

On the complete integrability and linearization of certain second order nonlinear ordinary differential equations

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A method of finding general solutions of second-order nonlinear ordinary differential equations by extending the Prolle-Singer (PS) method is briefly discussed. We explore integrating factors, integrals of motion and the general solution associated with several dynamical systems discussed in the current literature by employing our modifications and extensions of the PS method. In addition to the above we introduce a novel way of deriving linearizing transformations from the first integrals to linearize the second order nonlinear ordinary differential equations to free particle equation. We illustrate the theory with several potentially important examples and show that our procedure is widely applicable.

Keywords: Integrability, Integrating factor, Linearization, Equivalence problem

1. Introduction

Solving nonlinear ordinary differential equations (ODEs) is one of the classical but potentially important areas of research in the theory of dynamical systems (Arnold 1978; Jose & Saletan 2002; Wiggins 2003). Indeed the last century has witnessed considerable amount of research activity in this field. Progress has been made broadly in two different ways: one is through geometrical analysis and the other is through analytical studies. The modern geometrical theory was born with Poincaré and developed later vigorously by Arnold, Moser, Birkhoff and others (Percival & Richards 1982; Guckenheimer & Holmes 1983; Wiggins 2003). In parallel with the geometrical theories various analytical methods have also been devised to tackle nonlinear ODEs. The ideas developed by Kovalevskaya, Painlevé and his coworkers have been used to integrate a class of nonlinear ODEs and obtain their underlying solutions (Ince 1956). As a consequence of these studies, nonlinear dynamical systems are broadly classified into two categories, namely, (i) integrable and (ii) nonintegrable systems. Indeed one of the important current problems in nonlinear dynamics is to identify integrable dynamical systems (Lakshmanan & Rajasekar 2003). Of course, these methods have close connection with the group theoretical approach introduced by Sophus Lie in the nineteenth century and subsequently extended by Cartan and Tresse, to integrate ordinary and partial differential equations (see for example Olver (1995); Bluman & Anco (2002)).

In order to identify such integrable dynamical systems different techniques have been proposed, including Painlevé analysis (Conte 1999), Lie symmetry analysis

(Bluman & Anco 2002) and direct methods of finding involutive integrals of motion (Hietarinta 1987). Each method has its own advantages and disadvantages. For a detailed discussion about the underlying theory of each method, their limitations and applications we refer to Lakshmanan & Rajasekar (2003). Also certain nonlinear ODEs can be solved by transforming them to linear ODEs whose solutions are known. In fact, linearization of given nonlinear ODEs is one of the classical problems in the theory of ODEs whose origin dates back to Cartan. For the recent progress in this direction we refer Olver (1995).

In this direction sometime ago Prelle & Singer (1983) have proposed a procedure for solving first order ODEs that presents the solution in terms of elementary functions if such a solution exists. The attractiveness of the PS method is that if the given system of first order ODEs has a solution in terms of elementary functions then the method guarantees that this solution will be found. Very recently Duarte *et al.* (2001) modified the technique developed by Prelle & Singer (1983) and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exists at least one elementary first integral $I(t, x, \dot{x})$ whose derivatives are all rational functions of t , x and \dot{x} . For a class of systems these authors (Duarte *et al.* 2001) have deduced first integrals and in some cases for the first time through their procedure.

In this paper we show that the theory of Duarte *et al.* (2001) can be extended in different directions to isolate even two independent integrals of motion and obtain solutions explicitly. In the earlier study it has been shown that the theory can be used to derive only one integral. In this work we extend their theory and deduce general solution from the first integral. Our examples include those considered in their paper and certain important equations discussed in the recent literature whose solutions are not known. Our study was motivated by two reasons. Firstly, it is to show that apart from finding first integrals one can also deduce general solutions in a straightforward and simple manner. Here the method we propose is not confined to the PS method alone but can be treated as a *general one*, that is, suppose one has a first integral for a given second order ODE then our method provides the general solution in an algorithmic way at least for a class of equations. The reason for merging our procedure with PS method rather than any other method is due to the following facts. (1) For a given problem if the solution exists it has been conjectured that the PS method guarantees to provide first integrals. (2) The PS method not only gives the first integrals but also the underlying integrating factors, that is, multiplying the equation with these functions we can rewrite the equation as a perfect differentiable function which upon integration gives first integrals in a separate way. (3) The PS method can be used to solve nonlinear as well as linear second order ODEs. As the PS method is based on the equations of motion rather than Lagrangian or Hamiltonian the analysis is applicable to deal with both Hamiltonian and non-Hamiltonian systems.

Our second reason is to bring out a novel and straightforward way to construct linearizing transformations. Particularly we demonstrate that using our procedure one can also explore linearizing transformations in a simple and straightforward manner. The latter can be used to transform the given second order nonlinear ODEs to a linear one, in particular, to the free particle equation. As we illustrate below these transformations can be deduced from the first integral itself which is totally a *new technique* in the current literature. In a nutshell, once a first integral

is known then our procedure, at least for a class of problems, not only gives the general solutions but also provides linearizing transformations. The ideas proposed here can be applied to coupled system of second order ODEs as well as higher order ODEs, which will be presented separately.

The paper is organized as follows. In the following section we briefly describe the Prelle-Singer method applicable for second order ODEs and indicate certain new features in finding the integrals of motion. In §3, we have extended the theory in three different directions which indicates the novelty of the approach. The first significant application is that the second integral can be deduced straightforwardly from the method itself in many cases. The second one is to deduce the general solution from the first integral. Finally we propose a method to identify linearizing transformations. We emphasize the validity of the theory with several illustrative examples arising in different areas of physics in §4. In §5, we demonstrate the method of identifying linearizing transformation with three examples including the one studied in the recent literature. We present our conclusions in §6.

2. Prelle-Singer method for second order ODEs

In this section, we briefly discuss the theory introduced by Duarte *et al.* (2001) for second order ODEs and extend it suitably such that the general solutions can be deduced from the modifications. Let us consider the second order ODEs of the form

$$\ddot{x} = \frac{P}{Q}, \quad P, Q \in \mathbb{C}[t, x, \dot{x}], \quad (2.1)$$

where over dot denotes differentiation with respect to time and P and Q are polynomials in t , x and \dot{x} with coefficients in the field of complex numbers. Let us assume that the ODE (2.1) admits a first integral $I(t, x, \dot{x}) = C$, with C constant on the solutions, so that the total differential gives

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0, \quad (2.2)$$

where the subscript denotes partial differentiation with respect to that variable. Rewriting equation (2.1) in the form $\frac{P}{Q} dt - d\dot{x} = 0$ and adding a null term $S(t, x, \dot{x})\dot{x} dt - S(t, x, \dot{x})dx$ to the latter, we obtain that on the solutions the 1-form

$$\left(\frac{P}{Q} + S\dot{x}\right)dt - Sdx - d\dot{x} = 0. \quad (2.3)$$

Hence, on the solutions, the 1-forms (2.2) and (2.3) must be proportional. Multiplying (2.3) by the factor $R(t, x, \dot{x})$ which acts as the integrating factors for (2.3), we have on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0, \quad (2.4)$$

where $\phi \equiv P/Q$. Comparing equations (2.2) with (2.4) we have, on the solutions, the relations

$$\begin{aligned} I_t &= R(\phi + \dot{x}S), \\ I_x &= -RS, \\ I_{\dot{x}} &= -R. \end{aligned} \quad (2.5)$$

Then the compatibility conditions, $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, between the equations (2.5), require that

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \quad (2.6)$$

$$D[R] = -R(S + \phi_{\dot{x}}), \quad (2.7)$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \quad (2.8)$$

where

$$D = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial \dot{x}}.$$

Equations (2.6)-(2.8) can be solved in the following way. Substituting the given expression of ϕ into (2.6) and solving it one can obtain an expression for S . Once S is known then equation (2.7) becomes the determining equation for the function R . Solving the latter one can get an explicit form for R . Now the functions R and S have to satisfy an extra constraint, that is, equation (2.8). Once a compatible solution satisfying all the three equations have been found then the functions R and S fix the integral of motion $I(t, x, \dot{x})$ by the relation

$$\begin{aligned} I(t, x, \dot{x}) = & \int R(\phi + \dot{x}S)dt - \int \left(RS + \frac{d}{dx} \int R(\phi + \dot{x}S)dt \right) dx \\ & - \int \left\{ R + \frac{d}{d\dot{x}} \left[\int R(\phi + \dot{x}S)dt - \int \left(RS + \frac{d}{dx} \int R(\phi + \dot{x}S)dt \right) dx \right] \right\} d\dot{x}. \end{aligned} \quad (2.9)$$

Equation (2.9) can be derived straightforwardly by integrating the equations (2.5). Note that for every independent set (S, R) , equation (2.9) defines an integral.

Thus two independent sets, (S_i, R_i) , $i = 1, 2$, provide us two independent integrals of motion through the relation (2.9) which guarantees the integrability of equation (2.1). We noticed that since we are solving equations (2.6)-(2.7) first and check the compatibility of this solution with equation (2.8), one often meets the situation that all the solutions which satisfy equations (2.6)-(2.7) need not satisfy the constraint (2.8) since equations (2.6)-(2.8) constitute an overdetermined system for the unknowns R and S . In fact, for a class of problems one often gets a set (S_1, R_1) which satisfies all the three equations (2.6)-(2.8) and another set (S_2, R_2) which satisfies only the first two equations and not the third, namely, (2.8). In this situation, we find the interesting fact that one can use the first integral, derived from the set (S_1, R_1) , and deduce the second compatible solution (S_2, \hat{R}_2) . For example, let the set (S_2, R_2) be a solution of equations (2.6)-(2.7) and not of the constraint equation (2.8). After examining several examples we find that one can make the set (S_2, R_2) compatible by modifying the form of R_2 as

$$\hat{R}_2 = F(t, x, \dot{x})R_2, \quad (2.10)$$

where \hat{R}_2 satisfies equation (2.7), so that we have

$$(F_t + \dot{x}F_x + \phi F_{\dot{x}})R_2 + FD[R_2] = -FR_2(S_2 + \phi_{\dot{x}}). \quad (2.11)$$

Further, if F is a constant of motion (or a function of it), then the first term on the left hand side vanishes and one gets the same equation (2.7) for R_2 provided F is

non-zero. In other words, whenever F is a constant of motion or a function of it then the solution of equation (2.7) may provide only a factor of the complete solution \tilde{R}_2 without the factor F in equations (2.10). This general form of \tilde{R}_2 with S_2 can now form a complete solution to the equations (2.6)-(2.8). In a nutshell we describe the procedure as follows. First we determine S and R from equations (2.6)-(2.7). If the set (S, R) satisfies equation (2.8) then we take it as a compatible solution and proceed to construct the associated integral of motion. On the other hand if it does not satisfy (2.8) then we assume the modified form $\tilde{R}_2 = F(I_1)R_2$, where I_1 is the first integral which has already been derived through a compatible solution, and find the explicit form of $F(I_1)$ from equation (2.8), which in turn fixes the compatible solution (S_2, \tilde{R}_2) . This set (S_2, \tilde{R}_2) can be utilized to derive the second integral.

3. Generalization

(a) Identifying a second integral of motion

Duarte *et al.* (2001) have considered certain physically important systems and constructed first integrals. Further, they mentioned that applying the original PS algorithm to these first integrals (by treating them as first order ODEs) one can deduce the general solution. An interesting observation we make here is that there is no need to invoke the original PS procedure in order to deduce the general solution. In fact, as we show below, the general solution can be derived in a self-contained way. As the motivation of Duarte *et al.* (2001) was to construct only the first integral they reported only one set of solution (S, R) for the equations (2.6)-(2.8). However, we have observed that an additional independent set of solution, namely, (S_2, R_2) , of equations (2.6)-(2.8) may lead to another integral of motion, I_2 , and if the latter is an independent function of I_1 then one can write down the general solution for the given problem from these two integrals alone straightforwardly. Now the question is whether one will be able to find a second pair of solution for the system (2.6)-(2.8) and construct I_2 through the relation (2.9). After investigating several examples we observed the following. (i) For a class of equations, including that of harmonic oscillator, equation coming from general relativity and the generalized modified Emden equation with constant external forcing one can construct a second pair of solution (S_2, R_2) and deduce I_2 through the relation (2.9) straightforwardly. We call this class as Type I. (ii) For another class of equations we can find explicitly (S_2, R_2) from (2.6)-(2.8) but one is unable to integrate equation (2.9) exactly and obtain the second integral I_2 explicitly. We call this class as Type II. The examples included in this category are Helmholtz oscillator and Duffing oscillator. For this class of equations we identify an alternate way to derive the second integration constant. (iii) Besides the above there exists another category in which the systems do not even admit a second pair (S_2, R_2) of solution in simple rational forms for the equations (2.6)-(2.8) and we call this category as Type III. An example under this class is the Duffing - van der Pol oscillator which is one of the prototype examples for the study of nonlinear dynamics in many branches of science. For this class of equations also we identify an alternate way to obtain the second integral.

(b) *Method of deriving general solution*

To overcome the difficulties in constructing the second constant in Types II and III we propose the following procedure. As our aim is to derive the general solution for the given problem, we split the functional form of the first integral I into two terms such that one involves all the variables (t, x, \dot{x}) while the other excludes \dot{x} , that is,

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \quad (3.1)$$

Now let us split the function F_1 further in terms of two functions such that F_1 itself is a function of the product of the two functions, say, a perfect differentiable function $\frac{d}{dt}G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is,

$$I = F_1 \left(\frac{1}{G_2(t, x, \dot{x})} \frac{d}{dt}G_1(t, x) \right) + F_2(G_1(t, x)). \quad (3.2)$$

We note that while rewriting equation (3.1) in the form (3.2), we demand the function $F_2(t, x)$ in (3.1) automatically to be a function of $G_1(t, x)$. The reason for making such a specific decomposition is that in this case one can rewrite equation (3.2) as a simple first order ODE for the variable G_1 (see equation (3.4) below). Actually, we realized originally such a possibility for the case of the integrable force-free Duffing-van der Pol oscillator equation (Chandrasekar *et al.* 2004), which has now been generalized in the present case. Now identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t, x), \quad z = \int_0^t G_2(t', x, \dot{x}) dt', \quad (3.3)$$

one indeed obtains an explicit transformation to remove the time dependent part in the first integral (2.9). We note here that the integration on the right hand side of (3.3) leading to z can be performed provided the function G_2 is an exact derivative of t , that is, $G_2 = \frac{d}{dt}z(t, x) = \dot{x}z_x + z_t$, so that z turns out to be a function t and x alone. In terms of the new variables, equation (3.2) can be modified to the form

$$I = F_1 \left(\frac{dw}{dz} \right) + F_2(w). \quad (3.4)$$

In other words,

$$F_1 \left(\frac{dw}{dz} \right) = I - F_2(w). \quad (3.5)$$

Now rewriting equation (3.4) one obtains a separable equation

$$\frac{dw}{dz} = f(w), \quad (3.6)$$

which can lead to the solution after an integration. Now rewriting the solution in terms of the original variables one obtains a general solution for (2.1).

(c) *Method of deriving linearizing transformations*

Finally, one may note the following interesting point in the above analysis. Suppose $F_2(w)$ is zero in equation (3.4). Then one obtains the simple equation

$$\frac{dw}{dz} = \hat{I}, \quad (3.7)$$

where \hat{I} is a constant. In other words we have

$$\frac{d^2w}{dz^2} = 0, \quad (3.8)$$

which is nothing but the free particle equation. In this case the new variables z and w helps us to transform the given second order nonlinear ODE into a linear second order ODE which in turn leads to the solution by trivial integration. The new variables z and w turn out to be the linearizing transformations. We discuss in detail about this possibility separately in §5.

4. Applications

In this section, we demonstrate the theory discussed in the previous section with suitable examples. In particular, we consider several interesting examples, including those considered in Duarte *et al.* (2001) and derive general solutions and establish complete integrability of these dynamical systems. We split our analysis into three categories. In the first category we consider examples in which the I'_i s, $i = 1, 2$, can be derived straightforwardly from the relation (2.9). In the second and third categories we follow our own procedure detailed in §3 *b* and §3 *c* and deduce the second constant. We note that one can apply our procedure to a wide range of systems of second order of type (2.1) but for illustration purpose we consider only a few examples in the following.

(a) *Type - I Systems*

As mentioned earlier for certain equations one can also get the second pair of solution (S_2, R_2) , in an algorithmic way, from the determining equations (2.6)-(2.8) and construct I_2 through the relation (2.9). We observe that the Examples 1 and 2 discussed in Duarte *et al.* (2001) can be solved in this way and so we consider these two examples first and then a nontrivial example in the following.

Example 1 : An exact solution in general relativity

Duarte *et al.* (2001) considered the following equation, which was originally derived by Buchdahl (1967) in the theory of general relativity,

$$x\ddot{x} = 3\dot{x}^2 + \frac{x\dot{x}}{t}, \quad (4.1)$$

and deduced the first integral I through their procedure. In the following, we briefly discuss their results and then illustrate our ideas. Substituting $\phi = \frac{3\dot{x}^2}{x} + \frac{\dot{x}}{t}$ into

(2.6)-(2.8) we get

$$S_t + \dot{x}S_x + \frac{\dot{x}(3t\dot{x} + x)}{tx}S_{\dot{x}} = \frac{3\dot{x}^2}{x^2} + \frac{6t\dot{x} + x}{tx}S + S^2, \quad (4.2)$$

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x} + x)}{tx}R_{\dot{x}} = -RS - \frac{6\dot{x}t + x}{tx}R, \quad (4.3)$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \quad (4.4)$$

As mentioned in Sec. 2 let us first solve equation (4.2) and obtain an explicit form for the function S . To do so Duarte *et al.* (2001) consider an ansatz for S of the form

$$S = \frac{a(t, x) + b(t, x)\dot{x}}{c(t, x) + d(t, x)\dot{x}}, \quad (4.5)$$

where a , b , c and d are arbitrary functions of t and x . A rational form for S can be justified, since from (2.5) one may note that $S = \frac{I_x}{I}$. So we consider only rational forms in \dot{x} for S for all the examples which we consider in this paper. One may note that in certain examples, including the present one and examples 3 and 5 (given below), this form degenerates into a polynomial form in \dot{x} . However, for other examples like the examples 2 and 4 below, one requires a rational form such as (4.5). To be general, we carry out an analysis with the form (4.5).

Substituting (4.5) into (4.2) and equating the coefficients of different powers of \dot{x} to zero we get a set of partial differential equations for the variables a , b , c and d . Solving them one obtains

$$S_1 = -\frac{3\dot{x}}{x}, \quad S_2 = -\frac{\dot{x}}{x}. \quad (4.6)$$

We note that Duarte *et al.* (2001) have reported only the expression S_1 , as the solution for equation (4.2). However, we find S_2 also forms a solution for (4.2) and helps to deduce the general solution. Substituting these forms S_1 and S_2 into (4.3) and solving the latter one can get an explicit form for the function R . Let us first consider S_1 . Substituting the latter into (4.3) we get the following equation for R :

$$R_t + \dot{x}R_x + \frac{\dot{x}(3t\dot{x} + x)}{tx}R_{\dot{x}} = \frac{3\dot{x}}{x}R - \frac{6\dot{x}t + x}{tx}R. \quad (4.7)$$

In order to solve (4.7) again one has to make an ansatz. We assume that the following form for R , that is,

$$R = A(t, x) + B(t, x)\dot{x}, \quad (4.8)$$

where A and B are arbitrary functions of (t, x) . Since $R = -I_{\dot{x}}$ (vide equation (2.5)) the form of R may be a polynomial or rational in \dot{x} . So depending upon the problem one has to choose an appropriate ansatz. To begin with one can consider simple polynomial (in \dot{x}) for R and if it fails then one can go for rational forms. Let us start with equation (4.8). Now substituting (4.8) into (4.7) and equating the coefficients of different powers of \dot{x} to zero and solving the resultant equations one can obtain finally $R_1 = \frac{1}{tx^3}$. Now the solution $S_1 = -\frac{3\dot{x}}{x}$ and $R_1 = \frac{1}{tx^3}$ has to satisfy the equation (4.4) in order to be a compatible solution, which is indeed true. Once R

and S have been found the first integral I can be fixed easily using the expression (2.9) as

$$I_1 = \frac{\dot{x}}{tx^3}. \quad (4.9)$$

One can easily check that I_1 is constant on the solutions, that is, $\frac{dI_1}{dt} = 0$. This integral has been deduced in Duarte *et al.* (2001). However, the second expression, S_2 has been ignored by the authors since the corresponding R_2 coming out of (4.3) does not form a compatible solution, that is, it does not satisfy (4.4). But in the following we show how one can make it compatible and use it effectively to deduce the second integration constant.

Now substituting the expression $S_2 = -\frac{\dot{x}}{x}$ into (4.3) and solving it in the same way as outlined in the previous paragraph we obtain the following form for R , namely,

$$R_2 = \frac{1}{x^5 t}. \quad (4.10)$$

However, this set (S_2, R_2) does not satisfy the extra constraint (4.4). In fact, not all forms of R coming out from (2.7) satisfy (2.8) in general. As we explained in §3, the form of R_2 given in (4.10) may not be the 'complete form' but might be a factor of the complete form. To recover the complete form of R one may assume that

$$\hat{R} = F(I_1)R, \quad (4.11)$$

where $F(I_1)$ is a function of the first integral I_1 , and determine the form of $F(I_1)$ explicitly. For this purpose we proceed as follows. Substituting

$$\hat{R}_2 = F(I_1)R_2 = \frac{1}{tx^5}F(I_1) \quad (4.12)$$

into equation(4.4), we obtain an equation for F , that is,

$$I_1 F' + 2F = 0, \quad (4.13)$$

where prime denotes differentiation with respect to I_1 . Upon integrating (4.13) we get (after putting the constant of integration to zero)

$$F = \frac{1}{I_1^2} = \frac{t^2 x^6}{\dot{x}^2}, \quad (4.14)$$

which fixes the form of \hat{R}_2 as

$$\hat{R}_2 = \frac{1}{I_1^2} \frac{1}{x^5 t} = \frac{tx}{\dot{x}^2}. \quad (4.15)$$

Now one can easily check that this set $S_2 = -\frac{\dot{x}}{x}$ and $\hat{R}_2 = \frac{tx}{\dot{x}^2}$ is a compatible solution for the equations (4.2)-(4.4). Substituting S_2 and \hat{R}_2 into (2.9) we can get an explicit form for I_2 , namely,

$$I_2 = t\left(t + \frac{x}{\dot{x}}\right). \quad (4.16)$$

From the integrals I_1 and I_2 one can deduce the general solution directly (without performing any further integration) for the problem in the form

$$x = \sqrt{\frac{1}{I_1(I_2 - t^2)}}. \quad (4.17)$$

Of course the same result can be obtained solving equation (4.9) from the first integral. However, the point we want to emphasize here is that an independent second integral of motion can be deduced to find the solution without any further integration, which can be used profitably when the expression for I_1 cannot be solved straightforwardly.

Example 2 : Simple harmonic oscillator

To illustrate the above procedure also works for linear ODEs, in the following we consider the simple harmonic oscillator and derive the general solution. As the procedure of deriving the first integral has been discussed in detail in Duarte *et al.* (2001) we omit the details and provide only the essential expressions in the following.

The equation of motion for the simple harmonic oscillator is

$$\ddot{x} = -x \quad (4.18)$$

so that the equations (2.6)-(2.8) become

$$S_t + \dot{x}S_x - xS_{\dot{x}} = 1 + S^2, \quad (4.19)$$

$$R_t + \dot{x}R_x - xR_{\dot{x}} = -RS, \quad (4.20)$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (4.21)$$

As shown in Duarte *et al.* (2001) a simple solution for the equations (4.19)-(4.21) can be constructed of the form

$$S_1 = \frac{x}{\dot{x}}, \quad R_1 = \dot{x} \quad (4.22)$$

which in turn gives the first integral

$$I_1 = \dot{x}^2 + x^2, \quad (4.23)$$

through the relation (2.9). However, one can easily check that

$$S_2 = -\frac{\dot{x}}{x}, \quad R_2 = x \quad (4.24)$$

is also a solution for the set (4.19) and (4.20) (which has not been reported earlier) but does not satisfy the extra constraint (4.21). So, as before, let us seek an \hat{R}_2 of the form

$$\hat{R}_2 = F(I_1)R_2 = F(I_1)x, \quad (4.25)$$

where $F(I_1)$ is a function of I_1 . Substituting (4.25) into equation (4.21) and integrating the resultant equation, we get $F = \frac{1}{I_1}$. Thus \hat{R}_2 becomes

$$\hat{R}_2 = \frac{x}{I_1} = \frac{x}{x^2 + \dot{x}^2}. \quad (4.26)$$

Now one can check that (S_2, \hat{R}_2) satisfies all three equations (4.19)-(4.21) and furnishes the second integral through the relation (2.9) of the form

$$\begin{aligned} I_2 &= -t - \int \frac{\dot{x}}{\dot{x}^2 + x^2} dx - \int \left(\frac{x}{\dot{x}^2 + x^2} - \frac{d}{d\dot{x}} \int \frac{\dot{x}}{\dot{x}^2 + x^2} dx \right) d\dot{x}, \\ &= -t - \tan^{-1} \frac{\dot{x}}{x}. \end{aligned} \quad (4.27)$$

Using (4.23) and (4.27) we can write down the general solution for the simple harmonic oscillator directly in the form

$$x = \sqrt{I_1} \cos(t + I_2). \quad (4.28)$$

In a similar way one can deduce general solution for a class of physically important systems.

One may note that in the above two examples, I_2 can also be obtained trivially by simply integrating the expressions (4.9) and (4.23) without using the extended procedure. We stress that for certain equations one is not able to integrate and obtain the general solution in this simple way and has to follow the above said procedure in order to obtain the second integral. In the following we discuss one such example for which to our knowledge explicit solution was not known earlier.

Example 3 : Modified Emden type equation with linear term

It is known that the generalized Emden type equation with linear and constant external forcing is also linearizable since it admits an eight point Lie symmetry group (Mahomed & Leach 1989; Pandey *et al.* 2004). In the following we explore its general solution through the extended PS algorithm. Let us first consider the equation of the form

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x = 0, \quad (4.29)$$

where k and λ_1 are arbitrary parameters. To explore the general solution for the equation (4.29) we again use the PS method. In this case we have the following determining equations for the functions R and S ,

$$S_t + \dot{x}S_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x)S_{\dot{x}} = k\dot{x} + \frac{k^2}{3}x^2 + \lambda_1 - Skx + S^2, \quad (4.30)$$

$$R_t + \dot{x}R_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x)R_{\dot{x}} = -R(S - kx), \quad (4.31)$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (4.32)$$

As before, let us seek an ansatz for S of the form (4.5) to the first equation in (4.30)-(4.32). Substituting the ansatz (4.5) into (4.30) and equating the coefficients

of different powers of \dot{x} to zero we get

$$\begin{aligned}
db_x - bd_x - kd^2 &= 0, \\
db_t - bd_t + cb_x - bc_x + a_xd - ad_x - 2kcd - \left(\frac{k^2}{3}x^2 + \lambda_1\right)d^2 + kbdx - b^2 &= 0, \\
cb_t - bc_t + da_t - ad_t + ca_x - ac_x - kc^2 - 2\left(\frac{k^2}{3}x^2 + \lambda_1\right)cd + 2kadx - 2ab &= 0, \\
ca_t - ac_t - \left(\frac{k^2}{9}x^3 + \lambda_1x\right)(bc - ad) - \left(\frac{k^2}{3}x^2 + \lambda_1\right)c^2 + kacx - a^2 &= 0,
\end{aligned} \tag{4.33}$$

where subscripts denote partial derivative with respect to that variable. Solving equation (4.33) we can obtain two specific solutions,

$$S_1 = \frac{-\dot{x} + \frac{k}{3}x^2}{x}, \quad S_2 = \frac{kx + 3\sqrt{-\lambda_1}}{3} - \frac{k\dot{x}}{kx + 3\sqrt{-\lambda_1}}. \tag{4.34}$$

Putting the forms of S_1 and S_2 into (4.31) and solving it one can obtain the respective forms of R . To do so let us first consider S_1 . Substituting the latter into (4.31) we get the following equation for R :

$$R_t + \dot{x}R_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1x)R_{\dot{x}} = \left(\frac{\dot{x} - \frac{k}{3}x^2}{x} + kx\right)R. \tag{4.35}$$

Again to solve equation (4.35) we make an ansatz of the form

$$R = \frac{A(t, x) + B(t, x)\dot{x}}{C(t, x) + D(t, x)\dot{x} + E(t, x)\dot{x}^2}. \tag{4.36}$$

Now substituting (4.36) into (4.35) and equating the coefficients of different powers of \dot{x} to zero and solving the resultant equations we arrive at

$$R_1 = e^{-2\sqrt{-\lambda_1}t} \left(\frac{C_0x}{(3\dot{x} + kx^2 - 3\sqrt{\lambda_1}x)^2} \right), \tag{4.37}$$

where $C_0 = 18\sqrt{-\lambda_1}$. One can easily check that S_1 and R_1 satisfies equation (4.32) and as a consequence one obtains the first integral

$$I_1 = e^{-2\sqrt{-\lambda_1}t} \left(\frac{3\dot{x} + kx^2 + 3\sqrt{-\lambda_1}x}{3\dot{x} + kx^2 - 3\sqrt{-\lambda_1}x} \right). \tag{4.38}$$

We note that unlike the other two examples, equation (4.38) cannot be integrated straightforwardly to provide the second integral (though one can in fact explicitly solve the resultant Riccati equation after some effort). We follow the procedure adopted in the previous two examples and construct I_2 . Now substituting the expression S_2 into (4.31) and solving it in the same way as outlined above we obtain the following form for R , that is,

$$R_2 = C_0 \frac{kx + 3\sqrt{-\lambda_1}}{k(3\dot{x} + kx^2 - 3\sqrt{-\lambda_1}x)^2} e^{-3\sqrt{\lambda_1}t}. \tag{4.39}$$

However, this set (S_2, R_2) does not satisfy the extra constraint (4.32) and so to deduce the correct form of R_2 we assume that

$$\hat{R}_2 = F(I_1)R_2 = C_0 \frac{F(I_1)(kx + 3\sqrt{-\lambda_1})e^{-3\sqrt{-\lambda_1}t}}{k(3\dot{x} + kx^2 - 3\sqrt{-\lambda_1}x)^2}. \quad (4.40)$$

Substituting (4.40) into equation (4.32) we obtain $F = \frac{1}{I_1^2}$, which fixes the form of \hat{R}_2 as

$$\hat{R}_2 = C_0 \frac{kx + 3\sqrt{-\lambda_1}}{k(3\dot{x} + kx^2 + 3\sqrt{-\lambda_1}x)^2} e^{\sqrt{-\lambda_1}t}. \quad (4.41)$$

Now one can easily check that this set (S_2, \hat{R}_2) is a compatible solution for the set (4.30)-(4.32) which in turn provides I_2 through the relation (2.9),

$$I_2 = -\frac{2}{k} e^{\sqrt{-\lambda_1}t} \left(\frac{9\lambda_1 + 3k\dot{x} + k^2x^2}{3\dot{x} + kx^2 + 3\sqrt{-\lambda_1}x} \right). \quad (4.42)$$

Using the explicit form of the first integrals I_1 and I_2 , the solution can be deduced directly as

$$x = \left(\frac{3\sqrt{-\lambda_1}(I_1 e^{2\sqrt{-\lambda_1}t} - 1)}{kI_1 I_2 e^{\sqrt{-\lambda_1}t} + k(1 + I_1 e^{2\sqrt{-\lambda_1}t})} \right). \quad (4.43)$$

To our knowledge, the above explicit solution, (4.43), of the equation (4.29) is given for the first time. It has several interesting consequences for nonlinear dynamics, which will be discussed separately.

(b) Type - II Systems

In the previous category we considered examples which straightforwardly give the integrals I_1 and I_2 through the relation (2.9). In the present category we show that there are situations in which an explicit form of I_2 is difficult to obtain through the relation (2.9) even though one has a compatible solution for (2.6)-(2.8). So one has to go for an alternate way in order to obtain the general solution for the given problem. For this purpose, we make use of the method proposed in §3 b. In the following we give examples where such a possibility occurs and how to overcome this situation.

Example 4 : Helmholtz oscillator

Recently Almendral & Sanjuan (2003) studied the invariance and integrability properties of the Helmholtz oscillator with friction,

$$\ddot{x} + c_1\dot{x} + c_2x - \beta x^2 = 0, \quad (4.44)$$

where c_1 , c_2 and β are arbitrary parameters, which is a simple nonlinear oscillator having a quadratic nonlinearity. Using the Lie theory for differential equations Almendral & Sanjuan (2003) found a parameteric choice $c_2 = \frac{6c_1^2}{25}$ for which the

system is integrable and derived the general solution for this parametric value. In the following we solve this problem through the extended PS method.

Substituting $\phi = -(c_1\dot{x} + c_2x - \beta x^2)$ into equations (2.6)-(2.8) we obtain

$$S_t + \dot{x}S_x - (c_1\dot{x} + c_2x - \beta x^2)S_{\dot{x}} = c_2 - 2\beta x - c_1S + S^2, \quad (4.45)$$

$$R_t + \dot{x}R_x - (c_1\dot{x} + c_2x - \beta x^2)R_{\dot{x}} = -R(S - c_1), \quad (4.46)$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \quad (4.47)$$

Making the same form of an ansatz, vide equations (4.5) and (4.8), we find nontrivial solution exists for (4.45)-(4.46) only for the parametric restriction $c_2 = \frac{6c_1^2}{25}$. The respective solutions are

$$S_1 = \frac{(\frac{2c_1\dot{x}}{5} + \frac{4c_1^2x}{25} - \beta x^2)}{\dot{x} + \frac{2c_1}{5}x}, \quad R_1 = -(\dot{x} + \frac{2c_1}{5}x)e^{\frac{6}{5}c_1t}, \quad (4.48)$$

$$S_2 = \frac{(\frac{c_1\dot{x}}{5} + \frac{6c_1^2x}{25} - \beta x^2)}{\dot{x}}, \quad R_2 = -\dot{x}e^{c_1t}. \quad (4.49)$$

Now one can easily check that (S_1, R_1) satisfies the third equation (4.47) and as a consequence leads to the first integral of the form

$$I_1 = e^{\frac{6}{5}c_1t} \left(\frac{\dot{x}^2}{2} + \frac{2c_1x\dot{x}}{5} + \frac{2c_1^2x^2}{25} - \frac{\beta x^3}{3} \right). \quad (4.50)$$

However, the second set (S_2, R_2) does not satisfy the extra constraint (4.47) and so we take

$$\hat{R}_2 = F(I)R_2 = -F(I)\dot{x}e^{c_1t}, \quad (4.51)$$

which in turn gives $F = C_0I^{-\frac{5}{6}}$, where C_0 is an integration constant, so that

$$\hat{R}_2 = -\left(\frac{C_0}{I_1^{\frac{5}{6}}}\right)\dot{x}e^{c_1t} = -\frac{C_0\dot{x}}{(\frac{\dot{x}^2}{2} + \frac{2c_1x\dot{x}}{5} + \frac{2c_1^2x^2}{25} - \beta\frac{x^3}{3})^{\frac{5}{6}}}. \quad (4.52)$$

One can check that (S_2, \hat{R}_2) satisfy equations (4.45)- (4.47) and so one can proceed to deduce the second integration constant through the relation (2.9). However, upon substituting (S_2, \hat{R}_2) into (2.9) we arrive at

$$I_2 = \int \frac{c_1\dot{x} + \frac{6c_1^2x}{25} - \beta x^2}{(\frac{\dot{x}^2}{2} + \frac{2c_1x\dot{x}}{5} + \frac{2c_1^2x^2}{25} - \beta\frac{x^3}{3})^{\frac{5}{6}}} dx. \quad (4.53)$$

It is very difficult to evaluate the integral and so one is not able to obtain an explicit form of I_2 for this problem through this way. A similar form of I_2 has been also derived by Bluman & Anco (2002) and Jones *et al.* (1993) for the Duffing oscillator problem (that is cubic nonlinearity in equation (4.44)).

Unlike the other examples discussed in Type I the present example possesses difficulties in evaluating the second integration constant. In fact, for a class of equations one faces such complicated integrals. To overcome this one has to look for an alternate way such that the second constant can be deduced in a straightforward

and simple way. We tackle this situation in the following way. As we have seen, in most of the problems, we are able to deduce the first integral, that is, I_1 , straightforwardly and the first integral often admits explicit time dependent terms. A useful way to overcome this is to remove the explicit time dependent terms by transforming the resultant differential equation into an autonomous form and integrate the latter and obtain the solution. In order to do this one needs a transformation and the latter can often be constructed through ad-hoc way. However, as we have shown in the theory in §3 b, one can deduce the required transformation coordinates in a simple way from the first integral itself and the problem can be solved in a systematic way.

Rewriting the first integral I_1 given by equation (4.50), in the form (3.1), we get

$$I_1 = \frac{1}{2} \left(\dot{x} + \frac{2c_1 x}{5} \right)^2 e^{\frac{6}{5}c_1 t} - \frac{\beta x^3}{3} e^{\frac{6}{5}c_1 t}. \quad (4.54)$$

Now splitting the first term in equation (4.54) further in the form (3.2),

$$I_1 = e^{\frac{2c_1 t}{5}} \left(\frac{d}{dt} \left(\frac{1}{\sqrt{2}} x e^{\frac{2c_1 t}{5}} \right) \right)^2 - \frac{\beta}{3} (x e^{\frac{2}{5}c_1 t})^3, \quad (4.55)$$

and identifying the dependent and independent variables from (4.55) and the relations (3.3), we obtain the transformation

$$w = \frac{1}{\sqrt{2}} x e^{\frac{2c_1 t}{5}}, \quad z = -\frac{5}{c_1} e^{-\frac{c_1 t}{5}}. \quad (4.56)$$

One can easily check that equation (4.44) can be transformed to an autonomous form with the help of the transformation (4.56). We note that the transformation (4.56) exactly coincides with the earlier one which has been constructed via Lie symmetry analysis in Almindral & Sanjuan (2003).

Using the transformation (4.56) the first integral (4.54) can be rewritten in the form

$$\hat{I} = w'^2 - \frac{\hat{\beta}}{3} w^3 \quad (4.57)$$

which in turn leads to the solution by an integration. On the other hand the transformation changes the equation of motion (4.44) to

$$w'' = \hat{\beta} w^2, \quad (4.58)$$

where $\hat{\beta} = 2\sqrt{2}\beta$, which upon integration gives (4.57). From equation (4.57), we obtain

$$w'^2 = 4w^3 - g_3, \quad (4.59)$$

where $z = 2\sqrt{\frac{3}{\hat{\beta}}}\hat{z}$ and $g_3 = -\frac{12I_1}{\hat{\beta}}$. The solution of this differential equation can be represented in terms of Weierstrass function $\varrho(\hat{z}; 0, g_3)$ (Gradshteyn & Ryzhik 1980; Almindral & Sanjuan 2003).

(c) Type - III Systems

In the previous two categories we met the situation in which we are able to construct a pair of solutions (S_1, S_2) for the equations (2.6) from which R_1 and R_2 have been deduced. However, there are situations in which one is able to construct only one set of solution (R_1, S_1) and its corresponding first integral only and the second pair of solution (R_2, S_2) can not be obtained by simple rational form of ansatz. In this situation one can utilize our procedure and deduce the general solution for the given problem. In the following we illustrate this with a couple of examples.

Example 5 : Force free Duffing-van der Pol oscillator

One of the well-studied but still challenging equations in nonlinear dynamics is the Duffing-van der Pol oscillator equation. Its autonomous version (force-free) is

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \quad (4.60)$$

where over dot denotes differentiation with respect to time and α , β and γ are arbitrary parameters. Equation (4.60) arises in a model describing the propagation of voltage pulses along a neuronal axon and has received a lot of attention recently by many authors. A vast amount of literature exists on this equation, for details see for example Lakshmanan & Rajasekar (2003) and references therein. In this case we have

$$S_t + \dot{x}S_x - ((\alpha + \beta x^2)\dot{x} - \gamma x + x^3)S_{\dot{x}} = (2\beta x\dot{x} - \gamma + 3x^2) - (\alpha + \beta x^2)S + S^2, \quad (4.61)$$

$$R_t + \dot{x}R_x - ((\alpha + \beta x^2)\dot{x} - \gamma x + x^3)R_{\dot{x}} = (\alpha + \beta x^2 - S)R, \quad (4.62)$$

$$R_x = SR_{\dot{x}} + RS_{\dot{x}}. \quad (4.63)$$

To solve equation (4.61)-(4.63) we seek an ansatz for S and R of the form

$$S = \frac{a(t, x) + b(t, x)\dot{x}}{c(t, x) + d(t, x)\dot{x}}, \quad R = A(t, x) + B(t, x)\dot{x}. \quad (4.64)$$

Upon solving the equations (4.61)-(4.63) with the above ansatz we find nontrivial solution exists only for the choice $\alpha = \frac{4}{\beta}, \gamma = -\frac{3}{\beta^2}$ and the corresponding forms of S and R reads

$$S = \frac{1}{\beta} + \beta x^2, \quad R = e^{\frac{3t}{\beta}}. \quad (4.65)$$

For this set one can construct an invariant through the expression (2.9) which turns out to be (Senthilvelan & Lakshmanan 1995)

$$\dot{x} + \frac{1}{\beta}x + \frac{\beta}{3}x^3 = Ie^{-\frac{3}{\beta}t}. \quad (4.66)$$

To obtain a second pair of solutions for the equations (4.61)-(4.63) one may seek more general rational form of S and R by including higher polynomials in \dot{x} . However, they all lead to only functionally dependent integrals. As it is not possible to

seek the second pair of solution by simple ansatz one has to see an alternate way as indicated in §3*b*. We can deduce the required transformation coordinates from the first integral and transform the latter to an autonomous equation and integrate it.

Using our algorithm given in §3*b* one can deduce the transformation coordinates from the first integral itself which turns out to be (Chandrasekar *et al.* 2004)

$$w = -xe^{\frac{1}{\beta}t}, \quad z = e^{-\frac{2}{\beta}t}, \quad (4.67)$$

where w and z are new dependent and independent variables respectively. Substituting (4.67) into (4.60) with the parametric restriction $\alpha = \frac{4}{\beta}, \gamma = -\frac{3}{\beta^2}$, we get

$$w'' - \frac{\beta^2}{2}w^2w' = 0, \quad (4.68)$$

where prime denotes differentiation with respect to z . Equation (4.68) can be integrated trivially to yield

$$w' - \frac{\beta^2}{6}w^3 = I, \quad (4.69)$$

where I is the integration constant. Equivalently, the transformation (4.67) reduces (4.60) to this form. Solving (4.69), we obtain (Gradshteyn & Ryzhik 1980)

$$z - z_0 = \frac{a}{3I} \left[\frac{1}{2} \log \left(\frac{(w+a)^2}{w^2 - aw + a^2} \right) + \sqrt{3} \arctan \left(\frac{w\sqrt{3}}{2a - w} \right) \right], \quad (4.70)$$

where $a = \sqrt[3]{\frac{6I}{\beta^2}}$ and z_0 is the second integration constant. Rewriting w and z in terms of old variables one can get the explicit solution for the equation (4.60).

In the above, we have shown that the systems (4.44) and (4.60) are integrable for certain specific parametric restrictions only. One may also assume that the functions S and R involve higher degree rational functions in \dot{x} and repeat the analysis. However, such an analysis does not provide any new integrable choice. In fact, the present results coincide exactly with the results obtained through other methods, namely, Painlevé analysis, Lie symmetry analysis and direct methods (Senthilvelan & Lakshmanan 1995; Almindral & Sanjuan 2003; Lakshmanan & Rajasekar 2003).

5. Linearizable equations

In the previous section we discussed the complete integrability of nonlinear dynamical systems by constructing sufficient number of integrals of motion and obtaining the general solutions explicitly. Another way of solving nonlinear ODEs is to transform them to linear ODEs, in particular to a free particle equation and explore their underlying solutions. Eventhough this is one of the classical problems in the theory of ODEs, recently considerable progress has been made (Mahomed & Leach 1989*a*; Steeb 1993; Olver 1995; Harrison 2002). In this direction it has been shown that a necessary condition for a second order ODE to be linearizable is that it should be of the form (Mahomed & Leach 1989*a*)

$$\ddot{q} = D(t, q) + C(t, q)\dot{q} + B(t, q)\dot{q}^2 + A(t, q)\dot{q}^3, \quad (5.1)$$

where the functions A , B , C , and D are analytic. Sufficient condition for the above second order equation to be linearizable is (Mahomed & Leach 1989a),

$$\begin{aligned} 3A_{tt} + 3CA_t - 3DA_q + 3AC_t + C_{qq} - 6AD_q + BC_q - 2BB_t - 2B_{tq} &= 0, \\ B_{tt} + 6DA_t - 3DB_q + 3AD_t - 2C_{tq} - 3BD_q + 3D_{qq} + 2CC_q - CB_t &= 0, \end{aligned} \quad (5.2)$$

where the suffices refer to partial derivatives.

For a given second order nonlinear ODE one can easily check whether it can be linearizable or not by using the above necessary and sufficient conditions. However, the nontrivial problem is how to deduce systematically the linearizing transformations if the given equation is linearizable. As far our knowledge goes Lie symmetries are often used to extract the linearizing transformations (Mahomed & Leach 1985). As we pointed out in §3 the linearizing transformations can also be deduced from the first integral itself, whenever the system is linearizable, in a simple and straightforward way and we stress that our procedure is new to the literature. In fact, we use the same procedure discussed in §3c and deduce the linearizing transformations. The only difference is that in the case of linearizing transformations the function F_2 turns out to be zero in equation (3.2) and as a consequence the latter becomes $\frac{dw}{dz} = I$ and the transformation coordinates become the linearizing transformations. We illustrate the theory with certain new examples in the following.

Example 1 : General relativity

To illustrate the underlying ideas let us begin with a simple and physically interesting example, namely, the general relativity equation which we discussed as Example 1 in §4. We derived the solution (4.17) using the PS method. In the present section we linearize the system and derive its solution. Rewriting the first integral (4.9) in the form (3.1)

$$I = -\frac{1}{2t} \frac{d}{dt} \left(\frac{1}{x^2} \right), \quad (5.3)$$

and identifying (5.3) with (3.2), we get

$$G_1 = \frac{1}{x^2}, \quad G_2 = -2t, \quad F_2 = 0. \quad (5.4)$$

With the above choices, equation (3.3) furnishes the transformed variables,

$$w = \frac{1}{x^2}, \quad z = -t^2. \quad (5.5)$$

Substituting (5.5) into (4.1), the latter becomes the free particle equation, namely, $\frac{d^2w}{dz^2} = 0$, whose general solution is $w = I_1z + I_2$, where I_1 and I_2 are integration constants. Rewriting w and z in terms of x and t one gets exactly (4.17) which has been derived in a different way.

Example 2 : Modified Emden type equations

Recently several papers have been devoted to explore the invariance and integrable properties of the modified Emden type equations (Mahomed & Leach 1985;

Duarte *et al.* 1987),

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 = 0. \quad (5.6)$$

In fact, it is one of the rare second order nonlinear ODEs which admit eight Lie point symmetries and as a consequence is a linearizable one. Recently Pandey *et al.* (2004) have obtained the explicit forms of the Lie point symmetries associated with the more general equation

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x + \lambda_2 = 0, \quad (5.7)$$

where k , λ_1 and λ_2 are arbitrary parameters. They found that not only the Emden equation (5.6), but also its general form, that is, equation (5.7), admits eight Lie point symmetries. The authors have also reported that the explicit forms of the symmetry generators. However, due to the complicated forms of the symmetry generators it is difficult to derive the first integrals and linearizing transformations from the symmetries straightforwardly (though in principle this is always possible). Nevertheless, we discussed about the integrability of the case $\lambda_2 = 0$, $\lambda_1 \neq 0$ of equation (5.7) as Example 3 in §4 and deduced its general solution. In this section we transform the equation into free particle equation and deduce the general solution in an independent way. We divide our analysis into 2 cases, namely, (i) $\lambda_1 \neq 0$, $\lambda_2 = 0$ and (ii) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and construct linearizing transformations and general solutions for both the cases. As the procedure is same as given in the previous examples we give only the results.

Case(i) $\lambda_2 = 0$, $\lambda_1 \neq 0$: Modified Emden type equation with linear term

Restricting $\lambda_2 = 0$ in (5.7) we have

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1 x = 0. \quad (5.8)$$

Since the first integral is already derived, vide equation (4.38), we utilize it here to deduce the linearizing transformations. Rewriting the first integral (4.38) in the form

$$I_1 = -\frac{e^{-\sqrt{-\lambda_1}t} kx^2}{3\dot{x} + kx^2 - 3\sqrt{-\lambda_1}x} \left[\frac{d}{dt} \left(\left(\frac{3}{kx} + \frac{1}{\sqrt{-\lambda_1}} \right) e^{-\sqrt{-\lambda_1}t} \right) \right] \quad (5.9)$$

and identifying (5.9) with (3.2), we get

$$G_1 = \left(\frac{3}{kx} + \frac{1}{\sqrt{-\lambda_1}} \right) e^{-\sqrt{-\lambda_1}t}, \quad G_2 = -\frac{3\dot{x} + kx^2 - 3\sqrt{-\lambda_1}x}{kx^2} e^{\sqrt{-\lambda_1}t}. \quad (5.10)$$

With the above functions (3.3) furnishes

$$w = \left(\frac{3}{kx} + \frac{1}{\sqrt{-\lambda_1}} \right) e^{-\sqrt{-\lambda_1}t}, \quad z = \left(\frac{3}{kx} - \frac{1}{\sqrt{-\lambda_1}} \right) e^{\sqrt{-\lambda_1}t}, \quad (5.11)$$

which is nothing but the linearizing transformation. One may note that in this case also while rewriting the first integral I (equation (4.38)) in the form (3.1), the function F_2 disappears, and as a consequence we arrive at (vide equation (3.4))

$$\frac{dw}{dz} = I, \quad (5.12)$$

which in turn gives the free particle equation by differentiation or leads to the solution (4.43) by an integration. On the other hand vanishing of the function F_2 in this analysis is precisely the condition for the system to be transformed into the free particle equation.

Case(ii) $\lambda_1 \neq 0, \lambda_2 \neq 0$: Modified Emden type equation with linear term and constant external forcing

Finally, we consider the general case, that is,

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1x + \lambda_2 = 0. \quad (5.13)$$

To explore the first integrals associated with the system (5.13) let us seek the PS algorithm again. The determining equations for the functions R and S turn out to be

$$S_t + \dot{x}S_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1x + \lambda_2)S_{\dot{x}} = k\dot{x} + \frac{k^2}{3}x^2 + \lambda_1 - Skx + S^2, \quad (5.14)$$

$$R_t + \dot{x}R_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1x + \lambda_2)R_{\dot{x}} = (kx - S)R, \quad (5.15)$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0. \quad (5.16)$$

As before let us seek an ansatz for S to solve the equation (5.14), namely,

$$S = \frac{a(t, x) + b(t, x)\dot{x}}{c(t, x) + d(t, x)\dot{x}}. \quad (5.17)$$

Substituting (5.17) into (5.14) and equating the coefficients of different powers of \dot{x} to zero and solving the resultant equations we arrive at

$$S_1 = \frac{kx + 3\alpha}{3} - \frac{k\dot{x}}{kx + 3\alpha}, \quad S_2 = \frac{kx + 3\beta}{3} - \frac{k\dot{x}}{kx + 3\beta}, \quad (5.18)$$

where $\alpha^3 + \alpha\lambda_1 - \frac{k\lambda_2}{3} = 0$ and $\beta = \frac{-\alpha \pm \sqrt{-3\alpha^2 - 4\lambda_1}}{2}$. Putting the forms of S_1 into (5.15) we get

$$R_t + \dot{x}R_x - (kx\dot{x} + \frac{k^2}{9}x^3 + \lambda_1x + \lambda_2)R_{\dot{x}} = \left(\frac{k\dot{x}}{kx + 3\alpha} - \frac{kx + 3\alpha}{3} + kx \right) R. \quad (5.19)$$

Again to solve this equation we make an ansatz

$$R = \frac{A(t, x) + B(t, x)\dot{x}}{C(t, x) + D(t, x)\dot{x} + E(t, x)\dot{x}^2}. \quad (5.20)$$

Substituting (5.20) into (5.19) and solving it we obtain the following form of R , namely,

$$R_1 = \frac{C_0(kx + 3\alpha)e^{\mp\hat{\alpha}t}}{(3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2)^2}, \quad (5.21)$$

where C_0 is constant and $\hat{\alpha} = \sqrt{-3\alpha^2 - 4\lambda_1}$. We find that the solution (S_1, R_1) satisfies (5.16). Equations (5.18) and (5.21) fix the first integral of the form

$$I_1 = e^{\mp\hat{\alpha}t} \left(\frac{3k\dot{x} - 3\frac{(3\alpha \mp \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2} \right), \quad (5.22)$$

where $C_0 = 9k\hat{\alpha}$. Rewriting the first integral (5.22) in the form (3.1)

$$I_1 = -\frac{e^{-\frac{3\alpha \mp \hat{\alpha}}{2}t}(k_1x + 3\alpha)^2}{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2} \times \left[\frac{d}{dt} \left(\left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \pm \hat{\alpha}}{2(3\alpha^2 + \lambda_1)} \right) e^{\frac{3\alpha \mp \hat{\alpha}}{2}t} \right) \right] \quad (5.23)$$

and identifying (5.23) with (3.2), we get

$$\begin{aligned} G_1 &= \left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \pm \hat{\alpha}}{2(3\alpha^2 + \lambda_1)} \right) e^{\frac{3\alpha \mp \hat{\alpha}}{2}t}, \\ G_2 &= -\frac{3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2}{(kx + 3\alpha)^2} e^{\frac{3\alpha \pm \hat{\alpha}}{2}t}, \end{aligned} \quad (5.24)$$

so that (3.3) gives

$$\begin{aligned} w &= \left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \pm \hat{\alpha}}{2(3\alpha^2 + \lambda_1)} \right) e^{\frac{3\alpha \mp \hat{\alpha}}{2}t}, \\ z &= \left(\frac{-3}{kx + 3\alpha} + \frac{3\alpha \mp \hat{\alpha}}{2(3\alpha^2 + \lambda_1)} \right) e^{\frac{3\alpha \pm \hat{\alpha}}{2}t}, \end{aligned} \quad (5.25)$$

which is nothing but the linearizing transformation. Substituting (5.25) into (5.13) we get the free particle equation

$$\frac{d^2w}{dz^2} = 0 \quad (5.26)$$

whose general can be written as $w = I_1z + I_2$. Rewriting w and z in terms of the original variable x and t one obtains

$$x = -\frac{3\alpha}{k} + \frac{6}{k} \left(\frac{(3\alpha^2 + \lambda_1)(1 - I_1e^{\pm\hat{\alpha}t})}{3\alpha(1 - I_1e^{\pm\hat{\alpha}t}) - 2(3\alpha^2 + \lambda_1)I_2e^{-\frac{3\alpha \pm \hat{\alpha}}{2}t} \pm \hat{\alpha}(1 + I_1e^{\pm\hat{\alpha}t})} \right). \quad (5.27)$$

On the other hand the general solution can also be derived by extending the PS method itself. To do so one has to consider the function S_2 . Now substituting the expression S_2 into (5.15) and solving it in the same way as outlined in the previous paragraphs we obtain the following form for R , that is,

$$R_2 = \frac{C_0(kx + 3\beta)e^{\frac{3(\alpha \mp \hat{\alpha})}{2}t}}{(3k\dot{x} - 3\frac{(3\alpha \pm \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2)^2}. \quad (5.28)$$

However, this set, (S_2, R_2) , does not satisfy the extra constraint (5.16) and to recover the full form of the integrating factor we assume that

$$\hat{R}_2 = F(I_1)R_2. \quad (5.29)$$

Substituting (5.29) into equation (5.16) we obtain an equation for F , that is, $I_1 F' + 2F = 0$, where prime denotes differentiation with respect to I_1 . Upon integrating this equation we obtain $F = 1/I_1^2$, which fixes the form of \hat{R} as

$$\hat{R}_2 = \frac{C_0(kx + 3\beta)e^{\frac{3\alpha \pm \hat{\alpha}}{2}t}}{(3k\dot{x} - 3\frac{(3\alpha \mp \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2)}. \quad (5.30)$$

Now one can easily check that this set S_2 and \hat{R}_2 is a compatible solution for the equations (5.14)-(5.16). Substituting S_2 and \hat{R}_2 into (2.9), we can obtain an explicit form for the second integral I_2 , that is,

$$I_2 = -\left(\frac{2\hat{\alpha}(3k\dot{x} - 3\alpha kx + k^2x^2 + 9\alpha^2 + 9\lambda_1)e^{\frac{3\alpha \pm \hat{\alpha}}{2}t}}{(3\alpha \pm \hat{\alpha})(3k\dot{x} - 3\frac{(3\alpha \mp \hat{\alpha})}{2}(kx + 3\alpha) + (kx + 3\alpha)^2)} \right). \quad (5.31)$$

Rewriting equation (5.22) for \dot{x} and substituting it into (5.31) we get the same expression (5.27) as the general solution.

Example 3 : Generalized modified Emden type equation

Recently, Pandey *et al.* (2004) have considered the following Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad (5.32)$$

where f and g are arbitrary function of their arguments, and classified systematically all polynomial forms of f and g which admit eight Lie point symmetry generators with their explicit forms. They found that the most general nonlinear ODE which is linear in \dot{x} whose coefficients are functions of the dependent variable alone should be of the form

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1k_2}{3}x^2 + \lambda_1x + \lambda_2 = 0, \quad (5.33)$$

where k_i 's and λ_i 's, $i=1,2$, are arbitrary parameters, which is consistent with the criteria (5.1) and (5.2) given by Mahomed & Leach (1989a). Interestingly, equation (5.33) and all its sub cases possess $sl(3, R)$ symmetry algebra. For example, we discussed the integrability and linearization of equation (5.33) with $k_2 = 0$ in the previous example. As the linearizing transformations and the general solution of equation (5.33) are yet to be reported we include this equation as an example in the present work. As in the previous case we divide our analysis into three cases.

(i) $\lambda_1 = 0, \lambda_2 = 0$: Modified Emden type equation with quadratic and cubic nonlinearity

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1k_2}{3}x^2 = 0. \quad (5.34)$$

(ii) $\lambda_1 \neq 0, \lambda_2 = 0$: Modified Emden type equation with quadratic and linear terms

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1k_2}{3}x^2 + \lambda_1x = 0. \quad (5.35)$$

(iii) $\lambda_1 \neq 0, \lambda_2 \neq 0$: The full generalized modified Emden type equation (5.33).

We have derived the integrating factors, integrals of motion, linearizing transformations and the general solutions for all the cases. As the calculations are the similar to the one discussed in the previous case, we present the results in tabular form (Table I), where the results for the most general case (5.33) has been given from which the results for the limiting cases (5.34) and (5.35) can be deduced.

6. Conclusion

In this paper we have discussed the method of finding general solutions associated with second order nonlinear ODEs through a modified PS method. The method can be considered as a direct one, complimenting the well known method of Lie symmetries. In particular, we have extended the theory of Duarte *et al.* (2001) such that one can recover new integrating factors and their associated integrals of motion. These integrals of motion can be utilized to construct the general solution. In the situation when one is not able to recover the second integral of motion we introduced another approach to derive the second integration constant. Interestingly, we showed that in this case it can be derived from the first integral itself in a simple and elegant way. Apart from the above we introduced a technique which can be utilized to derive linearizing transformation from the first integral. We illustrated the theory with several new examples and explored their underlying solutions.

In this paper we concentrated our studies only on single second order ODEs. In principle the method can also be extended to third order ODEs and systems of second order ODEs. The results will be published elsewhere.

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References

- Ablowitz, M. J. & Clarkson, P. A. 1992 *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge: Cambridge University Press.
- Almendral, J. A. & Sanjuán, M. A. F. 2003 Integrability and symmetries for the Helmholtz oscillator with friction. *J. Phys. A* **36**, 695-710.
- Arnold, V. I. 1978 *Mathematical Methods of Classical Mechanics*. New York: Springer.
- Bluman, G. W. & Anco, S. C. 2002 *Symmetries and Integration Methods for Differential Equations*. New York: Springer.
- Buchdahl, H. A. 1964 A relativistic fluid spheres resembling the Emden polytrope of index 5. *Ap. J.* **140**, 1512-1516.
- Chandrasekar, V. K., Senthilvelan, M. & Lakshmanan, M. 2004 New aspects of integrability of force-free Duffing-van der Pol oscillator and related nonlinear systems. *J. Phys. A* **37**, 4527-4534.
- Conte, R. 1999 *The Painlevé Property: One Century Later*. New York, Springer.
- Duarte, L. G. S., Duarte, S. E. S. & Moreira, I. C. 1987 One-dimensional equations with the maximum number of symmetry generators. *J. Phys. A* **20**, L701-704.

- Duarte, L. G. S., Duarte, S. E. S., da Mota, A. C. P. & Skea, J. E. F. 2001 Solving the second-order ordinary differential equations by extending the Prelle-Singer method. *J. Phys. A* **34**, 3015-3024.
- Gradshteyn, I. S. & Ryzhik, I. M. 1980 *Table of Integrals, Series and Products*. London: Academic press.
- Guckenheimer, J. & Holmes, P. 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer.
- Harrison, B. K. 2002 An old problem newly treated with differential forms when and how can the equation $y'' = f(x, y, y')$ be linearized?. In *Proc. Int. Conf. on Symmetry in Nonlinear Mathematical Physics, Kyiv, Ukraine, 9 July 2001*, pp.27-35.
- Hietarinta, J. 1987 Direct methods for the search of the second invariant. *Phys. Rep.* **147**, 87-154.
- Ince, E. L. 1956 *Ordinary Differential Equations*. New York: Dover.
- Jones, S. E., Butson, G. J. & Lemmers, P. L. 1993 On Abel's identity, part II: The integration of a non-linear, non-conservative system. *J. Sound Vib.* **160**, 251-257.
- José, J. V. & Saletan, E. J. 2002 *Classical Dynamics: A Contemporary Approach*. Cambridge: Cambridge University Press.
- Lakshmanan, M. & Rajasekar, S. 2003 *Nonlinear Dynamics: Integrability, Chaos and Patterns*. New York: Springer-Verlag.
- Lichtenberg, A. J. & Leiberman, M. A. 1983 *Regular and Stochastic Motion*. New York: Springer.
- Mahomed, F. M. & Leach, P. G. L. 1985 The linear symmetries of a nonlinear differential equation. *Quaestiones Math.* **8**, 241-274.
- Mahomed, F. M. & Leach, P. G. L. 1989 The Lie algebra $sl(3, R)$ and linearization. *Quaestiones Math.* **12**, 121-139.
- Mahomed, F. M. & Leach, P. G. L. 1989a Lie algebras associated with scalar second-order ordinary differential equation. *J. Math. Phys.* **30**, 2770-2777.
- Olver, P. J. 1995 *Equivalence, Invariants, and Symmetry*. Cambridge: Cambridge University Press.
- Pandey, S. N., Senthilvelan, M. & Lakshmanan, M. 2004 Classification of Lie point symmetries for nonlinear dissipative systems (submitted for publication).
- Percival, I. & Richards, D. 1982 *Introduction to Dynamics*. Cambridge: Cambridge University Press.
- Prelle, M. & Singer, M. 1983 Elementary first integrals of differential equations *Trans. Am. Math. Soc.* **279**, 215-229.
- Senthilvelan, M. & Lakshmanan, M. 1995 Lie symmetries and infinite dimensional Lie algebras of certain nonlinear dissipative systems. *J. Phys. A: Math. Gen.* **28**, 1929-1942.
- Steeb, W. H. 1993 *Invertible Point Transformations and Nonlinear Differential Equations*. London: World Scientific.
- Wiggins, S. 2003 *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. New York: Springer-Verlag.

Table 1. Integrating factors, integrals of motion, linearizing transformations and the general solution of equation (5.33)

Null forms and Integrating factors

$$S_1 = \frac{k_1x + 3\alpha}{3} - \frac{k_1\dot{x}}{k_1x + 3\alpha}, \quad R_1 = \frac{C_0(k_1x + 3\alpha)e^{\mp\hat{\alpha}t}}{(3k_1\dot{x} - \frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x + 9\alpha) + (k_1x + 3\alpha)^2)}$$

$$S_2 = \frac{k_1x + 3\beta}{3} - \frac{k_1\dot{x}}{k_1x + 3\beta}, \quad R_2 = \frac{C_0(k_1x + 3\beta)e^{\frac{\hat{\beta}\pm\hat{\alpha}}{2}t}}{3k_1\dot{x} - \frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x + 9\alpha) + (k_1x + 3\alpha)^2}$$

$$\alpha^3 - k_2\alpha^2 + \alpha\lambda_1 - \frac{k_1\lambda_2}{3} = 0, \quad \hat{\alpha} = \sqrt{-3\alpha^2 + 2\alpha k_2 + k_2^2 - 4\lambda_1}, \quad \hat{\beta} = 3\alpha - k_2$$

$$\beta = \frac{-\alpha + k_2 \pm \hat{\alpha}}{2},$$

First Integrals

$$I_1 = e^{\mp\hat{\alpha}t} \left(\frac{3k_1\dot{x} - \frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x + 9\alpha) + (k_1x + 3\alpha)^2}{3k_1\dot{x} - \frac{\hat{\beta}\pm\hat{\alpha}}{2}(3k_1x + 9\alpha) + (k_1x + 3\alpha)^2} \right), \quad C_0 = 9k_1\hat{\alpha}$$

$$I_2 = \frac{-2\hat{\alpha}e^{\frac{\hat{\beta}\pm\hat{\alpha}}{2}t}}{\hat{\beta} \pm \hat{\alpha}} \left(\frac{3k_1\dot{x} - 3k_1x(\alpha - k_2) + k_1^2x^2 + 9\alpha^2 - 9\alpha k_2 + 9\lambda_1}{3k_1\dot{x} - \frac{\hat{\beta}\mp\hat{\alpha}}{2}(3k_1x + 9\alpha) + (k_1x + 3\alpha)^2} \right)$$

Linearizing Transformations

$$w = \left(\frac{-3}{k_1x + 3\alpha} + \frac{\hat{\beta} \pm \hat{\alpha}}{2(3\alpha^2 - 2\alpha k_2 + \lambda_1)} \right) e^{\frac{\hat{\beta}\mp\hat{\alpha}}{2}t},$$

$$z = \left(\frac{-3}{k_1x + 3\alpha} + \frac{\hat{\beta} \mp \hat{\alpha}}{2(3\alpha^2 - 2\alpha k_2 + \lambda_1)} \right) e^{\frac{\hat{\beta}\pm\hat{\alpha}}{2}t}$$

Solution

$$x = -\frac{3\alpha}{k_1} + \frac{1}{k_1} \left(\frac{6(3\alpha^2 - 2\alpha k_2 + \lambda_1)(1 - I_1 e^{\pm\hat{\alpha}t})}{\hat{\beta}(1 - I_1 e^{\pm\hat{\alpha}t}) \pm (\hat{\beta} \pm \hat{\alpha})I_1 I_2 e^{-\frac{\hat{\beta}\pm\hat{\alpha}}{2}t} \pm \hat{\alpha}(1 + I_1 e^{\pm\hat{\alpha}t})} \right)$$
