 \wedge \dots \wedge dx^D \sqrt{\det g} \tag{9.67}$$

under the change of coordinates $x'^j = A^j{}_i x^i$, the scalar density I transforms into I' such that

$$I' = I \, |\det A|^{-1} \tag{9.68}$$

and the D-form

$$dx'^1 \wedge \dots \wedge dx'^D = |\det A| \, dx^1 \wedge \dots \wedge dx^D \, . \tag{9.69}$$

Example 2: The electric current. In the 1930's the use of densities was often justified by the fact that, in a number of useful examples, it reduces the number of indicies. For example (9.60) the vector-density $\mathcal{T}$ of component

$$\mathcal{T}^l = \epsilon^{ijkl}_{1234} T_{ijk} \tag{9.70}$$

can replace the 3-form T. An axial vector in $\mathbb{R}^3$ can replace a 2-form.

Grassmann forms

The two following properties of forms on real variables remain true for forms on Grassmann variables:

$$dd\omega = 0 \tag{9.71}$$

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\tilde{\omega}\tilde{d}}\, \omega \wedge d\theta \tag{9.72}$$

where $\tilde{\omega}$ and $\tilde{d} = 1$ are the parities of ω and d, respectively. A form on Grassmann variables is a graded totally antisymmetric covariant tensor; this means that a Grassmann p-form is always even.

Since a Grassmann p-form is symmetric the ascending complex $\mathcal{A}^*(\mathbb{R}^{0|\nu})$ does not terminate at ν-forms.

Grassmann densities

The two following properties of densities on real variables remain true for densities $\mathcal{F}$ on Grassmann variables:

$$\nabla\nabla\mathcal{F} = 0\,. \tag{9.73}$$

Since a density is a tensor of weight 1, multiplication by a tensor of weight zero is the only possible product which maps a density into a density.

$$\nabla \cdot (XF) = (\nabla \cdot X) \cdot F + (-1)^{\tilde{X}\tilde{\nabla}}\, X\nabla \cdot F, \tag{9.74}$$

where X is a vector field.

Since a density on Grassmann variables is a symmetric contravariant tensor, the descending complex $\mathcal{D}_\bullet(\mathbb{R}^{0|\nu})$ of Grassmann densities with respect to ∇ does not terminate at ν-densities.

Volume elements

The purpose of introducing densities was to arrive at a definition of volume elements suitable both for ordinary and Grassmann variables. In example 1 (eqn. (9.67)) we showed how a scalar density enters a volume element on $\mathbb{M}^D$, and we gave the transformation (eqn. (9.69)) of a scalar density under a change of coordinates in $\mathbb{M}^D$. But in order to generalize scalar densities to Grassmann volume elements, we start from Pauli's definition ([11], p. 32) which follows Weyl's terminology ([12], p. 109). "If $\int \mathcal{F} dx$ is an invariant [under a change of coordinate system] then $\mathcal{F}$ is called a scalar density."

Under the change of variable $x' = f(x)$, the integrand $\mathcal{F}$ obeys the rule (9.69):

$$\mathcal{F} = \det\left(\partial x^j / \partial x^i\right) \mathcal{F}' \,. \tag{9.75}$$

If the Berezin integral

$$\int \delta\xi^\nu \ldots \delta\xi^1 \, f(\xi^1, \ldots, \xi^\nu) = \frac{\partial}{\partial \xi^\nu} \cdots \frac{\partial}{\partial \xi^1} \, f(\xi^1, \ldots, \xi^\nu)$$

is invariant under the change of coordinates $\theta(\xi)$, then f is a Grassmann scalar density. It follows from the formula for change of variable of integration (9.54) that a Grassmann scalar density obeys the rule

$$f = \det\left(\partial \theta^\lambda / \partial \xi^\kappa\right)^{-1} f' \tag{9.76}$$

with the inverse of the determinant.

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10

Grassmann Analysis: Applications

$$\xi^1 \xi^2 = -\xi^2 \xi^1$$

10.1 The Euler-Poincaré characteristic

A characteristic class is a topological invariant defined on a bundle over a base manifold $\mathbb{X}$. Let $\mathbb{X}$ be a $2n$-dimensional oriented compact, riemannian or pseudoriemannian manifold. Its *Euler number* $\chi(\mathbb{X})$ is the integral over $\mathbb{X}$ of the *Euler class* γ

$$\chi(\mathbb{X}) = \int_{\mathbb{X}} \gamma. \tag{10.1}$$

The *Euler-Poincaré characteristic is equal to the Euler number* $\chi(\mathbb{X})$. The definition of the Euler-Poincaré characteristic can start from the definition of the Euler class, or from the definition of the *Betti numbers* b_p (i.e. the dimension of the p-homology group of $\mathbb{X}$). Chern [1] called the Euler characteristic "the source and common cause of a large number of geometrical disciplines". See e.g. [2, p. 321] for a diagram connecting the Euler-Poincaré characteristic to more than half a dozen topics in geometry, topology, and combinatorics. In this section we compute $\chi(\mathbb{X})$ by means of a supersymmetric path integral [3][4][5].

Supertrace of $\exp(-\Delta)$

We recall some classic properties of the Euler number $\chi(\mathbb{X})$ beginning with its definition as an alternate sum of Betti numbers

$$\chi(\mathbb{X}) = \sum_{p=0}^{2n} (-1)^p \; b_p \; . \tag{10.2}$$

It follows from the Hodge theorem that the sum of the even Betti numbers is equal to the dimension d_{e} of the space of harmonic forms ω of even degrees, and similarly the sum of the odd Betti numbers is equal to the dimension d_{o} of the space of odd harmonic forms. Therefore

$$\chi(\mathbb{X}) = d_{\mathrm{e}} - d_{\mathrm{o}} \ . \tag{10.3}$$

By definition a form ω is said to be harmonic if

$$\Delta\omega = 0 \tag{10.4}$$

where Δ is the laplacian. On a compact manifold, Δ is a positive selfadjoint operator with discrete spectrum $\lambda_0 = 0, \lambda_1, \ldots, \lambda_n, \ldots$. Its trace in the Hilbert space of functions spanned by its normalized eigenvectors is

$$\operatorname{Tr} \exp(-\Delta) = \sum_{n=0}^{\infty} \nu_n \exp(-\lambda_n) \, , \tag{10.5}$$

where ν_n is the (finite) dimension of the space spanned by the eigenvectors corresponding to λ_n. Let $\mathcal{H}^+$ and $\mathcal{H}^-$ be the Hilbert spaces of even and odd forms on $\mathbb{X}$ respectively. Let $\mathcal{H}^{\pm}_{\lambda}$ be the eigenspaces of Δ corresponding to the eigenvalues $\lambda \geq 0$. Then

$$\chi(\mathbb{X}) = \operatorname{Tr} \ \exp(-\Delta)|_{\mathcal{H}^+} - \operatorname{Tr} \ \exp(-\Delta)|_{\mathcal{H}^-} =: \operatorname{Str} \exp(-\Delta) \ . \tag{10.6}$$

Proof Let d be the exterior derivative and δ the metric transpose, then†

$$\Delta = (d+\delta)^2 \ . \tag{10.7}$$

Let

$$Q = d + \delta \ . \tag{10.8}$$

Q is a selfadjoint operator such that

$$Q \ : \ \mathcal{H}^{\pm} \longrightarrow \mathcal{H}^{\mp} \tag{10.9}$$

$$Q \ : \ \mathcal{H}^{\pm}_{\lambda} \longrightarrow \mathcal{H}^{\mp}_{\lambda} \ . \tag{10.10}$$

Eq. (10.10) follows from $\Delta Q f = Q \Delta f = \lambda Q f$ together with (10.9)

$$Q^2 \big| \, \mathcal{H}^{\pm}_{\lambda} = \lambda \ . \tag{10.11}$$

If $\lambda \neq 0$, then $\lambda^{-1/2} \, Q | \, \mathcal{H}^{\pm}_{\lambda}$ and $\lambda^{-1} \, Q | \, \mathcal{H}^{\mp}_{\lambda}$ are inverse of each other, and‡

$$\dim \mathcal{H}^{+}_{\lambda} = \dim \mathcal{H}^{-}_{\lambda} \ . \tag{10.12}$$

† Note that (10.7) defines a positive operator; that is a laplacian with sign opposite to the usual definition $g^{ij} \, \partial_i \partial_j$.

‡ Each eigenvalue of λ has finite multiplicity, hence the spaces $\mathcal{H}^{+}_{\lambda}$ and $\mathcal{H}^{-}_{\lambda}$ have a finite dimension.

By definition

$$\begin{aligned}
\operatorname{Str}\exp(-\Delta) &= \sum_\lambda \exp(-\lambda)\left(\dim \mathcal{H}^+_\lambda - \dim \mathcal{H}^-_\lambda\right) \qquad (10.13)\\
&= \dim \mathcal{H}^+_0 - \dim \mathcal{H}^-_0\\
&= d_{\mathrm{e}} - d_{\mathrm{o}}
\end{aligned}$$

since a harmonic form is an eigenstate of Δ with 0 eigenvalue. □

The definition (10.2) of the Euler number belongs to a graded algebra. Expressing it as a supertrace (10.6) offers the possibility of computing it by a supersymmetric path integral.

Scale invariance

Since the sum (10.13) defining $\operatorname{Str}\exp(-\Delta)$ depends only on the term $\lambda = 0$,

$$\operatorname{Str}\exp(-\Delta) = \operatorname{Str}\exp(z\Delta) \qquad (10.14)$$

for any $z \in \mathbb{C}$. In particular, the laplacian scales like the inverse metric tensor, but according to (10.14) $\operatorname{Str}\exp(-\Delta)$ is scale invariant.

Supersymmetry

When Bose and Fermi systems are combined into a single system, new kind of symmetries and conservation laws can occur. The simplest model consists of combining a Bose and a Fermi oscillator. The action functional $S(x, \xi)$ for this model is an integral of the lagrangian

$$\begin{aligned}
L(x,\xi) &= \frac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right) + \frac{1}{2}\left(\xi^T\dot{\xi} + \omega\xi^T M \xi\right)\\
&= L_{\mathrm{bos}}(x) + L_{\mathrm{fer}}(\xi) \qquad (10.15)
\end{aligned}$$

The *Fermi oscillator* is described by two real a-type dynamical variables

$$\xi := \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \ \xi^T := (\xi_1, \xi_2), \ \text{and } M := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (10.16)$$

Remark: M is an even antisymmetric matrix;

$$\xi^T M \eta = \eta^T M \xi = \xi_1 \eta_2 + \eta_1 \xi_2$$

The dynamical equation for the Fermi trajectory is

$$\dot{\xi} + \omega M \xi = 0; \tag{10.17}$$

it implies

$$\ddot{\xi} + \omega^2 \xi = 0. \tag{10.18}$$

The general solution of (10.17) is

$$\xi(t) = f\, u(t) + f^*\, u^*(t) \tag{10.19}$$

where

$$u(t) := \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} e^{-i\omega t}, \tag{10.20}$$

and where f is an arbitrary complex a-number; upon quantization, the number f becomes an operator $\hat{f}$ satisfying the graded commutators

$$[\hat{f}, \hat{f}]_+ = 0\,, \qquad [\hat{f}, \hat{f}^\dagger]_+ = 1. \tag{10.21}$$

The Bose oscillator is described by one real c-type dynamical variable x. Its dynamical equation is

$$\ddot{x} + \omega^2 x = 0. \tag{10.22}$$

The general solution of (10.22) is

$$x(t) = \frac{1}{\sqrt{2\omega}} \left(b\, e^{-i\omega t} + b^*\, e^{i\omega t} \right) \tag{10.23}$$

where b is an arbitrary complex c-number; upon quantization it becomes an operator $\hat{b}$ satisfying the graded commutators

$$[\hat{b}, \hat{b}]_- = 0, \qquad [\hat{b}, \hat{b}^\dagger]_- = 1. \tag{10.24}$$

Bosons and fermions have vanishing graded commutators

$$[\hat{f}, \hat{b}]_- = 0, \qquad [\hat{f}, \hat{b}^\dagger]_- = 0. \tag{10.25}$$

The action functional $S(x, \xi)$ is invariant under the following infinitesimal changes of the dynamical variables generated by the real a-numbers $\delta\alpha = \begin{pmatrix} \delta\alpha_1 \\ \delta\alpha_2 \end{pmatrix}$:

$$\left\{ \begin{array}{lcl} \delta x & = & \tilde{\xi} M \delta\alpha \\ \delta\xi & = & (\dot{x}\mathbb{1}_2 - \omega x M)\delta\alpha \end{array} \right. . \tag{10.26}$$

The action functional $S(x, \xi)$ is called *supersymmetric* because it is invariant under the transformation (10.26) that defines δx by ξ and $\delta\xi$ by x. The supersymmetry occurs because the frequencies ω of the Bose and Fermi oscillators are equal.

The transformation (10.26) is a global supersymmetry because α is assured to be time-independent.

Remark: Global supersymmetry implemented by a unitary operator. Let us introduce new dynamical variables $b = (\omega x + i\dot{x}) / \sqrt{2\omega}$, $f = (\xi_1 - i\xi_2) / \sqrt{2}$ as well as as their complex conjugates b^* and f^*. The equation of motion reads as

$$\dot{b} = -i\omega b \,, \qquad \dot{f} = -i\omega f$$

and the hamiltonian is $H = \omega\,(b^* b + f^* f)$. Introducing the time independent Grassmann parameter $\beta = \sqrt{\omega}\,(\alpha_2 + i\alpha_1)$ and its complex conjugate $\beta^* = \sqrt{\omega}\,(\alpha_2 - i\alpha_1)$, the infinitesimal supersymmetric transformation is given now by

$$\delta b = f\delta\beta^* \,, \qquad \delta f = -f\delta\beta \,.$$

By taking complex conjugates, we get

$$\delta b^* = f^*\delta\beta \,, \qquad \delta f^* = -b^*\delta\beta^* \,.$$

After quantization, the dynamical variables b and f correspond to operators $\hat{b}$ and $\hat{f}$ obeying the commutation rules (10.21), (10.24), and (10.25). The global supersymmetry is implemented by the unitary operator

$$T = \exp\left(\hat{b}^\dagger \hat{f}\beta + \hat{f}^\dagger \hat{b}\beta^*\right) \,,$$

where β, β^* are complex conjugate Grassmann parameters commuting with $\hat{b}, \hat{b}^\dagger$, and anticommuting with $\hat{f}, \hat{f}^\dagger$. Hence any quantum dynamical variable $\hat{A}$ is transformed into $T\hat{A}T^{-1}$.

Equation (10.26) is modified when x and ξ are arbitrary functions of time. For the modified supersymmetric transformation see [5, p. 292].

In quantum field theory supersymmetry requires bosons to have fermion partners of the same mass.

A supersymmetric path integral†

Consider the following superclassical system on an $(m, 2m)$ supermanifold where the ordinary dynamical variables $x(t) \in \mathbb{X}$, an m-dimensional riemannian manifold:

$$S(x,\xi) = \int_{\mathbb{T}} dt \left(\frac{1}{2} g_{ij}\ \dot{x}^i\ \dot{x}^j + \frac{1}{2} g_{ij}\ \xi^i_\alpha\ \dot{\xi}^j_\alpha + \frac{1}{8} R_{ijkl}\ \xi^i_\alpha\ \xi^j_\alpha\ \xi^k_\beta\ \xi^l_\beta \right) \tag{10.27}$$

where $i, j \in \{1, \ldots, m\}$ and $\alpha \in \{1, 2\}$. Moreover g is a positive metric tensor, the Riemann tensor is

$$R^i{}_{jkl} = -\Gamma^i{}_{jk,l} + \Gamma^i{}_{jl,k} + \Gamma^i{}_{mk}\ \Gamma^m{}_{jl} - \Gamma^i{}_{ml}\ \Gamma^m{}_{jk}\,, \tag{10.28}$$

and $\Gamma^a{}_{bc}$ are the components of the connection ∇. Introducing two sets $\left(\xi^1_\alpha, \ldots, \xi^m_\alpha\right)$ for $\alpha \in \{1, 2\}$ is necessary for the contribution of the curvature term in (10.27) to be nonvanishing. This is obvious when $m = 2$. It can be proved by calculation in the general case.

The action (10.27) is invariant under the supersymmetric transformation generated by $\delta\eta = \begin{pmatrix} \delta\eta_1 \\ \delta\eta_2 \end{pmatrix}$ and δt, namely

$$\left\{ \begin{array}{lcl} \delta x^i & = & \dot{x}^i\ \delta t + \xi^i_\alpha\ \delta\eta_\alpha \\ \delta\xi^i_\alpha & = & \left(d\xi^i_\alpha/dt\right) \delta t + \dot{x}^i\ \delta\eta_\alpha + \Gamma^i{}_{jk}\ \xi^j_\alpha\ \xi^k_\beta\ \delta\eta_\beta \end{array} \right. \tag{10.29}$$

where summation convention applies also to the repeated greek indices

$$\xi_\alpha\, \delta\eta_\alpha = \xi_1\, \delta\eta_1 + \xi_2\, \delta\eta_2 \qquad \text{etc.} \tag{10.30}$$

and where δt and $\delta\eta$ are of compact support in $\mathbb{T}$. Therefore the action $S(x, \xi)$ is supersymmetric.

The hamiltonian H derived from the action functional (10.27) is precisely

† This section is extracted from [5, pp. 386-389] where the detailed calculations are carried out. For facilitating the use of [5], we label m (rather than D) the dimension of the riemannian manifold. In [5] the symbol $\delta\xi$ introduced in chapter 9 is written $d\xi$.

equal to half the laplacian operator Δ on forms [6, p. 319]. Therefore using (10.6) and (10.14),

$$\chi(\mathbb{X}) = \operatorname{Str}\exp(-iHt). \tag{10.31}$$

We shall show that for m even†

$$\operatorname{Str}\exp(-iHt) = \frac{1}{(8\pi)^{m/2}(m/2)!}\int_{\mathbb{X}} d^m x\ g^{-1/2}\ \epsilon^{i_1\ldots i_m}\ \epsilon^{j_1\ldots j_m} \times R_{i_1 i_2 j_1 j_2}\cdots R_{i_{m-1} i_m j_{m-1} j_m} \tag{10.32}$$

and

$$\operatorname{Str}\exp(-iHt) = 0 \quad \text{for } m \text{ odd}. \tag{10.33}$$

That is we shall show that the Gauss-Bonnet-Chern-Avez formula for the Euler-Poincaré characteristic can be obtained by computing a supersymmetric path integral. Since $\operatorname{Str}\exp(-iHt)$ is independent of the magnitude of t, we compute it for an infinitesimal time interval ϵ. The path integral reduces to an ordinary integral, and the reader may question the word "path integral" in the title of this section. The reason is that the path integral formalism simplifies the calculation considerably since it uses the action rather than the hamiltonian.

To spell out $\operatorname{Str}\exp(-iHt)$ one needs a basis in the super Hilbert space $\mathcal{H}$ on which the hamiltonian H i.e. the laplacian Δ, operates. A convenient basis is the coherent states basis defined as follows. For more on its property see [5, p. 381]. Let $|x', t\rangle$ be a basis for the bosonic sector

$$x^i(t)|x', t\rangle = x'^i|x', t\rangle. \tag{10.34}$$

Replace the a-type dynamical variables $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ by

$$z^i := \frac{1}{\sqrt{2}}\left(\xi_1^i - i\xi_2^i\right), \quad z^{i*} := \frac{1}{\sqrt{2}}\left(\xi_1^i + i\xi_2^i\right). \tag{10.35}$$

The superjacobian of this transformation is

$$\frac{\partial(z^*, z)}{\partial(\xi_1, \xi_2)} = \left(\operatorname{sdet}\begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}\right)^m = i^m. \tag{10.36}$$

Set

$$z_i = g_{ij}\ z^j, \quad z_i^* = g_{ij}\ z^{j*}. \tag{10.37}$$

† We set $g := \det(g_{\mu\nu})$.

The new variables satisfy

$$\left[z_i, z_j^*\right]_+ = g_{ij}, \tag{10.38}$$

the other graded commutators vanish.

Define supervectors in $\mathcal{H}$

$$\left|x', z', t\right\rangle := \exp\left(-\frac{1}{2} z_i'^* \, z'^i + z_i^*(t) \, z'^i\right) \left|x', t\right\rangle \tag{10.39}$$

and

$$\left\langle x', z'^*, t\right| := \left|x', z', t\right\rangle^\dagger = \left\langle x', t\right| \exp\left(\frac{1}{2} z_i'^* \, z'^i + z_i'^* \, z^i(t)\right). \tag{10.40}$$

This basis is called a coherent states basis because the $|x', z', t\rangle$ are right eigenvectors of the $z^i(t)$ while the $\langle x', z'^*, t|$ are left eigenvectors of the $z^{i*}(t)$. In terms of this basis

$$\begin{aligned} \operatorname{Str}\exp(-iHt) &= \frac{1}{(2\pi i)^m} \int d^m x' \prod_{j=1}^m \left(\delta z_j'^* \; \delta z_j'\right) \; g^{-1}(x') \\ &\qquad \times \langle x', z'^*, t' | e^{-iHt} | x', z', t' \rangle. \end{aligned} \tag{10.41}$$

We need not compute the hamiltonian, it is sufficient to note that it is a time translation operator; therefore

$$\langle x', z'^*, t' | \exp(-iHt) | x', z', t' \rangle = \langle x', z'^*, t' + t | x', z', t' \rangle. \tag{10.42}$$

Two circumstances simplify the path integral representation of this probability amplitude:

- It is a trace, therefore the paths are loops beginning and ending at the same point in the supermanifold.
- The supertrace is scale invariant, therefore, the time interval t can be taken arbitrarily small.

It follows that the only term in the action functional (10.27) contributing to the supertrace is the Riemann tensor integral,

$$\frac{1}{8} \int_{\mathbb{T}} dt \; R_{ijkl} \; \xi_\alpha^i \; \xi_\alpha^j \; \xi_\beta^k \; \xi_\beta^l = \frac{1}{2} \int_{\mathbb{T}} dt \; R_{ijkl} \; z^{i*} \, z^j \, z^{k*} \, z^l. \tag{10.43}$$

For an infinitesimal time interval ϵ,

$$\begin{aligned} &\operatorname{Str}\exp(-iH\epsilon) = (2\pi i\epsilon)^{-\frac{1}{2}m} (2\pi)^{-m} \\ &\quad \times \int_{\mathbb{R}^{m|2m}} \exp\left(\frac{1}{4} i R_{ijkl}(x') \; \xi_1^i \; \xi_1^j \; \xi_2^k \; \xi_2^l \; \epsilon\right) g^{-\frac{1}{2}}(x') \, d^m x' \prod_{i=1}^m \delta\xi_1^i \; \delta\xi_2^i \end{aligned} \tag{10.44}$$

The detour by the the z-variables was useful for constructing a supervector basis. The return to the ξ-variables simplifies the Berezin integrals. Expanding the exponent in (10.44), one sees that the integral vanishes for

m odd, and for m even is equal to†

$$\operatorname{Str}\exp(-iH\epsilon) = \frac{(2\pi)^{-3m/2}}{4^{m/2}(m/2)!}\int_{\mathbb{R}^{m|2m}} \Big(R_{ijkl}(x)\ \xi_1^i\ \xi_1^j\ \xi_2^k\ \xi_2^l\Big)^{m/2} \times g^{-1/2}(x)\, d^m x\ \delta\xi_1^1\ \delta\xi_2^1\ \ldots\ \delta\xi_1^m\ \delta\xi_2^m. \quad (10.45)$$

Finally

$$\chi(\mathbb{X}) \ = \ \operatorname{Str}\exp(-iH\epsilon) \ = \ \frac{1}{(8\pi)^{m/2}(m/2)!}\int_{\mathbb{X}} g^{-1/2}\,\epsilon^{i_1\ldots i_m}\ \epsilon^{j_1\ldots j_m} \times R_{i_1 i_2 j_1 j_2}\ \ldots\ R_{i_{m-1}i_m j_{m-1}j_m}\ d^m x. \quad (10.46)$$

□

In two dimensions, $m = 2$, one obtains the well-known formula for the Euler number

$$\begin{aligned}\chi(\mathbb{X}) &= \frac{1}{8\pi}\int_{\mathbb{X}} g^{-1/2}\,\epsilon^{ij}\ \epsilon^{kl}\ R_{ijkl}\ d^2x \qquad (10.47)\\ &= \frac{1}{8\pi}\int_{\mathbb{X}} g^{1/2}\left(g^{ik}\,g^{jl} - g^{il}\,g^{jk}\right) R_{ijkl} d^2x\\ &= \frac{s}{2\pi}\int g^{1/2}\,R\,d^2x.\end{aligned}$$

Again one can check that the r.h.s. is invariant under a scale transformation of the metric. When $g \to cg$ with a constant c, the Christoffel symbols are invariant, the Riemann scalar $R = R_{ij}\ g^{ij}$ goes into Rc^{-1} and $g^{1/2}$ goes into $c^{D/2}g^{1/2}$. Therefore in two dimensions $g^{1/2}R$ goes into $g^{1/2}R$.

Starting from a superclassical system more general than (10.27) A. Mostafazadeh [7] has used supersymmetric path integrals for deriving the index of the Dirac operator formula, and the Atiyah-Singer index theorem.

10.2 Supersymmetric Quantum Field Theory

It is often, but erroneously, stated that the "classical limits" of Fermi fields take their values in a Grassmann algebra because "their anticommutators

† The detailed calculation is carried out in [5, pp. 388-389]. There the normalization of the Berezin integral is not (9.51) but $C^2 = 2\pi i$; however it can be shown that (10.45) does not depend on the normalization.

vanish when $\hbar = 0$." Leaving aside the dubious concept of a "Fermi field's classical limit," we note that in fact the canonical anticommutators of Fermi fields do *not* vanish when $\hbar = 0$ because they do not depend on $\hbar$: given the canonical quantization

$$\begin{aligned} [\Phi(x), \Pi(y)]_- &= i\hbar\delta(x-y) \quad \text{for a bosonic system} && (10.48) \\ [\psi(x), \pi(y)]_+ &= i\hbar\delta(x-y) \quad \text{for a fermionic system} && (10.49) \end{aligned}$$

and the Dirac lagrangian

$$\mathcal{L} = \bar{\psi}(-p_\mu\gamma^\mu - mc)\psi = i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - mc\bar{\psi}\psi, \tag{10.50}$$

the conjugate momentum $\pi(x) = \delta\mathcal{L}/\delta\dot{\psi}$ is proportional to $\hbar$. The net result is that the graded commutator is independent of $\hbar$. □

Clearing up the above fallacy does not mean that Grassmann algebra plays no role in fermionic systems. Grassmann analysis *is* necessary for a consistent and unified functional approach to quantum field theory; the functional integrals are integrals over functions of Grassmann variables.

Supersymmetry in Quantum Field Theory is a symmetry that unites particles of integer and half-integer spin in common symmetry multiplets, called *supermultiplets*.

Supersymmetry in physics is too complex to be thoroughly addressed in this book. We refer the reader to works by Martin [ref: S. Martin, "A supersymmetry primer," `hep-ph/9709356`.], Weinberg [ref: S. Weinberg, *The Quantum Theory of Fields Volume III: Supersymmetry.* Cambridge University Press, 2000.], and Wess and Bagger [ref: J. Wess and J. Bagger, *Supersymmetry and Supergravity.* Princeton University Press, 2nd ed., 1992.].

Here we only mention supersymmetric Fock spaces, i.e. spaces of states which carry a representation of a supersymmetric algebra - that is, an algebra of bosonic and fermionic creation and annihilation operators.

Representations of fermionic and bosonic creation and annihilation operators are easily constructed on the ascending complex of forms (Section 9.4 and Section 9.2)

- on $\mathbb{M}^D$ for the fermionic case
- on $\mathbb{R}^{0|\nu}$ for the bosonic case.

They provide representations of operators on supersymmetric Fock spaces. The operator e defines creation operators, and the operator i defines annihilation operators. On $\mathbb{M}^D$, let f be a scalar function $f : \mathbb{M}^D \to \mathbb{R}$,

$$e(f) : \mathcal{A}^p \to \mathcal{A}^{p+1} \quad \text{by} \quad \omega \mapsto df \wedge \omega. \tag{10.51}$$

Let X be a vector field on $\mathbb{M}^D$,

$$i(X) : \mathcal{A}^p \to \mathcal{A}^{p-1} \quad \text{by contraction with the vector field } X. \tag{10.52}$$

Let ϕ be a scalar function on $\mathbb{R}^{0|\nu}$ such that $\phi : \mathbb{R}^{0|\nu} \to \mathbb{R}$,

$$e(\phi) : \mathcal{A}^p \to \mathcal{A}^{p+1} \quad \text{by} \quad \omega \mapsto d\phi \wedge \omega. \tag{10.53}$$

Let Ξ be a vector field on $\mathbb{R}^{0|\nu}$,

$$i(\Xi) : \mathcal{A}^p \to \mathcal{A}^{p-1} \quad \text{by contraction with the vector field } \Xi. \tag{10.54}$$

Representations of fermionic and bosonic creation and annihilation operators can also be constructed on descending complex of densities. They can be read off from Section 9.2 (a compendium of Grassmann analysis). They are given explicitly in [8].

A physical example of fermionic operators: Dirac fields

The second set of Maxwell's equations (see *e.g.* [6, p. 336])

$$\delta F + J = 0,$$

together with some initial data, gives the electromagnetic field F created by a current J. Dirac gave an expression for the current J:

$$J_\mu = ec\bar{\psi}\gamma_\mu\psi, \text{ with } \bar{\psi} \text{ such that } \bar{\psi}\psi \text{ is a scalar,} \tag{10.55}$$

e is the electric charge, the $\{\gamma_\mu\}$'s are the Dirac matrices, and ψ is a Dirac field which obeys Dirac's equation. The structural elements of Quantum Electrodynamics are the electromagnetic field and the Dirac fields. Their quanta are photons, electrons, and positrons, which are viewed as particles. The Dirac field ψ is an operator on a Fock space. It is a linear combination of an electron annihilation operator a and a positron creation operator $b^\dagger$, constructed so as to satisfy the causality principle, namely, the requirement that supercommutators of field operators vanish when the points at which the operators are evaluated are separated by a space-like interval (see [10, Vol. I] and [11]).

The Dirac field describes particles other than electrons and antiparticles other than positrons, generically called fermions and antifermions. There are several representations of Dirac fields that depend on the following:

- the signature of the metric tensor
- whether the fields are real [9] or complex
- the choice of Dirac pinors (one set of four complex components) or Weyl spinors (two sets of two complex components).

For example [11]

$$\begin{aligned}\Psi(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\, u^s(p)e^{-ip\cdot x} + b^{s\,\dagger}_{\mathbf{p}}\, v^s(p)e^{ip\cdot x}\right)\\ \bar{\Psi}(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(b^s_{\mathbf{p}}\, \bar{v}^s(p)e^{-ip\cdot x} + a^{s\,\dagger}_{\mathbf{p}}\, \bar{u}^s(p)e^{ip\cdot x}\right)\end{aligned}$$

represent a fermion and antifermion.

In Quantum Electrodynamics, the operator $a^{s\,\dagger}_{\mathbf{p}}$ creates electrons with energy $E_{\mathbf{p}}$, momentum $\mathbf{p}$, spin 1/2, charge +1 (in units of e), and polarization determined by the (s)pinor u^s. The operator $b^{s\,\dagger}_{\mathbf{p}}$ creates positrons with energy $E_{\mathbf{p}}$, momentum $\mathbf{p}$, spin 1/2, charge -1, and polarization opposite that of u^s.

The creation and annihilation operators are normalized so that

$$\{\Psi_a(\mathbf{x}), \Psi^\dagger_b(\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}$$

with all other anticommutators equal to zero. The (s)pinors u^s and v^s obey Dirac's equation. The term $d^3p/\sqrt{2E_{\mathbf{p}}}$ is Lorentz invariant; it is the positive energy part of $d^4p\,\delta(p^2-m^2)$.

There are many [8] constructions of supermanifolds and many representations of bosonic and fermionic algebras, that is many possibilities for a framework suitable for supersymmetric quantum field theory.

10.3 Dirac operator and Dirac matrices

The Dirac operator is the operator on a Pin bundle,

$$\partial\!\!\!/ := \gamma^a\partial_a,$$

where $\partial_a := \partial/\partial x^a$ and $\{\gamma^a\}$ are the Dirac matrices. The Dirac operator is the square root of the laplacian

$$\Delta = g^{ab}\,\partial_a\,\partial_b.$$

The operator Q on differential forms, (10.8)

$$Q = d + \delta,$$

is also a square root of the laplacian. We shall develop a connection between the Dirac operator $\not\partial$ and the Q-operator acting on superfunctions.

Consider a D-dimensional real vector space V with a scalar product. Introducing a basis $e_1, \ldots, e_D$ we represent a vector by its components $v = e_a\, v^a$. The scalar product reads

$$g(v, w) = g_{ab}\, v^a\, w^b. \tag{10.56}$$

Let $C(V)$ be the corresponding Clifford algebra generated by $\gamma_1, \ldots, \gamma_D$ satisfying the relations

$$\gamma_a\,\gamma_b + \gamma_b\,\gamma_a = 2g_{ab}. \tag{10.57}$$

The dual generators are given by $\gamma^a = g^{ab}\,\gamma_b$ and

$$\gamma^a\,\gamma^b + \gamma^b\,\gamma^a = 2g^{ab} \tag{10.58}$$

where $g^{ab}\,g_{bc} = \delta^a{}_c$ as usual.

We define now a representation of the Clifford algebra $C(V)$ by operators acting on a Grassmann algebra. Introduce Grassmann variables $\xi^1, \ldots, \xi^D$ and put†

$$\gamma^a = \xi^a + g^{ab}\frac{\partial}{\partial \xi^b}. \tag{10.59}$$

Then the relations (10.58) hold. In more intrinsic terms we consider the exterior algebra ΛV^* built on the dual V^* of V with a basis $(\xi^1, \ldots, \xi^D)$ dual to the basis $(e_1, ..., e_D)$ of V. The scalar product g defines an isomorphism $v \mapsto I_g v$ of V with V^* characterized by

$$\langle I_g v, w\rangle = g(v, w). \tag{10.60}$$

Then we define the operator $\gamma(v)$ acting on ΛV^* as follows

$$\gamma(v)\cdot\omega = I_g v \wedge \omega + i(v)\omega, \tag{10.61}$$

† Here again $\partial/\partial\xi^b$ denotes the left derivation operator, often denoted by $\overleftarrow{\partial}/\partial\xi^b$.

where the contraction operator $i(v)$ satisfies

$$i(v)(\omega_1 \wedge \ldots \wedge \omega_p) = \sum_{j=1}^{p} (-1)^{j-1} \langle \omega_j, v \rangle \omega_1 \wedge \ldots \wedge \hat{\omega}_j \wedge \ldots \wedge \omega_p \qquad (10.62)$$

for $\omega_1, \ldots, \omega_p$ in V^*. (The hat $\hat{}$ means that the corresponding factor is omitted.) An easy calculation gives

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = 2g(v, w). \qquad (10.63)$$

We recover $\gamma_a = \gamma(\mathrm{e}_a)$, hence $\gamma^a = g^{ab}\gamma_b$.

The representation constructed in this manner is not the spinor representation since it is of dimension 2^D. Assume D is even, $D = 2n$, for simplicity. Hence ΛV^* is of dimension $2^D = (2^n)^2$, and *the spinor representation should be a "square root" of* ΛV^*.

Consider the operator J on ΛV^* given by

$$J(\omega_1 \wedge \ldots \wedge \omega_p) = \omega_p \wedge \ldots \wedge \omega_1 = (-1)^{p(p-1)/2} \omega_1 \wedge \ldots \wedge \omega_p \qquad (10.64)$$

for $\omega_1, \ldots, \omega_p$ in V^*.

Introduce the operator

$$\gamma^o(v) = J\gamma(v)J. \qquad (10.65)$$

In components $\gamma^o(v) = v^a \gamma^o_a$ where $\gamma^o_a = J\gamma_a J$. Since $J^2 = 1$, they satisfy the Clifford relations

$$\gamma^o(v)\gamma^o(w) + \gamma^o(w)\gamma^o(v) = 2g(v, w). \qquad (10.66)$$

The interesting point is the commutation property †

$\gamma(v)$ and $\gamma^o(w)$ commute for all v, w.

According to the standard wisdom of quantum theory, the degrees of freedom associated with the γ_a decouple with the ones for the γ^o_a. Assume that the scalars are complex numbers, hence the Clifford algebra is isomorphic to the algebra of matrices of type $2^n \times 2^n$. Then ΛV^* can be decomposed as a tensor square

$$\Lambda V^* = S \otimes S \qquad (10.67)$$

with the $\gamma(v)$ acting on the first factor only, and the $\gamma^o(v)$ acting on the

† This construction is reminiscent of Connes' description of the standard model in [12]

second factor in the same way:

$$\gamma(v)(\psi \otimes \psi') = \Gamma(v)\psi \otimes \psi' \tag{10.68}$$

$$\gamma^o(v)(\psi \otimes \psi') = v \otimes \Gamma(v)\psi'. \tag{10.69}$$

The operator J is then the exchange

$$J(\psi \otimes \psi') = \psi' \otimes \psi. \tag{10.70}$$

The decomposition $S \otimes S = \Lambda V^*$ corresponds to the formula

$$c_{i_1 \dots i_p} = \bar{\psi}\gamma_{[i_1} \dots \gamma_{i_p]}\psi \qquad (0 \leq p \leq D) \tag{10.71}$$

for the currents † $c_{i_1 \dots i_p}$ (by $[\dots]$ we denote antisymmetrization).

In differential geometric terms, let $(\mathbb{M}^D, g)$ be a (pseudo-)riemannian manifold. The Grassmann algebra ΛV^* is replaced by the graded algebra $\mathcal{A}(\mathbb{M}^D)$ of differential forms. The Clifford operators are given by

$$\gamma(f)\omega = df \wedge \omega + i(\nabla f)\omega \tag{10.72}$$

(∇f is the gradient of f with respect to the metric g, a vector field). In components $\gamma(f) = \partial_\mu f \cdot \gamma^\mu$ with

$$\gamma^\mu = e(x^\mu) + g^{\mu\nu} i \left(\frac{\partial}{\partial x^\nu}\right). \tag{10.73}$$

The operator J satisfies

$$J(\omega) = (-1)^{p(p-1)/2}\omega \tag{10.74}$$

for a p-form ω. To give a spinor structure on the riemannian manifold $(\mathbb{M}^D, g)$ (in the case D even) is to give a splitting ‡

$$\Lambda T^*_{\mathbb{C}}\mathbb{M}^D \simeq S \otimes S \tag{10.75}$$

satisfying the analog of relations (10.68) and (10.70). The Dirac operator $\partial\!\!\!/$ is then characterized by the fact that $\partial\!\!\!/ \times 1$ acting on bispinor fields (sections of $S \otimes S$ on $\mathbb{M}^D$) corresponds to $d + \delta$ acting on (complex) differential forms, that is on (complex) superfunctions on $\Pi T\mathbb{M}^D$.

† For $n = 4$, this gives a scalar, a vector, a bivector, a pseudo-vector and a pseudo-scalar.

‡ $T^*_{\mathbb{C}}\mathbb{M}^D$ is the complexification of the cotangent bundle. We perform this complexification to avoid irrelevant discussions on the signature of the metric.

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11

Volume Elements, Divergences, Gradients

$$\int_{\mathbb{M}^D} \mathcal{L}_X \, \omega^D = 0 \qquad\qquad \mathcal{L}_X \, \omega = \operatorname{div}_\omega(X) \cdot \omega$$

11.1 Introduction. Divergences

So far we have constructed the following volume elements:

- Chapter 2. An integral definition of gaussian volume elements on Banach spaces (2.29), (2.30).
- Chapter 4. A class of ratios of infinite-dimensional determinants that are equal to finite determinants.
- Chapter 7. A mapping from spaces of pointed paths on $\mathbb{R}^D$ to spaces of pointed paths on riemannian manifolds $\mathbb{N}^D$ that makes it possible to use the results of Chapter 2 in non linearspaces.
- Chapter 9. A differential definition of volume elements, in terms of scalar densities, that is useful for integration over finite-dimensional spaces of Grassmann variables (Section 9.4).

In this chapter we exploit the triptych volume elements–divergences–gradients on nonlinear, infinite-dimensional spaces.

Lessons from finite-dimensional spaces

Differential calculus on Banach spaces and differential geometry on Banach manifolds are natural generalizations of their finite-dimensional counterparts. Therefore, we review differential definitions of volume elements on D-dimensional manifolds, which can be generalized to infinite-dimensional spaces.

Top-forms and divergences

Let ω be a D-form on $\mathbb{M}^D$, *i.e.*, a top-form†. Let X be a vector field on $\mathbb{M}^D$. Koszul [1] has introduced the following definition of divergence, henceforth abbreviated to "Div" which generalizes "div":

$$\mathcal{L}_X\,\omega =: \mathrm{Div}_\omega(\mathrm{X}) \cdot \omega\,. \tag{11.1}$$

According to this formula, the divergence of a vector X is the rate of change of a volume element ω under a transformation generated by the vector field X. The rate of change of a volume element is easier to comprehend than the volume element. This situation is reminiscent of other situations. For example it is easier to comprehend, and measure, energy differences than energy. Another example: ratios of infinite-dimensional determinants may have meaning even when each determinant alone is meaningless.

We shall show in (11.20) that this formula applies to the volume element ω_g on a riemannian manifold $(\mathbb{M}, g)$

$$\omega_g(x) := |\det g_{\alpha\beta}(x)|^{1/2}\, dx^1 \wedge \ldots \wedge dx^D\,, \tag{11.2}$$

and in (11.30) that it applies to the volume element ω_Ω on a symplectic manifold $(\mathbb{M}^{2N}, \Omega)$, where $2N = D$ and the symplectic form Ω is a nondegenerate closed 2-form of rank 1,

$$\omega_\Omega(x) = \frac{1}{N!}\,\Omega \wedge \ldots \wedge \Omega \qquad (N \text{ factors}). \tag{11.3}$$

In canonical coordinates (p, q), the symplectic form is

$$\Omega = \sum_\alpha dp_\alpha \wedge dq^\alpha \tag{11.4}$$

and the volume element in strict components (components with ordered indices) is

$$\omega_\Omega(p, q) = dp_1 \wedge dq^1 \wedge \ldots \wedge dp_N \wedge dq^N\,. \tag{11.5}$$

It is surprising that ω_g and ω_Ω obey equations with the same structure, namely

$$\mathcal{L}_X\,\omega = D(X) \cdot \omega\,, \tag{11.6}$$

† The set of top-forms on $\mathbb{M}^D$ is an interesting subset of the set of closed forms on $\mathbb{M}^D$. Top-forms satisfy a property not shared with arbitrary closed forms, namely,

$$\mathcal{L}_{fX}\,\omega = \mathcal{L}_X\, f\omega \qquad \text{where } f \text{ is a scalar function on } \mathbb{M}^D.$$

because riemannian and symplectic geometry are notoriously different [2]. For instance

Riemannian geometry	**Symplectic geometry**
line element	surface area
$\int ds$	$\int \Omega$
geodesics	minimal surface areas
$\mathcal{L}_X\, g = 0 \Rightarrow X$ is Killing	$\mathcal{L}_X\, \Omega = 0 \Rightarrow X$ is Hamiltonian
Killings are few	Hamiltonians are many

Riemannian manifolds $(\mathbb{M}^D, g)$ [3]

We want to show that the equation (11.6) with

$$D(X) := \frac{1}{2}\,\mathrm{Tr}\left(g^{-1}\mathcal{L}_X g\right) \tag{11.7}$$

characterizes the volume element ω_g up to a multiplication constant. Indeed, let

$$\omega(x) = \mu(x) d^d x\,. \tag{11.8}$$

By the Leibnitz rule

$$\mathcal{L}_X(\mu\, d^D\, x) = \mathcal{L}_X(\mu)\, d^D x + \mu\, \mathcal{L}_X(d^D x)\,. \tag{11.9}$$

Since $d^D x$ is a top form on $\mathbb{M}^D$,

$$\mathcal{L}_X(d^D x) = d(i_X\, d^D x) = \partial_\alpha\, X^\alpha\, d^D x = X^\alpha{}_{,\alpha}\; d^D x\,. \tag{11.10}$$

Hence

$$\mathcal{L}_X(\mu\, d^D x) = (X^\alpha \mu_{,\alpha} + \mu\, X^\alpha{}_{,\alpha}\,)\, \mu^{-1} \cdot \mu\, d^D x\,. \tag{11.11}$$

On the other hand

$$(\mathcal{L}_X\, g)_{\alpha\beta} = X^\gamma\, g_{\alpha\beta,\gamma} + g_{\gamma\beta}\, X^\gamma{}_{,\alpha} + g_{\alpha\gamma}\, X^\gamma{}_{,\beta}\ , \tag{11.12}$$

that is $(\mathcal{L}_X g)_{\alpha\beta} = X_{\alpha;\beta} + X_{\beta;\alpha}$. Hence

$$D(X) = \frac{1}{2}\,\mathrm{Tr}\,(g^{-1}\, \mathcal{L}_X\, g) = \frac{1}{2}\, g^{\beta\alpha}\, X^\gamma\, g_{\alpha\beta,\gamma} + X^\alpha{}_{,\alpha}\ . \tag{11.13}$$

The equation

$$\mathcal{L}_X\, \omega = D(X) \cdot \omega \tag{11.14}$$

is satisfied if and only if

$$(X^\gamma \mu_{,\gamma} + \mu\, X^\alpha{}_{,\alpha})\, \mu^{-1} = \frac{1}{2}\, g^{\alpha\beta}\, X^\gamma\, g_{\alpha\beta,\gamma} + X^\alpha{}_{,\alpha}\ , \tag{11.15}$$

i.e., if and only if

$$\begin{aligned} \partial_\gamma \ln \mu = \frac{\mu_{,\gamma}}{\mu} &= \frac{1}{2}\, g^{\alpha\beta}\, g_{\alpha\beta,\gamma} \\ &= \frac{1}{2}\, \partial_\gamma\ \ln |\det g|\,, \end{aligned} \tag{11.16}$$

that is

$$\mu(x) = C\, |\det g(x)|^{1/2}\,, \tag{11.17}$$

where C is a constant. The quantity $D(X) = \frac{1}{2}\,\mathrm{Tr}(g^{-1}\,\mathcal{L}_X\, g)$ is the covariant divergence

$$\mathrm{Div}_g(X) := X^\alpha{}_{;\,\alpha} := X^\alpha{}_{,\alpha} + \Gamma^\beta{}_{\beta\alpha}\, X^\alpha \tag{11.18}$$

$$= X^\alpha{}_{,\alpha} + \frac{1}{2}\, g^{\beta\gamma}\, g_{\gamma\beta,\alpha}\ X^\alpha$$

$$= \frac{1}{2}\,\mathrm{Tr}(g^{-1}\,\mathcal{L}_X\, g)\,. \tag{11.19}$$

In conclusion

$$\mathcal{L}_X\, \omega_g = X^\alpha{}_{;\,\alpha}\, \omega_g = \mathrm{Div}_g\, X \cdot \omega_g\,. \tag{11.20}$$

Remark: If X is a Killing vector field with respect to isometries, then $\mathcal{L}_X\, g = 0$, $\mathcal{L}_X\, \omega_g = 0$, $X_{\alpha;\,\beta} + X_{\beta;\,\alpha} = 0$, $X^\alpha{}_{;\,\alpha} = 0$, and eq. (11.20) is trivially satisfied.

Remark: On $\mathbb{R}^D$ the gaussian volume element $d\Gamma_Q$ has the same structure as ω_g

$$d\Gamma_Q(x) := |\det Q|^{1/2} \exp(-\pi Q(x))\, dx^1 \wedge \ldots \wedge dx^D\,. \tag{11.21}$$

In the infinite-dimensional version (2.19) of (11.21), we have regrouped the terms in order to introduce $\mathcal{D}_{s,Q(x)}$ a dimensionless translation invariant volume element on a Banach space

$$d\Gamma_{s,Q}(x) \overset{\int}{=} \mathcal{D}_{s,Q}(x) \exp\left(-\frac{\pi}{s}\, Q(x)\right)\,. \tag{11.22}$$

Remark: For historical reasons, different notation is used for volume elements. For instance, in the above remark we use different notations when we say "$d\Gamma$ has the same structure as ω". Why not use "$d\omega$"? Integrals were introduced with the notation $\int f(x)dx$. Much later, $f(x)dx$ was identified as a differential one-form, say ω, $\int f(x)dx = \int \omega$. The symbol $d\omega$ is used for the differential of ω, *i.e.*, for a two-form.

Symplectic manifolds $(\mathbb{M}^D)$, Ω, $D = 2N$

We shall show that the symplectic volume element ω_Ω satisfies the structural equation

$$\mathcal{L}_X \, \omega = D(X) \, \omega \tag{11.23}$$

with $D(X) = \mathrm{Div}_\Omega(X)$ defined by (11.29) if and only if†

$$\begin{aligned} \omega_\Omega &= \frac{1}{N!} \, \Omega^{\wedge N} \\ &= |\det \Omega_{\alpha\beta}|^{1/2} \, d^D x \; =: \mathrm{Pf}\,(\Omega_{\alpha\beta}) \, d^D x \end{aligned} \tag{11.24}$$

up to a multiplicative constant. Pf is a pfaffian. We define Ω^{-1} and calculate $\mathrm{Tr}\,(\Omega^{-1} \, \mathcal{L}_X \, \Omega)$.

- The symplectic form Ω defines an isomorphism from the tangent bundle TM to the cotangent bundle T^*M by

  $$\Omega : X \mapsto i_X \, \Omega \, . \tag{11.25}$$

  We can then define

  $$X_\alpha := X^\beta \, \Omega_{\beta\alpha} \, .$$

  The inverse $\Omega^{-1} : T^*M \to TM$ of Ω is given by

  $$X^\alpha = X_\beta \, \Omega^{\beta\alpha}$$

  where

  $$\Omega^{\alpha\beta} \, \Omega_{\beta\gamma} = \delta^\alpha_\gamma \, . \tag{11.26}$$

  Note that in strict components, *i.e.*, with $\Omega = \Omega_{AB} \, dx^A \wedge dx^B$ for $A < B$, X_A is not equal to $X^B \, \Omega_{BA}$.

† The symplectic form Ω is given in coordinates as $\Omega = \frac{1}{2} \Omega_{\alpha\beta} dx^\alpha \wedge dx^\beta$ with $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$.

- We compute

$$(\mathcal{L}_X \Omega)_{\alpha\beta} = X^\gamma \Omega_{\alpha\beta},_\gamma + \Omega_{\gamma\beta} X^\gamma,_\alpha + \Omega_{\alpha\gamma} X^\gamma,_\beta \quad (11.27)$$
$$= X_{\beta,\alpha} - X_{\alpha,\beta}$$

using $d\Omega = 0$, also written as $\Omega_{\beta\gamma},_\alpha + \Omega_{\gamma\alpha},_\beta + \Omega_{\alpha\beta},_\gamma = 0$. Hence

$$(\Omega^{-1} \mathcal{L}_X \Omega)^\gamma_\beta = \Omega^{\gamma\alpha}(X_{\beta,\alpha} - X_{\alpha,\beta}) \quad (11.28)$$

and

$$\frac{1}{2} \operatorname{Tr}(\Omega^{-1} \mathcal{L}_X \Omega) = \Omega^{\gamma\alpha} X_{\gamma,\alpha} \quad (11.29)$$
$$=: \operatorname{Div}_\Omega(X) .$$

- We compute $\mathcal{L}_X \omega_\Omega$. According to Darboux' theorem, there is a coordinate system (x^α) in which the volume form $\omega_\Omega = \frac{1}{N!} \Omega^{\wedge N}$ is

$$\omega_\Omega = dx^1 \wedge \ldots \wedge dx^{2N}$$

and $\Omega = \Omega_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$ with *constant* coefficients $\Omega_{\alpha\beta}$. The inverse matrix $\Omega^{\beta\alpha}$ of $\Omega_{\alpha\beta}$ is also made up of constants, hence $\Omega^{\beta\alpha},_\gamma = 0$. In these coordinates

$$\begin{aligned} \mathcal{L}_X \omega_\Omega &= X^\alpha,_\alpha \omega_\Omega \\ &= (X_\beta \Omega^{\beta\alpha}),_\alpha \omega_\Omega \\ &= (X_{\beta,\alpha} \Omega^{\beta\alpha} + X_\beta \Omega^{\beta\alpha},_\alpha) \omega_\Omega \\ &= X_{\beta,\alpha} \Omega^{\beta\alpha} \omega_\Omega \\ &= \operatorname{Div}_\Omega X \cdot \omega_\Omega , \end{aligned} \quad (11.30)$$

and equation (11.23) is satisfied.

Remark: If X is a hamiltonian vector field, then $\mathcal{L}_X \Omega = 0$, $\mathcal{L}_X \omega_\Omega = 0$, and $\operatorname{Div}_\Omega X = 0$. The basic equation (11.23) is trivially satisfied.

Supervector spaces $\mathbb{R}^{n|\nu}$ [4]

Let x be a point in the supervector space (Section 9.2) $\mathbb{R}^{n|\nu}$ with coordinates

$$x^A = (x^a, \xi^\alpha) \qquad \begin{cases} a \in \{1, \ldots, n\} \\ \alpha \in \{1, \ldots, \nu\} \end{cases} \quad (11.31)$$

where x^a is a bosonic variable, and ξ^α is a fermionic variable. Let X be a vector field on $\mathbb{R}^{n|\nu}$

$$X = X^A \, \partial/\partial x^A \,,$$

and let ω be a top-form. The divergence $\operatorname{Div} X$ defined, up to an invertible "volume density" f, by the Koszul formula

$$\mathcal{L}_X \, \omega = \operatorname{Div} X \cdot \omega \tag{11.32}$$

is [4]

$$\operatorname{Div} X = \frac{1}{f} \, (-1)^{\tilde{A}(1+\tilde{X})} \, \frac{\partial}{\partial X^A} \, (X^A \, f) \tag{11.33}$$

where $\tilde{A}$, $\tilde{X}$ are the parities of A and X respectively.

The general case $\mathcal{L}_X \, \omega = D(X) \cdot \omega$

Two properties of $D(X)$ dictated by properties of $\mathcal{L}_X \, \omega$:

(i)

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \, \mathcal{L}_Y - \mathcal{L}_Y \, \mathcal{L}_X \Leftrightarrow D([X,Y]) = \mathcal{L}_X D(Y) - \mathcal{L}_Y D(X) \tag{11.34}$$

since ω is a top-form

$$\mathcal{L}_X \, \omega = d i_X \, \omega$$

and

(ii)

$$\mathcal{L}_X \, (f \, \omega) = d i_X \, (f \, \omega) = d \, i_{fX} \, \omega = \mathcal{L}_{fX} \, \omega \,.$$

Therefore, when acting on a top-form (project 19.5)

$$\mathcal{L}_{fX} = f \mathcal{L}_X + \mathcal{L}_X(f) \Leftrightarrow D(fX) = f D(X) + X(f) \,. \tag{11.35}$$

11.2 Comparing volume elements

Volume elements and determinants of quadratic forms are offsprings of integration theory. Their values are somewhat elusive because they depend on the choice of coordinates. For instance, consider a quadratic form Q on some finite-dimensional space with coordinates $x^1, \dots, x^D$, namely

$$Q(x) = h_{ab} \, x^a \, x^b \,. \tag{11.36}$$

In another system of coordinates defined by

$$x^a = u^a_\ell \, \bar{x}^\ell \,, \tag{11.37}$$

the quadratic form

$$Q(x) = \bar{h}_{\ell m} \, \bar{x}^\ell \, \bar{x}^m \,, \tag{11.38}$$

introduces a new kernel

$$\bar{h}_{\ell m} = u^a_\ell \, u^b_m \, h_{ab} \,. \tag{11.39}$$

There is no such thing as "the determinant of a quadratic form" because it scales with a change of coordinates

$$\det \bar{h}_{\ell m} = \det h_{ab} \cdot (\det u^a_\ell)^2 \,. \tag{11.40}$$

Ratios of the determinants of these forms, on the other hand, have an intrinsic meaning. Consider two quadratic forms Q_0 and Q_1

$$Q_0 = h^{(0)}_{ab} \, x^a \, x^b \,, \; Q_1 = h^{(1)}_{ab} \, x^a \, x^b \,. \tag{11.41}$$

Denote by $\det(Q_1/Q_0)$ the ratio of the determinants of their kernels

$$\det(Q_1/Q_0) := \det(h^{(1)}_{ab}) / \det(h^{(0)}_{ab}) \,. \tag{11.42}$$

This ratio is invariant under a change of coordinates. Therefore, one expects that ratios of infinite-dimensional determinants can be defined using projective systems [5] or similar techniques.

Ratios of infinite-dimensional determinants

Consider two continuous quadratic forms Q_0 and Q_1 on a Banach space $\mathbb{X}$. Assume Q_0 to be invertible in the following sense. Let D_0 be a continuous, linear map from $\mathbb{X}$ into its dual $\mathbb{X}'$ such that

$$Q_0(x) = \langle D_0 \, x, x \rangle \,, \langle D_0 \, x, y \rangle = \langle D_0 \, y, x \rangle \,. \tag{11.43}$$

The form Q_0 is said to be invertible if the map D_0 is invertible, *i.e.*, if there exists a unique† inverse G of D_0

$$G \circ D_0 = 1\!\!1 \,. \tag{11.44}$$

Let D_1 be defined similarly, but without the invertibility requirement. There exists a unique continuous operator U on $\mathbb{X}$ such that

$$D_1 = D_0 \circ U \,, \tag{11.45}$$

† The inverse G of D_0 is uniquely determined either by restricting $\mathbb{X}$ or by choosing W in (2.2, 2.30).

that is $U = G \circ D_1$. If $U - 1$ is nuclear (see equations (11.47)–(11.63)), then the determinant of U is defined [6]. Let us denote $\det(Q_1/Q_0)$ the determinant of U. It can be calculated as follows. Let V be a finite-dimensional subspace of $\mathbb{X}$, and let $Q_{0,V}$ and $Q_{1,V}$ be the restrictions of Q_0 and Q_1 to V. Assume that V runs through an increasing sequence of subspaces, whose union is dense in $\mathbb{X}$, and that $Q_{0,V}$ is invertible for every V. Then

$$\mathrm{Det}(Q_1/Q_0) = \lim_V \det(Q_{1,V}/Q_{0,V})\,. \tag{11.46}$$

The fundamental trace/determinant relation

The fundamental relation between the trace and the determinant of a matrix A in $\mathbb{R}^D$ is

$$d \log \det A = \mathrm{tr}\,(A^{-1}\, dA) \tag{11.47}$$

also written

$$\det \exp A = \exp \mathrm{tr}\, A\,. \tag{11.48}$$

Indeed, the trace and the determinant of A are invariant under similarity transformations; the matrix A can be made triangular by a similarity transformation, and the above formula is easy to prove for triangular matrices [7, Part I, p. 174].

The fundamental relation (11.47) is valid for operators on nuclear spaces [6] [8].

Let $\mathbb{X}$ be a Banach space, and let Q be an invertible† positive definite quadratic form on $\mathbb{X}$. The quadratic form $Q(x)$ defines a norm on $\mathbb{X}$, namely

$$\|x\|^2 = Q(x)\,, \tag{11.49}$$

and a dual norm $\|x'\|$ on $\mathbb{X}'$, as usual.

According to Grothendieck, an operator T on $\mathbb{X}$ is *nuclear* if it admits a

† A positive-definite continuous quadratic form is not necessarily invertible. For instance, let $\mathbb{X}$ be the space ℓ^2 of sequences $(x_1, x_2, \ldots)$ of real numbers with $\sum_{n=1}^{\infty}(x_n)^2 < \infty$ and define the norm by $\|x\|^2 = \sum_{n=1}^{\infty}(x_n)^2$. We can identify $\mathbb{X}$ with its dual $\mathbb{X}'$, where the scalar product is given by $\sum_{n=1}^{\infty} x'_n x_n$. The quadratic form $Q(x) = \sum_{n=1}^{\infty}(x_n/n)^2$ corresponds to the map $D : \mathbb{X} \to \mathbb{X}'$ that takes $(x_1, x_2, \ldots)$ into $(x_1/1, x_2/2, \ldots)$. The inverse of D does not exist as a map from ℓ^2 into ℓ^2 since the sequence $1, 2, 3, \ldots$ is unbounded.

representation of the form

$$Tx = \sum_{n\geq 0} \langle x'_n, x\rangle\, x_n \tag{11.50}$$

with elements x_n in $\mathbb{X}$ and x'_n in $\mathbb{X}'$ such that $\sum_{n\geq 0} \|x_n\| \cdot \|x'_n\|$ is finite. The greatest lower bound of all such sums $\sum_n \|x_n\| \cdot \|x'_n\|$ is called the *nuclear norm* of T, denoted by $\|T\|_1$. The nuclear operators on $\mathbb{X}$ form a Banach space, denoted by $\mathcal{L}^1(\mathbb{X})$, with norm $\|\cdot\|_1$. On $\mathcal{L}^1(\mathbb{X})$, there exists a continuous linear form, the *trace*, such that

$$\mathrm{Tr}\,(T) = \sum_{n\geq 0} \langle x'_n, x_n\rangle \tag{11.51}$$

for an operator T given by (11.50).

We now introduce a power series in λ, namely:

$$\sum_{p\geq 0} \sigma_p(T)\,\lambda^p := \exp\left(\lambda\,\mathrm{Tr}\,(T) - \frac{\lambda^2}{2}\,\mathrm{Tr}\,(T^2) + \frac{\lambda^3}{3}\,\mathrm{Tr}\,(T^3) - \dots\right). \tag{11.52}$$

From Hadamard's inequality of determinants, we obtain the *basic estimate*

$$|\sigma_p(T)| \leq p^{p/2} \|T\|_1^p / p!\,. \tag{11.53}$$

It follows that the power series $\sum_{p\geq 0} \sigma_p(T)\,\lambda^p$ has an infinite radius of convergence. We can therefore define the determinant as

$$\mathrm{Det}\,(1+T) := \sum_{p\geq 0} \sigma_p(T) \tag{11.54}$$

for any nuclear operator T. Given the definition (11.52) of σ_p, we obtain the more general definition

$$\mathrm{Det}\,(1+\lambda\,T) = \sum_{p\geq 0} \sigma_p(T)\,\lambda^p\,. \tag{11.55}$$

The fundamental property of determinants is, as expected, the *multiplicative rule*

$$\mathrm{Det}(U_1 \circ U_2) = \mathrm{Det}\,(U_1)\,\mathrm{Det}\,(U_2)\,, \tag{11.56}$$

where U_i is of the form $1 + T_i$, where T_i is nuclear (for $i = 1, 2$). From equation (11.56) and the relation $\sigma_1(T) = \mathrm{Tr}\,(T)$, we find a *variation formula* (for nuclear $U - 1$ and δU)

$$\frac{\mathrm{Det}(U+\delta U)}{\mathrm{Det}(U)} = 1 + \mathrm{Tr}\,(U^{-1}\cdot\delta U) + \mathcal{O}(\|\delta U\|_1^2)\,. \tag{11.57}$$

In other words, if $U(\nu)$ is an operator of the form $1+T(\nu)$, where $T(\nu)$ is nuclear and depends smoothly on the parameter ν, we obtain the *derivation formula*

$$\frac{d}{d\nu} \log \operatorname{Det}(U(\nu)) = \operatorname{Tr}\left(U(\nu)^{-1} \frac{d}{d\nu} U(\nu)\right) . \tag{11.58}$$

Remark: For any other norm $\|\cdot\|^1$ defining the topology of $\mathbb{X}$, we have an estimate

$$C^{-1}\|x\| \leq \|x\|^1 \leq C\|x\| , \tag{11.59}$$

for a finite numerical constant $C > 0$. It follows easily from (11.59) that the previous definitions are independent of the choice of the particular norm $\|x\| = Q(x)^{1/2}$ in $\mathbb{X}$.

Explicit formulae

Introduce a basis $(e_n)_{n\geq 1}$ of $\mathbb{X}$ that is orthonormal for the quadratic form Q. Therefore $Q(\Sigma_n t_n e_n) = \Sigma_n t_n^2$. An operator T in $\mathbb{X}$ has a matrix representation (t_{mn}) such that

$$T e_n = \sum_m e_m \cdot t_{mn} . \tag{11.60}$$

Assume that T is nuclear. Then the series $\Sigma_n t_{nn}$ of diagonal terms in the matrix converges absolutely, and the trace $\operatorname{Tr}(T)$ is equal to $\Sigma_n t_{nn}$, as expected. Furthermore, $\sigma_p(T)$ is the sum of the series consisting of the principal minors of order p

$$\sigma_p(T) = \sum_{i_1<\cdots<i_p} \det(t_{i_\alpha,i_\beta})_{\substack{1\leq\alpha\leq p\\ 1\leq\beta\leq p}} . \tag{11.61}$$

The determinant of the operator $U = 1+T$, whose matrix has elements $u_{mn} = \delta_{mn} + t_{mn}$, is a *limit of finite-size determinants*

$$\operatorname{Det}(U) = \lim_{N=\infty} \det(u_{mn})_{\substack{1\leq m\leq N\\ 1\leq n\leq N}} . \tag{11.62}$$

As a special case, suppose that the basic vectors e_n are *eigenvectors* of T

$$T e_n = \lambda_n e_n . \tag{11.63}$$

Then

$$\mathrm{Tr}\,(T) = \sum_n \lambda_n\,, \quad \text{and} \quad \mathrm{Det}\,(1+T) = \prod_n (1+\lambda_n)\,,$$

where both the series and the infinite product converge absolutely.

The nuclear norm $\|T\|_1$ can also be computed as follows: there exists an orthonormal basis (e_n) such that the vectors $T\,e_n$ are mutually orthogonal (for the quadratic form Q). Then $\|T\|_1 = \Sigma_n \|T\,e_n\|$.

Remark: Let T be a continuous linear operator in $\mathbb{X}$. Assume that the series of diagonal terms $\Sigma_n\, t_{nn}$ converges absolutely for *every* orthonormal basis. Then T is nuclear. When T is symmetric and positive, it is enough to assume that this statement holds for *one* given orthonormal basis. Then it holds for all. Counterexamples exist for the case in which T is not symmetric and positive [6].

Comparing divergences

A divergence is a trace. For example†

$$\mathrm{div}_A(X) = \frac{1}{2}\,\mathrm{tr}\,(A^{-1}\,\mathcal{L}_X\,A) \tag{11.64}$$

regardless of whether A is the metric tensor g, or the symplectic form Ω (in both cases, an invertible bilinear form on the tangent vectors). It follows from the fundamental relation between trace and determinant

$$d\log\det A = \mathrm{tr}\,(A^{-1}\,dA)\,, \tag{11.65}$$

that

$$\mathrm{div}_A(X) - \mathrm{div}_B(X) = \mathcal{L}_X \log((\det A)^{1/2}/(\det B)^{1/2})\,. \tag{11.66}$$

Given Koszul's definition of divergence (11.1) in terms of volume elements, namely

$$\mathcal{L}_X\,\omega = \mathrm{Div}_\omega(X)\cdot\omega \tag{11.67}$$

equation (11.66) gives ratios of volume elements in terms of ratios of determinants.

† Notice the analogy between the formulae (11.2) and (11.24) for the volume elements

$$\begin{aligned} \omega_g &= |\det g_{\mu\nu}|^{1/2}\,dx^1\wedge\ldots\wedge dx^D \\ \omega_\Omega &= |\det \Omega_{\alpha\beta}|^{1/2} dx^1\wedge\ldots\wedge dx^D\,, \end{aligned}$$

for the metric $g = g_{\mu\nu}dx^\mu dx^\nu$ and the symplectic form $\Omega = \frac{1}{2}\Omega_{\alpha\beta}dx^\alpha\wedge dx^\beta = \Omega_{\alpha\beta}dx^\alpha\otimes dx^\beta$.

Example: Let Φ be a manifold equipped with two riemannian metrics g_A and g_B. Let

$$P : \Phi \to \Phi \,. \tag{11.68}$$

be a map transforming g_A into g_B. Let ω_A and ω_B be the corresponding volume elements on Φ such that by P,

$$\omega_A \mapsto \omega_B = \rho\, \omega_A \,. \tag{11.69}$$

Then

$$\mathrm{Div}_{\omega_B}(X) - \mathrm{Div}_{\omega_A}(X) = \mathcal{L}_X \, \log \rho \tag{11.70}$$

and

$$\omega_B / \omega_A = (\det g_B)^{1/2} / (\det g_A)^{1/2} \,. \tag{11.71}$$

$\rho(x)$ is the determinant of the jacobian matrix of the map P. The proof is given in Appendix F, equations (F.23–F.25). It can also be done by an explicit calculation of the change of coordinates defined by P.

11.3 Integration by parts

Integration theory is unthinkable without integration by parts, not only because it is a useful technique but also because it has its roots in the fundamental requirements of definite integrals, namely (see Section 9.3):

$$DI = 0 \,, \qquad ID = 0 \tag{11.72}$$

where D is a derivative operator and I an integral operator. We refer the reader to Chapter 9 for the meaning and some of the uses of the fundamental requirements (11.72).

Already in the early years of path integrals Feynman was promoting integration by parts in functional integration, jokingly and seriously. Here we use integration by parts for relating divergences and gradients; this relation completes the triptych "volume elements - divergences - gradients."

Divergences and gradients

In $\mathbb{R}^3$ the concept "gradient" is intuitive and easy to define: it measures the steepness of a climb. Mathematically, the gradient (or nabla $\nabla \equiv \nabla_{g^{-1}}$ is a *contravariant* vector:

$$\nabla^i := g^{ij} \, \frac{\partial}{\partial x^j} \,. \tag{11.73}$$

The divergence of a vector is its scalar product with the gradient vector:

$$(\nabla|V)_g = g_{ij}\,\nabla^i\,V^j = g_{ij}\,g^{ik}\,\frac{\partial}{\partial x^k}\,V^j = V^j{}_{,j}\ . \tag{11.74}$$

Divergence and gradient are related by integration by parts. Indeed

$$\begin{aligned}(V|\nabla f)_g\,(x) &= g_{ij}\,V^i\,g^{jk}\,\partial f/\partial x^k \;=\; V^i\,f_{,i}\,(x)\\ &= \mathcal{L}_V\,f(x) \qquad (11.75)\\ (\operatorname{div}V|f)\,(x) &= V^i{}_{,i}\;f(x) \qquad (11.76)\end{aligned}$$

and

$$\int d^3x\big((V|\nabla f)_g(x) + (\operatorname{div}V|f)(x)\big) = \int d^3x\frac{\partial}{\partial x^k}\left(fV^k\right)\,. \tag{11.77}$$

Assume that fV vanishes on the boundary of the domain of integration, and integrate the right-hand side by parts. The right-hand side vanishes because the volume element d^Dx is invariant under translation, hence

$$\int_{\mathbb{R}^D} d^Dx(V|\nabla f)(x) = -\int_{\mathbb{R}^D} d^Dx(\operatorname{div}V|f)(x)\,; \tag{11.78}$$

the functional scalar products satisfy modulo a sign the adjoint relation

$$(V|\nabla f) = -(\operatorname{div}V|f)\,. \tag{11.79}$$

The generalizations of (11.78) and (11.74) to spaces other than $\mathbb{R}^3$ face two difficulties:

- Contrary to d^3x, generic volume elements are not invariant under translation.
- Traces in infinite-dimensional spaces are notoriously sources of problems. Two examples:
  The infinite-dimensional matrix

  $$M = \operatorname{diag}(1, 1/2, 1/3, \ldots)$$

  has the good properties of a Hilbert-Schmidt operator (the sum of the squares of its elements is finite), but its trace is infinite.
  Closed loops in Quantum Field Theory introduce traces; they have to be set apart by one technique or another. For instance Wick products (see Appendix D) serve this purpose among others.

The definition of divergence provided by the Koszul formula (11.1)

$$\mathcal{L}_V\omega =: \operatorname{Div}_\omega V \tag{11.80}$$

bypasses both difficulties mentioned above: the definition (11.80) is not restricted to translation invariant volume elements and it is meaningful in infinite-dimensional spaces.

The definition (11.73) of the gradient vector as a contravariant vector requires the existence of a metric tensor g. Let $\mathcal{A}^p$ be the space of p-forms on $\mathbb{M}^D$ and $\mathcal{X}$ the space of contravariant vector fields on $\mathbb{M}^D$:

$$\mathcal{A}^0 \underset{\longleftarrow}{\xrightarrow{d}} \mathcal{A}^1 \underset{\xleftarrow[g]{}}{\xrightarrow{g^{-1}}} \mathcal{X}\,, \tag{11.81}$$

$$\nabla_{g^{-1}} = g^{-1} \circ d\,. \tag{11.82}$$

On the other hand the scalar product (11.75)

$$(V|\nabla_{g^{-1}} f)_g = \mathcal{L}_V f$$

is simply the Lie derivative with respect to V of the scalar function f, hence it is independent of the metric and easy to generalize to scalar functionals.

From the definitions (11.80) and (11.82), one sees that

- the volume element defines the divergence
- the metric tensor defines the gradient because it provides a canonical isomorphism between covariant and contravariant vectors.

If it happens that the volume element ω_g is defined by the metric tensor g, then one uses the explicit formulae (11.2) and (11.18).

With the Koszul definition (11.80) and the property (11.75), one can derive the grad/div relationship as follows

$$\begin{aligned} \int_{\mathbb{M}^D} (V|\nabla_{g^{-1}} f)_g(x) \cdot \omega &= \int_{\mathbb{M}^D} \mathcal{L}_V f(x) \cdot \omega \qquad (11.83) \\ &\qquad \text{by the property (11.75)} \\ &= -\int_{\mathbb{M}^D} f(x) \cdot \mathcal{L}_V \omega \qquad \text{by integration by parts} \\ &= -\int_{\mathbb{M}^D} f(x) \mathrm{Div}_\omega(V) \cdot \omega \qquad \text{by Koszul formula} \qquad (11.84) \end{aligned}$$

hence finally

$$(V|\nabla f)_\omega = -(\mathrm{Div}_\omega V|f)_\omega\,. \tag{11.85}$$

Divergence and gradient in function spaces

The basic ingredients in constructing the grad/div relationship are:
Lie derivatives, Koszul formula, scalar products of functions and scalar products of contravariant vectors. They can be generalized in function spaces $\mathbb{X}$, as follows:

- Lie derivatives. As usual on function spaces, one introduces a one-parameter family of paths $\{x_\lambda\}_\lambda$, $\lambda \in [0,1]$.

$$x_\lambda : \mathbb{T} \longrightarrow \mathbb{M}^D\,, \qquad x_\lambda(t) \equiv x(\lambda, t) \tag{11.86}$$

$$\dot{x}(\lambda, t) \;:=\; \frac{d}{dt} x(\lambda, t) \tag{11.87}$$

$$x'(\lambda, t) \;:=\; \frac{d}{d\lambda} x(\lambda, t)\,. \tag{11.88}$$

  For $\lambda = 0$, x_0 is abbreviated to x and

$$x'(\lambda, t)\big|_{\lambda=0} =: V_x(t)\,. \tag{11.89}$$

  V_x is a vector at $x \in \mathbb{X}$, tangent to the one-parameter family $\{x_\lambda\}$.

  Let F be a scalar functional on the function space $\mathbb{X}$, then

$$\mathcal{L}_V F(x) = \frac{d}{d\lambda} F(x_\lambda)\bigg|_{\lambda=0}\,, \quad x \in \mathbb{X}\,. \tag{11.90}$$

- The Koszul formula (11.1) defines the divergence of a vector field V as the rate of change of a volume element ω under the group of transformations generated by the vector field V:

$$\mathcal{L}_V \omega =: \mathrm{Div}_\omega(V) \cdot \omega\,.$$

  We adopt the Koszul formula as the definition of divergence in function space.
- Scalar products of real valued functionals

$$(F_1|F_2)_\omega = \int_{\mathbb{X}} \omega F_1(x) F_2(x)\,. \tag{11.91}$$

- Scalar products of contravariant vectors. In the finite-dimensional case, such a scalar product requires the existence of a metric tensor defining a canonical isomorphism between the dual spaces $\mathbb{R}^D$ and $\mathbb{R}_D$. In the infinite-dimensional case, a gaussian volume element on $\mathbb{X}$,

$$d\Gamma_Q(x) \overset{\int}{=} \mathcal{D}_Q(x) \cdot \exp\Big(-\frac{\pi}{s} Q(x)\Big) =: \omega_Q \tag{11.92}$$

defined by (2.30), does provide a canonical isomorphism between $\mathbb{X}$ and its dual $\mathbb{X}'$, namely the pair (D, G) defined by Q and W, respectively,

$$Q(x) = \langle Dx, x \rangle \quad \text{and} \quad W(x') = \langle x', Gx' \rangle\,. \tag{11.93}$$

$$\mathbb{X}' \underset{D}{\overset{G}{\rightleftarrows}} \mathbb{X}\,. \qquad GD = \mathbb{1}\,,\ DG = \mathbb{1}\,. \tag{11.94}$$

The role of the pair (G, D) as defining a canonical isomorphism between $\mathbb{X}$ and $\mathbb{X}'$ is interesting but is not necessary for generalizing the grad/div relation (11.85): indeed the scalar product of a contravariant vector V with the gradient of a scalar function F does not depend on the metric tensor (11.75); it is simply the Lie derivative of F in the V-direction.

- The functional grad/div relation
  At $x \in \mathbb{X}$,

$$(V|\nabla F)(x) = \mathcal{L}_V F(x)\,, \quad x \in \mathbb{X}\,; \tag{11.95}$$

upon integration on $\mathbb{X}$ with respect to the gaussian (11.92)

$$\begin{aligned} \int_{\mathbb{X}} \omega_Q \mathcal{L}_V F(x) &= -\int_{\mathbb{X}} \mathcal{L}_V \omega_Q \cdot F(x) \\ &= -\int_{\mathbb{X}} \mathrm{Div}_{\omega_Q}(V) \cdot \omega_Q F(x)\,. \end{aligned} \tag{11.96}$$

The functional grad/div relation is the global scalar product

$$(V|\nabla F)_{\omega_Q} = -\left(\mathrm{Div}_{\omega_Q}(V)|F\right)_{\omega_Q}\,. \tag{11.97}$$

Translation invariant symbols

The symbol "dx" for $x \in \mathbb{R}$ is translation invariant

$$d(x + a) = dx \qquad \text{for } a\text{, a fixed point in } \mathbb{R}. \tag{11.98}$$

Equivalently, one can characterize the translation invariance of dx by the following integral

$$\int_{\mathbb{R}} dx\, \frac{d}{dx} f(x) = 0\,, \tag{11.99}$$

provided f is a function vanishing on the boundary of the domain of integration. In order to generalize (11.99) on $\mathbb{R}^D$ introduce a vector field V in

$\mathbb{R}^D$; eq. (11.99) becomes

$$\int d^D x \; \partial_\alpha V^\alpha(x) = 0\,, \qquad \partial_\alpha = \partial/\partial x^\alpha$$

and

$$\int d^D x \; \partial_\alpha (f V^\alpha)(x) = \int d^D x \; \left(\partial_\alpha f \cdot V^\alpha + f \partial_\alpha V^\alpha\right) = 0\,.$$

Hence

$$\int d^D x \; (\mathcal{L}_V f + f \operatorname{div} V) = 0\,.$$

This calculation reproduces (11.78) in a more familiar notation.

Let $\mathcal{F}$ be a class of functionals F on the space $\mathbb{X}$ of functions x. We shall formally generalize the characterization of translation invariant symbols $\mathcal{D}$ on linear spaces $\mathbb{X}$, namely

$$\mathcal{D}(x + x_0) = \mathcal{D}x \text{ for } x_0 \text{ a fixed function,} \tag{11.100}$$

by the following requirement on $\mathcal{D}x$

$$\int_{\mathbb{X}} \mathcal{D}x \, \frac{\delta F}{\delta x(t)} = 0\,. \tag{11.101}$$

The characterization (11.101) is meaningful only for F in a class $\mathcal{F}$ such that

$$\int_{\mathbb{X}} \frac{\delta}{\delta x(t)} (\mathcal{D}x \, F) = 0\,. \tag{11.102}$$

Although work remains to be done for an operational definition of $\mathcal{F}$, we note that it is coherent with the fundamental requirement (11.75), $ID = 0$. We shall assume that F satisfies (11.102) and exploit the triptych

gradient – divergence – volume element

linked by the grad/div relation (11.97) and the Koszul equation (11.1).

The Koszul formula applied to the translation invariant symbol $\mathcal{D}x$ is a straightforward generalization of (11.10). Namely

$$\mathcal{L}_V \, d^D x = d(i_V \, d^D x) = V^i{}_{,i} \; d^D x = \operatorname{div}(V) d^D x \tag{11.103}$$

generalizes to

$$\mathcal{L}_V \, \mathcal{D}x = \int_{\mathbb{T}} dt \, \frac{\delta V(x,t)}{\delta x(t)} \mathcal{D}x =: \operatorname{div}(V) \mathcal{D}x\,, \tag{11.104}$$

where the Lie derivative $\mathcal{L}_V$ is defined by (11.90), *i.e.*, by a one-parameter family of paths $\{x_\lambda\}$. Recall that deriving the coordinate expression of $\mathrm{div}(V)$ by computing $\mathcal{L}_V\, d^D x$ is not a trivial exercise on D-forms. We take for granted its naive generalization

$$\mathrm{div}(V) = \int_{\mathbb{T}} dt \frac{\delta V(x,t)}{\delta x(t)} \tag{11.105}$$

and bypass the elusive concept "top-form on $\mathbb{X}$."

Example: The divergence $\mathrm{Div}_Q(V)$ of the gaussian volume element,

$$\omega_Q(x) := d\Gamma_Q(x) := \exp\left(-\frac{\pi}{s}Q(x)\right) \mathcal{D}_Q x\,, \tag{11.106}$$

is given by the Koszul formula

$$\begin{aligned} \mathcal{L}_V\, \omega_Q &= \mathcal{L}_V \left(\exp\left(-\frac{\pi}{s}Q(x)\right)\right) \mathcal{D}_Q x + \exp\left(-\frac{\pi}{s}Q(x)\right) \mathcal{L}_V\, \mathcal{D}_Q x \\ &=: \mathrm{Div}_Q(V)\omega_Q\,. \end{aligned}$$

The gaussian divergence is the sum of the "naive" divergence (11.105) and $-\frac{\pi}{s}\mathcal{L}_V\, Q(x)$, that is with the notation (11.93)

$$\mathrm{Div}_Q(V) = -\frac{2\pi}{s}\langle Dx, V(x)\rangle + \mathrm{div}(V)\,. \tag{11.107}$$

We recover a well-known result of Malliavin calculus [9].

Remark: Two translation invariant symbols are frequently used: The dimensionless volume elements $\mathcal{D}_Q x$ defined by (2.30) and the formal product

$$d[x] = \prod_t dx(t)\,. \tag{11.108}$$

In D-dimensions,

$$\mathcal{D}_Q(x) = |\det Q|^{1/2} dx^1 \dots dx^D\,; \tag{11.109}$$

in function spaces, we write symbolically

$$\mathcal{D}_Q(x) = |\mathrm{Det}\, Q|^{1/2} d[x]\,. \tag{11.110}$$

Application: The Schwinger variational principle

In the equation (11.101) which defines the translation invariant symbol $\mathcal{D}x$, replace $F(x)$ by

$$F(x)\mu(x)\exp\left(\frac{i}{\hbar}S_{\mathrm{cl}}(x)\right). \tag{11.111}$$

Then

$$\int_{\mathbb{X}} \mathcal{D}x \left(\frac{\delta F}{\delta x(t)} + F\frac{\delta \log \mu(x)}{\delta x(t)} + \frac{i}{\hbar}F\frac{\delta S_{\mathrm{cl}}}{\delta x(t)}\right)\mu(x)\exp\frac{i}{\hbar}S_{\mathrm{cl}}(x) = 0\,. \tag{11.112}$$

Translated in the operator formalism (6.80), this equation gives, for $\mathbb{X} = \mathcal{P}_{a,b}$,

$$\left\langle b \left| \widehat{\frac{\delta F}{\delta x(t)}} + F\widehat{\frac{\delta \log \mu}{\delta x(t)}} + F\frac{i}{\hbar}\widehat{\frac{\delta S_{\mathrm{cl}}}{\delta x(t)}} \right| a \right\rangle = 0 \tag{11.113}$$

where $\widehat{O}$ is the time-ordered product of operators corresponding to the functional O. For $F = 1$, equation (11.113)

$$\left\langle b \left| \widehat{\frac{\delta \log \mu}{\delta x(t)}} + \frac{i}{\hbar}\widehat{\frac{\delta S_{\mathrm{cl}}}{\delta x(t)}} \right| a \right\rangle = 0 \tag{11.114}$$

gives, for fixed initial and final states, the same quantum dynamics as the Schwinger variational principle

$$0 = \delta\langle b|a\rangle = \left\langle b \left| \frac{i}{\hbar}\delta S_{\mathrm{q}} \right| a \right\rangle \tag{11.115}$$

where the quantum action functional

$$S_{\mathrm{q}} = S_{\mathrm{cl}} + \frac{\hbar}{i}\log\mu$$

corresponds to the Dirac quantum action function [10] up to order $\hbar^2$.

Remark: In Chapter 6 we derived the Schwinger variational principle by varying the potential, *i.e.* by varying the action functional.

Group invariant symbols

LaChapelle [11] has investigated the use of integration by parts when there is a group action on the domain of integration other than translation. We have generalized [6] [12] the gaussian volume element defined in Section 2.3

on Banach spaces as follows. Let Θ and Z be two given continuous bounded functionals defined on a Banach space $\mathbb{X}$ and its dual $\mathbb{X}'$, respectively by

$$\Theta : \mathbb{X} \times \mathbb{X}' \to \mathbb{C}\,, \quad Z : \mathbb{X}' \to \mathbb{C}\,. \tag{11.116}$$

Define a volume element $\mathcal{D}_{\Theta,Z}$ by

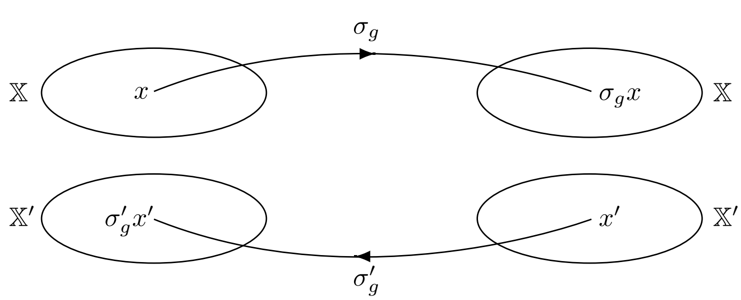
$$\int_{\mathbb{X}} \mathcal{D}_{\Theta,Z}(x)\Theta(x,x') = Z(x')\,. \tag{11.117}$$

There is a class $\mathcal{F}$ of functionals on $\mathbb{X}$ integrable with respect to $\mathcal{D}_{\Theta,Z}$ defined [12] as follows:

$$F_\mu \in \mathcal{F} \;\Leftrightarrow\; F_\mu(x) = \int_{\mathbb{X}'} \Theta(x,x')d\mu(x') \tag{11.118}$$

where μ is a bounded measure on $\mathbb{X}'$. Although μ is not necessarily defined by $F = F_\mu$, it can be proved that $\int_{\mathbb{X}} F(x)\mathcal{D}_{\Theta,Z}(x)$ is defined. Moreover it is not necessary to identify μ in order to compute $\int_{\mathbb{X}} F(x)\mathcal{D}_{\Theta,Z}(x)$. The class $\mathcal{F}$ generalizes the class chosen by Albeverio and Høegh-Krohn [13]. Let σ_g be a group action on $\mathbb{X}$ and σ'_g be a group action on $\mathbb{X}'$ such that

$$\langle \sigma'_g x', x\rangle = \langle x', \sigma_g x\rangle\,. \tag{11.119}$$



The volume element $\mathcal{D}_{\Theta,Z}$ defined by (11.117) is invariant under the group action if μ and Θ are invariant, namely if

$$\begin{aligned} Z(\sigma'_g x') &= Z(x') \\ \Theta(\sigma_g x, x') &= \Theta(x, \sigma'_g x')\,. \end{aligned} \tag{11.120}$$

Then

$$\int_{\mathbb{X}} F_\mu(\sigma_g x)\mathcal{D}_{\Theta,Z}(x) = \int_{\mathbb{X}} F_\mu(x)\mathcal{D}_{\Theta,Z}(x)\,. \tag{11.121}$$

Let V be an infinitesimal generator of the group of transformations $\{\sigma_g\}$,

then

$$\int_{\mathbb{X}} \mathcal{L}_V F_\mu(x) \cdot \mathcal{D}_{\Theta,Z}(x) = -\int_{\mathbb{X}} F_\mu(x) \cdot \mathcal{L}_V \mathcal{D}_{\Theta,Z}(x) = 0\,. \tag{11.122}$$

This equation is to the group invariant symbol $\mathcal{D}_{\Theta,Z}$ what the following equation is to the translation invariant symbol $\mathcal{D}_Q$

$$\int_{\mathbb{X}} \frac{\delta F}{\delta x(t)} \mathcal{D}_Q(x) = -\int F \frac{\delta}{\delta x(t)} \mathcal{D}_Q(x) = 0\,. \tag{11.123}$$

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