Geometry of Submanifolds and Its Applications

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CONTENTS

PREFACE

The present volume contains written versions of a series of talks on geometry of submanifolds and its applications delivered at Science University of Tokyo from January 16 to February 6, 1981. The material of these talks mainly bases on some of author's works and joint works done during the last few years.

Chapter I gives a brief review on the general theory of submanifolds for later use. However, results given in §3 are new. Chapter II modifies a recent paper of the author on sufaces. In Chaper III, a survey of recent results on total mean curvature is given. In Chapter IV, the theory of generic submanifolds is introduced. Some of its applications are given in this volume. In Chapter V, a series of author's papers on CR-submanifolds is presented in a simplified version. Chapter VI serves a brief expositary of (M_+, M_-) -method which was introduced by T. Nagano and the author in 1978. Indication of some of its applications is given. In the last chapter, some main results on totally umbilical submanifolds are summarized. Due to limitation of pages, proofs are given only to shorter ones or those appeared in less accessible papers.

The author would like to take this opportunity to express his hearty thanks to his teachers, professors Nagano and Otsuki, for their constant encouragement and guidance. He also like to express his many thanks to Professors Shibata and Yamaguchi and other colleagues at Science University of Tokyo for the valuable discussions and their hospitality while the author was a visiting professor there. Moreover, the author would like to express his thanks to professors at Tokyo Metropolitan University, University of Tokyo, Kūshyū University, Nagoya University, Osaka University, Tsukuba University, Tokyo Institute of Technology and Ochanomizu University in Japan and Soochow University and Tsinghua University in Taiwan for their kind invitations to visit and to give talks at their universities while the author was a visiting professor at Sicence University of Tokyo. During those visits the author learn much from them. In particular, the author is indebted to Prefessor Ogiue for his help which resulted in improvements of the presentation. Finally, the author wishes to thank miss Ikuko Fukui who typed the manuscript, for her patience and cooperation.

Bang-yen Chen
in Tokyo, February 20, 1981.

Chapter I: SUBMANIFOLDS

In this monograph manifolds and submanifolds are assumed to be connected and differentiable of class C^{∞} unless mentioned otherwise. In §§§1,2, and 4 we will recall some fundamental results and formulas for later use. (see, e.g. Chen (1973a) and Kobayashi and Nomizu (1963).) In §3, we will study submanifolds with planar normal sections. Applications to submanifolds with either parallel second fundamental form or planar geodesics are given.

§1. RIEMANNIAN MANIFOLDS

Let M be a Riemannian manifold with metric tensor g and Riemannian connection ∇. We have $\nabla g = 0$. For any vector fields X, Y, and Z tangent to M, the *curvature tensor* R of M is given by

$$(1.1)\qquad R(X,Y)Z = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

If $E_1,\cdots,E_n$ are local orthonormal vector fields on M, then

$$(1.2)\qquad S(X,Y) = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i)$$

defines a symmetric tensor field S of type $(0,2)$, called the *Ricci tensor of M*. Using the Ricci tensor S, the *scalar curvature* r is defined by

$$(1.3)\qquad r = \sum_{i=1}^{n} S(E_i,E_i).$$

For the curvature tensor R we have the following identities:

$$(1.4)\qquad R(X,Y) + R(Y,X) = 0,$$

$$(1.5)\qquad R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \quad \text{and}$$

$$(1.6)\qquad R(X,Y;Z,W) = R(Z,W:X,Y),$$

where $R(X,Y;Z,W) = \langle R(X,Y)Z,W\rangle$ and $\langle X,Y\rangle = g(X,Y)$.

Let X and Y be two linearly independent vectors at a point. The *sectional curvature* $K(X,Y)$ for the plane section spanned by X and Y is defined by

$$(1.7)\qquad K(X,Y) = \frac{R(X,Y;Y,X)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$

If the sectional curvature is equal to a constant c for all plane sections, M is called a *space of constant curvature* or a *real-space-form*. In this case we have

(1.8) $$R(X,Y;Z,W) = c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\}.$$

A Riemannian manifold M is said to be *Einsteinian* if the Ricci tensor S satisfies $S = \lambda g$ for some function λ on manifold M. If dim $M > 2$, λ is a constant. A Riemannian manifold M is called a *locally symmetric space* if its curvature tensor is covariant constant, i.e., $\nabla R = 0$. For a Riemannian manifold, we define the *conformal curvature tensor* C by

(1.9) $$C(X,Y;Z,W) = R(X,Y;Z,W) - \frac{1}{n-2}\{S(X,W)\langle Y,Z\rangle + S(Y,Z)\langle X,W\rangle - S(X,Z)\langle Y,W\rangle - S(Y,W)\langle X,Z\rangle\} + \frac{r}{(n-1)(n-2)}\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\}.$$

If dim $M \leqq 3$, $C \equiv 0$. And if dim $M \geqq 4$, M is *conformally flat* if and only if $C \equiv 0$.

In this book, by a *closed manifold* we mean a compact manifold without boundary.

§2. SUBMANIFOLDS

Let M be an n-dimensional manifold immersed in an m-dimensional Riemannian manifold $\tilde{M}$. We denote by g the metric tensor of $\tilde{M}$ as well as that induced on M. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on M and M, respectively. Then the Gauss and Weigarten formulas are given respectively by

(2.1) $$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

(2.2) $$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

respectively, where X and Y are vector fields tangent to M and ξ normal to M. Moreover, h is the *second fundamental form*, D the linear connection induced in the normal bundle $T^\perp M$, called the *normal connection*, and A_ξ the *second fundamental tensor* at ξ. From (2.1) and (2.2) we have

(2.3) $$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle.$$

We denote by R, $\tilde{R}$ and R^D the curvature tensors associated with ∇, $\tilde{\nabla}$ and D, respectively. For the second fundamental from h, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $(TM) \oplus (T^\perp M)$ by

$$(2.4)\qquad (\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. A submanifold is said to have *parallel second fundamental form* if $\bar{\nabla}h \equiv 0$. A geometric interpretation for such submanifolds of a Euclidean m-space E^m will be given in the next section.

The equations of Gauss, Codazzi, and Ricci are given respectively by

$$(2.5)\qquad R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle,$$

$$(2.6)\qquad (\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

$$(2.7)\qquad \tilde{R}(X,Y;\xi,\eta) = R^D(X,Y;\xi,\eta) - \langle [A_\xi, A_\eta]X, Y\rangle$$

for vectors X, Y, Z, W tangent to M, ξ, η normal to M, and $\perp$ in (2.6) denotes the normal component.

For an n-dimensional submanifold M in $\tilde{M}$. The *mean curvature vector* H is given by

$$(2.8)\qquad H = \frac{1}{n}\,\text{trace } h.$$

A submanifold M is said to be *minimal* (respectively, *totally geodesic*) if $H \equiv 0$ (respectively, $h \equiv 0$).

If we have

$$(2.9)\qquad h(X,Y) = \langle X,Y\rangle H,$$

for X, Y in TM, M is said to be *totally umbilical*.

A vector subbundle μ of $T^{\perp}M$ is said to be *parallel* if $D_X\xi \in \mu$ for any X in TM and any vector field ξ in μ.

§3. SUBMANIFOLDS WITH PLANAR NORMAL SECTIONS

Let M be an n-dimensional submanifold in a Euclidean m-space E^m. For any point p in M and any unit vector t at p tangent to M, the vector t and the normal space $T_p^{\perp}M$ of M at p determine an $(m-n+1)$-dimensional subspace $E(p,t)$ in E^m. The intersection of M and $E(p,t)$ gives a curve γ (in a neighborhood of p) which is called the *normal section* of M at p in the direction t. In general, the normal section γ is a (twisted) space curve in $E(p,t)$.

In this section we shall give a necessary and sufficient condition for a submanifold of E^m to have planar normal sections. Using this result we will obtain a geometric interpretation for submanifolds with parallel second fundament-at form. Moreover, we shall find a relation between submanifolds with planar normal sections and submanifolds with planar geodesics.

First we give the following

THEOREM 3.1. (Chen 1980a). *Let M be an n-dimensional ($n \geqq 2$) submanifold of E^m. Then M has planar normal sections if and only if h and $\overline{\nabla}h$ satisfy*

$$h(t,t) \wedge (\overline{\nabla}_t h)(t,t) = 0, \tag{3.1}$$

for any unit vector t tangent to M.

PROOF. Let M be an n-dimensional submanifold of E^m and $\gamma(s)$ any normal section through $p \in M$ with s as its arc length. We denote by T the unit tangent vector field along $\gamma(s)$ given by $T = \dot{\gamma}(s)$, $\dot{\gamma}(s) = \frac{d\gamma}{ds}$. Assume that $\gamma(0) = p$.

We choose a local field of orthonormal frame $e_1, \cdots, e_m$ in E^m in such a way that, restricted to M, the vector fields $e_1, \cdots, e_n$ are tangent to M, and hence, $e_{n+1}, \cdots, e_m$ are normal to M. And moreover, $e_1 = T$ along $\gamma(s)$. We denote by $\omega^1, \cdots, \omega^m$ the dual frame of $e_1, \cdots, e_m$. There exist m^2 1-forms ω_A^B; $A, B = 1, \cdots, m$, such that

$$\tilde{\nabla} e_A = \sum \omega_A^B e_B, \qquad \omega_A^B + \omega_B^A = 0.$$

We put

$$h_{ij}^r = \langle h(e_i, e_j), e_r \rangle, \qquad i,j = 1, \cdots, n; r = n+1, \cdots, m.$$

Because $T = e_1$ along $\gamma(s)$, we have

$$\ddot{\gamma}(s) = \frac{d^2\gamma}{ds^2} = \tilde{\nabla}_T T = \sum_{i=2}^{n} \omega_1^i(T) e_i + h(T,T), \tag{3.2}$$

$$\dddot{\gamma}(s) = \frac{d^3\gamma}{ds^3} = \tilde{\nabla}_T(\tilde{\nabla}_T T) = \sum_{i=2}^{n} T(\omega_1^i(T)) e_i + \sum_{i,j=1}^{n} \omega_1^i(T)\omega_i^j(T) e_j \tag{3.3}$$

$$+ \sum_{i=2}^{n} \omega_1^i(T) h(e_i, T) - \sum_{i,r} h_{11}^r h_{1i}^r e_i + D_T(h(T,T)).$$

At $\gamma(0) = p$, $\ddot{\gamma}(0)$ and $\dddot{\gamma}(0)$ lie in the $(m-n+1)$-space $E(p,t)$. Since $E(p,t)$ is spanned by t and $T_p^{\perp}M$, (3.2) and (3.3) give

$$\omega_1^i(t) = 0 \tag{3.4}$$

$$\ddot{\gamma}(0) = h(t,t) \tag{3.5}$$

$$\dddot{\gamma}(0) = -\|h(t,t)\|^2 t + (D_T(h(T,T))_{(0)}. \tag{3.6}$$

Combining (3.4), (3.5), (3.6), and the definition of $\overline{\nabla}h$, we have

$$\dddot{\gamma}(0) = -\kappa(0)^2 t + (\overline{\nabla}_t h)(t,t), \tag{3.7}$$

where $\kappa(0) = \|h(t,t)\|$ is the curvature of $\gamma(s)$ at $p=\gamma(0)$.

If the normal section $\gamma(s)$ is a planar curve, then $\dddot{\gamma}(0)$ is a linear combination of $\dot{\gamma}(0)$ and $\ddot{\gamma}(0)$. Hence we have (3.1), i.e., $h(t,t)$ is parallel to $(\bar{\nabla}_t h)(t,t)$.

Conversely, if (3.1) holds for any vector at any point p in M tangent to M, then along the normal section $\gamma(s)$, the first normal space spanned by $h(T,T)$ is either zero or parallel along $\gamma(s)$ with respect to the normal connection D. Thus, $\gamma(s)$ is a planar curve. This completes the proof of the theorem.

As an application of Theorem 3.1 we give the following simple geometric interpretation for submanifolds with parallel second fundamental form.

THEOREM 3.2 (Chen 1980a). *Let M be an n-dimensional $(n \geq 2)$ submanifold of E^m. Then the following three statements are equivalent.*

(a) *The second fundamental form h satisfies $(\bar{\nabla}_t h)(t,t) \equiv 0$.*

(b) *The second fundamental form h is parallel, i.e., $\bar{\nabla} h \equiv 0$.*

(c) *Normal section of M at any point $p \in M$ is a planar curve with p as one of its vertices.*

By a *vertex* of a planar curve we mean a point x on the curve such that $\frac{d\kappa^2}{ds} = 0$ at x.

PROOF. (a)$\Longrightarrow$(b). Since M in a submanifold of E^m, we have

$$(3.8) \qquad (\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z)$$

for any vectors X,Y,Z tangent to M.

Hence by putting $t = X+Y$ and $t = X-Y$, respectively, into $(\bar{\nabla}_t h)(t,t) = 0$, we obtain $(\bar{\nabla}_X h)(X,Y) = 0$. Thus, by linearity and (3.8), we may get $\bar{\nabla} h = 0$.

(b)$\Longrightarrow$(a). Trivial.

(b)$\Longrightarrow$(c). If $\bar{\nabla} h \equiv 0$, (3.1) holds automatically. Thus, by Theorem 3.1, the normal section $\gamma(s)$ of M at p in any given direction t is planar. From (3.2) the curvature $\kappa(s)$ of $\gamma(s)$ satisfies

$$(3.9) \qquad \kappa^2(s) = \|\ddot{\gamma}(s)\|^2 = \sum_{i=2}^{n} (\omega_1^i(T))^2 + \|h(T,T)\|^2$$

where $T = \dot{\gamma}(s)$. Thus

$$(3.10) \qquad \frac{d\kappa^2(s)}{ds} = 2\sum_{i=2}^{n} \omega_1^i(T)\, T(\omega_1^i(T)) + 2\langle D_T(h(T,T)), h(T,T)\rangle .$$

Hence, by (3.4), we obtain

$$\frac{d\kappa^2}{ds}(0) = 2\langle D_t(h(T,T)),\ h(t,t)\rangle$$

$$= 2\langle (\bar{\nabla}_t h)(t,t),\ h(t,t)\rangle$$

$$= 0$$

at $p = \gamma(0)$, that is, p is a vertex of the planar normal section $\gamma(s)$.

(c)$\Rightarrow$(a). If normal sections $\gamma(s)$ of M at any point $p \in M$ is a planar curve with p as one of its vertices, then, by Theorem 3.1, we have

$$(3.11)\qquad (\bar{\nabla}_t h)(t,t) = \lambda(t)h(t,t) \qquad \text{and}$$

$$(3.12)\qquad \frac{d\kappa^2}{ds}(0) = 0,$$

Therefore, by (3.4), (3.10), and (3.12), we have

$$\langle (\bar{\nabla}_t h)(t,t),\ h(t,t)\rangle = 0 \qquad .$$

from which, together with (3.11), we obtain $(\bar{\nabla}_t h)(t,t) = 0$. This completes the proof of the theorem.

REMARK 3.1. For a submanifold in a real-space-form, statements (a) and (b) of Theorem 3.2 are equivalent. However, these two statements are different if the ambient space is an arbitrary Riemannian manifold. This fact can be easily seen by comparing Theorems 3.3 and 3.4.

REMARK 3.2. For a submanifold M in a Riemannian manifold $\tilde{M}$, statement (a) is equivalent to the following statement (d). Every geodesic in M, regarded as a curve in $\tilde{M}$, has parallel κN, where N is the principal normal.

Given a point p in a Riemannian manifold $\tilde{M}$ and a small positive number ε, the *geodesic sphere* $G_\varepsilon(p)$ centered at p with radius ε is the hypersurface given by the image of the hypersphere $S_\varepsilon(p)$ under exp_p, where $S_\varepsilon(p)$ denotes the hypersphere of $T_p\tilde{M}$ centered at p and with radius ε.

THEOREM 3.3 (Chen and Vanhecke, 1978;1980). *For any n-dimensional* $(n \geqq 2)$ *Riemannian manifold* $\tilde{M}$, *the following three statements are equivalent.*

(1) *Locally, M is a Euclidean space or a rank one symmetric space,*

(2) *Sufficiently small geodesic spheres of M satisfy* $(\bar{\nabla}_t h)(t,t) \equiv 0$.

(3) *Every geodesic of a sufficiently small geodesic sphere of* $\tilde{M}$ *has constant curvature* κ *in* $\tilde{M}$.

THEOREM 3.4 (Chen and Vanhecke, 1978; 1980). *A Riemannian manifold $\tilde{M}$ of dimension ≥ 2 is a real-space-form if and only if sufficiently small geodesic spheres of $\tilde{M}$ have parallel second fundamental form.*

For the proof of these two theorems and further results on geodesic spheres, see Chen and Vanhecke (1978) and literatures mentioned in the bibliography of that paper.

Now we shall characterize submanifolds with planar geodesics in terms of planar normal sections.

THEOREM 3.5 (Chen 1980a). *An n-dimensional ($n \geq 2$) submanifold M of E^m has planar geodesics if and only if it has planar normal sections of the same constant curvature, i.e., they are either portions of straight lines or circles of the same radius.*

PROOF. Let $\gamma(s)$ be the normal section at $p \in M$ in the direction of a unit tangent vector t. By (3.2) and (3.5), we have

$$\kappa^2(s) = \sum_{i=2}^{n} (\omega_1^i(T))^2 + \|h(T,T)\|^2 \tag{3.13}$$

$$\kappa^2(0) = \|h(T,T)\|^2, \quad \omega_1^i(t) = 0. \tag{3.14}$$

If M has planar normal sections of the same constant curvature, then (3.13) and (3.14) give $\omega_1^i(T) = 0$ for $i=2,\cdots,n$. This shows that the normal section $\gamma(s)$ is in fact the geodesic in M through p with t as its initial velocity vector. Since M has planar normal sections, M has planar geodesics.

Conversely, if M has planar geodesics, then, by a result of Hong (1973), M is either contained in an n-plane or else all the geodesics are planar circles of the same radius. If the first case holds, every normal section is a portion of a straight line. If the second case holds, the curvature of every geodesic $\alpha(s)$ of M is constant. We put $X = \dot{\alpha}(s)$. We have $\nabla_X X = 0$. Therefore we find $\ddot{\alpha}(s) = \tilde{\nabla}_X X = h(X,X) = \kappa(s)N$, where $\kappa = \|h(X,X)\|$ is the curvature of $\alpha(s)$ and N its principal normal. Since $\alpha(s)$ is planar, $\tilde{\nabla}_X N = -\kappa X$. Thus $D_X N = 0$. Combining this with the constancy of κ, we get

$$(\bar{\nabla}_X h)(X,X) = D_X(h(X,X)) = 0.$$

Because this is true for every geodesic in M, we have $\bar{\nabla} h \equiv 0$. Theorem 3.2 implies that M has planar sections. Thus, after observing that the planar normal section given by $E(p,t)$ and the planar geodesic throngh p with t as its initial velocity vector are in fact the same curve, we conclude that normal sections of M are planar curves of the same curvature. This proves the theorem.

REMARK 3.3. For submanifolds in E^m, we put

$A = \{$submanifolds with planar normal sections$\}$,

$B = \{$submanifolds with $\bar{\nabla} h \equiv 0\}$,

$C = \{$submanifolds with planar geodesics$\}$,

then we have $A \supsetneq B \supsetneq C$.

Classification of submanifolds of class B (respectively, class C) has been obtained by Ferus (1974) (see also Takeuchi (1981)) (respectively, Hong (1973) and Little (1976)). It is obvious that class A contains all submanifolds of E^m of codimension one. In views of these, I would like to propose the following.

PROBLEM 3.1. Classify submanifolds of class A.

PROBLEM 3.2. Find topological conditions for higher dimensional closed submanifolds of E^m to have planar normal sections.

§4. THE FIRST AND SECOND VARIATIONS OF VOLUME

In this section, we shall give the first and second variation formulas of volume for later use.

Let $f: M \longrightarrow \tilde{M}$ be an immersion from a compact n-dimensional manifold into an m-dimensional Riemannian manifold $\tilde{M}$. Let $\{f_t\}$ be a 1-parameter family of immersions of $M \longrightarrow \tilde{M}$ with the property that $f_0 = f$ and that the map $F: M \times [0,1] \rightarrow \tilde{M}$ defined by $F(p,t) = f_t(p)$, be differentiable. Then $\{f_t\}$ is called a *variation* of f. If $\{f_t\}$ is a variation of f, it induces a vector field in $\tilde{M}$ defined along the image of M under f. We shall denote this field by η and it is constructed as follows. Let $\partial/\partial t$ be the standard vector field in $M \times [0,1]$. We set

$$\eta(p) = F_*(\frac{\partial}{\partial t}(p,0)),$$

then η gives rise to cross-sections η^T and η^N in TM and $T^{\perp}M$, respectively. If we have $\eta^T = 0$, $\{f_t\}$ in called a *normal variation* of f. For a given normal vector field u on M, $exp\, t\, u$ defines a normal variation $\{f_t\}$ induced from u. We denote by $\mathcal{V}(t)$ the volume of M under f_t with respect to the induced metric and by $\mathcal{V}'(u)$ and $\mathcal{V}''(u)$ the values $\frac{d}{dt}\mathcal{V}(t)$ and $\frac{d^2}{dt^2}\mathcal{V}(t)$ at $t = 0$ for the normal variation induced from u. The following formula is well-known (see, for example, p. 75 of Chen (1973a))

$$(4.1)\qquad \mathcal{V}'(u) = -n\int_M \langle u,H\rangle dV$$

where dV denotes the volume element of $f(M)$.

Let u be any normal vector field of $f(M)$ and $e_1,\cdots,e_n$ any orthonormal frame in TM. We put

$$(4.2)\qquad \overline{S}(u) = \sum_{i=1}^{n} \tilde{R}(u,e_i;e_i,u),$$

then (4.2) is well-defined. The second variation formula is then given by the following. (see, for example, Simons (1968)).

THEOREM 4.1. *Let* $\{f_t\}$ *be the normal variation induced from a normal vector field* u *of* $f:M\longrightarrow\tilde{M}$ *such that* $f_t|_{\partial M} = f|_{\partial M}$. *If* f *is minimal, we have*

$$(4.3)\qquad \mathcal{V}''(u) = \int_M \{\,\|Du\|^2 - \overline{S}(u) - \|A_u\|^2\}dV\ .$$

A minimal submanifold M of $\tilde{M}$ is said to be *stable* if $\mathcal{V}''(u) \geqq 0$ for any normal vector field u of M in $\tilde{M}$, otherwise, M is said to be *unstable*.

Chapter II: SURFACES WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR

§1. INTRODUCTION

In the classical theory of surfaces in an ordinary space E^3, the two most important curvatures are the Gauss curvature and the mean curvature $|H|$. The Gauss curvature is intrinsic and the mean curvature is extrinsic. For the Gauss curvature, the following result of Liebmann is well-known.

THEOREM 1.1. *A closed surface of constant Gauss curvature in E^3 is an ordinary sphere.*

Concerning mean curvature we have the following

THEOREM 1.2 (Hopf, 1951). *A closed surface of genus zero in E^3 is an ordinary sphere if it has constant mean curvature.*

It was conjectured by Hopf that spheres are the only closed immersed surface of E^3 with constant mean curvature.

For a surface M of E^m, $m > 3$, the mean curvature vector H plays more important rôle than the mean curvature $|H|$. Let ξ be the unit vector field in the direction of H, that is,

$$(1.1) \qquad H = \alpha\xi, \qquad \alpha = |H|,$$

then ξ is called the *normalized mean curvature vector*. It is obvious that a surface M has parallel mean curvature vector, i.e., $DH \equiv 0$, if and only if either M is minimal or the mean curvature is a nonzero constant and the normalized mean curvature vector ξ is parallel. In Ruh and Vilms (1970), we have

THEOREM 1.3 (Ruh and Vilms 1970). *A submanifold M of E^m has parallel mean curvature vector if and only if the Gauss map of M is harmonic in the sense of Eells and Sampson* (1965).

In views of these, it is interesting and natural to classify submanifolds with parallel mean curvature vector. For surfaces, this is done by the following.

THEOREM 1.4 (Chen 1972a and Yau 1974). *A surface M of E^m has parallel* *mean curvature vector if and only if M is one of the following surfaces*;

(a) *a minimal surface of* E^m,

(b) *a minimal surface of a hypersphere of* E^m,

(c) *a surface of* E^3 *with constant mean curvature*,

(d) *a surface of a 3-sphere in* E^4 *with constant mean curvature*.

Since the condition of parallel normalized mean curvature vector is much weaker than the condition of parallel mean curvature vector, it is natural to study surfaces satisfying the first condition and to find its relation with the second condition. The main purpose of this chapter is to deal with this problem.

§2. EXAMPLES

In this section, we give examples of surface with parallel normalized mean curvature vector.

EXAMPLE 2.1. Any minimal surface of E^{m-1} of E^m has parallel normalized mean curvature vector in E^m. This is simply due to the fact that the restriction of the hyperplane unit normal to the surface is a parallel normalized mean curvature vector.

EXAMPLE 2.2. Any surface of E^3 has parallel normalized mean curvature vector because the unit surface normal is always parallel.

EXAMPLE 2.3. Any minimal surface of a hypersphere of E^m has parallel normalized mean curvature vector. This follows immediately from Theorem 1.4.

EXAMPLE 2.4. D. S. P. Leung (1980) proved that there are many analytic surfaces of E^4 with parallel normalized mean curvature vector. Moreover, those surfaces do not lie in any hyperplane or hypersphere of E^4.

§3. SURFACES WITH PARALLEL NORMALIZED MEAN CURVATURE VECTOR

For simplicity we shall assume in this section that surfaces are of class C^ω. We give the following results of Chen (1980b).

THEOREM 3.1. *Let M be a surface of E^m with parallel normalized mean curvature vector. Then M is one of the following surfaces*;

(a) *a minimal surface of a hyperplane* E^{m-1} *of* E^m,

(b) *a minimal surface of a hypersphere of* E^m,

(c) *a surface of an affine* 4*-space* E^4 *of* E^m.

If M is minimal in E^m, then any unit normal vector field is a normalized mean curvature vector. In this case, the theorem follows from the following.

LEMMA 3.2. *A minimal surface* M *of* E^m *admits a parallel unit normal vector field if and only if either* $m=3$ *or* $m>3$ *and* M *lies in hyperplane of* E^m.

Let M be a minimal surface of E^m. If ξ is a parallel unit normal vector field, then we may choose m-2 orthonormal normal frame $\xi_3,\cdots,\xi_m$ such that $\xi_3=\xi$. Since $D\xi_3=0$, the Ricci equation implies $[A_3,A_r]=0$ for $r=4,\cdots,m$, where $A_r=A_{\xi_r}$. Since trace $A_r=0$, either $A_3=0$ or $R^D=0$ at any point $p\in M$. We put $U=\{p\in M|A_3=0 \text{ at } p\}$. Then we have $R^D\equiv 0$, on the closure of M-U. Now, let (x^1,x^2) be an isothermal coordinate system in M, we put $X_i=\partial/\partial x^i$ and

$$L=h(X_1,X_1),\quad M=h(X_1,X_2),\quad N=h(X_2,X_2).$$

Then since ξ_3 is parallel and trace $A_3=0$, Lemma 2.2 of p. 103 of Chen (1973a) shows that the following function

$$\phi=\left\langle\frac{L-N}{2},\ \xi_3\right\rangle-\langle M,\xi_3\rangle i\ ,\qquad i=\sqrt{-1}\ ,$$

is analytic in $z=x^1+ix^2$. Thus either $\phi\equiv 0$ or ϕ has only isolated zeros. In the first case, we have $A_3\equiv 0$. Since ξ_3 is parallel, M lies in a hyperplane of E^m with ξ_3 as the hyperplane normal. If ϕ has only isolated zero points, the normal curvature tensor R^D vanishes identically. Thus Lemma 3.2 follows immediately from the following (Chen (1973a), p. 115).

LEMMA 3.3. *If a minimal surface* M *of* E^m *has flat normal connection, then* M *is contained in an affine* 3*-space* E^3 *of* E^m.

If M is not minimal, then because M is analytic, H vanishes only at isolated points. We choose an orthonormal normal frame $\xi_3,\cdots,\xi_m$ in such a way that ξ_3 is the parallel normalized mean curvature vector. Put $M_1=\{p\in M|A_3=\lambda I \text{ at } p\}$. Then, by using the Ricci and Codazzi equations, we may conclude that $R^D\equiv 0$ on the closure of $M-M_1$ and $\langle H,H\rangle$ is constant on each component of $\operatorname{int}(M_1)$.

Form this we may conclude that either M has constant mean curvature or $R^D\equiv 0$ on M. If the first case occurs, Theorem 1.4 finishes the proof. If the second case holds, we put

$$M_2 = \{p \in M \mid \dim \operatorname{Im} h = 1 \text{ at } p\},$$
$$M_3 = \{p \in M \mid \dim \operatorname{Im} h = 2 \text{ at } p\}.$$

Then $M = M_2 \cup M_3$ and M_3 is an open subset of M. The next step is to show that the closure of each component of $\operatorname{int}(M_2)$ lies in an affine 3-space E^3 of E^m and the closure of each component of M_3 lies in an affine 4-space of E^m. Now, we apply Codazzi's equation and analytic function theory to show that either M_2 is the whole surface M or the closure of M_3 is M. If $M_2 = M$, M lies in an affine 3-space. If closure $(M_3) = M$, then M_2 consists of curves and points only. In this case, by applying analytic function theory and Codazzi equation again, we may prove that the whole surface lies in an affine 4-space. For the details, see Chen (1980b).

By applying Theorem 3.1 we may also prove the following.

THEOREM 3.4. *If a closed surface of genus zero in E^m has parallel normalized mean curvature vector, then either M is a minimal surface of a hypersphere of E^m or M lies in an affine 3-space E^3 of E^m.*

THEOREM 3.5. *If a closed surface M of E^m has parallel normalized mean curvature vector and constant Gauss curvature, then either M is a minimal surface of a hypersphere of E^m or M is the product surface of two plane circles in a E^4 of E^m.*

THEOREM 3.6. *If a flat surface of E^m has parallel normalized mean curvature vector, then M is one of the following surfaces*;

(a) *a flat minimal surface of hypersphere of E^m,*

(b) *an open portion of the product surface of two plane circles or two straight lines, or*

(c) *a flat surface of an affine 3-space E^3 of E^m.*

In views of these results we would like to recall the following.

PROPOSITION 3.7 (Chen 1972c; Yau 1974). *Every minimal surface of E^m is totally geodesic if it has constant Gauss curvature.*

The following problem seems to be interesting.

PROBLEM 3.1. Classify submanifolds of E^m with parallel mean curvature vector or with parallel normalized mean curvature vector.

Chapter III: TOTAL MEAN CURVATURE

§1. ROTATION INDEX, REGULAR HOMOTOPY, AND TOTAL CURVATURE

Let C be closed (smooth) oriented curve in E^2. As a point moves along C, the lines through a fixed point O and parallel to the tangent line of C rotate through an angle $2n\pi$ or rotate n times about O. This integer n is called the *rotation index* of C. It is known that if C is a simple curve, $n = \pm 1$.

Two closed curves are said to be *regularly homotopic* if one can be deformed to the other through a family of closed smooth curves. Because the rotation index is an integer and it varies continuously through the deformation, it must keep constant. Therefore, two closed smooth curves have the same rotation index if they are regularly homotopic. A theorem of Graustein and Whitney says that the converse of this is true. Thus, the only invariant of a regular homotopy class is the rotation index.

Let $(x(s),\, y(s))$ be the Cartesian coordinates of the closed curve in E^2 which is parameterized by its arc length s. Then we have

$$(1.1)\qquad x'' = -\kappa y', \qquad y'' = \kappa x',$$

where κ denotes the curvature of C. Let $\theta(s)$ denote the angle between the tangent line and the x-axis. Then we have

$$(1.2)\qquad d\theta = \frac{x'y'' - x''y'}{x'^2 + y'^2}\, ds = \kappa ds .$$

From this we obtain the following formula;

$$(1.3)\qquad \int \kappa\, ds = 2n\pi, \qquad n = \text{the rotation index.}$$

From (1.3) we conclude the following well-known result.

THEOREM 1.1. *Let C be a closed curve in E^2. Then the total absolute curvature of C in E^2 satisfies*

$$(1.4)\qquad \int |\kappa|\, ds \geqq 2\pi .$$

The equality holds if and only if C is a convex planar curve.

This result was generalized to closed curves in E^3 by Fenchel (1929) and to closed curves in E^m, $m > 3$, by Borsuk (1947). In 1950, Milnor obtained the following.

THEOREM 1.2. *If a closed curve C in E^m satisfies*

$$\int_C |\kappa|\, ds < 4\pi \tag{1.5}$$

then C is unknotted.

In a 3-dimensional space E^3, surfaces have far more important properties than curves. These important properties are usually related either to the Gauss curvature G or to the mean curvature $\alpha = |H|$.

For a closed oriented surface M of E^3, the integral of Gauss curvature gives the following famous Gauss-Bonnet formula;

$$\int_M G\, dV = 2\pi\chi(M) = 4\pi(1-g), \tag{1.6}$$

where χ and g denote the Euler characteristic and genus of M, respectively. On the other hand, by using Morse's theory, Chern and Lashof (1958) proved the following.

$$\int_M |G|\, dV \geqq 4\pi(1+g). \tag{1.7}$$

Let $M_+ = \{p \in M \mid G \geqq 0 \text{ at } p\}$. Then (1.6) and (1.7) give

$$\int_{M_+} G\, dV \geqq 4\pi. \tag{1.8}$$

From the definitions of Gauss and mean curvatures for surfaces in E^3 we have

$$\alpha^2 \geqq G, \tag{1.9}$$

where the equality holds if and only if M is totally umbilical. Thus, by combining (1.8) and (1.9), we obtain

$$\int_M \alpha^2\, dV \geqq 4\pi. \tag{1.10}$$

The equality holds if and only if M is an ordinary 2-sphere in E^3. This result was given in Willmore (1968). If M is a surface in a higher dimensional Euclidean space E^m $(m > 3)$, inequality (1.7) is not true in general. However, by using the notion of *Otsuki frame*, inequality (1.10) was obtained in Chen (1970) for any closed surface in any Euclidean space.

§2. TOTAL MEAN CURVATURE.

According to Nash (1956), every n-dimensional closed Riemannian manifold of class C^k $(3 \leqq k \leqq \infty)$ can be C^k-isometrically imbedded in E^m with $m=\frac{1}{2}n(3n+11)$. It is also well-known that most Riemannian manifolds cannot be isometrically imbedded in E^{n+1} as hypersurfaces. Thus, the theory of submanifolds of higher codimension is far richer than the theory of hypersurfaces, in particutar, than surfaces of E^3.

Concerning the total mean curvature, we have the following general result for any closed submanifolds of E^m.

THEOREM 2.1 (Chen 1971a). *Let M be an n-dimensional closed submanifold of E^m. Then we have*

$$\int_M \alpha^n \, dV \geqq c_n, \tag{2.1}$$

where $\alpha=|H|$ is the mean curvature and c_n the volume of unit n-sphere. The equality of (2.1) *holds if and only if M is imbedded as an ordinary n-sphere in an affine $(n+1)$-space when $n>1$ and as a convex planar curve when $n=1$.*

If $n=1$, this theorem is nothing but the famous Fenchel-Borsuk inequality.

In the 1973 Symposium on Differential Geometry held at Stanford University, the author proposed the following two problems (Chen, 1975).

PROBLEM 2.1. Let (M,g) be a closed Riemannian manifold and $f:M \longrightarrow E^m$ an isometric immersion from M into E^m. What can we say about the total mean curvature $\int \alpha^n dV$ of f and the Riemannian manifold (M,g)?

PROBLEM 2.2. Let M be a closed manifold and $f:M \longrightarrow E^m$ an immersion from M into E^m. What can we say about the total mean curvature of f and the differentiable manifold M (or $f(M)$)? (see also Willmore (1971 b)).

It is the main purpose of this chapter to summarize the recent results about these two problems. Some remarks and conjectures will be given in the last section, (For the older results in this direction, see Chapter VII of Chen (1973a) and Willmore (1971b)).

§3. TOTAL MEAN CURVATURE AND CONFORMAL GEOMETRY

Let $\tilde{M}$ be an m-dimensional Riemannian maniold with metric g and Riemannian connection $\tilde{\nabla}$. Let ρ be a positive fuction on $\tilde{M}$. Then $g^* = \rho^2 g$ defines a new metric on $\tilde{M}$. It is called a *conformal change of metric*. Let $\tilde{\nabla}^*$ denote the

Riemannian connection of g^*. Then we have

$$\tilde{\nabla}^*_X Y - \tilde{\nabla}_X Y = (X \log\rho) Y + (Y \log\rho) X - g(X,Y) U \tag{3.1}$$

where U is given by $g(U, X) = X \log\rho$.

Let M be an n-dimensional submanifold of $\tilde{M}$ and ∇ and ∇^* the covariant differentiations on M induced from g and g^*, respectively. For any vector fields X, Y tangent to M and ξ normal to M, we have ·

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \tag{3.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \qquad \tilde{\nabla}^*_X \xi = -A^*_\xi X + D^*_X \xi. \tag{3.3}$$

Form (3.1) we also have

$$\tilde{\nabla}^*_X \xi - \tilde{\nabla}_X \xi = (X \log\rho)\xi - (\xi \log\rho) X. \tag{3.4}$$

By substituting (3.3) into (3.4) we may obtain

$$D^*_X \xi = D_X \xi + (X \log\rho)\xi. \tag{3.5}$$

Thus if we denote by I the identity transformation, we get

$$D^*_X = D_X + (X \log\rho) I, \tag{3.6}$$

$$\begin{aligned} R^{D^*}(X,Y) = R^D(X,Y) &+ D_X((Y \log\rho) I) + (X \log\rho) D_Y \\ &- D_Y((X \log\rho) I) - (Y \log\rho) D_X - ([X,Y] \log \rho) I. \end{aligned} \tag{3.7}$$

By using the definition of Lie product we find

$$R^{D^*}(X,Y) = R^D(X,Y). \tag{3.8}$$

Consequently, we have the following results of Chen (1974a).

THEOREM 3.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$. Then the normal curvature tensor R^D is a conformal invariant, i.e., it is invariant under conformal change of metric on $\tilde{M}$.*

By substituting (3.2) into (3.1) we may obtain

$$h^*(X,Y) - h(X,Y) = g(X,Y)\, U^N, \tag{3.9}$$

where U^N denotes the normal component of U/M. Hence for any normal vector field ξ of M in $\tilde{M}$ we have

(3.10) $g(A^*_\xi X, Y) = g(A_\xi X, Y) + g(X,Y)g(U^N, \xi)$.

Let $e_1, \cdots, e_n$ be principal orthonormal directions of A_ξ with respect to g. Then $\rho^{-1}e_1, \cdots, \rho^{-1}e_n$ is an orthonormal frame of M which gives the principal directions of A^*_ξ with respect to g^*. If we denote by $k_1(\xi), \cdots, k_n(\xi)$ the prinicipal curvatures of A_ξ and $k^*_1(\xi), \cdots, k^*_n(\xi)$ that of A^*_ξ, then (3.10) gives

(3.11) $k^*_i(\xi) = k_i(\xi) + \lambda_\xi, \quad \lambda_\xi = g(U^N, \xi)$.

Since $A^*_\xi = \rho A^*_{\xi^*}$ and $\xi^* = \rho^{-1}\xi$ is a unit vector with respect to g^*, (3.11) implies

(3.12) $\rho\{k^*_i(\xi^*) - k^*_j(\xi^*)\} = k_i(\xi) - k_j(\xi)$.

Now, let $\xi_{n+1}, \cdots, \xi_m$ be an orthonormal normal frame with respect to g. Then the mean curvature vector H is given by

(3.13) $H = \dfrac{1}{n} \sum_r (\sum_i k_i(\xi_r))\xi_r$.

Moreover the following quantity G^e is well-defined.

(3.14) $G^e = \dfrac{2}{n(n-1)} \sum_r \sum_{i<j} k_i(\xi_r)k_j(\xi_r)$.

We call G^e the *extrinsic scalar curvature* (with respect to g). From (3.12), (3.13) and (3.14) we obtain the following.

THEOREM 3.2 (Chen 1974a). *If M is a submanifold of a Riemannian manifold $\tilde{M}$, then $(\alpha^2 - G^e)g$ is a conformal invariant, i.e., $(\alpha^2 - G^e)g$ is invariant under conformal changes of metric.*

In particular, Theorem 3.2 implies immediately that $(\alpha^2 - G^e)^{n/2}dV$ *is a conformal invariant*, where $n = \dim_R M$. For surfaces in E^m, the extrinsic scalar curvature G^e is nothing but the Gauss curvature. For such surfaces we also have the following (see, also White (1973) for the case $m = 3$).

THEOREM 3.3 (Chen 1973b). *Let $\phi(M)$ be the closed surface obtained from a closed surface M of E^m under a conformal mapping ϕ from E^m into E^m. Then*

(3.15) $\displaystyle\int_M \alpha^2 dV = \int_{\phi(M)} \alpha^2_\phi dV_\phi$

where α_ϕ and dV_ϕ denote the mean curvature and area element of $\phi(M)$, respectively.

§4. TOTAL MEAN CURVATURE, ORDER OF IMMERSION, AND SPECTRAL GEOMETRY

Let M be a closed Riemannian manifold and Δ the Laplace-Beltrami operator acting on differentiable functions in $C^{\infty}(M)$. It is well-known that Δ is an elliptic operator. The operator Δ has an infinite sequence

$$(4.1)\qquad 0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_p<\cdots\uparrow\infty$$

of eigenvalues. Let $V_i=\{f\in C^{\infty}(M)\,|\,\Delta f=\lambda_i f\}$ be the eigenspace with eigenvalue λ_i. Then the dimension of each V_i is finite. It is called the multiplicity of λ_i. The set of eigenvalues of Δ enumerated with multiplicity, denoted by spec(M), is called the *spectrum* of M.

If we define an inner product on $C^{\infty}(M)$ by

$$(4.2)\qquad \langle f,g\rangle=\int_M fg\,dV,$$

for $f,g\in C^{\infty}(M)$. Then the decomposition $\sum V_i$ is orthogonal with respect to this structure, moreover, $\sum V_i$ is dense in $C^{\infty}(M)$. Since M is closed, V_0 is 1-dimensional and it consists of constant functions. (For general results on spectrum, see Berger, Gauduchon and Mazet (1971)).

For each function $f\in C^{\infty}(M)$, let f_i be the projection of f onto the subspace V_i $(i=0,1,2,\cdots)$. We say that a function $f\in C^{\infty}(M)$ is of order p (respectively, of order $\geqq p$) if $f\in V_p$ (respectively, if $f_i=0$; $i=0,1,\cdots,p-1$). It is clear that the zero function is of order p for each p.

For an isometric immersion $x: M\longrightarrow E^m$. We put

$$x=(x_1,\cdots,x_m)$$

where x_i is the i-th coordinate function of M in E^m. Following Chen (1979a), we call an isometric immersion x is of *ordr* p (respectively, of *order* $\geqq p$) if each coordinate function x_i of x is of order p (respectively, of order $\geqq p$).

In the following theorem we will show that the total mean curvature of an isometric immersion is closely related with the *order of immersion* as well as its spectrum. The results of this section are given in Chen (1979 a,b)

THEOREM 4.1. *Let (M,g) be an n-dimensional $(n\geqq 2)$ closed Riemannian manifold. If $x: M\longrightarrow E^m$ is an isometric immersion of order $\geqq p$ $(p\geqq 1)$ from M into E^m, then the total mean curvature of M satisfies*

$$(4.3)\qquad \int_M \alpha^n\,dV\geqq\left(\frac{\lambda_p}{n}\right)^{\frac{n}{2}}v(M),$$

where $v(M)$ denotes the volume of (M,g). The equality holds if and only if x is an imbedding of order p.

PROOF. Suppose that $x : M \longrightarrow E^m$ is of order $\geqq p (\geqq 1)$. Then each coordinate function x_i is of order $\geqq p$. Since $(x_i)_t$ is the component of x_i in the eigenspace V_t, the inner product $<\ ,\ >$ on the pre-Hilbert space $C^\infty(M)$ satisfies

$$a_{i_t} = < x_i, \frac{(x_i)_t}{\| (x_i)_t \|} > = \left(\int_M x_i (x_i)_t dV \right) \left(\int_M |(x_i)_t^2| dV \right)^{-\frac{1}{2}}$$

where the first identity defines a_{i_t} and $\| \ . \ \|$ denotes the norm induced from $<\ ,\ >$ on $C^\infty(M)$. By a similar argument as given in Berger, Gauduchon and Mazet (1971, p. 186), we have

$$0 \leqq \| dx_i - \sum_{t \geqq p} \frac{a_{i_t} d(x_i)_t}{\|(x_i)_t\|} \|^2$$

$$= \|dx_i\|^2 - 2 \sum_{t \geqq p} \frac{a_{i_t}}{\|(x_i)_t\|} <dx_i, d(x_i)_t> + \sum_{t \geqq p} (a_{i_t})^2 \frac{\|d(x_i)_t\|^2}{\|(x_i)_t\|^2}$$

$$= \|dx_i\|^2 - 2 \sum_{t \geqq p} \frac{a_{i_t}}{\|(x_i)_t\|} < x_i, \Delta (x_i)_t >$$

$$+ \sum_{t \geqq p} \frac{(a_{i_t})^2}{\|(x_i)_t\|^2} < (x_i)_t, \Delta (x_i)_t >$$

$$= \|dx_i\|^2 - \sum_{t \geqq p} \lambda_t (a_{i_t})^2 .$$

From this we find

$$\|dx_i\|^2 \geqq \sum_{t \geqq p} \lambda_t (a_{i_t})^2 \geqq \lambda_p \left(\sum_{t \geqq p} (a_{i_t})^2 \right) = \lambda_p \|x_i\|^2 ,$$

i.e.,

(4.5) $$\int_M |dx_i|^2 dV \geqq \lambda_p \int_M x_i^2 \, dV ,$$

where $|dx_i|$ is the length of the 1-form dx_i on M. It is clear that the equality holds if and only if x_i is a function of order p. On the other hand, since

$$|dx|^2 = \sum_{i=1}^m |dx_i|^2 = n = \dim M,$$

(4.5) implies

(4.6) $$n\, v(M) \geqq \lambda_p \int_M |x|^2 dV ,$$

where $|x|$ is the length of x with respect to the Euclidean metric of E^m. From

(4.6) and the well-known Schwarz inequality, we have

$$(4.7)\qquad \begin{aligned} n\,v(M)\left(\int_M \alpha^2 dV\right) &\geqq \lambda_p\left(\int_M |x|^2 dV\right)\left(\int_M \alpha^2 dV\right) \\ &\geqq \lambda_p\left(\int_M \alpha|x|dV\right)^2 \geqq \lambda_p\left(\int_M \langle H,x\rangle dV\right)^2 \end{aligned}$$

where $\langle H,x\rangle$ is inner product of H and x in E^m. From Proposition 2.2 of Chen (1972 d) we have

$$(4.8)\qquad v(M) + \int_M \langle H,x\rangle dV = 0.$$

Thus, (4.7) and (4.8) give

$$(4.9)\qquad \int_M \alpha^2 dV \geqq \frac{\lambda_p}{n}\, v(M)\ .$$

Now, by using the Hölder inequality we find

$$\frac{\lambda_p}{n}\ v(M) \leqq \int_M \alpha^2 dV \leqq \left(\int_M \alpha^{2r} dV\right)^{1/r}\left(\int_M dV\right)^{1/s}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $r, s > 1$. Thus we have

$$(4.10)\qquad \left(\int_M \alpha^{2r} dV\right) \geqq \left(\frac{\lambda_p}{n}\right)^r v(M)\ .$$

If $n = 2$, (4.9) gives (4.3). If $n > 2$, then by setting $2r = n$, we get (4.3) from (4.10).

If the equality sign of (4.3) holds, the equalities above all become equalities. Therefore, each coordinate function x_i is of order p. From this we conclude that x is an imbedding of order p.

Conversely, if x is an imbedding of order p, we have

$$(4.11)\qquad \Delta x = \lambda_p x\ .$$

A theorem of Takahashi (1966) implies that M is imbedded in a hypersphere $S^{m-1}(r)$ of radius r centered at the origin as a minimal submanifold. Now, since $\Delta x = nH$ and M is minimal in $S^{m-1}(r)$, $x = r^2 H$. Thus (4.11) implies

$$\alpha^2 = \frac{1}{r^2} \qquad \text{and} \qquad \lambda_p = \frac{n}{r^2}\ .$$

From these we see that the equality sign of (4.3) holds (Q.E.D.)

For any immersion from M into E^m, if we choose the center of gravity to be the origin of E^m, then x is an immersion of order $\geqq 1$. Since inequality (4.3) is independent of the choice of the Euclidean coordinate system, Theorem

4.1 implies the following.

COROLLARY 4.2. *Let $x: M \longrightarrow E^m$ be an isometric immersion of a closed n-dimensional $(n \geqq 2)$ Riemannian manifold M into E^m. Then the total mean curvature of x satisfies*

$$(4.12) \qquad \int_M \alpha^n dV \geqq \left(\frac{\lambda_1}{n}\right)^{\frac{n}{2}} v(M) \quad .$$

The equality holds if and only if there is a vector c in E^m such that $x - c$ is an imbedding of order 1.

If $n = 2$, Corollary 4.2 is due to Bleecker and Weiner (1976) and Reilly (1977). By using Theorem 4.1 and Corollary 4.2 we may also obtain the following results.

THEOREM 4.3. *Let M be the product surface $S^1(a) \times S^1(b)$ of two plane circles of radii a and b, respectively. If $a \geqq b$, then*

(a) *M admits no isometric immersion of order $p < \frac{a}{b}$ in any Euclidean space, and*

(b) *every isometric immersion of order $\geqq \frac{a}{b}$ satisfies*

$$\int_M \alpha^2 dV \geqq \frac{2a\pi^2}{b} \quad .$$

The equality holds if and only if $\frac{a}{b}$ is an integer and the surface M is imbedded as of order $\frac{a}{b}$.

COROLLARY 4.4. *Let M be the Riemannian product of n plane circles of the same radius* 1. *Then every isometric immersion from M into E^m satisfies*

$$(4.13) \qquad \int_M \alpha^n dV \geqq \left(\frac{4\pi^2}{n}\right)^{n/2} \quad .$$

The equality holds if and only if M is imbedded in a hypersphere of radius $r = \sqrt{n}$ by an imbedding of order 1.

COROLLARY 4.5. *Let M be a flat Klein bottle. Then, for any isometric immersion from M into E^m, we have*

$$\int \alpha^2 dV > 2\pi^2.$$

COROLLARY 4.6. *Let $\boldsymbol{R}P^n$ be the n-dimensional real projective space with* *the standard metric. Then, for any isometric immersion from* $\mathbf{R}P^n$ *into* E^m, *we have*

$$(4.14) \qquad \int_{\mathbf{R}P^n} \alpha^n dV \geqq \left(\frac{2(n+1)}{n}\right)^{\frac{n}{2}} \frac{c_n}{2} \quad .$$

The equality holds if and only if $\mathbf{R}P^n$ *is imbedded in an affine* $\frac{1}{2}n(n+3)$-*space as a Veronese submanifold.*

COROLLARY 4.7. *Let* $\mathbf{C}P^n$ *be the complex n-dimensional complex projective space with the Fubini-Study metric. Then, for any isometric immersion from* $\mathbf{C}P^n$ *into* E^m, *we have*

$$(4.15) \qquad \int_{\mathbf{C}P^n} \alpha^{2n} dV \geqq \left[\frac{2(n+1)\pi}{n}\right]^n / n! \ .$$

The equality holds if and only if the immersion is an imbedding of order one.

COROLLARY 4.8. *Every isometric immersion from the quaternionic projective space* HP^n *with the standard metric into* E^m *satisfies*

$$(4.16) \qquad \int_{HP^n} \alpha^{4n} dV \geqq 2\left[\frac{(2n+3)\pi}{n}\right]^{2n} / (2n+1)! \ .$$

The equality holds if and only if the immersion is an imbedding of order one.

Let M be a submanifold of E^m of order $\geqq p$. A conformal mapping ϕ on E^m is said to *preserve the order* of M if $\phi(M)$ is order $\geqq p$ under a suitable translation of E^m, if it is necessary.

From Theorems 3.3 and 4.1 we may prove the following.

THEOREM 4.9. *Let* $x:M \longrightarrow E^m$ *be an imbedding of order p from a closed surface M into* E^m *and* ϕ *a conformal mapping on* E^m *which preserves the area and order of M. Then the p-th eigenvalues of the Laplace-Beltrami operator of M and* $\phi(M)$ *satisfy*

$$(4.17) \qquad \lambda_p(x) \geqq \lambda_p(\phi\cdot x) \ .$$

The equality holds if and only if ϕ *is a Euclidean motion.*

If $x:M \longrightarrow E^m$ is an immersion of order 1 from a closed surface M into E^m, then any conformal mapping on E^m preserves the order of M. Therefore, Theorem 4.9 gives the following *conformal inequality* for λ_1.

THEOREM 4.10. *Let* $x:M \longrightarrow E^m$ *be an imbedding of order one from a closed* *surface M into* E^m. *Then, for any conformal mapping* ϕ *on* E^m *with* $v(M) = v(\phi(M))$, *we have*

$$(4.18) \qquad \lambda_1(x) \geqq \lambda_1(\phi\cdot x).$$

The equality holds if and only if ϕ *is a Euclidean motion.*

A surface of E^m is called a *conformal Clifford torus* if it is the image of the Clifford torus $S^1(1)\times S^1(1) \subset E^4 \subset E^m$ under a conformal mapping on E^m. This class of surfaces includes *cyclides of Dupin* given by inversions of an anchor ring in E^3 with circles of radii a and b with ratio $\frac{a}{b} = \frac{1}{\sqrt{2}}$. From Theorem 4.10 we have the following.

COROLLARY 4.11. *If M is a conformal Clifford torus in* E^m *with area* $4\pi^2$, *then we have*

$$(4.19) \qquad \lambda_1 \leqq 1.$$

The equality holds if and only if M is a Clifford torus $S^1(1)\times S^1(1)$.

Let (x,y,z) be the Euclidean coordinates of E^3 and $(u^1,\cdots,u^5)$ be the Euclidean coordinates of E^5. We consider the mapping defined by $u^1 = \frac{1}{\sqrt{3}}yz$, $u^2 = \frac{1}{\sqrt{3}}xz$, $u^3 = \frac{1}{\sqrt{3}}xy$, $u^4 = \frac{1}{2\sqrt{3}}(x^2-y^2)$ and $u^5 = \frac{1}{6}(x^2+y^2-2z^2)$. This defines an isometric immersion of $S^2(1)$ into $S^4(1/\sqrt{3})$. Two points (x,y,z) and $(-x,-y,-z)$ of $S^2(1)$ are mapped into the same point. Thus, it defines an imbedding from $\boldsymbol{R}P^2$ into $S^4(1/\sqrt{3}) \subset E^5$. This real projective plane in E^5 is called the *Veronese surface*. It is known that it satisfies $v = 2\pi$ and $\lambda_1 = 6$. A surface of E^m is called a *conformal Veronese surface* if it is the image of the Veronese surface under a conformal mapping on E^m. From Theorem 4.10, we also have the following.

COROLLARY 4.12. *If M is a conformal Veronese surface in* E^m *with area* 2π, *then we have*

$$(4.20) \qquad \lambda_1 \leqq 6.$$

The equality holds if and only if M is a Veronese surface.

§5. TOTAL MEAN CURVATURE AND TOPOLOGY.

By studying total mean curvature, sometime we may obtain some informations about topological structure of the submanifold. In this section we would like to mention some results in this direction. Those results can be regarded as some partial solutions to Problem 2.2.

THEOREM 5.1 (Chen 1973c). *Let M be a closed surface of E^4 with Gauss curvature $G \geqq 0$ (or $G \leqq 0$). If we have*

$$\int_M \alpha^2 dV \leqq (2+\pi)\pi, \tag{5.1}$$

then M is diffeomorphic to a 2-sphere.

For the proof of this theorem see Chen (1973a) or (1973c).

Let $f: M \longrightarrow E^4$ be an immersion of an oriented closed surface M into E^4. By applying regular deformation to f if necessary, $f(M)$ intersects itself transversally. Hence, $f(M)$ intersects itself at isolated points. At each point p of self-intersection, we assign +1 if the direct sum orientation of the two complementary tangent planes equals the given orintation on E^4, and we assign -1 otherwise. Then the *self-intersection number* is defined as the sum of the local contributions from all the points of self-intersection. It is well-known that the self-intersection number q is the only immersion invariant up to regular homotopy from M in E^4.

THEOREM 5.2 (Wintgen 1979). *Let $x: M \longrightarrow E^4$ be an immersion from a closed oriented surface M into E^4. Then we have*

$$\int_M \alpha^2 dV \geqq 4\pi(1+|q| - g) \tag{5.2}$$

where q and g are the self-intersection number and genus of M, respectively.

PROOF. Let $x: M \longrightarrow E^4$ be an immersion of a closed oriented surface into E^4. We choose local field of orthonormal frame e_1, e_2, ξ_3, ξ_4 in E^4 such that, restricted to M, e_1, e_2 are tangent to M and ξ_3, ξ_4 are normal to M. By putting $h^r_{ij} = \langle h(e_i, e_j), \xi_r\rangle$, we see that the Gauss curvature G and *normal curvature* G^D are given respectively by

$$G = R(e_1, e_2; e_2, e_1) = \sum_{r=3}^{4} [h^r_{11} h^r_{22} - (h^r_{12})^2], \tag{5.3}$$

$$G^D = R^D(e_1, e_2; \xi_4, \xi_3) = h^3_{12}(h^4_{22} - h^4_{11}) - h^4_{12}(h^3_{22} - h^3_{11}). \tag{5.4}$$

Thus, the mean curvature $\alpha = |H|$ satisfies

$$\alpha^2 = \frac{1}{4}\{(h^3_{11}+h^3_{22})^2 + (h^4_{11}+h^4_{22})^2\}$$
$$= \left(\frac{h^3_{11}-h^3_{22}}{2}\right)^2 + \left(\frac{h^4_{11}-h^4_{22}}{2}\right)^2 + (h^3_{12})^2 + (h^4_{12})^2 + G$$
$$\geqq |h^4_{12}|\,|h^3_{11}-h^3_{22}| + |h^3_{12}|\,|h^4_{11}-h^4_{22}| + G \geqq |G^D| + G\ .$$

Hence, we have

$$\text{(5.5)} \qquad \int_M \alpha^2 dV \geqq 2\pi(\chi + |\chi^D|)$$

where χ and χ^D denote the Euler number of TM and $T^\perp M$, respectively. Since $\chi^D = 2q$ (see, Lashof and Smale (1958)), (5.5) gives inequality (5.2)

(Q.E.D.)

From (5.2) and a result of Smale (1959), we have immediately the following

THEOREM 5.3 (Wintgen 1979). *Let $f: S^2 \longrightarrow E^4$ be an immersion of a 2-sphere into E^4. If*

$$\int \alpha^2 dV < 8\pi,$$

then f is regularly homotopic to the standard imbedding of S^2 into an affine 3-space E^3.

If $x: M \longrightarrow E^4$ is an imbedding of a closed surface M into E^4, the fundmental group $\pi_1(E^4 - M)$ of $E^4 - M$ is called the *knot group* of x. The minimal number of generators of knot group is called the *knot number* of M.

We have the following relation between total mean curvature and knot number ρ.

THEOREM 5.4 (Wintgen 1978). *Let $f: M \longrightarrow E^4$ be an imbedding of a closed surface into E^4. Then we have*

$$\int_M \alpha^2 dV \geqq 4\pi\rho\ .$$

For the proof see Wintgen (1978).

§6. REMARKS AND CONJECTURES

REMARK 6.1. Corollary 4.4 shows that every isometric immersion from the Clifford torus, i.e., the product surface of two plane circles of the same radius, into E^m has total mean curvature $\geqq 2\pi^2$. For arbitrary flat surfaces we have the following best possible result.

THEOREM 6.1 (Chen 1981). *Let M be a closed flat surface. Then every isometric immersion from M into E^m satisfies*

$$(6.1) \qquad \int_M \alpha^2 dV \geqq 2\pi^2.$$

The equality holds if and only if M is imbedded in an affine 4-space of E^m as a Clifford torus.

CONJECTURE 6.1. Inequality (6.1) holds for any immersion from a 2-torus into E^m.

Willmore made this conjecture for $m = 3$.

CONJECTURE 6.2 (Wintgen 1979). For any closed oriented surface immersed in E^m we have

$$(6.2) \qquad \int_M \alpha^2 dV \geqq 4\pi(1 + |q|).$$

CONJECTURE 6.3 (Chen 1979a). For any immersion from $\mathbf{R}P^n$ into E^m we have (4.14).

CONJECTURE 6.4 (Chen 1979a). For any immersion from $\mathbf{C}P^n$ into E^m we have (4.15).

CONJECTURE 6.5 (Chen 1979a). For any immersion from HP^n into E^m we have (4.16).

REMARK 6.2. Lawson (1970) showed that for any positive integer g, there is a closed minimal surface M in the unit 3-sphere S^3 such that the genus of M is g and the area of M is $< 8\pi$. From this fact, we know that for any closed oriented surface M, there exists an immersion from M into E^3 whose total mean curvature is $< 8\pi$.

CONJECTURE 6.6. Let M be a closed surface which is not homeomorphic to

S^2, $\mathbf{R}P^2$ or $S^1 \times S^1$. Then for any immersion of M into E^m we have

$$\int_M \alpha^2 dV > 2\pi^2 .$$

REMARK 6.3. The relations between total mean curvature and the theory of variations have been studied in Chen (1972e, 1973a), Willmore and Jhaveri (1972). Chen and Houh (1975), Chen and Yano (1978), Weiner (1978), and others.

REMARK 6.4. Let M^n be a closed submanifold of E^m defined by some homogeneous polynomials. Then one may define the *degree* of M^n by using the degree of these polynomials. It seems to be interesting to *find a relation between total mean curvature and the degree of* M similar to Theorem 4.1. It follows from Theorem 2.1 that if $\int_M \alpha^n dV = c_n$, M is of degree 2.

Chapter IV: GENERIC SUBMANIFOLDS OF KAEHLER MANIFOLDS

In this chapter we shall study generic submanifolds of a Kaehler manifold. Results in this theory will be used to obtain some new results in older theories, e.g. theories of complex submanifolds and CR-submanifolds

§1. DIFFERENTIABILITY AND OPENNESS.

Let $(\tilde{M},J)$ be an almost complex manifold with almost complex structure J. Let N be a submanifold of $\tilde{M}$. For each point $x \in N$, we put

$$\mathscr{H}_x = T_x N \cap J(T_x N)$$

Then $\mathscr{H}_x$ is the maximal complex subspace of the tangent space $T_x\tilde{M}$ which is contained in $T_x M$. If the dimension of $\mathscr{H}_x$ is constant along N, N is called a *generic submanifold* of $(\tilde{M},J)$. For a generic submanifold N we denote by $\mathscr{H}$ the distribution defined by $\mathscr{H}_x$, $x \in N$. We call $\mathscr{H}$ the *holomorphic distribution* of N. The following result shows the differentiability of this distribution.

PROPOSITION 1.1. *For any generic submanifold N of an almost complex manifold $(\tilde{M},J)$, the holmorphic distribution is differentiable.*

PROOF. It is well-known that the Whitney sum $TN \oplus TN$ is a differentiable vector bundle over N. Let N be a generic submanifold of $(\tilde{M},J)$. We define a differentiable mapping

$$\phi : TN \oplus TN \longrightarrow T\tilde{M}$$

by $\phi(X,Y) = X - JY$. Because N is assumed to be generic, the implicit function theorem implies that the kernel of ϕ, $\phi^{-1}(N)$, is a differentiable submanifold of $TN \oplus TN$. Let $\psi : TN \oplus TN \longrightarrow TN$ be the projection given by $\psi(X,Y) = X$. Then $\psi \mid \phi^{-1}(N)$ is one-to-one and $\mathscr{H} = \psi(\phi^{-1}(N))$. This proves the proposition.

(Q.E.D.)

The author would like to express his thanks to Professors Nagano and Otsuki for giving the simplified proof of this proposition.

If $(\tilde{M},J,g)$ is an almost Hermitian manifold, then for each x in a generic submanifold N, we define $\mathscr{H}_x^{\perp}$ as the orthogonal complementary subspace of $\mathscr{H}_x$ in $T_x N$. Then, by Proposition 1.1, $\mathscr{H}_x$ defines another differentiable distribution on N. For this distribution $\mathscr{H}^{\perp}$, we have

$$J\mathscr{H}^{\perp} \cap \mathscr{H}^{\perp} = \{0\}. \tag{1.1}$$

We call this distribution the *purely real distribution*. A generic submanifold N in $(\tilde{M},J)$ is called a *complex* (respectively, *purely real*) *submanifold* if $\mathscr{H} = TN$ (respectively, $\mathscr{H} = \{0\}$).

Let N be a compact manifold and $\mathscr{D}^p(N, \boldsymbol{C}^m)$ the set of imbeddings from N into $\boldsymbol{C}^m$ with $\dim_{\boldsymbol{R}} \mathscr{H}_x \geqq \dim N - p$ for any $x \in N$. Then we have

$$\mathscr{D}^p(N, \boldsymbol{C}^m) \subset [C^\infty(N)]^{2m},$$

with respect to the usual Frechet topology, this gives $\mathscr{D}^p(N, \boldsymbol{C}^m)$ the induced topology. It can be proved that the set of generic imbeddings from N into $\boldsymbol{C}^m$ with $\dim_{\boldsymbol{R}} \mathscr{H}_x \equiv \dim N - p$ is *open in* $\mathscr{D}^p(N, \boldsymbol{C}^m)$.

From Proposition 1.1 we see that every real submanifold of an almost complex manifold is the closure of the union of some generic open submanifolds. Thus this theory is very general.

Let N be a generic submanifold of an almost Hermitian manifold $\tilde{M}$. For any vector field X tangent to N, we put

$$JX = PX + FX \tag{1.2}$$

where PX and FX are the tangential and normal components of JX, respectively. Then P is an endomorphism of TN and F a normal-bundle-valved 1-form on TN. For any vector field ξ normal to N, we put

$$J\xi = t\xi + f\xi \tag{1.3}$$

where $t\xi$ and $f\xi$ are the tangential and normal components of $J\xi$. Then t is a tangent-bundle-valued 1-form on $T^{\perp}N$ and f an endomorphism of $T^{\perp}N$. For a generic submanifold N in $(\tilde{M},g,J)$ we have

$$\mathscr{H}_x \perp \mathscr{H}_x^{\perp}, \quad P\mathscr{H}_x = \mathscr{H}_x, \text{ and } \quad P\mathscr{H}_x^{\perp} \subseteqq \mathscr{H}_x^{\perp}. \tag{1.4}$$

Moreover, $F: \mathscr{H}^{\perp} \longrightarrow F\mathscr{H}^{\perp}$ is an isomorphism.

Let ν_x be the vector subspace of $T_x^{\perp} N$ given by

$$\nu_x = T_x^{\perp} N \cap J(T_x^{\perp} N).$$

Then ν is a differentiable complex vector bundle over N. It is easy to verify that

$$T^{\perp}N = F\mathscr{H}^{\perp} \oplus \nu, \quad t(T^{\perp}N) = \mathscr{H}^{\perp}, \text{ and } \quad F\mathscr{H}^{\perp} \perp \nu. \tag{1.5}$$

Throughout this and the next chapters we always put

$$(1.6)\qquad h=\dim_{C}\mathscr{H},\qquad p=\dim_{R}\mathscr{H}^{\perp}.$$

§2. INTEGRABILITIES

Throughout this chapter we shall always assume that N is a generic submanifold of a Kaehler manifold $\tilde{M}$ we shall adopt the notations given in §1. The results given in this chapter are obtained in Chen (1980c).

PROPOSITION 2.1. *We have* $\langle Jh(X,U),\xi\rangle = \langle h(JX,U),\xi\rangle$ *for any vectors* $X\in\mathscr{H}$, $U\in TN$ *and* $\xi\in\nu$.

PROPOSITION 2.2. *The holomorphic distribution* $\mathscr{H}$ *is integrable if and only if* $\langle h(X,JY),FZ\rangle = \langle h(JX,Y),FZ\rangle$ *for* X,Y *in* $\mathscr{H}$ *and* Z *in* $\mathscr{H}^{\perp}$.

Proposition 2.1 follows from the Gauss formula and $\tilde{\nabla}J=0$. Proposition 2.2 follows from the fact that $h(X,JY)-h(JX,Y)=J[X,Y]+\nabla_Y JX-\nabla_X JY$, for X,Y in $\mathscr{H}$. (see, also, Blair and Chen (1979) and Bejancu (1978)).

PROPOSITION 2.3. *The purely real distribution* $\mathscr{H}^{\perp}$ *is integrable if and only if*

$$(2.1)\qquad \nabla_Z(PW)-\nabla_W(PZ)+A_{FZ}W-A_{FW}Z\in\mathscr{H}^{\perp}$$

for $Z,W\in\mathscr{H}^{\perp}$.

PROOF. For any Z,W in $\mathscr{H}^{\perp}$ we have

$$J\nabla_Z W+Jh(Z,W)=\nabla_Z(PW)+h(Z,PW)-A_{FW}Z+D_Z(FW),$$

from which we obtain

$$[Z,W]=P\{A_{FW}Z-A_{FZ}W+\nabla_W(PZ)-\nabla_Z(PW)\}$$
$$+t\{h(W,PZ)-h(Z,PW)+D_W(FZ)-D_Z(FW)\}\ .$$

Since $t(T^{\perp}N)=\mathscr{H}^{\perp}$, this proves the proposition.

PROPOSITION 2.4. *If* $\mathscr{H}$ *is integrable and its leaves are totally geodesic in* N, *then* $\langle h(\mathscr{H},\mathscr{H}),\ F\mathscr{H}^{\perp}\rangle=\{0\}$.

PROOF. Under the hypothesis, $\nabla_X Z\in\mathscr{H}^{\perp}$ for vector fields X in $\mathscr{H}$ and

$Z \ni \mathscr{H}^{\perp}$. Thus for Y in $\mathscr{H}$, we have

$$0 = \langle \nabla_X Z, JY\rangle = \langle A_{FZ}X, Y\rangle - \langle \nabla_X PZ, Y\rangle = \langle A_{FZ}X, Y\rangle.$$

This proves the proposition.

PROPOSITION 2.5. *If $\mathscr{H}^{\perp}$ is integrable and its leave are totally geodesic in N, then $\langle h(\mathscr{H},\mathscr{H}^{\perp}), F\mathscr{H}^{\perp}\rangle = \{0\}$.*

PROOF. For vector fields X in $\mathscr{H}$, Z,W in $\mathscr{H}^{\perp}$, we have

$$0 = \langle \nabla_Z X, W\rangle = \langle \tilde{\nabla}_Z JX, PW\rangle + \langle \tilde{\nabla}_Z JX, FW\rangle = \langle h(JX, Z), FW\rangle.$$

This proves the proposition.

These two propositions play important rôles in the theory of generic submanifolds because these two propositions tell us that if we impose suitable intrinsic conditions on the generic submanifold N, we obtain important extrinsic conclusions on N.

For the endomorphism $P : TN \longrightarrow TN$, if we put

$$(\nabla_U P)V = \nabla_U(PV) - P(\nabla_U V)$$

for vector fields U,V in TN, then we have

$$(\nabla_U P)V = th(U,V) + A_{FV}U \tag{2.2}$$

Thus we may obtain the following.

LEMMA 2.6. *P is parallel if and only if* (1) *$\mathscr{H}$ is integrable,* (2) $A_{FU}X = 0$ *for* $X \in \mathscr{H}$ *and* $U \in TN$, *and* (3) $A_{FU}V = A_{FV}U$ *for U,V in TN.*

For the normal bundle-valued 1-form F, if we put

$$(\nabla_U F)V = D_v(FV) - F(\nabla_U V),$$

for vector fields U,V tangent to N, we have

$$(\nabla_U F)V = fh(U,V) - h(U,PV). \tag{2.3}$$

Thus we obtain the following.

LEMMA 2.7. *F is parallel if and only if $A_{f\xi}U = -A_{\xi}PU$ for U in TN and ξ in $T^{\perp}N$.*

By using this lemma we may prove the following

LEMMA 2.8. *If F is parallel, then*

(1) *$\mathscr{H}$ is integrable and its leaves are totally geodesic in N,*

(2) *$\nabla_U X \in \mathscr{H}$ and $A_{FU}X = 0$,*

(3) *$F\mathscr{H}^{\perp}$ and ν are parallel in the normal bundle, and*

(4) *$(\nabla_X P)Z = (\nabla_U P)X = 0$,*

for X in $\mathscr{H}$, Z in $\mathscr{H}^{\perp}$, and U in TN,

Let $i : N \longrightarrow \tilde{M}$ be a generic submanifold of a Kaehler manifold $\tilde{M}$. If we denote by $\tilde{\Omega}$ and $\tilde{\gamma}_\alpha$ the fundamental 2-form (or the Kaehler form) and the α-th Chern form of $\tilde{M}$, respectively. And by $\Omega = i^*\tilde{\Omega}$ and $\gamma_\alpha = i^*\tilde{\gamma}_\alpha$ the induced forms on N. Then $d\Omega = d\gamma_\alpha = 0$. Thus, if N is closed, Ω and γ_α define cohomology classes $[\Omega]$ and $[\gamma_\alpha]$ in $H^2(N;\boldsymbol{R})$ and in $H^{2\alpha}(N;\boldsymbol{R})$, respectively. We call them the *fundamental class* and α*-th Chern class* of N, respectively. Such cohomology classes will be studied later.

§3. GENERIC SUBMANIFOLD OF COMPLEX-SPACE-FORMS

For a generic submanifold N of a Kaehler manifold $\tilde{M}$, if $\mathscr{H}$ is integrable and

$$\langle h(\mathscr{H}^{\perp},\mathscr{H}), F\mathscr{H}^{\perp}\rangle = \{0\}, \quad \text{i.e.,} \quad A_{F\mathscr{H}^{\perp}}\mathscr{H} \subseteq \mathscr{H}, \tag{3.1}$$

then, by the Codazzi equation, for X,Y in $\mathscr{H}$ and Z,W in $\mathscr{H}^{\perp}$, we have

$$\begin{aligned}\tilde{R}(X,Y;Z,FW) &- \langle D_X h(Y,Z) - D_Y h(X,Z), FW\rangle \\ &= \langle A_{FW}Y, J\tilde{\nabla}_X JZ\rangle - \langle A_{FW}X, J\tilde{\nabla}_Y JZ\rangle \\ &= \langle JA_{FW}X, \nabla_Y PZ\rangle - \langle JA_{FW}Y, \nabla_X PZ\rangle + \langle A_{FW}X, JA_{FZ}Y\rangle - \langle A_{FW}Y, JA_{FZ}X\rangle.\end{aligned} \tag{3.2}$$

Since $\mathscr{H}$ is integrable, (3.1) and Proposition 2.2 imply that, for $X \in \mathscr{H}$, $W \in \mathscr{H}^{\perp}$,

$$A_{FW}JX = -JA_{FW}X. \tag{3.3}$$

Thus, (3.2) and (3.3) give

$$\begin{aligned}\tilde{R}(X,JX;Z,FZ) &- \langle D_X h(JX,Z) - D_{JX} h(X,Z), FZ\rangle \\ &= \langle JA_{FZ}X, \nabla_{JX} PZ\rangle - \langle A_{FZ}X, \nabla_X PZ\rangle + 2\|A_{FZ}X\|^2 .\end{aligned} \tag{3.4}$$

On the other hand, we have $<JA_{FZ}X,PZ>=0$ and (3.1). Thus we obtain

$$(3.5)\qquad <JA_{FZ}X,\nabla_{JX}PZ> = <[JX,A_{FZ}X],JPZ> + <\tilde{\nabla}_{A_{FZ}X}X,PZ>$$

$$= <[A_{FZ}X,X],PZ> + <\tilde{\nabla}_X(A_{FZ}X),PZ> = -<A_{FZ}X,\nabla_X PZ>,$$

$$(3.6)\qquad \nabla_X PZ = A_{FZ}X + P\nabla_X Z.$$

Thus, by (3.4), (3.5) and (3.6), we have

$$(3.7)\qquad \tilde{R}(X,JX;Z,FZ) - <D_X h(JX,Z) - D_{JX}h(X,Z),FZ>$$

$$= 2\|A_{FZ}X\|^2 - 2\|\nabla_X PZ\|^2 + 2<P\nabla_X Z,\nabla_X PZ>.$$

From (3.6) we also have $\|A_{FZ}X\|^2 = \|\nabla_X PZ\|^2 + \|P\nabla_X Z\|^2 - 2<\nabla_X PZ,P\nabla_X Z>$. Substituting this into (3.7) we get

$$(3.8)\qquad \tilde{R}(X,JX;Z,FZ) - <D_X h(JX,Z) - D_{JX}h(X,Z),FZ>$$

$$= \|A_{FZ}X\|^2 - \|\nabla_X PZ\|^2 + \|P\nabla_X Z\|^2$$

If $\tilde{M}$ is a *complex-space-form* of constant holomorphic sectional curvature c, then we have

$$(3.9)\qquad \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \frac{c}{4}\{<\tilde{Y},\tilde{Z}>\tilde{X} - <\tilde{X},\tilde{Z}>\tilde{Y} + <J\tilde{Y},\tilde{Z}>J\tilde{X} - <J\tilde{X},\tilde{Z}>J\tilde{Y} + 2<\tilde{X},J\tilde{Y}>J\tilde{Z}\}$$

for $\tilde{X},\tilde{Y},\tilde{Z} \in T\tilde{M}$. Thus, for $X \in \mathscr{H}$ and $Z \in \mathscr{H}^\perp$, we obtain

$$(3.10)\qquad \tilde{R}(X,JX;Z,FZ) = -\frac{c}{2}<X,X><FZ,FZ>.$$

DEFINITION 3.1. A generic submanifold N in $\tilde{M}$ is called *mixed totally geodesic* if $h(\mathscr{H},\mathscr{H}^\perp) = \{0\}$.

From (3.8) and (3.10) we obtain the following.

LEMMA 3.1. *Let N be a generic submanifold of a complex-space-form $\tilde{M}(c)$. If $\mathscr{H}$ is integrable and N is either* (a) *mixed totally geodesic or* (b) *$F\mathscr{H}^\perp$ is parallel and* (3.1) *holds, then for unit vectors X in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$, we have*

$$(3.11)\qquad \|\nabla_X PZ\|^2 = \frac{c}{2}\|FZ\|^2 + \|P\nabla_X Z\|^2 + \|A_{FZ}X\|^2 .$$

By using Proposition 2.4. Lemma 3.1, formulas (2.2) and (3.12), we obtain

PROPOSITION 3.2. *Let N be a mixed totally geodesic generic submanifold of a complex-space-form $\tilde{M}(c)$, $c \neq 0$. Then $\mathscr{H}$ is integrable and its leaves are totally geodesic in N if and only if N is either a complex submanifold or a purely real submanifold.*

A generic submanifold is said to be *proper* if it is neither a complex submanifold nor a purely real submanifold.

THEOREM 3.3. *Let N be proper generic submanifold of a complex-space-form $\tilde{M}(c)$. If $\nabla F \equiv 0$, then $c = 0$.*

PROOF. If $\nabla F \equiv 0$, then, by Lemma 2.8 and Lemma 3.1, we have (3.11). Moreover, Lemma 2.8 also gives $A_{FZ}X = 0$ and $\nabla_X(PZ) = P\nabla_X Z$, for X in $\mathscr{H}$ and Z in $\mathscr{H}^{\perp}$. Thus (3.11) gives $c\|FZ\|^2 = 0$. Since $FZ \neq 0$, $c = 0$.

EXAMPLE 3.1. Let N^T be a complex submanifold of the complex number space $\boldsymbol{C}^r$ and $N^{\perp}$ any p-dimensional purely real submanifold of $\boldsymbol{C}^p$. Then the Riemannian product space $N^T \times N^{\perp}$ is a generic submanifold of $\boldsymbol{C}^{r+p}$ satisfying $\nabla P \equiv 0$ and $\nabla F \equiv 0$.

THEOREM 3.4. *Let N be proper generic submanifold of a complex-space-form $\tilde{M}^{h+p}(c)$. If $\nabla P \equiv 0$, $c = 0$.*

PROOF. Because $\dim_{\boldsymbol{C}} \tilde{M}^{h+p}(c) = \dim_{\boldsymbol{C}} \mathscr{H} + \dim_{\boldsymbol{R}} \mathscr{H}^{\perp}$, Lemma 2.6 implies that $A_{FZ}X = 0$ for $X \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$ and $\mathscr{H}$ is integrable. If $c \neq 0$, (3.11) gives $F\mathscr{H}^{\perp} = \{0\}$. This shows that N is not proper.

§4. GENERIC PRODUCTS

A real submanifold N of a Kaehler manifold $\tilde{M}$ is called a *generic product* if it is locally the Riemannian product of a complex submanifold N^T and a purely real submanifold $N^{\perp}$ of $\tilde{M}$.

Examples of generic product in $\boldsymbol{C}^m$ have been given in Example 3.1. In the following we give examples of generic product in CP^m.

EXAMPLE 4.1. Let CP^m be the complex m-dimensional complex projective space of constant holomorphic sectional curvature 4. The *Segre imbedding*

$$S_{hp} : CP^h \times CP^p \longrightarrow CP^{h+p+hp}$$

is defined by $(z_0, \cdots, z_h)\ (\omega_0, \cdots, \omega_p) \longrightarrow (z_0\omega_0, \cdots, z_i\omega_j, \cdots, z_h\omega_p)$, where (z_i) and (ω_j) are the homogeneous coordinates for CP^h and CP^p, respectively. It is easy to see that S_{hp} is a non-totally geodesic Kaehler imbedding such that each component is totally geodesic in CP^{h+p+hp}. Let $N^\perp$ be any p-dimensional purely real submanifold of CP^p. Then $CP^p \times N^\perp$ is a non-totally geodesic generic product in CP^{h+p+hp} in which CP^h is imbedded as a totally geodesic complex submanifold in CP^{h+p+hp}.

DEFINITION 4.1. A generic product $N = N^T \times N^\perp$ in a CP^m is called a *standard generic product* in CP^m if (1) N lies in a totally geodesic complex submanifold CP^{h+p+hp} of CP^m and (2) N^T is immersed in CP^m as a totally geodesic complex submanifold.

LEMMA 4.1. *If N is a generic product of a Kaehler manifold $\tilde{M}$, then* (1) $A_{FZ}X = 0$ *and* (2) $(\nabla_X P)Z = 0$ *for X in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$.*

This follows from Proposition 2.4, Proposition 2.5, and (3.12).

For a Kaehler manifold $\tilde{M}$, the *holomorphic bisectional curvature* $\tilde{H}_B$ of $\tilde{M}$ is defined by

$$(4.1) \qquad \tilde{H}_B(X,Y) = \tilde{R}(X,JX;JY,Y)$$

for orthonormal vectors X, Y tangent to $\tilde{M}$.

PROPOSITION 4.2. *If N is a generic product of a Kaehler manifold $\tilde{M}$, then, for unit vectors X in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$, we have*

$$(4.2) \qquad \tilde{H}_B(X,Z) = 2\|h(X,Z)\|^2.$$

PROOF. If N is a generic product, (3.8) holds. On the other hand, Proposition 2.2 and (3.1) give, for $X \in \mathscr{H}$, $Z \in \mathscr{H}^\perp$,

$$\begin{aligned}(4.3) \qquad \langle D_X h(JX,Z) - D_{JX} h(X,Z), FZ\rangle &= \langle h(X,Z), \tilde{\nabla}_{JX} FZ\rangle - \langle h(JX,Z), \tilde{\nabla}_X FZ\rangle \\ &= \langle h(X,Z), Jh(JX,Z)\rangle - \langle h(X,Z), h(JX,PZ)\rangle - \langle h(JX,Z), Jh(X,Z)\rangle \\ &\quad + \langle h(JX,Z), h(X,PZ)\rangle \\ &= -2\|h(X,Z)\|^2 + 2\langle Jh(X,Z), h(X,PZ)\rangle .\end{aligned}$$

By the equation of Gauss and (3.1) we also have

$$(4.4)\qquad \tilde{R}(X,JX;Z,PZ) = R(X,JX;Z,PZ) - 2\langle Jh(X,Z),h(X,PZ)\rangle .$$

Combining (3.8), (4.3) and (4.4) we obtain

$$(4.5)\qquad \tilde{H}_B(X,Z) = R(X,JX;PZ,Z) + 2\|h(X,Z)\|^2 + \|\nabla_X PZ\|^2 - \|P\nabla_X Z\|^2 - \|A_{FZ}X\|^2$$

for unit vectors $X\in\mathscr{H}$ and $Z\in\mathscr{H}^\perp$. Now, by Lemma 4.1, this gives

$$\tilde{H}_B(X,Z) = R(X,JX;PZ,Z) + 2\|h(X,Z)\|^2 .$$

Since N is the Riemannian product of N^T and $N^\perp$, $R(X,JX;PZ,Z) = 0$ This gives (4.2).

By using Proposition 4.2 we obtain the following.

THEOREM 4.3. *Let $\tilde{M}$ be any Kaehler manifold with negative holomorphic bisectional curvature. Then $\tilde{M}$ admits no proper generic products. In particular, the complex hyperbolic space admits no proper generic products.*

THEOREM 4.4. *Every complete, 1-connected, generic product in $\mathbf{C}^m$ is the Riemannian product of a complex submanifold of a $\mathbf{C}^N$ and a purely real submanifold of a $\mathbf{C}^{m-N}$.*

This theorem simply follows from Proposition 4.2 and a result of Moore (1971).

THEOREM 4.5. *Let N be a generic product of CP^m. Then*

$$(4.6)\qquad m \geqq h + p + hp.$$

If $m = h + p + hp$, then N is a standard generic product.

PROOF. From Proposition 4.2 we have, for unit $X\in\mathscr{H}$, $Z\in\mathscr{H}^\perp$,

$$(4.7)\qquad \|h(X,Z)\| = 1.$$

Thus by linearity we obtain

$$(4.8)\qquad \langle h(X_i,Z),h(X_j,Z)\rangle = 0, \qquad i\neq j ,$$

where $X_1,\cdots,X_{2h}$ and $Z_1,\cdots,Z_p$ are orthonormal bases for $\mathscr{H}$ and $\mathscr{H}^\perp$ respectively If $p=1$, then Lemma 4.1, (4.7) and (4.8) imply (4.6).

If $p \geqq 2$, then by applying linearity to (4.8) and using the equation of Gauss, we see that $\{h(X_i, Z_\alpha) \mid i=1,\cdots,2h;\ \alpha=1,\cdots,p\}$ forms an orthonormal basis for ν. Thus (4.6) follows.

Now, assume that N is a generic product of CP^m with $m=h+p+hp$. Then, for vectors X,Y in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$, (3.9) and the equation of Gauss give

$$(4.9) \qquad <h(X,Y),h(JX,Z)> = <h(X,Z),h(JX,Y)>.$$

On the other hand, Propositions 2.2 and 4.2 imply

$$<h(X,Z),h(JX,Y)> = <h(X,Z),Jh(X,Y)>$$

$$= -<Jh(X,Z),h(X,Y)> = -<Jh(X,Z),Jfh(X,Y)>$$

$$= -<h(JX,Z),h(X,Y)>.$$

Combining this with (4.9) we get

$$(4.10) \qquad 0 = <h(X,Z),h(JX,Y)> = <h(X,Z),h(X,JY)>$$

for X,Y in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$. Thus by linearity we find

$$<h(X,Z),h(Y,W)> + <h(Y,Z),h(X,W)> = 0$$

for X,Y,W in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$. Hence, by equation of Gauss, we have

$$(4.11) \qquad <h(X,Z),h(Y,W)> = 0$$

for X,Y,W in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$. Because $m=h+p+hp$, this gives

$$(4.12) \qquad h(\mathscr{H},\mathscr{H}) \subseteq F\mathscr{H}^\perp.$$

Combining this with Lemma 4.1 we obtain $h(\mathscr{H},\mathscr{H})=0$. Since N^T is totally geodesic in N, N^T is totally geodesic in CP^m. This proves the theorem.

By using Proposition 4.2 we may also prove the following.

THEOREM 4.6. *Let N be a generic product of CP^m. Then*

$$(4.13) \qquad ||h||^2 \geqq 4hp.$$

If the equality holds, locally, N is the Riemannian product of the totally *geodesic complex submanifold* $\boldsymbol{C}P^h$ *and a totally geodesic totally real submanifold* $\boldsymbol{R}P^h$ *of* $\boldsymbol{C}P^m$. *Moreover, the immersion is rigid.*

By a *totally real submanifold* N of a Kaehler manifold $\tilde{M}$ we mean a *purely real submanifold* with $J\mathscr{H}^{\perp} \subseteq T^{\perp}N$. (see, Chen and Ogiue (1974a))

For general results of complex submanifolds see Ogiue (1974).

§5. APPLICATION TO COMPLEX GEOMETRY

By using Theorem 4.5 we obtain easily the following converse theorem to Segre imbedding.

THEOREM 5.1. *Let* $M = M_1 \times M_2$ *be the Riemannian product of two Kaehler manifolds with* $\dim_{\boldsymbol{C}} M_1 = h$ *and* $\dim_{\boldsymbol{C}} M_2 = p$. *Then*

(1) *M admits no Kaehler immersion into* $\boldsymbol{C}P^m$ *for any* $m < h + p + hp$, *and*

(2) *If M admits a Kaehler immersion into* $\boldsymbol{C}P^{h+p+hp}$, *then we have*

(2.1) M_1 *and* M_2 *are open submanifolds of* $\boldsymbol{C}P^h$ *and* $\boldsymbol{C}P^p$, *respectively, and*

(2.2) *the Kaehler immersion is given by the Segre imbedding.*

PROOF. Let $N_2^{\perp}$ be any p-dimensional purely real submanifold of M_2. (Such submanifolds exist extensively). We put $N = M_1 \times N_2^{\perp}$. If $M = M_1 \times M_2$ admits a Kaehler immersion into $\boldsymbol{C}P^m$, then $N = M_1 \times N_2^{\perp} \subset M_1 \times M_2 \subset \boldsymbol{C}P^m$ is a generic product in $\boldsymbol{C}P^m$. Thus, by Theorem 4.5 we obtain $m \geqq h + p + hp$.

If M admits a Kaehler immersion into $\boldsymbol{C}P^{h+p+hp}$, then $N = M_1 \times N_2^{\perp}$ is a generic product in $\boldsymbol{C}P^{h+p+hp}$. Thus, by Theorem 4.5, M_1 is a totally geodesic complex submanifold of $\boldsymbol{C}P^{h+p+hp}$. Hence M_1 is an open submanifold of $\boldsymbol{C}P^h$. By applying the same argument to M_2, we conclude that M_2 is an open submanifold of $\boldsymbol{C}P^p$. Therefore, statement (2) follows from the Calabi local rigid theorem of Kaehler immersion. (Q.E.D.)

REMARK 5.1. Let M_1^h by any complex hypersurface of $\boldsymbol{C}P^{h+1}$. Then $M_1^h \times \boldsymbol{C}P^p$ admits a Kaehler imbedding into $\boldsymbol{C}P^{h+2p+hp+1}$. Hence, in order to conclude statement (2.1) of Theorem 5.1, the assumption on the dimension is necessary.

REMARK 5.2. Further applications of the theory of generic submanifolds will be given in Chapters V and VII.

Chapter V: *CR*-SUBMANIFOLDS OF KAEHLER MANIFOLDS

The main purpose of this chapter is to summarize some results obtained in Chen (1978a) and Chen (1980d). For the details and further results, see these two papers and furthcoming part of this series.

§1. INTEGRABILITY, MINIMALITY AND COHOMOLOGY.

A generic submanifold N of an almost Hermitian manifold $\tilde{M}$ is called a *CR-submanifold* if its purely real distribution $\mathscr{H}^{\perp}$ is totally real, i.e., $J\mathscr{H}_x^{\perp} \subseteq T_x^{\perp}N$, $x \in N$. This notion was first introduced in Bejancu (1978). It has been proved in Blair and Chen (1979) that every *CR*-submanifold is a *CR*-manifold in the sense of Andreotti and Hill (1972). It is clear that every generic submanifold with $\dim_{\mathbf{R}} \mathscr{H}^{\perp} = 1$ is a *CR*-submanifold. However, if $\dim_{\mathbf{R}} \mathscr{H}^{\perp} > 1$, generic submanifolds are not necessary *CR*-submanifolds.

We shall use the same notations given in previous chapters. Throughout this chapter, N is assumed to be a *CR*-submanifold of a Kaehler manifold $\tilde{M}$ unless mentioned otherwise.

LEMMA 1.1. *For vector fields X is $\mathscr{H}$, Z, W in $\mathscr{H}^{\perp}$, ξ in ν, and U in TN, we have*

(1.1) $\quad \langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle$,

(1.2) $\quad A_{JZ}W = A_{JW}Z$, *and*

(1.3) $\quad A_{J\xi}X = -A_{\xi}JX.$

This follows from the indentities, $J\nabla_U Z + Jh(U,Z) = -A_{JZ}U + D_U JZ$ and $\langle h(JX,Y),\xi\rangle = \langle \tilde{\nabla}_Y JX, \xi\rangle = \langle Jh(X,Y),\xi\rangle$.

For a *CR*-submanifold N, $P\mathscr{H}^{\perp} = \{0\}$. Thus, from Proposition IV, 2.3 (i.e., Proposition 2.3 of Chapter IV) and Lemma 1.1, we obtain the following fundamental result for *CR*-submanifolds.

THEOREM 1.2. *The totally real distribution $\mathscr{H}^{\perp}$ of a CR-submanifold of a Kaehler manifold is always integrable.*

This theorem was generalized to *CR*-submanifolds in a locally conformal almost Kaehler manifold by Blair and Chen (1979). Moreover, in Blair and Chen (1979), they have constructed *CR*-submanifolds in some Hermitian manifolds in which $\mathscr{H}^{\perp}$ is not integrable.

By using (1.1) and Proposition IV, 2.5, we find

PROPOSITION 1.3. *For a CR-submanifold N of a Kaehler manifold $\tilde{M}$, the leaves of $\mathscr{H}^\perp$ are totally geodesic in N if and only if* $\langle h(\mathscr{H},\mathscr{H}^\perp), J\mathscr{H}^\perp\rangle = \{0\}$.

This proposition can also be found in Bejancu, Kon and Yano (1980). Moreover, we have the following.

LEMMA 1.4. *If $\mathscr{H}$ is integrable and leaves of $\mathscr{H}^\perp$ are totally geodesic in N, then for any $X \in \mathscr{H}$ and $\xi \in J\mathscr{H}^\perp$, we have*

$$A_\xi JX = -JA_\xi X. \tag{1.4}$$

Let $\mathscr{D}$ be a differentiable distribution on a Riemannian manifold N. We put

$$\overset{\circ}{h}(X,Y) = (\nabla_X Y)^\perp \tag{1.5}$$

for any vector fields X, Y in $\mathscr{D}$, where $(\nabla_X Y)^\perp$ is the component of $\nabla_X Y$ in the orthogonal complementary distribution $\mathscr{D}^\perp$ of $\mathscr{D}$. Then $\overset{\circ}{h}$ is a well-defined $\mathscr{D}^\perp$-valued tensor of type (0,2). From the Frobenius theorem, we have

LEMMA 1.5. *The distribution $\mathscr{D}$ is integrable if and only if $\overset{\circ}{h}$ is symmetric on $\mathscr{D}\times\mathscr{D}$.*

Let $X_1,\cdots,X_r$ be an orthonormal basis of $\mathscr{D}$. We put

$$\overset{\circ}{H} = \frac{1}{r}\,\mathrm{trace}\,\overset{\circ}{h} = \frac{1}{r}\sum_i \overset{\circ}{h}(X_i,X_i). \tag{1.6}$$

Then $\overset{\circ}{H}$ is a well-defined vector field on N (up to sign). We call $\overset{\circ}{H}$ the *mean-curvature vector* of $\mathscr{D}$. If $\overset{\circ}{H} = 0$, $\mathscr{D}$ is called a *minimal distribution*. If $\overset{\circ}{h} = 0$, $\mathscr{D}$ is called a *totally geodesic distribution*.

THEOREM 1.6. *If N is a CR-submanifold of a Kaehler manifold $\tilde{M}$, then*

(a) *the holomorphic distribution $\mathscr{H}$ is always minimal and*

(b) *$\mathscr{H}$ is totally geodesic if and only if $\mathscr{H}$ is integrable and its leaves are totally geodesic in N.*

PROOF. For any vector fields X in $\mathscr{H}$ and Z in $\mathscr{H}^\perp$, Lemma 1.1 gives

$$\langle Z, \nabla_X X\rangle = \langle A_{JZ}X, JX\rangle\,. \tag{1.7}$$

Thus we find

$$(1.8) \qquad \langle Z,\nabla_{JX}JX\rangle = -\langle A_{JZ}X,JX\rangle .$$

From (1.7) and (1.8) we get $\langle\nabla_X X + \nabla_{JX}JX, Z\rangle = 0$. This implies (a). Statement (b) follows from (1.5) and Lemma 1.5. (Q.E.D.)

For a CR-submanifold N of a Kaehler manifold $\tilde{M}$, we choose $e_1,\cdots,e_h,Je_1,\cdots,Je_h$ an orthonormal local frame of $\mathscr{H}$. We let $\omega^1,\cdots,\omega^h,\cdots,\omega^{2h}$ be the $2h$ 1-forms on N satisfying $\omega^i(Z) = 0$ and $\omega^i(e_j) = \delta^i_j$ for $i,j=1,\cdots,2h$ and Z in $\mathscr{H}^\perp$, where $e_{h+j} = Je_j$. Then

$$(1.9) \qquad \omega = \omega^1\wedge\cdots\wedge\omega^{2h}$$

defines a $2h$-form on N. This form is a well-defined global form on N since $\mathscr{H}$ is orientable. It has been proved (see, for example, Tachibana 1973) that this form is closed if $\mathscr{H}$ is minimal and its orthogonal complemental distribution $\mathscr{H}^\perp$ is integrable. In our case, Theorems 1.2 and 1.6 say that these conditions hold, automatically. Thus we have the following

THEOREM 1.7. *For each closed CR-submanifold N of a Kaehler manifold $\tilde{M}$, there is a canonical deRham cohomology class given by*

$$(1.10) \qquad c(N) \equiv [\omega] \in H^{2h}(N;\mathbf{R}), \qquad h = \dim_{\mathbf{C}}\mathscr{H}.$$

Moreover, this cohomology class is nontrivial if $\mathscr{H}$ is integrable and $\mathscr{H}^\perp$ is minimal.

The last statement follows from the fact that if $\mathscr{H}$ is integrable and $\mathscr{H}^\perp$ is minimal, the form ω is harmonic. (see, e.g., Tachibana 1973). By using Theorem 1.7 we have the following

THEOREM 1.8. *Let N be a closed CR-submanifold of a Kaehler manifold $\tilde{M}$. If*

$$(1.11) \qquad H^{2k}(N;\mathbf{R}) = 0$$

for some $k \leqq \dim_{\mathbf{C}}\mathscr{H}$. Then either $\mathscr{H}$ is not integrable or $\mathscr{H}^\perp$ is not minimal.

PROOF. We choose a local field of orthonormal frame $e_1,\cdots,e_h,e_{h+1},\cdots,e_{h+p},e_{h+p+1},\cdots,e_m,Je_1,\cdots,Je_m$ in $\tilde{M}$ in such a way that, restricted to

N, $e_1,\cdots,e_h,Je_1,\cdots,Je_h$ are in $\mathscr{H}$ and $e_{h+1},\cdots,e_{h+p}$ are in $\mathscr{H}^\perp$. We denote by $\omega^1,\cdots,\omega^m,\omega^{1*},\cdots,\omega^{m*}$ the dual frame of $e_1,\cdots,e_m,Je_1,\cdots,Je_m$. We put $\theta^A=\omega^A+\sqrt{-1}\,\omega^{A*}$ and $\overline{\theta}^A=\omega^A-\sqrt{-1}\,\omega^{A*}$, $A=1,\cdots,m$. Restricting θ^A's and $\overline{\theta}^A$'s to N, we have $\theta^\alpha=\overline{\theta}^\alpha=\omega^\alpha$ for $\alpha=h+1,\cdots,h+p$, and $\theta^r=\overline{\theta}^r=0$ for $r=h+p+1,\cdots,m$. The fundamental form $\tilde{\Omega}$ of $\tilde{M}$ is given by $\tilde{\Omega}=\frac{\sqrt{-1}}{2}\sum\theta^A\wedge\overline{\theta}^A$. Thus the induced fundamental form $\Omega=i^*\tilde{\Omega}=\frac{\sqrt{-1}}{2}\sum_{i=1}^{h}\theta^i\wedge\overline{\theta}^i$. From this we find that the cononical class $c(N)$ and the fundamental class $[\Omega]$ of N satisfy

$$[\Omega]^h=(-1)^h(h!)c(N). \tag{1.12}$$

If $\mathscr{H}$ is integrable and the leaves of $\mathscr{H}^\perp$ are minimal in N, Theorem 1.7 and (1.12) imply $H^{2k}(N;\mathbf{R})\neq 0$ for $k=1,2,\cdots,h$. (Q.E.D.)

In Chen and Ogiue (1974a) the following result is proved.

PROPOSITION 1.9. *A submanifold N of a complex-space-form $\tilde{M}(c)$, $c\neq 0$, is a complex submanifold or a totally real submanifold if and only if $\tilde{R}(X,Y)TN\subset TN$ for X,Y in TN.*

For CR-submanifolds we have the following.

PROPOSITION 1.10 (Blair and Chen, 1979). *A generic submanifold N of a complex-space-form $\tilde{M}(c)$, $c\neq 0$, is a CR-submanifold if and only if $\tilde{R}(\mathscr{H},\mathscr{H};\mathscr{H}^\perp,\mathscr{H}^\perp)=\{0\}$.*

This proposition follows from formula (IV, 3.9).

§2. CR-PRODUCTS OF KAEHLER MANIFOLDS

DEFINITION 2.1. A CR-submanifold of a Kaehler manifold is called a *CR-product* if it is a generic product.

PROPOSITION 2.1. *A CR-submanifold of a Kaehler manifold is a CR-product if and only if P is parallel, i.e., $\nabla P\equiv 0$.*

PROOF. If $\nabla P\equiv 0$, equation (IV. 2.2) yields

$$th(U,V)=-A_{FV}U. \tag{2.1}$$

In particular, if $X \in \mathscr{H}$, then $FX = 0$. Thus we have

$$(2.2) \qquad A_{JZ}X = 0$$

for any $X \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$ Therefore, by Proposition IV. 2.2 and 1.3, $\mathscr{H}$ is integrable and each leaf $N^{\perp}$ of $\mathscr{H}^{\perp}$ is totally geodesic in N. Let N^T denote the leaf of $\mathscr{H}$. Then for $X, Y \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$, (2.2) and Lemma 1.1 give

$$0 = \langle A_{JZ}Y, X\rangle = \langle \nabla_Y Z, JX\rangle = -\langle Z, \nabla_Y JX\rangle.$$

From this we conclude that N is a CR-product of $\tilde{M}$.

Conversely, if N is a CR-product, we have $\nabla_U Y \in \mathscr{H}$ for $Y \in \mathscr{H}$ and $U \in TN$. Thus, we may obtain $Jh(U,Y) = h(U,JY)$. From this we may prove $(\nabla_U P)Y = 0$. Similarly, from $\nabla_U Z \in \mathscr{H}^{\perp}$ for $Z \in \mathscr{H}^{\perp}$, we may obtain $(\nabla_U P)Z = 0$.

Bejancu informed me that he also obtained this proposition independently. In Bejancu, Kon and Yano (1980), they proved that if $\nabla P = 0$ and N is anti-holomorphic, then N is a CR-product. By an *anti-holomorphic submanifold*, we mean a CR-submanifold with $J\mathscr{H}^{\perp} = T^{\perp}N$.

By using Proposition IV, 4.2 and Theorem IV, 4.6, we have

PROPOSITION 2.2. *If N is a minimal CR-product of $\mathbf{C}P^m$, then the scalar curvature r of N satisfies*

$$(2.3) \qquad r \leqq 4h^2 + 4h + p^2 - p \ .$$

The equality holds if and only if (a) N lies in a totally geodesic $\mathbf{C}P^{h+p+hp}$ of $\mathbf{C}P^m$, (b) locally, N is the Riemannian product of a totally geodesic, complex submanifold $\mathbf{C}P^h$ and a totally geodesic, totally real submanifold $\mathbf{R}P^p$ and (c) the immersion in induced from the Segre imbedding.

PROOF. Since N is a minimal CR-product, the Ricci tensor S of N satisfies

$$S(X,X) = (2h + p + 2)\|X\|^2 - \sum_r \|A_{\xi_r} X\|^2, \qquad X \in \mathscr{H},$$

$$S(Z,Z) = 2h + p - 1 - \sum_r \|A_{\xi_r} Z\|^2, \qquad Z \in \mathscr{H}^{\perp},$$

from these we obtain $r = 4h^2 + 4h + p^2 - p - \|h\|^2 + 4hp$. Thus the proposition follows from Theorem IV, 4.6. (Q.E.D.)

PROPOSITION 2.3. *Let N be a CR-product of a non-positively curved Kaehler manifold $\tilde{M}$. If N is anti-holomorphic, then*

(*a*) *the Ricci tensors of $\tilde{M}$ and N^T satisfy*

$$\tilde{S}(X,Y) = S^T(X,Y) \quad \text{for } X,Y \text{ tangent to } N^T \text{ and} \tag{2.4}$$

(*b*) *N^T is totally geodesic in $\tilde{M}$*

PROOF. Under the hypothesis, Proposition IV. 4.2 implies $\tilde{K}(X,Z) = \tilde{K}(X,JZ) = \|h(X,Z)\| = 0$ for $X \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$, where $\tilde{K}$ denotes the the sectional curvature of $\tilde{M}$. On the other hand,since N is anti-holomorphic and a CR-product, (2.2) gives $h(\mathscr{H},\mathscr{H}) = 0$. Thus N^T is totally geodesic in $\tilde{M}$. From this we find $\tilde{K}(X,Y) = K^T(X,Y)$ for $X,Y \in TN^T$. Since $\tilde{K}(X,Z)=\tilde{K}(X,JZ)=0$, we have (2.4). (Q.E.D.)

By using Proposition 2.3 and an argument which we used in Chen (1978b), we may prove the following.

THEOREM 2.4. *Let $\tilde{M}$ be a Hermitian symmetric space of non-compact type and N a complete CR-product in $\tilde{M}$. If N is anti-holomorphic, then*

(1) *N^T is a Hermitian symmetric space of non-compact type,*

(2) *there is a Hermitian symmetric space $M^{\perp}$ of non-compact type such that*

(2.1) *$\tilde{M}$ is the Riemannian product of N^T and $M^{\perp}$ and*

(2.2) *$N^{\perp}$ is a totally real submanifold of $M^{\perp}$.*

THEOREM 2.5. *Let N be a proper CR-product of an irreducible Hermitian symmetric space of non-compact type. Then N is not anti-holomorphic.*

REMARK 2.1. The rank 2 non-compact irreducible Hermitian symmetric space $SU(2,m)/S(U_2 \times U_m)$ admits a proper CR-product N for any $h = \dim_{\mathbf{C}} \mathscr{H}$ satisfying $0 < h < m$.

For the tangent-bundle-valued 1-form t defined in (IV, 1.3) if we put, for vector fields U in TN and ξ in $T^{\perp}N$,

$$(\nabla_U t)\xi = \nabla_U(t\xi) - tD_U\xi, \tag{2.5}$$

then we have $(\nabla_U t)\xi = A_{f\xi}U - PA_{\xi}U$. Therefore, $\nabla t \equiv 0$ if and only if for any vector fields U,V in TN and ξ in $T^{\perp}N$, we have

$$\langle D_X h(JX,Z) - D_{JX} h(X,Z), JZ\rangle = -2\|h(X,Z)\|^2$$

by using a similar argument as given in (IV, 4.3). Thus we obtain (3.1).

This lemma was also obtained by F. Urbana in 1980 independently. As an application of this lemma we obtain

THEOREM 3.2. *Let N be a CR-submanifold of CP^m. If $\mathscr{H}$ is integrable and the leaves of $\mathscr{H}^\perp$ are totally geodesic in N, then* $m \geqq 2h+p$.

PROOF. From (3.1) we have $\|h(X,Z)\|^2 = 1+\|A_{JZ}X\|^2$ for unit vectors $X \in \mathscr{H}$ and $Z \in \mathscr{H}^\perp$. Thus, for orthonormal basis $X_1,\cdots,X_{2h}$ of $\mathscr{H}$, we have

$$\langle h(X_i,Z), h(X_j,Z)\rangle = \langle A_{JZ}X_i, A_{JZ}X_j\rangle \qquad i \neq j. \tag{3.2}$$

For a given Z in $\mathscr{H}^\perp$, let $\bar{X}_1,\cdots,\bar{X}_{2h}$ be the eigenvectors of A_{JZ}. Then (3.2) gives $\langle h(\bar{X}_i,Z), h(\bar{X}_j,Z)\rangle = 0$, $i \neq j$. Since $\|h(X,Z)\| \geqq 1$ and $\langle h(X,Z), J\mathscr{H}^\perp\rangle = 0$, we obtain the theorem.

DEFINITION 3.1. A CR-submanifold N of a Kaehler manifold $\tilde{M}$ is said to be *mixed foliate* if (1) $\mathscr{H}$ is integrable and (2) N is mixed totally geodesic, i.e., $h(\mathscr{H},\mathscr{H}^\perp) = 0$.

PROPOSITION 3.3. *Let N be a mixed foliate CR-submanifold of a Kaehler manifold $\tilde{M}$. For unit vectors $X \in \mathscr{H}$, $Z \in \mathscr{H}^\perp$, we have*

$$\tilde{H}_B(X,Z) = -2\|A_{JZ}X\|^2. \tag{3.3}$$

This proposition follows immediately from Lemma 3.1. From this we obtain the following.

THEOREM 3.4. *If $\tilde{M}$ is a Kaehler manifold of positive holomorphic bisectional curvature, then $\tilde{M}$ admits no mixed foliate proper CR-submanifolds.*

From this we obtain immediately the following.

COROLLARY 3.5 (Bejancu, Kon and Yano 1980). *CP^m admits no mixed foliate proper CR-submanifolds.*

REMARK 3.1. The geodesic sphere $G_\varepsilon(p)$ of CP^m is a mixed totally geodesic

CR-hypersurface, but its holomorphic distribution $\mathscr{H}$ is not integrable.

THEOREM 3.6. *A CR-submanifold of* $\mathbf{C}^m$ *is mixed foliate if and only if it is a CR-product.*

This theorem follows from Proposition 3.3 and Theorem 2.1. For mixed foliate CR-submanifolds in a complex hyperbolic space $H^m = \tilde{M}^m(-4)$, we have the following.

THEOREM 3.7. *If* N *is a mixed foliate proper CR-submanifold of* H^m, *then*

(a) *each leaf* N^T *of* $\mathscr{H}$ *lies in a complex* $(h+p)$*-dimensional totally geodesic complex submanifold* H^{h+p} *of* H^m,

(b) *each leaf* N^T *is an Einstein-Kaehler submanifold of* H^{h+p} *with Ricci tensor given by* $S^T = -2(h+p+1)g$,

(c) $h+1 \geqq p \geqq 2$; $h \geqq 2$,

(d) *the leaves of* $\mathscr{H}^{\perp}$ *are totally geodesic in* N, *and*

(e) $\dim_{\mathbf{R}} N \geqq 6$.

For the proof of this theorem, see Chen (1980d). For Hermitian symmetric spaces of compact type we have the following.

THEOREM 3.8. *Let* N *be a mixed foliate CR-submanifold of a Hermitian symmetric space of compact type. Then*

(1) N *is a CR-product, and*

(2) $\tilde{K}(X,Z) = \tilde{K}(X,JZ) = 0$ *for unit* $X \in \mathscr{H}$ *and* $Z \in \mathscr{H}^{\perp}$.

This theorem follows from Proposition 3.3. The compact irreducible Hermitian symmetric space $SU(2+m)/S(U_2 \times U_m)$ admits a mixed foliate CR-submanifold for any h, $0 < h < m$. From Theorem 3.8 we may also obtain the following.

THEOREM 3.9. *Let* N *be a complete mixed foliate anti-holomorphic submanifold of a Hermitian symmetric space* $\tilde{M}$ *of compact type. Then* $\tilde{M} = N^T \times M^{\perp}$ *where* N^T *and* $M^{\perp}$ *are Hermitian symmetric spaces of compact type. Moreover* $N = N^T \times N^{\perp}$ *where* $N^{\perp}$ *lies in* $M^{\perp}$ *as a totally real submanifold.*

DEFINITION 3.2. A CR-submanifold N of a complex-space-form $\tilde{M}(c)$ is said to have *semi-flat normal connection* if its normal curvature tensor R^D satisfies

(4.1) $$R^D(X,Y;\xi,\eta) = \frac{c}{2}\langle X,PY\rangle\langle J\xi,\eta\rangle$$

for vectors X,Y in TN and ξ,η in $T^{\perp}N$

For such submanifolds we have the following classification theorem. For its proof, see Chen (1980d).

THEOREM 3.10. *A CR-submanifold N of a complex-space-form $\tilde{M}^m(c)$, $c \neq 0$, has semi-flat normal connection if and only if N is one of the following submanifolds:*

(1) a totally geodesic complex submanifold of $\tilde{M}^m(c)$,

(2) a flat totally real submanifold of a totally geodesic submanifold $\tilde{M}^p(c)$ of $\tilde{M}^m(c)$,

(3) a proper anti-holomorphic submanifold with flat normal connection in a totally geodesic complex submanifold $\tilde{M}^{h+p}(c)$ of $\tilde{M}^m(c)$,

(4) a space of positive constant sectional curvature immersed in a totally geodesic complex submanifold $\tilde{M}^{p+1}(c)$ of $\tilde{M}^m(c)$ as a totally real submanifold with flat normal connection.

Combining Proposition 2.6 and Lemma 3.1 we obtain immediately the following.

PROPOSITION 3.11. *For any CR-submanifold N of any Kaehler manifold $\tilde{M}$, if $\nabla F \equiv 0$, we have $\tilde{H}_B(\mathscr{H}^{\perp},\mathscr{H}) = \{0\}$.*

§4. STABILITY OF TOTALLY REAL SUBMANIFOLDS

Let N be a p-dimensional closed totally real minimal submanifold of a complex p-dimensional Kaehler manifold $\tilde{M}$. Then, for any normal vector field u along N, we consider the normal variation of N in $\tilde{M}$ induced from u. By Theorem I,4.1, we have

(4.1) $$\mathscr{V}''(u) = \int_N \{\|Du\|^2 - \sum_{i=1}^{p} \tilde{R}(e_i,u;u,e_i) - \|A_u\|^2\}dV,$$

where $e_1,\cdots,e_p$ is an orthonormal frame of TN.

Since N is totally real in $\tilde{M}$ with the smallest possible codimension, there is a tangent vector field X of N such that $JX = u$. Using $\tilde{\nabla}J = 0$, we have

(4.2) $$D_Y JX = J\nabla_Y X.$$

On the other hand, by the equation of Gauss, we find

$$(4.3)\qquad \sum \tilde{R}(e_i,u;u,e_i) = \sum \tilde{R}(Je_i,X;X,Je_i)$$
$$= \tilde{S}(X,X) - \sum \tilde{R}(e_i,X;X,e_i)$$
$$= \tilde{S}(X,X) - S(X,X) - \sum \|h(X,e_i)\|^2.$$

Moreover, from (1.2) of Lemma 1.1, we have $\|A_u\|^2 = \sum \|h(e_i,X)\|^2$. Thus (4.1), (4.2) and (4.3) imply

$$(4.4)\qquad \mathcal{V}''(u) = \int_N \{\|\nabla X\|^2 - \tilde{S}(X,X) + S(X,X)\}dV.$$

We put

$$W = \nabla_X X + (div\ X)X,$$

where $div\ X$ denotes the divergence of X. Let ξ be the 1-form associated with X. Then, by computing the divergence of W, we get (see, for example, Yano and Bochner (1953))

$$(4.5)\qquad 0 = \int_N (div\ W)dV = \int_N \{S(X,X) + \|\nabla X\|^2 - \frac{1}{2}\|d\xi\|^2 - (\delta\xi)^2\}dV.$$

Combining (4.4) and (4.5) we obtain

$$(4.6)\qquad \mathcal{V}''(u) = \int_M \{\frac{1}{2}\|d\xi\|^2 + (\delta\xi)^2 - \tilde{S}(X,X)\}dV.$$

If the Kaehler manifold $\tilde{M}$ has positive Ricci tensor $\tilde{S}$ and $H^1(N,\mathbf{R}) \neq 0$, then there is a harmonic 1-form β on N, Thus $d\beta = \delta\beta = 0$. Let $\bar{u} = JY$, where Y is the vector field on N associated with β. Then, for this normal vector field $\bar{u}$, we have $\mathcal{V}''(\bar{u}) < 0$. Thus, N is unstable.

If the Ricci tensor $\tilde{S}$ of $\tilde{M}$ is nonpositive, then (4.6) shows that $\mathcal{V}''(u) \geq 0$ for any normal vector field on N. Therefore, N is always stable. Consequently, we have the following result.

THEOREM 4.1 (Chen, Leung, and Nagano 1980). *Let N be a closed, totally real, minimal submanifold of a Kaehler manifold $\tilde{M}$ with* $\dim_{\mathbf{R}} N = \dim_{\mathbf{C}} \tilde{M}$.

(1) *If $\tilde{M}$ has positive Ricci tensor and $H^1(N;\mathbf{R}) \neq 0$, then N is unstable, and*

(2) *If $\tilde{M}$ has nonpositive Ricci tensor, then N is stable.*

There exist many totally real submanifolds. In fact, we have the following.

THEOREM 4.2 (Chen and Nagano 1978). *Let B be a totally geodesic submanifold of a locally Hermitian symmetric space. If B is irreducible and non-Hermitian, then B is totally real.*

REMARK 4.1. In Chen, Leung and Nagano (1980) a general method to determine stability of totally geodesic submanifolds of symmetric spaces is established by using representation theory. Moreover, they have used this method to determine stability of the basic totally geodesic submanifolds M_+, M_- of the next chapter.

Chapter VI: (M_+, M_-)-METHOD AND ITS APPLICATIONS

In this chapter we will briefly discuss the (M_+,M_-)-method of Chen and Nagano (1978) and indicate some of its applications. *Results obtained in this chapter are joint works with Professor Tadashi Nagano unless mentioned otherwise.* For the details, please refer to Chen and Nagano (1977, 1978) and forthcoming parts of this series.

§1. TOTALLY GEODESIC SUBMANIFOLDS AND (M_+,M_-)-METHOD

A submanifold B of a Riemannian manifold M is a totally geodesic submanifold if its second fundamental form vanishes. It is well-known that B is totally geodesic in M if and only if geodesics of B are geodesics of M. In other words, B is totally geodesic in M if and only if bridges and tunnels are not need if one wants to travel in shortest way between any two nearby points in B. The following problems are fundamental.

PROBLEM 1.1. For a given Riemannian manifold M, find all totally geodesic submanifolds of M.

PROBLEM 1.2. Give two Riemannian manifolds, when there is a totally geodesic immersion from one into the other?

PROBLEM 1.3. Suppose the space we live is the ordinary n-sphere S^n. When our space S^n can be realized in a Riemannian manifold M as a totally geodesic submanifold?

It is known for a longtime that totally geodesic submanifolds of E^m and S^m are linear subspaces and great spheres, respectively. It is somewhat surprising that totally geodesic submanifolds of all rank one symmetric spaces are not classified until 1963 by Wolf.

Concerning Problem 1.1 for symmetric spaces of higher rank, Chen and Lue (1975) classified totally geodesic surfaces of the complex quadric $Q_m = SO(m+2)/SO(2)\times SO(m)$. The complete classification of totally geodesic submanifolds of Q_m was done in Chen and Nagano (1977). However the methods used in the works of Wolf (1963), Chen and Lue (1975), and Chen and Nagano (1977) are not unified and difficult. So, we introduced the (M_+,M_-)-method to solve Problems 1.1, 1.2, and 1.3. However, due to the simplicity of this method, we may also apply this new method to solve some other problems in mathematics.

§2. GENERAL THEORY

An isometry s of a Riemannian manifold is called involutive if its iterate $s^2 = s \cdot s$ is the identity map. A Riemannian manifold M is called a *symmetric space* if, for each point q of M, there exists an involutive isometry s_q of M such that q is an isolated fixed point of s_q. We call such s_q the *symmetry* of M at q. We denote by G_M or simply G the closure of the group of isometries generated by $\{s_q \mid q \in M\}$ in the compact-open topology. Then G is a Lie group which acts transitively on the symmetric space M; hence the typical isotropy subgroup H, say at O, is compact and $M = G/H$. (For the general theory of symmetric spaces, see Kobayashi and Nomizu (1963, vol.II) and Helgason (1978)).

For each closed smooth geodesic of a compact symmetric space M, a *circle* for short, c through O, we consider the antipodal point p of O on c. We denote by $M_+(p)$ the orbit $H(p)$. We have the following.

LEMMA 2.1. *$M_+(p)$ is a totally geodesic submanifold of the symmetric space $M = G/H$.*

LEMMA 2.2 *The normal space $T_p^{\perp} M_+(p)$ of $M_+(p)$ at p in M is the tangent space of a complete connected totally geodesic submanifold $M_-(p)$.*

It is well-known that every complete totally geodesic submanifold of a symmetric space is a symmetric space. For a symmetric space M the dimension of a maximal flat totally geodesic submanifold of M is called the *rank* of M, denoted by $rk\ M$. From the equation of Gauss, it follows that $rk\ B \leqq rk\ M$ if B is totally geodesic in M.

LEMMA 2.3. *The symmetric space $M_-(p)$ has the same rank as M.*

LEMMA 2.4. $M_+(p) = H(p)$ *is connected.*

For each point p in M, we denote by σ_p the involution of G which corresponds to s_p, i.e., $\sigma_p(g) = s_p\, g\, s_p^{-1}$. Then σ_o leaves H invariat. Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of H and G, respectively. Then it is known that $\mathfrak{h}$ is the eigenspace with eigenvalue 1 of the involutive automorphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ induced from $\sigma_o : G \longrightarrow G$. Let $\mathfrak{m}$ denote the eigenspace of σ with eigenvalue -1. Then we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, called the *Cartan decomposition*, Moreover, $\mathfrak{m}$ can be regarded as the tangent space of M at o.

THEOREM 2.5. *Let O be a point fixed by H in a compact symmetric space $M = G/H$. Then*

(i) *the fixed point set $F(s_O, M) = \{q \in M | s(q) = q\}$ less O is the set of all points p which are antipodal points of the circles passing through O,*

(ii) *to each such point p there corresponds an inner involutive automorphism $ad(b) \neq 1$ of G, $b \in H \cap \exp \mathfrak{m}$ such that*

(ii-a) *$M_+(p) = H(p)$ is a covering space of $H/F(ad(b), H)$,*

(ii-b) *$M_-(p)$, the connected component containing p of $F(b, M)$, is locally isometric with $F(ad(b), G)/F(ad(b), H)$ and*

(ii-c) *the tangent spaces to the totally geodesic submanifolds $M_+(p)$ and $M_-(p)$ at p are the orthogonal complements of each others in T_pM and finally,*

(iii) *as to the ranks, one has $rk\, M_-(p) = rk\, M$ and if H is connected,* $\mathrm{rk}\, F(ad(b), H) = \mathrm{rk}\, H$.

Given a pair of antipodal points (O, p) on a circle in a compact symmetric space M, we have the system $(O, p, M_+(p), M_-(p))$ as considered above. The isometry group $G = G_M$ acts on the set of all such systems in the natural fashion. We denote the orbit set by $P(M)$. Then $P(M)$ is a finite set and the cardinal number $\#P(M)$ of $P(M)$ gives us a *global invariant*. This number gives us many information about M and it plays an important rôle in the theory of symmetric spaces.

THEOREM 2.6. *For a compact symmetric space M we have*

$$\#P(M) \leqq 2^{rk\, M} - 1. \tag{2.1}$$

If, in addition, M is irreducible, we have

$$\#P(M) \leqq rk\, M, \tag{2.2}$$

A Riemannian manifold M is called a *two-point homogeneous space* if for any two pairs of points (p, q) and (p', q') of M with the same distance, there is an isometry of M which carries one pair into the other. It is known that two-point homogeneous spaces are E^n or rank one symmetric spaces (Wang 1952; Tits 1955). It follows from Theorem 2.6 the following.

COROLLARY 2.7. *An irreducible compact symmetric space M is a two-point homogeneous space if and only if* $\#P(M) = 2^{rk\,M} - 1$.

For a compact symmetric space M of rank one, any two pairs of antipodal points have the same distance and, moreover, there is an isometry of M which carries one pair into the other. In the following we call a compact Riemannian manifold M an *equal-antipodal-pair space* if any two pairs of antipodal points have the same property. A compact symmetric space M is an equal-antipodal-pair space if and only if $\#P(M) = 1$.

We have the following result for equal-antipodal pair spaces.

PROPOSITION 2.8. *A compact symmetric space M is an equal-antipodal-pair space if and only if M is either a rank one symmetric space or one of the following spaces* G_2, GI, *and* EIV .

Any isometric totally geodesic imbedding $f : B \longrightarrow M$ gives rise to a mapping $P(f) : P(B) \longrightarrow P(M)$ induced by the mapping carring $(o,p,B_+(p),B_-(p))$ into $(f(o),f(p),M_+(f(p)),M_-(f(p)))$. $P(f)$ is well-defined since every isometry ϕ in G_B "extends" to an isometry ϕ' in G_M so that we have $f \cdot \phi = \phi' \cdot f$. It is easy to see from Theorem 2.5 that $f(B_+(p)) \subset M_+(f(p))$ and $f(B_-(p)) \subset M_-(f(p))$ as totally geodesic submanifolds. The later one follows from (ii-c). Since this is an important fact, we express it by saying that $P(f)$ is a pairwise totally geodesic immersion. We record this as the next Theorem.

THEOREM 2.9. *Every isometric totally geodesic imbedding $f : B \longrightarrow M$ of a compact symmetric space into another induces a pairwise totally geodesic immersion* $P(f) : P(B) \longrightarrow P(M)$.

We also have the following Theorems.

THEOREM 2.10. *If a totally geodesic submanifold B of a compact symmetric space M has the same rank as M, then*

(i) $P(f) : P(B) \longrightarrow P(M)$ *is surjective, where f is the inclusion. In particular, we have* $\#P(B) \geqq \#P(M)$,

(ii) *the Weyl group $W(B)$ of B is a subgroup of $W(M)$, and*

(iii) *if the Weyl group $W(B)$ is isomorphic with $W(M)$ by the natural homomorphism, then $P(f)$ is bijective.*

THEOREM 2.11. *M is globally determined by $P(M)$, i.e., the set of the global isomorphism classes of compact irreducible symmetric spaces is in one-to-one correspondence with the set of the corresponding $P(M)$.*

It should be noted that the Satake diagram and the Dynkin diagram for symmetric spaces do not distinguish symmetric spaces globally, for example, in their diagrams the sphere S^n and the real projective space $\mathbf{R}P^n$ have the same diagram. However, $P(S^n)$ and $P(\mathbf{R}P^n)$ are quite different as we seen in Table IV. For the Satake deagram and Dynkin diagram, see, for instances, Araki (1962) and Helgason (1978).

In the following we will discuss how the set of the pairs $(M_+(p), M_-(p))$ is related to the corresponding set of any other locally isometric space. Thus we assume in addition that $M = G/H$ is 1-connected and G is the connected isometry group (which acts on M effectively). Then H is the identity component of $F(\sigma_0, G)$. This $\sigma = \sigma_0$ can act on the adjoint group $ad\ G = G/C$, where C is the center of G. And we obtain another symmetric space $ad\ G/F(\sigma, ad\ G)$, denoted by M^* throughout. It is known that M^* is characterized by the property that every locally isometric space to M is a covering Riemannian manifold of M^*. Thus, there is a locally isometric projection $\pi : M \longrightarrow M^*$. Let c^* be a circle in M^* which passes through the origin $0^* = \pi(0)$. Let c be the lift of c^* which starts at 0, then the k-time extension kc of c will be a circle if k denotes the order of the homotopy class of c^* in $\pi_1(M^*)$. We have

THEOREM 2.12. *If k is even, the antipodal point q of 0 on kc is a fixed point of H so that $(M_+(q), M_-(q)) = (\{q\}, M)$. If k is odd, q and the pair $(M_+(q), M_-(q))$ project to the antipodal point p^* and $(M_+^*(p^*), M_-^*(p^*))$, respectively.*

PROPOSITION 2.13. *$M_+^*(p^*)$ cannot be a singleton $\{p^*\}$.*

By using Theorem 2.5 and Proposition 2.13 we have the following

THEOREM 2.14. *Any symmetric space M of dimension $\geqq 2$ contains a totally geodesic submanifold B satisfies*

$$\frac{1}{2} \dim M \leqq \dim B < \dim M.$$

This estimate is best possible because the maximal dimension of totally geodesic submanifolds of the 16-dimensional Cayley plane FII is 8.

§3. DETERMINATION OF THE PAIRS

We will determine $P(M)$ in this section. In view of §2, we may assume $M = M^* = ad\ G/F(ad\ s_0, G)$ and G is simple. The results will be listed in Tables I, II and III, in which only the local isomorphism classes of symmetric spaces in each member $[(0, p, M_+(p), M_-(p))]$ of $P(M)$ will be indicated. The classification can be accomplished by means of Theorem 2.5 as principle. In practice, additional use of known facts on symmetric spaces will be helpful. A tool among others we used to find and crosschecked is the root system $R(M)$ of M. Theorem 2.5 tells us that one obtains those pairs in a similar way to the algorithm of Borel and Siebenthal (1949) for finding the maximal subalgebras of the same rank. Namely, one expresses the highest root as a linear combination $\sum m_i \alpha_i$ of the simple roots α_i. Pick up the vertices in the extended Dynkin diagram which correspond to $m_i = 2$ or the vertices in the Dynkin diagram which correspond to $m_i = 1$. Then do as Borel-Siebenthal say. When $R(M)$ is not reduced, i.e., the diagram is of BC-type, one first removes every root λ such that 2λ is also a root (this already gives us one pair), before one applies the above method. The multiplicities of the roots are determined in each case.

TABLE I

M	M_+	M_-	$\#P(M)$
$A_n\ SU(n+1), n \geq 1$		$S(U(k) \times U(n+1-k))$	n
$B_n\ SO(2n+1), n \geq 2$		$SO(k) \times SO(2n+1-k)$	n
$C_n\ Sp(n), n \geq 3$		$U(n), Sp(k) \times Sp(n-k)$	n
$D_n\ SO(2n), n \geq 4$	$M_+ = M/M_-$	$U(n), SO(2k) \times SO(2n-2k)$	n
$E_6\ \ E_6$		$SU(6) \times SU(2), \mathrm{Spin}(10) \times T$	2
$E_7\ \ E_7$		$SU(8), SO(12) \times SU(2), E_6 \times T$	3
$E_8\ \ E_8$		$SO(16), E_7 \times SU(2)$	2
$F_4\ \ F_4$		$\mathrm{Spin}(9), Sp(3) \times SU(2)$	2
$G_2\ \ G_2$		$SU(2) \times SU(2)$	1

TABLE II

M		M_+	M_-	Remark	$\#P(M)$
$AI(n)$	$SU(n)/SO(n)$	$G^{\mathbf{R}}(k, n-k)$	$T \times AI(k) \times AI(n-k)$	$0 < k \leqq n-1$	$n-1$
$AII(n)$	$SU(2n)/Sp(n)$	$G^H(k, n-k)$	$T \times AII(k) \times AII(n-k)$	$0 < k \leqq n-1$	$n-1$
$AIII$	$G^C(p,q)$	$G^C(h, p-h) \times G^C(h, q-h)$	$G^C(h,h) \times G^C(p-h, q-h)$	$0 < h \leqq p \leqq q$	p
		$U(p)$	$U(p)$	(*)	
BDI	$G^{\mathbf{R}}(p,q)$	$G^R(h, p-h) \times G^R(h, q-h)$	$G^R(h,h) \times G^R(p-h, q-h)$	$0 < h \leqq p \leqq q$	p
		$SO(p)$	$T \times AI(p)$	(*)	
$CI(n)$	$Sp(n)/U(n)$	$G^C(k, n-k)$	$CI(k) \times CI(n-k)$	$0 < k \leqq n-1$	n
		$T \times AI(n)$	$T \times AI(n)$		
CII	$G^H(p,q)$	$G^H(h, p-h) \times G^H(h, q-h)$	$G^H(h,h) \times G^H(p-h, q-h)$	$0 < h \leq p \leq q$	p
		$Sp(p)$	$T \times AII(p)$	(*)	
$DIII(n)$	$SO(2n)/U(n)$	$G^C(k, n-k)$	$DIII(k) \times DIII(n-k)$	$0 < k < n$	$[\frac{n}{2}]$
				k = even	
		$T \times AII(\frac{n}{2})$	$T \times AII(\frac{n}{2})$	n = even	
				on M^* only	

(*) There should replace the pairs for $h = p$ where $p = q$.

TABLE III

M	M_+	M_-	Remark	$\#P(M)$
EI	$G^H(2,2)$	$G^R(5,5)\times T$		2
	CI(4)	AI(6) $\times S^2$		
EII	$G^C(4,2)$	$G^R(6,4)$		2
	$S^2\times G^C(3,3)$	$S^2\times G^C(3,3)$		
EIII	DIII(5)	$S^2\times G^C(5,1)$		2
	$G^R(8,2)$	$G^R(8,2)$		
EIV	FII	$T\times S^9$		1
EV	$G^C(4,4)$	$G^R(6,6)\times S^2$		3
	AII(4)	$T\times E$I	on M^* only	
	AI(8)	AI(8)	on M^* only	
EVI	$G^R(8,4)$	$G^R(8,4)$		2
	$S^2\times D$III(6)	$S^2\times D$III(6)		
EVII	EIII	$S^2\times G^R(10,2)$		2
	$T\times E$IV	$T\times E$IV		
EVIII	$G^R(8,8)$	$G^R(8,8)$		2
	DIII(8)	$S^2\times E$V		
EIX	EIV	$G^R(12,4)$		2
	$S^2\times E$VII	$S^2\times E$VII		
FI	$S^2\times C$I(3)	$S^2\times C$I(3)		2
	$G^H(1,2)$	$G^R(5,4)$		
FII	S^8	S^8		1
GI	$S^2\times S^2$	$S^2\times S^2$		1

For compact symmetric spaces of rand one by using the informations on π_1, we have the following.

TABLE IV

M	M_+	M_-
S^n	{a point}	S^n
RP^n	RP^n	S^1
CP^n	CP^n	S^2
HP^n	HP^n	S^4
F	S^8	S^8

§4. APPLICATIONS

I. APPLICATION TO TOTALLY GEODESIC SUBMANIFOLDS

For compact Lie groups, the following results gives a complete answer to Problem 1.1.

THEOREM 4.1. *Let M be a compact Lie group. Then the local isomorphism classes of totally geodesic submanifold of M are exactly those of symmetric space* $B = G_B/H_B$ *such that* G_B *are subgroups of* $G_M = M \times M$.

For a general compact symmetric space M, if B is a complete totally geodesic submanifold of M, Theorem 2.9 tells us that for any pair (B_+, B_-) in $P(B)$, there is a pair (M_+, M_-) in $P(M)$ such that B_+ and B_- are totally geodesic in M_+ and M_-, respectively. By applying this argument to $B_+ \subset M_+$, $B_- \subset M_-, \cdots$, and so on, we obtain a sequence of totally geodesic submanifolds as follows which gives us a sequence of conditions for the original totally geodesic imbedding;

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & \cdots\cdots \\
 & & (B_+)_+ \subset (M_+)_+ \arrow[urr] \arrow[drr] & & \\
 & & & & \cdots\cdots \\
 & B_+ \subset M_+ \arrow[uur] \arrow[ddr] & & & \cdots\cdots \\
 & & (B_+)_- \subset (M_+)_- \arrow[urr] \arrow[drr] & & \\
B \subset M \arrow[uuur] \arrow[ddddr] & & & & \cdots\cdots \\
 & & & & \cdots\cdots \\
 & & (B_-)_+ \subset (M_-)_+ \arrow[urr] \arrow[drr] & & \\
 & & & & \cdots\cdots \\
 & B_- \subset M_- \arrow[uur] \arrow[ddr] & & & \cdots\cdots \\
 & & (B_-)_- \subset (M_-)_- \arrow[urr] \arrow[drr] & & \\
 & & & & \cdots\cdots
\end{tikzcd}$$

For example, by using Table IV and this argument to rank one symmetric spaces, we obtain the following results of Wolf (1963) very easily.

THEOREM 4.2. *The maximal totally geodesic submanifold of* S^n *is* S^{n-1}; *of* $\boldsymbol{R}P^n$ *is* $\boldsymbol{R}P^{n-1}$; *of* $\boldsymbol{C}P^n$ *are* $\boldsymbol{C}P^{n-1}$ *and* $\boldsymbol{R}P^n$; *of* $\boldsymbol{H}P^n$ *are* $\boldsymbol{H}P^{n-1}$ *and* $\boldsymbol{C}P^n$; *of* FII *are* $\boldsymbol{H}P^2$ *and* S^8.

In Chen and Nagano (1978), such method was used to obtain the classication of totally geodesic submanifolds of symmetric spaces of higher rank.

Concerning Problem 1.2, Theorem 2.10 provides an easy method to solve Problem 1.2 by using arithmetic or group theory. Just to give one simplest example, from Theorem 2.10, we conclude immediately that the 8-dimensional rank two symmetric space $GI = G_2/SU(2) \times SU(2)$ cannot be isometrically imbedded in any rank two Grassmann manifold M of any dimension as totally geodesic submanifold. This important fact follows simply from the following inequality: $\#P(GI) = 1 < 2 = \#P(M)$.

We would like to mention the following best possible result which follows easily from the (M_+, M_-)-method. Using induction argument on dimension and rank of B_+, B_-, M_+ and M_-, we have.

THEOREM 4.3. *Let M be an irreducible symmetric space and B a totally geodesic submanifold of M. Then*

(4.1) $\quad \operatorname{codim} B \geqq \operatorname{rk} M.$

Concerning Problem 1.3, we may again use the (M_+, M_-)-method to give the following answer.

THEOREM 4.4. *Let M be an irreducible symmetric space. Then an n-sphere can be isometrically immersed in M as a totally geodesic submanifold if and only if $n \leqq \lambda$, where λ is the integer given in the following table.*

TABLE V.

M	λ	M	λ
$AI(n), n \geqq 3$	$n-1$	EVI	8
$AII(n)$, $n = 3,4,5,6$	5,6,6,6	$EVII$	10
$n > 6$	$n-1$	$EVIII$	8
$G^C(1,q)$	$\max(2,q)$	EIX	12
$G^C(p,q), 2 \geqq p \geqq q$	$\max(4,q)$	FI	5
$G^R(1,q)$	q	FII	8
$G^R(p,q), 2 \geqq p \geqq q$	q	GI	2
$CI(n), n \geqq 3$	$n-1$	A_n	$\max(2,n)$
$G^H(p,q), p \leqq q$	$\max(4,q)$	$B_n, n \geqq 2$	2
$DIII(n), n = 4,5,6$	6	$C_n, n \geqq 3$	$\max(4,n-1)$
$n > 6$	$\max(3,n-1)$	$D_n, n \geqq 4$	$2n-1$
EI	5	E_6	9
EII	6	E_7	11
$EIII$	8	E_8	15
EIV	9	E_4	8
EV	7	G_2	3

Theorems 4.3 and 4.4 may have important impacts to the theory of submanifolds of symmetric spaces. The simplest application is to obtain the following.

THEOREM 4.5. *Every submanifold* N *with parallel second fundamental form in an irreducible symmetric space* M *satisfies* codim $N \geqq rk\ M$.

This theorem follows immediately from Theorem 4.3, the equation of Codazzi, and Lemma VII, 3.4. Another important application of Theorem 4.3 is to prove results of the following type.

THEOREM 4.6 (Chen and Verstraelen 1980). *Spheres, real and complex projective spaces and their noncompact duals are the only irreducible symmetric spaces in which one can find tubular hypersurfaces.*

By a *tubular hyperurface* we mean a hypersurface N on which the second fundamental tensor has a constant eigenvalue of multiplicity $\geqq \dim N - 1$.

Let N be a hypersurface of a Kaehler manifold $\tilde{M}$. Let ξ be a unit normal vector field of N in $\tilde{M}$ and η the 1-form on N associated with $J\xi$. Then N is called an η-hypersurface of $\tilde{M}$ if the second fundamental form h of N has the form; $h = (ag + b\eta \otimes \eta)\xi$, for some functions a and b on N. For η-hypersurfaces of a Hermitian symmetric space we have the following

THEOREM 4.7 (Chen and Verstraelen 1980). *The following statements hold*

(1) *The only irreducible Hermitian symmetric spaces which admit* η-*hypersurfaces are the complex projective spaces and their noncompact duals.*

(2) *A hypersurface of a complex projective space or its noncompact dual of dimension* $\geqq 4$ *is an* η-*hypersurface if and only if it is a geodesic hypersphere.*

Since the curvature tensor of a general symmetric space is very difficult to handle, the non-existence theorems of certain submanifolds are very difficult to obtain by using the standard methods, e.g., by using the fundamental equations. However, by applying the (M_+, M_-)-method, if one can obtain a conclusion about the existence of a totally geodesic submanifold (or more general submanifold) of certain codimension, Theorem 4.3 (or Theorem 4.5 or Theorem VII, 3.7) will automatically reduce the class of ambient spaces to a class of small ranks. This is the essential idea used in Chen and Verstraelen (1980). For the proof of Theorems 4.6 and 4.7 and further results in this direction, see Chen and Verstraelen (1980).

More applications of Theorems 4.3 and 4.4 will be given in Application II and Chapter VII.

II. APPLICATIONS TO LIE GROUPS

Since a closed subgroup of a compact Lie group M is a totally geodesic submanifold of M, the (M_+,M_-)-method provides a new method to the theory of Lie groups and their subgroups. We mention two applications in this direction. First, by using Theorem 4.3 we may determine the codimension of closed subgroups as follows.

THEOREM 4.7. *Let H be a closed subgroup of a compact simple Lie group G. Then* $\operatorname{codim} H \geqq \operatorname{rk} G$.

Second, we may use the new global invariant $\#P(M)$ to distinguish classical simple Lie groups and exceptional simple Lie groups.

III. APPLICATIONS TO TOPOLOGY AND OTHER SUBJECTS

It is well-known that many important global invariants, such as the Hirzebruch index, Lefschetz number, and spectrum of the Laplace-Betrami operator Δ, are closely related with the fixed point set of an isometry (see, e.g., Atiyah and Singer (1968), Atiyah and Bott (1968), Donnelly (1976), and Donnelly and Patodi (1977)).

For a symmetric space M, the simplest and most natural isometries are symmetries. In fact, from Theorem 2.5, we know that the union of $\{0\}$ and the $M_+'s$ is nothing but the fixed point set of s_0 on M. Since $M_+'s$ are lower dimensional manifolds and the $M_+'s$ of a compact symmetric space M have been completely determined, it is possible to apply the (M_+,M_-)-method to determine some global invariants of M. For example, by applying a result of Atiyah and Singer (1968), index for compact symmetric spaces can be determined by using this new method in a unified and simpler way. For some special symmetric spaces, index has been determined by various authors by using various different and difficult methods. For GI and FII see Borel and Hirzebruch (1958); for CP^n see Atiyah and Singer (1968); for $G^{\mathbf{C}}(p,q)$ and $G^H(p,q)$ see Mong (1975) and also Connolly and Nagano (1977); and for $G^R(p,q)$ see Shanahan (1979).

Chapter VII: TOTALLY UMBILICAL SUBMANIFOLDS

The main purpose of this chapter is to classify totally umbilical submanifolds of a locally symmetric space. The results are obtained in a series of papers given in Chen (1976a), (1977a,b,c), (1978a), (1979c,d,e) and (1980e,f,g) unless mentioned otherwise.

§1. FIXED POINT SET AND TWISTED PRODUCTS

As we already defined in Chapter I, a *totally umbilical submanifold* N of a Rimannian manifold M is a submanifold whose first fundamental form (i.e., the metric tensor) and second fundamental form are proportional. An ordinary hypersphere S^n of an affine $(n+1)$-space E^{n+1} of E^m is the best known example of totally umbilical submanifold. In fact, from differential geometric point of views, totally umbilical submanifolds are the simplest submanifolds next to totally geodesic submanifolds. In this section we will give many examples of such submanifolds by using conformal mappings and the twisted products.

For a Riemannian manifold M with metric tensor g, we denote by $C(M,g)$ the group of conformal transformations of (M,g) and by $C_0(M,g)$ the connected component of the identity. We denote by $I(M,g)$ the group of isometries of (M,g). According to Obata (1971), a subgroup G of $C(M,g)$ is called *essential* if there is no function $\rho > 0$ such that G is a subgroup of $I(M,g^*)$, $g^* = \rho^2 g$.

PROPOSITION 1.1. *Let (M,g) be a compact Riemannian manifold and $\phi \in C_0(M,g)$. Then every connected component of the fixed point set $F(\phi,M)$ is totally umbilical in M.*

PROOF. This Proposition is trivial if the dimension of M is $\leqq 2$. So we may assume that $\dim M = m \geqq 3$.

If the group $C_0(M,g)$ is inessential, there is a conformal change of metric given by $g^* = \rho^2 g$ such that $C_0(M,g) \subset I(M,g^*)$. From Lemma 2.1 of Chen and Nagano (1977), we see that every connected component of $F(\phi,M)$ is totally geodesic in (M,g^*). In particular, it is totally umbilical in (M,g^*). Thus, by (III, 3.11), we see that each component of $F(\phi,M)$ is totally umbilical in (M,g).

If $C_0(M,g)$ is essential, M is conformally diffeomorphic to an ordinary m-sphere (S^m, g_0) with the standard metric g_0. (Obata 1971). Thus, there is a conformal mapping f from (M,g) into (S^n,g_0). For any ϕ in $C_0(M,g)$ we consider the conformal transformation $\overline{\phi}=f\cdot\phi\cdot f^{-1}$ on (S^m,g_0). It is well-known that each connected component of $F(\overline{\phi},S^m)$ is a totally umbilical submanifold of (S^m,g_0). Since $F(\phi,M)=f^{-1}(F(\overline{\phi},S^m))$ and f is conformal, each connected component of $F(\phi,M)$ is totally umbilical in (M,g). (Q.E.D.)

REMARK 1.1. If $C_0(M,g)$ acts transitively on M, the assumption on compactness of M can be removed.

In the following, we shall simply denote a totally umbilical submanifold by a *t. u. submanifold*. In order to construct more examples of *t. u.* submanifolds with various properties, we recall the notion of *twisted products* introduced in Chen (1977c).

Let B and F be Riemannian manifolds and f a positive function on $B\times F$. Consider the product manifold $B\times F$ with projections $\pi : B\times F\longrightarrow B$ and $\pi' : B\times F\longrightarrow F$. The *twisted product* $M=B\times_f F$ is by definition the manifold $B\times F$ with the Riemannian structure given by

$$(1.1)\qquad ||X||^2=||\pi_*X||^2+(f(b,p))^2||\pi'_*X||^2$$

for vector X tangent to M at (b,p). If f depends on B only, twisted product becomes the so-called *warped product* of B and F in the sense of Bishop and O'Neill (1969). If B is a singleton $\{b\}$, the twisted product is nothing but a conformal change of metric on F. Furthermore, if we regard $\pi : B\times F\longrightarrow B$ as a submersion, then the fibers are conformally related with each other. This gives us a *conformal submersion*.

It follows trivially from (1.1) that each $\pi'^{-1}(p)$, $p\in F$, is totally geodesic in M and each $\pi^{-1}(b)$, $b\in B$ is totally umbilical in M (see, also, Adati (1963)).

By identifying the tangent space $T_{(b,p)}M$, $(b,p)\in B\times F$, with T_bB+T_pF, we say that a vector $X\in TM$ is *horizontal* if $X\in T_bB$ and X is *vertical* if $X\in T_pF$. For simplicity, we shall denote by Df the horizontal component of the gradient of f. We give the following result.

PROPOSITION 1.2. *Let* $M=B\times_f F$ *be a twisted product of* B *and* F. *Then*

(1) *for each* $b\in B$, *the fiber* $F_b=\{b\}\times F$ *is* t. u. *in* M *with* $-Df/f$ *as its mean curvature vector,*

(2) *fibers have constant mean curvature if and only if* $||D\log f||$ *is a function of* B, *and*

(3) *fibers have parallel mean curvature vector if and only if f is the product of two positive functions $\lambda(b)$ and $\mu(p)$ of B and F, respectively.*

PROOF. From (1.1) we get

$$(1.2) \qquad <X, V> = 0$$

for horizontal vector X and vertical vector V, where $<\ ,\ >$ denote the inner product on M. Since $[X, V] = 0$ for such X and V, we have

$$(1.3) \qquad \nabla_X V = \nabla_V X,$$

where ∇ denotes the covariant differentiation on M. Let V and W be any two vertical vector fields on M, we have

$$(1.4) \qquad X<V, W> = X(f^2 <V, W>_F) = 2\frac{(Xf)}{f} <V, W>$$

for horizonal vector field X, where $<\ ,\ >_F$ denotes the inner product on F induced from the metric of F. On the other hand, (1.3) implies

$$(1.5) \qquad X<V, W> = 2<\nabla_V X, W>$$

Consecuently, from (1.4) and (1.5), we find that the second fundamental form h of fibers is given by

$$h(V, W) = \frac{-Df}{f} <V, W>$$

This shows (1) and (2). Now, let $E_1, \cdots, E_r$ be an orthonormal basis of horizontal space, then we have

$$H = -\sum_{i=1}^{r} \left(\frac{E_i f}{f}\right) E_i = -\sum_i (E_i \log f) E_i.$$

Thus, for any vertical vector V, we have

$$(1.6) \qquad \nabla_V H = -\sum V(E_i \log f) E_i - \sum (E_i \log f) \nabla_V E_i.$$

Since $\pi^{-1}(p)$ are totally geodesic in M, the last term of (1.6) is vertical. Thus (1.6) shows that the mean curvature vector H of fibers is parallel if and only if

$$(1.7) \qquad V(X \log f) = 0$$

for all horizontal vector X and vertical vector V. This proves (3).

(Q.E.D.)

§2. REDUCIBLE TOTALLY UMBILICAL SUBMANIFOLDS

First we recall the following general results.

PROPOSITION 2.1 (Schouten 1954). *Every t. u. submanifold of dimension $\geqq 4$ in a conformally flat space is conformally flat.*

PROPOSITION 2.2 (Miyazawa and Chūman, 1972). *Let N be a t. u. submanifold of dimension $\geqq 4$ of a locally symmetric space M. Then each component of $\{x \in N | H \neq 0 \text{ at } x\}$ is conformally flat.*

PROPOSITION 2.3 (Miyazawa and Chūman, 1972). *A t. u. submanifold of a locally symmetric space is locally symmetric if and only if the mean curvature is constant.*

We give the following classification of reducible t. u. submanifolds.

THERorem 2.4. *If N is a reducible t. u. submanifold of a locally symmetric space M, then N is one of the following locally symmetric spaces:*

(1) *a totally geodesic submanifold,*

(2) *a local Riemannian product of a curve and a real-space-form $N_2(c)$ of contant curvature c,*

(3) *a local Riemannian product of two real-space-forms $N_1(c)$ and $N_2(-c)$, $c \neq 0$.*

PROOF. If N is a reducible t. u. submanifold of a locally symmetric space M, then locally $N = N_1 \times N_2$ as a Riemannian product. For any Z in TN, we put $Z = Z_1 + Z_2$, where Z_1 and Z_2 are tangent to N_1 and N_2, respectively. Then we have

$$(2.1) \qquad R(X_1, Y_2; Y_2, X_1) = 0.$$

By using equation (I,2.5) of Gauss we get for unit vectors X_1 and Y_2,

$$(2.2) \qquad \tilde{R}(X_1, Y_2; Y_2, X_1) = -\alpha^2, \qquad \alpha^2 = \langle H, H\rangle.$$

For any U tangent to N, (2.2) gives

$$(2.3) \qquad -U\alpha^2 = 2\tilde{R}(\nabla_U X_1, Y_2; Y_2, X_1) + 2\langle U, X_1\rangle \tilde{R}(H, Y_2; Y_2, X_1)$$
$$+ 2\tilde{R}(X_1, \nabla_U Y_2; Y_2, X_1) + 2\langle U, Y_2\rangle \tilde{R}(X_1, H; Y_2, X_1).$$

Because $N = N_1 \times N_2$, locally, and N is t. u. in M, equations (I,2.5) and (I,2.6) imply

$$(2.4) \qquad \tilde{R}(\nabla_U X_1, Y_2; Y_2, X_1) = -\langle Y_2, Y_2\rangle\langle\nabla_U X_1, X_1\rangle\alpha^2 = 0,$$

$$(2.5) \qquad \tilde{R}(H, Y_2; Y_2, X_1) = \frac{1}{2} X_1\alpha^2,$$

for unit vector fields X_1 tangent to N_1 and Y_2 tangent to N_2. Combinining (2.3), (2.4) and (2.5) we obtain

$$(2.6) \qquad -U\alpha^2 = \langle U, X_1\rangle(X_1\alpha^2) + \langle U, Y_2\rangle(Y_2\alpha^2).$$

From this we conclude that N has constant mean curvature. If $\alpha = 0$, N is totally geodesic. If $\alpha \neq 0$, N is locally symmetric and conformally flat. From these we may conclude that N is one of those spaces given in (2) or (3). (Q.E.D.)

REMARK 2.1. The symmetric space $\mathbf{R} \times S^n$ admits irreducible t. u. hypersurfaces which are not locally symmetric, and hence with nonconstant mean corvature.

REMARK 2.2. Some locally symmetric spaces admit reducible non-totally geodesic, t. u. submanifolds.

§3. CODIMENSION OF TOTALLY UMBILICAL SUBMANIFOLDS.

By an *extrinsic sphere* we mean a t. u. submanifold with nonzero parallel mean curvature vector. Extrinsic spheres have been characterized by Nomizu and Yano (1974). We recall the following. (Chen, 1979 d)

THEOREM 3.1. *Let N be an n-dimensional $(n \geqq 2)$ submanifold of a locally symmetric space $\tilde{M}$. Then N is an extrinsic sphere of $\tilde{M}$ if and only if N is an extrinsic hypersphere of an $(n+1)$-dimensional totally geodesic submanifold $\overline{M}$ of constant sectional curvature.*

As an application of (M_+, M_-)-theory, we obtain from Theorem VI, 4.3 and Theorem 3.1 the following.

THEOREM 3.2. *The maximal dimension of extrinsic spheres of an irreducible symmetric space M is given by $\lambda - 1$, where λ is the integer* *given in Theorem* VI, 4.3.

THEOREM 3.3. *Let N be a t. u. submanifold of a locally symmetric space M. If*

(3.1) $\quad \operatorname{codim} N \leqq \operatorname{rk} M - 2,$

then either N is totally geodesic or N is an extrinsic sphere.

PROOF. Since our study is local, we may assume that M is a symmetric space. For any fixed point $x \in N$, we regard x as the origin of M and we have $M = G/H$ where H is the isotropy subgroup at x. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$. Then $\mathfrak{m}$ can be identified with T_xM. A well-known result of É. Cartan says that the curvature tensor $\tilde{R}$ of M at x satifies (see, e.g,., Kobayashi and Nomizu (1963)),

(3.2) $\quad \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = -[[\tilde{X},\tilde{Y}],\tilde{Z}] \qquad \text{for} \quad \tilde{X},\tilde{Y},\tilde{Z} \in m.$

The famous criterion of Cartan is given by the following.

LEMMA 3.4. *A linear subspace π of T_xM is the tangent space to some totally geodesic submanifold B of M if and only if π satisfies $[[\pi,\pi],\pi] \subset \pi$. Moreover, B is flat if and only if $[\pi,\pi] = 0$.*

Now, for any vectors X, Y, and Z in TN at x, we have from (I,2.4), (I,2.6), and (I,2.9) that

(3.3) $\quad (\tilde{R}(X,Y)Z)^{\perp} = \langle Y,Z\rangle\, D_XH - \langle X,Z\rangle D_YH.$

For any given vector X in T_xN, there exists a flat totally geodesic submanifold B through x such that $X \in T_xB$ and $\dim B = \operatorname{rk} M$. If (3.1) holds, there is a unit vector Y in $T_xN \cap T_xB$ such that $\langle X,Y\rangle = 0$. Hence, (3.3) gives

(3.4) $\quad (\tilde{R}(X,Y)Y)^{\perp} = D_XH$

On the other hand, since B is flat, totally geodesic in M, T_xB forms an abelian linear subspace of $\mathfrak{m}$. Thus, we have $[X,Y] = 0$. Consequently, (3.2) and (3.4) give $D_XH = 0$. Since X can be chosen to be any vector tangent to N at x and x can be chosen to be any point in N, the mean curvature vector is parallel.

Combining Theorems VI, 4.3 and 3.1 we obtain the following

PROPOSITION 3.5. *There is no t. u. submanifold N in any irreducible locally symmetric space M with* codim $N \leqq$ rk $M-2$.

Since the rank 2 symmetric space $\boldsymbol{R} \times S^n$ admits a t. u. hypersurface which is neither totally geodesic nor an extrinsic hypersphere, Theorem 3.3 is best possible. For irreducible locally symmetric spaces, we also have the following.

PROPOSITION 3.6 (Chen and Verheyen 1980a). *There is no t. u. submanifold N in any irreducible locally symmetric space M with* codim $N =$ rk $M-1$.

By combining Propositions 3.5 and 3.6, we obtain the following best possible result for irreducible case.

THEOREM 3.7. *Let N be a t. u. submanifold of an irreducible locally symmetric space M. Then*

(3.5) codim $N \geqq$ rk M.

§4. TOTALLY UMBILICAL HYPERSURFACES.

In this section we shall classify locally symmetric spaces which admit a t. u. hypersurface. It follows from Theorem 3.7 that such spaces are either reducible or of rank one. Totally umbilical submanifolds of rank one symmetric spaces have been studied and classified. For complex-space-forms it was done in Chen and Ogiue (1974b); for quaternion-space-forms it was done in Chen (1978c); and for the Cayley place FII, it was studies in Chen (1977a).

THEOREM 4.1. *A locally symmetric space M admits a non-totally geodesic, t. u. hypersurface if and only if M is one of the following.*

(a) *a real-space-form $M(c)$ of constant curvature c,*

(b) *a local Riemannian product of a line and a real-space-form,*

(c) *a local Riemannian product of two real-space-forms $M_1(c)$ and $M_2(-c)$.*

If dim $M \geqq 4$, *M is conformally flat.*

PROOF. If *dim* $M<4$, this Theorem is trivial. Thus we may assume that dim $M \geqq 4$. Since N is t. u. and non-totally geodesic, the conformal curvature tensor of N vanishes. This is trivial if dim $N = 3$. From (I,1.9) we have

$$(4.1) \qquad R(X,Y;Z,W) = 0$$

for any orthogonal vectors X, Y, Z and W in TN. This formula is trivial if dim $N = 3$. Because N is *t. u.* in M, (4.1) and the equation (I,2.5) of Gauss give

$$(4.2) \qquad \tilde{R}(X,Y;Z,W) = 0$$

for orthogonal vectors X, Y, Z and W in TN. From (I,2.6) and (I,2.9), we find

$$(4.3) \qquad \tilde{R}(X,Y;Z,\xi) = 0$$

for any Z in TN perpendicular to any X, Y in TN and ξ in $T^{\perp}N$. Let U be any vector in TN. Because M is locally symmetric, (4.3) yields

$$(4.4) \qquad 0 = \langle X,U\rangle\tilde{R}(H,Y;Z,\xi) + \tilde{R}(\nabla_U X,Y;Z,\xi) + \langle Y,U\rangle\tilde{R}(X,H;Z,\xi)$$
$$+ \tilde{R}(X,\nabla_U Y;Z,\xi) + \tilde{R}(X,Y;\nabla_U Z,\xi) - \alpha\tilde{R}(X,Y;Z,U)$$

where ξ is the unit normal vector such that $H = \alpha\xi$. By (I,2.6) and (4.4) we may obtain

$$(4.5) \qquad \tilde{R}(X,Y;Z,U) = \langle X,U\rangle\tilde{R}(\xi,Y;Z,\xi) - \langle Y,U\rangle\tilde{R}(\xi,X;Z,\xi)$$

for Z perpendicular to X and Y and U in TN. Let $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ be orthogonal vectors tangent to M. If $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ are tangent to N, we have $\tilde{R}(\tilde{X},\tilde{Y};\tilde{Z},\tilde{W}) = 0$ by (4.2). If $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ are not all tangent to N, there exist orthogonal vectors $\tilde{X}', \tilde{Y}', \tilde{Z}'$, and $\tilde{W}'$ satisfying

$$(4.6) \qquad \tilde{X} \wedge \tilde{Y} = \tilde{X}' \wedge \tilde{Y}', \qquad \tilde{Z} \wedge \tilde{W} = \tilde{Z}' \wedge \tilde{W}', \qquad \text{and}$$

$$(4.7) \qquad \langle \tilde{Y}',\xi\rangle = \langle \tilde{Z}',\xi\rangle = 0.$$

For such $\tilde{X}', \tilde{Y}', \tilde{Z}', \tilde{W}'$ we have $\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = \tilde{R}(\tilde{X}',\tilde{Y}';\tilde{Z}',\tilde{W}')$. We put

$$(4.8) \qquad \tilde{X}' = (\cos\theta)X + (\sin\theta)\xi, \qquad \tilde{W}' = (\cos\phi)W + (\sin\phi)W,$$

where X and W are tangent to N. Since $\langle \tilde{X}',\tilde{W}'\rangle = 0$, we obtain

$$(4.9) \qquad \cos\theta \ \cos\phi \langle X,W\rangle + \sin\theta \ \sin\phi = 0.$$

From (4.3) and (4.8) we find

$$\tilde{R}(\tilde{X},\tilde{Y};\tilde{Z},\tilde{W}) = \cos\theta\cos\phi\ \tilde{R}(X,\tilde{Y}';\tilde{Z}',W) + \sin\theta\sin\phi\ \tilde{R}(\xi,\tilde{Y}';\tilde{Z}',\xi).$$

Combining this with (4.9) we find

$$\tilde{R}(\tilde{X},\tilde{Y};\tilde{Z},\tilde{W}) = \cos\theta\cos\phi\,\{\tilde{R}(X,\tilde{Y}';\tilde{Z}',W) - \langle X,W\rangle\tilde{R}(\xi,\tilde{Y}';\tilde{Z}',\xi)\}. \tag{4.10}$$

On the other hand, since $\tilde{X}',\tilde{Y}',\tilde{Z}'$ and $\tilde{W}'$ are orthogonal, and $\tilde{Y}',\tilde{Z}'$ are tangent to N, $\langle\tilde{Y},W\rangle = \langle\tilde{Y}',\tilde{W}'\rangle = 0$. Thus (4.5) yields

$$\tilde{R}(X,\tilde{Y}';\tilde{Z}',W) = \langle X,W\rangle\tilde{R}(\xi,\tilde{Y}';\tilde{Z}',\xi). \tag{4.11}$$

From (4.10) and (4.11) we get $\tilde{R}(\tilde{X},\tilde{Y};\tilde{Z},\tilde{W}) = 0$ for orthogonal vectors $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}$ in $T_x\tilde{M}$, $x \in N$. Since M is locally symmetric, the same property holds at every point of M. Thus, by a result of Schouten (1954), M is conformally flat. Because M is locally symmetric, M is one of those spaces given in (a), (b) or (c). The converse is trivial. (Q.E.D.)

From Theorem 4.1 we may obtain the following result which says that non-totally geodesic, t. u. hypersurfaces of a locally symmetric space are in fact given by fixed point sets of some conformal mappings.

THEOREM 4.2. *Let M be a locally symmetric space of dimension ≥ 4. If N is a non-totally, t. u. hypersurface of M, then, for any point p in N, there is a neighborhood U of p in M and a conformal mapping ϕ from U into M such that $U \cap N$ lies in the fixed point set of ϕ.*

From Theorems 4.1, VI, 4.2 and VI, 4.3 we obtain the following

COROLLARY 4.3. *The only irreducible locally symmetric spaces which admit t. u. hypersurfaces are real-space-forms.*

From Theorem 4.1 we may also prove the following.

THEOREM 4.4. *Let $\tilde{M}$ be a Hermitian symmetric space and N a t. u. hypersurface of $\tilde{M}$. Then*

(a) *If N is totally geodesic, $\tilde{M}$ is the Riemannian product of two Hermitian symmetric spaces M_1 and M_2 of complex dimension $m-1$ and 1, respectively. Moreover, N is the Riemannian product of M_1 and a geodesic of M_2, locally.*

(b) *If N is non-totally geodesic, then one of the following three statments holds.*

(b.1) *$\tilde{M}$ is a $\mathbf{C}P^1(c)$, $\mathbf{C}^1$, or $D^1(-c)$ and N any curve in $\tilde{M}$. Here c and $-c$ denote the sectional curvature of $\mathbf{C}P^1$ and D^1, respectively.*

(b.2) *N is a t. u. hypersurface of $\tilde{M}$ and $\tilde{M} = \mathbf{C}P^1(c) \times D^1(-c)$.*

(b.3) *$\tilde{M}$ is $\mathbf{C}^m$ and N is an open portion of an ordinary hypersphere of $\mathbf{C}^m$.*

If statement (b.2) *holds, locally, N is the conformal image of a hypersphere S^3 or a hyperplane E^3 of $\mathbf{C}^2$ under a conformal mapping from $\mathbf{C}^2$ into M.*

T. u. hypersurfaces of $\mathbf{C}P^1(c) \times D^1(-c)$ do not have constant mean curvature in general. However, for higher dimensional Kaehler manifolds, we have the following surprising result.

THEOREM 4.5. *If $\tilde{M}$ is any Kaehler manifold of complex dimension $\geqq 3$, then every t. u. hypersurface of $\tilde{M}$ is either totally geodesic or an extrinsic hypersphere.*

The following result follows easily from the equation of Codazzi (see, Chen (1979c) and Kawalski (1972)).

PROPOSITION 4.5. *Every t. u. hypersurface N of an Einstein space M of dimension $\geqq 3$ is either totally geodesic or an extrinsic sphere.*

PROOF. Since N is t. u. in M, the equation of Codazzi gives

$$\tilde{R}(E_1,E_i;E_i,H) = \frac{1}{2} E_1\alpha^2. \tag{4.12}$$

where $E_1.\cdots.E_n$ is an orthonormal frame of TN. Thus the Ricci tensor $\tilde{S}$ of M satisfies $0 = \tilde{S}(X,H) = (\frac{n-1}{2}) X\alpha^2$ for any X in TN. This proves the Proposition.

In views of Theorem 4.1 the author would like to ask the following.

PROBLEM 4.1. Let N be an n-dimensional ($n \geqq 3$) *t. u.* submanifold of a locally symmetric space M. If the mean curvature $\alpha \neq 0$ everywhere on N, does N lie in a conformally flat totally geodesic submanifold of M?

Theorem 4.1 tells us that the answer to this problem is affirmative if the codimension of N is one.

§5. IRREDUCIBLE TOTALLY UMBILICAL SUBMANIFOLDS

Reducible *t. u.* submanifolds of a locally symmetric space were studied in §2. In this section we will treat irreducible case. Contrary to reducible case, irreducible t. u. submanifolds do not have constant mean curature in general (see, REMARK 2.1). For hypersurface case, a t. u. hypersurface of a locally symmetric space M is locally the fixed point set of a conformal mapping on M. From the equation of Gauss, it follows trivially that every t. u. submanifold of a (locally) symmetric space of compact type is either totally geodesic or irreducible. For irreducible t. u. submanifolds in a general locally symmetric space, we have the following.

THEOREM 5.1. *If N is an irreducible t. u. submanifold with constant mean curvature of a locally symmetric space M, then*

(1) *N is either totally geodesic or a real-space-form and*

(2) *if $\dim N > 2$ and the mean curvature vector H is not parallel, then $\dim N < \frac{1}{2} \dim M$.*

PROOF. If N is a t. u. submanifold with constant mean curvature of a locally symmetric space, then N is either totally geodesic, or conformally flat. Moreover, N is locally symmetric. Since N is irreducible by assumption, N is Einsteinian. Thus, by (I, 1.9), we conclude that either N is totally geodesic or N is of constant sectional curvature, this proves (1).

If H is not parallel, N is a real-space form $N(c)$ of constant curvature, say c. Since N has constant mean curvature, (I,2.4), (I,2.6) and (I,2.9) give

$$\tilde{R}(X,Y;Z,H) = 0 \tag{5.1}$$

for vector fields X,Y,Z tangent to N. By using $\tilde{\nabla}\tilde{R} \equiv 0$, (I,2.1), (I,2.6), (I,2.9), and (5.1), we may find

$$\alpha^2\tilde{R}(X,Y;Z,U) = \langle U,X\rangle\tilde{R}(H,Y;Z,H) - \langle U,Y\rangle\tilde{R}(H,X;Z,H) \tag{5.2}$$
$$+ \langle Y,Z\rangle\langle D_X H, D_U H\rangle - \langle X,Z\rangle\langle D_Y H, D_U H\rangle$$

for U tangent to N. Let $X = U$, $Y = Z$ be orthonormal vectors tangent to N. Then we find, from (I,2.5) and (5.2), that

$$(5.3) \qquad \alpha^2\tilde{K}(H,Y) = -|D_X H|^2 + \alpha^2 c - \alpha^4,$$

where $\tilde{K}$ denotes the sectional curvature of M. Because (5.3) holds for all orthonormal vectors X,Y in TN, and $\dim N > 2$, $|D_X H|$ is independent of X. In particular, $|D_X H|$ is nonzero for any unit vector X in $T_x N$ for some $x \in N$. By setting $U = X = E_i$ and summing on i for (5.2), we find

$$(5.4) \qquad (n-1)\tilde{R}(H,Y;Z,H) = \alpha^2 S(Y,Z) - (n-1)\alpha^4 <Y,Z> + <D_Y H, D_Z H> - <Y,Z>|DH|^2,$$

where $|DH|^2 = \sum_{i=1}^{n} |D_{E_i} H|^2$. Substituting (5.4) into (5.2) we obtain

$$\begin{aligned}(5.5) \qquad (n-1)\alpha^2 R(X,Y;Z,U) = {} & |DH|^2\{<Y,U><X,Z> - <X,U><Y,Z>\} \\ & + <U,X>\{\alpha^2 S(Y,Z) + <D_Y H, D_Z H>\} \\ & - <U,Y>\{\alpha^2 S(X,Z) + <D_X H, D_Z H>\} \\ & + (n-1)\{<Y,Z><D_X H, D_U H> - <X,Z><D_Y H, D_U H>\}.\end{aligned}$$

By setting $Y = Z = E_i$ and summing on i, we get

$$(5.6) \qquad (n-2)<D_X H, D_U H> = \alpha^2 S(X,U) + \frac{n-2}{n} <X,U>|DH|^2 - \frac{\alpha^2}{n} r <X,U>.$$

Since N is of constant curvature, (5.6) gives $<D_X H, D_U H> = 0$ for orthogonal X and U. Because $D_X H \neq 0$ for any unit vector X at $T_x N$, H, $D_{E_1} H, \cdots, D_{E_n} H$ are orthogonal and they span an $(n+1)$-dimensional linear subspace of $T_x^\perp N$. This proves (2). (Q.E.D.)

From the proof we have the following.

COROLLARY 5.2. *Let N be a t. u. submanifold of a locally symmetric space M. If N is of constant sectional curvature, then N is either totally geodesic, or an extrinsic sphere, or of dimension* $< \frac{1}{2} \dim M$.

PROBLEM 5.1. Determine whether there exists a t. u. submanifold of constant sectional curvature in a locally symmetric space with nonparallel mean curvature vector.

§6. TOTALLY UMBILICAL SUBMANIFOLDS OF A KAEHLER MANIFOLD

In this section we will apply the theory of generic submanifolds to the theory of totally umbilical submanifolds. We shall use the same notations as before. The results obtained in this section are given in Chen and Verheyen (1980b) unless mentioned otherwise.

LEMMA 6.1. *Let N be t. u. generic submanifold of a Kaehler manifold M. Then*

(1) *the purely real distribution $\mathscr{H}^{\perp}$ is integrable and its leaves are totally geodesic in N,*

(2) *if N is not purely real, $H \in F\mathscr{H}^{\perp}$, and*

(3) *if $H \in F\mathscr{H}^{\perp}$, both $F\mathscr{H}^{\perp}$ and ν are parallel.*

PROOF. For vector fields U in TN and ξ in $T^{\perp}N$, we have

$$-JA_{\xi}U + JD_{U}\xi = \nabla_{U}t\xi + h(U,t\xi) - A_{f\xi}U + D_{U}f\xi,$$

from which we find

$$\nabla_{U}t\xi = \langle f\xi,H\rangle U - \langle \xi,H\rangle PU + tD_{U}\xi, \tag{6.1}$$

$$D_{U}f\xi = -FA_{\xi}U + fD_{U}\xi - h(U,t\xi). \tag{6.2}$$

In particular, if $Z,W \in \mathscr{H}^{\perp}$, these give $\nabla_{Z}W \in \mathscr{H}^{\perp}$. This proves (1). Let η be any vector field in ν and X in $\mathscr{H}$, we have $\langle X,X\rangle\langle H,\eta\rangle = \langle J\tilde{\nabla}_{X}X,J\eta\rangle = \langle h(X,JX),J\eta\rangle = 0$. This proves (2).

By (6.2) and statement (2) we have $D_{U}J\xi = fD_{U}\xi$ for ξ in ν. On the other hand, $JD_{U}\xi = J\tilde{\nabla}_{U}\xi = \tilde{\nabla}_{U}J\xi = D_{U}J\xi$. Therefore, we obtain $tD_{U}\xi = 0$. Thus ν is parallel. Since $F\mathscr{H}^{\perp}$ is the orthogonal complementary distribution of ν in $T^{\perp}N$, $F\mathscr{H}^{\perp}$ is also parallel.

LEMMA 6.2. *Let N be a t. u. submanifold of a Hermitian symmetric space M. Then, for any unit vectors $X \in \mathscr{H}_{x}$, $Z \in \mathscr{H}_{x}^{\perp}$ at $x \in N$ with $\alpha(x) \neq 0$, we have $\tilde{K}(X,Z) = 0$.*

PROOF. Under the hypothesis, we have $\tilde{K}(X,Z) = \tilde{R}(Z,X;JX,PZ) + \tilde{R}(Z,X;JX,FZ)$. By using this and the equations of Gauss and Codazzi, we may obtain $\tilde{K}(X,Z) = R(Z,X;JZ,PZ)$. Since N is t. u. with $\alpha(x) \neq 0$ in M, N is conformally flat in a neighborhood of $x \in N$. Because Z,X,JX and PZ are orthogonal, $R(Z,X;JX,PZ) = 0$. (Q.E.D.)

PROPOSITION 6.3. *Let N be a t. u. generic submanifold of a positively (or negatively) curved Kaehler manifold M. Then N is either a complex totally* *geodesic submanifold or a purely real submanifold.*

PROOF. Assume that N is a proper generic t. u. submanifold. For any vector $X \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$, equation (I,2.6) implies

$$(6.3) \qquad 0 = \tilde{R}(JX,Z;X,FZ) = \tilde{R}(FZ,X;X,FZ) + \tilde{R}(PZ,X;X,FZ),$$

$$(6.4) \qquad |FZ|^2\tilde{K}(X,FZ) = -\langle D_{PZ}H,FZ\rangle.$$

On the other hand, we have $\tilde{K}(X,JZ) = \tilde{R}(X,PZ;JX,Z) = \tilde{R}(PZ,X;X,PZ) + \langle D_{PZ}H,FZ\rangle$. Combining this with (6.4) we get

$$(6.5) \qquad |FZ|^2\{\tilde{K}(X,JZ) + \tilde{K}(X,FZ)\} = |PZ|^2\{\tilde{K}(X,PZ) - \tilde{K}(X,JZ)\}.$$

If M is positively curved, this gives $\tilde{K}(X,PZ) > \tilde{K}(JX,Z)$ for unit vectors $X \in \mathscr{H}$ and $Z \in \mathscr{H}^{\perp}$. Replacing Z be PZ we have

$$(6.6) \qquad \tilde{K}(X,P^2Z) > \tilde{K}(JX,PZ) > \tilde{K}(X,Z).$$

Now, for Z, $W \in \mathscr{H}^{\perp}$, we have $\langle Z,PW\rangle = \langle Z,JW\rangle = -\langle PZ,W\rangle$. Thus $(P/\mathscr{H}^{\perp})^2$ is a symmetric endomorphism of $\mathscr{H}^{\perp}$. If $P^2 \equiv 0$ on $\mathscr{H}^{\perp}$, $\langle PZ,PZ\rangle = -\langle P^2Z,Z\rangle = 0$ for $Z \in \mathscr{H}^{\perp}$. Thus (6.5) gives $FZ = 0$. This is a contradiction. Consequently, $P^2 \not\equiv 0$ on $\mathscr{H}^{\perp}$. Thus, there is a unit vector $Z \in \mathscr{H}^{\perp}$ such that $P^2Z = \lambda Z$, $\lambda \neq 0$. This contradicts (6.6). Similar argument applies to negatively curved Kaehler manifolds.

By using Proposition 6.3 we obtain the following two results.

THEOREM 6.4. *Let N be a t. u. submanifold of a positively (or negatively) curved Kaehler manifold M. If* $\dim N \geqq 3$, *then N is either a totally geodesic complex submanifold or purely real.*

THEOREM 6.5. *There is no extrinsic sphere of dimension $\geqq m$ in any positively (or negatively) curved Kaehler manifold M of complex dimension m for $m \geqq 2$.*

Theorem 6.4 follows immediately from Proposition 6.3 and Theorem 6.5 follows from Theorem 3.1 and Proposition 6.3.

REMARK 6.1. CP^m and its non-compact dual are positively curved and negatively curved Kaehler manifolds which admit an extrinsic sphere of dimension $m-1$. Thus, Theorem 6.5 is best possible. This result improves a result of Chen (1977b).

By using the equation of Codazzi and (IV, 1.5) we have the following.

LEMMA 6.6. *Let N be a t. u. generic submanifold of a Kaehler manifold M. Then*

$$\tilde{R}(\mathscr{H}^{\perp},\mathscr{H}^{\perp};\mathscr{H},T^{\perp}N)=\{0\}, \tag{6.7}$$

$$\tilde{R}(\mathscr{H}^{\perp},\mathscr{H}^{\perp};\mathscr{H}^{\perp},\mathscr{H})=\{0\}. \tag{6.8}$$

By using Lemmas 6.1 and 6.6 we may also prove the following.

LEMMA 6.7. *Let N be a t. u. generic submanifold of a locally Hermitian symmetric space M. If $\dim_{\boldsymbol{R}}\mathscr{H}^{\perp}=p\geqq 2$, then*

$$\tilde{R}(\mathscr{H}^{\perp},\mathscr{H};\mathscr{H}^{\perp},H)=\{0\}. \tag{6.9}$$

PROOF. For Z,U,V,W in $\mathscr{H}^{\perp}$ and X in $\mathscr{H}$, Lemma 6.6 gives

$$0=U\tilde{R}(Z,W\,;\,V,X)=\langle U,Z\rangle\tilde{R}(H,W\,;\,V,X)+\langle U,W\rangle\tilde{R}(Z,H\,;\,V,X)$$

$$+\tilde{R}(\nabla_U Z,W\,;\,V,X)+\tilde{R}(Z,\nabla_U W\,;\,V,X)+\tilde{R}(Z,W\,;\,\nabla_U V,X)+\tilde{R}(Z,W\,;\,V,\nabla_U X).$$

By applying Lemmas 6.1 and 6.6 this gives $0=\langle U,Z\rangle\tilde{R}(H,W;V,X)+\langle U,W\rangle\tilde{R}(Z,H;V,X)$. If $p\geqq 2$, this yields (6.9).

PROPOSITION 6.8. *Let N be a t. u. generic submanifold of a locally Hermitian symmetric space M. If $\dim_{\boldsymbol{R}}N\geqq 4$, then either N is purely real or N has constant mean curvature.*

PROOF. If $p\geqq 2$, (I,2.6) and Lemma 6.7 give $0=\tilde{R}(Z,X;Z,H)=-\langle Z,Z\rangle\langle D_X H,H\rangle$ for $Z\in\mathscr{H}^{\perp}$ and $X\in\mathscr{H}$. If N is proper, this implies $X\alpha^2=0$. Now, we put $N'=\{x\in N\mid \alpha\neq 0 \text{ at } x\}$. Then each component of N' is an open submanifold of N. If N' is nonempty and N is not purely real, N' is proper generic. By Lemma 6.2, we get $\tilde{R}(Z,X;X,Z)=0$ for $X\in\mathscr{H}$ and $Z\in\mathscr{H}^{\perp}$. Thus, by linearity, we get $\tilde{R}(Z,X;Y,Z)=\tilde{R}(Z,X;X,W)=0$ for vector fields X,Y in $\mathscr{H}$ and Z,W in $\mathscr{H}^{\perp}$ on N'. From this we get $0=(1/2)W\tilde{R}(Z,X;X,Z)=\langle Z,W\rangle\tilde{R}(H,X;X,Z)$ by Lemma 6.1. Combining this with (I,2.6) we get $Z\alpha^2=0$ for $Z\in\mathscr{H}^{\perp}_x$, $x\in N'$. Therefore

α^2 is constant on each component of N'. Since α^2 is continuous, α^2 is constant on N.

If $p=1$, this proposition follows from the following.

THEOREM 6.9 (Chen 1980g). *Let N be a t. u. generic submanifold of a locally Hermitian symmetric space M. If* $\dim_{\mathbf{R}} N \geqq 5$ *and* $\dim_{\mathbf{R}} \mathscr{H}^{\perp} = 1$, *then*

(a) *the mean curvature vector H of N is parallel, and*

(b) *if N is not totally geodesic, N is locally isometric to a sphere of radius $\frac{1}{\alpha}$ and* $\operatorname{rk} M > \dim_{\mathbf{R}} N$. *Moreover, N is a t. u. hypersurface of a flat totally geodesic submanifold of M.*

REMARK 6.2. For t. u. CR-submanifolds we have the following

THEOREM 6.10 (Chen, 1980g). *Let N be a t. u. CR-submanifold of a Kaehler manifold M. Then*

(i) *N is totally geodesic, or*

(ii) *the totally real distribution is one-dimensional, or*

(iii) *N is totally real.*

If (iii) *occurs,* $\dim_{\mathbf{R}} M \geqq 2 \dim_{\mathbf{R}} N$. *In particular, if N is not totally geodesic and* $\dim_{\mathbf{R}} N \geqq 2$, *then* $\dim_{\mathbf{R}} M \geqq 2 \dim_{\mathbf{R}} N + 2$.

By using Proposition 6.8, we may obtain the following.

THEOREM 6.11. *Let N be a t. u. generic submanifold of a locally Hermitian symmetric space M. If* $\dim_{\mathbf{R}} N \geqq 4$, *then one of the following statements holds*

(a) *N is purely real,*

(b) *N is totally geodesic,*

(c) $\operatorname{rk} M > \dim_{\mathbf{R}} N$ *and N lies in a maximal flat totally geodesic submanifold of M as an extrinsic sphere.*

It is easy to see that all of these three cases actually occur.

For the proof see Chen and Verheyen (1980b). By applying this theorem we obtain the following two Theorems.

THEOREM 6.12. *Let N be a t. u. submanifold of a locally Hermitian symmetric space M. If* $\dim_{\mathbf{R}} N > \dim_{\mathbf{C}} M$ *and* $\dim_{\mathbf{R}} N \geqq 4$, *then either*

(a) *N is totally geodesic in M or*

(b) rk $M > \dim_{\mathbf{R}} N$ *and N lies in a maximal flat totally geodesic submanifold of M as an extrinsic sphere.*

THEOREM 6.13. *Let N be a t. u. submanifold of a locally Hermitian symmetric space M of compact or non-compact type. If* $\dim_{\mathbf{R}} N > \dim_{\mathbf{C}} M$ *and* $\dim_{\mathbf{R}} N \geqq 4$, *then N is totally geodesic.*

REMARK 6.3. If $\dim_{\mathbf{R}} N \leqq 3$, Theorems 6.11 and 6.12 are not ture in general. Moreover, if M is of Euclidean type, Theorem 6.13 is false.

REMARK 6.4. Since $S^1 \times S^{n-1}$ can be isometrically immersed in the irreducible rank two Hermitian symmetric space $Q_n = SO(n+2)/SO(2) \times SO(n)$ (Chen and Nagano, 1977). Q_n admits a purely real t. u. submanifold of dimension $n-1$ with non-constant mean curvature.

§7. EXTRINSIC SPHERES OF KAEHLER MANIFOLDS.

Since extrinsic spheres have the same extrinsic properties as ordinary spheres in a Euclidean space, it is natural to ask when an extrinsic sphere is an ordinary sphere. It follows from Proposition 1.2 that every Riemannian manifold F can be an extrinsic sphere in the twisted product $M = B \times_{\lambda\mu} F$. Here λ and μ are positive functions on B and F, respectively. Hence in order to conclude that an extrinsic sphere is isometric to an ordinary sphere we need to impose some suitable conditions on the ambient space. For extrinsic spheres in a Kaehler manifold, we have the following.

THEOREM 7.1. *A complete, 1-connected, even-dimensional extrinsic sphere N of a Kaehler manifold M is isometric to an ordinary sphere if its normal connection is flat.*

PROOF. Since N is 1-connected and its normal connection of N in M is flat, there exist $2m - 2n$ mutually orthogonal unit parallel normal vector fields defined globally on N, where $\dim_{\mathbf{R}} N = 2n$ and $\dim_{\mathbf{C}} M = m$. Because, N is an extrinsic sphere of M. We have

$$(7.1) \qquad h(X,Y) = \langle X,Y\rangle H, \qquad D_X H = 0, \quad \text{and} \quad H \neq 0.$$

Since H is parallel, the mean curvature $\alpha = |H|$ is constant. We put $\bar{\xi} = H/\alpha$.

Then $\bar{\xi}$ is parallel. Now, suppose that $\xi_1, \cdots, \xi_{2m-2n}$ are mutually orthogonal unit normal vector fields on N. Then we may assume that $\xi_1 = \bar{\xi}$. We put $\Phi_r = \langle J\bar{\xi}, \xi_r\rangle$, $r = 2, \cdots, 2m-2n$. Then we have

$$(7.2) \qquad \tilde{\nabla}_X \xi_r = -A_r X + D_X \xi_r = 0.$$

Thus we get

$$(7.3) \qquad X\Phi_r = \langle J\tilde{\nabla}_X \bar{\xi}, \xi_r\rangle = \alpha\langle X, J\xi_r\rangle.$$

From (7.1), (7.2), and (7.3) we obtain $XY\Phi_r = \alpha X\langle Y, J\xi_r\rangle = \alpha\langle\tilde{\nabla}_X Y, J\xi_r\rangle$ $= \alpha\langle\nabla_X Y, J\xi_r\rangle + \alpha\langle h(X,Y), J\xi_r\rangle = (\nabla_X Y)\Phi_r - \alpha^2\Phi_r\langle X, Y\rangle$, from which we find

$$(7.4) \qquad \nabla_X d\Phi_r = -\alpha^2\Phi_r X, \qquad r = 2, \cdots, 2m-2n.$$

Now, we shall claim that at least one of the functions Φ_r, $r = 2 \cdots, 2m-2n$, is nonconstant. If all of the Φ_r are constant, then (7.1) and (7.2) imply

$$0 = X\Phi_r = \langle J\tilde{\nabla}_X\bar{\xi}, \xi_r\rangle = -\alpha\langle JX, \xi_r\rangle = \alpha\langle X, J\xi_r\rangle, \qquad r = 2, \cdots, 2m-2n.$$

Thus, the subspace spanned by $\xi_2, \cdots, \xi_{2m-2n}$, $J\xi_2, \cdots . J\xi_{2m-2n}$ is a complex normal subspace. Thus, it is even-dimensional and of dimension greater than $2m-2n-1$. Hence, it is the whole normal subspace $T^{\perp}N$. This implies that N is a complex submanifold. Hence, it is totally geodesic. This is a contradiction. Thus, there is a nonconstant function Φ defined on N and satisfying the differential equation $\nabla_X d\Phi = -\alpha^2\Phi X$ for $X \in TN$. Therefore, by a result of Obata (1962), N is isometric to an ordinary $2n$-sphere of radius $1/\alpha$. (Q.E.D.)

In the following we shall give an odd-dimensional example of such an extrinsic sphere which is not an ordinary sphere, in fact, not even a homotopy sphere.

Let $F_1, \cdots, F_r$ be r irreducible homogeneous polynomials in m complex variables $z_1, \cdots, z_m$. The set $\tilde{M} = \tilde{M}(F_1, \cdots, F_r)$ of all common zeros of $F_1, \cdots, F_r$ less the origin is a complex variety. If the r hypersurfaces given by $F_i(z_1, \cdots, z_m) = 0$, $i = 1, \cdots, r$, are in general position, this variety is nonsingular and is called a *complete intersection in* $\mathbf{C}^m$. Clearly the natural Kaehler structure of $\mathbf{C}^m$ induces a Kaehler structure on $\tilde{M}$.

Let $S^{2m-1}(1)$ be the unit hypersphere of $\mathbf{C}^m$ centered at the origin 0 and $M = \tilde{M} \cap S^{2m-1}(1)$. Then we have the following.

PROPOSITION 7.2. *The intersection* $M = \tilde{M}(F_1, \cdots, F_r) \cap S^{2m-1}(1)$ *is a closed* *extrinsic hypersphere of* $\tilde{M}(F_1, \cdots, F_r)$.

If we consider $z_1, \cdots, z_m$ as the homogeneous complex coordinates of $\mathbf{C}P^{m-1}$, the homogeneous equations $F_1 = \cdots = F_r = 0$ define an algebraic manifold $A(M)$ of $\mathbf{C}P^{m-1}$. We have

PROPOSITION 7.3. *The homotopy groups of* $M = \tilde{M}(F_1, \cdots, F_r) \cap S^{2m-1}(1)$ *and the associated algebraic manifold* $A(M)$ *satisfy*

$$(7.5) \qquad \pi_k(M) \cong \pi_k(A(M)), \qquad k \geqq 3.$$

Moreover, if $\pi_1(A(M)) = 0$ *and* $\dim_{\mathbf{C}} A(M) \geqq 3$, *then either* $\pi_1(M) = \pi_2(M) = 0$ *or* $\pi_1(M) = \pi_2(M) = Z$.

From this Proposition we see that in general the extrinsic sphere $M = \tilde{M}(F_1, \cdots, F_r) \cap S^{2m-1}(1)$ is not a homotopy sphere. In particular, if $F_1 = a_1 z_1^d + a_2 z_2^d + \cdots + a_m z_m^d$. $F_2 = z_2, \cdots, F_r = Z_r$, then the extrinsic sphere M is 1-connected and its normal connection in $\tilde{M}(F_1, \cdots, F_r)$ is flat. Thus, we may obtain the following.

THEOREM 7.4. *For each positive odd integer* k ; $k \leqq 2m - 3$, *there is a Kaehler manifold* $\overline{M}$ *of complex dimension* m *and a submanifold* M *of real dimension* $2m - k$ *in* $\overline{M}$ *such that*

(1) *M is closed and 1-connected,*

(2) *M is an extrinsic sphere of $\overline{M}$,*

(3) *the normal connection of M in $\overline{M}$ is flat, and*

(4) *M is not a homotopy sphere.*

As we already know from Theorem 7.1, there is no such submanifold if k is even.

In Blair and Chen (1980), we have also the following.

THEOREM 7.5. *Let* M *be a complete extrinsic sphere of a complete intersection* $\tilde{M}(F_1, \cdots, F_r)$ *in* $\mathbf{C}^m$. *Then either* M *is isometric to an ordinary sphere or* M *is an extrinsic sphere of* $\tilde{M}$ *obtained in the way mentioned above.*

For extrinsic spheres in a locally Hermitian symmetric space we have the following stronger result than Theorem 6.11.

THEOREM 7.6. *Let* N *be an extrinsic sphere of a locally Hermitian* *symmetric space* $\tilde{M}$, *then either* N *is totally real in* $\tilde{M}$ *or* $\mathrm{rk}\,\tilde{M} > \dim_{\boldsymbol{R}} N$ *and* N *lies in a flat totally geodesic submanifold of* $\tilde{M}$ *as an extrinsic sphere.*

PROOF. From Theorem 3.1 we known that N lies in a totally geodesic submanifold $\overline{M}$ of constant sectional curvature c of $\tilde{M}$ as an extrinsic sphere. If $c = 0$, i.e., $\overline{M}$ is flat, it is done. If $c \neq 0$, $\overline{M}$ is an irreducible, non-Hermition, locally symmetric space immersed in $\tilde{M}$ as a totally geodesic submanifold. Thus, by applying Theorem V, 4.2 of Chen and Nagano (1978), we conclude that $\overline{M}$ is totally real in $\tilde{M}$. In particular, N is totally real in $\tilde{M}$. (Q.E.D.)

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