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Characteristics of Linear Differential Operators Over Commutative Algebras

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Abstract. Three definitions for characteristics of linear differential operators in the category of modules over a commutative unitary algebra are given. These definitions are compared with each other and some basic fact concerning their properties are proved. It is shown that for algebras without zero divisors the characteristic ideal is involutive and is the support of the symbolic module.

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1. Introduction and Notations

In this paper, we construct a pure algebraic theory for characteristics of linear differential operator in the framework of [4]. We show that relations between the characteristic ideal and the symbolic module of a differential operator in the algebraic setting is the same as in the geometric one (cf. [2, 3]). It is also shown that junior symbols of differential operators can be introduced as module morphisms over appropriate characteristic ideals.

In what follows, we consider a commutative unitary algebra A over a commutative ring $\mathcal{R}$. We deal with the functors of differential calculus arising in the category $\mathcal{M}(A)$ of modules over A (see [4]). In this section, we recall some essential definitions from [4] and [5].

Let P and Q be two A-modules and consider the algebra $\mathrm{Diff}_*^{(+)}(P,Q)$ of linear differential operators acting from P to Q. Due to embeddings $\mathrm{Diff}_{k-1}^{(+)}(P,Q) \subset \mathrm{Diff}_k^{(+)}(P,Q)$ one can define quotient modules

$$S_k(P,Q) = \frac{\mathrm{Diff}_k^{(+)}(P,Q)}{\mathrm{Diff}_{k-1}^{(+)}(P,Q)}.$$

Note that two different A-module structures (the left and the right ones) coincide in $S_k(P,Q)$.

DEFINITION 1.1. A-module $S(P,Q)=\sum_{k\geqslant 0} S_k(P,Q)$ is called the module of symbols of differential operators acting from P to Q. We also use the notations

$$S(P)=\sum_{k\geqslant 0} S_k(P) \quad \text{for } S(A,P) \quad \text{and} \quad S=\sum_{k\geqslant 0} S_k \quad \text{for } S(A).$$

Let $\mathrm{Diff}_k^{(+)}(P,Q)\to S_k(P,Q)$ be the natural projection and consider $\Delta\in\mathrm{Diff}_k^{(+)}(P,Q)$. The image of Δ under this projection is denoted by $|\Delta|_k=|\Delta|$ and is called *the symbol of the operator* Δ.

For any A-modules P,Q,R one has an A-module homomorphism

$$S_k(P,Q)\otimes_A S_l(Q,R)\longrightarrow S_{k+l}(P,R)$$

induced by the composition of differential operators:

$$|\nabla|_l\cdot|\Delta|_k=|\nabla\circ\Delta|_{k+l}.$$

In particular, $S(P,P)$ is an associative A-algebra.

Consider two operators $\Delta_i\in\mathrm{Diff}_{k_i}^{(+)}(A,A)$. Evidently, their commutator $[\Delta_1,\Delta_2]$ is an operator of order $\leqslant k_1+k_2-1$ and, consequently, $S(A)$ is a commutative algebra. Consider now two elements $s=|\Delta|\in S_k$ and $t=|\nabla|\in S_l$ and define

$$\{s,t\}\stackrel{\text{def}}{=}|\Delta\circ\nabla-\nabla\circ\Delta|_{k+l-1}.$$

PROPOSITION 1.2. *For any elements $s,t,r\in S$ one has*

(i) $\{s,t\}+\{t,s\}=0$,
(ii) $\{s,\{t,r\}\}+\{t,\{r,s\}\}+\{r,\{s,t\}\}=0$,
(iii) $\{s,tr\}=\{s,t\}r+t\{s,r\}$.

The operation $\{\cdot,\cdot\}\colon S\otimes_{\mathcal{R}} S\to S$ is called *the Poisson bracket* in S.

Remark 1.3. For the case $A=C^\infty(M)$, where M is a smooth manifold, S plays the role of the algebra of function on $T^*(M)$ polynomial along the fibers of $\pi\colon T^*(M)\to M$. This justifies the notation $T^*(A)$ for the space $\mathrm{Spec}_{\mathrm{gr}}A$ of graded primitive ideals of A.

If $\Delta\in\mathrm{Diff}_i^{(+)}(P,Q)$ is a differential operator, we define its symbolic map (or simply, the symbol of Δ)

$$s(\Delta)=\mathrm{smbl}(\Delta)\colon S(P)\longrightarrow S(Q)$$

by

$$s(\Delta)(t)=|\Delta\circ\nabla|_{k+i} \quad \text{for } t=|\nabla|_k\in S_k(P),\ \nabla\in\mathrm{Diff}_k^{(+)}(A,P).$$

In what follows we use standard concepts and notations from commutative algebra [1]. Namely, let A be a commutative algebra with a unit and P be an A-module. Then:

(i) $\mathrm{Ann}(P) \subset A$ denotes the annihilator of P in A,
(ii) $\mathrm{Supp}(P) \subset \mathrm{Spec}(A)$ is the support of P,
(iii) if $\mu \subset A$ is a multiplicative set, then by P_μ corresponding localization is denoted,
(iv) in particular, if $p \subset A$ is a primitive ideal, then by P_p localization with respect to $A \setminus p$ is denoted,
(v) the quotient ring of A/p is denoted by $\langle A_p \rangle$.

2. Characteristic Ideal

Let $\Delta \in \mathrm{Diff}_i^{(+)}(P, Q)$ be a differential operator of the order i.

DEFINITION 2.1. An element $a \in A$ is called a characteristic of the operator $\Delta \in \mathrm{Diff}_i^{(+)}(P, Q)$ if the kernel of the map

$$\delta_a^i(\Delta) = \underbrace{[a, [a, \ldots [a}_{i \text{ times}}, \Delta] \ldots]] \in \hom_A(P, Q)$$

is nontrivial.

The set of characteristics of the operator Δ is denoted by $\mathrm{char}(\Delta) \subset A$.

PROPOSITION 2.2. *If* $\Delta_1, \Delta_2 \in \mathrm{Diff}_i^{(+)}(P, Q)$ *are differential operators such that* $|\Delta_1| = |\Delta_2|$, *then* $\mathrm{char}(\Delta_1) = \mathrm{char}(\Delta_2)$.

Hence, $\mathrm{char}(\Delta)$ is determined by the symbol of Δ only and for any $s \in S_i(P, Q)$ the set $\mathrm{char}(s) \subset A$ is well-defined.

PROPOSITION 2.3. *Let* $\Delta_1 \in \mathrm{Diff}_i^{(+)}(P, Q), \Delta_2 \in \mathrm{Diff}_j^{(+)}(Q, R)$ *be two differential operators. Then one has the following embeddings*

$$\mathrm{char}(\Delta_1) \subset \mathrm{char}(\Delta_2 \circ \Delta_1) \subset \mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2).$$

Proof. It follows from the identity

$$\delta_a^{i+j}(\Delta_2 \circ \Delta_1) = \delta_a^j(\Delta_2) \circ \delta_a^i(\Delta_1)$$

which is valid for any $a \in A$.

COROLLARY 2.4. *For any two operators* $\Delta_1, \Delta_2 \in \mathrm{Diff}_*^{(+)}(A, A)$ *one has*

$$\mathrm{char}(\Delta_1 \circ \Delta_2) = \mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2).$$

Proof. Due to Proposition 2.3, one has

$$\mathrm{char}(\Delta_1) \subset \mathrm{char}(\Delta_2 \circ \Delta_1) \subset \mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2).$$

On the other hand, since $\mathrm{char}(\Delta)$ depends on $|\Delta|$ only, one has $\mathrm{char}(\Delta_1 \circ \Delta_2) = \mathrm{char}(\Delta_2 \circ \Delta_1)$. Consequently

$$\mathrm{char}(\Delta_2) \subset \mathrm{char}(\Delta_2 \circ \Delta_1) \subset \mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2)$$

and

$$\mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2) \subset \mathrm{char}(\Delta_2 \circ \Delta_1) \subset \mathrm{char}(\Delta_1) \cup \mathrm{char}(\Delta_2).$$

COROLLARY 2.5. *For any operator* $\Delta \in \mathrm{Diff}_*^{(+)}(A, A)$ *and* $n \geqslant 1$ *one has*

$$\mathrm{char}(\Delta^n) = \mathrm{char}(\Delta).$$

From here till the end of Section 2 we suppose that A has no zero divisors. Note that in this case an element $a \in A$ lies in $\mathrm{char}(\Delta), \Delta \in \mathrm{Diff}_i^{(+)}(A, A)$ if and only if $\delta_a^i(\Delta) = 0$. Note also that the absence of zero divisors in A implies the same for property for $S(A)$.

LEMMA 2.6. *For any two operators* $\Delta_1, \Delta_2 \in \mathrm{Diff}_i^{(+)}(A, P)$ *one has*

$$\mathrm{char}(\Delta_1) \cap \mathrm{char}(\Delta_2) \subset \mathrm{char}(\Delta_1 + \Delta_2).$$

Proof. Let $a \in \mathrm{char}(\Delta_1) \cap \mathrm{char}(\Delta_2)$. It means that there exist two nonzero elements $a_1, a_2 \in A$ such that

$$\delta_a^i(\Delta_j)(a_j) = a_j p_j = 0, \quad j = 1, 2,$$

where $p_j = \delta_a^i(\Delta_j)(1)$. Then $a_1 a_2 \neq 0$ and consequently

$$\delta_a^i(\Delta_1 + \Delta_2)(a_1 a_2) = a_1 a_2 (p_1 + p_2) = 0.$$

Consider an operator $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$ and the sets $J_i \subset S_i$ defined by

$$J_i = J_i(\Delta) = \{s \in S_i \mid \mathrm{char}(\Delta) \subset \mathrm{char}(s)\}.$$

From Lemma 2.6 it follows that these sets are additively closed, while from Proposition 2.3 one has that $J_\Delta = \sum_{i \geqslant 0} J_i$ is an ideal in S.

DEFINITION 2.7. The ideal J_Δ is called the characteristic ideal of the operator $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$.

PROPOSITION 2.8. *The characteristic ideal coincides with its radical, i.e.* $J_\Delta = r(J_\Delta)$.

Proof. Let $s \in S$ and $s = s_1 + \cdots + s_k$ be a decomposition of s in homogeneous elements satisfying $\deg(s_1) < \deg(s_2) < \cdots < \deg(s_k)$. Suppose that $s^n \in J_\Delta$ for some $n > 0$ and prove by induction with respect to k that $s \in J_\Delta$.

For $k = 1$ it follows from Corollary 2.5.

Let now $k > 1$ and suppose that the statement is valid for $k - 1$. Then in the decomposition

$$(s_1 + \cdots + s_k)^n = \sum_{i_1 + \cdots + i_k = n} \frac{i_1! \cdot \ldots \cdot i_k!}{n!} s_1^{i_1} \cdots s_k^{i_k}$$

the summand s_k^n is of the maximal order (which equals to $n \cdot \deg(s_k)$). Hence, $s_k^n \in J_\Delta$ which means that s_k lies in J_Δ as well. Consequently, $(s_1 + \cdots + s_{k-1})^n \in J_\Delta$, from where it follows that $s_1 + \cdots + s_{k-1} \in J_\Delta$ by the induction hypothesis.

PROPOSITION 2.9. *The characteristic ideal is involutive, i.e. it is closed with respect to the Poisson bracket in* $S, \{J_\Delta, J_\Delta\} \subset J_\Delta$.

Proof. It is sufficient to prove the statement for homogeneous elements only. Let $a \in \mathrm{char}(\Delta)$ and $|\Delta_1| \in J_{i_1},\ |\Delta_2| \in J_{i_2}$. Then $\delta_a^{i_1}(\Delta_1) = \delta_a^{i_2}(\Delta_2) = 0$ and consequently

$$\delta_a^{i_1+i_2-1}([\Delta_1, \Delta_2]) = \binom{i_1 + i_2 - 1}{i_1} [\delta_a^{i_1}(\Delta_1), \delta_a^{i_2-1}(\Delta_2)] + \\ + \binom{i_1 + i_2 - 1}{i_1 - 1} [\delta_a^{i_1-1}(\Delta_1), \delta_a^{i_2}(\Delta_2)] = 0.$$

But, by definition, $\{|\Delta_1|, |\Delta_2|\} = |[\Delta_1, \Delta_2]|_{i_1+i_2-1}$.

For any element $a \in A$ define the sets

$$m_i(a) = \{s \in S_i \mid a \in \mathrm{char}(s)\}$$

and $m(a) = \sum_{i \geqslant 0} m_i(a)$.

From Proposition 2.3 and Lemma 2.6 it follows that $m(a)$ is an ideal in S.

PROPOSITION 2.10. *The ideal* $m(a)$ *is primitive for any* $a \in A$.

Proof. Consider elements $s, t \in S$ and let $s = s_1 + \cdots + s_k,\ t = t_1 + \cdots + t_l$ be their decompositions in homogeneous elements such that $\deg(s_1) < \deg(s_2) < \cdots < \deg(s_k)$ and $\deg(t_1) < \deg(t_2) < \cdots < \deg(t_l)$. Suppose that $s \cdot t \in m(a)$ and prove by induction with respect to $k + l$ that either $s \in m(a)$ or $t \in m(a)$.

For $k + l = 2$ it follows from Corollary 2.4.

Let $k + l > 2$ and suppose that the statement is valid for $k + l - 1$. Then in the decomposition $s \cdot t = \sum_{i,j} s_i t_j$ the summand $s_k t_l$ is of the maximal order (and equals to $\deg(s_k) + \deg(t_l)$). Hence, $s_k t_l$ belongs to $m(a)$. From here it follows that either s_k or t_l belong to $m(a)$. Let, say, $s_k \in m(a)$. Then

$$(s_1 + \cdots + s_{k-1})(t_1 + \cdots + t_l) \in m(a)$$

and consequently either $s_1 + \cdots + s_{k-1}$ or $t_1 + \cdots + t_l$ lies in $m(a)$ by the induction hypothesis.

Remark 2.11. The absence of zero divisors in A is necessary for the proof of Lemma 2.6. This Lemma, in its turn, is used to prove that J_i and $m_i(a)$ are closed with respect to summation. Multiplicative properties of these sets are independent of the existence of zero divisors in A. In particular, both J_Δ and $m(a)$ are stable with respect to multiplication by any element of the symbolic ring, while $S \setminus m(a)$ is closed with respect to multiplication.

Remark 2.12. Let $a \in A$. Consider the map $\delta_a^*\colon\ S \to A$ defined by

$$\delta_a^*(s) = \delta_a^i(\Delta), \quad s \in S_i,\ |\Delta| = s.$$

Obviously, δ_a^* is an epimorphism. Denote its kernel by $\overline{m}(a)$. If A possesses no zero divisors, one has $\overline{m}(a) = m(a)$. The map δ_a^* is an algebraic analog of the section of the cotangent bundle corresponding to the form da, while $\overline{m}(a)$ correspond to a Lagrangian submanifold in $T^*(M)$ related to this form (cf. [5]). The ideal $\overline{m}(a)$ is closed with respect to the Poisson bracket. It can be proved in the same way as it was done in Proposition 2.9. On the other hand, the ideal J_Δ plays the role of the Hamilton–Jacobi equation, and in the rest of this Section we deal with the algebraic viewpoint on the relations between characteristicity and solutions of the Hamilton–Jacobi equation corresponding to the operator Δ.

PROPOSITION 2.13. *For any* $a \in \mathrm{char}(\Delta)$ *the ideal* $m(a)$ *contains* J_Δ. *Moreover, one has the following equality* $J_\Delta = \bigcap_{a\in\mathrm{char}(\Delta)} m(a)$.

Proof. It is sufficient to prove the statement for homogeneous components. Let $s \in J_\Delta$ be a homogeneous element and $a \in \mathrm{char}(\Delta)$. Then $a \in \mathrm{char}(s)$, i.e. $s \in m(a)$. Hence, $J_\Delta \subset \bigcap_{a\in\mathrm{char}(\Delta)} m(A)$.

Conversely, if $a \notin J_\Delta$, then $\mathrm{char}(\Delta) \not\subset \mathrm{char}(s)$. Hence, there exists an element $a \in \mathrm{char}(\Delta)$, such that $a \notin \mathrm{char}(s)$. Therefore, $s \notin m(a)$.

PROPOSITION 2.14. *The following statements are equivalent:*

(i) $\mathrm{char}(\Delta) \subset \bigcap_{s\in J_\Delta} \mathrm{char}(s)$

and

(ii) $a \in \mathrm{char}(\Delta)$ *if and only if* $J_\Delta \subset m(a)$.

Remark 2.15. Condition (i) of the previous Proposition holds, for example, in the following situation. Suppose that the operator Δ is such that the module $P' = \bigcap_{a\in\mathrm{char}(\Delta)} \ker \delta_a^i(\Delta)$ is nontrivial. Suppose further that for any nontrivial $q \in Q$ there exists $q^* \in \hom_A(Q, A)$ such that $q^*(q) \neq 0$. Note that the embedding $\mathrm{char}(\Delta) \subset \bigcap_{s\in J_\Delta} \mathrm{char}(s)$ is always valid. Let now $a \notin \mathrm{char}(\Delta)$. Then $\delta_a^i(\Delta)\colon\ P \to Q$ if a monomorphism. Consider an element $p \in P'$ and define a homomorphism $\varphi_p\colon\ A \to P$ by setting $\varphi_p(1) = p$. Choose an element $q^* \in \hom_A(Q, A)$ such that $(q^* \circ \delta_a^i(\Delta))(p) \neq 0$. Then for the operator $\partial = q \circ \Delta \circ \varphi_p$ one has

$$\delta_a^i(\partial) = q^* \circ \delta_a^i(\Delta) \circ \varphi_p \neq 0,$$

i.e. $a \notin \mathrm{char}(\partial)$. On the other hand, if $a' \in \mathrm{char}(\Delta)$, then

$$(\delta_{a'}^i \partial)(1) = (q^* \circ \delta_{a'}^i(\Delta))(p) = 0,$$

i.e. $\partial \in J_\Delta$.

The result obtained allows us to generalize the definition of the characteristics of linear differential operators and to give the following

DEFINITION 2.16. Let $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$. A generalized characteristic of the operator Δ is a graded primitive ideal p of the symbolic ring S, such that $J_\Delta \subset p$. A characteristic manifold of the operator Δ is the space $\mathrm{Spec_{gr}}(S/J_\Delta) \subset \mathrm{Spec_{gr}} S = T^*(A)$.

3. Symbolic Module

Here we define the concept of symbolic module (cf. [2]) which makes it possible to get rid of the absence of zero divisors condition and to define the notion of characteristics in a general algebraic setting.

Consider an operator $\Delta \in \mathrm{Diff}_i^{(+)}(P, Q)$. Then one can define an S-module homomorphism $\sigma(\Delta)\colon\ S(Q, A) \to S(P, A)$ in the following way. For $|\nabla| \in S_k(Q, A)$ we set

$$\sigma(\Delta)(|\nabla|) = |\nabla \circ \Delta|_{k+i} \in S_{k+i}(Q, A).$$

Co-kernel of this homomorphism is called *the symbolic module* of the operator Δ and is denoted by M_Δ: $\mathrm{coker}(\sigma(\Delta)) = M_\Delta$.

Note that an element $s = |\partial|,\ \deg(s) = r$, lies in $\mathrm{Ann}(M_\Delta)$ if and only if for any $\nabla \in \mathrm{Diff}_{i+k}^{(+)}(P, A)$ there exist $\nabla' \in \mathrm{Diff}_{r+k}^{(+)}(Q, A)$ and $\varepsilon \in \mathrm{Diff}_{i+r+k-1}^{(+)}$ (Q, A) such that

$$\partial \circ \nabla = \nabla' \circ \Delta + \varepsilon. \qquad (1)$$

From (1) it follows that if $s \in \mathrm{Ann}(M_\Delta)$, then

$$\mathrm{char}(\Delta) \subset \mathrm{char}(s) \cup \left(\bigcap_{s' \in S(P,A)} \mathrm{char}(s') \right).$$

Hence, if $\bigcap_{s' \in S(P,A)} \mathrm{char}(s')$ is empty, then one has the following

PROPOSITION 3.1. *The annihilator of* M_Δ *lies in* J_Δ, $\mathrm{Ann}(M_\Delta) \subset J_\Delta$.

Due to Proposition 2.8, one has also the following

COROLLARY 3.2. *The radical of* $\mathrm{Ann}(M_\Delta)$ *lies in* J_Δ, $r(\mathrm{Ann}(M_\Delta)) \subset J_\Delta$.

In the sequel we restrict ourselves to the operators Δ for which the symbolic module M_Δ is of finite type. Then

$$\mathrm{Supp}(M_\Delta) = \mathrm{Spec}(S/\mathrm{Ann}(M_\Delta)) \subset T^*(A)$$

and hence, due to Proposition 3.1, for any generalized characteristic $p \in \mathrm{Spec}_{\mathrm{gr}}(A/J_\Delta)$ localization $(M_\Delta)_p$ is nontrivial. It means that the set of generalized characteristics of the operator Δ lies in $\mathrm{Supp}(M_\Delta)$. These remarks motivate another.

DEFINITION 3.3. Let p be a primitive ideal of the algebra A. Denote by $\langle S_p \rangle$ the quotient field of the algebra S/p. A graded primitive ideal p is a generalized characteristic of the operator $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$, if the map

$$s_p(\Delta)\colon\ S(P)/(pS(P)) \otimes_{S/p} \langle S_p \rangle \longrightarrow S(Q)/(pS(Q)) \otimes_{S/p} \langle S_p \rangle,$$

induced by the homomorphism $s(\Delta)\colon\ S(P) \to S(Q)$, possesses a nontrivial kernel.

Denote by $\Lambda^1(A)$ the A-module of 1-forms of the algebra A (see [4]).

THEOREM 3.4. *Suppose that the modules* $\Lambda^1(A), P, Q$ *are projective and of finite type. Let the operator* $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$ *be such that* M_Δ *is of finite type as well. Then* a *graded primitive ideal* $p \in \mathrm{Spec}_{\mathrm{gr}}(S)$ *is a generalized characteristic of* Δ *in the sense of Definition* 3.3 *if and only if* $(M_\Delta)_p \neq 0$.

Proof. Consider a homomorphism $\eta\colon\ S(P) \to \hom_S(S(P, A), S)$ defined by

$$\eta(|\nabla|)(|\partial|) = |\nabla \circ \partial|, \qquad \nabla \in \mathrm{Diff}_*^{(+)}(A, P), \quad \partial \in \mathrm{Diff}_*^{(+)}(P, A).$$

Then η induces a homomorphism

$$\eta_p\colon\ S(P)/(pS(P)) \otimes_{S/p} \langle S_p \rangle \longrightarrow \hom_S(S(P, A), \langle S_p \rangle)$$

for any $p \in \mathrm{Spec}(S)$. Moreover, for any $\Delta \in \mathrm{Diff}_*^{(+)}(P, Q)$ one has a commutative diagram

$$\begin{array}{ccc}
S(P)/(pS(P)) \otimes_{S/p} \langle S_p \rangle & \xrightarrow{s_p(\Delta)} & S(Q)/(pS(Q)) \otimes_{S/p} \langle S_p \rangle \\
\downarrow{\scriptstyle \eta_p} & & \downarrow{\scriptstyle \eta_p} \\
\hom_S(S(P, A), \langle S_p \rangle) & \xrightarrow{\sigma^*(\Delta)} & \hom_S(S(Q, A), \langle S_p \rangle)
\end{array}$$

where $s_p(\Delta)$ is induced by the symbolic map smbl(Δ): $S(P) \to S(Q)$.

The functor $\hom_S(\cdot, \langle S_p\rangle)$ is exact from the right. Hence, the kernel of $s_p(\Delta)$ is trivial if and only if $\hom_S(M_\Delta, \langle S_p\rangle) = 0$. But

$$\hom_S(M_\Delta, \langle S_p\rangle) \simeq \hom_{\langle S_p\rangle}(M_\Delta \otimes_S \langle S_p\rangle, \langle S_p\rangle).$$

Since $\langle S_p\rangle$ is a field, the map $s_p(\Delta)$ is a monomorphism if and only if $M_\Delta \otimes_S \langle S_p\rangle = 0$. On the other hand,

$$M_\Delta \otimes_S \langle S_p\rangle \simeq M_\Delta \otimes_S S_p \otimes_{S_p} \langle S_p\rangle \simeq (M_\Delta)_p \otimes_{S_p} \langle S_p\rangle.$$

Let p_0 be the maximal ideal of the local ring S_p. Then $\langle S_p\rangle \simeq S_p/p_0$ and

$$(M_\Delta)_p \otimes_{S_p} \langle S_p\rangle \simeq \frac{(M_\Delta)_p}{p_0(M_\Delta)_p}.$$

Due to the Nakayama lemma [1], it is equivalent to triviality of the module $(M_\Delta)_p$.

COROLLARY 3.5. *If an ideal* $p \in \mathrm{Spec_{gr}}(S)$ *is a characteristic of the operator* Δ *in the sense of Definition* 2.16, *then* p *is a characteristic in the sense of Definition* 3.3.

To prove the equivalence of Definitions 2.16 and 3.3, we suppose additionally that

(a) the modules P, Q are free,

(b) S possesses the following property of divisibility: if $s, s_1, \ldots, s_r \in S$ and $s \in \bigcap_{i=1}^r(\mathrm{char}(s_i))$, then there exist elements $t_1, \ldots, t_r \in S$, such that

$$s = \sum_{i=1}^r t_i s_i, \tag{2}$$

(c) the number of generators in Q is greater or equal than the one in P, i.e. Δ is not an underdetermined operator (naturally, it can always be done by embedding Q in some R with sufficient number of generator).

Choose free bases in P and Q and represent the operator Δ by the matrix $\|\partial_{jk}\|$, where $\partial_{jk} \in \mathrm{Diff}_*^{(+)}(P, Q)$. Denote by $\nabla_1, \ldots, \nabla_r$ the leading minors of this matrix. Note that though these minors are not well-defined, their symbols are uniquely determined. Hence, there characteristics are well-defined.

PROPOSITION 3.6. $J_\Delta = \sum_{l=1}^r J_{\nabla_l}$.

Proof. Let i be the order of Δ and n be the dimension of the leading minors. Let $a \in A$. Since $\ker \delta_a^i(\Delta) \neq 0$ if and only if $\delta_a^{ni}(\nabla_l) \neq 0$ for all $l = 1, \ldots, r$, one has $\mathrm{char}(\Delta) = \bigcap_l \mathrm{char}(\nabla_l)$. Our result follows now from the divisibility condition formulated above.

Under conditions (a), (b), (c) one also has the following

PROPOSITION 3.7. *The equality* $\mathrm{char}(\Delta) = \bigcap_{s\in J_\Delta} \mathrm{char}(s)$ *holds.*

Thus in the case under consideration, an element $a \in A$ is a characteristic of Δ if and only if $J_\Delta \subset m(a)$ (see Proposition 2.14).

We shall also need the following result valid in terms of conditions (a), (b), (c) formulated above.

LEMMA 3.8. *If* $|\nabla| \in S$*, then* J_∇ *is the principal ideal generated by* $|\nabla|$: $J_\nabla = (|\nabla|)$.

Consider again the localization $s_p(\Delta)$ of the symbol of the operator Δ. Its kernel is nontrivial if and only if all its leading minors (in some coordinate representation) vanish. In other words, the kernel is nontrivial if and only if for each ∇_l there exists an element $t_l \in S \setminus p$, such that $t_l|\nabla_l| \in p$. Since p is a primitive ideal, it is equivalent to the fact that $|\nabla_l| \in p$. This, together with Proposition 3.6 and Lemma 3.8, gives the following

THEOREM 3.9. *If the symbolic module of the operator* Δ *is of finite type and if the conditions* (a), (b), (c) *hold, then Definitions* 2.16 *and* 3.3 *are equivalent.*

COROLLARY 3.10. *Under conditions of Theorem* 3.9 *one has*

$$J_\Delta = r(\mathrm{Ann}\, M_\Delta).$$

4. Characteristics of Morphisms and Junior Symbols

Let, as before, $\mathcal{R}$ be a commutative ring with a unit and B be a commutative unitary $\mathcal{R}$-algebra. Consider B-modules P and Q.

DEFINITION 4.1. A primitive ideal $p \in \mathrm{Spec}\, B$ is called a characteristic of a morphism $\varphi \in \mathrm{hom}_B(P, Q)$, if the kernel of the map

$$\varphi_p\colon\ P/(pP) \otimes_{B/p} \langle B_p\rangle \longrightarrow Q/(pQ) \otimes_{B/p} \langle B\rangle$$

is nontrivial.

Denote the set of characteristics of φ by $\mathrm{char}(\varphi)$. We call $J_\varphi = \bigcap_{p\in \mathrm{char}(\varphi)} p$ *the characteristic ideal* of φ.

EXAMPLE 4.2. Let V be a finite-dimensional space over a field $\mathbf{k}$ and $L \in \mathrm{End}_{\mathbf{k}}(V)$. Consider a $\mathbf{k}[x]$-module structure in V defined in the standard way: $x \cdot v = L(v),\ v \in V$. Consider the trivial endomorphism $\mathbf{o}$ of this module. Then $J_{\mathbf{o}} = r(\det(L - xE))$. If $\mathbf{k}$ is algebraically closed, then $\mathrm{char}(\mathbf{o}) = \mathrm{Spec}(B/J_{\mathbf{o}})$.

Consider relations between $\mathrm{Spec}(B/J_\varphi)$ and the set $\mathrm{char}(\varphi)$.

DEFINITION 4.3. Let $\varphi^*\colon\ Q^* \to P^*$ be a morphism dual to φ. The module $M_\varphi = \operatorname{coker}(\varphi^*)$ is called the symbolic module of φ.

PROPOSITION 4.4. *If P and Q are finitely generated modules, then one has* $\operatorname{char}(\varphi) = \operatorname{Supp}(M_\varphi)$.

Proof. Consider the natural isomorphism

$$\eta_p\colon\ P/(pP) \otimes_{B/p} \langle B_p\rangle \longrightarrow \hom_B(P^*, \langle B_p\rangle),$$

where $p \in \operatorname{Spec}(B)$. Then one has a commutative diagram

$$\begin{array}{ccc} P/(pP) \otimes_{B/p} \langle B_p\rangle & \xrightarrow{\ \varphi_p\ } & Q/(pQ) \otimes_{B/p} \langle B_p\rangle \\ \Big\downarrow{\scriptstyle \eta_p} & & \Big\downarrow{\scriptstyle \eta_p} \\ \hom_B(P^*, \langle B_p\rangle) & \xrightarrow{\ \varphi_p^{**}\ } & \hom_B(Q^*, \langle B_p\rangle) \end{array}$$

The rest of the proof literary repeats the proof of Theorem 3.4.

COROLLARY 4.5. $J_\varphi = r(\operatorname{Ann} M_\varphi)$.

THEOREM 4.6. *Let the conditions of Proposition* 4.4 *be valid. Then a primitive ideal p lies in* $\operatorname{char}(\varphi)$ *if and only if $J_\varphi \subset p$, i.e.*

$$\operatorname{char}(\varphi) = \operatorname{Spec}(B/J_\varphi).$$

Proof. Let $J_\varphi \subset p$. Then $\operatorname{Ann}(M_\varphi) \subset p$ and, consequently, $p \in \operatorname{char}(\varphi)$.

Note that the result of previous subsections remain valid in the category of graded modules over a graded algebra B. Suppose additionally that the graded objects $B,\ P$, and Q are associated with filtered ones, $B',\ P',\ Q'$, i.e. $B' = \bigcap_i B'_i,\ B'_{i-1} \subset B'_i,\ B_i = B'_i/B'_{i-1}$, etc. Let $\varphi = \varphi_n\colon\ P \to Q$ be a morphism of the grading $n \geqslant 0$ associated with a filtered morphism $\varphi' = \varphi'_n\colon\ P' \to Q'$. Denote the characteristic ideal of φ by $J^n_\varphi = J^n$. Obviously, $\varphi(J^n P) \subset J^n Q$. Consider the natural homomorphism of graded B/J^n-modules

$$\varphi_{J^n}\colon\ P/J^n P \longrightarrow Q/J^n Q.$$

Let $a \in P,\ \deg(a) = i$. Then the class aJ^nP lies in $\ker(\varphi_{J^n})$ if and only if $\varphi'(a') = b' + b''$, where $|a'| = a,\ |b'| = b \in J^n Q,\ \deg(b'') = n + i - 1$ (as before, by $|s|$ we denote an element of the graded object corresponding to the element s from the filtered object). Since the element a' is defined up to a sum of elements $a'',\ |a''| \in J^n P$, and $a''',\ \deg(a''') = i - 1$, then one has a well-defined morphism $\varphi_{n-1}\colon\ \ker(\varphi_{J^n}) \to \operatorname{coker}(\varphi_{J^n})$ of the grading $n-1$. Hence, one can define the set

$$\operatorname{char}(\varphi_{n-1}) \subset \operatorname{Spec}(B/J^n) \subset \operatorname{Spec}(B)$$

and the ideal $J_{\varphi_{n-1}}$. Denote by $J^{n-1} = J^{n-1}_\varphi$ the inverse image of $J_{\varphi_{n-1}}$ under the natural projection $B \to B/J^n$.

PROPOSITION 4.7. *For all* $j = n-1, \dots, 0$ *one has well-defined morphisms*

$$\varphi_j\colon\ \ker(\varphi_{J^{j+1}}) \longrightarrow \operatorname{coker}(\varphi_{J^{j+1}}),$$

where ideals J^{j+1} *and morphisms*

$$\varphi_{J^{j+1}}\colon\ \frac{\ker(\varphi_{J^{j+2}})}{J_{\varphi_{j+1}}\ker(\varphi_{J^{j+2}})} \longrightarrow \frac{\operatorname{coker}(\varphi_{J^{j+2}})}{J_{\varphi_{j+1}}\operatorname{coker}(\varphi_{J^{j+2}})}$$

are naturally constructed using the morphisms φ_{j+1}.

Proof. Use the inverse induction by j. For $j = n-1$ the statement has been proved already. Let now $j < n-1$ and suppose that φ_j has been constructed. Consider natural projections

$$\pi_k\colon\ \operatorname{coker}(\varphi_{J^k}) \longrightarrow \frac{\operatorname{coker}(\varphi_{J^k})}{J_{\varphi_{k-1}}\operatorname{coker}(\varphi_{J^k})}$$

and

$$\rho_k\colon\ \frac{\operatorname{coker}(\varphi_{J^k})}{J_{\varphi_{k-1}}\operatorname{coker}(\varphi_{J^k})} \longrightarrow \operatorname{coker}(\varphi_{J^{k-1}}),$$

where $k = n+1, \dots, j$. Then it can be easily seen that an element $a' \in P'_i$ determines an element $\overline{a} \in \ker(\varphi_{J^i})$ if and only if

$$\varphi'(a') = b' + \varphi'(a'') + b'',$$

where

- b' is of filtration $i+n$,
- $(\rho_{j+2} \circ \pi_{j+2} \circ \dots \circ \rho_{n+1} \circ \pi_{n+1})(|b'|) \in J_{\varphi_j}\operatorname{coker}(\varphi_{J^{j+1}})$,
- a'' is of filtration $i-1$,
- b'' is of filtration $j-1$.

Hence there exists a well-defined element

$$\varphi_{j-1}(\overline{a}) = (\rho_{j+1} \circ \pi_{j+1} \circ \dots \circ \rho_{n+1} \circ \pi_{n+1})(|b''|).$$

DEFINITION 4.8. The morphism

$$\varphi_j\colon\ \ker(\varphi_{J^{j+1}}) \longrightarrow \operatorname{coker}(\varphi_{J^{j+1}})$$

defined above is called the j-th symbol of the morphism φ, $j = n-1, \dots, 0$.

Let $J^j_\varphi = J^j$ be the inverse image of J_{φ_j} in B. Obviously, one has a sequence of embeddings

$$J^n \subset J^{n-1} \subset \dots \subset J^1 \subset J^0 \subset B.$$

DEFINITION 4.9. The ideal J^j_φ is called the characteristic ideal of the rank j for the morphism φ. A primitive graded ideal

$$p \in \mathrm{Spec}_{\mathrm{gr}}(B/J^j) \subset \mathrm{Spec}_{\mathrm{gr}}(B)$$

is called a characteristic of the rank j for the morphism φ.

Denote the set of such characteristics by $\mathrm{char}_j(\varphi)$. Obviously, one has a sequence of embeddings

$$\mathrm{char}(\varphi) = \mathrm{char}_0(\varphi) \subset \mathrm{char}_1(\varphi) \subset \cdots \subset \mathrm{char}_n(\varphi) \subset \mathrm{Spec}_{\mathrm{gr}}(B).$$

THEOREM 4.10. *Let* $\ker(\varphi_{J^{j+1}})$ *and* $\mathrm{coker}(\varphi_{J^{j+1}})$ *be projective* B/J^{j+1} *modules of finite type. Then a primitive graded ideal* $p \in \mathrm{Spec}_{\mathrm{gr}}(B)$ *belongs to* $\mathrm{char}_j(\varphi)$ *if and only iff the kernel of the map*

$$(\varphi_j)_p\colon\ \frac{\ker(\varphi_{J^{j+1}})}{p\ker(\varphi_{J^{j+1}})} \otimes_{B/p} \langle B_p\rangle \longrightarrow \frac{\mathrm{coker}(\varphi_{J^{j+1}})}{p\,\mathrm{coker}(\varphi_{J^{j+1}})} \otimes_{B/p} \langle B_p\rangle$$

is nontrivial.

Remark 4.11. Let B, C be unitary commutative algebras over a commutative ring $\mathcal{R}$. Let $f\colon\ B \to C$ be a homomorphism due to which C can be considered as an B-algebra. If P is an B-module, then $C \otimes_B P$ is a C-module. This construction is an algebraic analog of the induced bundle. In particular, if J is an ideal in B, then an B/J-module $P/(JP) \simeq (B/J) \otimes_B P$ geometrically corresponds to the restriction onto a submanifold determined by the ideal J. Thus one can see that the $(n-1)$st symbol is defined on submanifold of degeneration of the senior symbol, the next one is defined on the submanifold of degeneration of the $(n-1)$-st, etc.

Remark 4.12. Junior symbols of a differential operator $\Delta\colon\ P \to Q$ arise, when one considers S for B and an S-module homomorphism $\varphi = \mathrm{smbl}(\Delta)\colon\ S(P) \to S(Q)$ and applies the results of this section to φ.

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