Solving First Order ODEs using Linear Transformations

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One of the most attractive aspects of Lie's method of symmetries for differential equations (DEs) is its generality: roughly speaking, all solving methods for DEs can be correlated to particular forms of the corresponding symmetry generators [1, 2]. However, for first order ordinary differential equations (ODEs), Lie's method appears in principle not as useful as in the higher order case. The problem is that the determining PDE - whose solution gives by the infinitesimals of the symmetry group - has the original first order ODE in its characteristic strip. Hence, the finding of these infinitesimals requires the solving of the original ODE, which in turn is what we want to solve using these infinitesimals; thus invalidating the approach in the first order ODE case.

In the higher order case, the strategy consists of restricting the cases handled to the universe of ODEs having *point* symmetries, so that the infinitesimals depend on just two variables, and then the determining PDE splits into a system of PDEs. Although few second or higher order ODEs have point symmetries, and the solving of such a PDE system for the infinitesimal may be a major problem in itself [3], the hope is that one will be able to solve it by taking advantage of the fact that it is overdetermined. One of the motivating aspects in this approach is also that point transformations form a group.

This approach however is not useful in the first order ODEs case for which point symmetries already represent the most general case. The alternatives left then, roughly speaking, consists of: looking for particular solutions to the determining PDE [4], or restricting the form of the infinitesimals, such that the problem can be formulated in terms of an *overdetermined* linear PDE system [5, 6]. The question in this latter approach, however, is: what would be an "appropriate restriction" on the symmetries leading to

- An ODE problem which is non-trivial and nevertheless solvable in that the determination of the symmetries of the types considered, when they exist, is systematic;
- The invariant ODE families related to these restricted infinitesimals include a reasonable variety of cases typically arising in mathematical physics;
- The related finite transformations form a group.

Bearing this in mind, this paper develops an approach for finding symmetries of the form

$$\xi = F(x),\ \eta = P(x) + y(x)\,Q(x) \tag{1}$$

where $\{\xi,\ \eta\}$ are the infinitesimals, the symmetry generator is given by $\xi\,\frac{\partial}{\partial x} + \eta\,\frac{\partial}{\partial y}$, and $\{x, y(x)\}$ are respectively the independent and dependent variables. This symmetry pattern has interesting features. The finite transformation related to (1) is actually of the same form

$$x = f(t),\ y(x) = p(t)\,u(t) + q(t) \tag{2}$$

where $\{t, u(t)\}$ are respectively the new independent and dependent variables, and $\{f,\ p,\ q\}$ are arbitrary functions of t. These transformations also form a group so that any two ODEs having a symmetry of the form (1) can be transformed into each other by means of (2).

Also, the invariant ODE family associated to (1) includes all of the separable, linear, Bernoulli and homogeneous ODEs. For polynomial ODEs, e.g. of Riccati or Abel type, (2) defines their respective classes. In the case of Abel ODEs, this fact plus the fact that separable ODEs have symmetries of the form (1) means that there are Abel ODE classes, all of whose members can be transformed into separable ODEs by means of (2)[1]. For polynomial ODEs of degree higher than 3, the equivalent problem was discussed by Chini [9] at the beginning of the twentieth century. Riccati type ODEs are also a subset of the invariant family implied by (1). Actually, despite the simple form of (1), its related invariant family includes as particular cases most of the first order ODE types classified or solvable found in the literature.

For the purpose of this work, the key observation is that *"all first order ODE having a symmetry of the form (1) can be transformed into separable ODEs by means of a linear transformation of the form (2)"*. In connection, this work presents a systematic algorithm for both determining if a given ODE has a symmetry of the form (1)[2]; and if so, explicitly finding a transformation of the form (2) leading to a separable ODE.

Both the existence of this symmetry and the determination of $\{f,\ p,\ q\}$ such that (2) transforms the input ODE into separable are systematic and performed without solving any auxiliary differential equations. Due to its simplicity and the varied ODE problems covered at once, this method appeared to us appropriate for a computer algebra implementation in the framework of ODE solvers and we have implemented it in Maple R5. This implementation alone is able to successfully solve $\approx 70\%$ of Kamke's first order examples as well as various ODE families not solved by ODE-solvers of other popular computer algebra systems. Some estatistics and examples illustrating the type of ODE problem which can be solved using this method are also shown.

References

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