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Algorithms for Differential Invariants of Symmetry Groups of Differential Equations

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Abstract. We develop new computational algorithms, based on the method of equivariant moving frames, for classifying the differential invariants of Lie symmetry pseudo-groups of differential equations and analyzing the structure of the induced differential invariant algebra. The Korteweg–deVries (KdV) and Kadomtsev–Petviashvili (KP) equations serve to illustrate examples. In particular, we deduce the first complete classification of the differential invariants and their syzygies of the KP symmetry pseudo-group.

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1. Introduction

Differential invariants play a central role in a wide variety of problems arising in geometry, differential equations, mathematical physics, and applications. These include equivalence problems for geometric structures [**26**], classification of invariant differential equations and invariant variational problems [**15**], [**25**], [**32**], [**34**], integration of ordinary differential equations [**25**], [**32**], equivalence and symmetry properties of solutions (or submanifolds) [**26**], the construction of particular solutions to systems of partial differential equations [**22**], [**23**], [**24**], [**32**], [**37**], the invariant variational bicomplex and its cohomology [**2**], [**33**], [**36**], the construction of moduli spaces of solutions [**14**], object recognition in computer vision [**3**], and the design of invariant numerical methods [**27**]. Understanding the structure of the underlying algebra of differential invariants for a given Lie group or pseudo-group action is an essential first step in developing these areas of application. The goal of this paper is to develop constructive computational algorithms that expose the differential and algebraic structure of the space of differential invariants of Lie group and pseudo-group actions. While our constructions are completely general, the focus of this paper will be on symmetry (pseudo-)groups of systems of differential equations.

Given a system of differential equations, the moving frame techniques developed in [**5**], [**29**], [**30**], [**31**] are used to obtain the structure equations of its symmetry groups directly from the infinitesimal determining equations. In this paper we further develop and implement the moving frame calculus for analyzing and classifying the associated differential invariants, and illustrate our algorithms in the context of two representative examples: the symmetry (pseudo-)groups of the Korteweg–deVries (KdV) and Kadomtsev–Petviashvili (KP) equations. The first has a finite-dimensional symmetry group, and so could be treated by the finite-dimensional moving frame methods developed earlier in [**12**]. However, every finite-dimensional Lie group action is also an example of a pseudo-group action, and so this example was chosen because (a) the calculations are relatively simple, and (b) it serves to compare and contrast the two algorithms. The KP equation is technically more challenging, and we defer the analysis of its symmetry pseudo-group until the end of the paper, where we deduce the first complete classification of its differential invariants and their syzygies. It should be emphasized that these two examples were chosen due to their familiarity and interest in applications, and *not* because of their integrability or remarkable soliton properties. (Indeed, we make no use of the higher-order symmetries that underlie their integrability [**6**], [**8**], [**25**].) Our algorithms are completely general, and can be readily applied to arbitrary systems of differential equations, possessing either finite- or infinite-dimensional symmetry groups.

In general, we will study the action of a Lie pseudo-group $\mathcal{G}$—either finite- or infinite-dimensional—on an m-dimensional manifold M and the induced action on submanifolds $N \subset M$ of a fixed dimension $0 < p < m$. We focus on the case when $\mathcal{G}$ is the symmetry group of some system of differential equations in p

independent variables and $q = m - p$ dependent variables; the submanifolds represent the graphs of candidate solutions. As stressed by Cartan, local equivalence and symmetry properties of submanifolds (solutions) are entirely prescribed by the differential invariants of the pseudo-group action, and so their classification is an essential first step in the detailed analysis of the induced pseudo-group action.

In a seminal paper, Tresse [**35**] outlined a proof of a Basis Theorem, stating that, under some vague hypotheses, the algebra[1] $\mathbf{I}_{\mathcal{G}}$ of differential invariants is finitely generated. More concretely, there exist a finite number of generating differential invariants $I_1, \ldots, I_k$, along with invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, such that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives: $\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_\nu} I_\kappa$, $\kappa = 1, \ldots, k$, $\nu = \#J \geq 0$. The underlying structure of $\mathbf{I}_{\mathcal{G}}$ is subject to the following complications:

- While the number p of independent invariant differential operators is fixed a priori by the dimension of the submanifolds (or, equivalently, by the number of independent variables), the number k of generating differential invariants, and their precise orders, depend on the pseudo-group and are difficult to predict in advance.
- The invariant differential operators $\mathcal{D}_j$ do not necessarily commute. Thus, effective computations in $\mathbf{I}_{\mathcal{G}}$ will, of necessity, rely on the methods from noncommutative differential algebra [**13**], [**22**].
- In general, the differentiated invariants are not necessarily functionally independent, and are subject to certain functional relations or *syzygies*

$$\mathcal{S}(\ldots, \mathcal{D}_J I_\kappa, \ldots) \equiv 0.$$

  A well-known example of a differential invariant syzygy is the Codazzi equation relating derivatives of the principal curvatures (or, equivalently, the Gauss and mean curvatures), which are the generating differential invariants in the geometry of surfaces $S \subset M = \mathbb{R}^3$ under the action of the Euclidean group [**4**], [**15**]. Finding and classifying these syzygies is essential to understanding the structure of, as well as computing in, the differential invariant algebra $\mathbf{I}_{\mathcal{G}}$.

Rigorous formulations and proofs of the Basis Theorem in the case of finite-dimensional Lie group actions can be found in [**26**], [**32**]. For infinite-dimensional pseudo-groups, rigorous modern formulations, based on the machinery of Spencer cohomology, can be found in Kumpera [**17**] and, more recently, in Kruglikov and Lychagin [**16**]. Both versions impose Spencer cohomological growth bounds on the

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[1] In our geometric approach to the subject, the term "algebra" is to be taken in a loose sense. We classify differential invariants up to functional dependency [**25**]. Keep in mind that differential invariants may only be locally defined, and so functional combinations must respect the various domains of definition. A more technically precise development would recast everything in the language of sheaves [**17**], [**38**]. However, as our primary target audience is oriented toward applications, we will refrain from this additional technicality, and proceed to work locally on suitable open subsets of the indicated manifolds and bundles.

prolonged pseudo-group action, and are purely existential. In [**31**] we will present a fully constructive proof of the Basis Theorem for free actions (as defined below), based on moving frames and Gröbner basis techniques applied to the symbol module of the determining equations for the pseudo-group. As a consequence, our algorithms will identify the generating differential invariants and produce their differential syzygies, terminating in finitely many steps.

Our approach to the subject is founded on a new, equivariant formulation of Cartan's method of *moving frames* that was initiated in [**11**], [**12**], and then developed through a series of papers, including [**15**], [**29**], [**30**]. The construction of moving frames for finite-dimensional group actions can be effectively extended to infinite-dimensional pseudo-groups by adopting the pseudo-group jet bundle coordinates in the role of the group parameters. Once a moving frame is fixed, the task of explicit construction, via *invariantization*, of differential invariants of all orders, as well as invariant differential forms, invariant differential operators, etc., becomes a routine algorithmic procedure. The resulting *recurrence formulas*, relating normalized and differentiated invariants, can then be used to prescribe a minimal generating set of differential invariants, and, once the commutation formulas for the invariant differential operators have been established, to complete classification of the syzygies among the differentiated invariants. This procedure relies essentially on the associated Maurer–Cartan forms which, for Lie pseudo-groups (and groups), are realized as invariant contact forms on the pseudo-group jet bundle [**29**], [**5**]. Importantly, as opposed to the results in [**11**], [**12**], the new algorithms divulge the structure of the symmetry group of an arbitrary system of PDEs, and the recurrence relations and syzygies among the differential invariants of the symmetry group action. Moreover, in contrast to the methods in [**11**], [**12**], the algorithms we develop, strikingly, require only linear algebra and differentiation, and do not require any explicit formulas for either the moving frame, or the differential invariants and invariant differential operators, or even the Maurer–Cartan forms!

Our methods require that the prolonged pseudo-group action be locally free at sufficiently high order; see Theorem 4.1. Local freeness can be immediately checked by computing the dimension of the space of prolonged infinitesimal generators, and is a much simpler geometric counterpart of Kumpera's more technical Spencer cohomological growth bounds. A significant challenge is to extend our methods to nonfree actions that possess nontrivial differential invariants.

2. Preliminaries

Throughout, we will use the basic framework and notation of [**25**], [**26**] without further comment. We are concerned with the point[2] symmetry group of a system

[2] In this paper we restrict our attention to point symmetries. Adapting our methods to, say, projectable (fiber-preserving) or contact symmetry groups [**26**] is straightforward. (However, higher-order symmetries remain an unexplored challenge.)

of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k, \tag{1}$$

involving p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, and their derivatives u^α_J up to some finite order n. We regard $z = (x, u)$ as local coordinates on the *total space* M, a manifold of dimension $m = p + q$, and so the system defines a subvariety $\mathcal{S}_\Delta \subset \mathrm{J}^n(M, p)$ of the nth-order (extended) jet bundle of p-dimensional submanifolds of M, that is, graphs of functions $u = f(x)$. To avoid unnecessary technicalities, the system (1) is assumed to be locally solvable [**25**] and define a regular submanifold of $\mathrm{J}^n(M, p)$.

Let $\mathcal{X}(M)$ denote the space of smooth vector fields

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{2}$$

on M. Let

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \widehat{\varphi}^\alpha_J \frac{\partial}{\partial u^\alpha_J} \tag{3}$$

denote the nth-order prolongation of the vector field to $\mathrm{J}^n(M, p)$, whose coefficients[3] are given by the well-known prolongation formula [**25**],

$$\widehat{\varphi}^\alpha_J = D_J\left(\varphi^\alpha - \sum_{i=1}^{p} u^\alpha_i \xi^i\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\xi^i, \tag{4}$$

obtained by repeatedly applying the total derivatives $D_i = D_{x^i}, i = 1, \ldots, p$, to its characteristic. Observe that each $\widehat{\varphi}^\alpha_J$ is a certain linear function of the derivatives $\xi^i_A = \partial^{\#A}\xi^i/\partial z^A$, $\varphi^\alpha_A = \partial^{\#A}\varphi^\alpha/\partial z^A$, of the vector field coefficients with respect to all variables $z = (x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ whose coefficients are certain well-prescribed polynomials of the derivative coordinates u^β_K.

A vector field $\mathbf{v} \in \mathcal{X}(M)$ is an *infinitesimal symmetry* of the system of differential equations (1) if and only if it satisfies the infinitesimal invariance condition

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \quad \text{on } S_\Delta \qquad \text{for all} \quad \nu = 1, 2, \ldots, k. \tag{5}$$

When expanded out, this forms an overdetermined system of homogeneous linear partial differential equations for the coefficients ξ^i, φ^α of the vector field (2). We let

$$\mathcal{L}(\ldots, x^i, \ldots, u^\alpha, \ldots, \xi^i_A, \ldots, \varphi^\alpha_A, \ldots) = 0 \tag{6}$$

[3] We place hats on the prolonged vector field coefficients $\hat{\varphi}^\alpha_J$ in order to distinguish them from partial derivatives of the vector field coefficients with respect to the independent and dependent variables, to be denoted by φ^α_A.

denote the completion of the system of *infinitesimal determining equations*, which includes the original determining equations along with all equations obtained by repeated differentiation.

The solution space $\mathfrak{g} \subset \mathcal{X}(M)$ to the infinitesimal determining equations (6) is the Lie algebra[4] of infinitesimal symmetries of the system (1), and can be either finite- or infinite-dimensional. In [**5**] we developed new algorithms for directly determining the structure of the symmetry algebra $\mathfrak{g}$ that completely avoided integration of the determining equations. The goal of the present paper is to develop analogous computational algorithms for studying the structure of its differential invariant algebra $\mathbf{I}_{\mathcal{G}}$.

2.1. *The KdV Equation*

Our running example, chosen for its simplicity, will be the celebrated Korteweg–deVries (KdV) equation [**1**], [**25**],

$$u_t + u_{xxx} + u u_x = 0. \tag{7}$$

The total space $M = \mathbb{R}^3$ has coordinates (t, x, u), and its solutions $u = f(t, x)$ define $p = 2$-dimensional submanifolds of M. The prolongation of a vector field

$$\mathbf{v} = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \varphi(t, x, u)\frac{\partial}{\partial u}$$

on M to $\mathrm{J}^n(M, 2)$ has the form

$$\mathbf{v}^{(n)} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \sum_{\#J \geq 0} \widehat{\varphi}^J \frac{\partial}{\partial u_J},$$

whose coefficients, in view of (4), are given by the explicit formulas

$$\begin{aligned} \widehat{\varphi}^t &= \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u, \\ \widehat{\varphi}^x &= \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u, \\ &\vdots \end{aligned} \tag{8}$$

The vector field $\mathbf{v}$ is an infinitesimal symmetry of the KdV equation if and only if

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \widehat{\varphi}^t + \widehat{\varphi}^{xxx} + u\,\widehat{\varphi}^x + u_x\,\widehat{\varphi} = 0 \qquad \text{whenever} \quad u_t + u_{xxx} + u u_x = 0.$$

[4] Technically, since they may only be locally defined, only those symmetry vector fields in $\mathfrak{g}$ that are defined on a common open subset will form a bona fide lie algebra. Since we work locally anyway, we can ignore this minor complication.

Substituting the prolongation formulas (8), and equating the coefficients of the independent derivative monomials to zero, leads to the infinitesimal determining equations which together with their differential consequences reduce to the system

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0, \qquad \varphi = \xi_t - \tfrac{2}{3} u \tau_t, \qquad \varphi_u = -\tfrac{2}{3}\tau_t = -2\xi_x, \tag{9}$$

while all the derivatives of the components of order two or higher vanish. The general solution

$$\tau = c_1 + 3c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2c_4 u,$$

defines the four-dimensional KdV symmetry algebra with the basis given by

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = t\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3t\partial_t + x\partial_x - 2u\partial_u. \tag{10}$$

3. Structure of Lie Pseudo-Groups

Each vector field in the symmetry algebra $\mathfrak{g}$ generates a one-parameter local transformation group. These combine to form the (connected component of) the symmetry pseudo-group $\mathcal{G}$ of the system, which forms a sub-pseudo-group of the pseudo-group $\mathcal{D} = \mathcal{D}(M)$ of all local diffeomorphisms of the total space M. Let us briefly discuss the structure and geometry of the diffeomorphism and symmetry pseudo-groups, referring the reader to [**5**], [**29**] for details.

For $0 \le n \le \infty$, let $\mathcal{D}^{(n)} \to M$ be the subbundle of $\mathrm{J}^n(M, M)$ consisting of the nth-order jets, $\mathrm{j}_n\psi$, of local diffeomorphisms $\psi : M \to M$. Local coordinates $(x, u, X^{(n)}, U^{(n)})$ on $\mathcal{D}^{(n)}$ consist of the source (base) coordinates x^i, u^α on M, the corresponding target coordinates[5] X^i, U^α, along with their derivatives X^i_A, U^α_A, $1 \le \#A \le n$, with respect to the source coordinates. We view the jet coordinates X^i_A, U^α_A as representing group parameters of the diffeomorphism pseudo-group $\mathcal{D}$.

The local coordinate expressions for the prolonged action of a local diffeomorphism of M on the submanifold jet bundle $\mathrm{J}^n(M, p)$ are obtained by implicit differentiation. In view of the chain rule, this action only depends on nth-order derivatives of the diffeomorphism at the base point, and so factors through $\mathcal{D}^{(n)}$. To formalize the process, we introduce the *lifted horizontal coframe*

$$\mathrm{d}_H X^i = \sum_{j=1}^{p} (D_j X^i)\, \mathrm{d}x^j = \sum_{j=1}^{p} \left(X^i_{x^j} + \sum_{\alpha=1}^{q} u^\alpha_j X^i_{u^\alpha} \right) \mathrm{d}x^j, \quad i = 1, 2, \ldots, p, \tag{11}$$

where d_H denotes the horizontal differential. Their coefficients depend on the first-order jet coordinates $X^i_{x^j}$, $X^i_{u^\alpha}$ on $\mathcal{D}^{(1)}$, along with the first-order jet coordinates

[5] Throughout, we adopt Cartan's convention that source coordinates are denoted with lower-case letters, while target coordinates of diffeomorphisms and their jets are denoted by the corresponding upper-case letters.

u^α_j on $\mathrm{J}^1(M, p)$. Thus, strictly speaking, the lifted horizontal coframe consists of p one-forms on the bundles $\mathcal{E}^{(n)} \to \mathrm{J}^n(M, p)$ obtained by forming the pull-back bundle of $\mathcal{D}^{(n)} \to M$ under the usual jet projection $\pi^n : \mathrm{J}^n(M, p) \to M$. Coordinates on $\mathcal{E}^{(n)}$ consist of the submanifold jet coordinates x^i, u^α_J along with the diffeomorphism jet coordinates (or group parameters) X^i_A, U^α_A.

The dual *implicit total differential operators*, denoted $D_{X^1}, \ldots, D_{X^p}$, are defined so that

$$\mathrm{d}_H F = \sum_{j=1}^{p} (D_{X^i} F)\, \mathrm{d}_H X^i \qquad \text{for any function} \quad F : \mathcal{E}^{(n)} \to \mathbb{R}. \tag{12}$$

The prolonged action of a diffeomorphism jet $(X^{(n)}, U^{(n)}) \in \mathcal{D}^{(n)}$ maps the submanifold jet $(x, u^{(n)}) \in \mathrm{J}^n(M, p)$ to the target jet $(X, \widehat{U}^{(n)}) \in \mathrm{J}^n(M, p)$, whose components[6]

$$\widehat{U}^\alpha_J = D_{X^{j_1}} \cdots D_{X^{j_k}} U^\alpha, \qquad 0 \le k = \#J \le n, \quad \alpha = 1, \ldots, q, \tag{13}$$

are obtained by repeatedly applying the implicit differential operators to the target dependent variables $U^\alpha = \widehat{U}^\alpha$.

Warning: In these formulas, as in (11), the total derivatives $D_i = D_{x^i}$ act on both the submanifold jet coordinates u^α_J and the diffeomorphism jet coordinates X^i_A, U^α_A in a natural manner. See [**29**], [**30**] for full details.

The symmetry group $\mathcal{G}$ forms a sub-pseudo-group of the diffeomorphism pseudo-group $\mathcal{D}$, and hence its nth-order jets determine a subbundle[7] $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$. When $n < \infty$, we let r_n be the fiber dimension of the subbundle $\mathcal{G}^{(n)}$, which can be identified with the pseudo-group dimension at order n. Clearly,

$$0 \le r_0 \le r_1 \le r_2 \le \cdots . \tag{14}$$

In the finite-dimensional case when the pseudo-group $\mathcal{G}$ represents the (local) action of a Lie group G, the fiber dimensions stabilize: $r_n = \widehat{r}$ for $n \gg 0$, where $\widehat{r} \le r = \dim G$, with equality under the mild restriction that G acts locally effectively on subsets of M [**28**]. On the other hand, for infinite-dimensional pseudo-group actions, the fiber dimensions continue to increase without bound as $n \to \infty$. Local coordinates on $\mathcal{G}^{(n)}$ consist of the source coordinates x^i, u^α on M along with r_n group parameters $\lambda^{(n)} = (\lambda_1, \ldots, \lambda_{r_n})$ that serve to parametrize the fibers.

The prolonged action of the pseudo-group $\mathcal{G}$ on the submanifold jets $\mathrm{J}^n(M, p)$ is then given by restricting the prolonged diffeomorphism action (13) to $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$. Alternatively, once a parametrization of the pseudo-group subbundle is specified, one can directly apply the induced implicit differentiation operators, as in (13).

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[6] As before, we place hats over the transformed submanifold jet coordinates to avoid confusing them with the diffeomorphism jet coordinates U^α_A.

[7] As always, we are assuming regularity of the symmetry pseudo-group.

3.1. *The KdV Symmetry Pseudo-Group*

When $M = \mathbb{R}^3$ has coordinates (t, x, u), the induced coordinates on the diffeomorphism jet bundle $\mathcal{D}^{(n)}$ are denoted by

$$(t, x, u, T, X, U, T_t, T_x, T_u, X_t, X_x, X_u, U_t, U_x, U_u,\\ T_{tt}, T_{tx}, T_{xx}, T_{tu}, T_{xu}, T_{uu}, X_{tt}, X_{tx}, X_{xx}, \ldots).$$

By integrating the infinitesimal symmetries (10), we recover the action of the KdV symmetry group $\mathcal{G}_{\mathrm{KdV}}$ on M, which can be obtained by composing the flows of the symmetry algebra basis and is given by

$$\begin{aligned}(T, X, U) &= \exp(\lambda_4 \mathbf{v}_4) \circ \exp(\lambda_3 \mathbf{v}_3) \circ \exp(\lambda_2 \mathbf{v}_2) \circ \exp(\lambda_1 \mathbf{v}_1)(t, x, u)\\ &= (e^{3\lambda_4}(t + \lambda_1),\ e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2),\ e^{-2\lambda_4}(u + \lambda_3)),\end{aligned} \tag{15}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the group parameters. A parametrization of the subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is obtained by repeatedly differentiating T, X, U with respect to t, x, u, which yields the expressions

$$\begin{gathered} T = e^{3\lambda_4}(t + \lambda_1), \quad X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2), \quad U = e^{-2\lambda_4}(u + \lambda_3),\\ T_t = e^{3\lambda_4}, \quad T_x = 0, \quad T_u = 0, \quad X_t = \lambda_3 e^{\lambda_4}, \quad X_x = e^{\lambda_4}, \quad X_u = 0,\\ U_t = 0, \quad U_x = 0, \quad U_u = e^{-2\lambda_4}, T_{tt} = 0, \quad T_{tx} = 0, \quad T_{xx} = 0,\\ T_{tu} = 0, \quad T_{xu} = 0, \quad T_{uu} = 0, \quad X_{tt} = 0, \quad X_{tx} = 0, \quad X_{xx} = 0, \quad \ldots, \end{gathered} \tag{16}$$

implying that the fiber dimension of $\mathcal{G}^{(n)}$ is $r_n = 4 = r = \dim \mathcal{G}$ for all $n \geq 1$.

The lifted horizontal coframe (11), when restricted to $\mathcal{G}$, is

$$\begin{aligned} \mathrm{d}_H T &= (T_t + u_t T_u)\,\mathrm{d}t + (T_x + u_x T_u)\,\mathrm{d}x = e^{3\lambda_4}\,\mathrm{d}t,\\ \mathrm{d}_H X &= (X_t + u_t X_u)\,\mathrm{d}t + (X_x + u_x X_u)\,\mathrm{d}x = \lambda_3 e^{\lambda_4}\,\mathrm{d}t + e^{\lambda_4}\,\mathrm{d}x, \end{aligned} \tag{17}$$

with dual implicit differentiation operators

$$D_T = e^{-3\lambda_4} D_t - \lambda_3 e^{-3\lambda_4} D_x, \qquad D_X = e^{-\lambda_4} D_x, \tag{18}$$

where now D_t, D_x are the usual total derivative operators on $\mathrm{J}^\infty(M, 2)$. A repeated application of these to $\widehat{U} = U = e^{-2\lambda_4}(u + \lambda_3)$, as in (13), produces the explicit formulas for prolonged action of $\mathcal{G}$ on the submanifold jet space $\mathrm{J}^n(M, 2)$.

Specifically, we have

$$\begin{aligned}
&T = e^{3\lambda_4}(t+\lambda_1), \qquad X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2), \qquad \widehat{U} = U = e^{-2\lambda_4}(u+\lambda_3),\\
&\widehat{U}_T = D_T\widehat{U} = e^{-5\lambda_4}(u_t - \lambda_3 u_x), \qquad \widehat{U}_X = D_X\widehat{U} = e^{-3\lambda_4}u_x,\\
&\widehat{U}_{TT} = D_T^2\widehat{U} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}),\\
&\widehat{U}_{TX} = D_X D_T\widehat{U} = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}), \widehat{U}_{XX} = D_X^2\widehat{U} = e^{-4\lambda_4}u_{xx},\\
&\widehat{U}_{TTT} = D_T^3\widehat{U} = e^{-11\lambda_4}(u_{ttt} - 3\lambda_3 u_{ttx} + 3\lambda_3^2 u_{txx} - \lambda_3^3 u_{xxx}),\\
&\widehat{U}_{TTX} = D_X D_T^2\widehat{U} = e^{-9\lambda_4}(u_{ttx} - 2\lambda_3 u_{txx} + \lambda_3^2 u_{xxx}),\\
&\widehat{U}_{TXX} = D_X^2 D_T\widehat{U} = e^{-7\lambda_4}(u_{txx} - \lambda_3 u_{xxx}),\\
&\widehat{U}_{XXX} = D_X^3\widehat{U} = e^{-5\lambda_4}u_{xxx}, \quad \ldots .
\end{aligned} \tag{19}$$

4. Moving Frames and Invariantization

In the finite-dimensional theory [**12**] a moving frame is defined to be an equivariant map from (an open subset of) the jet bundle $\mathrm{J}^n(M,p)$ back to the Lie group G. In the more general context of pseudo-groups [**29**], [**30**], the role of the group is played by the pseudo-group jet bundles (or, more accurately, groupoids) $\mathcal{G}^{(n)}$. Let $\mathcal{H}^{(n)} \to \mathrm{J}^n(M,p)$ be the pull-back of $\mathcal{G}^{(n)} \to M$ along the usual jet projection $\pi^n : \mathrm{J}^n(M,p) \to M$, which, assuming regularity, forms a subbundle $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$. Local coordinates on $\mathcal{H}^{(n)}$ have the form $(x, u^{(n)}, \lambda^{(n)})$, where $(x, u^{(n)})$ are jet coordinates on $\mathrm{J}^n(M,p)$ while the fiber coordinates $\lambda^{(n)}$ represent the pseudo-group parameters of order $\leq n$.

Since $\mathcal{G}$ acts on $\mathrm{J}^n(M,p)$ by prolongation, and on $\mathcal{G}^{(n)}$ through right jet multiplication, $\mathcal{G}$ also acts on $\mathcal{H}^{(n)}$. The key definition was first proposed in [**30**].

Definition 4.1. An nth-order *moving frame* for a pseudo-group $\mathcal{G}$ acting on p-dimensional submanifolds $N \subset M$ is a locally $\mathcal{G}$-equivariant section $\rho^{(n)} : \mathrm{J}^n(M,p) \to \mathcal{H}^{(n)}$.

As in the finite-dimensional version, local freeness of the pseudo-group action is both necessary and sufficient for the existence of a locally equivariant moving frame.

Theorem 4.1. *A locally equivariant moving frame exists in a neighborhood of a jet $(x, u^{(n)}) \in \mathrm{J}^n(M,p)$ if and only if $\mathcal{G}$ acts locally freely at $(x, u^{(n)})$. In this case, the $\mathcal{G}$-orbits near $(x, u^{(n)})$ form a foliation whose leaves have dimension r_n equal to the fiber dimension of the pseudo-group jet bundle $\mathcal{G}^{(n)} \to M$ or, equivalently, of its pull-back $\mathcal{H}^{(n)} \to \mathrm{J}^n$.*

Remark. In the case of finite-dimensional group actions, local freeness in the usual sense (discrete isotropy) implies local freeness as a pseudo-group, but not conversely [**30**]. Thus the methods presented in this paper provide a potentially important generalization of the moving frames constructions developed in [**11**], [**12**]. In practice, the existence of moving frames can be verified through direct (and successful) implementation of the normalization procedure, rather than a priori checking the condition of local freeness of the action.

A practical way to construct a moving frame $\rho^{(n)}$ is through the *normalization* procedure based on the choice of a cross-section to the $\mathcal{G}$-orbits. The computational algorithm proceeds as follows:

(i) First, explicitly write out the local coordinate formulas (13) for the prolonged pseudo-group action on $\mathrm{J}^n(M, p)$ in terms of the jet coordinates $(x, u^{(n)})$ and the r_n independent pseudo-group parameters $\lambda^{(n)}$:

$$(X, \widehat{U}^{(n)}) = P^{(n)}(x, u^{(n)}, \lambda^{(n)}). \tag{20}$$

(ii) Set r_n of the coordinate functions (20) to constants,

$$P_\kappa(x, u^{(n)}, \lambda^{(n)}) = c_\kappa, \qquad \kappa = 1, 2, \ldots, r_n, \tag{21}$$

suitably chosen so as to form a cross-section[8] to the pseudo-group orbits.

(iii) Solve the *normalization equations* (21) for the independent group parameters

$$\lambda^{(n)} = h^{(n)}(x, u^{(n)}) \tag{22}$$

in terms of the submanifold jet coordinates. (A local solution is guaranteed, by the Implicit Function Theorem, through the requirement that (21) define a bona fide cross-section, intersecting the pseudo-group orbits transversally.) The induced moving frame section $\rho^{(n)} : \mathrm{J}^n(M, p) \to \mathcal{H}^{(n)}$ has the explicit form

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, h^{(n)}(x, u^{(n)})). \tag{23}$$

From here on, we assume that the pseudo-group acts locally freely. According to [**28**], all finite-dimensional Lie groups that act locally effectively on subsets of M act locally freely in $\mathrm{J}^n(M, p)$ for $n \gg 0$. For infinite-dimensional pseudo-groups, it can be proved [**31**] that local freeness at order n automatically implies local freeness at all higher orders; the minimal such n will be called the *order of freeness* of the pseudo-group. In general, pseudo-groups that act freely admit a *moving frame of infinite order*, that is, a hierarchy of mutually compatible moving frames. In practice, compatibility is assured by fixing all lower-order cross-section normalizations when proceeding to the next higher order. See [**30**], [**31**] for details, as well as an alternative Taylor series-based algorithm that can simultaneously implement the moving frame normalizations at all orders.

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[8] Thus, we restrict our attention here, for simplicity, to coordinate cross-sections. Moving frames based on general cross-sections can be treated by adapting the methods presented by Mansfield [**22**].

4.1. *A Moving Frame for the KdV Equation*

As noted earlier, the KdV symmetry group has dimension 4. Let us choose the coordinate cross-section to the $\mathcal{G}$-orbits in $\mathrm{J}^n(M,2)$, for any $n \geq 1$, defined by the four normalization equations

$$\begin{aligned} T &= e^{3\lambda_4}(t+\lambda_1) = 0, & X &= e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0, \\ \widehat{U} &= e^{-2\lambda_4}(u+\lambda_3) = 0, & \widehat{U}_T &= e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1. \end{aligned} \tag{24}$$

On the subset[9] $\mathcal{V} = \{u_t + uu_x > 0\}$, the normalization equations can be solved for the group parameters

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + uu_x), \tag{25}$$

thereby prescribing the compatible moving frames $\rho^{(n)} : \mathrm{J}^n(M,2) \to \mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ for all $n \geq 1$. Namely, by substituting into (16), $\rho^{(n)}$ maps the point $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots) \in \mathrm{J}^n(M,2)$ to the pseudo-group jet in $\mathcal{H}^{(n)}$ with fiber coordinates

$$\begin{gathered} T = 0, \quad X = 0, \quad U = 0, \quad T_t = (u_t + uu_x)^{3/5}, \quad T_x = 0, \quad T_u = 0, \\ X_t = -u(u_t + uu_x)^{1/5}, \quad X_x = (u_t + uu_x)^{1/5}, \quad X_u = 0, \\ U_t = 0, \quad U_x = 0, \quad U_u = (u_t + uu_x)^{-2/5}, \quad T_{tt} = 0, \quad T_{tx} = 0, \quad T_{xx} = 0, \\ T_{tu} = 0, \quad T_{xu} = 0, \quad T_{uu} = 0, \quad X_{tt} = 0, \quad X_{tx} = 0, \quad X_{xx} = 0, \quad \ldots . \end{gathered} \tag{26}$$

By Theorem 4.1, the existence of a moving frame implies that the action of $\mathcal{G}$ is locally free on the subset $\mathcal{V} = \{u_t + uu_x > 0\} \subset \mathrm{J}^n(M,2)$ for all $n \geq 1$.

Once a moving frame is fixed, the induced *invariantization process* ι associates to each object on $\mathrm{J}^n(M,p)$—differential function, differential form, differential operator, etc.—a uniquely prescribed invariant counterpart with the property that the object and its invariantization coincide when restricted to the cross-section. In local coordinates, this is accomplished by writing out the transformed version of the object, and then replacing all occurrences of the pseudo-group parameters by their moving frame expressions (22). In particular, invariantizing the nth-order jet coordinates $(x, u^{(n)})$ leads to the normalized differential invariants

$$H^i = \iota(x^i), \qquad I^\alpha_J = \iota(u^\alpha_J). \tag{27}$$

These naturally split into two classes: Those that correspond to the r_n coordinate functions used in the normalization equations (21) are equal to the corresponding normalization constants c_κ, and are known as the *phantom differential invariants*. The remaining $s_n = \dim \mathrm{J}^n(M,p) - r_n$ differential functions form a complete

[9] One can define alternative moving frames that include jets where $u_t + uu_x = 0$ by employing different cross-sections. For brevity, in this paper we only deal with this particular choice of moving frame.

system of functionally independent differential invariants, in the sense that any differential invariant of order $\leq n$ can be locally uniquely written as a function of the nonphantom differential invariants (27).

Theorem 4.2. *Suppose the pseudo-group $\mathcal{G}$ admits a mutually compatible hierarchy of moving frames defined on suitable open subsets of* $\mathrm{J}^n(M, p)$ *for* $n \gg 0$. *Then the nonphantom normalized differential invariants* (27) *of all orders* $n \geq 0$ *are functionally independent and generate the differential invariant algebra* $\mathbf{I}_{\mathcal{G}}$.

Invariantization is clearly an algebra morphism, so

$$\iota(\Phi(F_1, \ldots, F_k)) = \Phi(\iota(F_1), \ldots, \iota(F_k))$$

for any function Φ of the differential functions $F_1, \ldots, F_k$. Moreover, it defines a projection, meaning that $\iota \circ \iota = \iota$; see [**12**], [**30**]. In particular, ι does not affect differential invariants, which implies the elementary, but salient Replacement Theorem [**12**].

Theorem 4.3. *If*

$$I(x, u^{(n)}) = I(\ldots, x^i, \ldots, u^\alpha_J, \ldots)$$

is any differential invariant, then

$$I(x, u^{(n)}) = \iota(I(x, u^{(n)})) = I(\ldots, H^i, \ldots, I^\alpha_J, \ldots)$$

on the intersection of the domains of definition of the differential invariant and the moving frame. Similarly, any invariant system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

can be rewritten[10] *in terms of the normalized differential invariants by invariantization*:

$$\iota(\Delta(x, u^{(n)})) = \Delta(\ldots, H^i, \ldots, I^\alpha_J, \ldots) = 0.$$

The alternative, more traditional means of generating higher-order differential invariants is by invariant differentiation. A basis for the *invariant differential operators* $\mathcal{D}_1, \ldots, \mathcal{D}_p$ can be obtained by invariantizing the total differential operators $D_1, \ldots, D_p$. More explicitly, we let

$$\omega^i = \iota(\mathrm{d}x^i), \qquad i = 1, 2, \ldots, p, \tag{28}$$

[10] The left-hand side of the original system may only be relatively invariant, whereas the left-hand side of its invariantization is fully invariant, and so, in such cases, will include an additional nonvanishing multiplier [**26**].

be the *contact-invariant*[11] *horizontal coframe* obtained by invariantizing the horizontal coordinate coframe. In practice, the one-forms (28) are found by substituting for the pseudo-group parameters in the lifted horizontal coframe (11) in accordance with the moving frame formulas (22). The invariant differential operators are the dual total differential operators, defined so that

$$\mathrm{d}_H F = \sum_{i=1}^{p} (\mathcal{D}_i F)\omega^i \quad \text{for any differential function} \quad F : \mathrm{J}^n(M, p) \to \mathbb{R}. \tag{29}$$

The invariant differential operators $\mathcal{D}_i$ can also be obtained directly by replacing the pseudo-group parameters in the implicit differential operators D_{X^i} by their moving frame expressions.

4.2. *Differential Invariants for the KdV Equation*

For the KdV symmetry group, the differential invariants are obtained by invariantizing the jet coordinates $t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots$, which is equivalent to substituting the moving frame expressions (25) into the prolonged action formulas (19). The constant phantom differential invariants

$$H^1 = \iota(t) = 0, \quad H^2 = \iota(x) = 0, \quad I_0 = \iota(u) = 0, \quad I_{10} = \iota(u_t) = 1, \tag{30}$$

result from our particular choice of normalization (24). Invariantizing the remaining coordinate functions yields a complete system of functionally independent normalized differential invariants:

$$\begin{aligned}
&I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}, \\
&I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}, \qquad I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}, \\
&I_{30} = \iota(u_{ttt}) = \frac{u_{ttt} + 3uu_{ttx} + 3u^2 u_{txx} + u^3 u_{xxx}}{(u_t + uu_x)^{11/5}}, \\
&I_{21} = \iota(u_{ttx}) = \frac{u_{ttx} + 2uu_{txx} + u^2 u_{xxx}}{(u_t + uu_x)^{9/5}}, \\
&I_{12} = \iota(u_{txx}) = \frac{u_{txx} + uu_{xxx}}{(u_t + uu_x)^{7/5}}, \qquad I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x}, \qquad \ldots .
\end{aligned} \tag{31}$$

The Replacement Theorem 4.3 allows us to immediately rewrite the KdV equation in terms of the differential invariants by applying the invariantization process to it:

$$0 = \iota(u_t + uu_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + uu_x + u_{xxx}}{u_t + uu_x}.$$

[11] These one-forms are invariant if the pseudo-group acts projectably, but only invariant modulo contact forms in general, see [**15**], [**30**]. A familiar example is the arc length form $\omega = \mathrm{d}s$ in Euclidean curve geometry, which is only contact-invariant under general Euclidean motions [**26**].

Note the appearance of a nonzero multiplier indicating that the KdV equation is initially defined by a relative differential invariant. The invariant horizontal coframe

$$\omega^1 = (u_t + uu_x)^{3/5}\,\mathrm{d}t, \qquad \omega^2 = -u(u_t + uu_x)^{1/5}\,\mathrm{d}t + (u_t + uu_x)^{1/5}\,\mathrm{d}x, \quad (32)$$

is obtained by substituting (25) into the lifted horizontal coframe (17), while the corresponding invariant differential operators

$$\mathcal{D}_1 = (u_t+uu_x)^{-3/5} D_t + u(u_t+uu_x)^{-3/5} D_x, \qquad \mathcal{D}_2 = (u_t+uu_x)^{-1/5} D_x, \quad (33)$$

can be found either by invoking duality (29), or by directly substituting the moving frame expressions (25) into the implicit total derivative operators (18). The invariant horizontal one-forms ω^1, ω^2 satisfy the structure equations

$$\mathrm{d}_H\omega^1 = -\tfrac{3}{5}(I_{11} + I_{01}^2)\omega^1 \wedge \omega^2, \qquad \mathrm{d}_H\omega^2 = \tfrac{1}{5}(I_{20} + 6I_{01})\omega^1 \wedge \omega^2, \qquad (34)$$

where d_H is the horizontal derivative. By duality, equations (34) imply the commutation formula

$$[\mathcal{D}_1, \mathcal{D}_2] = \tfrac{3}{5}(I_{11} + I_{01}^2)\mathcal{D}_1 - \tfrac{1}{5}(I_{20} + 6I_{01})\mathcal{D}_2. \qquad (35)$$

Higher-order differential invariants can now be constructed by repeatedly applying the invariant differential operators to the lower-order differential invariants, and hence can be expressed in terms of the normalized differential invariants. For example,

$$\mathcal{D}_1 I_{01} = -\tfrac{3}{5}I_{01}^2 + I_{11} - \tfrac{3}{5}I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = -\tfrac{3}{5}I_{01}^3 + I_{02} - \tfrac{3}{5}I_{01}I_{11},$$

as can be checked by a somewhat tedious explicit calculation. Similarly, the commutation formula (35) can be used to derive syzygies among the differentiated invariants. In the next section, we will develop an algorithm for constructing the recurrence formulas and syzygies in a much simpler, direct fashion.

5. The Algebra of Differential Invariants

Unlike the normalized differential invariants obtained from Theorem 4.2, the *differentiated invariants* are typically not functionally independent. Thus, it behooves us to establish the *recurrence formulas* relating the normalized and differentiated invariants, which will, in turn, enable us to write down a finite generating system of differential invariants as well as a complete system of *syzygies* or functional dependencies among the differentiated invariants. The required recurrence formulas rely on the Maurer–Cartan forms for the pseudo-group, and so we begin by briefly reviewing their construction, as developed in [**5**], [**29**].

5.1. *The Maurer–Cartan Forms*

First, the Maurer–Cartan forms for the diffeomorphism pseudo-group $\mathcal{D}$ are explicitly realized as the right-invariant contact forms on the infinite jet bundle $\mathcal{D}^{(\infty)}$. A basis is labeled by the fiber coordinates X^i_A, U^α_A on $\mathcal{D}^{(\infty)}$, and we use the symbols

$$\chi^i_A, \ \ \mu^\alpha_A, \qquad \text{for} \quad i = 1, \dots p, \quad \alpha = 1, \dots, q, \quad \#A \geq 0, \tag{36}$$

to denote the corresponding basis Maurer–Cartan forms. Their explicit formulas, along with the complete system of diffeomorphism structure equations, will not be required here, but can be found in [**5**], [**29**].

The Maurer–Cartan forms for a Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms[12] (36) to the subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$. Of course, the resulting differential forms are no longer (pointwise) linearly independent. But remarkably, the complete system of linear dependencies among the restricted forms can be immediately described in terms of the infinitesimal determining equations for the pseudo-group.

Theorem 5.1. *The restricted Maurer–Cartan forms satisfy the* lifted determining equations

$$\mathcal{L}(\dots, X^i, \dots, U^\alpha, \dots, \chi^i_A, \dots, \mu^\alpha_A, \dots) = 0 \tag{37}$$

that are obtained by applying the following replacement rules to the infinitesimal determining equations (6):

$$x^i \mapsto X^i, \qquad u^\alpha \mapsto U^\alpha, \qquad \xi^i_A \mapsto \chi^i_A, \qquad \varphi^\alpha_A \mapsto \mu^\alpha_A, \qquad \textit{for all} \quad i, \alpha, A. \tag{38}$$

As discussed in [**29**] (see also [**5**]), the structure equations for the pseudo-group $\mathcal{G}^{(\infty)}$ can simply be obtained by imposing the dependencies (37) on the structure equations of the diffeomorphism pseudo-group.

In the construction of recurrence formulas, the most important forms are not the Maurer–Cartan forms per se, but rather their pull-backs under the moving frame map. In what follows, we will only need the horizontal components of the resulting invariantized forms, as specified by the splitting of coordinates on M into independent and dependent variables [**26**], [**30**].

Definition 5.1. Given a moving frame $\rho^{(n)} : \mathrm{J}^n(M, p) \to \mathcal{H}^{(n)}$, we define the *invariantized Maurer–Cartan forms* to be the horizontal components of the pull-backs

$$\beta^i_A = \pi_H[\,(\rho^{(n)})^* \chi^i_A\,], \qquad \zeta^\alpha_A = \pi_H[\,(\rho^{(n)})^* \mu^\alpha_A\,]. \tag{39}$$

Remark 5.1.1.. In general, the pull-backs $(\rho^{(n)})^* \chi^i_A$ and $(\rho^{(n)})^* \mu^\alpha_A$ are one-forms on $\mathrm{J}^n(M, p)$ with nontrivial vertical or contact components. Only the horizontal

[12] To avoid unnecessary clutter, we will retain the same notation for the restricted forms.

components are required in the analysis of the algebraic structure of differential invariants. The contact components *are* important in the study of invariant variational problems and the invariant variational bicomplex [**15**], and will be the focus of future research.

Applying the moving frame pull-back map to (37) and then extracting the horizontal components of the resulting linear system, we deduce the corresponding dependencies among the invariantized Maurer–Cartan forms.

Theorem 5.2. *The invariantized Maurer–Cartan forms satisfy the* invariantized determining equations

$$\mathcal{L}(\ldots, H^i, \ldots, I^\alpha, \ldots, \beta^i_A, \ldots, \zeta^\alpha_A, \ldots) = 0. \tag{40}$$

We next extend the invariantization process to include, besides differential functions and forms, the derivatives (jet coordinates) of vector field coefficients[13] by setting

$$\iota(\xi^i_A) = \beta^i_A, \qquad \iota(\varphi^\alpha_A) = \zeta^\alpha_A, \tag{41}$$

to be the corresponding invariantized Maurer–Cartan forms (39). The *invariantization* of any linear differential function[14]

$$\sum_{i,A} F^i_A(x, u^{(n)})\xi^i_A + \sum_{\alpha,A} F^\alpha_A(x, u^{(n)})\varphi^\alpha_A,$$

on the space of vector fields $\mathcal{X}(M)$, is the corresponding invariant linear combination

$$\sum_{i,A} F^i_A(H, I^{(n)})\beta^i_A + \sum_{\alpha,A} F^\alpha_A(H, I^{(n)})\zeta^\alpha_A \tag{42}$$

of invariantized Maurer–Cartan forms. In other words, to invariantize, we replace jet coordinates x^i, u^α_J by the corresponding normalized differential invariants H^i, I^α_J, while derivatives of the vector field coefficient are replaced by the corresponding invariantized Maurer–Cartan forms for the pseudo-group. In particular, applying the invariantization process ι to the infinitesimal determining equations (6) yields the linear dependencies (40) among the invariantized Maurer–Cartan forms.

5.2. *Maurer–Cartan Forms for the KdV Symmetry Group*

Let us apply our constructions to the KdV symmetry group. Its Maurer–Cartan forms are obtained by restricting the diffeomorphism Maurer–Cartan forms to the

[13] Each derivative ξ^i_A, φ^α_A serves to define a linear function on the space of vector fields $\mathcal{X}(M)$, and so the fact that its invariantization is a differential form should not come as a complete surprise.

[14] All sums are finite.

pseudo-group subbundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$. Let ν_A, χ_A, μ_A be the diffeomorphism Maurer–Cartan forms (36) corresponding to the target jet coordinates T_A, X_A, U_A. According to Theorem 5.1, the restricted forms satisfy the linear equations

$$\begin{aligned} &\nu_X = \nu_U = \chi_U = \mu_T = \mu_X = 0, \qquad \mu - \chi_T + \tfrac{2}{3}\, U \nu_T = 0, \\ &\mu_U = -\tfrac{2}{3}\nu_T = -2\chi_X, \qquad \nu_{TT} = \nu_{TX} = \cdots = 0, \end{aligned} \tag{43}$$

obtained from the determining equations (9) by using the replacement rules (38). From these equations we see that the forms ν, χ, μ, ν_T form a basis for the Maurer–Cartan forms for the four-dimensional symmetry group $\mathcal{G}$ of the KdV equation. In [**5**] this fact was used to establish the structure of the KdV symmetry group directly without having to solve the determining equations.

We now pull back the Maurer–Cartan forms by our moving frame map. The resulting (horizontal) invariantized Maurer–Cartan forms are denoted by

$$\iota(\tau_A) = \alpha_A, \qquad \iota(\xi_A) = \beta_A, \qquad \iota(\varphi_A) = \gamma_A. \tag{44}$$

They are subject to the equations obtained by invariantization of the determining equations (9), and so, in view of the normalizations (30),

$$\begin{aligned} &\alpha_X = \alpha_U = \beta_U = \gamma_T = \gamma_X = 0, \qquad \gamma - \beta_T = 0, \\ &\gamma_U = -\tfrac{2}{3}\alpha_T = -2\beta_X, \qquad \alpha_{TT} = \alpha_{TX} = \cdots = 0. \end{aligned} \tag{45}$$

As with the lifted forms, we can use these linear relations to write all of the invariantized Maurer–Cartan forms as linear combinations of

$$\alpha = \iota(\tau), \qquad \beta = \iota(\xi), \qquad \gamma = \iota(\varphi), \qquad \zeta = \alpha_T = \iota(\tau_t). \tag{46}$$

5.3. *The Recurrence Formulas*

According to the prolongation formula (4), the coefficients $\widehat{\varphi}^{\alpha}_J$ of a prolonged vector field are certain well-prescribed linear combinations of the derivatives ξ^i_A, φ^α_A, $\#A \le \#J$, of its original coefficients. Let

$$\widehat{\psi}^{\alpha}_J = \iota(\widehat{\varphi}^{\alpha}_J) \tag{47}$$

denote their invariantizations, which, in accordance with the general procedure (42), are linear combinations of invariantized Maurer–Cartan forms β^i_A, ζ^α_A defined in (41) whose coefficients are differential invariants; in fact, they are certain universal polynomial functions of the normalized differential invariants I^α_J. These particular invariant differential forms provide the crucial correction terms in the recurrence relations for the differentiated invariants. See [**30**] for a proof of this key result.

Theorem 5.3. *The* recurrence formulas *for the normalized differential invariants* (27) *are*

$$\begin{aligned} \mathrm{d}_H H^j &= \sum_{i=1}^{p} (\mathcal{D}_i H^j)\omega^i = \omega^j + \beta^j, \\ \mathrm{d}_H I^\alpha_J &= \sum_{i=1}^{p} (\mathcal{D}_i I^\alpha_J)\omega^i = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}^{\,\alpha}_J. \end{aligned} \tag{48}$$

The recurrence formulas (48) split into two types: First, whenever H^j or I^α_J is a phantom (constant) differential invariant, its differential is identically zero, and so the left-hand side of the phantom recurrence equation in (48) vanishes. As we will prove in [**31**], under the assumption that the pseudo-group acts locally freely at order n, the phantom recurrence equations can always be solved for all the independent invariantized Maurer–Cartan forms of order $\#A \leq n$. We then substitute the resulting expressions for the invariantized Maurer–Cartan forms into the remaining nonphantom recurrence equations in (48). Identifying the induced coefficients of the invariant horizontal coframe $\omega^1, \ldots, \omega^p$ results in a complete system of recurrence formulas relating the differentiated and normalized invariants.

Thus, the fundamental recurrence formulas have the form

$$\mathcal{D}_i H^j = \delta^j_i + \widehat{R}^j_i, \qquad \mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + R^\alpha_{J,i}, \tag{49}$$

where δ^i_j is the usual Kronecker delta, and the explicit formulas for the *correction terms* $\widehat{R}^j_i$, $R^\alpha_{J,i}$ are deduced by applying the preceding algorithm. Iterating, we establish the general *recurrence formulas*

$$\mathcal{D}_K I^\alpha_J = I^\alpha_{J,K} + R^\alpha_{J,K}, \tag{50}$$

valid for any multi-indices J, K. In computations, the *correction terms* $R^\alpha_{J,K}$ are rewritten in terms of the generating differential invariants and their invariant derivatives.

The most striking fact is that the preceding algorithm establishes the recurrence formulas, without any need to explicitly compute the Maurer–Cartan forms or their pull-backs in advance, nor the explicit formulas for the differential invariants and invariant differential forms, nor the infinitesimal generators or symmetry group transformations. Once the cross-section normalizations have been chosen, the algorithm is entirely based on the standard prolongation formula, and the resulting infinitesimal determining equations for the symmetry group!

With the recurrence formulas (49), (50) in hand, the generating set of differential invariants and the syzygies can, at least in relatively simple examples, be found by inspection along the same lines as in the finite-dimensional version presented in [**12**]. A more sophisticated approach relies on the subtle algebraic structure of the differential invariant algebra discovered in [**31**], which we now briefly summarize.

Let $\mathbb{R}[x]$ be the ring of real-valued polynomials $p(x) = \sum_J c_J x^J$ in the independent variables $x^1, \ldots, x^p$. Let $\mathbb{R}[x; u]$ be the $\mathbb{R}[x]$ module consisting of polynomials $q(x,u)=\sum_J c_{J,\alpha}x^J u^\alpha$ which are linear in the dependent variables $u^1,\ldots,u^q$. By Dickson's Lemma [**7**], any *monomial submodule* $\mathcal{N}\subset\mathbb{R}[x;u]$ is generated by a finite number of monomials $x^{J_1}u^{\alpha_1},\ldots,x^{J_k}u^{\alpha_k}$. We call a subspace $\mathcal{N}\subset\mathbb{R}[x;u]$ an *eventual monomial module* of *order n* if it is spanned by monomials, and its "high degree" component $\mathcal{N}_{>n}$, that is spanned by all monomials $x^J u^\alpha$ of degree $\#J>n$ in $\mathcal{N}$, forms a module. A *generating set* for an eventual monomial module of order n is given by a Gröbner basis for $\mathcal{N}_{>n}$ along with all monomials $x^I u^\beta\in\mathcal{N}$ of degree $\#I\le n$. Note that generators $x^{J_1}u^{\alpha_1},\ldots,x^{J_k}u^{\alpha_k}$ for $\mathcal{N}_{>n}$ guaranteed by Dickson's lemma automatically form a Gröbner basis for $\mathcal{N}_{>n}$.

Given a moving frame (of infinite order) based on compatible coordinate cross-sections, we identify each nonphantom normalized differential invariant $I^\alpha_J=\iota(u^\alpha_J)$ as in (27) with the monomial $x^J u^\alpha$. We let $\mathcal{N}\subset\mathbb{R}[x;u]$ be the subspace spanned by these nonphantom monomials. An infinite-order moving frame is called *algebraic* of order n if $\mathcal{N}$ is an eventual monomial module of order n. Moving frames for finite-dimensional Lie group actions are always algebraic; indeed, if the moving frame has order n, then $\mathcal{N}_{>n}$ contains *all* monomials of degree $>n$, and so is trivially a module. Assuming that an infinite-dimensional pseudo-group acts freely at some order n, then it admits an algebraic moving frame of order n; see [**31**] for complete details.

The following results will be established in [**31**]. For simplicity, we shall assume that the pseudo-group acts transitively on the independent variables, and that the cross-section has been chosen so that all $H^i=\iota(x^i)=c_i$, $i=1,\ldots,p$, are phantom differential invariants. (Including cases when some of the independent variables lead to nonphantom differential invariants requires only technical modifications of the results.) We are now able to formulate a constructive version of Tresse's Basis Theorem.

Theorem 5.4. *Suppose that $\mathcal{G}$ admits an algebraic moving frame $\rho^{(\infty)}:\mathrm{J}^\infty(M,p)\to\mathcal{H}^{(\infty)}$. Then, the nonphantom differential invariants I^α_J, corresponding to the elements in a generating set for its order n eventual monomial module $\mathcal{N}$, generate its differential invariant algebra $\mathbf{I}_\mathcal{G}$.*

Furthermore, in [**31**] we apply Gröbner basis techniques to determine a complete system of syzygies amongst the generating system constructed in Theorem 5.4. As in the finite-dimensional theory [**12**], under suitable regularity assumptions, the generating syzygies fall into two classes. The first one consists of syzygies of the form

$$\mathcal{D}_K I^\alpha_J=c^\alpha_{JK}+M^\alpha_{J,K},\tag{51}$$

where I^α_J is a generating differential invariant and $I^\alpha_{JK}=c^\alpha_{JK}$ is a phantom differential invariant, while the second one consists of all equations of the form

$$\mathcal{D}_J I^\alpha_{LK}-\mathcal{D}_K I^\alpha_{LJ}=M^\alpha_{LK,J}-M^\alpha_{LJ,K},\tag{52}$$

where I^α_{LK} and I^α_{LJ} are generating differential invariants, the multi-indices $K \cap J = \emptyset$ are disjoint and nonzero, and L is an arbitrary multi-index. Note that the first type of syzygy (51) only arises when I^α_J has order $\le n$, and usually does not occur. All other syzygies amongst the generating differential invariants are invariant linear combinations of the invariant derivatives of the generating syzygies.

Fine details of the algorithm are illustrated in the course of the following examples.

5.4. *Recurrence Formulas for the KdV Equation*

In the case of the KdV equation, the prolongation of the general infinitesimal symmetry generator

$$\mathbf{v} = (c_1 + 3c_4 t)\partial_t + (c_2 + c_3 t + c_4 x)\partial_x + (c_3 - 2c_4 u)\partial_u$$

has

$$\widehat{\varphi}^{jk} = -jc_3 u_{t^{j-1}x^{k+1}} - (3j+k+2)c_4 u_{t^j x^k}, \qquad j+k \ge 1, \tag{53}$$

as the coefficient of $\partial/\partial u_{t^j x^k}$. Identifying $c_3 = \xi_t$, $c_4 = \frac{1}{3}\tau_t$, the corresponding invariantized forms are

$$\begin{gathered} \alpha = \iota(\tau), \qquad \beta = \iota(\xi), \qquad \psi = \iota(\varphi) = \gamma, \\ \widehat{\psi}_{jk} = -jI_{j-1,k+1}\,\gamma - \frac{3j+k+2}{3}\,I_{j,k}\,\zeta, \qquad j+k \ge 1. \end{gathered} \tag{54}$$

Thus, according to (48),

$$\begin{aligned} 0 &= \mathrm{d}_H H^1 = \omega^1 + \alpha, \\ 0 &= \mathrm{d}_H H^2 = \omega^2 + \beta, \\ 0 &= \mathrm{d}_H I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \gamma, \\ 0 &= \mathrm{d}_H I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \widehat{\psi}_{10} = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\gamma - \tfrac{5}{3}\zeta, \\ \mathrm{d}_H I_{01} &= I_{11}\omega^1 + I_{02}\omega^2 + \widehat{\psi}_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\zeta, \\ \mathrm{d}_H I_{20} &= I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\gamma + \widehat{\psi}_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\gamma - \tfrac{8}{3}I_{20}\zeta, \\ \mathrm{d}_H I_{11} &= I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\gamma + \widehat{\psi}_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\gamma - 2I_{11}\zeta, \\ \mathrm{d}_H I_{02} &= I_{12}\omega^1 + I_{03}\omega^2 + \widehat{\psi}_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\zeta, \\ &\;\;\vdots \end{aligned} \tag{55}$$

The left-hand sides of the first four recurrence formulas in (55) are all zero since they are the differentials of the phantom invariants (30). Thus we can solve those phantom recurrence equations to establish the explicit formulas for the independent invariantized Maurer–Cartan forms in terms of the invariant horizontal coframe:

$$\begin{gathered} \alpha = -\omega^1, \qquad \beta = -\omega^2, \qquad \gamma = -\omega^1 - I_{01}\omega^2, \\ \zeta = \tfrac{3}{5}(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}(I_{11} + I_{01}^2)\omega^2. \end{gathered} \tag{56}$$

Substituting these results into the recurrence formulas for the differentials

$$d_H I = (\mathcal{D}_1 I)\omega^1 + (\mathcal{D}_2 I)\omega^2$$

of nonphantom invariants, and equating the coefficients of ω^1, ω^2 on both sides, yields the complete collection of recurrence formulas for the differentiated invariants:

$$\begin{aligned}
\mathcal{D}_1 I_{01} &= I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, \\
\mathcal{D}_2 I_{01} &= I_{02} - \tfrac{3}{5} I_{01}^3 - \tfrac{3}{5} I_{01} I_{11}, \\
\mathcal{D}_1 I_{20} &= I_{30} + 2 I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, \\
\mathcal{D}_2 I_{20} &= I_{21} + 2 I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20}, \\
\mathcal{D}_1 I_{11} &= I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, \\
\mathcal{D}_2 I_{11} &= I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2, \\
\mathcal{D}_1 I_{02} &= I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, \\
\mathcal{D}_2 I_{02} &= I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11},
\end{aligned} \tag{57}$$

and so on.

In general, the expressions (54) yield the recurrence formulas

$$\begin{aligned}
\mathcal{D}_1 I_{j,k} &= I_{j+1,k} - \frac{3j+k+2}{5}(I_{20} + I_{01}) I_{j,k} + j\, I_{j-1,k+1}, \\
\mathcal{D}_2 I_{j,k} &= I_{j,k+1} - \frac{3j+k+2}{5}(I_{01}^2 + I_{11}) I_{j,k} + j\, I_{01} I_{j-1,k+1},
\end{aligned} \qquad \text{for all} \quad i,\, j \geq 0, \tag{58}$$

for the normalized differential invariants, where, by convention, we set $I_{-1,k} = 0$. As a consequence, we conclude that every normalized differential invariant can be obtained from the two fundamental differential invariants

$$I_{01} = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2 u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}, \tag{59}$$

by invariant differentiation, and hence I_{01} and I_{20} generate the KdV differential invariant algebra $\mathbf{I}_{\mathrm{KdV}}$. This is in accordance with Theorem 5.4, since the module corresponding to the nonphantom differential invariants induced by our choice of cross-section is generated by the monomials $xu \sim I_{01}$ and $t^2 u \sim I_{20}$. Finally, according to (51), (52), there is one fundamental syzygy, namely,

$$\begin{aligned}
\mathcal{D}_1^2 I_{01} + \tfrac{3}{5} I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + (\tfrac{1}{5} I_{20} + \tfrac{19}{5} I_{01}) \mathcal{D}_1 I_{01} \\
- \mathcal{D}_2 I_{01} - \tfrac{6}{25} I_{01} I_{20}^2 - \tfrac{7}{25} I_{01}^2 I_{20} + \tfrac{24}{25} I_{01}^3 = 0.
\end{aligned}$$

It should be emphasized that while, for the sake of transparency, the methods for constructing Maurer–Cartan forms, differential invariants, recurrence formulas, and syzygies are illustrated above by the relatively simple symmetry algebra of the KdV equation, the algorithms described in this paper are general and apply to arbitrary systems of differential equations with finite- or infinite-dimensional symmetry groups.

5.5. *The Algebra of Differential Invariants for the KP Equation*

In this example, we will show how to obtain the structure of the algebra of differential invariants, including a set of generators and a complete list of basic syzygies, for the symmetry pseudo-group $\mathcal{G}_{\mathrm{KP}}$ of the KP equation

$$u_{tx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{1}{4}u_{xxxx} + \tfrac{3}{4}\varepsilon u_{yy} = 0, \qquad \varepsilon = \pm 1, \tag{60}$$

without having to establish its (prolonged) action in advance. Earlier work on its symmetry group and differential invariants can be found in [**6**], [**8**], [**9**], [**10**], [**18**], [**19**], [**20**], [**21**]. The underlying total space is $M = \mathbb{R}^4$ with coordinates (t, x, y, u). Applying the standard Lie algorithm [**25**], we find that a vector field

$$\mathbf{v} = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \eta(t,x,y,u)\frac{\partial}{\partial y} + \varphi(t,x,y,u)\frac{\partial}{\partial u}$$

is an infinitesimal symmetry of the KP equation if and only if its coefficients satisfy the infinitesimal symmetry determining equations

$$\begin{gathered}\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \xi_x - \tfrac{1}{3}\tau_t = 0, \quad \xi_y + \tfrac{2}{3}\varepsilon\eta_t = 0, \quad \xi_u = 0,\\ \eta_x = 0, \quad \eta_y - \tfrac{2}{3}\tau_t = 0, \quad \eta_u = 0, \quad \varphi - \tfrac{2}{3}\xi_t + \tfrac{2}{3}u\tau_t = 0,\end{gathered} \tag{61}$$

along with all their differential consequences.

Our actual choice of cross-section that defines the moving frame will be deferred until we acquire some familiarity with the structure of the recurrence formulas. First, invariantization of the determining equations (61) implies the complete system of linear dependencies among the invariantized Maurer–Cartan forms

$$\alpha_{ijkl} = \iota(\tau_{ijkl}), \qquad \beta_{ijkl} = \iota(\xi_{ijkl}), \qquad \gamma_{ijkl} = \iota(\eta_{ijkl}), \qquad \zeta_{ijkl} = \iota(\varphi_{ijkl}),$$

namely,

$$\begin{gathered}\alpha_X = 0, \quad \alpha_Y = 0, \quad \alpha_U = 0, \quad \beta_X = \tfrac{1}{3}\alpha_T, \quad \beta_Y = -\tfrac{2}{3}\varepsilon\gamma_T, \quad \beta_U = 0,\\ \gamma_X = 0, \quad \gamma_Y = \tfrac{2}{3}\alpha_T, \quad \gamma_U = 0, \quad \zeta = \tfrac{2}{3}\beta_T - \tfrac{2}{3}I_{000}\alpha_T,\end{gathered} \tag{62}$$

and so on. Here we denote the corresponding normalized differential invariants by

$$H^1 = \iota(t), \qquad H^2 = \iota(x), \qquad H^3 = \iota(y), \qquad I_{ijk} = \iota(u_{ijk}),$$

some of which will be phantom, i.e., constant, once the moving frame is fixed. As in [**5**], a basis of the invariantized Maurer–Cartan forms can be obtained by invariantization of the involutive completion of the lifted determining equations (61), and, for example, is provided by the forms

$$\alpha_{T^n}, \quad \beta_{T^n}, \quad \gamma_{T^n}, \quad n \geq 0. \tag{63}$$

The remaining invariantized Maurer–Cartan forms can now easily be expressed in terms of the basis forms (63). We have

$$\begin{aligned}
&\alpha_{T^n X^p Y^q U^r} = 0, && \text{if } (p,q,r) \neq (0,0,0);\\
&\beta_{T^n X} = \tfrac{1}{3}\alpha_{T^{n+1}}, \quad \beta_{T^n Y} = -\tfrac{2}{3}\varepsilon\gamma_{T^{n+1}}, &&\\
&\beta_{T^n YY} = -\tfrac{4}{9}\varepsilon\beta_{T^{n+2}}, \quad \gamma_{T^n Y} = \tfrac{1}{3}\alpha_{T^{n+1}}, &&\\
&\beta_{T^n X^p Y^q U^r} = 0, \quad \gamma_{T^n X^p Y^q U^r} = 0, && \text{for all other choices of } (p,q,r);\\
&\zeta_{T^n} = \tfrac{2}{3}\beta_{T^{n+1}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s00}\alpha_{T^{n-s+1}}, &&\\
&\zeta_{T^n X} = \tfrac{2}{9}\alpha_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s10}\alpha_{T^{n-s+1}}, &&\\
&\zeta_{T^n Y} = -\tfrac{4}{9}\varepsilon\gamma_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s01}\alpha_{T^{n-s+1}}, &&\\
&\zeta_{T^n YY} = -\tfrac{4}{27}\varepsilon\alpha_{T^{n+3}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s02}\alpha_{T^{n-s+1}}, &&\\
&\zeta_{T^n X^p Y^q} = -\tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{spq}\alpha_{T^{n-s+1}}, && \text{for all other choices of } (p,q),\\
&\zeta_{T^n U} = -\tfrac{2}{3}\alpha_{T^{n+1}}, \quad \zeta_{T^n X^p Y^q U^r} = 0, && \text{if } r \geq 2.
\end{aligned} \tag{64}$$

Let $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ be the invariant differential operators dual to the invariantized horizontal coframe

$$\omega^1 = \iota(\mathrm{d}t), \qquad \omega^2 = \iota(\mathrm{d}x), \qquad \omega^3 = \iota(\mathrm{d}y). \tag{65}$$

As above, the explicit formulas are not required at the moment.

It follows from (62) that the correction terms $\widehat{\psi}^{\alpha}_{J}$ in equation (47) for the KP symmetry algebra are precisely the coefficients of the invariantization of the vector field obtained by first prolonging the vector field

$$\mathbf{w} = \tau(t)\frac{\partial}{\partial t} + \xi(t,x,y)\frac{\partial}{\partial x} + \eta(t,y)\frac{\partial}{\partial y} + \left(-\tfrac{2}{3}u\tau_t(t) + \tfrac{2}{3}\xi_t(t,x,y)\right)\frac{\partial}{\partial u} \tag{66}$$

and then applying the relations

$$\xi_x = \tfrac{1}{3}\tau_t, \qquad \xi_y = -\tfrac{2}{3}\varepsilon\eta_t, \qquad \eta_y = \tfrac{2}{3}\tau_t \tag{67}$$

and their differential consequences to express the resulting coefficient functions solely in terms of the repeated t-derivatives of τ, ξ, and η. This yields the expression

$$\begin{aligned}\widehat{\psi}_{pqr} \;=\; & \tfrac{2}{9}\delta_{q1}\delta_{r0}\alpha_{T^{p+2}} - \tfrac{4}{9}\delta_{q0}\delta_{r1}\varepsilon\gamma_{T^{p+2}} - \tfrac{8}{27}\delta_{q0}\delta_{r2}\varepsilon\alpha_{T^{p+3}} \\ & -\sum_{s=0}^{p}\binom{p}{s}\left(\frac{2+q+2r}{3}+\frac{p-s}{s+1}\right) I_{p-s,q,r}\alpha_{T^{s+1}} \\ & +\tfrac{2}{9}\varepsilon r(r-1)\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-2}\alpha_{T^{s+2}} \\ & -\sum_{s=1}^{p}\binom{p}{s} I_{p-s,q+1,r}\beta_{T^s} - \sum_{s=1}^{p}\binom{p}{s} I_{p-s,q,r+1}\gamma_{T^s} \\ & +\tfrac{2}{3}\varepsilon r\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-1}\gamma_{T^{s+1}} \end{aligned} \tag{68}$$

for the correction terms $\widehat{\psi}_{pqr} = \widehat{\psi}_{T^pX^qY^r}$, where δ_{ij} denotes the Kronecker delta.

With (68), equations (48) directly yield the recurrence formulas

$$\begin{aligned}
&\mathrm{d}_H H^1 = \omega^t + \alpha, \qquad \mathrm{d}_H H^2 = \omega^x + \beta, \qquad \mathrm{d}_H H^3 = \omega^y + \gamma, \\
&\mathrm{d}_H I_{000} = I_{100}\omega^t + I_{010}\omega^x + I_{001}\omega^y - \tfrac{2}{3}I_{000}\alpha_T + \tfrac{2}{3}\beta_T, \\
&\mathrm{d}_H I_{100} = I_{200}\omega^t + I_{110}\omega^x + I_{101}\omega^y - \tfrac{5}{3}I_{100}\alpha_T - \tfrac{2}{3}I_{000}\alpha_{TT} - I_{010}\beta_T \\
&\qquad + \tfrac{2}{3}\beta_{TT} - I_{001}\gamma_T, \\
&\mathrm{d}_H I_{010} = I_{110}\omega^t + I_{020}\omega^x + I_{011}\omega^y - I_{010}\alpha_T + \tfrac{2}{9}\alpha_{TT}, \\
&\mathrm{d}_H I_{001} = I_{101}\omega^t + I_{011}\omega^x + I_{002}\omega^y - \tfrac{4}{3}I_{001}\alpha_T + \tfrac{2}{3}\,\varepsilon I_{010}\gamma_T - \tfrac{4}{9}\,\varepsilon\gamma_{TT}, \\
&\mathrm{d}_H I_{200} = I_{300}\omega^t + I_{210}\omega^x + I_{201}\omega^y - \tfrac{8}{3}I_{200}\alpha_T - \tfrac{7}{3}I_{100}\alpha_{TT} - \tfrac{2}{3}I_{000}\alpha_{TTT} \\
&\qquad - 2I_{110}\beta_T - I_{010}\beta_{TT} + \tfrac{2}{3}\beta_{TTT} - 2I_{101}\gamma_T - I_{001}\gamma_{TT}, \\
&\mathrm{d}_H I_{110} = I_{210}\omega^t + I_{120}\omega^x + I_{111}\omega^y - 2I_{110}\alpha_T - I_{010}\alpha_{TT} + \tfrac{2}{9}\alpha_{TTT} \\
&\qquad - I_{020}\beta_T - I_{011}\gamma_T, \\
&\mathrm{d}_H I_{101} = I_{201}\omega^t + I_{111}\omega^x + I_{102}\omega^y - \tfrac{7}{3}I_{101}\alpha_T - \tfrac{4}{3}I_{001}\alpha_{TT} - I_{011}\beta_T \\
&\qquad + (\tfrac{2}{3}\,\varepsilon I_{110} - I_{002})\gamma_T + \tfrac{2}{3}\,\varepsilon I_{010}\gamma_{TT} - \tfrac{4}{9}\,\varepsilon\gamma_{TTT}, \\
&\mathrm{d}_H I_{020} = I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y - \tfrac{4}{3}I_{020}\alpha_T, \\
&\mathrm{d}_H I_{011} = I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y - \tfrac{5}{3}I_{011}\alpha_T + \tfrac{2}{3}\,\varepsilon I_{020}\gamma_T, \\
&\mathrm{d}_H I_{002} = I_{102}\omega^t + I_{012}\omega^x + I_{003}\omega^y - 2I_{002}\alpha_T + \tfrac{4}{9}\,\varepsilon I_{010}\alpha_{TT} - \tfrac{8}{27}\,\varepsilon\alpha_{TTT} \\
&\qquad - \tfrac{4}{3}\,\varepsilon I_{011}\gamma_T, \\
&\qquad \vdots
\end{aligned} \tag{69}$$

In general, a specification of normalization equations defines a valid cross-section to the pseudo-group orbits if and only if the resulting phantom recurrence equations (69) can be solved for the basis (63) of invariantized Maurer–Cartan forms. For this, we choose the normalizations

$$\begin{aligned}
&H^1 \mapsto 0, \quad H^2 \mapsto 0, \quad H^3 \mapsto 0, \quad I_{000} \mapsto 0, \quad I_{100} \mapsto 0, \; I_{010} \mapsto 0,\\
&I_{001} \mapsto 0, \quad I_{200} \mapsto 0, \quad I_{101} \mapsto 0, \quad I_{020} \mapsto 1, \quad I_{011} \mapsto 0, \; I_{002} \mapsto 0,\\
&I_{i,0,0} \mapsto 0, \; I_{i-1,0,1} \mapsto 0, \; I_{i-2,0,2} \mapsto 0, \; \text{for all } i \geq 3,
\end{aligned} \tag{70}$$

which, when substituted into equations (69), yield the expressions

$$\begin{aligned}
\alpha &= -\omega^t, \qquad \beta = -\omega^x, \qquad \gamma = -\omega^y,\\
\alpha_T &= \tfrac{3}{4}(I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y), \qquad \alpha_{TT} = \tfrac{9}{2}(I_{110}\omega^t + \omega^x),\\
\alpha_{TTT} &= \tfrac{27}{8}\varepsilon(I_{012}\omega^x + I_{003}\omega^y), \quad \ldots \quad ,\\
\beta_T &= 0, \qquad \beta_{TT} = -\tfrac{3}{2}I_{110}\omega^x, \qquad \beta_{TTT} = -\tfrac{3}{2}I_{210}\omega^x, \quad \ldots \quad ,\\
\gamma_T &= -\tfrac{3}{2}\varepsilon(I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y), \quad \gamma_{TT} = 0,\\
\gamma_{TTT} &= \tfrac{9}{4}\varepsilon(-I_{110}I_{111}\omega^t + (I_{111} - I_{110}I_{021})\omega^x - I_{110}I_{012}\omega^y), \quad \ldots \quad ,
\end{aligned} \tag{71}$$

for the basic invariant forms. The higher-order invariantized Maurer–Cartan forms can be recursively determined from the equations

$$\begin{aligned}
\alpha_{T^{p+3}} \;=\;& \tfrac{27}{8}\varepsilon(I_{p12}\omega^x + I_{p03}\omega^y) + \tfrac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-2,1,0}\alpha_{T^{s+2}}\\
&- \tfrac{27}{8}\varepsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,1,2}\beta_{T^s} + \tfrac{9}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,1}\gamma_{T^{s+1}}\\
&- \tfrac{27}{8}\varepsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,0,3}\gamma_{T^s},
\end{aligned} \tag{72}$$

$$\beta_{T^{p+1}} \;=\; -\tfrac{3}{2}I_{p10}\omega^x + \frac{3}{2}\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,0}\beta_{T^s},$$

$$\gamma_{T^{p+2}} \;=\; \tfrac{9}{4}\varepsilon I_{p11}\omega^x - \tfrac{9}{4}\varepsilon\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,1}\beta_{T^s} + \frac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,0}\gamma_{T^{s+1}}.$$

Next we substitute expressions (71) for the invariantized Maurer–Cartan forms into the equations for the nonphantom variables in (69) to derive the recurrence formulas between the differentiated and normalized invariants

$$
\begin{aligned}
\mathcal{D}_1 I_{110} &= I_{210} - \tfrac{3}{2} I_{110} I_{120}, \\
\mathcal{D}_2 I_{110} &= I_{120} - \tfrac{3}{2} I_{110} I_{030} + \tfrac{3}{4} \varepsilon I_{012}, \\
\mathcal{D}_3 I_{110} &= I_{111} - \tfrac{3}{2} I_{110} I_{021} + \tfrac{3}{4} \varepsilon I_{003}, \\
\mathcal{D}_1 I_{210} &= I_{310} - \tfrac{9}{4} I_{210} I_{120} + \tfrac{3}{2} \varepsilon I_{111}^2 + \tfrac{9}{8} I_{111} I_{003} + 12 I_{110}^2, \\
\mathcal{D}_2 I_{210} &= I_{220} - \tfrac{9}{4} I_{210} I_{030} + \tfrac{3}{4} \varepsilon I_{112} + \tfrac{3}{2} \varepsilon I_{111} I_{021} + \tfrac{9}{8} I_{003} I_{021} + \tfrac{27}{2} I_{110}, \\
\mathcal{D}_3 I_{210} &= I_{211} - \tfrac{9}{4} I_{210} I_{021} + \tfrac{3}{2} \varepsilon I_{111} I_{012} + \tfrac{3}{4} \varepsilon I_{103} + \tfrac{9}{8} I_{012} I_{003}, \\
\mathcal{D}_1 I_{120} &= I_{220} + \tfrac{3}{2} \varepsilon I_{111} I_{021} - \tfrac{7}{4} I_{120}^2 + 6 I_{110}, \\
\mathcal{D}_2 I_{120} &= I_{130} - \tfrac{7}{4} I_{120} I_{030} + \tfrac{3}{2} \varepsilon I_{021}^2 + 6, \\
\mathcal{D}_3 I_{120} &= I_{121} - \tfrac{7}{4} I_{120} I_{021} + \tfrac{3}{2} \varepsilon I_{021} I_{012}, \\
\mathcal{D}_1 I_{111} &= I_{211} - (3 I_{120} - \tfrac{3}{2} \varepsilon I_{012}) I_{111}, \\
\mathcal{D}_2 I_{111} &= I_{121} - (I_{120} - \tfrac{3}{2} \varepsilon I_{012}) I_{021} - 2 I_{111} I_{030}, \\
\mathcal{D}_3 I_{111} &= I_{112} - (I_{120} - \tfrac{3}{2} \varepsilon I_{012}) I_{012} - 2 I_{111} I_{021}, \\
\mathcal{D}_1 I_{030} &= I_{130} - \tfrac{5}{4} I_{030} I_{120}, \\
\mathcal{D}_2 I_{030} &= I_{040} - \tfrac{5}{4} I_{030}^2, \\
\mathcal{D}_3 I_{030} &= I_{031} - \tfrac{5}{4} I_{030} I_{021}, \\
\mathcal{D}_1 I_{021} &= I_{121} - I_{030} I_{111} - \tfrac{3}{2} I_{120} I_{021}, \\
\mathcal{D}_2 I_{021} &= I_{031} - \tfrac{5}{2} I_{030} I_{021}, \\
\mathcal{D}_3 I_{021} &= I_{022} - I_{030} I_{012} - \tfrac{3}{2} I_{021}^2, \\
\mathcal{D}_1 I_{012} &= I_{112} - 2 I_{021} I_{111} - \tfrac{7}{4} I_{120} I_{012} - 2 I_{110}, \\
\mathcal{D}_2 I_{012} &= I_{022} - 2 I_{021}^2 - \tfrac{7}{4} I_{030} I_{012} - 2, \\
\mathcal{D}_3 I_{012} &= I_{013} - \tfrac{15}{4} I_{021} I_{012}, \\
\mathcal{D}_1 I_{003} &= I_{103} - 3 I_{012} I_{111} - 2 I_{003} I_{120}, \\
\mathcal{D}_2 I_{003} &= I_{013} - 3 I_{012} I_{021} - 2 I_{003} I_{030}, \\
\mathcal{D}_3 I_{003} &= I_{004} - 3 I_{012}^2 - 2 I_{003} I_{021}, \\
&\ \ \vdots
\end{aligned}
\tag{73}
$$

The commutation relations among the invariant differential operators $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, are established as in the KdV example using the methods of [**30**]:

$$
\begin{aligned}
[\mathcal{D}_1, \mathcal{D}_2] &= \tfrac{3}{4} I_{030} \mathcal{D}_1 - \tfrac{1}{4} I_{120} \mathcal{D}_2 - \tfrac{3}{2} \varepsilon I_{021} \mathcal{D}_3, \\
[\mathcal{D}_1, \mathcal{D}_3] &= \tfrac{3}{4} I_{021} \mathcal{D}_1 - I_{111} \mathcal{D}_2 - (\tfrac{1}{2} I_{120} + \tfrac{3}{2} \varepsilon I_{012}) \mathcal{D}_3, \\
[\mathcal{D}_2, \mathcal{D}_3] &= -\tfrac{3}{4} I_{021} \mathcal{D}_2 - \tfrac{1}{2} I_{030} \mathcal{D}_3.
\end{aligned}
\tag{74}
$$

We can then express higher-order normalized invariants in terms of the lower-order invariants by repeated application of the recurrence formulas (73). For example, the first four equations in (73) yield the expressions

$$\begin{aligned} I_{310} = {} & \mathcal{D}_1^2 I_{110} + (\tfrac{3}{2} I_{120} + \tfrac{9}{4}\mathcal{D}_2 I_{110} + \tfrac{27}{8} I_{110} I_{030} - \tfrac{27}{16}\varepsilon I_{012})\mathcal{D}_1 I_{110} \\ & + \tfrac{27}{8} I_{110} I_{120} \mathcal{D}_2 I_{110} - \tfrac{3}{2}\varepsilon(\mathcal{D}_3 I_{110})^2 - (\tfrac{9}{2}\varepsilon I_{110} I_{021} - \tfrac{9}{8} I_{003})\mathcal{D}_3 I_{110} \\ & + \tfrac{3}{2} I_{110}\mathcal{D}_1 I_{120} + (\tfrac{81}{16} I_{120} I_{030} - \tfrac{27}{8}\varepsilon I_{021}^2) I_{110}^2 - \tfrac{81}{32}\varepsilon I_{110} I_{120} I_{012} \\ & + \tfrac{27}{16} I_{110} I_{021} I_{003} - 12 I_{110}^2. \end{aligned} \tag{75}$$

Moreover, in light of the results in [**31**], we derive the following fundamental syzygies amongst the basic differential invariants I_{110}, I_{030}, I_{021}, I_{012}, I_{003}:

$$\begin{aligned} & \mathcal{D}_3 I_{012} - \mathcal{D}_2 I_{003} + \tfrac{3}{4} I_{012} I_{021} - 2 I_{030} I_{003} = 0, \\ & \mathcal{D}_2 I_{021} - \mathcal{D}_3 I_{030} + \tfrac{5}{4} I_{021} I_{030} = 0, \\ & \mathcal{D}_3 I_{021} - \mathcal{D}_2 I_{012} - \tfrac{1}{2} I_{021}^2 - \tfrac{3}{4} I_{012} I_{030} - 2\varepsilon = 0, \\ & \mathcal{D}_2\mathcal{D}_2 I_{110} - \mathcal{D}_1 I_{030} - \tfrac{3}{4}\varepsilon\mathcal{D}_3 I_{021} + \tfrac{3}{2} I_{110}\mathcal{D}_2 I_{030} + 2 I_{030}\mathcal{D}_2 I_{110} \\ & \qquad - \tfrac{9}{8}\varepsilon I_{021}^2 + \tfrac{3}{16}\varepsilon I_{030} I_{012} + \tfrac{3}{4} I_{030}^2 I_{110} - \tfrac{9}{2} = 0, \\ & \mathcal{D}_2\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{021} + I_{030}\mathcal{D}_3 I_{110} + I_{021}\mathcal{D}_2 I_{110} + \tfrac{3}{2} I_{110}\mathcal{D}_3 I_{030} \\ & \qquad - \tfrac{3}{4}\varepsilon\mathcal{D}_2 I_{003} - \tfrac{9}{8} I_{110} I_{030} I_{021} - \tfrac{3}{4}\varepsilon I_{030} I_{003} - \tfrac{9}{8}\varepsilon I_{021} I_{012} = 0, \\ & \mathcal{D}_2\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{021} + I_{030}\mathcal{D}_3 I_{110} + I_{021}\mathcal{D}_2 I_{110} + \tfrac{3}{2} I_{110}\mathcal{D}_2 I_{021} \\ & \qquad + \tfrac{3}{4} I_{110} I_{030} I_{021} - \tfrac{3}{4}\varepsilon\mathcal{D}_2 I_{003} - \tfrac{9}{8}\varepsilon I_{021} I_{012} - \tfrac{3}{4}\varepsilon I_{030} I_{003} = 0, \\ & \mathcal{D}_3\mathcal{D}_3 I_{110} - \mathcal{D}_1 I_{012} + \tfrac{3}{2} I_{021}\mathcal{D}_3 I_{110} - \tfrac{3}{4} I_{012}\mathcal{D}_2 I_{110} \\ & \qquad + (\tfrac{3}{2}\mathcal{D}_2 I_{012} + \tfrac{3}{4} I_{021}^2 + \varepsilon) I_{110} - \tfrac{3}{4}\varepsilon\mathcal{D}_3 I_{003} - \tfrac{15}{16}\varepsilon I_{012}^2 = 0, \\ & \mathcal{D}_3\mathcal{D}_3\mathcal{D}_3 I_{110} - \mathcal{D}_1\mathcal{D}_2 I_{003} - 2 I_{030}\mathcal{D}_1 I_{003} + 3 I_{021}\mathcal{D}_1 I_{012} - 2 I_{003}\mathcal{D}_1 I_{030} \\ & \quad + (2\mathcal{D}_3 I_{021} - \tfrac{7}{4} I_{021}^2)\mathcal{D}_3 I_{110} + (\tfrac{21}{16} I_{012} I_{021} - \tfrac{5}{4}\mathcal{D}_3 I_{012})\mathcal{D}_2 I_{110} \\ & \quad + (\tfrac{3}{2}\mathcal{D}_3\mathcal{D}_3 I_{021} - \tfrac{15}{4} I_{021}\mathcal{D}_3 I_{021} - \tfrac{15}{8}\mathcal{D}_3 I_{012} I_{030} + \tfrac{63}{32} I_{021} I_{012} I_{030} \\ & \qquad + \tfrac{3}{4} I_{021}^3 + 6\varepsilon I_{021}) I_{110} - \tfrac{3}{4}\varepsilon\mathcal{D}_3\mathcal{D}_3 I_{003} + \tfrac{9}{8}\varepsilon I_{021}\mathcal{D}_3 I_{003} \\ & \quad - \tfrac{57}{16}\varepsilon I_{012}\mathcal{D}_3 I_{012} + \tfrac{3}{4}\varepsilon I_{003}\mathcal{D}_3 I_{021} - \tfrac{3}{8}\varepsilon I_{021}^2 I_{003} - \tfrac{3}{2} I_{003} + \tfrac{9}{64}\varepsilon I_{021} I_{012}^2 = 0. \end{aligned} \tag{76}$$

These allow us to further reduce the number of generating differential invariants:

Theorem 5.5. *The differential invariants I_{110}, I_{021}, I_{003} form a generating set for the algebra $\mathbf{I}_{\mathrm{KP}}$ of differential invariants for the* KP *symmetry pseudo-group.*

Computations indicate that I_{110}, I_{021}, I_{003} form, in fact, a minimal generating set. However, a few technical details remain to be overcome. Indeed, a significant issue is to devise a general differential-algebraic theory for pinpointing a minimal generating set for a prescribed differential invariant algebra.

The recurrence formulas (73), the generating syzygies (76), along with the commutation relations (74), serve to completely specify the structure of the KP differential invariant algebra $\mathbf{I}_{\mathrm{KP}}$. Observe that so far we have only used the infinitesimal determining equations and choice of cross-section normalization to completely reveal this intricate structure! However, to derive explicit formulas for the moving frame, the differential invariants and the invariant differential operators, we require the explicit formulas for the KP symmetry pseudo-group transformations. The standard algorithm [**25**] for constructing a group action from the infinitesimal generators yields the explicit KP symmetry transformations:

$$\begin{aligned}
T &= F(t),\\
X &= xF'(t)^{1/3} - \tfrac{2}{9}\varepsilon y^2 F'(t)^{-2/3}F''(t) - \tfrac{2}{3}\varepsilon y F'(t)^{-1/3}H'(t) + G(t),\\
Y &= yF'(t)^{2/3} + H(t),\\
U &= uF'(t)^{-2/3} + \tfrac{2}{9}xF'(t)^{-5/3}F''(t) - \tfrac{4}{27}y^2(\varepsilon F'(t)^{-5/3}F'''(t)\\
&\quad + \tfrac{4}{3}F'(t)^{-8/3}F''(t)^2) + \tfrac{4}{9}\varepsilon y(F'(t)^{-7/3}F''(t)h'(t) - F'(t)^{-4/3}H''(t))\\
&\quad + \tfrac{2}{9}\varepsilon F'(t)^{-2}H'(t)^2 + \tfrac{2}{3}F'(t)^{-1}G'(t),
\end{aligned} \tag{77}$$

where $F(t)$ is an arbitrary smooth, invertible function and $G(t)$, $H(t)$ are arbitrary smooth functions; see also [**20**]. Thus the prolonged action of the KP symmetry algebra on submanifold jets can be obtained by applying the differential operators

$$\begin{aligned}
D_X &= F'(t)^{-1/3}D_x,\\
D_Y &= F'(t)^{-2/3}D_y + \varepsilon(\tfrac{4}{9}yF'(t)^{-5/3}F''(t) + \tfrac{2}{3}F'(t)^{-4/3}G'(t))D_x,\\
D_T &= F'(t)^{-1}D_t + (-\tfrac{1}{3}xF'(t)^{-2}F''(t) + \varepsilon y^2(\tfrac{2}{9}F'(t)^{-2}\ F'''(t)\\
&\qquad - \tfrac{4}{9}F'(t)^{-3}F''(t)^2) + \varepsilon y(\tfrac{2}{3}F'(t)^{-5/3}G''(t)\\
&\qquad - \tfrac{10}{9}F'(t)^{-8/3}F''(t)G'(t)) - F'(t)^{-4/3}H'(t)\\
&\qquad - \tfrac{2}{3}\varepsilon F'(t)^{-7/3}G'(t)^2))D_x + (-\tfrac{2}{3}yF'(t)^{-2}F''(t)\\
&\qquad - F'(t)^{-5/3}G'(t))D_y,
\end{aligned} \tag{78}$$

to U in (77). Now normalizations (70) yield the expressions

$$\begin{aligned}
I_{110} &= u_{xx}^{-3/2}(u_{tx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}\varepsilon u_{yy}),\\
I_{030} &= u_{xx}^{-5/4}u_{xxx},\\
I_{021} &= u_{xx}^{-5/2}(u_{xx}u_{xxy} - u_{xy}u_{xxx}),\\
I_{012} &= u_{xx}^{-15/4}(u_{xx}^2u_{xyy} - 2u_{xx}u_{xy}u_{xxy} - 2\varepsilon u_x u_{xx}^3 + u_{xy}^2u_{xxx}),\\
I_{003} &= u_{xx}^{-5}(u_{xx}^3u_{yyy} - 3u_{xx}^2u_{xy}u_{xyy} + 3u_{xx}u_{xy}^2u_{xxy} - u_{xy}^3u_{xxx}),
\end{aligned} \tag{79}$$

for the basic differential invariants for the KP symmetry algebra as well as the expressions

$$\begin{aligned}
\mathcal{D}_1 &= u_{xx}^{-3/4}D_t + \tfrac{3}{4}u_{xx}^{-11/4}(2uu_{xx}^2 - \varepsilon u_{xy}^2)D_x + \tfrac{3}{2}\varepsilon u_{xy}u_{xx}^{-7/4}D_y,\\
\mathcal{D}_2 &= u_{xx}^{-1/4}D_x,\\
\mathcal{D}_3 &= -u_{xx}^{-3/2}u_{xy}D_x + u_{xx}^{-1/2}D_y,
\end{aligned} \tag{80}$$

for the invariant differential operators. To our knowledge, the results of this section provide the first complete classification of the differential invariants of the KP symmetry algebra; for earlier partial results, see [**10**], [**21**].

Additionally, by applying the invariantization map as in Theorem 4.3, we see that the KP equation (60) can be written in terms of the normalized differential invariants as

$$I_{110} + \tfrac{1}{4} I_{040} = u_{xx}^{-3/2}(u_{tx} + \tfrac{3}{2} u_x^2 + \tfrac{3}{4} \varepsilon u_{yy} + \tfrac{3}{2} u u_{xx}) + \tfrac{1}{4} u_{xx}^{-3/2} u_{xxxx} = 0. \quad (81)$$

Indeed, using invariantization on the known, incomplete system of differential invariants in [**10**] is a useful tool for facilitating and verifying the accuracy of these intricate explicit computations.

The KP symmetry algebra is known to possess a Kac–Moody–Virasoro structure [**6**], [**8**], [**9**]. The infinitesimal generators of a pseudo-group form the general solution to the infinitesimal determining equations $\mathcal{L}$, while the lifted version of $\mathcal{L}$ determines the structure of the Maurer–Cartan forms, which play a key role in the construction of the recurrence formulas between normalized and differentiated invariants. It would be an interesting problem, which now can be systematically studied by our methods, to investigate to what extent the Lie algebra structure of a symmetry algebra determines the structure of its differential invariant algebra.

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