MAURER–CARTAN EQUATIONS FOR LIE SYMMETRY PSEUDO-GROUPS OF DIFFERENTIAL EQUATIONS

JEONGOO CHEH, PETER J. OLVER, JUHA POHJANPELTO

ABSTRACT. A new method of constructing structure equations of Lie symmetry pseudo-groups of differential equations, dispensing with explicit solutions of the (infinitesimal) determining systems of the pseudo-groups, is presented, and illustrated by the examples of the Kadomtsev–Petviashvili and Korteweg–de-Vries equations.

1. INTRODUCTION

The theory of continuous groups of transformations created by Sophus Lie in the late nineteenth century has evolved to become one of the most important tools for geometric and algebraic study of general nonlinear partial differential equations. Lie himself made no essential distinction between finite-dimensional Lie group actions and infinite-dimensional pseudo-group actions. However, since his time, the two subjects have developed in very different directions. The theoretical foundations of finite-dimensional Lie groups and Lie algebras were well-established in the early twentieth century. In contrast, despite its evident importance in both mathematics and applications, the basic theory for infinite-dimensional Lie pseudo-groups remains in relatively primitive shape. Unlike Lie groups, to this day, there is no generally accepted abstract object that represents an infinite-dimensional pseudo-group, and so, like Lie and Cartan, [3], we can only study them in the context of their action on a manifold. This makes the subject considerably more difficult than the finite-dimensional case, and a significant effort has been made in establishing a proper rigorous foundation for pseudo-groups, [9, 12, 13, 14, 25, 28].

Lie pseudo-groups appear in gauge theories, Hamiltonian mechanics, symplectic and Poisson geometry, conformal geometry of surfaces, conformal field theory, and geometry of real hypersurfaces, as symmetry groups of both linear and nonlinear partial differential equations arising in fluid mechanics, solitons, relativity, etc., and as foliation-preserving groups of transformations. In general, a Lie pseudo-group $\mathcal{G}$ is defined in terms of a system $\mathcal{R}$ of (typically nonlinear) differential equations, called its *determining system*, whose solutions are the local diffeomorphisms constituting the pseudo-group. One immediate issue is to determine their local structure, which is usually expressed in the form of Maurer–Cartan structure equations, as in the case of finite-dimensional Lie groups. Both Lie's attempt to use his infinitesimal method based on the infinitesimal determining system obtained by linearizing the determining system, and Cartan's method using intricate recursive prolongation of exterior differential systems are either limited in scope or impractical from the standpoint of applications. Along this

Date: April 13, 2005.

line of research in the last decade, G. Reid, *et al.*, [15, 16, 17, 26, 27], developed methods for determining Cartan structure equations of Lie pseudo-groups, which depended only on algebraic and differential manipulation, without any integration, of infinitesimal determining systems, hence increasing the feasibility of their computer algebra implementation. Their algorithms were successfully applied to certain types of Lie symmetry pseudo-groups of differential equations. A major drawback, however, is that their methods were based on ad hoc series expansions and became significantly more complicated, requiring more case-by-case analyses, if they worked at all, when it came to intransitive pseudo-group actions.

More recently, the second and third authors developed a theory, [23], where the invariant contact forms on the diffeomorphism jet bundle were interpreted as the Maurer–Cartan forms of the Lie pseudo-group. (For finite-dimensional symmetry groups, Morozov, [21], has introduced a related approach based on the method of moving coframes, [8].) As a result, a very efficient method for constructing the structure equations of the Maurer–Cartan forms was discovered. This method bypasses the troublesome process of integrating either the determining system $\mathcal{R}$ or its linearization, or the complicated Cartan prolongation process. Moreover, the algorithm directly applies to completely general Lie pseudo-group actions, whether finite- or infinite-dimensional, transitive or intransitive, and can be easily implemented in computer algebra systems.

The goal of this paper is to show how to use the method to directly construct structure equations for Lie (point) symmetry pseudo-groups of differential equations. Our algorithm works for any general Lie symmetry pseudo-group, and it will also give us better understanding of known local symmetry structures, as well as revealing those of a wide range of differential equations that still wait to be investigated. We also wrote some Mathematica routines, [4], to facilitate the computations needed for the implementation of our method on specific differential equations. To illustrate our algorithm, we will use the Kadomtsev–Petviashvili (KP) equation,

$$(u_t + \tfrac{3}{2} u u_x + \tfrac{1}{4} u_{xxx})_x + \tfrac{3}{4} \epsilon u_{yy} = 0, \qquad \epsilon = \pm 1, \tag{1}$$

and the Korteweg–de-Vries (KdV) equation

$$u_t + u_{xxx} + u u_x = 0, \tag{2}$$

both of which are integrable soliton equations, possessing, respectively, infinite- and finite-dimensional Lie symmetry pseudo-groups, [1, 5, 6, 7, 18, 19, 20].

Let us recall how the classical Lie symmetry method, [22], works in the context of the KP equation. Let $M = \mathbb{R}^4$ with coordinates t, x, y, u given by the independent and dependent variables in the differential equation, and let $J^\infty(M, 3)$ stand for the jet bundle of equivalence classes of three-dimensional submanifolds $u = f(t, x, y)$ of M under the equivalence relation of infinite-order contact. The infinitesimal symmetry algebra $\mathfrak{g}$ of the KP equation consists of the local vector fields

$$\mathbf{v} = \tau \partial_t + \xi \partial_x + \eta \partial_y + \phi \partial_u \tag{3}$$

on M such that their prolongations $\mathbf{v}^{(\infty)}$ are tangent to the variety in $J^\infty(M, 3)$ defined by the equation (1). This characterizing condition yields the system of partial differential equations

$$
\begin{aligned}
&\eta_u = 0, \quad \eta_x = 0, \quad \xi_u = 0, \quad \eta_t + \tfrac{3}{2}\epsilon\xi_y = 0, \quad 3\xi_x - \tau_t = 0,\\
(4) \qquad &\tfrac{3}{2}\phi - \xi_t + u\tau_t = 0, \quad \tfrac{3}{2}\eta_y - \tau_t = 0, \quad \tau_u = 0, \quad \tau_x = 0, \quad \tau_y = 0,\\
&\tau_t + \tfrac{3}{2}\phi_u = 0, \quad \tfrac{2}{3}\tau_{tt} - 3\phi_x = 0, \quad \phi_{tx} + \tfrac{3}{2}u\phi_{xx} + \tfrac{1}{4}\phi_{xxxx} + \tfrac{3}{4}\epsilon\phi_{yy} = 0,
\end{aligned}
$$

for the coefficient functions of the infinitesimal symmetries $\mathbf{v}$. These equations are the minimal *infinitesimal* (or *linearized*) *determining system* of the Lie symmetry pseudo-group $\mathcal{G}$ of the KP equation. Once the system (4) is completed to involution, all the higher order equations are obtained by differentiation with respect to t, x, y, u. The key point of this paper is to determine the local structure of $\mathcal{G}$ directly from its infinitesimal determining system (4). This set-up has an obvious counterpart for the KdV equation (2), too, which, for brevity, we will not elaborate on until the end of the paper.

Our subsequent discussions in the paper are organized in such a way that the results of each section are applied to the case of the KP equation within that section, and, starting from the current section, consistent notation without further comment will be used for the examples. It is worth emphasizing that both finite-dimensional and infinite-dimensional symmetry pseudo-groups are handled, on an equal footing, by precisely the same algorithms to be presented here.

2. Maurer–Cartan Forms for the Pseudo-group of Local Diffeomorphisms

Let M be a smooth manifold of dimension m, and $\mathcal{D} = \mathcal{D}(M)$ the pseudo-group of local diffeomorphisms on M. For each $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M)$ be the bundle of n-jets of maps in $\mathcal{D}$. The bundle $\mathcal{D}^{(n)}$ is double-fibered over M with fibrations being the source projection

$$\boldsymbol{\sigma}^{(n)} \colon \mathcal{D}^{(n)} \longrightarrow M, \qquad \boldsymbol{\sigma}^{(n)}(j_z^n\phi) = z,$$

and the target projection

$$\boldsymbol{\tau}^{(n)} \colon \mathcal{D}^{(n)} \longrightarrow M, \qquad \boldsymbol{\tau}^{(n)}(j_z^n\phi) = \phi(z),$$

where $z \in M$ and $\phi \in \mathcal{D}$. Composition of local diffeomorphisms turns $\mathcal{D}^{(n)}$ into a Lie groupoid with multiplication law

$$(j_z^n\phi)\cdot(j_w^n\psi) := j_w^n(\phi\circ\psi) \qquad \text{provided} \qquad \boldsymbol{\sigma}^{(n)}(j_z^n\phi) = z = \psi(w) = \boldsymbol{\tau}^{(n)}(j_w^n\psi).$$

Local coordinates of the base space M and the total space $\mathcal{D}^{(n)}$ are denoted by $z = (z^i)$ and $(z, Z^{(n)}) = (z^i, Z^a_J)$, respectively, where $Z = (Z^a)$ and Z^a_J, with J a symmetric multi-index, represents the derivative $\partial^J Z^a/\partial z^J$. The natural identification, obtained by viewing maps in terms of their graphs, of $\mathcal{D}^{(\infty)}$ with an open subbundle of the jet bundle $J^\infty(M\times M)$ of infinite jets of local sections of the trivial bundle

$$M\times M \longrightarrow M, \qquad (z, Z) \longmapsto z,$$

induces a variational bicomplex structure, [2, 10, 11], on the cotangent bundle $T^*\mathcal{D}^{(\infty)}$, where the horizontal subbundle is spanned by the horizontal forms $dz^1, \ldots, dz^m$, and the vertical subbundle is spanned by the basic contact forms

$$\Upsilon^a_J := dZ^a_J - \sum_{i=1}^m Z^a_{J,i}\,dz^i, \qquad a = 1, \ldots, m, \quad \#J \geq 0.$$

Accordingly, the exterior differential $d = d_M + d_G$ on $\mathcal{D}^{(\infty)}$ splits into the horizontal differential[1] d_M and the contact (or vertical) differential d_G, so that

$$dF = d_M F + d_G F = \sum_{i=1}^{m} (\mathbb{D}_{z^i} F) dz^i + \sum_{a=1}^{m} \sum_{\#J \geq 0} (\partial_{Z^a_J} F) \Upsilon^a_J,$$

for any smooth function $F\colon \mathcal{D}^{(\infty)} \to \mathbb{R}$. Here, $\mathbb{D}_{z^i}$ is the total differential operator with respect to the base coordinate function z^i, and $\partial_{Z^a_J}$ is the partial differential operator with respect to the fiber coordinate function Z^a_J.

Let

$$R_\psi(j^n_z \phi) := (j^n_z \phi) \cdot (j^n_z \psi)^{-1} = j^n_{\psi(z)}(\phi \circ \psi^{-1})$$

denote the right action of the diffeomorphism pseudo-group on its jets. A differential form μ on $\mathcal{D}^{(n)}$ is said to be *right-invariant* if $R^*_\psi \mu = \mu$, whenever defined, for all $\psi \in \mathcal{D}$. In particular, the target coordinate functions Z^a are right-invariant. Since the right action preserves the splitting of $T^*\mathcal{D}^{(\infty)}$, the horizontal and contact differentials of invariant forms are invariant. Thus,

$$\sigma^a := d_M Z^a = \sum_{i=1}^{m} Z^a_i dz^i, \qquad \mu^a := d_G Z^a = \Upsilon^a, \qquad a = 1, \ldots, m,$$

are, respectively, *invariant horizontal forms* and *invariant contact forms*. The *invariant differential operators*, denoted $\mathbb{D}_{Z^a}$, are dual to the invariant horizontal forms σ^a, so

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F) \sigma^a \qquad \text{for all} \qquad F \in C^\infty(\mathcal{D}^{(\infty)}).$$

Lie derivatives of invariant differential forms with respect to the invariant differential operators are invariant. Thus, the *higher order invariant contact forms*

$$\mu^a_J := \mathbb{D}^J_Z \mu^a := \mathbb{D}_{Z^{j_1}} \mathbb{D}_{Z^{j_2}} \ldots \mathbb{D}_{Z^{j_k}} \mu^a$$

are obtained by repeated Lie differentiation. As argued in [23, 24], these invariant contact forms play the role of the *Maurer–Cartan forms* for $\mathcal{D}$.

Let us present the explicit formulas in the case relevant to the KP equation. Since the KdV case is completely similar, except with one fewer independent variable, it will not be explicitly presented.

KP equation 1. The KP equation has independent variables t, x, y and dependent variable u, which we regard as coordinates on the total space $M = \mathbb{R}^4$. We denote the corresponding local coordinates of the diffeomorphism groupoid $\mathcal{D}^{(\infty)}$ by

$$(t, x, y, u, T, X, Y, U, T_t, T_x, T_y, T_u, X_t, X_x, X_y, X_u, \ldots).$$

The horizontal forms on $\mathcal{D}^{(\infty)}$ are dt, dx, dy, du, and the contact forms are

$$\begin{aligned}
&\Upsilon^t := d_G T = dT - T_t dt - T_x dx - T_y dy - T_u du,\\
&\Upsilon^x := d_G X = dX - X_t dt - X_x dx - X_y dy - X_u du,\\
&\Upsilon^y := d_G Y = dY - Y_t dt - Y_x dx - Y_y dy - Y_u du,\\
&\Upsilon^u := d_G U = dU - U_t dt - U_x dx - U_y dy - U_u du,\\
&\Upsilon^t_t := \mathbb{D}_t \Upsilon^t = d_G T_t = dT_t - T_{tt} dt - T_{tx} dx - T_{ty} dy - T_{tu} du,\\
&\ldots \qquad \text{and, in general,} \qquad \Upsilon^a_{h,k,l,n} := \mathbb{D}^h_t \mathbb{D}^k_x \mathbb{D}^l_y \mathbb{D}^n_u \Upsilon^a,
\end{aligned}$$

[1]We retain the notation adopted in [23].

where we use a to signify either t, x, y or u, and $h, k, l, n \geq 0$, with $\mathbb{D}_t$, $\mathbb{D}_x$, $\mathbb{D}_y$, $\mathbb{D}_u$ denoting the total differential operators. The invariant horizontal forms are

$$\begin{aligned} \sigma^t &:= d_M T = T_t dt + T_x dx + T_y dy + T_u du, \\ \sigma^x &:= d_M X = X_t dt + X_x dx + X_y dy + X_u du, \\ \sigma^y &:= d_M Y = Y_t dt + Y_x dx + Y_y dy + Y_u du, \\ \sigma^u &:= d_M U = U_t dt + U_x dx + U_y dy + U_u du, \end{aligned} \tag{5}$$

and the invariant contact forms are

$$\mu^t := d_G T = \Upsilon^t, \quad \mu^x := d_G X = \Upsilon^x, \quad \mu^y := d_G Y = \Upsilon^y, \quad \mu^u := d_G U = \Upsilon^u,$$

$$\mu^t_T := \mathbb{D}_T \mu^t, \quad \mu^t_X := \mathbb{D}_X \mu^t, \quad \ldots \quad \text{and, in general,} \quad \mu^a_{h,k,l,n} := \mathbb{D}^h_T \mathbb{D}^k_X \mathbb{D}^l_Y \mathbb{D}^n_U \mu^a,$$

where $a = t, x, y$ or u, and

$$\begin{bmatrix} \mathbb{D}_T \\ \mathbb{D}_X \\ \mathbb{D}_Y \\ \mathbb{D}_U \end{bmatrix} = \begin{bmatrix} T_t & X_t & Y_t & U_t \\ T_x & X_x & Y_x & U_x \\ T_y & X_y & Y_y & U_y \\ T_u & X_u & Y_u & U_u \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{D}_t \\ \mathbb{D}_x \\ \mathbb{D}_y \\ \mathbb{D}_u \end{bmatrix} \tag{6}$$

are the invariant differential operators.

3. Structure Equations of the Diffeomorphism Pseudo-group

The invariant coframe for the diffeomorphism pseudo-group are the invariant horizontal and contact forms σ^a, μ^a_J. The structure equations amount to writing their differentials $d\sigma^a, d\mu^a_J$ as linear combinations of wedge products of the invariant differential forms. A concise way to write down the structure equations, as first described in [23], rests on a formal power series expansion[2]. To this end, we define the vector-valued formal power series

$$\mu[\![H]\!] := (\ \mu^a[\![H\ ]\!]) = \left(\sum_{\#J \geq 0} \frac{1}{J!} \mu^a_J H^J \right) \tag{7}$$

whose coefficients are the invariant contact forms on $\mathcal{D}^{(\infty)}$. In particular, if we set $H = (H^a) = 0$ in (7), then $\mu[\![0]\!] = \mu := (\mu^a)$. The key result, proved in [23], is that the structure equations of the invariant coframe can be read off from certain matrix identities.

Lemma 3.1. *Let*

$$\nabla \mu[\![H]\!] := \left(\frac{\partial \mu^a[\![H]\!]}{\partial H^b} \right)$$

denote the Jacobian matrix of the vector $\mu[\![H]\!]$ of power series in the variables $H = (H^a)$. Then

$$d\mu[\![H]\!] = \nabla \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ). \tag{8}$$

The structure equations of the invariant horizontal forms are given by

$$d\sigma = -d\mu \tag{9}$$

where $\sigma := (\sigma^a)$.

---

[2]Unlike Reid, *et al.*, [16, 17, 26, 27], we are *not* using power series to expand the determining equations. They are merely a convenient notational device.

The coefficients of the powers H^J in equations (8), along with (9), form the complete system of structure equations for the diffeomorphism pseudo-group $\mathcal{D}$.

KP equation 2. We use the capital letters $H, K, L, N \in \mathbb{R}$ to denote variables for power series. The structure equations for the invariant contact forms on $\mathcal{D}^{(\infty)}$ are given by the identity

$$d\mu[\![H,K,L,N]\!] = \nabla\mu[\![H,K,L,N]\!] \wedge (\mu[\![H,K,L,N]\!] - dZ), \tag{10}$$

where

$$d\mu[\![H,K,L,N]\!] = \sum_{h,k,l,n\geq 0} \frac{H^h K^k L^l N^n}{h!\,k!\,l!\,n!} \begin{bmatrix} d\mu^t_{h,k,l,n} \\ d\mu^x_{h,k,l,n} \\ d\mu^y_{h,k,l,n} \\ d\mu^u_{h,k,l,n} \end{bmatrix},$$

$$\nabla\mu[\![H,K,L,N]\!] =$$

$$\sum_{h,k,l,n\geq 0} \frac{H^h K^k L^l N^n}{h!\,k!\,l!\,n!} \begin{bmatrix} \mu^t_{h+1,k,l,n} & \mu^t_{h,k+1,l,n} & \mu^t_{h,k,l+1,n} & \mu^t_{h,k,l,n+1} \\ \mu^x_{h+1,k,l,n} & \mu^x_{h,k+1,l,n} & \mu^x_{h,k,l+1,n} & \mu^x_{h,k,l,n+1} \\ \mu^y_{h+1,k,l,n} & \mu^y_{h,k+1,l,n} & \mu^y_{h,k,l+1,n} & \mu^y_{h,k,l,n+1} \\ \mu^u_{h+1,k,l,n} & \mu^u_{h,k+1,l,n} & \mu^u_{h,k,l+1,n} & \mu^u_{h,k,l,n+1} \end{bmatrix},$$

$$\mu[\![H,K,L,N]\!] - dZ = -\begin{bmatrix} \sigma^t \\ \sigma^x \\ \sigma^y \\ \sigma^u \end{bmatrix} + \sum_{\substack{h,k,l,n\geq 0 \\ h+k+l+n\geq 1}} \frac{H^h K^k L^l N^n}{h!\,k!\,l!\,n!} \begin{bmatrix} \mu^t_{h,k,l,n} \\ \mu^x_{h,k,l,n} \\ \mu^y_{h,k,l,n} \\ \mu^u_{h,k,l,n} \end{bmatrix}.$$

Once the structure equations for the invariant contact forms are established, those for the invariant horizontal forms are immediately obtained by

$$\begin{bmatrix} d\sigma^t \\ d\sigma^x \\ d\sigma^y \\ d\sigma^u \end{bmatrix} = -\begin{bmatrix} d\mu^t \\ d\mu^x \\ d\mu^y \\ d\mu^u \end{bmatrix}.$$

4. MAURER–CARTAN EQUATIONS FOR LIE SYMMETRY PSEUDO-GROUPS

Let $\mathcal{X} = \mathcal{X}(M)$ denote the space of locally defined vector fields

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a \partial_{z^a}$$

on M, i.e., the space of local sections of its tangent bundle TM. Given a sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$, let $\mathfrak{g} \subset \mathcal{X}$ denote its local Lie algebra of infinitesimal generators, and $\mathfrak{g}^{(n)}$ their jets. With each sufficiently large n, the subbundle $\mathfrak{g}^{(n)} \subset J^n TM$ is characterized by the linearized (or infinitesimal) determining equations

$$L^{(n)}(z^i, \zeta^a_J) = 0 \tag{11}$$

for the pseudo-group $\mathcal{G}$. Here $\zeta^a_J = \partial^J_z \zeta^a$ are the jet coordinates of a vector field $\mathbf{v}$. If $\mathcal{G}$ is the symmetry group of a system of differential equations, then (11) are (the involutive completion of) the usual determining equations obtained through Lie's infinitesimal symmetry method.

A complete, though certainly not minimal, system of invariant differential one-forms for $\mathcal{G}$ is obtained by restricting the invariant coframe $\{\sigma^a, \mu^a_J\}$ to the Lie subgroupoid $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ given by the jets of pseudo-group diffeomorphisms. For simplicity, we will not explicitly employ the pull-back notation on these restricted forms. The resulting dependencies among the restricted forms are elucidated by the following theorem, [23].

Theorem 4.1. *The invariant forms μ^a_J on $\mathcal{G}^{(n)}$ satisfy the linear system*

$$L^{(n)}(Z^i, \mu^a_J) = 0 \tag{12}$$

obtained by replacing z^i by Z^i and ζ^a_J by μ^a_J in the linearized determining equations (11).

In accordance with [23], we refer to (12) as the *lifted determining equations* for the pseudo-group.

Theorem 4.2. *The structure equations of the invariant coframe for a Lie pseudo-group $\mathcal{G}$ are obtained by restricting the diffeomorphism structure equations (8–9) to the space of solutions of the lifted determining equations (12).*

Since the target coordinates $Z = (Z^a)$ are right-invariant, the individual fibers of the target fibration $\boldsymbol{\tau}^{(n)} : \mathcal{G}^{(n)} \to M$ are invariant under the right action of $\mathcal{G}$ and, for that matter, $\mathcal{G}^{(n)}$. The Cartan structure equations of a Lie pseudo-group, [3], are obtained by restricting the invariant coframe to a single fiber $\mathcal{G}^{(n)}|_Z = (\boldsymbol{\tau}^{(n)})^{-1}(Z)$, where $Z \in M$ is fixed. Since $0 = dZ^a = \sigma^a + \mu^a$ when restricted to a fiber $\mathcal{G}^{(n)}|_Z$, we can replace σ^a by $-\mu^a$, and hence only the independent invariant contact forms μ^a_J will appear in the resulting structure equations. For example, if the pseudo-group is defined by the (local) action of a finite-dimensional Lie transformation group G on M, then, under mild regularity assumptions, $\mathcal{G}^{(n)} \to M$ has the structure of a principal G-bundle for n sufficiently large,. Each fiber can be identified with a copy of the Lie group, and the restrictions of the independent invariant contact forms to $\mathcal{G}^{(n)}|_Z \simeq G$ are a system of classical (right-invariant) Maurer–Cartan forms for the group G.

These results form the foundation for a general, intrinsic algorithm for directly determining the structure of the symmetry group of a system of differential equations, as well as the structure of its algebra of differential invariants, as fixed by a choice of a moving frame, [8, 10]. The key point is that the required computations rely exclusively on linear differential algebra, and so can be readily implemented in any standard symbolic computation package. In this paper, we have concentrated on the first part of the method, the determination of the structure of the symmetry (pseudo-)group. The second part, on the structure of the differential invariant algebra and the invariant variational bicomplex, [10, 11], will be explained in more detail in a subsequent publication.

KP equation 3. Let $\mathcal{G}$ denote the infinite-dimensional symmetry pseudo-group of the KP equation (1). We begin by writing out the lifted determining equations

(13)

$$\mu^y_U = 0, \quad \mu^y_X = 0, \quad \mu^x_U = 0, \quad \mu^y_T + \tfrac{3}{2}\epsilon\mu^x_Y = 0, \quad 3\mu^x_X - \mu^t_T = 0,$$
$$\tfrac{3}{2}\mu^u - \mu^x_T + U\mu^t_T = 0, \quad \tfrac{3}{2}\mu^y_Y - \mu^t_T = 0, \quad \mu^t_U = 0, \quad \mu^t_X = 0, \quad \mu^t_Y = 0,$$
$$\mu^t_T + \tfrac{3}{2}\mu^u_U = 0, \quad \tfrac{2}{3}\mu^t_{TT} - 3\mu^u_X = 0, \quad \mu^u_{TX} + \tfrac{3}{2}U\mu^u_{XX} + \tfrac{1}{4}\mu^u_{XXXX} + \tfrac{3}{4}\epsilon\mu^u_{YY} = 0,$$

and so on, which are obtained from the KP symmetry determining equations (4) by replacing t, x, y, u by T, X, Y, U and τ, ξ, η, ϕ by the invariant contact forms $\mu^t, \mu^x, \mu^y, \mu^u$, respectively. These and their higher order counterparts obtained by Lie differentiation with respect to the invariant differential operators (6) form a complete system of linear dependencies among the invariant contact forms when restricted to the symmetry groupoid $\mathcal{G}^{(\infty)}$.

We solve the lifted system of equations (13) (or, equivalently, the original system (4) of infinitesimal determining equations prior to the lifting) through cross-differentiations and Gaussian Elimination to determine the following basis of linearly independent Maurer–Cartan forms

$$\begin{aligned} &\omega^1 := \mu^t, \quad \omega^2 := \mu^x, \quad \omega^3 := \mu^y, \quad \omega^4 := \mu^u, \quad \omega^5 := \mu^t_T = \mu^t_{1,0,0,0}, \\ &\omega^6 := \mu^y_T = \mu^y_{1,0,0,0}, \quad \alpha^i := \mu^u_{i,0,0,0}, \quad \beta^i := \mu^u_{i-1,1,0,0}, \quad \gamma^i := \mu^u_{i-1,0,1,0}, \end{aligned} \tag{14}$$

for $i = 1, 2, 3, \ldots$. For example,

$$\begin{aligned} &\mu^x_T = \tfrac{3}{2}\omega^4 + U\omega^5, \qquad \mu^t_X = 0, \qquad \mu^x_X = \tfrac{1}{3}\omega^5, \\ &\mu^y_X = 0, \qquad \mu^t_Y = 0, \qquad \mu^x_Y = -\tfrac{2}{3}\epsilon\omega^6, \qquad \mu^y_Y = \tfrac{2}{3}\omega^5, \\ &\mu^t_U = 0, \qquad \mu^x_U = 0, \qquad \mu^y_U = 0, \qquad \mu^u_U = -\tfrac{2}{3}\omega^5, \\ &\mu^t_{TT} = \tfrac{9}{2}\beta^1, \qquad \mu^x_{TT} = \tfrac{3}{2}\alpha^1 + \tfrac{9}{2}U\beta^1, \qquad \mu^y_{TT} = -\tfrac{9}{4}\epsilon\gamma^1, \\ &\mu^t_{TX} = 0, \qquad \mu^x_{TX} = \tfrac{3}{2}\beta^1, \qquad \mu^y_{TX} = 0, \qquad \mu^t_{TY} = 0, \qquad \ldots \end{aligned} \tag{15}$$

The independent invariant contact forms (14) together with the restricted invariant horizontal forms $\{\sigma^t, \ \sigma^x, \ \sigma^y, \ \sigma^u\}$ form an invariant coframe on the Lie groupoid $\mathcal{G}^{(\infty)}$. The structure equations of this coframe are obtained by imposing the dependence relation (15) on the structure equations (10) for the full diffeomorphism groupoid $\mathcal{D}^{(\infty)}$. The resulting structure equations are

$$\begin{aligned} d\sigma^t &= \omega^5 \wedge \sigma^t, \\ d\sigma^x &= \tfrac{3}{2}\omega^4 \wedge \sigma^t + U\omega^5 \wedge \sigma^t + \tfrac{1}{3}\omega^5 \wedge \sigma^x - \tfrac{2}{3}\epsilon\omega^6 \wedge \sigma^y, \\ d\sigma^y &= \tfrac{2}{3}\omega^5 \wedge \sigma^y + \omega^6 \wedge \sigma^t, \\ d\sigma^u &= -\tfrac{2}{3}\omega^5 \wedge \sigma^u + \alpha^1 \wedge \sigma^t + \beta^1 \wedge \sigma^x + \gamma^1 \wedge \sigma^y, \\ d\omega^1 &= -\omega^5 \wedge \sigma^t, \\ d\omega^2 &= -\tfrac{3}{2}\omega^4 \wedge \sigma^t - U\omega^5 \wedge \sigma^t - \tfrac{1}{3}\omega^5 \wedge \sigma^x + \tfrac{2}{3}\epsilon\omega^6 \wedge \sigma^y_, \\ d\omega^3 &= -\tfrac{2}{3}\omega^5 \wedge \sigma^y - \omega^6 \wedge \sigma^t, \\ d\omega^4 &= \tfrac{2}{3}\omega^5 \wedge \sigma^u - \alpha^1 \wedge \sigma^t - \beta^1 \wedge \sigma^x - \gamma^1 \wedge \sigma^y, \\ d\omega^5 &= -\tfrac{9}{2}\beta^1 \wedge \sigma^t, \\ d\omega^6 &= -\tfrac{1}{3}\omega^5 \wedge \omega^6 - 3\beta^1 \wedge \sigma^y + \tfrac{9}{4}\epsilon\gamma^1 \wedge \sigma^t, \\ d\alpha^1 &= -\tfrac{3}{2}\omega^4 \wedge \beta^1 - \tfrac{5}{3}\omega^5 \wedge \alpha^1 - U\omega^5 \wedge \beta^1 - \omega^6 \wedge \gamma^1 + 3\beta^1 \wedge \sigma^u \\ &\qquad - \alpha^2 \wedge \sigma^t - \beta^2 \wedge \sigma^x - \gamma^2 \wedge \sigma^y, \\ d\beta^1 &= -\omega^5 \wedge \beta^1 - \beta^2 \wedge \sigma^t, \end{aligned} \tag{16}$$

$$
\begin{aligned}
d\gamma^1 &= -\tfrac{4}{3}\omega^5\wedge\gamma^1 + \tfrac{2}{3}\epsilon\omega^6\wedge\beta^1 + \tfrac{4}{3}\epsilon\beta^2\wedge\sigma^y - \gamma^2\wedge\sigma^t,\\
d\alpha^2 &= -3\omega^4\wedge\beta^2 - \tfrac{8}{3}\omega^5\wedge\alpha^2 - 2U\omega^5\wedge\beta^2 - 2\omega^6\wedge\gamma^2 + 9\alpha^1\wedge\beta^1\\
&\qquad + 3\beta^2\wedge\sigma^u - \alpha^3\wedge\sigma^t - \beta^3\wedge\sigma^x - \gamma^3\wedge\sigma^y,\\
d\beta^2 &= -2\omega^5\wedge\beta^2 - \beta^3\wedge\sigma^t,\\
d\gamma^2 &= -\tfrac{7}{3}\omega^5\wedge\gamma^2 + 2\epsilon\omega^6\wedge\beta^2 - \tfrac{9}{2}\beta^1\wedge\gamma^1 - \gamma^3\wedge\sigma^t + \tfrac{4}{3}\epsilon\beta^3\wedge\sigma^y,\\
&\ \vdots
\end{aligned}
$$

After restricting the equations (16) to a target fiber $(\boldsymbol{\tau}^{(\infty)})^{-1}(T,X,Y,U)$, i.e., fixing the values of the target coordinates T,X,Y,U, we find the Maurer–Cartan equations for the KP symmetry pseudo-group $\mathcal{G}$ to be

$$
\begin{aligned}
d\omega^1 &= -\omega^1\wedge\omega^5,\\
d\omega^2 &= -\tfrac{3}{2}\omega^1\wedge\omega^4 - U\omega^1\wedge\omega^5 - \tfrac{1}{3}\omega^2\wedge\omega^5 + \tfrac{2}{3}\epsilon\omega^3\wedge\omega^6,\\
d\omega^3 &= -\omega^1\wedge\omega^6 - \tfrac{2}{3}\omega^3\wedge\omega^5,\\
d\omega^4 &= -\omega^1\wedge\alpha^1 - \omega^2\wedge\beta^1 - \omega^3\wedge\gamma^1 + \tfrac{2}{3}\omega^4\wedge\omega^5,\\
d\omega^5 &= -\tfrac{9}{2}\omega^1\wedge\beta^1,\\
d\omega^6 &= \tfrac{9}{4}\epsilon\omega^1\wedge\gamma^1 - 3\omega^3\wedge\beta^1 - \tfrac{1}{3}\omega^5\wedge\omega^6,\\
d\alpha^1 &= -\omega^1\wedge\alpha^2 - \omega^2\wedge\beta^2 - \omega^3\wedge\gamma^2 + \tfrac{3}{2}\omega^4\wedge\beta^1 - \tfrac{5}{3}\omega^5\wedge\alpha^1 - U\omega^5\wedge\beta^1\\
&\qquad - \omega^6\wedge\gamma^1,\\
d\beta^1 &= -\omega^1\wedge\beta^2 - \omega^5\wedge\beta^1,\\
d\gamma^1 &= -\omega^1\wedge\gamma^2 + \tfrac{4}{3}\epsilon\omega^3\wedge\beta^2 - \tfrac{4}{3}\omega^5\wedge\gamma^1 + \tfrac{2}{3}\epsilon\omega^6\wedge\beta^1,\\
d\alpha^2 &= -\omega^1\wedge\alpha^3 - \omega^2\wedge\beta^3 - \omega^3\wedge\gamma^3 - \tfrac{8}{3}\omega^5\wedge\alpha^2 - 2U\omega^5\wedge\beta^2\\
&\qquad - 2\omega^6\wedge\gamma^2 + 9\alpha^1\wedge\beta^1,\\
d\beta^2 &= -\omega^1\wedge\beta^3 - 2\omega^5\wedge\beta^2,\\
d\gamma^2 &= -\omega^1\wedge\gamma^3 + \tfrac{4}{3}\epsilon\omega^3\wedge\beta^3 - \tfrac{7}{3}\omega^5\wedge\gamma^2 + 2\epsilon\omega^6\wedge\beta^2 - \tfrac{9}{2}\beta^1\wedge\gamma^1,\\
&\ \vdots
\end{aligned}
\tag{17}
$$

The structure equations for a slightly different variant of the KP equation obtained by G. Reid, *et al.*, [15, 16, 17], involve nine basic Maurer–Cartan forms $\{\boldsymbol{\omega}^i \mid i = 1,2,\ldots,9\}$. The Maurer–Cartan equations that our algorithm finds for the particular target fiber $U=0$ can be mapped to theirs by the scaling correspondence

$$
\begin{aligned}
&\omega^3 = \mu^y \longmapsto p\,\boldsymbol{\omega}^1, \qquad \omega^2 = \mu^x \longmapsto \frac{p^2}{q}\,\boldsymbol{\omega}^2, \qquad \omega^1 = \mu^t \longmapsto q\,\boldsymbol{\omega}^3,\\
&\omega^4 = \mu^u \longmapsto -\frac{p^2}{q^2}\,\boldsymbol{\omega}^4 \qquad \gamma^1 = \mu^u_Y \longmapsto -\frac{p}{q^2}\,\boldsymbol{\omega}^5, \qquad \beta^1 = \mu^u_X \longmapsto -\frac{1}{q}\,\boldsymbol{\omega}^6,\\
&\alpha^1 = \mu^u_T \longmapsto -\frac{p^2}{q^3}\,\boldsymbol{\omega}^7, \qquad \omega^5 = \mu^t_T \longmapsto -\frac{3}{2}\,\boldsymbol{\omega}^8 \qquad \omega^6 = \mu^y_T \longmapsto -\frac{2p}{q}\,\boldsymbol{\omega}^9,\\
&\beta^2 = \mu^u_{TX} \longmapsto \frac{1}{q^2}\,\boldsymbol{\pi}^1, \qquad \gamma^2 = \mu^u_{TY} \longmapsto -\frac{p}{q^3}\,\boldsymbol{\pi}^2, \qquad \alpha^2 = \mu^u_{TT} \longmapsto -\frac{p^2}{q^4}\,\boldsymbol{\pi}^3,
\end{aligned}
\tag{18}
$$

where p, q are any nonzero constants. Moreover, the other invariant forms $\{\boldsymbol{\pi}^1, \boldsymbol{\pi}^2, \boldsymbol{\pi}^3\}$ appearing in their list of structure equations will correspond to rescalings of our next three second-order Maurer–Cartan forms $\alpha^2, \beta^2, \gamma^2$.

KdV equation Now let $\mathcal{G}$ denote the symmetry group of the KdV equation (2). Applying Lie's algorithm, the infinitesimal symmetries $\mathbf{v} = \tau\partial_t + \xi\partial_x + \phi\partial_u$ must satisfy the (minimal) determining equations

$$\xi_u = 0, \qquad 3\xi_x - \tau_t = 0, \qquad \phi - \xi_t + \tfrac{2}{3}u\tau_t = 0, \qquad \tau_u = 0,$$
$$\tau_x = 0, \qquad \phi_{uu} = 0, \qquad \phi_{xu} = 0, \qquad \phi_t + u\phi_x + \phi_{xxx} = 0.$$

When this system is completed to involution, all the higher order equations are obtained by differentiation. The corresponding lifted determining equations are

$$\mu^x_U = 0, \qquad 3\mu^x_X - \mu^t_T = 0, \qquad \mu^u - \mu^x_T + \tfrac{2}{3}U\mu^t_T = 0, \qquad \mu^t_U = 0,$$
$$\mu^t_X = 0, \qquad \mu^u_{UU} = 0, \qquad \mu^u_{XU} = 0, \qquad \mu^u_T + U\mu^u_X + \mu^u_{XXX} = 0,$$

and so on, where the higher order equations are obtained by repeated Lie differentiation with respect to $\mathbb{D}_T, \mathbb{D}_X, \mathbb{D}_U$. Restricting to the symmetry groupoid $\mathcal{G}^{(\infty)}$, there are precisely 4 independent invariant contact forms:

$$\omega^1 := \mu^t, \qquad \omega^2 := \mu^x, \qquad \omega^3 := \mu^u, \qquad \omega^4 := \mu^t_T,$$

which reflects the fact that the symmetry group of the KdV equation is a four-dimensional Lie group. The structure equations of the coframe are

$$\begin{aligned}
d\sigma^t &= \omega^4 \wedge \sigma^t, \\
d\sigma^x &= \omega^3 \wedge \sigma^t + \tfrac{2}{3}U\omega^4 \wedge \sigma^t + \tfrac{1}{3}\omega^4 \wedge \sigma^x, \\
d\sigma^u &= -\tfrac{2}{3}\omega^4 \wedge \sigma^u, \\
d\omega^1 &= -\omega^4 \wedge \sigma^t, \\
d\omega^2 &= -\omega^3 \wedge \sigma^t - \tfrac{2}{3}U\omega^4 \wedge \sigma^t - \tfrac{1}{3}\omega^4 \wedge \sigma^x, \\
d\omega^3 &= \tfrac{2}{3}\omega^4 \wedge \sigma^u, \\
d\omega^4 &= 0,
\end{aligned}$$

where $\sigma^t, \sigma^x, \sigma^u$ are the invariant horizontal forms. The Maurer–Cartan equations for the Lie symmetry pseudo-group $\mathcal{G}$ are obtained by restricting to a target fiber where T, X, U are fixed, whence

$$\begin{aligned}
d\omega^1 &= -\omega^1 \wedge \omega^4, \\
d\omega^2 &= -\omega^1 \wedge \omega^3 - \tfrac{2}{3}U\omega^1 \wedge \omega^4 - \tfrac{1}{3}\omega^2 \wedge \omega^4, \\
d\omega^3 &= \tfrac{2}{3}\omega^3 \wedge \omega^4, \\
d\omega^4 &= 0.
\end{aligned}$$

5. Discussion

An efficient method for finding the local structure of Lie symmetry pseudo-groups of differential equations was explained, and was demonstrated for the particular cases of the KP and KdV equations. The algorithm can be straightforwardly applied to any system of differential equations, irrespective of whether its symmetry group is finite-dimensional or infinite-dimensional. To apply our method to other more complicated differential equations, we should optimize our computational procedure and develop more efficient symbolic algebra routines.

The next stage of applications of our method is to develop the moving frame algorithms for pseudo-group actions on submanifolds, [24], which will construct complete systems of differential invariants and invariant differential forms, classify their syzygies and recurrence relations, analyze invariant variational principles, [10, 11], and solve equivalence and symmetry problems arising in geometry and physics.

6. Acknowledgements

The first author would like to thank Evelyne Hubert and Elizabeth Mansfield for their helpful advice received in his effort to understand the computational aspect of the project. The research of the first and second authors was supported in part by NSF Grant DMS 01–03944.

References

[1] Ablowitz, M.J. and Clarkson, P.A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991

[2] Anderson, I.M., *The Variational Bicomplex*, Utah State Technical Report, 1989

[3] Cartan, É., La structure des groupes infinis, *Oeuvres Complètes*, part. II. **2**, Gauthier–Villars, Paris, (1953) 1335–1384

[4] Cheh, J., Mathematica routines for Maurer–Cartan equations, to be available

[5] David, D., Kamran, N., Levi, D. and Winternitz, P., Subalgebras of loop algebras and symmetries of the KP equation, Phys. Rev. Lett. **55** (1985) 2111–2114

[6] David, D., Kamran, N., Levi, D. and Winternitz, P., Symmetry reduction for the KP equation using a loop algebra, J. Math. Phys. **27** (1986) 1579–1591

[7] David, D., Levi, D. and Winternitz, P., Equations invariant under the symmetry group of the Kadomtsev–Petviashvili equation, Phys. Lett. A **129** (1988) 161–164

[8] Fels, M. and Olver, P.J., Moving coframes. II. Regularization and theoretical foundations, Acta Appl. Math. **55** (1999) 127–208

[9] Kamran, N. and Robart, T., A manifold structure for analytic isotropy Lie pseudo-groups of infinite type, J. Lie Theory 11 (2001) 57–80

[10] Kogan, I.A., and Olver, P.J., Invariant Euler–Lagrange equations and the invariant variational bicomplex, Acta Appl. Math. **76** (2003) 137–193

[11] Kogan, I.A. and Olver, P.J., The invariant variational bicomplex, Contemp. Math. **285** (2001) 131–144

[12] Kumpera, A., Invariants différentiels d'un pseudo-groupe de Lie, J. Diff. Geom. **10** (1975) 289–416

[13] Kuranishi, M., On the local theory of continuous infinite pseudo groups I, Nagoya Math. J. **15** (1959) 225–260

[14] Kuranishi, M., On the local theory of continuous infinite pseudo groups II, Nagoya Math. J. **19** (1961) 55–91

[15] Lisle, I.G. and Reid, G.J., Cartan structure of infinite Lie pseudo-groups, in: *Geometrical Approaches to Differential Equations*, P.J. Vassiliou and I.G. Lisle, eds., 2000, pp. 116–145

[16] Lisle, I.G. and Reid, G.J., Geometry and structure of Lie pseudogroups from infinitesimal defining systems, J. Symb. Comp. **26** (1998) 355–379

[17] Lisle, I.G., Reid, G.J. and Boulton, A., Algorithmic determination of the structure of infinite symmetry groups of differential equations, in Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation, 1995

[18] Lou, S., Symmetries of the Kadomtsev–Petviashvili equation, J. Phys. A: Math. Gen. **26** (1993) 4387–4394

[19] Lou, S. and Hu, X., Infinitely many Lax pairs and symmetry constraints of the KP equation, J. Math. Phys. **38** (1997) 6401–6427

[20] Lou, S.Y. and Tang, X., Equations of arbitrary order invariant under the Kadomtsev–Petviashvili symmetry group, J. Math. Phys. **45** (2004) 1020–1030

[21] Morozov, O.I., Symmetries of differential equations and Cartan's equivalence method, Proceedings of Institute of Mathematics of NAS of Ukraine **50** (2004) 196–203
[22] Olver, P.J., *Applications of Lie Groups to Differential Equations*, second edition, Springer Verlag, 1993
[23] Olver, P.J. and Pohjanpelto, J., Moving frames for pseudo-groups. I. The Maurer–Cartan forms, preprint, University of Minnesota, 2003
[24] Olver, P.J. and Pohjanpelto, J., Moving frames for pseudo-groups. II. Differential invariants for submanifolds, preprint, University of Minnesota, 2003
[25] Pommaret, J.F., *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach Science Publishers, 1978
[26] Reid, G.J., Lisle, I.G., Boulton, A. and Wittkopf, A.D., Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs, in: *Proceedings of the 1992 International Symposium on Symbolic and Algebraic Computation*, Academic Press, 1992, 63–68
[27] Reid, G.J., Finding abstract Lie symmetry algebras of differential equations without integrating determining equations, Euro. J. Appl. Math. **2** (1991) 319–340
[28] Singer, I.M., Sternberg, S., The infinite groups of Lie and Cartan. Part I (the transitive groups), J. Analyse Math. **15** (1965) 1–114

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA