Multivariable and Vector Analysis

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This set of notes has been organized in such a way to create a single volume suitable for an introduction to some of the basic ideas in multivariable and vector analysis.

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SECTION A --- MULTIVARIABLE ANALYSIS

SECTION B --- VECTOR ANALYSIS

MULTIVARIABLE AND VECTOR ANALYSIS

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Chapter 1

FUNCTIONS OF SEVERAL VARIABLES

1.1. Basic Definitions

In this chapter, we consider functions of the form

$$f : A \to \mathbb{R}^m : \mathbf{x} \mapsto f(\mathbf{x}), \tag{1}$$

where the domain $A \subseteq \mathbb{R}^n$ is a set in the n-dimensional euclidean space, and where the codomain $\mathbb{R}^m$ is the m-dimensional euclidean space. For each $\mathbf{x} \in A$, we can write

$$\mathbf{x} = (x_1, \ldots, x_n),$$

where $x_1, \ldots, x_n \in \mathbb{R}$. In other words, we think of the function (1) as a function of n real variables $x_1, \ldots, x_n$. If $n > 1$, then we say that the function (1) is a function of several (real) variables. On the other hand, we can write

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})),$$

where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \in \mathbb{R}$. We say that the function (1) is a vector valued function. If $m = 1$, then we also say that the function (1) is a real valued function.

EXAMPLE 1.1.1. We are familiar with the case $n = m = 1$ (real valued functions of a real variable) and the case $n = 2$ and $m = 1$ (real valued functions of two real variables).

EXAMPLE 1.1.2. The area of a rectangular box in 3-dimensional euclidean space can be given by a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$ consists of all triples $\mathbf{x} = (x, y, z)$, with real numbers $x, y, z \geq 0$ describing the lengths of the sides of the rectangular box. Here $f(\mathbf{x}) = xyz$.

EXAMPLE 1.1.3. To describe the flow of air, we may consider a function of the form $f : A \to \mathbb{R}^3$, where $A \subseteq \mathbb{R}^4$ consists of all 4-tuples $\mathbf{x} = (x, y, z, t)$, with the triple (x, y, z) describing the position and the real number t describing time. Here the image $f(\mathbf{x})$ describes the velocity of air at position (x, y, z) and time t.

EXAMPLE 1.1.4. We can define $f : \mathbb{R}^7 \to \mathbb{R}^5$ by writing

$$f(x_1, \ldots, x_7) = (x_1 + x_2, x_3x_4x_5, x_2x_6, x_1x_7, x_2x_3 + x_5)$$

for every $(x_1, \ldots, x_7) \in \mathbb{R}^7$.

1.2. Open Sets

In the next section, we shall develop the concept of continuity. To do this, we need the notion of a limit. However, to understand limits, we must first study open sets.

DEFINITION. For every $\mathbf{x}_0 \in \mathbb{R}^n$ and every real number $r > 0$, the set

$$D(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < r\}$$

is called the open disc or open ball with centre $\mathbf{x}_0$ and radius r. Here $\|\mathbf{x} - \mathbf{x}_0\|$ denotes the euclidean distance between $\mathbf{x}$ and $\mathbf{x}_0$.

REMARK. More precisely, for every $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the quantity $\|\mathbf{y}\|$ denotes the norm of the vector $\mathbf{y}$, and is given by

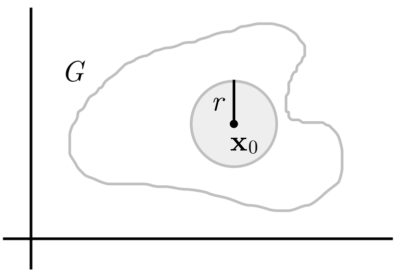
$$\|\mathbf{y}\| = \sqrt{y_1^2 + \ldots + y_n^2}.$$

EXAMPLE 1.2.1. Suppose that $x_0 \in \mathbb{R}$. Then $D(x_0, r)$ denotes the open interval $(x_0 - r, x_0 + r)$.

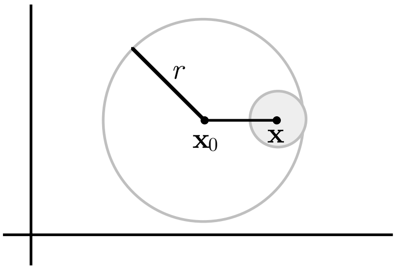
EXAMPLE 1.2.2. For $n = 2$, $D(\mathbf{0}, r)$ is the open disc $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2\}$.

EXAMPLE 1.2.3. For $n = 3$, $D(\mathbf{0}, r)$ is the open ball $\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < r^2\}$.

DEFINITION. A set $G \subseteq \mathbb{R}^n$ is said to be an open set if, for every $\mathbf{x}_0 \in G$, there exists $r > 0$ such that the open disc $D(\mathbf{x}_0, r) \subseteq G$. In other words, a set G is open if every point of G is the centre of some open disc contained in G.



EXAMPLE 1.2.4. For every $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$, the open disc $D(\mathbf{x}_0, r)$ is open. To see this, we shall show that for every $\mathbf{x} \in D(\mathbf{x}_0, r)$, we can find some $s > 0$ such that $D(\mathbf{x}, s) \subseteq D(\mathbf{x}_0, r)$. The picture below in the case $n = 2$ should convince you that $s = r - \|\mathbf{x} - \mathbf{x}_0\|$ is a suitable choice.



To prove that $D(\mathbf{x}, s) \subseteq D(\mathbf{x}_0, r)$, note that for every $\mathbf{y} \in D(\mathbf{x}, s)$, we have

$$\|\mathbf{y} - \mathbf{x}_0\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{x}_0\| \le \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < s + \|\mathbf{x} - \mathbf{x}_0\|,$$

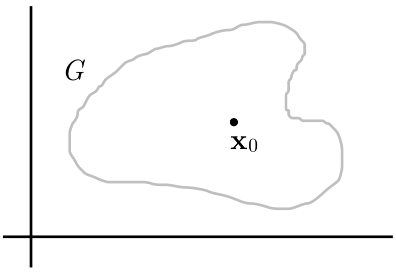
using the Triangle inequality for vectors in $\mathbb{R}^n$. It now follows from our choice of s that $\|\mathbf{y} - \mathbf{x}_0\| < r$, so that $\mathbf{y} \in D(\mathbf{x}_0, r)$.

EXAMPLE 1.2.5. The set $G = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1 \text{ and } |x_2| < 1\}$ in $\mathbb{R}^2$ is open.

EXAMPLE 1.2.6. The set $G = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ in $\mathbb{R}^3$ is open.

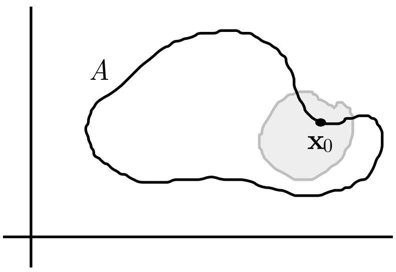
Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$ is given. On many occasions, we do not need to consider open discs centred at $\mathbf{x}_0$. Very often, it may be sufficient to consider some open set containing $\mathbf{x}_0$. We therefore make the following definition for convenience.

DEFINITION. Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$. Then any open set G such that $\mathbf{x}_0 \in G$ is called a neighbourhood of the point $\mathbf{x}_0$. In other words, any open set containing $\mathbf{x}_0$ is a neighbourhood of $\mathbf{x}_0$.



To complete our preparation before introducing the idea of a limit, we make two more definitions.

DEFINITION. Suppose that $A \subseteq \mathbb{R}^n$ is given. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is said to be a boundary point of A if every neighbourhood of $\mathbf{x}_0$ contains a point of A as well as a point not in A.



REMARK. Note that a boundary point of a set A does not necessarily belong to A.

DEFINITION. Suppose that $A \subseteq \mathbb{R}^n$ is given. The set $\overline{A}$, containing precisely all the points of A and all the boundary points of A, is called the closure of A.

EXAMPLE 1.2.7. In $\mathbb{R}$, the intervals $(0,1)$, $(0,1]$, $[0,1)$ and $[0,1]$ all have boundary points 0 and 1, and closure $[0,1]$.

EXAMPLE 1.2.8. The boundary points of an open disc in $\mathbb{R}^2$ are precisely all the points of a circle. The closure of an open disc in $\mathbb{R}^2$ is the open disc together with its boundary circle.

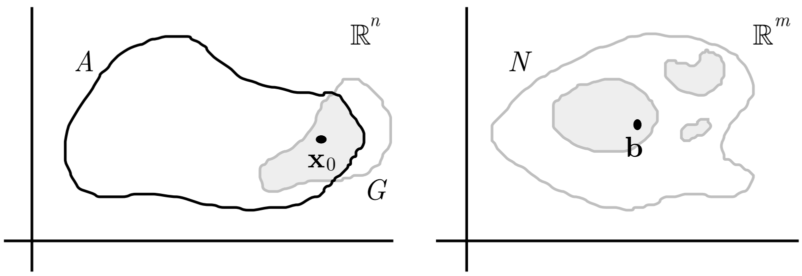
1.3. Limits and Continuity

In this section, we shall use the idea of neighbourhoods to study limits. The interested reader may wish also to study the ϵ-δ approach discussed in the next section.

DEFINITION. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}_0 \in \overline{A}$ and $\mathbf{b} \in \mathbb{R}^m$. We say that the function f has a limit $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, denoted by $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$ or

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b},$$

if, given any neighbourhood N of $\mathbf{b}$, there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$.



EXAMPLE 1.3.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

Then f has no limit as x approaches 2.

EXAMPLE 1.3.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Then $f(x) \to 0$ as $x \to 2$.

EXAMPLE 1.3.3. In Example 1.3.1, if we change the domain of the function to $A = (-\infty, 2)$, then $f(x) \to 0$ as $x \to 2$.

Below we state three results. The interested reader may refer to the next section for the proofs.

THEOREM 1A. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$, and that $f(\mathbf{x}) \to \mathbf{b}_1$ and $f(\mathbf{x}) \to \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$, where $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^m$. Then $\mathbf{b}_1 = \mathbf{b}_2$. In other words, the limit, if it exists, is unique.*

THEOREM 1B. *Suppose that $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$. Then as $\mathbf{x} \to \mathbf{x}_0$,*
(a) if $f(\mathbf{x}) \to \mathbf{b}$, then $(cf)(\mathbf{x}) \to c\mathbf{b}$ for every $c \in \mathbb{R}$;
(b) if $f(\mathbf{x}) \to \mathbf{b}_1$ and $g(\mathbf{x}) \to \mathbf{b}_2$, then $(f + g)(\mathbf{x}) \to \mathbf{b}_1 + \mathbf{b}_2$; and
(c) if $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$, then $f(\mathbf{x}) \to \mathbf{b}$ if and only if $f_i(\mathbf{x}) \to b_i$ for every $i = 1, \ldots, m$, where $\mathbf{b} = (b_1, \ldots, b_m)$.

THEOREM 1C. *Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$. Then as $\mathbf{x} \to \mathbf{x}_0$,*
(a) if $f(\mathbf{x}) \to b_1$ and $g(\mathbf{x}) \to b_2$, then $(fg)(\mathbf{x}) \to b_1 b_2$; and
(b) if $f(\mathbf{x}) \to b \neq 0$, then $(1/f)(\mathbf{x}) \to 1/b$.

We can now define continuity in terms of limits.

DEFINITION. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. The function f is said to be continuous at $\mathbf{x}_0 \in A$ if $f(\mathbf{x}) \to f(\mathbf{x}_0)$ as $\mathbf{x} \to \mathbf{x}_0$. Furthermore, we say that f is continuous in A if f is continuous at every $\mathbf{x}_0 \in A$.

Corresponding to Theorems 1B and 1C, we deduce immediately the following two results.

THEOREM 1D. *Suppose that $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$.*
(a) If f is continuous at $\mathbf{x}_0$, then cf is also continuous at $\mathbf{x}_0$.
(b) If f and g are continuous at $\mathbf{x}_0$, then $f + g$ is also continuous at $\mathbf{x}_0$.
(c) If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$, then f is continuous at $\mathbf{x}_0$ if and only if $f_1, \ldots, f_m$ are all continuous at $\mathbf{x}_0$.

THEOREM 1E. *Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$.*
(a) If f and g are continuous at $\mathbf{x}_0$, then fg is also continuous at $\mathbf{x}_0$.
(b) If f is continuous at $\mathbf{x}_0$ and $f(\mathbf{x}_0) \neq 0$, then $1/f$ is also continuous at $\mathbf{x}_0$.

EXAMPLE 1.3.4. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$, given by

$$f(x_1, x_2, x_3) = \left(x_1^2 x_2 + x_3, \frac{x_2 x_3}{1 + x_1^2} \right).$$

We shall show that f is continuous in $\mathbb{R}^3$. Using Theorem 1D(c), it suffices to show that both components

$$(x_1, x_2, x_3) \mapsto x_1^2 x_2 + x_3 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto \frac{x_2 x_3}{1 + x_1^2} \tag{2}$$

are continuous in $\mathbb{R}^3$. Clearly all three functions

$$(x_1, x_2, x_3) \mapsto x_1 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto x_2 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto x_3,$$

as well as the constant function $(x_1, x_2, x_3) \mapsto 1$, are continuous in $\mathbb{R}^3$. It follows from Theorem 1E(a) and Theorem 1D(b) that the function on the left hand side of (2) is continuous in $\mathbb{R}^3$. It follows from Theorem 1E and Theorem 1D(b) that the function on the right hand side of (2) is also continuous in $\mathbb{R}^3$.

We also have the following result on compositions of functions.

THEOREM 1F. *Suppose that $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^p$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Suppose further that $f(A) \subseteq B$, so that $g \circ f : A \to \mathbb{R}^p$ is well defined. If f is continuous at $\mathbf{x}_0 \in A$ and g is also continuous at $\mathbf{y}_0 = f(\mathbf{x}_0) \in B$, then $g \circ f$ is continuous at $\mathbf{x}_0$.*

1.4. Limits and Continuity: Proofs

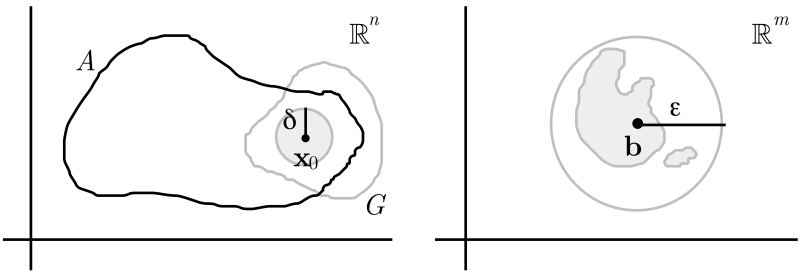
The purpose of this section is to illustrate the equivalence between the neighbourhood approach and the ϵ-δ approach, and to establish Theorems 1A, 1B, 1C and 1F. The material is optional for students not proceeding beyond the current unit of study. However, other students are advised to study the proofs carefully.

THEOREM 1G. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$ if and only if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.*

PROOF. ($\Rightarrow$) Suppose that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. Then given any neighbourhood N of $\mathbf{b}$, there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$. Let $\epsilon > 0$ be given. Clearly

$$N = D(\mathbf{b}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{b}\| < \epsilon\},$$

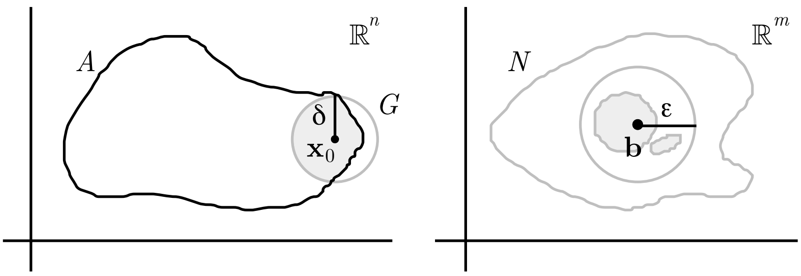
the open disc with centre $\mathbf{b}$ and radius ϵ, is a neighbourhood of $\mathbf{b}$. It follows that there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in D(\mathbf{b}, \epsilon)$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$; in other words, $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$.



Since $G \subseteq \mathbb{R}^n$ is an open set, there exists $\delta > 0$ such that $D(\mathbf{x}_0, \delta) \subseteq G$. We therefore conclude that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in D(\mathbf{x}_0, \delta) \cap A$; in other words, for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

($\Leftarrow$) Suppose that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Let N be a neighbourhood of $\mathbf{b}$. Since $N \subseteq \mathbb{R}^m$ is an open set, there exists $\epsilon > 0$ such that

$$D(\mathbf{b}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{b}\| < \epsilon\} \subseteq N.$$



It follows that there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Let

$$G = B(\mathbf{x}_0, \delta) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < \delta\}.$$

Then clearly $f(\mathbf{x}) \in D(\mathbf{b}, \epsilon) \subseteq N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$. It follows that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. ◯

PROOF OF THEOREM 1A. Since $f(\mathbf{x}) \to \mathbf{b}_1$ and $f(\mathbf{x}) \to \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$, it follows from Theorem 1G that given any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}_1\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1,$$

and

$$\|f(\mathbf{x}) - \mathbf{b}_2\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then by the Triangle inequality,

$$\|\mathbf{b}_1 - \mathbf{b}_2\| \leq \|f(\mathbf{x}) - \mathbf{b}_1\| + \|f(\mathbf{x}) - \mathbf{b}_2\| < 2\epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Since $\epsilon > 0$ is arbitrary and $\|\mathbf{b}_1 - \mathbf{b}_2\|$ is independent of $\mathbf{x}$, we must have $\|\mathbf{b}_1 - \mathbf{b}_2\| = 0$, so that $\mathbf{b}_1 = \mathbf{b}_2$. ◯

PROOF OF THEOREM 1B. We use Theorem 1G to enable us to use the ϵ-δ approach.

(a) The result is trivial if $c = 0$, so we shall assume that $c \neq 0$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}\| < \frac{\epsilon}{c} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

It follows that

$$\|(cf)(\mathbf{x}) - c\mathbf{b}\| = c\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(b) Given any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}_1\| < \frac{\epsilon}{2} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1,$$

and

$$\|g(\mathbf{x}) - \mathbf{b}_2\| < \frac{\epsilon}{2} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then by the Triangle inequality,

$$\|(f + g)(\mathbf{x}) - (\mathbf{b}_1 - \mathbf{b}_2)\| \leq \|f(\mathbf{x}) - \mathbf{b}_1\| + \|g(\mathbf{x}) - \mathbf{b}_2\| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(c) Suppose first of all that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

For every $i = 1, \ldots, m$, it is clear that

$$|f_i(\mathbf{x}) - b_i| \leq \|f(\mathbf{x}) - \mathbf{b}\|,$$

so it follows easily that $|f_i(\mathbf{x}) - b_i| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Hence $f_i(\mathbf{x}) \to b_i$ as $\mathbf{x} \to \mathbf{x}_0$ for every $i = 1, \ldots, m$. Suppose now that $f_i(\mathbf{x}) \to b_i$ as $\mathbf{x} \to \mathbf{x}_0$ for every $i = 1, \ldots, m$. Then given any $\epsilon > 0$ and any $i = 1, \ldots, m$, there exists $\delta_i > 0$ such that

$$|f_i(\mathbf{x}) - b_i| < \frac{\epsilon}{m} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_i.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_m\}$. Then

$$\|f(\mathbf{x}) - \mathbf{b}\| \leq \sum_{i=1}^{m} |f_i(\mathbf{x}) - b_i| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Hence $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. ◯

Proof of Theorem 1C. We again use Theorem 1G to enable us to use the ϵ-δ approach.

(a) We use the inequality

$$|f(\mathbf{x})g(\mathbf{x}) - b_1b_2| = |f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})b_2 + f(\mathbf{x})b_2 - b_1b_2| \leq |f(\mathbf{x})||g(\mathbf{x}) - b_2| + |b_2||f(\mathbf{x}) - b_1|.$$

Since $f(\mathbf{x}) \to b_1$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that (with $\epsilon = 1$) there exists $\delta_1 > 0$ such that

$$|f(\mathbf{x}) - b_1| < 1 \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

It follows from the Triangle inequality that

$$|f(\mathbf{x})| \leq |f(\mathbf{x}) - b_1| + |b_1| < 1 + |b_1| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

Let $\epsilon > 0$ be given. Since $f(\mathbf{x}) \to \mathbf{b}_1$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_2 > 0$ such that

$$|f(\mathbf{x}) - b_1| < \frac{\epsilon}{2(1 + |b_2|)} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Since $g(\mathbf{x}) \to b_2$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_3 > 0$ such that

$$|g(\mathbf{x}) - b_2| < \frac{\epsilon}{2(1 + |b_1|)} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_3.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then it is easily seen that $|f(\mathbf{x})g(\mathbf{x}) - b_1b_2| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(b) We use the identity

$$\left| \frac{1}{f(\mathbf{x})} - \frac{1}{b} \right| = \frac{|f(\mathbf{x}) - b|}{|f(\mathbf{x})||b|}.$$

Since $b \neq 0$ and $f(\mathbf{x}) \to b$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that (with $\epsilon = \frac{1}{2}|b|$) there exists $\delta_1 > 0$ such that

$$|f(\mathbf{x}) - b| < \frac{1}{2}|b| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

It follows from the Triangle inequality that

$$|f(\mathbf{x})| \geq +|b| - |f(\mathbf{x}) - b| > \frac{1}{2}|b| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

Let $\epsilon > 0$ be given. Since $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_2 > 0$ such that

$$|f(\mathbf{x}) - b| < \frac{1}{2}|b|^2\epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it is easily seen that

$$\left| \frac{1}{f(\mathbf{x})} - \frac{1}{b} \right| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. ◯

PROOF OF THEOREM 1F. Since g is continuous at $\mathbf{y}_0 = f(\mathbf{x}_0)$, it follows that given any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\|g(\mathbf{y}) - g(f(\mathbf{x}_0))\| = \|g(\mathbf{y}) - g(\mathbf{y}_0)\| < \epsilon \qquad \text{for every } \mathbf{y} \in B \text{ satisfying } \|\mathbf{y} - f(\mathbf{x}_0)\| < \eta.$$

Since f is also continuous at $\mathbf{x}_0$, it follows that given any $\eta > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \eta \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

Suppose now that $\mathbf{x} \in A$ satisfies $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Then it follows that $f(\mathbf{x}) \in f(A) \subseteq B$ and satisfies $\|f(\mathbf{x}) - \mathbf{y}_0\| < \eta$, so that

$$\|(g \circ f)(\mathbf{x}) - (g \circ f)(\mathbf{x}_0)\| = \|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))\| < \epsilon$$

as required. ◯

PROBLEMS FOR CHAPTER 1

1. Draw each of the following sets in $\mathbb{R}^2$, and determine heuristically whether it is open:
   a) $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$
   b) $A = \{(x, y) \in \mathbb{R}^2 : 0 \le x^2 + y^2 < 4\}$
   c) $A = \{(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : 0 \le \theta < \pi/4 \text{ and } \theta^2 < r < \theta\}$

2. Suppose that G and H are both neighbourhoods of a point $\mathbf{x}_0 \in \mathbb{R}^n$. Show that both $G \cap H$ and $G \cup H$ are neighbourhoods of $\mathbf{x}_0$.
   [REMARK: You may assume that both $G \cap H$ and $G \cup H$ are open. Alternatively, you may take it as a challenge to prove these two assertions first.]

3. Use the arithmetic of limits to evaluate

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{xy}{x^2 + y^2 + 2}, \frac{x^2 + 3y^2 + 4yz}{x + y + 1} \right).$$

4. Use the ϵ-δ definition of a limit to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = 0.$$

5. For each of the following two functions, investigate the behaviour on each of the two lines given, and explain why the limit does not exist as $(x, y) \to (0, 0)$:

   a) $f(x, y) = \dfrac{x^2}{x^2 + y^2}$; lines $x = 0$ and $y = 0$

   b) $f(x, y) = \dfrac{(x - y)^2}{x^2 + y^2}$; lines $y = x$ and $y = -x$

6. Consider the function $h : \mathbb{R}^3 \to \mathbb{R}$, given by

$$h(x, y, z) = \begin{cases} \dfrac{1 - \cos(x + y + z)}{(x + y + z)^2} & \text{if } x + y + z \neq 0, \\ \dfrac{1}{2} & \text{if } x + y + z = 0. \end{cases}$$

   By writing down suitable functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ which are continuous at $(0, 0, 0)$ and $f(0, 0, 0)$ respectively, use Theorem 1F to show that

$$\lim_{(x,y,z)\to(0,0,0)} h(x, y, z) = \frac{1}{2}.$$

7. Use Theorem 1F to show that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1.$$

8. Consider the open disc $D(\mathbf{0}, 1) \in \mathbb{R}^2$, centred at the origin $\mathbf{0} = (0, 0)$ and of radius 1. Suppose that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in D(\mathbf{0}, 1)$. Show that $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to (1, 0)$.
   [Remark: Note that $(1, 0) \notin D(\mathbf{0}, 1)$.]

9. Find a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0, 0) = 1$, $f(1, 0) = 0$, and $0 \leq f(x, y) \leq 1$ for every $(x, y) \in \mathbb{R}^2$.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 2

DIFFERENTIATION

2.1. Partial Derivatives

The notion of differentiability for vector valued functions of several variables is more complicated than one might expect. As a first step, we consider partial derivatives of real valued functions of several variables.

Definition. Consider a function of the form $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set. For every $\mathbf{x} = (x_1, \ldots, x_n) \in A$ and every $j = 1, \ldots, n$, the limit

$$\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h},$$

if it exists, is called the j-th partial derivative of f.

Remark. If for every $j = 1, \ldots, n$, we write

$$\mathbf{e}_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{n-j}),$$

then

$$\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h}.$$

Example 2.1.1. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = \sin xy + x \cos y$ for every $(x, y) \in \mathbb{R}^2$. Then

$$\frac{\partial f}{\partial x} = y \cos xy + \cos y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x \cos xy - x \sin y.$$

EXAMPLE 2.1.2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$, where

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$$

for every $(x,y) \in \mathbb{R}^2$. Then

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

EXAMPLE 2.1.3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x,y) = x^{1/5}y^{1/5}$ for every $(x,y) \in \mathbb{R}^2$. Then to obtain the partial derivatives at $(0,0)$, we cannot simply differentiate and substitute $(x,y) = (0,0)$. In fact, we can work from first definitions that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

Let us study the restriction of the function to the plane $x = y$. Try to draw a picture to convince yourself that the function $f(x,x) = x^{2/5}$ is not differentiable at $x = 0$. It follows that the graph of the function $f(x,y) = x^{1/5}y^{1/5}$ does not have a tangent plane at $(x,y) = (0,0)$. This suggests that differentiability of a function at a point has to be more than just the existence of partial derivatives at that point.

2.2. Total Derivatives

Next, we turn to the idea of differentiability at a point. To motivate this, let us consider the following two examples.

EXAMPLE 2.2.1. Suppose that $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, is differentiable at a point x_0. Consider the tangent to the curve of f at the point $(x_0, f(x_0))$. The slope of this tangent is $f'(x_0)$, and it is easy to see that the equation of the tangent is the line

$$y = f(x_0) + f'(x_0)(x - x_0).$$

It follows that as $x \to x_0$, the quantity $f(x_0) + f'(x_0)(x - x_0)$ is a good approximation of the function $f(x)$. On the other hand, note that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

so that

$$\lim_{x\to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| = 0,$$

whence

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} = 0.$$

EXAMPLE 2.2.2. Suppose that $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$. Suppose further that $(x_0, y_0) \in A$ and that there is a tangent plane to the surface of f at $(x_0, y_0, f(x_0, y_0))$. The equation of a non-vertical plane is of the form $z = ax + by + c$, and it is not difficult to show that the tangent plane must be of the form

$$z = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y - y_0).$$

It follows that as $(x, y) \to (x_0, y_0)$, the quantity

$$f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0)$$

is a good approximation of the function $f(x, y)$. If we take into consideration Example 2.2.1, we may perhaps wish to say that f is differentiable at (x_0, y_0) if $\partial f/\partial x$ and $\partial f/\partial y$ exist at (x_0, y_0) and if

$$\text{(1)} \qquad \lim_{(x,y)\to(x_0,y_0)} \frac{\left| f(x, y) - f(x_0, y_0) - \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) - \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0)\right|}{\|(x, y) - (x_0, y_0)\|} = 0.$$

Now write

$$(\mathbf{D}f)(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

as a row matrix. Then with a slight abuse of notation, we have

$$f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) = f(x_0, y_0) + (\mathbf{D}f)(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

and so (1) can be written in the form

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\left| f(x, y) - f(x_0, y_0) - (\mathbf{D}f)(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}\right|}{\|(x, y) - (x_0, y_0)\|} = 0.$$

We now try to generalize our discussion so far to the case of real valued functions of several real variables.

Definition. Consider a function of the form $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set. We say that f is differentiable at $\mathbf{x}_0 \in A$ if all partial derivatives

$$\frac{\partial f}{\partial x_j}, \qquad \text{where } j = 1, \ldots, n,$$

exist and if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0,$$

where $(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ denotes the matrix product of

$$(\mathbf{D}f)(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) & \ldots & \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}$$

with the vector $\mathbf{x} - \mathbf{x}_0$ regarded as a column matrix.

The generalization to vector valued functions is now rather straightforward.

Definition. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose that $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$. Then we say that f is differentiable at $\mathbf{x}_0 \in A$ if $f_i : A \to \mathbb{R}$ is differentiable at $\mathbf{x}_0 \in A$ for every $i = 1, \ldots, m$. We say that $f : A \to \mathbb{R}^m$ is differentiable if f is differentiable at every $\mathbf{x}_0 \in A$.

DEFINITION. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose that $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$. Then the total derivative of f at $\mathbf{x}_0 \in A$ is defined to be the $m \times n$ matrix

$$\mathbf{T} = (\mathbf{D}f)(\mathbf{x}_0) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \ldots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \ldots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix} \tag{2}$$

if all the partial derivatives exist. The matrix $\mathbf{T}$ is also called the matrix of partial derivatives.

REMARK. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. It can be shown that f is differentiable at $\mathbf{x}_0 \in A$ if and only if all partial derivatives

$$\frac{\partial f_i}{\partial x_j}, \qquad \text{where } i = 1, \ldots, m \text{ and } j = 1, \ldots, n,$$

exist and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0, \tag{3}$$

where $\mathbf{T}(\mathbf{x} - \mathbf{x}_0)$ denotes the matrix product of $\mathbf{T}$ given by (2) with the vector $\mathbf{x} - \mathbf{x}_0$ regarded as a column matrix. To see this, note first of all that the existence of the partial derivatives is obvious. Next, note that for any vector $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we have $|y_i| \le \|\mathbf{y}\|$ for every $i = 1, \ldots, m$, and $\|\mathbf{y}\| \le |y_1| + \ldots + |y_m|$. Note now that

$$f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0) = \begin{pmatrix} f_1(\mathbf{x}) - f_1(\mathbf{x}_0) - (\mathbf{D}f_1)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ \vdots \\ f_m(\mathbf{x}) - f_m(\mathbf{x}_0) - (\mathbf{D}f_m)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \end{pmatrix},$$

so that

$$|f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\mathbf{D}f_i)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)| \le \|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|$$

for every $i = 1, \ldots, m$, and

$$\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| \le \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\mathbf{D}f_i)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|.$$

EXAMPLE 2.2.3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x \sin y, x + \sin z)$. In this case, we have $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$, where $f_1(x, y, z) = x \sin y$ and $f_2(x, y, z) = x + \sin z$ for every $(x, y, z) \in \mathbb{R}^3$. It follows that

$$\mathbf{D}f = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin y & x \cos y & 0 \\ 1 & 0 & \cos z \end{pmatrix}.$$

Example 2.2.4. Consider the function $f : A \to \mathbb{R}^2 : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, where $A = \{(r,\theta) \in \mathbb{R}^2 : r > 0\}$. In this case, we have $f(r,\theta) = (f_1(r,\theta), f_2(r,\theta))$, where $f_1(r,\theta) = r\cos\theta$ and $f_2(r,\theta) = r\sin\theta$ for every $(r,\theta) \in A$. It follows that

$$\mathbf{D}f = \begin{pmatrix} \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial \theta} \\ \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Remark. Consider the special case $m = 1$. Then

$$\mathbf{T} = (\mathbf{D}f)(\mathbf{x}_0) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \dots & \dfrac{\partial f}{\partial x_n} \end{pmatrix}.$$

The corresponding vector

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

is called the gradient of f and denoted by ∇f or $\operatorname{grad} f$. For $f : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$, we can use the special notation

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \qquad \text{and} \qquad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

respectively. Here **i**, **j** and **k** denote, as usual, unit vectors in the x, y and z directions.

2.3. Consequences of Differentiability

We know that in the theory of real valued functions of one real variable, a function f is continuous whenever it is differentiable. The purpose of this section is to extend this result to vector valued functions of several variables.

THEOREM 2A. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose further f is differentiable at $\mathbf{x}_0 \in A$. Then f is continuous at $\mathbf{x}_0$.*

† Proof. Since (3) holds, it follows that there exists $\delta_1 > 0$ such that

$$\frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} < 1,$$

and so

$$\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| < \|\mathbf{x} - \mathbf{x}_0\|$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. It follows from the Triangle inequality that

$$\begin{aligned} \|f(\mathbf{x}) - f(\mathbf{x}_0)\| &= \|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0) + \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| \\ &\le \|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| + \|\mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| \\ &< \|\mathbf{x} - \mathbf{x}_0\| + \|\mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| \end{aligned}$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. Next, note that

$$\|\mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| = \left\| \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \ldots & \dfrac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \ldots & \dfrac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix} (\mathbf{x} - \mathbf{x}_0) \right\| = \left\| \begin{pmatrix} (\nabla f_1)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\ \vdots \\ (\nabla f_m)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \end{pmatrix} \right\|$$
$$= \left(\sum_{i=1}^m ((\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0))^2 \right)^{1/2} \leq \left(\sum_{i=1}^m \|(\nabla f_i)(\mathbf{x}_0)\|^2 \|\mathbf{x} - \mathbf{x}_0\|^2 \right)^{1/2}$$
$$= \left(\sum_{i=1}^m \|(\nabla f_i)(\mathbf{x}_0)\|^2 \right)^{1/2} \|\mathbf{x} - \mathbf{x}_0\|,$$

where the inequality follows from the Cauchy-Schwarz inequality in $\mathbb{R}^m$. Now let

$$M = \left(\sum_{i=1}^m \|(\nabla f_i)(\mathbf{x}_0)\|^2 \right)^{1/2} + 1.$$

Then

$$\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < M \|\mathbf{x} - \mathbf{x}_0\|$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. Suppose that $\epsilon > 0$ is given. Then let $\delta = \min\{\delta_1, \epsilon/M\}$. It is now easily seen that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. It follows that f is continuous at $\mathbf{x}_0$. ○

Remark. The Cauchy-Schwarz inequality in $\mathbb{R}^m$ states that for every two vectors $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ in $\mathbb{R}^m$, we have $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$, where $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_m y_m$.

If we look at the proof of Theorem 2A carefully, then it is easy to see that we have also proved the following result.

THEOREM 2B. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose further f is differentiable at $\mathbf{x}_0 \in A$. Then there exist positive real numbers M and δ_1 such that*

$$\|f(\mathbf{x}) - f(\mathbf{x}_0)\| \leq M \|\mathbf{x} - \mathbf{x}_0\|$$

for every $\mathbf{x} \in A$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta_1$.

2.4. Conditions for Differentiability

In the theory of real valued functions of one real variable, we have many examples of continuous functions that are not differentiable. On the other hand, the example

$$f(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 0, \\ 0 & \text{otherwise,} \end{cases}$$

shows that while the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0, 0)$, the function is not continuous at $(0, 0)$ and so not differentiable at $(0, 0)$.

On the other hand, the definition for differentiability is very difficult to use, since it is practically impossible to verify condition (3) in most instances. The following result eases the pain somewhat.

THEOREM 2C. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose further that all partial derivatives*

$$\frac{\partial f_i}{\partial x_j}, \qquad \textit{where } i = 1, \ldots, m \textit{ and } j = 1, \ldots, n,$$

exist and are continuous in a neighbourhood of a point $\mathbf{x}_0 \in A$. Then f is differentiable at $\mathbf{x}_0$.

† Proof. We need to establish (3). To do this, note first of all that

$$f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0) = \begin{pmatrix} f_1(\mathbf{x}) - f_1(\mathbf{x}_0) - (\nabla f_1)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\ \vdots \\ f_m(\mathbf{x}) - f_m(\mathbf{x}_0) - (\nabla f_m)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \end{pmatrix}.$$

It is then easy to see that

$$\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\| \le \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|.$$

It follows that to prove (3), it suffices to show that for every $i = 1, \ldots, m$,

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{|f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0. \tag{4}$$

Our proof will depend on the Mean value theorem for real valued functions of a real variable. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}_0 = (X_1, \ldots, X_n)$. Then

$$\begin{aligned} f_i(\mathbf{x}) - f_i(\mathbf{x}_0) &= f_i(x_1, \ldots, x_n) - f_i(X_1, \ldots, X_n) \\ &= f_i(x_1, \ldots, x_n) - f_i(X_1, x_2, \ldots, x_n) \\ &\quad + f_i(X_1, x_2, \ldots, x_n) - f_i(X_1, X_2, x_3 \ldots, x_n) \\ &\quad + \ldots \\ &\quad + f_i(X_1, \ldots, X_{n-2}, x_{n-1}, x_n) - f_i(X_1, \ldots, X_{n-1}, x_n) \\ &\quad + f_i(X_1, \ldots, X_{n-1}, x_n) - f_i(X_1, \ldots, X_n) \\ &= \frac{\partial f_i}{\partial x_1}(\mathbf{y}_1)(x_1 - X_1) + \ldots + \frac{\partial f_i}{\partial x_n}(\mathbf{y}_n)(x_n - X_n) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{y}_j)(x_j - X_j) \end{aligned}$$

by the Mean value theorem, where for every $j = 1, \ldots, n$,

$$\mathbf{y}_j = (X_1, \ldots, X_{j-1}, y_j, x_{j+1}, \ldots, x_n)$$

for some y_j between x_j and X_j. On the other hand,

$$(\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)(x_j - X_j).$$

It follows that

$$\begin{aligned} |f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)| &= \left| \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}(\mathbf{y}_j) - \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) \right) (x_j - X_j) \right| \\ &\le \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial x_j}(\mathbf{y}_j) - \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) \right| |x_j - X_j| \end{aligned}$$

by the Triangle inequality. Note also that $|x_j - X_j| \le \|\mathbf{x} - \mathbf{x}_0\|$ for every $j = 1, \ldots, n$, so that

$$\frac{|f_i(\mathbf{x}) - f_i(\mathbf{x}_0) - (\nabla f_i)(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \le \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial x_j}(\mathbf{y}_j) - \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) \right|.$$

Clearly the right hand side converges to zero as $\mathbf{x} \to \mathbf{x}_0$, in view of the continuity of the partial derivatives. (4) follows immediately. ○

EXAMPLE 2.4.1. Consider the function $f : A \to \mathbb{R}^2$, given by

$$f(x, y) = \left(\frac{\sin x + \mathrm{e}^y}{x^2 + y^2}, \frac{1}{x^2 + y^2 - 1} \right)$$

for $(x, y) \in A$, where $A \subseteq \mathbb{R}^2$ is some suitable domain. We can write $f(x, y) = (f_1(x, y), f_2(x, y))$, where

$$f_1(x, y) = \frac{\sin x + \mathrm{e}^y}{x^2 + y^2} \qquad \text{and} \qquad f_2(x, y) = \frac{1}{x^2 + y^2 - 1}.$$

Now

$$\frac{\partial f_1}{\partial x} = \frac{(x^2 + y^2)\cos x - 2x(\sin x + \mathrm{e}^y)}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial f_1}{\partial y} = \frac{(x^2 + y^2)\mathrm{e}^y - 2y(\sin x + \mathrm{e}^y)}{(x^2 + y^2)^2}$$

are continuous except when $x = y = 0$. On the other hand,

$$\frac{\partial f_2}{\partial x} = -\frac{2x}{(x^2 + y^2 - 1)^2} \qquad \text{and} \qquad \frac{\partial f_2}{\partial y} = -\frac{2y}{(x^2 + y^2 - 1)^2}$$

are continuous except when $x^2 + y^2 = 1$. It follows from Theorem 2C that f is differentiable at every (x, y) such that $x^2 + y^2 \neq 0$ or 1.

2.5. Properties of the Derivative

The first two results in this section concern the arithmetic of derivatives, and are stated without proof. The interested reader is invited to write the proofs which proceed in a similar way as in the case of real valued functions of one real variable.

THEOREM 2D. *Suppose that $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$ is an open set. Suppose further that $\mathbf{x}_0 \in A$.*
(a) If f is differentiable at $\mathbf{x}_0$, then cf is also differentiable at $\mathbf{x}_0$ for every $c \in \mathbb{R}$, and

$$(\mathbf{D}(cf))(\mathbf{x}_0) = c(\mathbf{D}f)(\mathbf{x}_0).$$

(b) If f and g are differentiable at $\mathbf{x}_0$, then $f + g$ is also differentiable at $\mathbf{x}_0$, and

$$(\mathbf{D}(f + g))(\mathbf{x}_0) = (\mathbf{D}f)(\mathbf{x}_0) + (\mathbf{D}g)(\mathbf{x}_0).$$

THEOREM 2E. *Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose further that f and g are differentiable at $\mathbf{x}_0 \in A$.*
(a) Then fg is also differentiable at $\mathbf{x}_0$, and

$$(\mathbf{D}(fg))(\mathbf{x}_0) = g(\mathbf{x}_0)(\mathbf{D}f)(\mathbf{x}_0) + f(\mathbf{x}_0)(\mathbf{D}g)(\mathbf{x}_0).$$

(b) If $g(\mathbf{x}_0) \neq 0$, then f/g is also differentiable at $\mathbf{x}_0$, and

$$\left(\mathbf{D}\left(\frac{f}{g}\right)\right)(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)(\mathbf{D}f)(\mathbf{x}_0) - f(\mathbf{x}_0)(\mathbf{D}g)(\mathbf{x}_0)}{g^2(\mathbf{x}_0)}.$$

EXAMPLE 2.5.1. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are given by $f(x, y) = x^2 + y^2$ and $g(x, y) = x + y$ for every $(x, y) \in \mathbb{R}^2$. Let

$$h(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{x^2 + y^2}{x + y}.$$

Then

$$(\mathbf{D}h)(x, y) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = \left(\frac{x^2 + 2xy - y^2}{(x + y)^2}, \frac{y^2 + 2xy - x^2}{(x + y)^2}\right)$$

whenever $x + y \neq 0$. On the other hand,

$$\begin{aligned}
\frac{g(x, y)(\mathbf{D}f)(x, y) - f(x, y)(\mathbf{D}g)(x, y)}{g^2(x, y)} &= \frac{g(x, y)\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) - f(x, y)\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)}{g^2(x, y)} \\
= \frac{(x + y)(2x, 2y) - (x^2 + y^2)(1, 1)}{(x + y)^2} &= \frac{(2x(x + y), 2y(x + y)) - (x^2 + y^2, x^2 + y^2)}{(x + y)^2} \\
= \frac{(x^2 + 2xy - y^2, y^2 + 2xy - x^2)}{(x + y)^2} &= \left(\frac{x^2 + 2xy - y^2}{(x + y)^2}, \frac{y^2 + 2xy - x^2}{(x + y)^2}\right)
\end{aligned}$$

whenever $x + y \neq 0$. This verifies the Quotient rule in Theorem 2E.

It is in the Chain rule where there is substantial difference from the case of a real valued function of one real variable. Here, our notation helps to provide some visual resemblance at least.

THEOREM 2F. *Suppose that $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^p$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are open sets. Suppose further that $f(A) \subseteq B$, so that $g \circ f : A \to \mathbb{R}^p$ is well defined. If f is differentiable at $\mathbf{x}_0 \in A$ and g is differentiable at $\mathbf{y}_0 = f(\mathbf{x}_0) \in B$, then $g \circ f$ is differentiable at $\mathbf{x}_0$, and*

$$(\mathbf{D}(g \circ f))(\mathbf{x}_0) = (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{D}f)(\mathbf{x}_0),$$

where the right hand side represents the matrix product of $(\mathbf{D}g)(\mathbf{y}_0)$ and $(\mathbf{D}f)(\mathbf{x}_0)$.

EXAMPLE 2.5.2. Suppose that $f : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (x(t), y(t))$ and $g : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto g(x, y)$ are both differentiable. Then

$$(\mathbf{D}f)(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \qquad \text{and} \qquad (\mathbf{D}g)(x, y) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix},$$

so that we obtain the familiar formula

$$(\mathbf{D}(g \circ f))(t) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

EXAMPLE 2.5.3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2 : (s,t) \mapsto (x(s,t), y(s,t))$ and $g : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto g(x,y)$ are both differentiable. Then

$$(\mathbf{D}f)(s,t) = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} \qquad \text{and} \qquad (\mathbf{D}g)(x,y) = \begin{pmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix},$$

so that

$$(\mathbf{D}(g \circ f))(s,t) = \begin{pmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial g}{\partial y}\dfrac{\partial y}{\partial s} & \dfrac{\partial g}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial g}{\partial y}\dfrac{\partial y}{\partial t} \end{pmatrix}.$$

This gives the change of variable formulae in functions of two variables.

EXAMPLE 2.5.4. Suppose that

$$f : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (x^2, x^2y, \mathrm{e}^z) \qquad \text{and} \qquad g : \mathbb{R}^3 \to \mathbb{R} : (u,v,w) \mapsto u^2 + v^2 - w^2.$$

We can write $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$, where

$$f_1(x,y,z) = x^2 \qquad \text{and} \qquad f_2(x,y,z) = x^2y \qquad \text{and} \qquad f_3(x,y,z) = \mathrm{e}^z$$

for every $(x,y,z) \in \mathbb{R}^3$. Note that

$$(\mathbf{D}g)(u,v,w)(\mathbf{D}f)(x,y,z) = \begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} & \dfrac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 2u & 2v & -2w \end{pmatrix} \begin{pmatrix} 2x & 0 & 0 \\ 2xy & x^2 & 0 \\ 0 & 0 & \mathrm{e}^z \end{pmatrix}$$

$$= \begin{pmatrix} 2x^2 & 2x^2y & -2\mathrm{e}^z \end{pmatrix} \begin{pmatrix} 2x & 0 & 0 \\ 2xy & x^2 & 0 \\ 0 & 0 & \mathrm{e}^z \end{pmatrix} = \begin{pmatrix} 4x^3(1+y^2) & 2x^4y & -2\mathrm{e}^{2z} \end{pmatrix}.$$

Now consider

$$h = g \circ f : \mathbb{R}^3 \to \mathbb{R} : (x,y,z) \mapsto x^4(1+y^2) - \mathrm{e}^{2z}.$$

It is easily seen that

$$(\mathbf{D}h)(x,y,z) = \begin{pmatrix} \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} 4x^3(1+y^2) & 2x^4y & -2\mathrm{e}^{2z} \end{pmatrix}.$$

This verifies the Chain rule.

EXAMPLE 2.5.5. Suppose that in Example 2.5.4, we would like to compute the derivative of $g \circ f$ at $(1,2,0)$. Note first of all that $f(1,2,0) = (1,2,1)$. Then by the Chain rule,

$$(\mathbf{D}(g \circ f))(1,2,0) = (\mathbf{D}g)(1,2,1)(\mathbf{D}f)(1,2,0) = \begin{pmatrix} 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 4 & -2 \end{pmatrix}.$$

† Proof of Theorem 2F. We need to show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|g(f(\mathbf{x})) - g(f(\mathbf{x}_0)) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0. \tag{5}$$

Note first of all that writing $\mathbf{y} = f(\mathbf{x})$, we have

$$\begin{aligned} &\|g(f(\mathbf{x})) - g(f(\mathbf{x}_0)) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\| \\ &\quad = \|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0) + (\mathbf{D}g)(\mathbf{y}_0)(f(\mathbf{x}) - f(\mathbf{x}_0)) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\| \\ &\quad \le \|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\| + \|(\mathbf{D}g)(\mathbf{y}_0)(f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0))\| \end{aligned}$$

by the Triangle inequality. As in the proof of Theorem 2A, there exists a positive constant M', depending only on $(\mathbf{D}g)(\mathbf{y}_0)$, such that $\|(\mathbf{D}g)(\mathbf{y}_0)\mathbf{u}\| \le M'\|\mathbf{u}\|$ for every $\mathbf{u} \in \mathbb{R}^m$. It follows that

$$\|(\mathbf{D}g)(\mathbf{y}_0)(f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0))\| \le M'\|f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\|,$$

and so

$$\begin{aligned} &\|g(f(\mathbf{x})) - g(f(\mathbf{x}_0)) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\| \\ &\quad \le \|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\| + M'\|f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\| \end{aligned}$$

Since f is differentiable at $\mathbf{x}_0$, it follows that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0.$$

It follows that to prove (5), it suffices to show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\|}{\|\mathbf{x}-\mathbf{x}_0\|} = 0. \tag{6}$$

To do this, we first of all recall Theorem 2B. Since f is differentiable at $\mathbf{x}_0$, there exist positive real numbers M and δ_1 such that

$$\|f(\mathbf{x}) - f(\mathbf{x}_0)\| \le M\|\mathbf{x}-\mathbf{x}_0\|$$

for every $\mathbf{x} \in A$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \delta_1$. On the other hand, since g is differentiable at $\mathbf{y}_0$, it follows that

$$\lim_{\mathbf{y}\to\mathbf{y}_0} \frac{\|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\|}{\|\mathbf{y}-\mathbf{y}_0\|} = 0.$$

Hence given any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\| < \frac{\epsilon}{M}\|\mathbf{y}-\mathbf{y}_0\|$$

for every $\mathbf{y} \in B$ satisfying $\|\mathbf{y}-\mathbf{y}_0\| < \eta$. Also, f is continuous at $\mathbf{x}_0$, so there exists $\delta_2 > 0$ such that

$$\|\mathbf{y}-\mathbf{y}_0\| = \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \eta$$

for every $\mathbf{x} \in A$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \delta_2$. Now let $\delta = \min\{\delta_1, \delta_2\}$. Then clearly

$$\|g(\mathbf{y}) - g(\mathbf{y}_0) - (\mathbf{D}g)(\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0)\| < \epsilon\|\mathbf{x}-\mathbf{x}_0\|$$

for every $\mathbf{x} \in A$ satisfying $\|\mathbf{x}-\mathbf{x}_0\| < \delta$. (6) follows immediately. ○

2.6. Gradients and Directional Derivatives

Recall the last remark in Section 2.2. Suppose that $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$ is an open set. Suppose further that f is differentiable at $\mathbf{x}_0 \in A$. Then the vector in $\mathbb{R}^3$ given by

$$(\nabla f)(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x}(\mathbf{x}_0), \frac{\partial f}{\partial y}(\mathbf{x}_0), \frac{\partial f}{\partial z}(\mathbf{x}_0)\right)$$

is called the gradient of f at $\mathbf{x}_0$.

In this section, we shall use this to obtain a formula for tangent planes to surfaces in $\mathbb{R}^3$.

Example 2.6.1. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

for every $(x, y, z) \in \mathbb{R}^3$. Then

$$\begin{aligned}(\nabla f)(x_0, y_0, z_0) &= \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0)\right) \\ &= \left(\frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}, \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}, \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\right) \\ &= \frac{(x_0, y_0, z_0)}{\|(x_0, y_0, z_0)\|}\end{aligned}$$

is the unit vector in the direction of (x_0, y_0, z_0).

Example 2.6.2. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(x, y, z) = \mathrm{e}^{xy} + z$$

for every $(x, y, z) \in \mathbb{R}^3$. Then

$$(\nabla f)(x_0, y_0, z_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0)\right) = (y_0\mathrm{e}^{x_0y_0}, x_0\mathrm{e}^{x_0y_0}, 1).$$

For every $\mathbf{x}_0 \in \mathbb{R}^3$ and every unit vector $\mathbf{n} \in \mathbb{R}^3$, the mapping

$$L : \mathbb{R} \to \mathbb{R}^3 : t \mapsto \mathbf{x}_0 + t\mathbf{n}$$

represents the line through $\mathbf{x}_0$ in the direction $\mathbf{n}$. It follows that for every function of the form $f : \mathbb{R}^3 \to \mathbb{R}$, the composite function

$$f \circ L : \mathbb{R} \to \mathbb{R} : t \mapsto f(\mathbf{x}_0 + t\mathbf{n})$$

represents the function f restricted to the line L.

Definition. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$. Then the limit

$$\lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{n}) - f(\mathbf{x}_0)}{t},$$

if it exists, is called the directional derivative of f at $\mathbf{x}_0$ in the direction $\mathbf{n}$.

REMARK. Note that

$$\lim_{t\to 0}\frac{f(\mathbf{x}_0+t\mathbf{n})-f(\mathbf{x}_0)}{t}=\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{x}_0+t\mathbf{n})\bigg|_{t=0}.$$

THEOREM 2G. *Suppose that $f:\mathbb{R}^3\to\mathbb{R}$ is differentiable. Then all directional derivatives of f exist. Furthermore, for every $\mathbf{x}_0\in\mathbb{R}^3$ and every unit vector $\mathbf{n}\in\mathbb{R}^3$, the directional derivative of f at $\mathbf{x}_0$ in the direction $\mathbf{n}=(n_1,n_2,n_3)$ is given by the scalar product*

$$(\nabla f)(\mathbf{x}_0)\cdot\mathbf{n}=\left(\frac{\partial f}{\partial x}(\mathbf{x}_0)\right)n_1+\left(\frac{\partial f}{\partial y}(\mathbf{x}_0)\right)n_2+\left(\frac{\partial f}{\partial z}(\mathbf{x}_0)\right)n_3.$$

REMARK. Note that

$$(\mathbf{D}f)(\mathbf{x}_0)\mathbf{n}=(\nabla f)(\mathbf{x}_0)\cdot\mathbf{n},$$

where the left hand side denotes the matrix product of the total derivative $(\mathbf{D}f)(\mathbf{x}_0)$ and the column matrix $\mathbf{n}$.

PROOF OF THEOREM 2G. Consider the functions

$$L:\mathbb{R}\to\mathbb{R}^3:t\mapsto\mathbf{x}_0+t\mathbf{n}=(x_0+tn_1,y_0+tn_2,z_0+tn_3)=(L_1(t),L_2(t),L_3(t))$$

and

$$g=f\circ L:\mathbb{R}\to\mathbb{R}:t\mapsto f(\mathbf{x}_0+t\mathbf{n}).$$

Note that

$$\lim_{t\to 0}\frac{f(\mathbf{x}_0+t\mathbf{n})-f(\mathbf{x}_0)}{t}=\lim_{t\to 0}\frac{g(t)-g(0)}{t}=\frac{\mathrm{d}g}{\mathrm{d}t}(0)=(\mathbf{D}g)(0).$$

By the Chain rule, we have

$$(\mathbf{D}g)(0)=(\mathbf{D}f)(L(0))(\mathbf{D}L)(0).$$

Since

$$(\mathbf{D}f)(L(0))=(\mathbf{D}f)(\mathbf{x}_0)=\begin{pmatrix}\dfrac{\partial f}{\partial x}(\mathbf{x}_0) & \dfrac{\partial f}{\partial y}(\mathbf{x}_0) & \dfrac{\partial f}{\partial z}(\mathbf{x}_0)\end{pmatrix}$$

and

$$(\mathbf{D}L)(0)=\begin{pmatrix}\dfrac{\mathrm{d}L_1}{\mathrm{d}t}(0)\\ \dfrac{\mathrm{d}L_2}{\mathrm{d}t}(0)\\ \dfrac{\mathrm{d}L_3}{\mathrm{d}t}(0)\end{pmatrix}=\begin{pmatrix}n_1\\ n_2\\ n_3\end{pmatrix},$$

it follows that

$$(\mathbf{D}g)(0)=(\nabla f)(\mathbf{x}_0)\cdot\mathbf{n}.$$

The result follows immediately. ○

Example 2.6.3. We continue our discussion in Example 2.6.2, where $f(x, y, z) = e^{xy} + z$ for every $(x, y, z) \in \mathbb{R}^3$. The rate of change of the function f at $(1, 0, 1)$ in the direction of the unit vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ is given by

$$(\nabla f)(1, 0, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = (0, 1, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \frac{2}{\sqrt{3}}.$$

We now investigate the directional derivative further. Suppose that $(\nabla f)(\mathbf{x}_0) \neq \mathbf{0}$. Then for every unit vector $\mathbf{n} \in \mathbb{R}^3$, we have

$$(\nabla f)(\mathbf{x}_0) \cdot \mathbf{n} = \|(\nabla f)(\mathbf{x}_0)\| \cos \theta,$$

where θ is the angle between $(\nabla f)(\mathbf{x}_0)$ and $\mathbf{n}$. This is maximum if $\theta = 0$ and minimum if $\theta = \pi$, corresponding respectively to $(\nabla f)(\mathbf{x}_0)$ and $\mathbf{n}$ being in the same direction and being in opposite directions. In other words, $(\nabla f)(\mathbf{x}_0)$ is in the direction in which f increases the fastest, while $-(\nabla f)(\mathbf{x}_0)$ is in the direction in which f decreases the fastest.

Example 2.6.4. We continue our discussion in Examples 2.6.2 and 2.6.3, where $f(x, y, z) = e^{xy} + z$ for every $(x, y, z) \in \mathbb{R}^3$. The maximum rate of change of the function f at $(1, 0, 1)$ is in the direction of the vector $(\nabla f)(1, 0, 1) = (0, 1, 1)$. Since $(0, 1/\sqrt{2}, 1/\sqrt{2})$ is the unit vector in this direction, it follows that the maximum rate of change of the function f at $(1, 0, 1)$ is equal to

$$(\nabla f)(1, 0, 1) \cdot \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = (0, 1, 1) \cdot \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \sqrt{2}.$$

Suppose now that $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable. Consider a surface of the form

$$S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\},$$

where $c \in \mathbb{R}$ is a constant. Suppose further that $(x_0, y_0, z_0) \in S$ and $(\nabla f)(x_0, y_0, z_0) \neq (0, 0, 0)$ (the reader is advised to draw a picture). Let $g : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function such that $g(\mathbb{R}) \subseteq S$ and $g(0) = (x_0, y_0, z_0)$; in other words, g is a path on S that passes through the point (x_0, y_0, z_0) when $t = 0$. Then clearly

$$(f \circ g)(t) = c$$

for every $t \in \mathbb{R}$, so it follows from the Chain rule that

$$0 = (\mathbf{D}(f \circ g))(0) = (\mathbf{D}f)(x_0, y_0, z_0)(\mathbf{D}g)(0) = (\nabla f)(x_0, y_0, z_0) \cdot \mathbf{v},$$

where $\mathbf{v} = (\mathbf{D}g)(0)$ is a tangent vector to the path $g(t)$ at $t = 0$, and so is tangent to the surface S at (x_0, y_0, z_0). It follows that $(\nabla f)(x_0, y_0, z_0)$ must be normal to the surface S at (x_0, y_0, z_0). It also follows that

$$(\nabla f)(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

is the equation of the tangent plane to the surface S at (x_0, y_0, z_0).

Example 2.6.5. Consider the surface $2xy + z^3 = 3$ at the point $(1, 1, 1)$. Here $f(x, y, z) = 2xy + z^3$. It is easy to check that $(\nabla f)(1, 1, 1) = (2, 2, 3)$. Hence the equation of the tangent plane to the surface at $(1, 1, 1)$ is $(2, 2, 3) \cdot (x - 1, y - 1, z - 1) = 0$; in other words, $2x + 2y + 3z = 7$.

Problems for Chapter 2

1. Compute the matrix of partial derivatives of each of the following functions:
   a) $f : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (e^{x+y} + z, \sin(x + y + z) - \cos(x - y))$
   b) $f : \mathbb{R}^2 \to \mathbb{R}^4 : (x, y) \mapsto (x + y, x - y, 2x + y^2, y)$
   c) $f : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x^2 + 2y + 3z^2, \sin(x^2 + y^2), \cos z)$
   d) $f : \mathbb{R}^2 \to \mathbb{R}^5 : (x, y) \mapsto (\sin x, \cos y, \sin y, \cos x, e^{xy})$

2. For each of the following functions, determine precisely where the function is differentiable, and find the total derivatives at these points:
   a) $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^3 - y^3$
   b) $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (|x|, e^{x+y})$

3. Suppose that $A \subseteq \mathbb{R}^n$ is an open set, and that $\mathbf{x}_0 \in A$. Suppose further that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both differentiable at $\mathbf{x}_0$, and that $g(\mathbf{x}) > 0$ for every $\mathbf{x} \in A$. Explain why the function $h : A \to \mathbb{R}$, defined by
$$h(\mathbf{x}) = \frac{f^3(\mathbf{x}) + f(\mathbf{x})g^2(\mathbf{x})}{f^2(\mathbf{x}) + g(\mathbf{x})}$$
for every $\mathbf{x} \in A$, is differentiable at $\mathbf{x}_0$, and find $(\mathbf{D}h)(\mathbf{x}_0)$.

4. Suppose that $g_1 : \mathbb{R}^4 \to \mathbb{R}$, $g_2 : \mathbb{R}^4 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions. Suppose further that $f : \mathbb{R}^4 \to \mathbb{R}$ is defined by
$$f(\mathbf{x}) = h(g_1(\mathbf{x}), g_2(\mathbf{x}))$$
for every $\mathbf{x} \in \mathbb{R}^4$. Express $(\nabla f)(\mathbf{x})$ in terms of the partial derivatives of g_1, g_2 and h.
[Remark: It is convenient to write $\mathbf{x} = (x_1, x_2, x_3, x_4)$.]

5. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are both differentiable at $\mathbf{x}_0$. Prove that
$$(\nabla(fg))(\mathbf{x}_0) = f(\mathbf{x}_0)(\nabla g)(\mathbf{x}_0) + g(\mathbf{x}_0)(\nabla f)(\mathbf{x}_0).$$

6. Let n be a fixed positive integer. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$f(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   a) For what positive integer values of n is f differentiable at 0?
   b) For what positive integer values of n is the derivative of f continuous at 0?

7. Consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$, given by
$$f(x, y) = (x^2y^3, e^{x-y}) \qquad \text{and} \qquad g(u, v) = (v \sin u, e^u v^2).$$
   a) Show that the function $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $(1, 1)$, and find $(\mathbf{D}h)(1, 1)$.
   b) Find $(\mathbf{D}f)(1, 1)$ and $(\mathbf{D}g)(1, 1)$.
   c) Explain why $(\mathbf{D}h)(1, 1) = (\mathbf{D}g)(1, 1)(\mathbf{D}f)(1, 1)$.

8. Consider the functions $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^4$, defined by $f(x, y, z) = (x^2y, z \sin y)$ and $g(u, v) = (uv, u^2v^2, u + v, u - v^2)$.
   a) Find the total derivatives $\mathbf{D}f$ and $\mathbf{D}g$.
   b) Find the composition $h = g \circ f : \mathbb{R}^3 \to \mathbb{R}^4$.
   c) Find the total derivative $\mathbf{D}h$.
   d) Verify the Chain rule.

9. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   a) Show that the partial derivatives of f exist at every $(x_0, y_0) \in \mathbb{R}^2$, and evaluate them in terms of x_0 and y_0, taking extra care when $(x_0.y_0) = (0,0)$.
   b) Explain why f is not differentiable at $(0,0)$.

10. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$, given by $f(u) = (au, bu)$, where $a, b \in \mathbb{R}$ are fixed. Consider also the function $g : \mathbb{R}^2 \to \mathbb{R}$, given by

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   a) Find $(\mathbf{D}f)(0)$.
   b) Show that $\partial g/\partial x$ and $\partial g/\partial y$ exist at $(0,0)$, and find $(\mathbf{D}g)(0,0)$.
   c) Consider the function $h = g \circ f : \mathbb{R} \to \mathbb{R}$. Show that h is differentiable, and find $h'(0)$.
   d) Explain why $(\mathbf{D}g)(0,0)(\mathbf{D}f)(0) \neq h'(0)$.

11. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   a) Find the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at $(x_0, y_0) \in \mathbb{R}^2$, taking extra care in the case when $(x_0, y_0) = (0,0)$.
   b) Is f differentiable at $(0,0)$? Justify your assertion.

12. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is an even function, so that $f(\mathbf{x}) = f(-\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$. Suppose further that f is differentiable. Find $(\mathbf{D}f)(\mathbf{0})$.

13. Compute the directional derivative of the function $f(x,y,z) = xyz$ at the point $(1,0,1)$ and in the direction of the vector $(-1,0,1)$.

14. For each of the following, find an equation of the tangent plane of the graph of the function f at the point $(x_0, y_0, f(x_0, y_0))$:
   a) $f(x,y) = \sqrt{x^2+y^2}$ at $(x_0, y_0) = (1,1)$
   b) $f(x,y) = 4x^2 + y^2$ at $(x_0, y_0) = (1,2)$

15. Consider the surface $2x^2 + 3y^2 + 4z^2 = 9$. Suppose that a particle leaves the surface at the point $(1,1,1)$ along the normal directed towards the xy-plane, and with the constant speed of 1 unit per second. How long does it take for the particle to reach the xy-plane?

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 3

IMPLICIT AND INVERSE FUNCTION THEOREMS

3.1. Implicit Function Theorem

The Implicit function theorem, in its generality, is rather complicated to state and difficult to prove. Here, we shall begin by stating this result in full, and then continue to examine only special cases of it. Throughout this section, $(\mathbf{x},\mathbf{z}) \in \mathbb{R}^n \times \mathbb{R}^m$ denotes $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\mathbf{z} = (z_1,\ldots,z_m) \in \mathbb{R}^m$. For functions $F_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, where $i = 1,\ldots,m$, we write

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(z_1,\ldots,z_m)} = \det \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix}$$

whenever the partial derivatives in the matrix exist.

THEOREM 3A. (IMPLICIT FUNCTION THEOREM) *Suppose that for every $i = 1,\ldots,m$, the function $F_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ has continuous partial derivatives. Suppose further that there exists a point $(\mathbf{x}_0,\mathbf{z}_0) \in \mathbb{R}^n \times \mathbb{R}^m$ such that*

$$F_i(\mathbf{x}_0,\mathbf{z}_0) = 0 \qquad \textit{for every } i = 1,\ldots,m,$$

and

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(z_1,\ldots,z_m)}(\mathbf{x}_0,\mathbf{z}_0) \neq 0. \tag{1}$$

Then the following three conclusions hold:

(a) *There exist neighbourhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $V \subseteq \mathbb{R}^m$ of $\mathbf{z}_0$ such that there are unique functions*

$$z_i = g_i(\mathbf{x}) = g_i(x_1, \ldots, x_n), \qquad \text{where } i = 1, \ldots, m, \tag{2}$$

defined for every $\mathbf{x} \in U$ and $\mathbf{z} \in V$ and satisfying

$$F_i(\mathbf{x}, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})) = 0 \qquad \text{for every } i = 1, \ldots, m. \tag{3}$$

(b) *If $\mathbf{x} \in U$ and $\mathbf{z} \in V$ satisfy $F_i(\mathbf{x}, \mathbf{z}) = 0$ for every $i = 1, \ldots, m$, then (2) must hold.*

(c) *The functions (2) are continuously differentiable, and for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we have*

$$\frac{\partial g_i}{\partial x_j} = -\frac{\partial(F_1, \ldots, F_m)}{\partial(z_1, \ldots, z_{i-1}, x_j, z_{i+1}, \ldots, z_m)} \bigg/ \frac{\partial(F_1, \ldots, F_m)}{\partial(z_1, \ldots, z_m)}. \tag{4}$$

Remark. Note that

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(z_1, \ldots, z_m)}(\mathbf{x}_0, \mathbf{z}_0) = \det \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1}(\mathbf{x}_0, \mathbf{z}_0) & \ldots & \dfrac{\partial F_1}{\partial z_m}(\mathbf{x}_0, \mathbf{z}_0) \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1}(\mathbf{x}_0, \mathbf{z}_0) & \ldots & \dfrac{\partial F_m}{\partial z_m}(\mathbf{x}_0, \mathbf{z}_0) \end{pmatrix}.$$

Note also that

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(z_1, \ldots, z_{i-1}, x_j, z_{i+1}, \ldots, z_m)} = \det \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1} & \ldots & \dfrac{\partial F_1}{\partial z_{i-1}} & \dfrac{\partial F_1}{\partial x_j} & \dfrac{\partial F_1}{\partial z_{i+1}} & \ldots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \ldots & \dfrac{\partial F_m}{\partial z_{i-1}} & \dfrac{\partial F_m}{\partial x_j} & \dfrac{\partial F_m}{\partial z_{i+1}} & \ldots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix}.$$

We are interested in the special case $m = 1$. Here, $(\mathbf{x}, z) \in \mathbb{R}^n \times \mathbb{R}$ denotes $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

THEOREM 3A'. *Suppose that the function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ has continuous partial derivatives. Suppose further that there exists a point $(\mathbf{x}_0, z_0) \in \mathbb{R}^n \times \mathbb{R}$ such that*

$$F(\mathbf{x}_0, z_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then the following three conclusions hold:

(a) *There exist neighbourhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $V \subseteq \mathbb{R}$ of z_0 such that there is a unique function*

$$z = g(\mathbf{x}) = g(x_1, \ldots, x_n) \tag{5}$$

defined for $\mathbf{x} \in U$ and $z \in V$ and satisfying $F(\mathbf{x}, g(\mathbf{x})) = 0$.

(b) *On the other hand, if $\mathbf{x} \in U$ and $z \in V$ satisfy $F(\mathbf{x}, z) = 0$, then (5) must hold.*

(c) Furthermore, the function (5) *is continuously differentiable, and for every* $j = 1, \ldots, n$*, we have*

$$\frac{\partial g}{\partial x_j} = -\frac{\partial F}{\partial x_j} \Bigg/ \frac{\partial F}{\partial z}. \tag{6}$$

EXAMPLE 3.1.1. Consider the function

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1.$$

Here $n = 2$ and $m = 1$. It is easy to see that $F(x_0, y_0, z_0) = 0$ on the surface of a sphere centred at the origin $(0, 0, 0)$ and with radius 1. On the other hand,

$$\frac{\partial F}{\partial z}(x_0, y_0, z_0) = 2z_0 \neq 0$$

whenever $z_0 \neq 0$; in other words, this partial derivative does not vanish on the surface of the sphere except on the "equator" $z_0 = 0$ and $x_0^2 + y_0^2 = 1$. Clearly the function

$$z = g(x, y) = \sqrt{1 - x^2 - y^2}$$

satisfies the requirements in a sufficiently small neighbourhood of (x_0, y_0, z_0) if $z_0 > 0$, and the function

$$z = g(x, y) = -\sqrt{1 - x^2 - y^2}$$

satisfies the requirements in a sufficiently small neighbourhood of (x_0, y_0, z_0) if $z_0 < 0$. On the other hand, in a neighbourhood of (x_0, y_0, z_0) on the "equator", it is not clear whether we should take the positive or negative root. But then the partial derivative above vanishes here, so the theorem does not apply in this case. Let us investigate the case $z_0 > 0$ further. We have

$$\frac{\partial g}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - y^2}} \qquad \text{and} \qquad -\frac{\partial F}{\partial x} \Bigg/ \frac{\partial F}{\partial z} = -\frac{x}{z} = -\frac{x}{\sqrt{1 - x^2 - y^2}}.$$

We also have

$$\frac{\partial g}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}} \qquad \text{and} \qquad -\frac{\partial F}{\partial y} \Bigg/ \frac{\partial F}{\partial z} = -\frac{y}{z} = -\frac{y}{\sqrt{1 - x^2 - y^2}}.$$

EXAMPLE 3.1.2. Consider the functions

$$F_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (x, y, z, w) \mapsto x^2 + y^2 + z^2 + w^2 - 2,$$
$$F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (x, y, z, w) \mapsto x^2 - y^2 + z^2 - w^2.$$

Here $n = 2$ and $m = 2$. Note that

$$\frac{\partial(F_1, F_2)}{\partial(z, w)}(z_0, w_0) = \det \begin{pmatrix} \dfrac{\partial F_1}{\partial z}(z_0, w_0) & \dfrac{\partial F_1}{\partial w}(z_0, w_0) \\ \dfrac{\partial F_2}{\partial z}(z_0, w_0) & \dfrac{\partial F_2}{\partial w}(z_0, w_0) \end{pmatrix} = \det \begin{pmatrix} 2z_0 & 2w_0 \\ 2z_0 & -2w_0 \end{pmatrix} = -8z_0 w_0 \neq 0$$

if $z_0 \neq 0$ and $w_0 \neq 0$. Suppose now that $F_1(x_0, y_0, z_0, w_0) = F_2(x_0, y_0, z_0, w_0) = 0$ with $z_0 > 0$ and $w_0 > 0$. Then it is easy to check that the functions

$$z = g_1(x, y) = \sqrt{1 - x^2} \qquad \text{and} \qquad w = g_2(x, y) = \sqrt{1 - y^2}$$

satisfy $F_1(x,y,g_1(x,y),g_2(x,y))=0$ and $F_2(x,y,g_1(x,y),g_2(x,y))=0$. Furthermore, it is easily checked (the reader is advised to fill in the details) that

$$\frac{\partial g_1}{\partial x}=-\frac{\partial(F_1,F_2)}{\partial(x,w)}\bigg/\frac{\partial(F_1,F_2)}{\partial(z,w)}=-\frac{x}{\sqrt{1-x^2}},$$
$$\frac{\partial g_1}{\partial y}=-\frac{\partial(F_1,F_2)}{\partial(y,w)}\bigg/\frac{\partial(F_1,F_2)}{\partial(z,w)}=0,$$
$$\frac{\partial g_2}{\partial x}=-\frac{\partial(F_1,F_2)}{\partial(z,x)}\bigg/\frac{\partial(F_1,F_2)}{\partial(z,w)}=0,$$
$$\frac{\partial g_2}{\partial y}=-\frac{\partial(F_1,F_2)}{\partial(z,y)}\bigg/\frac{\partial(F_1,F_2)}{\partial(z,w)}=-\frac{y}{\sqrt{1-y^2}}.$$

† Sketch of Proof of Theorem 3A'. We shall show here how one may prove the special case $n=2$ and $m=1$. For convenience, we shall use the notation $\mathbf{x}=(x,y)$ and $\mathbf{x}_0=(x_0,y_0)$. Since

$$\frac{\partial F}{\partial z}(\mathbf{x}_0,z_0)\neq 0,$$

we shall assume that it is positive (otherwise we simply consider $-F$ instead of F). By the continuity of the partial derivative $\partial F/\partial z$, there exist $a>0$ and $b>0$ such that

$$\frac{\partial F}{\partial z}(\mathbf{x},z)>b \qquad \text{whenever } \|\mathbf{x}-\mathbf{x}_0\|<a \text{ and } |z-z_0|<a.$$

In view of continuity, we may also assume that there exists $M>0$ such that

$$\left|\frac{\partial F}{\partial x}(\mathbf{x},z)\right|<M \qquad \text{and} \qquad \left|\frac{\partial F}{\partial y}(\mathbf{x},z)\right|<M$$

in the same region. Note next that since $F(\mathbf{x}_0,z_0)=0$, it follows that

$$F(\mathbf{x},z)=(F(\mathbf{x},z)-F(\mathbf{x}_0,z))+(F(\mathbf{x}_0,z)-F(\mathbf{x}_0,z_0)).$$

To study the term $F(\mathbf{x},z)-F(\mathbf{x}_0,z)$, we note that the line segment in $\mathbb{R}^3$ from $(\mathbf{x}_0,z)$ to $(\mathbf{x},z)$ can be described by the function

$$L:[0,1]\to\mathbb{R}^3:t\mapsto(t\mathbf{x}+(1-t)\mathbf{x}_0,z)=(tx+(1-t)x_0,ty+(1-t)y_0,z).$$

If we let $h=F\circ L:[0,1]\to\mathbb{R}$, then

$$F(\mathbf{x},z)-F(\mathbf{x}_0,z)=h(1)-h(0)=h'(\theta)$$

for some $\theta\in(0,1)$, in view of the Mean value theorem. On the other hand, it follows from the Chain rule that

$$\begin{aligned}h'(\theta)=(\mathbf{D}F)(L(\theta))(\mathbf{D}L)(\theta)&=\begin{pmatrix}\frac{\partial F}{\partial x}(L(\theta)) & \frac{\partial F}{\partial y}(L(\theta)) & \frac{\partial F}{\partial z}(L(\theta))\end{pmatrix}\begin{pmatrix}x-x_0\\ y-y_0\\ 0\end{pmatrix}\\ &=\left(\frac{\partial F}{\partial x}(\theta\mathbf{x}+(1-\theta)\mathbf{x}_0,z)\right)(x-x_0)+\left(\frac{\partial F}{\partial y}(\theta\mathbf{x}+(1-\theta)\mathbf{x}_0,z)\right)(y-y_0).\end{aligned}$$

To study the term $F(\mathbf{x}_0,z)-F(\mathbf{x}_0,z_0)$, we note that the line segment in $\mathbb{R}^3$ from $(\mathbf{x}_0,z_0)$ to $(\mathbf{x}_0,z)$ can be described by the function

$$L:[0,1]\to\mathbb{R}^3:t\mapsto(\mathbf{x}_0,tz+(1-t)z_0)=(x_0,y_0,tz+(1-t)z_0).$$

If we let $h = F \circ L : [0,1] \to \mathbb{R}$, then

$$F(\mathbf{x}_0, z) - F(\mathbf{x}_0, z_0) = h(1) - h(0) = h'(\phi)$$

for some $\phi \in (0,1)$, in view of the Mean value theorem. On the other hand, it follows from the Chain rule that

$$\begin{aligned} h'(\phi) = (\mathbf{D}F)(L(\phi))(\mathbf{D}L)(\phi) &= \begin{pmatrix} \dfrac{\partial F}{\partial x}(L(\phi)) & \dfrac{\partial F}{\partial y}(L(\phi)) & \dfrac{\partial F}{\partial z}(L(\phi)) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ z - z_0 \end{pmatrix} \\ &= \left(\frac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi) z_0) \right) (z - z_0). \end{aligned}$$

Hence

$$\begin{aligned} F(\mathbf{x}, z) = & \left(\frac{\partial F}{\partial x}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (x - x_0) + \left(\frac{\partial F}{\partial y}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (y - y_0) \\ & + \left(\frac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi) z_0) \right) (z - z_0) \end{aligned}$$

for some $\theta, \phi \in (0,1)$. We now choose

$$a_0 \in (0, a) \qquad \text{and} \qquad \delta < \min \left\{ a_0, \frac{b a_0}{2M} \right\}.$$

If $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then it is easily seen that

$$\left| \left(\frac{\partial F}{\partial x}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (x - x_0) + \left(\frac{\partial F}{\partial y}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (y - y_0) \right| < b a_0,$$

so that

$$F(\mathbf{x}, z_0 + a_0) > 0 \qquad \text{and} \qquad F(\mathbf{x}, z_0 - a_0) < 0.$$

It now follows from the Intermediate value theorem that there exists $z \in (z_0 - a_0, z_0 + a_0)$ such that $F(\mathbf{x}, z) = 0$. Furthermore, this value of z is unique, since a function with positive derivative cannot have more than one zero. In other words, if we take $U = D(\mathbf{x}_0, \delta)$ and $V = (z_0 - a_0, z_0 + a_0)$, then for every $\mathbf{x} \in U$, there exists a unique $z \in V$ such that $F(\mathbf{x}, z) = 0$. We can write $z = g(x, y)$ and this completes the proof of the first two parts. To prove the last part, note that since $F(\mathbf{x}, z) = 0$, we have

$$g(\mathbf{x}) - g(\mathbf{x}_0) = z - z_0 = - \frac{\left(\dfrac{\partial F}{\partial x}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (x - x_0) + \left(\dfrac{\partial F}{\partial y}(\theta \mathbf{x} + (1-\theta)\mathbf{x}_0, z) \right) (y - y_0)}{\dfrac{\partial F}{\partial z}(\mathbf{x}_0, \phi z + (1-\phi) z_0)}.$$

In particular,

$$\frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h} = - \frac{\partial F}{\partial x}(x_0 + \theta h, y_0, z) \bigg/ \frac{\partial F}{\partial z}(x_0, y_0, \phi z + (1-\phi) z_0).$$

Letting $h \to 0$, we have $x \to x_0$ and $z \to z_0$, and so

$$\frac{\partial g}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{g(x_0 + h, y_0) - g(x_0, y_0)}{h} = - \frac{\partial F}{\partial x}(x_0, y_0, z_0) \bigg/ \frac{\partial F}{\partial z}(x_0, y_0, z_0).$$

Similarly

$$\frac{\partial g}{\partial y}(x_0,y_0) = -\frac{\partial F}{\partial y}(x_0,y_0,z_0)\Big/\frac{\partial F}{\partial z}(x_0,y_0,z_0).$$

The argument can be repeated for every $(x,y,z) \in U \times V$. This completes the proof. ◯

REMARKS. (1) Suppose that it has been established that the functions (2) are continuously differentiable. We shall show here how we may deduce (4) by the use of the Chain rule. Consider the functions

$$G : U \to \mathbb{R}^n \times \mathbb{R}^m : \mathbf{x} \mapsto (\mathbf{x}, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$$

and

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m : (\mathbf{x},\mathbf{z}) \mapsto (F_1(\mathbf{x},\mathbf{z}), \ldots, F_m(\mathbf{x},\mathbf{z})).$$

In view of (3), it is clear that the composite function $H = F \circ G : U \to \mathbb{R}^m$ is identically zero, so that $(\mathbf{D}H)(\mathbf{x})$ is the zero $m \times n$ matrix. On the other hand, we have, by the Chain rule, that

$$(\mathbf{D}H)(\mathbf{x}) = (\mathbf{D}F)(\mathbf{x},\mathbf{z})(\mathbf{D}G)(\mathbf{x}),$$

where $\mathbf{z} = (z_1,\ldots,z_m) = (g_1(\mathbf{x}),\ldots,g_m(\mathbf{x}))$. It follows that

$$(\mathbf{D}H)(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \ldots & \dfrac{\partial F_1}{\partial x_n} & \dfrac{\partial F_1}{\partial z_1} & \ldots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \ldots & \dfrac{\partial F_m}{\partial x_n} & \dfrac{\partial F_m}{\partial z_1} & \ldots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x_1}{\partial x_1} & \ldots & \dfrac{\partial x_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial x_1} & \ldots & \dfrac{\partial x_n}{\partial x_n} \\ \dfrac{\partial g_1}{\partial x_1} & \ldots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial g_m}{\partial x_1} & \ldots & \dfrac{\partial g_m}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \ldots & \dfrac{\partial F_1}{\partial x_n} & \dfrac{\partial F_1}{\partial z_1} & \ldots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \ldots & \dfrac{\partial F_m}{\partial x_n} & \dfrac{\partial F_m}{\partial z_1} & \ldots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 \\ \dfrac{\partial g_1}{\partial x_1} & & \ldots & & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & & & & \vdots \\ \dfrac{\partial g_m}{\partial x_1} & & \ldots & & \dfrac{\partial g_m}{\partial x_n} \end{pmatrix}.$$

Hence for every $j = 1, \ldots, n$, the j-th (zero) column of $(\mathbf{D}H)(\mathbf{x})$ is equal to

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial F_m}{\partial x_j} \end{pmatrix} + \begin{pmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix} \begin{pmatrix} \dfrac{\partial g_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial g_m}{\partial x_j} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

so that

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{pmatrix} \begin{pmatrix} \dfrac{\partial g_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial g_m}{\partial x_j} \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial F_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial F_m}{\partial x_j} \end{pmatrix}. \tag{7}$$

We can now deduce (4) from (7) by applying Cramer's rule.

(2) Note that in the special case when $m = 1$, then (4) reduces to (6).

3.2. Inverse Function Theorem

In the theory of real valued functions of one real variable, we appreciate the importance of inversion. For example, we know that the exponential function has as its inverse the logarithmic function. In this last section, we shall study this problem more closely. More precisely, we shall deduce the following result from the Implicit function theorem. Throughout this section, $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^n \times \mathbb{R}^n$ denotes $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$. For functions $f_i : \mathbb{R}^n \to \mathbb{R}$, where $i = 1, \ldots, n$, we write

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)} = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial z_1} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial z_1} & \cdots & \dfrac{\partial f_n}{\partial z_n} \end{pmatrix} \tag{8}$$

whenever the partial derivatives in the matrix exist. We also write

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_{i-1}, \mathbf{e}_j, z_{1+1}, \ldots, z_n)}$$

for the determinant when the i-th column of the matrix in (8) is replaced by the vector

$$\mathbf{e}_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{n-j}),$$

written as a column.

THEOREM 3B. (INVERSE FUNCTION THEOREM) *Suppose that for every $i = 1, \ldots, n$, the function $f_i : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives. Suppose further that the system of n equations*

$$
\begin{aligned}
f_1(z_1, \ldots, z_n) &= x_1 \\
&\vdots \\
f_n(z_1, \ldots, z_n) &= x_n
\end{aligned}
\tag{9}
$$

has a solution $(\mathbf{x}_0, \mathbf{z}_0) \in \mathbb{R}^n \times \mathbb{R}^n$, where

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}(\mathbf{z}_0) \neq 0.$$

Then the following two conclusions hold:

(a) There exist neighbourhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $V \subseteq \mathbb{R}^n$ of $\mathbf{z}_0$ such that the system (9) can be solved uniquely as $\mathbf{z} = g(\mathbf{x})$ for $\mathbf{x} \in U$ and $\mathbf{z} \in V$. In other words, there exist unique functions

$$z_i = g_i(\mathbf{x}) = g_i(x_1, \ldots, x_n), \qquad \text{where } i = 1, \ldots, n, \tag{10}$$

such that the equations (9) hold.

(b) Furthermore, the functions (10) are continuously differentiable, and for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$, we have

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_{i-1}, \mathbf{e}_j, z_{1+1}, \ldots, z_n)} \bigg/ \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}. \tag{11}$$

Remark. Note that

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}(\mathbf{z}_0) = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial z_1}(\mathbf{z}_0) & \ldots & \dfrac{\partial f_1}{\partial z_n}(\mathbf{z}_0) \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial z_1}(\mathbf{z}_0) & \ldots & \dfrac{\partial f_n}{\partial z_n}(\mathbf{z}_0) \end{pmatrix}.$$

Note also that

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_{i-1}, \mathbf{e}_j, z_{1+1}, \ldots, z_n)} = \det \begin{pmatrix}
\dfrac{\partial f_1}{\partial z_1} & \ldots & \dfrac{\partial f_1}{\partial z_{i-1}} & 0 & \dfrac{\partial f_1}{\partial z_{i+1}} & \ldots & \dfrac{\partial f_1}{\partial z_n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\dfrac{\partial f_{j-1}}{\partial z_1} & \ldots & \dfrac{\partial f_{j-1}}{\partial z_{i-1}} & 0 & \dfrac{\partial f_{j-1}}{\partial z_{i+1}} & \ldots & \dfrac{\partial f_{j-1}}{\partial z_n} \\
\dfrac{\partial f_j}{\partial z_1} & \ldots & \dfrac{\partial f_j}{\partial z_{i-1}} & 1 & \dfrac{\partial f_j}{\partial z_{i+1}} & \ldots & \dfrac{\partial f_j}{\partial z_n} \\
\dfrac{\partial f_{j+1}}{\partial z_1} & \ldots & \dfrac{\partial f_{j+1}}{\partial z_{i-1}} & 0 & \dfrac{\partial f_{j+1}}{\partial z_{i+1}} & \ldots & \dfrac{\partial f_{j+1}}{\partial z_n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\dfrac{\partial f_n}{\partial z_1} & \ldots & \dfrac{\partial f_n}{\partial z_{i-1}} & 0 & \dfrac{\partial f_n}{\partial z_{i+1}} & \ldots & \dfrac{\partial f_n}{\partial z_n}
\end{pmatrix}.$$

SKETCH OF PROOF OF THEOREM 3B. We use Theorem 3A with $m = n$ and

$$F_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{z}) \mapsto f_i(\mathbf{z}) - x_i, \qquad \text{where } i = 1, \ldots, n.$$

Note now that the conditions for F_i in Theorem 3A are satisfied. Note also that for $i, j = 1, \ldots, n$, we have

$$\frac{\partial F_i}{\partial z_j} = \frac{\partial f_i}{\partial z_j} \qquad \text{and} \qquad \frac{\partial F_i}{\partial x_j} = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is easy to see that (11) now follows from (4). ○

DEFINITION. The determinant given by (8) is called the Jacobian determinant of $f = (f_1, \ldots, f_n)$.

EXAMPLE 3.2.1. Consider the equations

$$\frac{z^2 + w^2}{z} = x \qquad \text{and} \qquad \mathrm{e}^z \sin w = y.$$

Here the functions

$$f_1(z, w) = \frac{z^2 + w^2}{z} \qquad \text{and} \qquad f_2(z, w) = \mathrm{e}^z \sin w$$

have continuous partial derivatives except when $z = 0$. Now

$$\frac{\partial(f_1, f_2)}{\partial(z, w)}(z_0, w_0) = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial z}(z_0, w_0) & \dfrac{\partial f_1}{\partial w}(z_0, w_0) \\ \dfrac{\partial f_2}{\partial z}(z_0, w_0) & \dfrac{\partial f_2}{\partial w}(z_0, w_0) \end{pmatrix} = \det \begin{pmatrix} 1 - \dfrac{w_0^2}{z_0^2} & \dfrac{2w_0}{z_0} \\ \mathrm{e}^{z_0} \sin w_0 & \mathrm{e}^{z_0} \cos w_0 \end{pmatrix}$$
$$= \mathrm{e}^{z_0} \left(\left(1 - \frac{w_0^2}{z_0^2} \right) \cos w_0 - \frac{2w_0}{z_0} \sin w_0 \right).$$

It follows that we can solve for z and w in terms of x and y near any point (z_0, w_0) for which

$$\left(1 - \frac{w_0^2}{z_0^2} \right) \cos w_0 \neq \frac{2w_0}{z_0} \sin w_0.$$

EXAMPLE 3.2.2. Consider the equations

$$r \cos \theta = x \qquad \text{and} \qquad r \sin \theta = y.$$

Here the functions

$$f_1(r, \theta) = r \cos \theta \qquad \text{and} \qquad f_2(r, \theta) = r \sin \theta$$

have continuous partial derivatives everywhere. Now

$$\frac{\partial(f_1, f_2)}{\partial(r, \theta)}(r_0, \theta_0) = \det \begin{pmatrix} \dfrac{\partial f_1}{\partial r}(r_0, \theta_0) & \dfrac{\partial f_1}{\partial \theta}(r_0, \theta_0) \\ \dfrac{\partial f_2}{\partial r}(r_0, \theta_0) & \dfrac{\partial f_2}{\partial \theta}(r_0, \theta_0) \end{pmatrix} = \det \begin{pmatrix} \cos \theta_0 & -r_0 \sin \theta_0 \\ \sin \theta_0 & r_0 \cos \theta_0 \end{pmatrix} = r_0.$$

It follows that we can solve for r and θ in terms of x and y near any point (r_0, θ_0) for which $r_0 \neq 0$. Furthermore, we have

$$\frac{\partial r}{\partial x} = \frac{\det\begin{pmatrix} 1 & \dfrac{\partial f_1}{\partial \theta} \\ 0 & \dfrac{\partial f_2}{\partial \theta} \end{pmatrix}}{\dfrac{\partial(f_1, f_2)}{\partial(r, \theta)}} = \ldots = \cos\theta \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{\det\begin{pmatrix} 0 & \dfrac{\partial f_1}{\partial \theta} \\ 1 & \dfrac{\partial f_2}{\partial \theta} \end{pmatrix}}{\dfrac{\partial(f_1, f_2)}{\partial(r, \theta)}} = \ldots = \sin\theta,$$

as well as

$$\frac{\partial \theta}{\partial x} = \frac{\det\begin{pmatrix} \dfrac{\partial f_1}{\partial r} & 1 \\ \dfrac{\partial f_2}{\partial r} & 0 \end{pmatrix}}{\dfrac{\partial(f_1, f_2)}{\partial(r, \theta)}} = \ldots = -\frac{\sin\theta}{r} \qquad \text{and} \qquad \frac{\partial \theta}{\partial y} = \frac{\det\begin{pmatrix} \dfrac{\partial f_1}{\partial r} & 0 \\ \dfrac{\partial f_2}{\partial r} & 1 \end{pmatrix}}{\dfrac{\partial(f_1, f_2)}{\partial(r, \theta)}} = \ldots = \frac{\cos\theta}{r}.$$

These can be checked by using the formulae

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x}.$$

Problems for Chapter 3

1. Consider the function

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} : (x, y, z) \mapsto x^3 z^2 - z^3 yx.$$

   a) Explain why there is no neighbourhoods U of $(0, 0)$ and V of 0 such that there exists a function $z = g(x, y)$ defined for $(x, y) \in U$ and $z \in V$ and satisfying $F(x, y, g(x, y)) = 0$.
   b) Explain why the equation is soluble for z as a function of (x, y) near the point $(1, 1, 1)$. Compute the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ at this point by using the partial derivatives of F.

2. Consider the functions

$$\begin{aligned} F_1 &: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} : (x_1, x_2, x_3, z_1, z_2, z_3) \mapsto 3x_1 + 2x_2 + x_3^2 + z_1 + z_2^2, \\ F_2 &: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} : (x_1, x_2, x_3, z_1, z_2, z_3) \mapsto 4x_1 + 3x_2 + x_3 + z_1^2 + z_2 + z_3 + 2, \\ F_3 &: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} : (x_1, x_2, x_3, z_1, z_2, z_3) \mapsto x_1 + x_3 + z_1^2 + z_3 + 2. \end{aligned}$$

   Discuss the solubility of the system of equations

$$F_i(x_1, x_2, x_3, z_1, z_2, z_3) = 0, \qquad \text{where } i = 1, 2, 3,$$

   for z_1, z_2, z_3 in terms of x_1, x_2, x_3 in a neighbourhood of the point $(0, 0, 0, 0, 0, -2)$.

3. In $\mathbb{R}^2$, rectangular coordinates (x, y) and polar coordinates (r, θ) are related by $x = r\cos\theta$ and $y = r\sin\theta$. Discuss when we can solve for (r, θ) in terms of (x, y).

4. Spherical coordinates r, ϕ, θ and rectangular coordinates x, y, z in $\mathbb{R}^3$ are related by

$$\begin{aligned} x(r,\phi,\theta) &= r\sin\phi\cos\theta, \\ y(r,\phi,\theta) &= r\sin\phi\sin\theta, \\ z(r,\phi,\theta) &= r\cos\phi. \end{aligned}$$

a) Show that

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = r^2\sin\phi.$$

b) When can we solve for r, ϕ, θ in terms of x, y, z?

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 4

HIGHER ORDER DERIVATIVES

4.1. Iterated Partial Derivatives

In this chapter, we shall be concerned with functions of the type $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. We shall consider iterated partial derivatives of the form

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right),$$

where $i, j = 1, \ldots, n$. An immediate question that arises is whether

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

when $i \neq j$.

EXAMPLE 4.1.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It is easily seen that

$$\frac{\partial f}{\partial x} = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2}$$

whenever $(x,y) \neq (0,0)$. Furthermore,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0.$$

Note, however, that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x\to 0} \frac{\dfrac{\partial f}{\partial y}(x,0) - \dfrac{\partial f}{\partial y}(0,0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x-0} = 1,$$

while

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\dfrac{\partial f}{\partial x}(0,y) - \dfrac{\partial f}{\partial x}(0,0)}{y-0} = \lim_{y\to 0} \frac{-y-0}{y-0} = -1,$$

so that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

It can further be checked that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

whenever $(x,y) \neq (0,0)$. Clearly at least one of these two iterated second partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}$$

is not continuous at $(0,0)$.

THEOREM 4A. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$ is an open set, has continuous iterated second partial derivatives. Then*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

holds everywhere in A.

† PROOF. Suppose that $(x_0, y_0) \in A$ is chosen. Since A is open, there exists an open disc $D(x_0, y_0, r) \subseteq A$. For every $(x,y) \in D(x_0, y_0, r)$, consider the expression

$$S(x,y) = f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0).$$

For every fixed y, write

$$g_y(x) = f(x,y) - f(x,y_0),$$

so that

$$S(x,y) = g_y(x) - g_y(x_0).$$

By the Mean value theorem on g_y, there exists $\widetilde{x}$ between x_0 and x such that

$$g_y(x) - g_y(x_0) = (x - x_0)\frac{\partial g_y}{\partial x}(\widetilde{x}) = (x-x_0)\left(\frac{\partial f}{\partial x}(\widetilde{x}, y) - \frac{\partial f}{\partial x}(\widetilde{x}, y_0)\right).$$

By the Mean value theorem on $\partial f/\partial x$, there exists $\widetilde{y}$ between y_0 and y such that

$$\frac{\partial f}{\partial x}(\widetilde{x}, y) - \frac{\partial f}{\partial x}(\widetilde{x}, y_0) = (y - y_0)\frac{\partial^2 f}{\partial y \partial x}(\widetilde{x}, \widetilde{y}).$$

Hence

$$\frac{\partial^2 f}{\partial y \partial x}(\widetilde{x},\widetilde{y}) = \frac{S(x,y)}{(x-x_0)(y-y_0)}.$$

Since $\partial^2 f/\partial y\partial x$ is continuous at (x_0,y_0), and since $(\widetilde{x},\widetilde{y}) \to (x_0,y_0)$ as $(x,y) \to (x_0,y_0)$, we must have

$$\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{S(x,y)}{(x-x_0)(y-y_0)}.$$

A similar argument with the roles of the two variables x and y reversed gives

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \lim_{(x,y)\to(x_0,y_0)} \frac{S(x,y)}{(x-x_0)(y-y_0)}.$$

Hence

$$\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)$$

as required. ○

4.2. Taylor's Theorem

Recall that in the theory of real valued functions of one real variable, Taylor's theorem states that for a smooth function,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_k(x), \tag{1}$$

where the remainder term

$$R_k(x) = \int_{x_0}^{x} \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,\mathrm{d}t \tag{2}$$

satisfies

$$\lim_{x\to x_0} \frac{R_k(x)}{(x-x_0)^k} = 0.$$

REMARK. We usually prove this result by first using the Fundamental theorem of integral calculus to obtain

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,\mathrm{d}t.$$

Integrating by parts, we obtain

$$\int_{x_0}^{x} f'(t)\,\mathrm{d}t = \Big[(t-x)f'(t)\Big]_{x_0}^{x} - \int_{x_0}^{x} (t-x)f''(t)\,\mathrm{d}t = f'(x_0)(x-x_0) + \int_{x_0}^{x} (x-t)f''(t)\,\mathrm{d}t.$$

Hence

$$f(x) - f(x_0) = f'(x_0)(x-x_0) + \int_{x_0}^{x} (x-t)f''(t)\,\mathrm{d}t,$$

proving (1) and (2) for $k = 1$. The proof is now completed by induction on k and using integrating by parts on the integral (2).

Our goal in this section is to obtain Taylor approximations for functions of the type $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set. Suppose first of all that $\mathbf{x}_0 \in A$, and that f is differentiable at $\mathbf{x}_0$. For any $\mathbf{x} \in A$, let

$$R_1(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0),$$

where $(\mathbf{D}f)(\mathbf{x}_0)$ denotes the total derivative of f at $\mathbf{x}_0$, and where $\mathbf{x} - \mathbf{x}_0$ is interpreted as a column matrix. Since f is differentiable at $\mathbf{x}_0$, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0;$$

in other words,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|R_1(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

Note that

(3) $$(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\right)(x_i - X_i),$$

where $\mathbf{x}_0 = (X_1, \ldots, X_n)$.

We have therefore proved the following result on first-order Taylor approximations.

THEOREM 4B. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set, is differentiable at $\mathbf{x}_0 \in A$. Then for every $\mathbf{x} \in A$, we have*

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\right)(x_i - X_i) + R_1(\mathbf{x}),$$

where $\mathbf{x}_0 = (X_1, \ldots, X_n)$, and where

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|R_1(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

For second-order Taylor approximation, we have the following result.

THEOREM 4C. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set, has continuous iterated second partial derivatives. Suppose further that $\mathbf{x}_0 \in A$. Then for every $\mathbf{x} \in A$, we have*

(4) $$f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\right)(x_i - X_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right)(x_i - X_i)(x_j - X_j) + R_2(\mathbf{x}),$$

where $\mathbf{x}_0 = (X_1, \ldots, X_n)$, and where

(5) $$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{|R_2(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0.$$

† Sketch of Proof. We shall attempt to demonstrate the theorem by making the extra assumption that f has continuous iterated third partial derivatives. Consider the function

$$L : [0, 1] \to \mathbb{R}^n : t \mapsto (1 - t)\mathbf{x}_0 + t\mathbf{x};$$

here L denotes the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$, and we shall make the extra assumption that this line segment lies in A. Then consider the composition $g = f \circ L : [0,1] \to \mathbb{R}$, where $g(t) = f((1-t)\mathbf{x}_0 + t\mathbf{x})$ for every $t \in [0,1]$. We now apply (1) and (2) to the function g to obtain

$$g(1) = g(0) + g'(0) + \frac{g''(0)}{2} + R_2,$$

where

$$R_2 = \int_0^1 \frac{(t-1)^2}{2} g'''(t)\,\mathrm{d}t.$$

Applying the Chain rule, we have

$$\begin{aligned} g'(t) = (\mathbf{D}f)(L(t))(\mathbf{D}L)(t) &= \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(L(t)) & \ldots & \dfrac{\partial f}{\partial x_n}(L(t)) \end{pmatrix} \begin{pmatrix} x_1 - X_1 \\ \vdots \\ x_n - X_n \end{pmatrix} \\ &= \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(L(t)) \right) (x_i - X_i), \end{aligned}$$

so that

$$g'(0) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(L(0)) \right) (x_i - X_i) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right) (x_i - X_i).$$

Note that

$$g'(t) = \sum_{j=1}^n \left(\left(\frac{\partial f}{\partial x_j} \circ L \right)(t) \right) (x_j - X_j).$$

It follows from the Chain rule and the arithmetic of derivatives that

$$\begin{aligned} g''(t) &= \sum_{j=1}^n \left(\left(\mathbf{D}\frac{\partial f}{\partial x_j} \right)(L(t))(\mathbf{D}L)(t) \right) (x_j - X_j) \\ &= \sum_{j=1}^n \left(\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_j}(L(t)) & \ldots & \dfrac{\partial^2 f}{\partial x_n \partial x_j}(L(t)) \end{pmatrix} \begin{pmatrix} x_1 - X_1 \\ \vdots \\ x_n - X_n \end{pmatrix} \right) (x_j - X_j) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(L(t)) \right) (x_i - X_i) \right) (x_j - X_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(L(t)) \right) (x_i - X_i)(x_j - X_j), \end{aligned}$$

so that

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(L(0)) \right) (x_i - X_i)(x_j - X_j) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) \right) (x_i - X_i)(x_j - X_j).$$

Note that

$$g''(t) = \sum_{j=1}^n \sum_{k=1}^n \left(\left(\frac{\partial^2 f}{\partial x_j \partial x_k} \circ L \right)(t) \right) (x_j - X_j)(x_k - X_k).$$

It can be shown, using the Chain rule and the arithmetic of derivatives, that

$$\begin{aligned} g'''(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(L(t))\right)(x_i - X_i)(x_j - X_j)(x_k - X_k) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}((1-t)\mathbf{x}_0 + t\mathbf{x})\right)(x_i - X_i)(x_j - X_j)(x_k - X_k). \end{aligned}$$

Writing $R_2 = R_2(\mathbf{x})$, we have established (4), where

$$R_2(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^1 \frac{(t-1)^2}{2}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}((1-t)\mathbf{x}_0 + t\mathbf{x})\right)(x_i - X_i)(x_j - X_j)(x_k - X_k)\,\mathrm{d}t.$$

The function

$$\frac{(t-1)^2}{2}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}((1-t)\mathbf{x}_0 + t\mathbf{x})\right)$$

is continuous, and hence bounded by M, say, in $[0,1]$. Also

$$|x_i - X_i|, |x_j - X_j|, |x_k - X_k| \le \|\mathbf{x} - \mathbf{x}_0\|,$$

so $|R_2(\mathbf{x})| \le n^3 M \|\mathbf{x} - \mathbf{x}_0\|^3$, and so (5) follows. ○

The second-order term that arises in Theorem 4C is of particular importance in the determination of the nature of stationary points later.

Definition. The quadratic function

$$\mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right)(x_i - X_i)(x_j - X_j) \tag{6}$$

is called the Hessian of f at $\mathbf{x}_0$.

Remark. The expression (4) can be rewritten in the form

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + R_2(\mathbf{x}), \tag{7}$$

where $(\mathbf{D}f)(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$, the matrix product of the total derivative $(\mathbf{D}f)(\mathbf{x}_0)$ with the column matrix $\mathbf{x} - \mathbf{x}_0$, is given by (3), and where the Hessian $\mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ is given by (6).

Example 4.2.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = x^2y + 3y - 2$ for every $(x,y) \in \mathbb{R}^2$, near the point $(x_0, y_0) = (1, -2)$. Clearly $f(1,-2) = -10$. We have

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 + 3.$$

Also

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x.$$

Hence

$$\frac{\partial f}{\partial x}(1,-2) = -4 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(1,-2) = 4.$$

Also

$$\frac{\partial^2 f}{\partial x^2}(1,-2) = -4 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(1,-2) = 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}(1,-2) = \frac{\partial^2 f}{\partial y \partial x}(1,-2) = 2.$$

Since

$$\begin{aligned} f(x,y) &= f(1,-2) + \left(\left(\frac{\partial f}{\partial x}(1,-2)\right)(x-1) + \left(\frac{\partial f}{\partial y}(1,-2)\right)(y+2)\right) \\ &\quad + \frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(1,-2)\right)(x-1)^2 + \left(\frac{\partial^2 f}{\partial x \partial y}(1,-2)\right)(x-1)(y+2)\right. \\ &\quad \left. + \left(\frac{\partial^2 f}{\partial y \partial x}(1,-2)\right)(x-1)(y+2) + \left(\frac{\partial^2 f}{\partial y^2}(1,-2)\right)(y+2)^2\right) + R_2(x,y) \\ &= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + R_2(x,y), \end{aligned}$$

it follows that the second-order Taylor approximation of f at $(1,-2)$ is given by

$$-10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2),$$

and the Hessian of f at $(1,-2)$ is given by

$$-2(x-1)^2 + 2(x-1)(y+2).$$

EXAMPLE 4.2.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \mathrm{e}^x \cos y$ for every $(x,y) \in \mathbb{R}^2$, near the point $(x_0, y_0) = (0,0)$. Clearly $f(0,0) = 1$. We have

$$\frac{\partial f}{\partial x} = \mathrm{e}^x \cos y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -\mathrm{e}^x \sin y.$$

Also

$$\frac{\partial^2 f}{\partial x^2} = \mathrm{e}^x \cos y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = -\mathrm{e}^x \cos y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\mathrm{e}^x \sin y.$$

Hence

$$\frac{\partial f}{\partial x}(0,0) = 1 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,0) = 0.$$

Also

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = -1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0.$$

Since

$$\begin{aligned} f(x,y) &= f(0,0) + \left(\left(\frac{\partial f}{\partial x}(0,0)\right)(x-0) + \left(\frac{\partial f}{\partial y}(0,0)\right)(y-0)\right) \\ &\quad + \frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(0,0)\right)(x-0)^2 + \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)(x-0)(y-0)\right. \\ &\quad \left. + \left(\frac{\partial^2 f}{\partial y \partial x}(0,0)\right)(x-0)(y-0) + \left(\frac{\partial^2 f}{\partial y^2}(0,0)\right)(y-0)^2\right) + R_2(x,y) \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + R_2(x,y), \end{aligned}$$

it follows that the second-order Taylor approximation of f at $(0,0)$ is given by

$$1+x+\frac{1}{2}x^2-\frac{1}{2}y^2,$$

and the Hessian of f at $(0,0)$ is given by

$$\frac{1}{2}x^2-\frac{1}{2}y^2.$$

EXAMPLE 4.2.3. Consider the function $f:\mathbb{R}^3\to\mathbb{R}$, defined by $f(x,y,z)=x^2y+xz^3+y^2z^2$ for every $(x,y,z)\in\mathbb{R}^3$, near the point $(x_0,y_0,z_0)=(1,1,1)$. Clearly $f(1,1,1)=3$. We have

$$\frac{\partial f}{\partial x}=2xy+z^3 \qquad\text{and}\qquad \frac{\partial f}{\partial y}=x^2+2yz^2 \qquad\text{and}\qquad \frac{\partial f}{\partial z}=3xz^2+2y^2z.$$

Also

$$\frac{\partial^2 f}{\partial x^2}=2y \qquad\text{and}\qquad \frac{\partial^2 f}{\partial y^2}=2z^2 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial z^2}=6xz+2y^2.$$

Furthermore,

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=2x \qquad\text{and}\qquad \frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial z\partial x}=3z^2 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}=4yz.$$

Hence

$$\frac{\partial f}{\partial x}(1,1,1)=3 \qquad\text{and}\qquad \frac{\partial f}{\partial y}(1,1,1)=3 \qquad\text{and}\qquad \frac{\partial f}{\partial z}(1,1,1)=5.$$

Also

$$\frac{\partial^2 f}{\partial x^2}(1,1,1)=2 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial y^2}(1,1,1)=2 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial z^2}(1,1,1)=8.$$

Furthermore,

$$\frac{\partial^2 f}{\partial x\partial y}(1,1,1)=2 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial x\partial z}(1,1,1)=3 \qquad\text{and}\qquad \frac{\partial^2 f}{\partial y\partial z}(1,1,1)=4.$$

Since

$$\begin{aligned}
f(x,y,z)&=f(1,1,1)+\left(\left(\frac{\partial f}{\partial x}(1,1,1)\right)(x-1)+\left(\frac{\partial f}{\partial y}(1,1,1)\right)(y-1)+\left(\frac{\partial f}{\partial z}(1,1,1)\right)(z-1)\right)\\
&\qquad+\frac{1}{2}\Bigg(\left(\frac{\partial^2 f}{\partial x^2}(1,1,1)\right)(x-1)^2+\left(\frac{\partial^2 f}{\partial y^2}(1,1,1)\right)(y-1)^2+\left(\frac{\partial^2 f}{\partial z^2}(1,1,1)\right)(z-1)^2\\
&\qquad\qquad+2\left(\frac{\partial^2 f}{\partial x\partial y}(1,1,1)\right)(x-1)(y-1)+2\left(\frac{\partial^2 f}{\partial x\partial z}(1,1,1)\right)(x-1)(z-1)\\
&\qquad\qquad+2\left(\frac{\partial^2 f}{\partial y\partial z}(1,1,1)\right)(y-1)(z-1)\Bigg)+R_2(x,y,z)\\
&=3+3(x-1)+3(y-1)+5(z-1)+(x-1)^2+(y-1)^2+4(z-1)^2\\
&\qquad+2(x-1)(y-1)+3(x-1)(z-1)+4(y-1)(z-1)+R_2(x,y,z),
\end{aligned}$$

it follows that the second-order Taylor approximation of f at $(1,1,1)$ is given by

$$3+3(x-1)+3(y-1)+5(z-1)+(x-1)^2+(y-1)^2+4(z-1)^2+2(x-1)(y-1)+3(x-1)(z-1)+4(y-1)(z-1),$$

and the Hessian of f at $(1,1,1)$ is given by

$$(x-1)^2+(y-1)^2+4(z-1)^2+2(x-1)(y-1)+3(x-1)(z-1)+4(y-1)(z-1).$$

4.3. Stationary Points

In this section, we study stationary points using an approach which allows us to generalize our technique for functions of two real variables. Throughout this section, we shall consider functions of the type $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set. We shall assume that f has continuous iterated second partial derivatives.

DEFINITION. A point $\mathbf{x}_0 \in A$ is said to be a stationary point of f if the total derivative $(\mathbf{D}f)(\mathbf{x}_0) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero $1 \times n$ matrix.

REMARK. In other words, $\mathbf{x}_0 \in A$ is a stationary point of f if

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = 0$$

for every $i = 1, \ldots, n$.

DEFINITION. A point $\mathbf{x}_0 \in A$ is said to be a (local) maximum of f if there exists a neighbourhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) \le f(\mathbf{x}_0)$ for every $\mathbf{x} \in U$.

DEFINITION. A point $\mathbf{x}_0 \in A$ is said to be a (local) minimum of f if there exists a neighbourhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) \ge f(\mathbf{x}_0)$ for every $\mathbf{x} \in U$.

DEFINITION. A stationary point $\mathbf{x}_0 \in A$ that is not a maximum or minimum of f is said to be a saddle point of f.

Our first task is to show that if f is differentiable, then every maximum or minimum of f is a stationary point of f. Note that this may not be the case if the function f is not differentiable, as can be observed for the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto |x|$ at the point $x = 0$.

THEOREM 4D. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set, is differentiable. Suppose further that $\mathbf{x}_0 \in A$ is a maximum or minimum of f. Then $\mathbf{x}_0$ is a stationary point of f.*

† PROOF. Suppose that $\mathbf{x}_0 \in A$ is a maximum of f. Consider the restriction of f to a line through $\mathbf{x}_0$. More precisely, consider the points $\mathbf{x}_0 + t\mathbf{h} \in \mathbb{R}^n$, where $\mathbf{0} \ne \mathbf{h} \in \mathbb{R}^n$ is fixed. Since A is open, there exists an open interval I containing $t = 0$ and such that $\{\mathbf{x}_0 + t\mathbf{h} : t \in I\} \subseteq A$. Consider now the line segment

$$L : I \to \mathbb{R}^n : t \mapsto \mathbf{x}_0 + t\mathbf{h}.$$

Since the function f has a maximum at $\mathbf{x}_0$, it follows that the function

$$g = f \circ L : I \to \mathbb{R},$$

where $g(t) = f(\mathbf{x}_0 + t\mathbf{h})$ for every $t \in I$, has a maximum at $t = 0$. By the Chain rule, g is differentiable. Since

$$g'(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t - 0},$$

it clearly follows that

$$g'(0) = \lim_{t\to 0+} \frac{g(t)-g(0)}{t-0} \le 0 \qquad \text{and} \qquad g'(0) = \lim_{t\to 0-} \frac{g(t)-g(0)}{t-0} \ge 0,$$

and so $g'(0) = 0$, whence $(\mathbf{D}g)(0) = 0$. Again, by the Chain rule, we have

$$(\mathbf{D}g)(0) = (\mathbf{D}f)(L(0))(\mathbf{D}L)(0).$$

It is easy to check that $(\mathbf{D}L)(0) = \mathbf{h}$, and so $(\mathbf{D}f)(L(0))\mathbf{h} = 0$. Since $\mathbf{h} \ne \mathbf{0}$ is arbitrary, we must have $(\mathbf{D}f)(\mathbf{x}_0) = (\mathbf{D}f)(L(0)) = \mathbf{0}$. The case when $\mathbf{x}_0 \in A$ is a minimum of f can be studied by considering the function $-f$. ◯

It is a consequence of (7) that if f has a stationary point at $\mathbf{x}_0 \in A$, then

(8) $$f(\mathbf{x}) = f(\mathbf{x}_0) + \mathbf{H}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + R_2(\mathbf{x}).$$

It follows that the Hessian $\mathbf{H}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)$ plays a crucial role in the determination of the nature of the stationary point. Recall that the Hessian is given by (6). Let us write $\mathbf{h} = \mathbf{x}-\mathbf{x}_0$ and $\mathbf{h} = (h_1, \ldots, h_n)$. Then (6) becomes

$$\mathbf{H}f(\mathbf{x}_0)(\mathbf{h}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right) h_i h_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} h_i h_j,$$

where, for every $i, j = 1, \ldots, n$, we have

$$\alpha_{ij} = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0).$$

A function of the type

(9) $$g(\mathbf{h}) = g(h_1, \ldots, h_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} h_i h_j$$

is called a quadratic function. Note that if we write

$$B = \begin{pmatrix} \beta_{11} & \ldots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{n1} & \ldots & \beta_{nn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix},$$

then

$$g(\mathbf{h}) = \mathbf{h}^t B \mathbf{h}.$$

Clearly for any real number $\lambda \in \mathbb{R}$, we have

(10) $$g(\lambda\mathbf{h}) = (\lambda\mathbf{h})^t B(\lambda\mathbf{h}) = \lambda^2 \mathbf{h}^t B\mathbf{h} = \lambda^2 g(\mathbf{h});$$

hence the term "quadratic".

Definition. A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $g(\mathbf{0}) = 0$ and $g(\mathbf{h}) > 0$ for every non-zero $\mathbf{h} \in \mathbb{R}^n$.

Definition. A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be negative definite if $g(\mathbf{0}) = 0$ and $g(\mathbf{h}) < 0$ for every non-zero $\mathbf{h} \in \mathbb{R}^n$.

THEOREM 4E. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set, has continuous iterated second partial derivatives. Suppose further that $\mathbf{x}_0 \in A$ is a stationary point of f.*
(a) If the Hessian $\mathbf{H}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) = \mathbf{H}f(\mathbf{x}_0)(\mathbf{h})$ is positive definite, then f has a minimum at $\mathbf{x}_0$.
(b) If the Hessian $\mathbf{H}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) = \mathbf{H}f(\mathbf{x}_0)(\mathbf{h})$ is negative definite, then f has a maximum at $\mathbf{x}_0$.

EXAMPLE 4.3.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = x^3 + y^3 - 3x - 12y + 4.$$

Then

$$(\mathbf{D}f)(x,y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2 - 3 & 3y^2 - 12 \end{pmatrix}.$$

For stationary points, we need $3x^2 - 3 = 0$ and $3y^2 - 12 = 0$, so there are four stationary points $(\pm 1, \pm 2)$. Now

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = 6y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

At the stationary point $(1,2)$, we have

$$\frac{\partial^2 f}{\partial x^2}(1,2) = 6 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(1,2) = 12 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}(1,2) = 0.$$

Hence the Hessian of f at $(1,2)$ is given by

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(1,2)\right)(x-1)^2 + \left(\frac{\partial^2 f}{\partial y^2}(1,2)\right)(y-2)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}(1,2)\right)(x-1)(y-2)\right) \\ &\quad = 3(x-1)^2 + 6(y-2)^2 \end{aligned}$$

and is positive definite. It follows that f has a minimum at $(1,2)$. At the stationary point $(-1,-2)$, we have

$$\frac{\partial^2 f}{\partial x^2}(-1,-2) = -6 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(-1,-2) = -12 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}(-1,-2) = 0.$$

Hence the Hessian of f at $(-1,-2)$ is given by

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(-1,-2)\right)(x+1)^2 + \left(\frac{\partial^2 f}{\partial y^2}(-1,-2)\right)(y+2)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}(-1,-2)\right)(x+1)(y+2)\right) \\ &\quad = -3(x+1)^2 - 6(y+2)^2 \end{aligned}$$

and is negative definite. It follows that f has a maximum at $(-1,-2)$. At the stationary point $(1,-2)$, we have

$$\frac{\partial^2 f}{\partial x^2}(1,-2) = 6 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(1,-2) = -12 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}(1,-2) = 0.$$

Hence the Hessian of f at $(1,-2)$ is given by

$$\begin{aligned} &\frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(1,-2)\right)(x-1)^2 + \left(\frac{\partial^2 f}{\partial y^2}(1,-2)\right)(y+2)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}(1,-2)\right)(x-1)(y+2)\right) \\ &\quad = 3(x-1)^2 - 6(y+2)^2. \end{aligned}$$

Let us investigate the function

$$\mathbf{H}f(1,-2)(h_1,h_2) = 3h_1^2 - 6h_2^2$$

more closely. Note that

$$\mathbf{H}f(1,-2)(h_1,0) = 3h_1^2 \geq 0 \qquad \text{and} \qquad \mathbf{H}f(1,-2)(0,h_2) = 0 - 6h_2^2 \leq 0.$$

In this case, Theorem 4E does not give any conclusion. In fact, both stationary points $(1,-2)$ and $(-1,2)$ are saddle points.

Remark. To prove Theorem 4E, we need the following result in linear algebra. Suppose that a quadratic function of the type (9) is positive definite. Then there exists a constant $M > 0$ such that for every $\mathbf{h} \in \mathbb{R}^n$, we have

$$g(\mathbf{h}) \geq M\|\mathbf{h}\|^2. \tag{11}$$

To see this, consider the restriction

$$g_S : S \to \mathbb{R} : \mathbf{h} \mapsto g(\mathbf{h})$$

of the function g to the unit sphere $S = \{\mathbf{h} \in \mathbb{R}^n : \|\mathbf{h}\| = 1\}$. The function g_S is continuous in S and has a minimum value $M > 0$, say. Then for any non-zero $\mathbf{h} \in \mathbb{R}^n$, we have, noting (10), that

$$g(\mathbf{h}) = g\left(\|\mathbf{h}\|\frac{\mathbf{h}}{\|\mathbf{h}\|}\right) = \|\mathbf{h}\|^2 g\left(\frac{\mathbf{h}}{\|\mathbf{h}\|}\right) = \|\mathbf{h}\|^2 g_S\left(\frac{\mathbf{h}}{\|\mathbf{h}\|}\right) \geq M\|\mathbf{h}\|^2.$$

Hence (11) holds for any non-zero $\mathbf{h} \in \mathbb{R}^n$. Clearly it also holds for $\mathbf{h} = \mathbf{0}$.

† Sketch of Proof of Theorem 4E. We shall attempt to demonstrate the theorem by making the extra assumption that iterated third partial derivatives exist and are continuous. At a stationary point $\mathbf{x}_0$, the expression (8) is valid, and can be rewritten in the form

$$f(\mathbf{x}) - f(\mathbf{x}_0) = \mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + R_2(\mathbf{x}),$$

where $R_2(\mathbf{x})$ satisfies (5). Suppose that $\mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ is positive definite. Then by our remark on linear algebra, there exists $M > 0$ such that

$$\mathbf{H}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \geq M\|\mathbf{x} - \mathbf{x}_0\|^2$$

for every $\mathbf{x} \in \mathbb{R}^n$. On the other hand, it follows from (5) that

$$|R_2(\mathbf{x})| \leq \frac{1}{2}M\|\mathbf{x} - \mathbf{x}_0\|^2,$$

provided that $\|\mathbf{x} - \mathbf{x}_0\|$ is sufficiently small. Hence

$$f(\mathbf{x}) - f(\mathbf{x}_0) \geq \frac{1}{2}M\|\mathbf{x} - \mathbf{x}_0\|^2 \geq 0,$$

provided that $\|\mathbf{x} - \mathbf{x}_0\|$ is sufficiently small, whence f has a minimum at $\mathbf{x}_0$. The negative definite case can be studied by considering the function $-f$. ◯

4.4. Functions of Two Variables

We now attempt to link the Hessian to the discriminant. Suppose that a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$ is an open set, has continuous iterated second partial derivatives. Suppose further that f has a stationary point at (x_0, y_0). Then the Hessian is given by

$$\frac{1}{2}\left(\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\right)(x-x_0)^2+\left(\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)\right)(x-x_0)(y-y_0)\right.$$
$$\left.+\left(\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right)(x-x_0)(y-y_0)+\left(\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\right)(y-y_0)^2\right)$$
$$=\frac{1}{2}\begin{pmatrix} x-x_0 & y-y_0 \end{pmatrix}\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y\partial x}(x_0,y_0) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}.$$

REMARK. We need the following result in linear algebra. The quadratic function

$$g(x,y)=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$, is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. To see this, note that

$$g(x,y) = ax^2 + 2bxy + cy^2.$$

Suppose first of all that $a > 0$ and $ac - b^2 > 0$. Completing squares, we have

$$g(x,y)=a\left(x+\frac{b}{a}y\right)^2+\left(c-\frac{b^2}{a}\right)y^2 \geq 0, \tag{12}$$

with equality only when

$$y=0 \qquad \text{and} \qquad x+\frac{b}{a}y=0;$$

in other words, when $(x, y) = (0, 0)$. Suppose now that $a = 0$. Then $g(x, y) = 2bxy + cy^2$ clearly cannot be positive definite (why?). It follows that if $g(x, y)$ is positive definite, then $a \neq 0$ and (12) holds, with strict inequality whenever $(x, y) \neq (0, 0)$. Setting $y = 0$, we conclude that we must have $a > 0$. Setting $x = -by/a$, we conclude that we must have $ac - b^2 > 0$.

We have essentially proved the following result.

THEOREM 4F. *Suppose that the function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$ is an open set, has continuous iterated second partial derivatives. Suppose further that $(x_0, y_0) \in A$ is a stationary point of f.*
(a) If

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0)>0 \qquad \textit{and} \qquad \Delta=\det\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y\partial x}(x_0,y_0) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix}>0,$$

then f has a minimum at (x_0, y_0).
(b) If

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0)<0 \qquad \textit{and} \qquad \Delta=\det\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y\partial x}(x_0,y_0) \\ \dfrac{\partial^2 f}{\partial x\partial y}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix}>0,$$

then f has a maximum at (x_0, y_0).

Remark. The reader may wish to re-examine Example 4.3.1 using this result.

4.5. Constrained Maxima and Minima

In this last section, we consider the problem of finding maxima and minima of functions of n variables, where the veriables are not always independent of each other but are subject to some constraints. In the case of one constraint, we have the following useful result.

THEOREM 4G. *Suppose that the functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open set, have continuous partial derivatives. Suppose next that $c \in \mathbb{R}$ is fixed, and $S = \{\mathbf{x} \in A : g(\mathbf{x}) = c\}$. Suppose further that the function $f|_S$, the restriction of f to S, has a maximum or minimum at $\mathbf{x}_0 \in S$, and that $(\nabla g)(\mathbf{x}_0) \neq \mathbf{0}$. Then there exists a real number $\lambda \in \mathbb{R}$ such that $(\nabla f)(\mathbf{x}_0) = \lambda(\nabla g)(\mathbf{x}_0)$.*

Remarks. (1) The restriction of f to $S \subseteq A$ is the function $f|_S : S \to \mathbb{R} : \mathbf{x} \mapsto f(\mathbf{x})$.

(2) The number λ is called the Lagrange multiplier.

(3) Note that $(\nabla g)(\mathbf{x}_0)$ is a vector which is orthogonal to the surface S at $\mathbf{x}_0$. It follows that if f has a maximum or minimum at $\mathbf{x}_0$, then $(\nabla f)(\mathbf{x}_0)$ must be orthogonal to the surface S at $\mathbf{x}_0$.

† Sketch of Proof of Theorem 4G. We shall only consider the case $n = 3$. Suppose that $I \subseteq \mathbb{R}$ is an open interval containing the number 0. Suppose further that

$$L : I \to \mathbb{R}^3 : t \mapsto L(t) = (L_1(t), L_2(t), L_3(t))$$

is a path on S, with $L(0) = \mathbf{x}_0$, so that $L(t) \in S$ for every $t \in I$. Consider first of all the function $h = g \circ L : I \to \mathbb{R}$. Clearly $h(t) = g(L(t)) = c$ for every $t \in I$. It follows that

$$(\mathbf{D}h)(0) = \frac{\mathrm{d}h}{\mathrm{d}t}(0) = 0.$$

On the other hand, it follows from the Chain rule that

$$(\mathbf{D}h)(0) = (\mathbf{D}g)(L(0))(\mathbf{D}L)(0) = (\nabla g)(\mathbf{x}_0) \cdot (L_1'(0), L_2'(0), L_3'(0)),$$

so that $(\nabla g)(\mathbf{x}_0)$ is perpendicular to $(L_1'(0), L_2'(0), L_3'(0))$, a tangent vector to S at $\mathbf{x}_0$. Since L is arbitrary, it follows that $(\nabla g)(\mathbf{x}_0)$ must be perpendicular to the tangent plane to S at $\mathbf{x}_0$. Consider next the function $k = f \circ L : I \to \mathbb{R}$. If $f|_S$ has a maximum or minimum at $\mathbf{x}_0$, then clearly k has a maximum or minimum at $t = 0$. It follows that

$$(\mathbf{D}k)(0) = \frac{\mathrm{d}k}{\mathrm{d}t}(0) = 0.$$

On the other hand, it follows from the Chain rule that

$$(\mathbf{D}k)(0) = (\mathbf{D}f)(L(0))(\mathbf{D}L)(0) = (\nabla f)(\mathbf{x}_0) \cdot (L_1'(0), L_2'(0), L_3'(0)),$$

so that $(\nabla f)(\mathbf{x}_0)$ is perpendicular to $(L_1'(0), L_2'(0), L_3'(0))$. Since L is arbitrary, it follows as before that $(\nabla f)(\mathbf{x}_0)$ must also be perpendicular to the tangent plane to S at $\mathbf{x}_0$. Since $(\nabla f)(\mathbf{x}_0)$ and $(\nabla g)(\mathbf{x}_0) \neq \mathbf{0}$ are perpendicular to the same plane, there exists a real number $\lambda \in \mathbb{R}$ such that $(\nabla f)(\mathbf{x}_0) = \lambda(\nabla g)(\mathbf{x}_0)$. ○

EXAMPLE 4.5.1. We wish to find the distance from the origin to the plane $x-2y-2z=3$. To do this, we consider the function

$$f:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto x^2+y^2+z^2,$$

which represents the square of the distance from the origin to a point $(x,y,z)\in\mathbb{R}^3$. The points (x,y,z) under consideration are subject to the constraint $g(x,y,z)=3$, where

$$g:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto x-2y-2z.$$

We now wish to minimize f subject to this constraint. Using the Lagrange multiplier method, we know that the minimum is attained at a point (x,y,z) which satisfies

$$(\nabla f)(x,y,z)=\lambda(\nabla g)(x,y,z)$$

for some real number $\lambda\in\mathbb{R}$. Note that

$$(\nabla f)(x,y,z)=(2x,2y,2z)\qquad\text{and}\qquad(\nabla g)(x,y,z)=(1,-2,-2).$$

Hence we need to solve the equations

$$(2x,2y,2z)=\lambda(1,-2,-2)\qquad\text{and}\qquad x-2y-2z=3.$$

Substituting the former into the latter, we obtain $\lambda=2/3$. This gives $(x,y,z)=(1/3,-2/3,-2/3)$. Clearly $f(x,y,z)=1$ at this point. Hence the minimum distance is equal to 1, the square root of $f(x,y,z)$ at this point.

EXAMPLE 4.5.2. We wish to find the volume of the largest rectangular box with edges parallel to the coordinate axes and inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{13}$$

Clearly the box is given by $[-x,x]\times[-y,y]\times[-z,z]$ for some positive $x,y,z\in\mathbb{R}$ satisfying (13), with volume equal to $8xyz$. We therefore wish to maximize the function

$$f:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto 8xyz,$$

subject to the constraint $g(x,y,z)=1$, where

$$g:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}.$$

Using the Lagrange multiplier method, we know that the maximum is attained at a point (x,y,z) which satisfies

$$(\nabla f)(x,y,z)=\lambda(\nabla g)(x,y,z)$$

for some real number $\lambda\in\mathbb{R}$. Note that

$$(\nabla f)(x,y,z)=(8yz,8xz,8xy)\qquad\text{and}\qquad(\nabla g)(x,y,z)=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right).$$

Hence we need to solve the equations (13) and

$$(8yz,8xz,8xy)=\lambda\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right). \tag{14}$$

Since $x,y,z>0$, it follows from (14) that

$$8xyz=\frac{2\lambda x^2}{a^2}=\frac{2\lambda y^2}{b^2}=\frac{2\lambda z^2}{c^2}, \tag{15}$$

so that combining with (13), we have

$$24xyz=\frac{2\lambda x^2}{a^2}+\frac{2\lambda y^2}{b^2}+\frac{2\lambda z^2}{c^2}=2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=2\lambda,$$

whence $\lambda=12xyz$. Substituting this into the left hand side of (15), we deduce that

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3},$$

giving

$$(x,y,z)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)\qquad\text{and}\qquad f(x,y,z)=\frac{8abc}{3\sqrt{3}}.$$

Remark. In Example 4.5.2, we clearly have constrained minima at points such as $(a,0,0)$. Note, however, that we have dispensed with such trivial cases by considering only positive values of x,y,z. Note that (15) is obtained only under such a specialization.

In the case of more than one constraint, we have the following generalized version of Theorem 4G.

THEOREM 4H. *Suppose that the functions $f:A\to\mathbb{R}$ and $g_i:A\to\mathbb{R}$, where $A\subseteq\mathbb{R}^n$ is an open set and $i=1,\ldots,k$, have continuous partial derivatives. Suppose next that $c_1,\ldots,c_k\in\mathbb{R}$ are fixed, and $S=\{\mathbf{x}\in A:g_i(\mathbf{x})=c_i \text{ for every } i=1,\ldots,k\}$. Suppose further that the function $f|_S$, the restriction of f to S, has a maximum or minimum at $\mathbf{x}_0\in S$, and that $(\nabla g_1)(\mathbf{x}_0),\ldots,(\nabla g_k)(\mathbf{x}_0)$ are linearly independent over $\mathbb{R}$. Then there exist real numbers $\lambda_1,\ldots,\lambda_k\in\mathbb{R}$ such that*

$$(\nabla f)(\mathbf{x}_0)=\lambda_1(\nabla g_1)(\mathbf{x}_0)+\ldots+\lambda_k(\nabla g_k)(\mathbf{x}_0).$$

Example 4.5.3. We wish to find the distance from the origin to the intersection of $xy=12$ and $x+2z=0$. To do this, we consider the function

$$f:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto x^2+y^2+z^2,$$

which represents the square of the distance from the origin to a point $(x,y,z)\in\mathbb{R}^3$. The points (x,y,z) under consideration are subject to the constraints $g_1(x,y,z)=12$ and $g_2(x,y,z)=0$, where

$$g_1:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto xy\qquad\text{and}\qquad g_2:\mathbb{R}^3\to\mathbb{R}:(x,y,z)\mapsto x+2z.$$

We now wish to minimize f subject to these constraints. Using the Lagrange multiplier method, we know that the minimum is attained at a point (x,y,z) which satisfies

$$(\nabla f)(x,y,z)=\lambda_1(\nabla g_1)(x,y,z)+\lambda_2(\nabla g_2)(x,y,z)$$

for some real numbers $\lambda_1,\lambda_2\in\mathbb{R}$. Note that

$$(\nabla f)(x,y,z)=(2x,2y,2z)\qquad\text{and}\qquad(\nabla g_1)(x,y,z)=(y,x,0)\qquad\text{and}\qquad(\nabla g_2)(x,y,z)=(1,0,2).$$

Hence we need to solve the equations

$$(2x,2y,2z)=\lambda_1(y,x,0)+\lambda_2(1,0,2)\qquad\text{and}\qquad xy=12\qquad\text{and}\qquad x+2z=0.$$

Eliminating λ_1 and λ_2 from this system of five equations, we conclude (after a fair bit of calculation) that

$$(x,y,z) = \left(\pm 2\sqrt[4]{\frac{36}{5}}, \pm 6\sqrt[4]{\frac{5}{36}}, \mp\sqrt[4]{\frac{36}{5}}\right) \qquad \text{and} \qquad f(x,y,z) = 12\sqrt{5}.$$

Hence the minimum distance is equal to $\sqrt{12\sqrt{5}}$, the square root of $f(x,y,z)$ at this point.

Problems for Chapter 4

1. For each of the following functions, find the second-order Taylor approximation at the given point:
   a) $f(x,y) = x\cos(xy) + y\sin(xy)$; $(x_0,y_0) = (0,0)$
   b) $f(x,y,z) = \mathrm{e}^{xz}y^2 + \sin y\cos z + x^2 z$; $(x_0,y_0,z_0) = (0,\pi,0)$

2. Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = x^3 - y^3 - 3xy + 4.$$

   a) Show that the total derivatives $(\mathbf{D}f)(-1,1) = 0$ and $(\mathbf{D}f)(0,0) = 0$.
   b) Find the second-order Taylor approximations to $f(x,y)$ at the points $(-1,1)$ and $(0,0)$.
   c) Find the Hessians $(\mathbf{H}f)(-1,1)$ and $(\mathbf{H}f)(0,0)$.
   d) Is $(\mathbf{H}f)(-1,1)$ positive definite? Negative definite? Comment on the result.
   e) Is $(\mathbf{H}f)(0,0)$ positive definite? Negative definite?
   f) Find the discriminant of f at $(0,0)$.
   g) Comment on your observations in (e), (f) and the second part of (a).

3. Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \frac{x^4 - 4x^3 + 4x^2 - 3}{1+y^2}.$$

   a) Find the total derivative $(\mathbf{D}f)(x,y)$.
   b) Show that the three stationary points are $(0,0)$, $(1,0)$ and $(2,0)$.
   c) Evaluate the partial derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y$, and find the Hessian of f at each of the stationary points.
   d) Show that the Hessian of f at $(0,0)$ and at $(2,0)$ are positive definite.
   e) Find the discriminant of f at $(1,0)$.
   f) Classify the stationary points.

4. Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = x^3 + y^3 + 9x^2 + 9y^2 + 12xy$.
   a) Show that $(0,0)$, $(-10,-10)$, $(-4,2)$ and $(2,-4)$ are stationary points.
   b) Find the Hessian of f at $(0,0)$ and show that it is positive definite.
   c) Find the Hessian of f at $(-10,-10)$ and show that it is negative definite.
   d) Classify the stationary points $(0,0)$ and $(-10,-10)$.
   e) Find the discriminant of f at the other two stationary points, and classify these stationary points.

5. Consider the function $f:\mathbb{R}^3 \to \mathbb{R}$, defined by $f(x,y,z) = x^2 + y^2 + z^2 - 6xy + 8xz - 10yz$.
   a) Show that $(\mathbf{D}f)(x,y,z) = \mathbf{0}$ leads to a system of three linear equations with unique solution $(x,y,z) = (0,0,0)$.
   b) Without any calculation, can you write down the Hessian of f at $(0,0,0)$?
   c) If you cannot do (b), then proceed to calculate the Hessian of f at $(0,0,0)$. Then try to understand the surprise (assuming that your calculation is correct).

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = 4x^2 - 12xy + 9y^2$.
   a) Show that f has infinitely many stationary points.
   b) Show that the Hessian at any stationary point of f is given by the same function $(2x - 3y)^2$.
   c) Can you classify these stationary points?
   [HINT: Dispense with the theorems and have some fun instead.]

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = (y - x^2)(y - 2x^2)$.
   a) Show that $(0, 0)$ is a stationary point of f.
   b) Find the Hessian of f at $(0, 0)$. Is it positive definite? Negative definite?
   c) Show that on any line through the origin, f has a minimum at $(0, 0)$.
   [HINT: Consider three cases: $y = 0$, $x = 0$ and $y = \alpha x$ where α is any non-zero real number.]
   d) Draw a picture of the two parabolas $y = x^2$ and $y = 2x^2$ on the plane. Note that $f(x, y)$ is a product of two factors which are non-zero at any point (x, y) not on the parabolas. Shade in one colour the region in $\mathbb{R}^2$ for which $f(x, y) > 0$, and in another colour the region in $\mathbb{R}^2$ for which $f(x, y) < 0$. Convince yourself that f has a saddle point at $(0, 0)$.

8. Follow the steps indicated below to find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$. Write $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 + 8xy + 7y^2$. We shall minimize $f(x, y)$ subject to the constraint $g(x, y) = 225$.
   a) Let λ be a Lagrange multiplier. Show that the equation $(\nabla f)(x, y) - \lambda(\nabla g)(x, y) = 0$ can be rewritten as a system of two homogeneous linear equations in x and y, where some of the coefficients depend on λ.
   b) Clearly $(x, y) \neq (0, 0)$. It follows that the system of homogeneous linear equations in (a) has non-trivial solution, and so the determinant of the corresponding matrix is zero. Use this fact to find two roots for λ.
   c) Show that one of the roots λ in (b) leads to no real solution of the system, while the other root λ leads to a solution. Use this solution to minimize $f(x, y)$.
   d) What is the shortest distance from the origin to the hyperbola?

9. Find the point on the paraboloid $z = x^2 + y^2$ which is closest to the point $(3, -6, 4)$.

10. Find the extreme values of z on the surface $2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35$.

MULTIVARIABLE AND VECTOR ANALYSIS

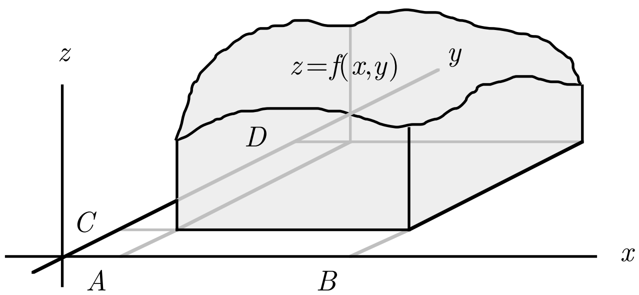
W W L CHEN

Chapter 5

DOUBLE AND TRIPLE INTEGRALS

5.1. Introduction

Consider a real valued function $f(x, y)$, defined over a rectangle $R = [A, B] \times [C, D]$. Suppose, for simplicity, that $f(x, y) \geq 0$ for every $(x, y) \in R$. We would like to evaluate the volume of the region in $\mathbb{R}^3$ above R on the xy-plane (between the planes $x = A$ and $x = B$, and between the planes $y = C$ and $y = D$) and under the surface $z = f(x, y)$.

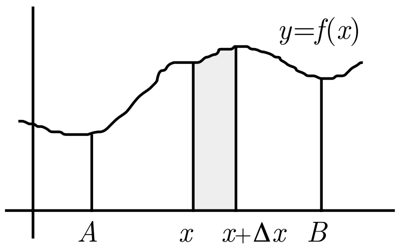


Let this volume be represented by

$$\iint_R f(x, y)\, \mathrm{d}x\mathrm{d}y.$$

The purpose of this chapter is to investigate the properties of this "integral".

We shall first of all take a very cavalier approach to the problem. Consider the simpler case of a function $f(x)$ defined over an interval $[A,B]$. Suppose, for simplicity, that $f(x) \geq 0$ for every $x \in [A,B]$.



Let us split the interval $[A,B]$ into a large number of very short intervals. Consider now one such interval $[x, x+\Delta x]$, where Δx is very small. Then the region in $\mathbb{R}^2$ above the interval $[x, x+\Delta x]$ on the x-axis and under the curve $y = f(x)$ is roughly a rectangle with base Δx and height $f(x)$, and so has area roughly equal to $f(x)\Delta x$. Hence the area of the region in $\mathbb{R}^2$ above the interval $[A,B]$ on the x-axis and under the curve $y = f(x)$ is roughly

$$\sum_{\Delta x} f(x)\Delta x,$$

where the summation is over all these very short intervals making up the interval $[A,B]$. As $\Delta x \to 0$, we have, with any luck,

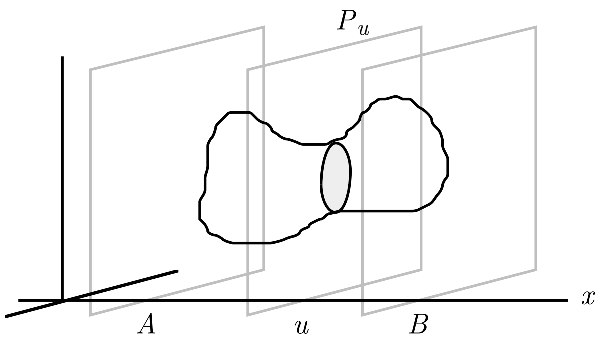
$$\sum_{\Delta x} f(x)\Delta x \to \int_A^B f(x)\,\mathrm{d}x.$$

We next extend this approach to the problem of finding the volume of an object in 3-space. Consistent with our lack of rigour so far, the following seems plausible.

CAVALIERI'S PRINCIPLE. *Suppose that S is a solid in 3-space, and that for $u \in [\alpha, \beta]$, P_u is a family of parallel planes such that the solid S lies between the planes P_α and P_β. Suppose further that for every $u \in [\alpha, \beta]$, the area of the intersection of S with the plane P_u is given by $a(u)$. Then the volume of S is given by*

$$\int_\alpha^\beta a(u)\,\mathrm{d}u.$$

Let us now apply Cavalieri's principle to our original problem. For every $u \in [A,B]$, let P_u denote the plane $x = u$.



Clearly the region in question lies between the planes P_A and P_B. On the other hand, if $a(u)$ denotes the area of the intersection between the region in question and the plane $x = u$, then

$$a(u) = \int_C^D f(u, y)\,\mathrm{d}y.$$

By Cavalieri's principle, the volume of the region in question is now given by

$$\int_A^B a(u)\,\mathrm{d}u = \int_A^B \left(\int_C^D f(u, y)\,\mathrm{d}y \right) \mathrm{d}u = \int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Similarly, for every $u \in [C, D]$, let P_u denote the plane $y = u$. Clearly the region in question lies between the planes P_C and P_D. On the other hand, if $a(u)$ denotes the area of the intersection between the region in question and the plane $y = u$, then

$$a(u) = \int_A^B f(x, u)\,\mathrm{d}x.$$

By Cavalieri's principle, the volume of the region in question is now given by

$$\int_C^D a(u)\,\mathrm{d}u = \int_C^D \left(\int_A^B f(x, u)\,\mathrm{d}x \right) \mathrm{d}u = \int_C^D \left(\int_A^B f(x, y)\,\mathrm{d}x \right) \mathrm{d}y.$$

We therefore conclude that, with any luck,

$$(1) \qquad \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x = \int_C^D \left(\int_A^B f(x, y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Unfortunately, the identity (1) does not hold all the time.

Example 5.1.1. Consider the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$, given by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational,} \end{cases}$$

Then

$$\int_0^1 f(x, y)\,\mathrm{d}y = \begin{cases} \displaystyle\int_0^1 \mathrm{d}y = 1 & \text{if } x \text{ is rational,} \\ \displaystyle\int_0^1 2y\,\mathrm{d}y = 1 & \text{if } x \text{ is irrational,} \end{cases}$$

so that

$$\int_0^1 \left(\int_0^1 f(x, y)\,\mathrm{d}y \right) \mathrm{d}x = 1.$$

On the other hand, the integral

$$\int_0^1 f(x, y)\,\mathrm{d}x$$

does not exist except when $y = 1/2$, so

$$\int_0^1 \left(\int_0^1 f(x, y)\,\mathrm{d}x \right) \mathrm{d}y$$

does not exist.

5.2. Double Integrals over Rectangles

Suppose that the function $f : R \to \mathbb{R}^2$ is bounded in R, where $R = [A, B] \times [C, D]$ is a rectangle. Suppose further that

$$\Delta : \; A = x_0 < x_1 < x_2 < \ldots < x_n = B, \; C = y_0 < y_1 < y_2 < \ldots < y_m = D$$

is a dissection of the rectangle $R = [A, B] \times [C, D]$.

Definition. The sum

$$s(\Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y)$$

is called the lower Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

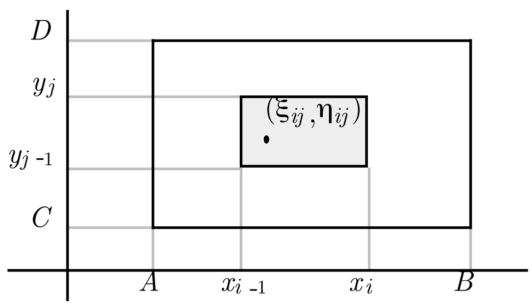
Definition. The sum

$$S(\Delta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y)$$

is called the upper Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

Remark. Strictly speaking, the above definitions are invalid, since the minimum or maximum may not exist. The correct way is to replace the minimum and maximum with infimum and supremum respectively. However, since we have not discussed infimum and supremum, we shall be somewhat economical with the truth and simply use minimum and maximum.

Definition. Suppose that for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the point (ξ_{ij}, η_{ij}) lies in the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$.



Then the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) f(\xi_{ij}, \eta_{ij})$$

is called a Riemann sum of $f(x, y)$ corresponding to the dissection Δ.

Remarks. (1) It is clear that

$$\min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y) \le f(\xi_{ij}, \eta_{ij}) \le \max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x, y).$$

It follows that every Riemann sum is bounded below by the corresponding lower Riemann sum and bounded above by the corresponding upper Riemann sum; in other words,

$$s(\Delta) \le \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1})f(\xi_{ij},\eta_{ij}) \le S(\Delta).$$

(2) It can be shown that for any two dissections Δ' and Δ'' of the rectangle $R = [A,B]\times[C,D]$, we have $s(\Delta') \le S(\Delta'')$; in other words, a lower Riemann sum can never exceed an upper Riemann sum.

Definition. We say that

$$\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y = L$$

if, given any $\epsilon > 0$, there exists a dissection Δ of $R = [A,B]\times[C,D]$ such that

$$L - \epsilon < s(\Delta) \le S(\Delta) < L + \epsilon.$$

In this case, we say that the function $f(x,y)$ is Riemann integrable over the rectangle $R = [A,B]\times[C,D]$ with integral L.

Remark. In other words, if the lower Riemann sums and upper Riemann sums can get arbitrarily close, then their common value is the integral of the function.

The following result follows easily from our definition. The proof is left as an exercise.

THEOREM 5A. *Suppose that the functions $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ are Riemann integrable over the rectangle $R = [A,B]\times[C,D]$.*
(a) Then the function $f+g$ is Riemann integrable over R, and

$$\iint_R (f(x,y)+g(x,y))\,\mathrm{d}x\mathrm{d}y = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint_R g(x,y)\,\mathrm{d}x\mathrm{d}y;$$

(b) On the other hand, for any $c \in \mathbb{R}$, the function cf is Riemann integrable over R, and

$$\iint_R cf(x,y)\,\mathrm{d}x\mathrm{d}y = c\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

(c) Suppose further that $f(x,y) \ge g(x,y)$ for every $(x,y) \in R$. Then

$$\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y \ge \iint_R g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

We also state without proof the following result.

THEOREM 5B. *Suppose that $Q = R_1 \cup \ldots \cup R_p$ is a rectangle, where, for $k = 1,\ldots,p$, the rectangles $R_k = [A_k,B_k]\times[C_k,D_k]$ are pairwise disjoint, apart possibly from boundary points. Suppose further that the function $f : Q \to \mathbb{R}$ is bounded in Q. Then f is Riemann integrable over Q if and only if it is Riemann integrable over R_k for every $k = 1,\ldots,p$. In this case, we also have*

$$\iint_Q f(x,y)\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{p}\iint_{R_k} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

5.3. Conditions for Integrability

The following important result will be discussed in Section 5.5.

THEOREM 5C. (FUBINI'S THEOREM) *Suppose that the function $f : R \to \mathbb{R}$ is continuous in the rectangle $R = [A, B] \times [C, D]$. Then f is Riemann integrable over R. Furthermore, the identity* (1) *holds.*

EXAMPLE 5.3.1. Consider the integral

$$\iint_R (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y,$$

where $R = [0, 2] \times [0, 1]$ is a rectangle. Since the function $f(x, y) = x^2 + y^2$ is continuous in R, the integral exists by Fubini's theorem. Furthermore,

$$\int_0^2 \left(\int_0^1 (x^2 + y^2)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^2 \left(x^2 + \frac{1}{3} \right) \mathrm{d}x = \frac{10}{3}.$$

On the other hand,

$$\int_0^1 \left(\int_0^2 (x^2 + y^2)\,\mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(\frac{8}{3} + 2y^2 \right) \mathrm{d}y = \frac{10}{3}.$$

By Fubini's theorem,

$$\iint_R (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y = \frac{10}{3}.$$

EXAMPLE 5.3.2. Consider the integral

$$\iint_R \sin x \cos y\,\mathrm{d}x\mathrm{d}y,$$

where $R = [0, \pi] \times [-\pi/2, \pi/2]$ is a rectangle. Since the function $f(x, y) = \sin x \cos y$ is continuous in R, the integral exists by Fubini's theorem. Furthermore,

$$\int_0^\pi \left(\int_{-\pi/2}^{\pi/2} \sin x \cos y\,\mathrm{d}y \right) \mathrm{d}x = \int_0^\pi 2 \sin x\,\mathrm{d}x = 4.$$

On the other hand,

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^\pi \sin x \cos y\,\mathrm{d}x \right) \mathrm{d}y = \int_{-\pi/2}^{\pi/2} 2 \cos y\,\mathrm{d}y = 4.$$

By Fubini's theorem,

$$\iint_R \sin x \cos y\,\mathrm{d}x\mathrm{d}y = 4.$$

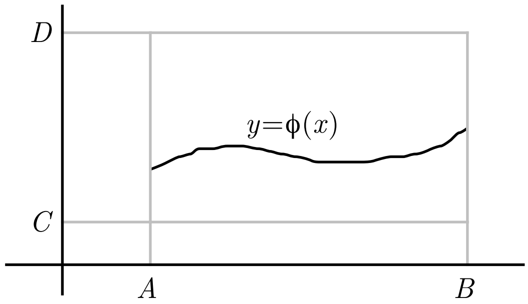
5.4. Double Integrals over Special Regions

The purpose of this section is to study Riemann integration over regions R which are not necessarily rectangles of the form $[A, B] \times [C, D]$. Our argument hinges on the following generalization of Fubini's theorem.

DEFINITION. Consider the rectangle $R = [A, B] \times [C, D]$. A set of the form

$$\{(x, \phi(x)) : x \in [A, B]\} \subseteq R$$

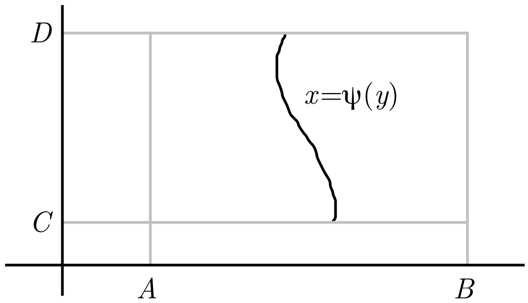
is said to be a curve of type 1 in R if the function $\phi : [A, B] \to \mathbb{R}$ is continuous in the interval $[A, B]$.



A set of the form

$$\{(\psi(y), y) : y \in [C, D]\} \subseteq R$$

is said to be a curve of type 2 in R if the function $\psi : [C, D] \to \mathbb{R}$ is continuous in the interval $[C, D]$.



THEOREM 5D. *Suppose that the function $f : R \to \mathbb{R}$ is bounded in the rectangle $R = [A, B] \times [C, D]$. Suppose further that f is continuous in R, except possibly at points contained in a finite number of curves of type 1 or 2 in R. Then f is Riemann integrable over R. Furthermore, if the integral*

$$\int_C^D f(x, y)\,\mathrm{d}y$$

exists for every $x \in [A, B]$, then the integral

$$\int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x$$

exists, and

$$\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_C^D f(x, y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Similarly, if the integral

$$\int_A^B f(x, y)\,\mathrm{d}x$$

exists for every $y \in [C,D]$*, then the integral*

$$\int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

exists, and

$$\iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_C^D \left(\int_A^B f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

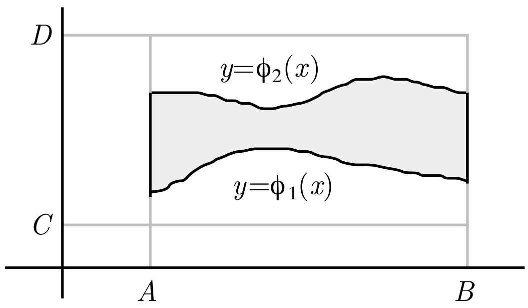
Thus, the identity (1) *holds if all the integrals exist.*

In view of Theorem 5D, we can now study integrals over some special regions.

Definition. Suppose that $\phi_1 : [A,B] \to \mathbb{R}$ and $\phi_2 : [A,B] \to \mathbb{R}$ are continuous in the interval $[A,B]$. Suppose further that $\phi_1(x) \le \phi_2(x)$ for every $x \in [A,B]$. Then we say that

(2) $$S = \{(x,y) \in \mathbb{R}^2 : x \in [A,B] \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$$

is an elementary region of type 1.



Suppose now that S is an elementary region of type 1, of the form (2). Since $\phi_1 : [A,B] \to \mathbb{R}$ and $\phi_2 : [A,B] \to \mathbb{R}$ are continuous in $[A,B]$, there exist $C, D \in \mathbb{R}$ such that $C \le \phi_1(x) \le \phi_2(x) \le D$ for every $x \in [A,B]$. It follows that $S \subseteq R$, where $R = [A,B] \times [C,D]$. Suppose that the function $f : S \to \mathbb{R}$ is continuous in S. Then it is bounded in S. Hence the function $f^* : R \to \mathbb{R}$, defined by

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in S, \\ 0 & \text{if } (x,y) \notin S, \end{cases}$$

is bounded in R. Furthermore, it is continuous in R, except possibly at points contained in two curves of type 1 in R. It follows from Theorem 5D that f^* is Riemann integrable over R. We can now define

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_R f^*(x,y)\,\mathrm{d}x\mathrm{d}y.$$

On the other hand, for any $x \in [A,B]$, the function $f^*(x,y) = f(x,y)$ is a continuous function of y in the interval $[\phi_1(x), \phi_2(x)]$. Also, $f^*(x,y) = 0$ for every $y \in [C, \phi_1(x)) \cup (\phi_2(x), D]$. Hence

$$\int_C^D f^*(x,y)\,\mathrm{d}y = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y,$$

and so

$$\int_A^B \left(\int_C^D f^*(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_A^B \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

We have proved the following result.

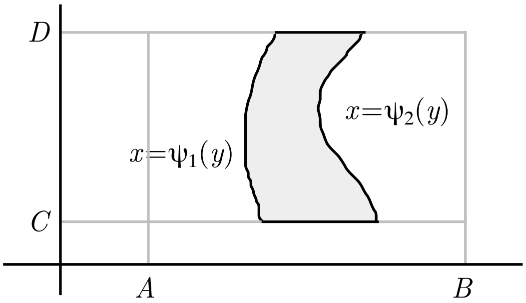
THEOREM 5E. *Suppose that S is an elementary region of type 1, of the form (2). Suppose further that the function $f : S \to \mathbb{R}$ is continuous in S. Then*

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_A^B \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

DEFINITION. Suppose that $\psi_1 : [C,D] \to \mathbb{R}$ and $\psi_2 : [C,D] \to \mathbb{R}$ are continuous in the interval $[C,D]$. Suppose further that $\psi_1(y) \leq \psi_2(y)$ for every $y \in [C,D]$. Then we say that

(3) $$S = \{(x,y) \in \mathbb{R}^2 : y \in [C,D] \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$$

is an elementary region of type 2.



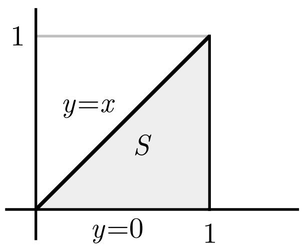
Analogous to Theorem 5E, we have the following result.

THEOREM 5F. *Suppose that S is an elementary region of type 2, of the form (3). Suppose further that the function $f : S \to \mathbb{R}$ is continuous in S. Then*

$$\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_C^D \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

EXAMPLE 5.4.1. The function $f(x,y) = x^2 + y^2$ is continuous in the triangle S with vertices $(0,0)$, $(1,0)$ and $(1,1)$. It is easy to see that S is an elementary region of type 1, of the form

$$S = \{(x,y) \in \mathbb{R}^2 : x \in [0,1] \text{ and } 0 \leq y \leq x\}.$$

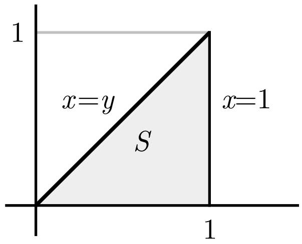


By Theorem 5E,

$$\iint_S (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_0^x (x^2+y^2)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \frac{4}{3}x^3\,\mathrm{d}x = \frac{1}{3}.$$

On the other hand, it is also easy to see that S is an elementary region of type 2, of the form

$$S = \{(x,y) \in \mathbb{R}^2 : y \in [0,1] \text{ and } y \leq x \leq 1\}.$$

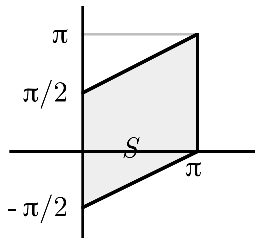


By Theorem 5F,

$$\iint_S (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_y^1 (x^2+y^2)\,\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \left(\frac{1}{3}+y^2-\frac{4}{3}y^3\right)\mathrm{d}y = \frac{1}{3}.$$

EXAMPLE 5.4.2. The function $f(x,y) = \sin x \cos y$ is continuous in the parallelogram S with vertices $(\pi,0)$, (π,π) and $(0,\pm\pi/2)$. It is easy to see that S is an elementary region of type 1, of the form

$$S = \left\{(x,y) \in \mathbb{R}^2 : x \in [0,\pi] \text{ and } \frac{x}{2}-\frac{\pi}{2} \leq y \leq \frac{x}{2}+\frac{\pi}{2}\right\}.$$



By Theorem 5E,

$$\begin{aligned}\iint_S \sin x \cos y\,\mathrm{d}x\mathrm{d}y &= \int_0^\pi \left(\int_{\frac{x}{2}-\frac{\pi}{2}}^{\frac{x}{2}+\frac{\pi}{2}} \sin x \cos y\,\mathrm{d}y\right)\mathrm{d}x = \int_0^\pi \sin x\left(\sin\left(\frac{x}{2}+\frac{\pi}{2}\right)-\sin\left(\frac{x}{2}-\frac{\pi}{2}\right)\right)\mathrm{d}x \\ &= \int_0^\pi \sin x\left(\sin\frac{x}{2}\cos\frac{\pi}{2}+\cos\frac{x}{2}\sin\frac{\pi}{2}-\sin\frac{x}{2}\cos\frac{\pi}{2}+\cos\frac{x}{2}\sin\frac{\pi}{2}\right)\mathrm{d}x \\ &= 2\int_0^\pi \sin x \cos\frac{x}{2}\,\mathrm{d}x = 4\int_0^\pi \sin\frac{x}{2}\cos^2\frac{x}{2}\,\mathrm{d}x = \frac{8}{3}.\end{aligned}$$

On the other hand, it is also easy to see that S is an elementary region of type 2, of the form

$$S = \left\{(x,y) \in \mathbb{R}^2 : y \in \left[-\frac{\pi}{2},\pi\right] \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\right\},$$

where

$$\psi_1(y) = \begin{cases} 0 & \text{if } y \in [-\pi/2,\pi/2], \\ 2y-\pi & \text{if } y \in [\pi/2,\pi], \end{cases} \qquad \text{and} \qquad \psi_2(y) = \begin{cases} 2y+\pi & \text{if } y \in [-\pi/2,0], \\ \pi & \text{if } y \in [0,\pi]. \end{cases}$$

By Theorem 5F,

$$
\begin{aligned}
&\iint_S \sin x \cos y \,\mathrm{d}x\mathrm{d}y \\
&= \int_{-\pi/2}^{0} \left(\int_0^{2y+\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y + \int_0^{\pi/2} \left(\int_0^{\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y + \int_{\pi/2}^{\pi} \left(\int_{2y-\pi}^{\pi} \sin x \cos y \,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-\pi/2}^{0} (\cos y - \cos(2y+\pi)\cos y)\,\mathrm{d}y + \int_0^{\pi/2} 2\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} (\cos y + \cos(2y-\pi)\cos y)\,\mathrm{d}y \\
&= \int_{-\pi/2}^{0} \cos y \,\mathrm{d}y + 2\int_0^{\pi/2} \cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} \cos y \,\mathrm{d}y - \int_{-\pi/2}^{0} \cos(2y+\pi)\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} \cos(2y-\pi)\cos y \,\mathrm{d}y \\
&= 2 - \int_{-\pi/2}^{0} (\cos 2y \cos \pi - \sin 2y \sin \pi)\cos y \,\mathrm{d}y + \int_{\pi/2}^{\pi} (\cos 2y \cos \pi + \sin 2y \sin \pi)\cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} \cos 2y \cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} \cos 2y \cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} (1 - 2\sin^2 y)\cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} (1 - 2\sin^2 y)\cos y \,\mathrm{d}y \\
&= 2 + \int_{-\pi/2}^{0} \cos y \,\mathrm{d}y - \int_{\pi/2}^{\pi} \cos y \,\mathrm{d}y - 2\int_{-\pi/2}^{0} \sin^2 y \cos y \,\mathrm{d}y + 2\int_{\pi/2}^{\pi} \sin^2 y \cos y \,\mathrm{d}y = \frac{8}{3}.
\end{aligned}
$$

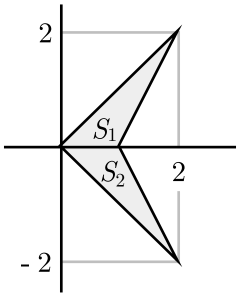
Note that the calculation is much simpler if we think of S as an elementary region of type 1.

We can extend our study further. Suppose that $T = S_1 \cup \ldots \cup S_p$ is a finite region in $\mathbb{R}^2$, where, for every $k = 1, \ldots, p$, S_k is an elementary region of type 1 or 2. Suppose further that $S_1, \ldots, S_p$ are pairwise disjoint, apart possibly from boundary points. Then we can define

$$
\iint_T f(x,y)\,\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{p} \iint_{S_k} f(x,y)\,\mathrm{d}x\mathrm{d}y,
$$

provided that every integral on the right hand side exists.

EXAMPLE 5.4.3. Consider the finite region T bounded by the four lines $y = x$, $y = -x$, $y = 2x - 2$ and $y = 2 - 2x$ and with vertices $(0,0)$, $(2,-2)$, $(1,0)$ and $(2,2)$.



Note that $T = S_1 \cup S_2$, where S_1 is the triangle with vertices $(0,0)$, $(1,0)$ and $(2,2)$ and S_2 is the triangle with vertices $(0,0)$, $(1,0)$ and $(2,-2)$. It is easy to see that S_1 and S_2 are disjoint, apart from boundary points. Clearly the function $f(x,y) = x + y^2$ is continuous in T, and so continuous in S_1 and S_2. Hence

$$
\iint_T (x + y^2)\,\mathrm{d}x\mathrm{d}y = \iint_{S_1} (x + y^2)\,\mathrm{d}x\mathrm{d}y + \iint_{S_2} (x + y^2)\,\mathrm{d}x\mathrm{d}y.
$$

To study the first integral on the right hand side, it is easier to interpret S_1 as an elementary region of type 2, of the form

$$S_1 = \left\{(x,y) \in \mathbb{R}^2 : y \in [0,2] \text{ and } y \le x \le 1 + \frac{y}{2}\right\}.$$

Then

$$\begin{aligned}\iint_{S_1} (x+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_0^2 \left(\int_y^{1+\frac{y}{2}} (x+y^2)\,\mathrm{d}x\right)\mathrm{d}y = \int_0^2 \left(\frac{1}{2}\left(1+\frac{y}{2}\right)^2 + \left(1+\frac{y}{2}\right)y^2 - \frac{1}{2}y^2 - y^3\right)\mathrm{d}y \\ &= \int_0^2 \left(\frac{1}{2} + \frac{1}{2}y + \frac{5}{8}y^2 - \frac{1}{2}y^3\right)\mathrm{d}y = \frac{5}{3}.\end{aligned}$$

To study the second integral on the right hand side, it is again easier to interpret S_2 as an elementary region of type 2, of the form

$$S_2 = \left\{(x,y) \in \mathbb{R}^2 : y \in [-2,0] \text{ and } -y \le x \le 1 - \frac{y}{2}\right\}.$$

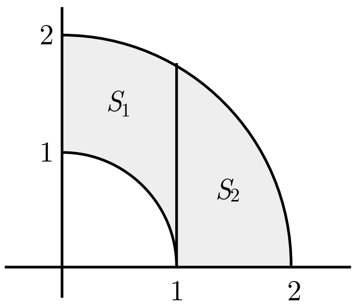
Then

$$\begin{aligned}\iint_{S_2} (x+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_{-2}^0 \left(\int_{-y}^{1-\frac{y}{2}} (x+y^2)\,\mathrm{d}x\right)\mathrm{d}y = \int_{-2}^0 \left(\frac{1}{2}\left(1-\frac{y}{2}\right)^2 + \left(1-\frac{y}{2}\right)y^2 - \frac{1}{2}y^2 + y^3\right)\mathrm{d}y \\ &= \int_{-2}^0 \left(\frac{1}{2} - \frac{1}{2}y + \frac{5}{8}y^2 + \frac{1}{2}y^3\right)\mathrm{d}y = \frac{5}{3}.\end{aligned}$$

Hence

$$\iint_T (x+y^2)\,\mathrm{d}x\mathrm{d}y = \frac{10}{3}.$$

Example 5.4.4. Consider the region T in the first quadrant between the two circles of radii 1 and 2 and centred at $(0,0)$.



Note that $T = S_1 \cup S_2$, where

$$S_1 = \{(x,y) \in \mathbb{R}^2 : x \in [0,1] \text{ and } (1-x^2)^{1/2} \le y \le (4-x^2)^{1/2}\}$$

and

$$S_2 = \{(x,y) \in \mathbb{R}^2 : x \in [1,2] \text{ and } 0 \le y \le (4-x^2)^{1/2}\}$$

are elementary regions of type 1 and disjoint, apart from boundary points. The function $f(x,y) = x^2+y^2$ is clearly continuous in T, and so continuous in S_1 and S_2. Hence

$$\iint_T (x^2+y^2)\,\mathrm{d}x\mathrm{d}y = \iint_{S_1} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y + \iint_{S_2} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y.$$

We have

$$\begin{aligned}\iint_{S_1} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_{(1-x^2)^{1/2}}^{(4-x^2)^{1/2}} (x^2+y^2)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \left(x^2(4-x^2)^{1/2} + \frac{1}{3}(4-x^2)^{3/2} - x^2(1-x^2)^{1/2} - \frac{1}{3}(1-x^2)^{3/2} \right) \mathrm{d}x \\ &= \frac{4}{3}\int_0^1 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_0^1 x^2(4-x^2)^{1/2}\,\mathrm{d}x - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\iint_{S_2} (x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_1^2 \left(\int_0^{(4-x^2)^{1/2}} (x^2+y^2)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_1^2 \left(x^2(4-x^2)^{1/2} + \frac{1}{3}(4-x^2)^{3/2} \right) \mathrm{d}x = \frac{4}{3}\int_1^2 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_1^2 x^2(4-x^2)^{1/2}\,\mathrm{d}x.\end{aligned}$$

Hence

$$\begin{aligned}&\iint_T (x^2+y^2)\,\mathrm{d}x\mathrm{d}y \\ &\quad= \frac{4}{3}\int_0^2 (4-x^2)^{1/2}\,\mathrm{d}x + \frac{2}{3}\int_0^2 x^2(4-x^2)^{1/2}\,\mathrm{d}x - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x \\ &\quad= \frac{16}{3}\int_0^1 (1-z^2)^{1/2}\,\mathrm{d}z + \frac{32}{3}\int_0^1 z^2(1-z^2)^{1/2}\,\mathrm{d}z - \frac{1}{3}\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x - \frac{2}{3}\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x \\ &\quad= 5\int_0^1 (1-x^2)^{1/2}\,\mathrm{d}x + 10\int_0^1 x^2(1-x^2)^{1/2}\,\mathrm{d}x = \frac{15\pi}{8}.\end{aligned}$$

Sometimes, we may be faced with repeated integrals that are extremely difficult to handle. Occasionally, a change in the order of integration may be helpful. We illustrate this point by the following two examples.

EXAMPLE 5.4.5. Consider the repeated integral

$$\int_0^{\sqrt{2}} \left(\int_{y/2}^{1/\sqrt{2}} \cos(\pi x^2)\,\mathrm{d}x \right) \mathrm{d}y. \tag{4}$$

Here the inner integral

$$\int_{y/2}^{1/\sqrt{2}} \cos(\pi x^2)\,\mathrm{d}x$$

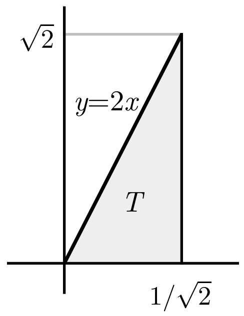
is rather hard to evaluate. However, we may treat the integral (4) as a double integral of the form

$$\iint_S \cos(\pi x^2)\,\mathrm{d}x\mathrm{d}y.$$

To make any progress, we must first of all find out what the region S is. Clearly

$$S = \left\{ (x,y) \in \mathbb{R}^2 : y \in [0,\sqrt{2}] \text{ and } \frac{y}{2} \le x \le \frac{1}{\sqrt{2}} \right\}$$

is a triangle as shown below.



Interchanging the order of integration, we may interpret the region S as an elementary region of type 1, of the form

$$S = \left\{(x,y) \in \mathbb{R}^2 : x \in \left[0, \frac{1}{\sqrt{2}}\right] \text{ and } 0 \le y \le 2x\right\},$$

so that

$$\iint_S \cos(\pi x^2)\,\mathrm{d}x\mathrm{d}y = \int_0^{1/\sqrt{2}} \left(\int_0^{2x} \cos(\pi x^2)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^{1/\sqrt{2}} 2x\cos(\pi x^2)\,\mathrm{d}x = \frac{1}{\pi}.$$

Example 5.4.6. Suppose that we are asked to evaluate

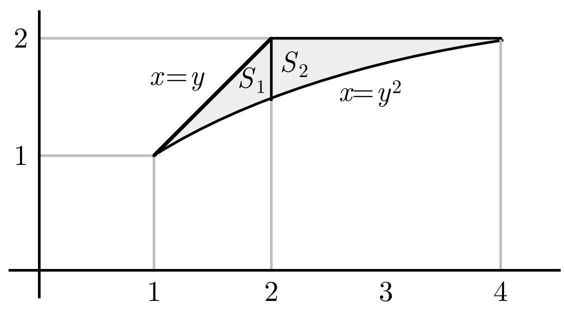
$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\mathrm{d}x + \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\mathrm{d}x. \tag{5}$$

Here the inner integrals are rather hard to evaluate. Instead, we write

$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\mathrm{d}x = \iint_{S_1} \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\mathrm{d}x = \iint_{S_2} \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y,$$

where

$$S_1 = \{(x,y) \in \mathbb{R}^2 : x \in [1,2] \text{ and } \sqrt{x} \le y \le x\} \quad \text{and} \quad S_2 = \{(x,y) \in \mathbb{R}^2 : x \in [2,4] \text{ and } \sqrt{x} \le y \le 2\}.$$



If we write $S = S_1 \cup S_2$, then

$$S = \{(x,y) \in \mathbb{R}^2 : y \in [1,2] \text{ and } y \le x \le y^2\}$$

is an elementary region of type 2, and the sum (5) is equal to

$$\iint_S \sin\frac{\pi x}{2y}\,\mathrm{d}x\mathrm{d}y = \int_1^2 \left(\int_y^{y^2} \sin\frac{\pi x}{2y}\,\mathrm{d}x\right)\mathrm{d}y = -\int_1^2 \frac{2y}{\pi}\cos\frac{\pi y}{2}\,\mathrm{d}y = \frac{4(\pi+2)}{\pi^3},$$

where the last step involves integration by parts.

5.5. Fubini's Theorem

In this section, we briefly indicate how continuity of a function f in a rectangle $R = [A,B]\times[C,D]$ leads to integrability of f in R. Note, however, that our discussion here falls well short of a proof of Fubini's theorem. We shall only discuss the following result.

THEOREM 5G. *Suppose that the function $f : R \to \mathbb{R}$ is continuous in the rectangle $R = [A,B]\times[C,D]$. Then for every $\epsilon > 0$, there exists a dissection*

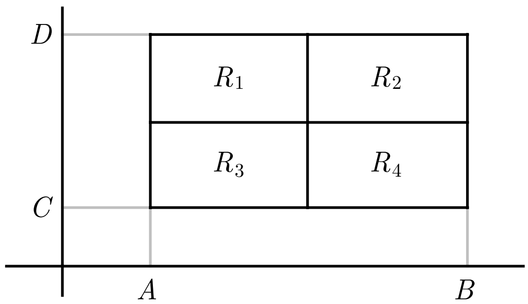
$$\Delta:\ A = x_0 < x_1 < x_2 < \ldots < x_n = B,\ C = y_0 < y_1 < y_2 < \ldots < y_m = D$$

of R such that for every $i = 1,\ldots,n$ and $j = 1,\ldots,m$, we have

$$\max_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) - \min_{(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]} f(x,y) < \epsilon, \tag{6}$$

so that $S(\Delta) - s(\Delta) < (B-A)(D-C)\epsilon$.

† Sketch of Proof. Suppose on the contrary that there exists $\epsilon > 0$ for which no dissection of R will satisfy (6). We now dissect the rectangle R into four similar rectangles as shown.



Then for at least one of the four rectangles R_k $(k = 1,2,3,4)$, we cannot find a dissection of R_k which will achieve an inequality of the type (6). We now dissect this rectangle R_k into four smaller and similar rectangles in the same way. Note that each dissection quarters the area of the rectangle, so this process eventually collapses to a point $(\alpha,\beta) \in R$. Since f is continuous at (α,β) (with a slightly modified argument if (α,β) is on the boundary of R), there exists $\delta > 0$ such that $|f(x,y) - f(\alpha,\beta)| < \epsilon/2$ for every $(x,y) \in R$ satisfying $\|(x,y) - (\alpha,\beta)\| < \delta$. It follows that for every

$$(x_1,y_1),(x_2,y_2) \in \{(x,y) \in R : \|(x,y) - (\alpha,\beta)\| < \delta\},$$

we have

$$|f(x_1,y_1) - f(x_2,y_2)| \le |f(x_1,y_1) - f(\alpha,\beta)| + |f(x_2,y_2) - f(\alpha,\beta)| < \epsilon.$$

However, our dissection will result in a rectangle contained in $\{(x,y) \in R : \|(x,y) - (\alpha,\beta)\| < \delta\}$, giving a contradiction. ◯

5.6. Mean Value Theorem

Suppose that $S \subseteq \mathbb{R}^2$ is an elementary region. Suppose further that a function $f : S \to \mathbb{R}$ is continuous in S. Then there exist $(x_1, y_1), (x_2, y_2) \in S$ such that

$$f(x_1, y_1) \le f(x, y) \le f(x_2, y_2)$$

for every $(x, y) \in S$; in other words, f has a minimum value and a maximum value in S. It follows that

$$f(x_1, y_1)A(S) = \iint_S f(x_1, y_1)\,\mathrm{d}x\mathrm{d}y \le \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y \le \iint_S f(x_2, y_2)\,\mathrm{d}x\mathrm{d}y = f(x_2, y_2)A(S),$$

where

$$A(S) = \iint_S \mathrm{d}x\mathrm{d}y$$

denotes the area of S, so that

$$f(x_1, y_1) \le \frac{1}{A(S)} \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y \le f(x_2, y_2).$$

Since f is continuous in S, it follows from the Intermediate value theorem that there exists $(x_0, y_0) \in S$ such that

$$f(x_0, y_0) = \frac{1}{A(S)} \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

We have sketched a proof of the following result.

THEOREM 5H. (MEAN VALUE THEOREM) *Suppose that $S \subseteq \mathbb{R}^2$ is an elementary region. Suppose further that a function $f : S \to \mathbb{R}$ is continuous in S. Then there exists $(x_0, y_0) \in S$ such that*

$$\iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y = f(x_0, y_0)A(S),$$

where $A(S)$ denotes the area of S.

EXAMPLE 5.6.1. The function $f(x, y) = \cos(x+y)$ has maximum value $f(0, 0) = 1$ and minimum value $f(\pi/4, \pi/4) = 0$ in the rectangle $R = [0, \pi/4] \times [0, \pi/4]$. On the other hand,

$$\begin{aligned}
\iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y &= \int_0^{\pi/4} \left(\int_0^{\pi/4} \cos(x + y)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^{\pi/4} \left(\sin\left(x + \frac{\pi}{4}\right) - \sin x \right) \mathrm{d}x \\
&= \int_0^{\pi/4} \left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4} - \sin x \right) \mathrm{d}x = \int_0^{\pi/4} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x - \sin x \right) \mathrm{d}x \\
&= \int_0^{\pi/4} \left(\frac{1}{\sqrt{2}} \cos x - \left(1 - \frac{1}{\sqrt{2}}\right) \sin x \right) \mathrm{d}x = \frac{1}{\sqrt{2}} \left(\sin\frac{\pi}{4} - \sin 0 \right) + \left(1 - \frac{1}{\sqrt{2}}\right) \left(\cos\frac{\pi}{4} - \cos 0 \right) \\
&= \frac{1}{2} + \left(1 - \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}} - 1 \right) = \sqrt{2} - 1.
\end{aligned}$$

It follows that

$$\frac{1}{A(R)} \iint_R f(x, y)\,\mathrm{d}x\mathrm{d}y = \frac{16}{\pi^2}(\sqrt{2} - 1).$$

Simple calculation gives

$$0<\frac{16}{\pi^2}(\sqrt{2}-1)<1,$$

so that there exists $u_0\in[0,\pi/4]$, and so $(u_0,u_0)\in[0,\pi/4]\times[0,\pi/4]$, such that

$$f(u_0,u_0)=\cos 2u_0=\frac{16}{\pi^2}(\sqrt{2}-1).$$

EXAMPLE 5.6.2. The function $f(x,y)=x^2+y^2+2$ has maximum value $f(1,1)=3$ and minimum value $f(0,0)=2$ in the disc $S=\{(x,y)\in\mathbb{R}^2:x^2+y^2\le 1\}$. Hence

$$2\pi\le\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y\le 3\pi.$$

5.7. Triple Integrals

In this last section, we extend our discussion so far to the case of triple integrals. Triple integrals over rectangular boxes $[A,B]\times[C,D]\times[P,Q]$ can be studied via Riemann sums if we extend the argument in Section 5.2 to the 3-dimensional case, although it is harder to visualize geometrically the graphs of real valued functions of three real variables.

We have the following 3-dimensional version of Theorem 5C.

THEOREM 5J. (FUBINI'S THEOREM) *Suppose that the function $f:R\to\mathbb{R}$ is continuous in the rectangular box $R=[A,B]\times[C,D]\times[P,Q]$. Then f is Riemann integrable over R. Furthermore,*

$$\begin{aligned}\iiint_R f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z&=\int_A^B\left(\int_C^D\left(\int_P^Q f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x=\int_A^B\left(\int_P^Q\left(\int_C^D f(x,y,z)\,\mathrm{d}y\right)\mathrm{d}z\right)\mathrm{d}x\\&=\int_C^D\left(\int_A^B\left(\int_P^Q f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y=\int_C^D\left(\int_P^Q\left(\int_A^B f(x,y,z)\,\mathrm{d}x\right)\mathrm{d}z\right)\mathrm{d}y\\&=\int_P^Q\left(\int_C^D\left(\int_A^B f(x,y,z)\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z=\int_P^Q\left(\int_A^B\left(\int_C^D f(x,y,z)\,\mathrm{d}y\right)\mathrm{d}x\right)\mathrm{d}z.\end{aligned}$$

EXAMPLE 5.7.1. The function $f(x,y,z)=\cos x-\sin y-\sin z$ is continuous in the rectangular box $R=[0,\pi/2]\times[0,\pi/2]\times[0,\pi/2]$. It follows from Fubini's theorem that

$$\iiint_R f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

exists, and is equal to

$$\begin{aligned}&\int_0^{\pi/2}\left(\int_0^{\pi/2}\left(\int_0^{\pi/2}(\cos x-\sin y-\sin z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x=\int_0^{\pi/2}\left(\int_0^{\pi/2}\left(\frac{\pi}{2}\cos x-\frac{\pi}{2}\sin y-1\right)\mathrm{d}y\right)\mathrm{d}x\\&\quad=\int_0^{\pi/2}\left(\frac{\pi^2}{4}\cos x-\frac{\pi}{2}-\frac{\pi}{2}\right)\mathrm{d}x=\int_0^{\pi/2}\left(\frac{\pi^2}{4}\cos x-\pi\right)\mathrm{d}x=\frac{\pi^2}{4}-\frac{\pi^2}{2}=-\frac{\pi^2}{4}.\end{aligned}$$

EXAMPLE 5.7.2. We have

$$\int_0^1 \left(\int_0^3 \left(\int_0^2 (x+y)z\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\int_0^3 2(x+y)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 (6x+9)\,\mathrm{d}x = 12.$$

We can also extend the integral to elementary regions. Instead of giving the definitions of a number of different types of elementary regions, we shall simply illustrate the technique with two examples. Our discussion here closely follows the ideas introduced in Section 5.4.

EXAMPLE 5.7.3. Consider the set

$$S = \{(x,y,z) \in \mathbb{R}^3 : x,y,z \ge 0 \text{ and } x^2+y^2+z^2 \le 1\}$$

in $\mathbb{R}^3$. We shall find its volume by evaluating the triple integral

$$V(S) = \iiint_S \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

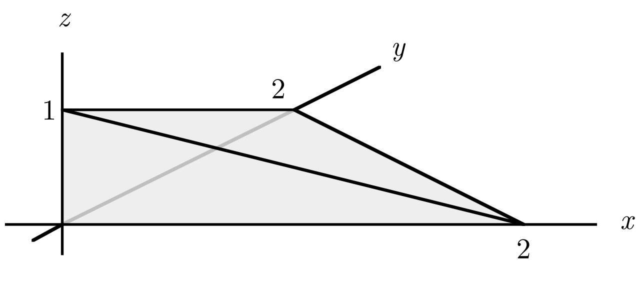
The region S can be described by

$$S = \{(x,y,z) \in \mathbb{R}^3 : x \in [0,1],\ y \in [0,\sqrt{1-x^2}] \text{ and } z \in [0,\sqrt{1-x^2-y^2}]\}.$$

Hence

$$\begin{aligned} V(S) &= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} \mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left[\frac{1}{2}\left(y\sqrt{1-x^2-y^2} + (1-x^2)\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}} \mathrm{d}x \\ &= \frac{\pi}{4}\int_0^1 (1-x^2)\,\mathrm{d}x = \frac{\pi}{6}. \end{aligned}$$

EXAMPLE 5.7.4. Consider the region S bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+2z=2$, with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(0,0,1)$.



Consider also the function $f(x,y,z) = 4xy+8z$. We can write

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x \in [0,2],\ y \in [0,2-x] \text{ and } z \in \left[0, \frac{2-x-y}{2}\right]\right\}.$$

Then

$$\int_0^2 \left(\int_0^{2-x} \left(\int_0^{(2-x-y)/2} (4xy+8z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x$$
$$= \int_0^2 \left(\int_0^{2-x} (2xy(2-x-y)+(2-x-y)^2)\,\mathrm{d}y \right) \mathrm{d}x$$
$$= \int_0^2 \left(\int_0^{2-x} (6xy-2x^2y-2xy^2+4+x^2+y^2-4x-4y)\,\mathrm{d}y \right) \mathrm{d}x$$
$$= \int_0^2 \left(3x(2-x)^2 - x^2(2-x)^2 - \frac{2}{3}x(2-x)^3 + 4(2-x) + x^2(2-x) \right.$$
$$\left. + \frac{1}{3}(2-x)^3 - 4x(2-x) - 2(2-x)^2 \right) \mathrm{d}x$$
$$= \int_0^2 \left(\frac{8}{3} - \frac{4}{3}x - 2x^2 + \frac{5}{3}x^3 - \frac{1}{3}x^4 \right) \mathrm{d}x = \frac{28}{15}.$$

We can also write

$$S = \{(x,y,z) \in \mathbb{R}^3 : z \in [0,1],\ x \in [0,2-2z] \text{ and } y \in [0,2-2z-x]\}.$$

Then

$$\int_0^1 \left(\int_0^{2-2z} \left(\int_0^{2-2z-x} (4xy+8z)\,\mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z$$
$$= \int_0^1 \left(\int_0^{2-2z} (2x(2-2z-x)^2 + 8z(2-2z-x))\,\mathrm{d}x \right) \mathrm{d}z$$
$$= \int_0^1 \left(\int_0^{2-2z} (8x+8xz^2+2x^3-24xz-8x^2+8x^2z+16z-16z^2)\,\mathrm{d}x \right) \mathrm{d}z$$
$$= \int_0^1 \left(\frac{8}{3} + \frac{16}{3}z - 16z^2 + \frac{16}{3}z^3 + \frac{8}{3}z^4 \right) \mathrm{d}z = \frac{28}{15}.$$

Problems for Chapter 5

1. Evaluate each of the following integrals, where $R = [0,1] \times [0,1]$:

   a) $\displaystyle\iint_R (x^2+y^3)\,\mathrm{d}x\mathrm{d}y$ b) $\displaystyle\iint_R \log((x+1)(y+1))\,\mathrm{d}x\mathrm{d}y$

2. Evaluate the integral $\displaystyle\int_{-1}^1 \left(\int_{-1}^1 \mathrm{e}^y y^2 \sin 2xy\,\mathrm{d}y \right) \mathrm{d}x$.

3. Suppose that the function f is continuous in the interval $[A,B]$, and the function g is continuous in the interval $[C,D]$. Suppose further that $R = [A,B] \times [C,D]$. Explain why the integral

$$\iint_R f(x)g(y)\,\mathrm{d}x\mathrm{d}y$$

   exists and is equal to

$$\left(\int_A^B f(x)\,\mathrm{d}x \right) \left(\int_C^D g(y)\,\mathrm{d}y \right).$$

4. Suppose that the function f is continuous in a rectangle $R = [A, B] \times [C, D]$, and $f(x, y) \geq 0$ for every $(x, y) \in R$. Show that
$$\iint_R f(x, y)\, \mathrm{d}x\mathrm{d}y = 0$$
if and only if $f(x, y) = 0$ for every $(x, y) \in R$.

5. Sketch the region S on the xy-plane bounded by the lines $x = 2$, $y = 1$ and the parabola $y = x^2$, and evaluate the double integral
$$\iint_S (x^3 + y^2)\, \mathrm{d}x\mathrm{d}y.$$

6. Consider the function $f(x, y) = \dfrac{x - y}{(x + y)^3}$, defined for every $(x, y) \in R = [0, 1] \times [0, 1]$, except at the point $(x, y) = (0, 0)$.
   a) Show that $\displaystyle\int_0^1 \left(\int_0^1 f(x, y)\, \mathrm{d}y \right) \mathrm{d}x = \frac{1}{2}$, and deduce that $\displaystyle\int_0^1 \left(\int_0^1 f(x, y)\, \mathrm{d}x \right) \mathrm{d}y = -\frac{1}{2}$.
   b) Show that $f(x, y)$ does not have a limit as $(x, y) \to (0, 0)$.
   c) Does the double integral $\displaystyle\iint_R f(x, y)\, \mathrm{d}x\mathrm{d}y$ exist as a Riemann integral? Give your reason(s).
   d) Comment on the results.

7. Consider the integral $\displaystyle\int_0^3 \left(\int_1^{\sqrt{4-y}} (x + y)\, \mathrm{d}x \right) \mathrm{d}y$.
   a) Sketch the area S on the xy-plane such that the integral is equal to $\displaystyle\iint_S (x + y)\, \mathrm{d}x\mathrm{d}y$.
   b) Interchange the order of integration, and evaluate the integral.

8. Show that the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, $y = -a$ and $y = a$, where $a > 0$, is given by $8a^4/3$.

9. Find each of the following integrals:
   a) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x + y + z)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$
   b) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x + y + z)^2\, \mathrm{d}x\mathrm{d}y\mathrm{d}z$

10. Let S be the region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$, with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Show that
$$\iiint_S z\, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{24}.$$

11. Let S be a "pyramid" with top vertex $(0, 0, 1)$ and base vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$. Show that
$$\iiint_S (1 - z^2)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{10}.$$
[Hint: Cut S by the plane $x = y$.]

12. Consider the region $S \subseteq \mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$, where $a > 0$ is fixed.
    a) Make a rough sketch of the region S.
    b) Show that $\displaystyle\iiint_S (x^2 + y^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{a^5}{20}$.

13. Show that $\displaystyle\int_0^1 \left(\int_0^1 \left(\int_{\sqrt{x^2+y^2}}^2 xyz\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x = \frac{3}{8}$.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

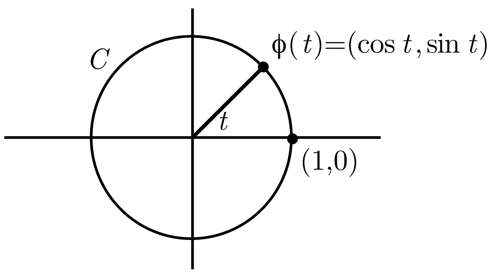
Chapter 7

PATHS

7.1. Introduction

In this chapter, we discuss paths in $\mathbb{R}^n$; in particular, we are interested in paths in $\mathbb{R}^2$ and $\mathbb{R}^3$. Before we give any formal definition, let us consider two examples.

Example 7.1.1. Consider the unit circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.



At time $t = 0$, a particle at the point $(1, 0) \in C$ starts to move at constant speed along C in the anticlockwise direction and returns for the first time to this initial position at time $t = 2\pi$. It is easy to see that at any time $t \in [0, 2\pi]$, the position of the particle may be given by $\phi(t) = (\cos t, \sin t)$. Here we are interested in the function

$$\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t). \tag{1}$$

Note that $C = \phi([0, 2\pi]) = \{\phi(t) : t \in [0, 2\pi]\}$ is the range of the function ϕ.

EXAMPLE 7.1.2. Consider a particle moving away from the origin $\mathbf{0} = (0,0,0)$ at time $t = 0$ in the direction of the unit vector $\mathbf{u} \in \mathbb{R}^3$ with constant acceleration a, and hence speed ta at any given time $t \geq 0$. In this case, the distance of the particle from the origin at time t is given by $\frac{1}{2}t^2a$, and so its position is given by $\phi(t) = \frac{1}{2}t^2a\mathbf{u}$. Suppose that we trace the movement of this particle from $t = 0$ to $t = T$. Then we are interested in the function

$$\phi : [0,T] \to \mathbb{R}^3 : t \mapsto \tfrac{1}{2}t^2a\mathbf{u}. \tag{2}$$

The range of this function is given by $\phi([0,T]) = \{\frac{1}{2}t^2a\mathbf{u} : t \in [0,T]\}$, and is a line segment joining the origin $\mathbf{0}$ and the point $\frac{1}{2}T^2a\mathbf{u}$.

Note that the functions (1) and (2) above do not just trace out curves. They also give the position of the particles at any time within the time interval.

DEFINITION. By a path in $\mathbb{R}^n$, we mean a function of the type

$$\phi : [A,B] \to \mathbb{R}^n,$$

where $A, B \in \mathbb{R}$ and $A < B$. The range

$$\phi([A,B]) = \{\phi(t) : t \in [A,B]\} \subseteq \mathbb{R}^n$$

of the function ϕ is called a curve, with initial point $\phi(A)$ and terminal point $\phi(B)$. Suppose that for every $t \in [A,B]$, we have $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, where $\phi_1(t), \ldots, \phi_n(t) \in \mathbb{R}$. Then the functions $\phi_i : [A,B] \to \mathbb{R}$, where $i = 1, \ldots, n$, are called the components of the path ϕ.

REMARKS. (1) We usually write $\phi(t) = (x(t), y(t))$ and $\phi(t) = (x(t), y(t), z(t))$ in the cases $n = 2$ and $n = 3$ respectively.

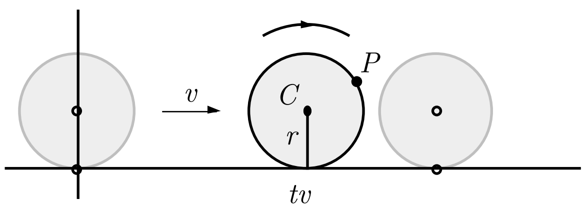
(2) Note the distinction between a path and a curve. Quite often, distinct paths may share the same curve. For example, the three distinct paths

$$\begin{aligned}
&\phi : [0,2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t) \\
&\psi : [0,1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \\
&\eta : [0,1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2)
\end{aligned}$$

satisfy $\phi([0,2\pi]) = \psi([0,1]) = \eta([0,1]) = C$, the unit circle in $\mathbb{R}^2$.

(3) Often, we refer to the path $\phi(t)$ without specifying the domain of definition of the function ϕ. This is a convenient abuse of rigour.

EXAMPLE 7.1.3. Consider a circular disc of radius r standing on a level surface. Let C denote the centre of the disc, and let P denote a fixed point on the rim of the disc. Suppose that at time $t = 0$, the point P touches the surface, and is therefore directly below the point C. For convenience, let us assume that this point where the disc touches the surface at time $t = 0$ is the origin $(0,0)$.



The disc now starts rolling to the right at constant speed v. We now wish to describe the path taken by the point P. Clearly the point C is at position $(0,r)$ at time $t = 0$. Its position at time t is given by

(tv, r). Note next that the circumference of the disc is $2\pi r$, and so the disc will complete one revolution at time $t = 2\pi r/v$. It follows that the angular speed of the disc is v/r. Now let $\psi(t)$ denote the relative position of P with respect to C. Clearly P rotates around C in a clockwise direction with angular speed v/r, so it follows that

$$\psi(t) = \left(r\cos\left(-\frac{vt}{r} + \theta\right), r\sin\left(-\frac{vt}{r} + \theta\right)\right),$$

where $\theta \in \mathbb{R}$ is a constant. Clearly $\psi(0) = (0, -r)$, so that $\cos\theta = 0$ and $\sin\theta = -1$, whence $\theta = -\pi/2$. Hence

$$\psi(t) = \left(r\cos\left(\frac{vt}{r} + \frac{\pi}{2}\right), -r\sin\left(\frac{vt}{r} + \frac{\pi}{2}\right)\right) = \left(-r\sin\frac{vt}{r}, -r\cos\frac{vt}{r}\right).$$

It follows that the actual position of P at time t is given by

$$\phi(t) = (tv, r) + \psi(t) = \left(tv - r\sin\frac{vt}{r}, r - r\cos\frac{vt}{r}\right).$$

Suppose that $v = r = 1$. Then

$$\phi(t) = (t - \sin t, 1 - \cos t).$$

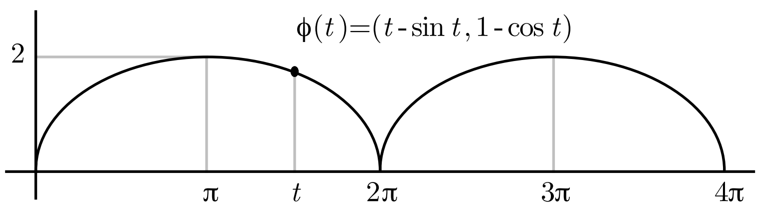
Clearly the point P touches the surface when $t = 2k\pi$, where k is a non-negative integer. The image curve of the path

$$\phi : [A, B] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$$

is called a cycloid. Note that we have not specified the range for t in our discussion. We can consider any interval $[A, B] \subseteq \mathbb{R}^2$, although to get a full picture, the interval should have length at least 2π. A picture of the path

$$\phi : [0, 4\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$$

is given below.



7.2. Differentiable Paths

Definition. We say that a path $\phi : [A, B] \to \mathbb{R}^n$ is differentiable if the limit

$$\lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h}$$

exists for every $t \in [A, B]$, with the obvious restriction to one-sided limits at the endpoints of the interval $[A, B]$. In this case, the vector

$$\phi'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = \lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h}$$

is called the velocity vector of the path ϕ, and the quantity $\|\phi'(t)\|$ is called the speed of the path ϕ.

Remarks. (1) Note that we have borrowed some terminology from physics. This is entirely natural, as this area of mathematics is, to a large extent, motivated by the study of various problems in physics.

(2) Note that if the path is given by $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, then the velocity vector is given by $\phi'(t) = (\phi_1'(t), \ldots, \phi_n'(t))$ and the speed is given by $\|\phi(t)\| = (|\phi_1'(t)|^2 + \ldots + |\phi_n'(t)|^2)^{1/2}$.

(3) Note the special notation in the cases $n = 2$ and $n = 3$.

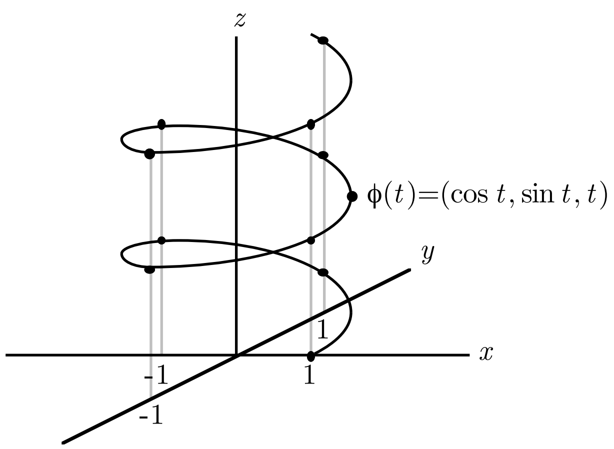
(4) The velocity vector $\phi'(t)$ is a vector tangent to the path $\phi(t)$ at time t. If C is the curve of the path $\phi(t)$ and $\phi'(t) \neq \mathbf{0}$, then $\phi'(t)$ is a vector tangent to the curve C at the point $\phi(t) \in C$.

Example 7.2.1. For the cycloid $\phi(t) = (t - \sin t, 1 - \cos t)$ described in Example 7.1.3, the velocity vector is given by $\phi'(t) = (1 - \cos t, \sin t)$. Note that $1 - \cos t = 0$ implies that $\sin t = 0$, so the velocity is never vertical. The speed of the path is

$$\|\phi'(t)\| = ((1 - \cos t)^2 + \sin^2 t)^{1/2} = (2 - 2\cos t)^{1/2}.$$

This is minimum and zero when $\cos t = 1$, when the point P touches the surface. The speed is maximum when $\cos t = -1$, when the point P is at the maximum height.

Example 7.2.2. To study the path $\phi(t) = (\cos t, \sin t, t)$ in $\mathbb{R}^3$, we first of all consider the first two components, and study the path $\psi(t) = (\cos t, \sin t)$ in $\mathbb{R}^2$. This path describes a circle on the plane, followed in the anticlockwise direction. The third component t describes an increase in height with time if we think of the third component as the vertical component. It follows that if we consider the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$, then the path $\phi(t)$ wraps round this cylinder in an anticlockwise direction with the third component increasing if we look from above. The curve of the path $\phi(t)$ is called a helix.



The path has velocity vector $\phi'(t) = (-\sin t, \cos t, 1)$ and speed $\|\phi'(t)\| = (\sin^2 t + \cos^2 t + 1)^{1/2} = \sqrt{2}$, so the path has constant speed.

Suppose that $\phi(t)$ is a differentiable path. We have already indicated that if $\phi'(t_0) \neq \mathbf{0}$, then it is a vector tangent to the path at the point $\phi(t_0)$. It follows immediately that

THEOREM 7A. *Suppose that $\phi(t)$ is a differentiable path in $\mathbb{R}^n$. Then the tangent line to the path at the point $\phi(t_0)$ is given by*

$$L(\lambda) = \phi(t_0) + \lambda \phi'(t_0),$$

provided that $\phi'(t_0) \neq \mathbf{0}$.

Example 7.2.3. The equation of the tangent line to the helix $\phi(t) = (\cos t, \sin t, t)$ at the point $\phi(t_0)$ is given by

$$L(\lambda) = \phi(t_0) + \lambda\phi'(t_0) = (\cos t_0, \sin t_0, t_0) + \lambda(-\sin t_0, \cos t_0, 1).$$

Suppose that $t_0 = 2\pi$. Then $\phi(2\pi) = (1, 0, 2\pi)$, and the tangent line becomes $L(\lambda) = (1, 0, 2\pi)+\lambda(0, 1, 1)$. Writing $L(\lambda) = (x, y, z)$, we have $x = 1$, $y = \lambda$ and $z = 2\pi + \lambda$. It follows that the tangent line to the helix at the point $(1, 0, 2\pi)$ is given by $x = 1$ and $z = y + 2\pi$. Try to visualize this from the picture in Example 7.2.2.

Example 7.2.4. The equation of the tangent line to the cycloid $\phi(t) = (t - \sin t, 1 - \cos t)$ at the point $\phi(t_0)$ is given by

$$L(\lambda) = \phi(t_0) + \lambda\phi'(t_0) = (t_0 - \sin t_0, 1 - \cos t_0) + \lambda(1 - \cos t_0, \sin t_0).$$

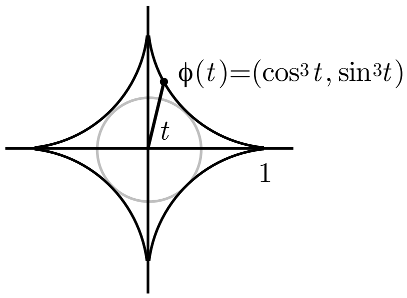
Suppose that $t_0 = 2\pi$. Then $\phi(2\pi) = (2\pi, 0)$, and $L(\lambda) = (2\pi, 0) + \lambda(0, 0) = (2\pi, 0)$, clearly not the equation of a line. Observe that since $\phi'(2\pi) = (0, 0)$, Theorem 7A does not apply in this case. In fact, the tangent line is vertical.

Example 7.2.5. Let us return to the helix discussed in Examples 7.2.2 and 7.2.3. Suppose that a particle follows the helix from $t = 0$ to $t = 2\pi$ and then flies off at constant velocity on a tangent at $t = 2\pi$. We wish to determine the position of the particle at $t = 4\pi$. Note that the particle is at position $\phi(2\pi) = (1, 0, 2\pi)$ when $t = 2\pi$, with tangential velocity $\phi'(2\pi) = (0, 1, 1)$. It follows that its position at $t = 4\pi$ must be given by

$$\phi(2\pi) + (4\pi - 2\pi)\phi'(2\pi) = (1, 0, 2\pi) + 2\pi(0, 1, 1) = (1, 2\pi, 4\pi).$$

Example 7.2.6. Consider the hypocycloid of four cusps

$$\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t).$$



This path has velocity vector

$$\phi'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t)$$

and speed

$$\|\phi'(t)\| = (9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t)^{1/2} = 3|\cos t \sin t|.$$

Note that while the hypocycloid is a differentiable path, its curve has cusps. Note, however, that the velocity and speed are zero at these cusps.

We state without proof the following two theorems. The proofs are not difficult, and follow by applying the usual differentiation rules to the components.

THEOREM 7B. *Suppose that $\phi(t)$ and $\psi(t)$ are differentiable paths in $\mathbb{R}^n$. Suppose further that $a(t)$ and $b(t)$ are differentiable real valued functions. Then*

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\phi(t)+\psi(t)) = \phi'(t)+\psi'(t)$;

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(a(t)\phi(t)) = a(t)\phi'(t)+a'(t)\phi(t)$;

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\phi(t)\cdot\psi(t)) = \phi(t)\cdot\psi'(t)+\phi'(t)\cdot\psi(t)$; *and*

(d) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\phi(a(t))) = a'(t)\phi'(a(t))$.

The above represent the sum rule, scalar multiplication rule, dot product rule and chain rule respectively. Note also the vector product rule below which is valid only in $\mathbb{R}^3$.

THEOREM 7C. *Suppose that $\phi(t)$ and $\psi(t)$ are differentiable paths in $\mathbb{R}^3$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi(t)\times\psi(t)) = \phi(t)\times\psi'(t)+\phi'(t)\times\psi(t).$$

7.3. Arc Length

In this section, we are interested in calculating the length of the curve followed by a path. To motivate this, note that the speed $\|\phi'(t)\|$ of a path $\phi(t)$ is the rate of change of distance with respect to time.

DEFINITION. Suppose that $\phi : [A,B] \to \mathbb{R}^n$ is a differentiable path. The velocity differential is given by

$$\mathrm{d}\mathbf{s} = \phi'(t)\,\mathrm{d}t = (\phi_1'(t),\ldots,\phi_n'(t))\,\mathrm{d}t.$$

The corresponding arc length differential is given by

$$\mathrm{d}s = \|\phi'(t)\|\,\mathrm{d}t = (|\phi_1'(t)|^2+\ldots+|\phi_n'(t)|^2)^{1/2}\,\mathrm{d}t.$$

REMARKS. (1) The velocity differential describes an infinitesimal displacement of a particle following the path ϕ. The arc length differential describes the magnitude of this infinitesimal displacement.

(2) In $\mathbb{R}^2$ and $\mathbb{R}^3$, we have velocity differential

$$\mathrm{d}\mathbf{s} = \left(\frac{\mathrm{d}x}{\mathrm{d}t},\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathrm{d}t \qquad\text{and}\qquad \mathrm{d}\mathbf{s} = \left(\frac{\mathrm{d}x}{\mathrm{d}t},\frac{\mathrm{d}y}{\mathrm{d}t},\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t$$

and arc length differential

$$\mathrm{d}s = \left(\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|^2+\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|^2\right)^{1/2}\mathrm{d}t \qquad\text{and}\qquad \mathrm{d}s = \left(\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|^2+\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|^2+\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right|^2\right)^{1/2}\mathrm{d}t$$

respectively.

DEFINITION. Suppose that $\phi : [A,B] \to \mathbb{R}^n$ is a continuously differentiable path. Then the quantity

$$\ell = \int_A^B \|\phi'(t)\|\,\mathrm{d}t$$

is called the arc length of the path ϕ.

REMARK. Note that if $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, then

$$\ell = \int_A^B (|\phi_1'(t)|^2 + \ldots + |\phi_n'(t)|^2)^{1/2} \mathrm{d}t.$$

EXAMPLE 7.3.1. The cycloid $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$ has arc length

$$\ell = \int_0^{2\pi} \|\phi'(t)\| \,\mathrm{d}t = \int_0^{2\pi} (2 - 2\cos t)^{1/2} \,\mathrm{d}t = 8.$$

EXAMPLE 7.3.2. The helix $\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$ has arc length

$$\ell = \int_0^{2\pi} \|\phi'(t)\| \,\mathrm{d}t = \int_0^{2\pi} \sqrt{2} \,\mathrm{d}t = 2\pi\sqrt{2}.$$

EXAMPLE 7.3.3. The hypocycloid of four cusps $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t)$ has arc length

$$\ell = \int_0^{2\pi} \|\phi'(t)\| \,\mathrm{d}t = \int_0^{2\pi} 3|\cos t \sin t| \,\mathrm{d}t = 12 \int_0^{\pi/2} \sin t \cos t \,\mathrm{d}t = 6.$$

PROBLEMS FOR CHAPTER 7

1. Sketch the curve of each of the following paths:
   a) $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\sin t, 3\cos t)$
   b) $\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (2t + 1, t + 2, t)$

2. For each of the following paths, find the equation of the tangent line at the point $\phi(t_0)$:
   a) $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\mathrm{e}^t, \cos t)$
   b) $\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (t^3, t^2, t)$

3. A particle follows the path $\phi(t) = (\sin \mathrm{e}^t, t, 4 - t^3)$ in $\mathbb{R}^3$ from time $t = 0$ to time $t = 1$, and then flies off at a tangent at constant velocity. Determine its position at time $t = 3$.

4. Consider the path $\phi : [0, 1] \to \mathbb{R}^2 : t \mapsto (t\cos t, t\sin t)$.
   a) Determine the velocity vector $\phi'(\pi/6)$.
   b) Find the equation of the tangent line to the path at $\phi(\pi/6)$, if it exists.
   c) Determine the speed $\|\phi'(t)\|$ for every $t \in [0, 1]$.
   d) Determine the arc length of the path ϕ.
   [HINT: You may need a substitution and integration by parts.]

5. For each of the following paths, determine its arc length:
   a) $\phi : [0, \pi] \to \mathbb{R}^3 : t \mapsto (t, t\cos t, t\sin t)$
   b) $\phi : [0, 4\pi] \to \mathbb{R}^3 : t \mapsto \begin{cases} (2\cos t, t, 2\sin t) & \text{if } t \in [0, 2\pi] \\ (2, t, t - 2\pi) & \text{if } t \in [2\pi, 4\pi] \end{cases}$

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 8

VECTOR FIELDS

8.1. Introduction

In this chapter, we consider functions of the form

(1) $$F : A \to \mathbb{R}^n : \mathbf{x} \mapsto F(\mathbf{x}),$$

where the domain $A \subseteq \mathbb{R}^n$ is a set in the n-dimensional euclidean space, and where the codomain is also the n-dimensional euclidean space $\mathbb{R}^n$. For each $\mathbf{x} \in A$, we can write

$$\mathbf{x} = (x_1, \ldots, x_n),$$

where $x_1, \ldots, x_n \in \mathbb{R}$. We can also write

(2) $$F(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_n(\mathbf{x})),$$

where $F_1(\mathbf{x}), \ldots, F_n(\mathbf{x}) \in \mathbb{R}$.

DEFINITION. A function of the type (1), where $A \subseteq \mathbb{R}^n$, is called a vector field in $\mathbb{R}^n$. The functions $F_i : A \to \mathbb{R}$, defined for $i = 1, \ldots, n$ by (2), are called the component scalar fields of F.

REMARK. In the special cases $n = 2$ and $n = 3$, we usually write $\mathbf{x} = (x, y)$ and $\mathbf{x} = (x, y, z)$, so that

$$F(x, y) = (F_1(x, y), F_2(x, y)) \qquad \text{and} \qquad F(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$$

respectively.

EXAMPLE 8.1.1. Suppose that the real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Define the function $F : \mathbb{R}^n \to \mathbb{R}^n$ by writing

$$F(\mathbf{x}) = (\nabla f)(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right)$$

for every $\mathbf{x} \in \mathbb{R}^n$. Recall that this is the gradient of f studied in Chapter 2. This vector field F is sometimes called a gradient vector field.

EXAMPLE 8.1.2. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$. There is no continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $F = \nabla f$. To see this, note that if there were, then

$$F(x, y) = (\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right),$$

so that

$$\frac{\partial f}{\partial x} = -y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x.$$

It would then follow that

$$\frac{\partial^2 f}{\partial y \partial x} = -1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y} = 1$$

which is impossible. This vector field F is an example of a non-gradient vector field.

EXAMPLE 8.1.3. Newton's law of gravitation states that the force acting on a point mass m at position $\mathbf{x} \in \mathbb{R}^3$ due to a point mass M at the origin $\mathbf{0}$ is given by

$$F(\mathbf{x}) = -\frac{\epsilon M m}{\|\mathbf{x}\|^3} \mathbf{x},$$

where $\epsilon > 0$ is a proportionality constant. This is an attractive force field. Note that $F = \nabla f$, where

$$f : \mathbb{R}^3 \to \mathbb{R} : \mathbf{x} \mapsto \frac{\epsilon M m}{\|\mathbf{x}\|},$$

so F is a gradient vector field.

EXAMPLE 8.1.4. Coulomb's law in electrostatics states that the force acting on a point charge q at position $\mathbf{x} \in \mathbb{R}^3$ due to a point charge Q at the origin $\mathbf{0}$ is given by

$$F(\mathbf{x}) = \frac{\epsilon Q q}{\|\mathbf{x}\|^3} \mathbf{x},$$

where $\epsilon > 0$ is a proportionality constant. This is a repulsive force field. Note that $F = \nabla f$, where

$$f : \mathbb{R}^3 \to \mathbb{R} : \mathbf{x} \mapsto -\frac{\epsilon Q q}{\|\mathbf{x}\|},$$

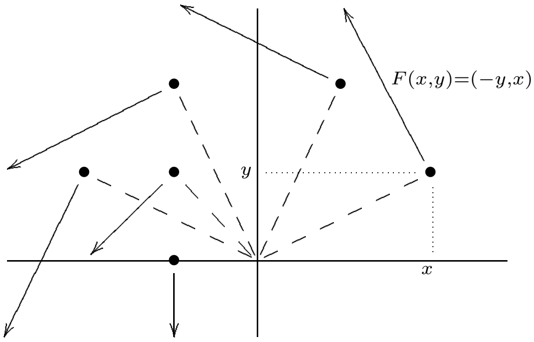
so F is a gradient vector field.

REMARK. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field. By a flow line of F, we mean a path $\phi(t)$ in $\mathbb{R}^n$ such that $\phi'(t) = F(\phi(t))$; in other words, F yields the velocity vector of the path $\phi(t)$. Flow lines are useful in understanding some of the properties of vector fields, as we shall see in the following examples.

EXAMPLE 8.1.5. For the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$, the path $\phi(t) = (\cos t, \sin t)$ is a flow line, for clearly

$$\phi'(t) = (-\sin t, \cos t) \qquad \text{and} \qquad F(\phi(t)) = F(\cos t, \sin t) = (-\sin t, \cos t).$$

Similarly, it can be shown that for any real number $c \in \mathbb{R}$, the path $\phi(t) = (c\cos t, c\sin t)$ is a flow line of F. Here the flow is circular, anticlockwise about the origin.



Example 8.1.6. Let us return to Example 8.1.4, and consider again Coulomb's law of electrostatics, where

$$F(\mathbf{x}) = \frac{\epsilon Qq}{\|\mathbf{x}\|^3}\mathbf{x}.$$

Let $\mathbf{u} \in \mathbb{R}^3$ be a fixed unit vector. The path $\phi(t) = (3\epsilon Qqt)^{1/3}\mathbf{u}$ is a flow line of F, for clearly

$$\phi'(t) = \frac{\epsilon Qq}{(3\epsilon Qqt)^{2/3}}\mathbf{u} \qquad \text{and} \qquad F(\phi(t)) = \frac{\epsilon Qq}{(3\epsilon Qqt)}(3\epsilon Qqt)^{1/3}\mathbf{u}.$$

This shows that the flow lines are radial and away from the origin.

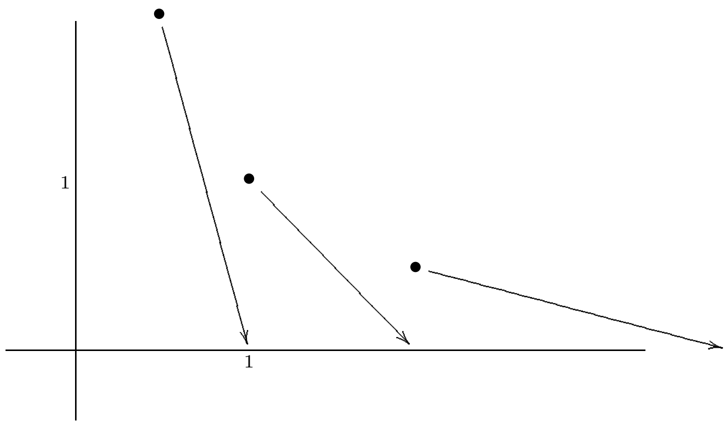
Example 8.1.7. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (x,-y)$. For a path $\phi(t) = (\phi_1(t), \phi_2(t))$ to be a flow line of F, we must have $\phi'(t) = F(\phi(t))$, so that

$$(\phi_1'(t), \phi_2'(t)) = F(\phi_1(t), \phi_2(t)) = (\phi_1(t), -\phi_2(t)),$$

whence $\phi_1'(t) = \phi_1(t)$ and $\phi_2'(t) = -\phi_2(t)$; in other words, we need

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = \phi_1 \qquad \text{and} \qquad \frac{\mathrm{d}\phi_2}{\mathrm{d}t} = -\phi_2.$$

These two differential equations have solutions $\phi_1(t) = C_1\mathrm{e}^t$ and $\phi_2(t) = C_2\mathrm{e}^{-t}$, where $C_1, C_2 \in \mathbb{R}$ are constants. It follows that the flow lines of F are of the form $\phi(t) = (C_1\mathrm{e}^t, C_2\mathrm{e}^{-t})$, where $C_1, C_2 \in \mathbb{R}$ are constants. Note that the curve of the path $\phi(t) = (C_1\mathrm{e}^t, C_2\mathrm{e}^{-t})$ is given by the hyperbola $xy = C_1C_2$. The picture below shows $F(x,y)$ at some points along the same flow line $xy = 1$.



For the sake of convenience, we make the following definition.

DEFINITION. The ∇ operator in $\mathbb{R}^n$ is given by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

REMARKS. (1) In the special cases $n = 2$ and $n = 3$, we have respectively

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

(2) Note that for any real valued function $f(x_1, \ldots, x_n)$, the gradient of f is equal to

$$\nabla f = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

8.2. Divergence of a Vector Field

Suppose that F is the vector field of a gas or fluid. Then we may wish to discuss the rate of expansion of volume under flow. This is a scalar valued function of a vector field.

DEFINITION. Suppose that $F = (F_1, \ldots, F_n)$ is a vector field in $\mathbb{R}^n$. Then the divergence of F is the scalar field

$$\operatorname{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, \ldots, F_n) = \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}.$$

EXAMPLE 8.2.1. For the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (yz, xz, xy)$, we have

$$\operatorname{div} F = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0.$$

EXAMPLE 8.2.2. For the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, 0)$, we have $\operatorname{div} F = 1$. Consider next the flow lines of this vector field. Any flow line must be a path $\phi(t) = (\phi_1(t), \phi_2(t))$ satisfying $\phi'(t) = F(\phi(t))$, so that

$$(\phi_1'(t), \phi_2'(t)) = F(\phi_1(t), \phi_2(t)) = (\phi_1(t), 0).$$

It follows that $\phi_1(t) = C_1 \mathrm{e}^t$ and $\phi_2(t) = C_2$, where $C_1, C_2 \in \mathbb{R}$ are constants. The flow is therefore in the x-direction. If we think of F as a velocity field, then the speed is greater as we move further away from the line $x = 0$. This corresponds to an expansion which is consistent with $\operatorname{div} F > 0$.

EXAMPLE 8.2.3. For the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-x, -y)$, we have $\operatorname{div} F = -2$. Consider next the flow lines of this vector field. Any flow line must be a path $\phi(t) = (\phi_1(t), \phi_2(t))$ satisfying $\phi'(t) = F(\phi(t))$, so that

$$(\phi_1'(t), \phi_2'(t)) = F(\phi_1(t), \phi_2(t)) = (-\phi_1(t), -\phi_2(t)).$$

It follows that $\phi_1(t) = C_1 \mathrm{e}^{-t}$ and $\phi_2(t) = C_2 \mathrm{e}^{-t}$, where $C_1, C_2 \in \mathbb{R}$ are constants. The flow is therefore radial and towards the origin. This corresponds to a contraction which is consistent with $\operatorname{div} F < 0$.

EXAMPLE 8.2.4. For the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$, we have shown in Example 8.1.5 that the paths of the type $\phi(t) = (c \cos t, c \sin t)$, where $c \in \mathbb{R}$, are flow lines of F. It can actually be shown that these are all the flow lines of F. It follows that the flow is circular and anticlockwise around the origin, with no expansion or contraction. Note now that $\operatorname{div} F = 0$.

8.3. Curl of a Vector Field

While the divergence of a vector field is related to expansion or contraction, so the curl of a vector field is related to rotation – there are beaches in Sydney named after this operator! Indeed, a vector field with zero curl will be called irrotational.

DEFINITION. Suppose that $F = (F_1, F_2, F_3)$ is a vector field in $\mathbb{R}^3$. Then the curl of F is the vector field

$$\operatorname{curl} F = \nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F_1, F_2, F_3) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

REMARKS. (1) We can write

$$\operatorname{curl} F = \nabla \times F = \det \begin{pmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathrm{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathrm{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{k},$$

where $\mathrm{i} = (1, 0, 0)$, $\mathrm{j} = (0, 1, 0)$ and $\mathrm{k} = (0, 0, 1)$.

(2) Unlike gradient and divergence which are valid in euclidean space $\mathbb{R}^n$ for any natural number $n \in \mathbb{N}$, curl is only defined in $\mathbb{R}^3$.

(3) Suppose that F is a vector field in $\mathbb{R}^2$. While we cannot define $\operatorname{curl} F$, we can nevertheless regard F as a vector field in $\mathbb{R}^3$ for which the third component is zero and the two other components are independent of z. Then

$$\nabla \times F = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

The function

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

is sometimes called the scalar curl of F.

EXAMPLE 8.3.1. For the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (yz, xz, xy)$, we have

$$\operatorname{curl} F = \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz), \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz) \right) = \mathbf{0}.$$

Here, note that if we consider the function $f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto xyz$, then $F = \nabla f$. We shall show later in Theorem 8G that $\nabla \times (\nabla f) = \mathbf{0}$ for any twice continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$.

EXAMPLE 8.3.2. For the vector field $F:\mathbb{R}^3\to\mathbb{R}^3:(x,y,z)\mapsto(x^2,(x+y)^2,(x+y+z)^2)$, we have

$$\begin{aligned}\operatorname{curl} F &= \left(\frac{\partial}{\partial y}(x+y+z)^2-\frac{\partial}{\partial z}(x+y)^2,\frac{\partial}{\partial z}x^2-\frac{\partial}{\partial x}(x+y+z)^2,\frac{\partial}{\partial x}(x+y)^2-\frac{\partial}{\partial y}x^2\right)\\ &= (2(x+y+z),-2(x+y+z),2(x+y)).\end{aligned}$$

Hence

$$\operatorname{div}(\operatorname{curl} F)=\frac{\partial}{\partial x}(2(x+y+z))+\frac{\partial}{\partial y}(-2(x+y+z))+\frac{\partial}{\partial z}(2(x+y))=0.$$

We shall show later in Theorem 8F that $\nabla\cdot(\nabla\times F)=0$ for any twice continuously differentiable function $F:\mathbb{R}^3\to\mathbb{R}^3$.

EXAMPLE 8.3.3. Consider again the vector field $F:\mathbb{R}^2\to\mathbb{R}^2:(x,y)\mapsto(-y,x)$. We have shown in Examples 8.1.5 and 8.2.4 that the flow is circular and anticlockwise around the origin. Note now that the scalar curl of F is equal to

$$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=2.$$

8.4. Basic Identities of Vector Analysis

The first three theorems do not involve curl and are therefore valid in $\mathbb{R}^n$ for any natural number $n\in\mathbb{N}$. The first two of these theorems are easy to prove.

THEOREM 8A. *For any continuously differentiable functions $f:A\to\mathbb{R}$ and $g:A\to\mathbb{R}$, where $A\subseteq\mathbb{R}^n$, and for any fixed real number $c\in\mathbb{R}$, we have*
(a) $\nabla(f+g)=\nabla f+\nabla g$;
(b) $\nabla(cf)=c\nabla f$;
(c) $\nabla(fg)=f\nabla g+g\nabla f$; *and*
(d) $\nabla(f/g)=(g\nabla f-f\nabla g)/g^2$ *at any point* $\mathbf{x}\in A$ *for which* $g(\mathbf{x})\neq 0$.

THEOREM 8B. *For any continuously differentiable functions $F:A\to\mathbb{R}^n$ and $G:A\to\mathbb{R}^n$, where $A\subseteq\mathbb{R}^n$, and for any fixed real number $c\in\mathbb{R}$, we have*
(a) $\operatorname{div}(F+G)=\operatorname{div} F+\operatorname{div} G$; *and*
(b) $\operatorname{div}(cF)=c\operatorname{div} F$.

THEOREM 8C. *For any continuously differentiable functions $F:A\to\mathbb{R}^n$ and $f:A\to\mathbb{R}$, where $A\subseteq\mathbb{R}^n$, we have* $\operatorname{div}(fF)=f\operatorname{div} F+F\cdot\nabla f$.

PROOF. Let $F=(F_1,\ldots,F_n)$. Then

$$\begin{aligned}\operatorname{div}(fF)=\operatorname{div}(fF_1,\ldots,fF_n)&=\frac{\partial}{\partial x_1}(fF_1)+\ldots+\frac{\partial}{\partial x_n}(fF_n)\\ &=\left(f\frac{\partial F_1}{\partial x_1}+F_1\frac{\partial f}{\partial x_1}\right)+\ldots+\left(f\frac{\partial F_n}{\partial x_n}+F_n\frac{\partial f}{\partial x_n}\right)\\ &=\left(f\frac{\partial F_1}{\partial x_1}+\ldots+f\frac{\partial F_n}{\partial x_n}\right)+\left(F_1\frac{\partial f}{\partial x_1}+\ldots+F_n\frac{\partial f}{\partial x_n}\right)\\ &=f\left(\frac{\partial F_1}{\partial x_1}+\ldots+\frac{\partial F_n}{\partial x_n}\right)+(F_1,\ldots,F_n)\cdot\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)\\ &=f\operatorname{div} F+F\cdot\nabla f.\end{aligned}$$

This completes the proof. ○

We also have the following four theorems which involve curl and are therefore restricted to $\mathbb{R}^3$.

THEOREM 8D. *For any continuously differentiable functions $F : A \to \mathbb{R}^3$ and $G : A \to \mathbb{R}^3$, where $A \subseteq \mathbb{R}^3$, and for any fixed real number $c \in \mathbb{R}$, we have*
(a) $\operatorname{curl}(F + G) = \operatorname{curl} F + \operatorname{curl} G$;
(b) $\operatorname{curl}(cF) = c \operatorname{curl} F$; *and*
(c) $\operatorname{div}(F \times G) = G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G$.

PROOF. Parts (a) and (b) are easy to check. To prove (c), let $F = (F_1, F_2, F_3)$ and $G = (G_1, G_2, G_3)$. Then

$$\begin{aligned}\operatorname{div}(F \times G) &= \operatorname{div}(F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1)\\ &= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1).\end{aligned}$$

Using the sum and product rules to the terms on the right and rearranging, we obtain

$$\begin{aligned}\operatorname{div}(F \times G) &= G_1\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + G_2\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + G_3\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &\quad - F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\\ &= (G_1, G_2, G_3) \cdot \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &\quad - (F_1, F_2, F_3) \cdot \left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\\ &= G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G.\end{aligned}$$

This completes the proof. ◯

THEOREM 8E. *For any continuously differentiable functions $F : A \to \mathbb{R}^3$ and $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$, we have* $\operatorname{curl}(fF) = f \operatorname{curl} F - F \times \nabla f$.

PROOF. Let $F = (F_1, F_2, F_3)$. Then

$$\operatorname{curl}(fF) = \operatorname{curl}(fF_1, fF_2, fF_3) = \left(\frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2), \frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3), \frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1)\right).$$

Using the sum and product rules to the terms on the right and rearranging, we obtain

$$\begin{aligned}\operatorname{curl}(fF) &= f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &\quad + \left(F_3\frac{\partial f}{\partial y} - F_2\frac{\partial f}{\partial z}, F_1\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial x}, F_2\frac{\partial f}{\partial x} - F_1\frac{\partial f}{\partial y}\right)\\ &= f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) + \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \times (F_1, F_2, F_3)\\ &= f \operatorname{curl} F - F \times \nabla f.\end{aligned}$$

This completes the proof. ◯

THEOREM 8F. *For any twice continuously differentiable function* $F : A \to \mathbb{R}^3$*, where* $A \subseteq \mathbb{R}^3$*, we have* $\operatorname{div}(\operatorname{curl} F) = 0$.

PROOF. Let $F = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\operatorname{div}(\operatorname{curl} F) &= \operatorname{div}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\ &= \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}\right) = 0,\end{aligned}$$

in view of Theorem 4A. ○

THEOREM 8G. *For any twice continuously differentiable function* $f : A \to \mathbb{R}$*, where* $A \subseteq \mathbb{R}^3$*, we have* $\operatorname{curl}(\nabla f) = \mathbf{0}$.

PROOF. We have

$$\begin{aligned}\operatorname{curl}(\nabla f) &= \operatorname{curl}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = \mathbf{0},\end{aligned}$$

in view of Theorem 4A. ○

EXAMPLE 8.4.1. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x, y, z)$. It is easily checked that $\operatorname{div} F = 3$. It follows that there is no function $G : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F = \operatorname{curl} G$, for otherwise $\operatorname{div} F = 0$ by Theorem 8F.

EXAMPLE 8.4.2. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y, -x, 0)$. It is easily checked that $\operatorname{curl} F = (0, 0, -2)$. It follows that there is no function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$, for otherwise $\operatorname{curl} F = \mathbf{0}$ by Theorem 8G.

For any twice continuously differentiable function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}.$$

DEFINITION. The Laplace operator ∇^2 in $\mathbb{R}^n$ is defined as the divergence of the gradient, so that

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

EXAMPLE 8.4.3. A function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, is said to satisfy Laplace's equation if $\nabla^2 f = 0$. An example of such a function is given in the case $n = 3$ by

$$f(\mathbf{x}) = f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{1/2}} = \frac{1}{\|\mathbf{x}\|}.$$

THEOREM 8H. *For any twice continuously differentiable functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have*
(a) $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$; *and*
(b) $\mathrm{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

PROOF. Note that

$$\begin{aligned}
\nabla^2(fg) &= \frac{\partial^2(fg)}{\partial x_1^2} + \ldots + \frac{\partial^2(fg)}{\partial x_n^2} \\
&= \left(f\frac{\partial^2 g}{\partial x_1^2} + 2\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + g\frac{\partial^2 f}{\partial x_1^2}\right) + \ldots + \left(f\frac{\partial^2 g}{\partial x_n^2} + 2\frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_n} + g\frac{\partial^2 f}{\partial x_n^2}\right) \\
&= f\left(\frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}\right) + g\left(\frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}\right) + 2\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_n}\right) \\
&= f\left(\frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}\right) + g\left(\frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}\right) + 2\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)\cdot\left(\frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n}\right) \\
&= f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g).
\end{aligned}$$

This gives (a). On the other hand,

$$\begin{aligned}
\mathrm{div}(f\nabla g - g\nabla f) &= \mathrm{div}\left(f\left(\frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n}\right) - g\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)\right) \\
&= \mathrm{div}\left(f\frac{\partial g}{\partial x_1} - g\frac{\partial f}{\partial x_1},\ldots,f\frac{\partial g}{\partial x_n} - g\frac{\partial f}{\partial x_n}\right) \\
&= \frac{\partial}{\partial x_1}\left(f\frac{\partial g}{\partial x_1} - g\frac{\partial f}{\partial x_1}\right) + \ldots + \frac{\partial}{\partial x_n}\left(f\frac{\partial g}{\partial x_n} - g\frac{\partial f}{\partial x_n}\right) \\
&= \left(f\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_1} - g\frac{\partial^2 f}{\partial x_1^2}\right) + \ldots + \left(f\frac{\partial^2 g}{\partial x_n^2} + \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_n} - \frac{\partial g}{\partial x_n}\frac{\partial f}{\partial x_n} - g\frac{\partial^2 f}{\partial x_n^2}\right) \\
&= f\left(\frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}\right) - g\left(\frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}\right) \\
&= f\nabla^2 g - g\nabla^2 f.
\end{aligned}$$

This gives (b). ○

Finally, we have the following result. The proof is left as an exercise to test the reader's grasp of the techniques.

THEOREM 8J. *For any twice continuously differentiable functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$, we have* $\mathrm{div}(\nabla f \times \nabla g) = 0$.

PROBLEMS FOR CHAPTER 8

1. Consider the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (4x, 2y)$.
   a) Is this a gradient vector field? Justify your assertion.
   b) Suppose that $\phi(t) = (\phi_1(t), \phi_2(t))$ is a flow line of the vector field F. Show that ϕ satisfies the two differential equations

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = 4\phi_1 \qquad \text{and} \qquad \frac{\mathrm{d}\phi_2}{\mathrm{d}t} = 2\phi_2.$$

   c) By solving the differential equations in part (b), show that the flow lines are parabolas.
   d) Draw a picture to support your observation in part (c).

2. Consider the functions $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (2xz, xyz, y^3xz)$ and $f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x^2yz$. Compute each of the following:
   a) ∇f
   b) $\operatorname{div} F$
   c) $\operatorname{curl} F$
   d) $F \times \nabla f$
   e) $F \cdot \nabla f$
   f) $\nabla(F \cdot \nabla f)$
   g) $\operatorname{div}(\operatorname{curl} F)$

3. Discuss the question of whether $\operatorname{curl} F$ has to be perpendicular to F.

4. Consider the vector field $F(x, y) = \left(\dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right)$.
   a) Let $\mathbf{u} \in \mathbb{R}^2$ be a unit vector. Show that $\phi(t) = t\mathbf{u}$ is a flow line of F.
   b) Show that $\operatorname{div} F > 0$ whenever $(x, y) \neq (0, 0)$.
   c) Show that the scalar curl of F is zero.
   d) Draw a picture of the vector field F and briefly explain the observations in parts (b) and (c).

5. Consider the vector field $F(x, y, z) = (2x, z, -z^2)$.
   a) Let $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ be a flow line of F. Show that $\phi_1(t), \phi_2(t), \phi_3(t)$ satisfy three ordinary differential equations.
   b) Solve two of the differential equations in part (a) for $\phi_1(t)$ and $\phi_3(t)$. Then solve the remaining differential equation for $\phi_2(t)$.
   c) Check that $\phi(t) = (\mathrm{e}^{2t}, \log t, 1/t)$, where $t > 0$, is one of the solutions.
   d) Find $\operatorname{div} F$ and $\operatorname{curl} F$.

6. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (3x^2y, x^3 + y^3, 0)$.
   a) Verify that $\operatorname{curl} F = \mathbf{0}$.
   b) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 9

INTEGRALS OVER PATHS

9.1. Integrals of Scalar Functions over Paths

Suppose that the path

$$\phi : [A,B] \to \mathbb{R}^n : t \mapsto \phi(t) = (x_1(t), \ldots, x_n(t))$$

is continuously differentiable. For any real valued function $f(x_1, \ldots, x_n)$ such that the composition function

$$f \circ \phi : [A,B] \to \mathbb{R} : t \mapsto f(x_1(t), \ldots, x_n(t))$$

is continuous, we define

$$\int_\phi f \,\mathrm{d}s = \int_\phi f(x_1, \ldots, x_n) \,\mathrm{d}s = \int_A^B f(\phi(t)) \|\phi'(t)\| \,\mathrm{d}t.$$

REMARKS. (1) We are mainly interested in the special cases $n = 2$ and $n = 3$, and write respectively

$$\int_\phi f \,\mathrm{d}s = \int_\phi f(x,y) \,\mathrm{d}s \qquad \text{and} \qquad \int_\phi f \,\mathrm{d}s = \int_\phi f(x,y,z) \,\mathrm{d}s.$$

(2) Suppose that $f = 1$ identically. Then the integral simply represents the arc length of ϕ.

(3) Note that f has only to be defined on the image curve $C = \phi([A,B])$ of the path ϕ for our definition to make sense. The continuity of the composition function $f \circ \phi$ on the closed interval $[A,B]$ ensures the existence of the integral.

(4) Sometimes, ϕ may only be piecewise continuously differentiable; in other words, there exists a dissection $A = t_0 < t_1 < \ldots < t_k = B$ of the interval $[A, B]$ such that ϕ is continuously differentiable in $[t_{i-1}, t_i]$ for each $i = 1, \ldots, k$. In this case, we define

$$\int_\phi f \, ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\phi(t)) \|\phi'(t)\| \, dt.$$

In other words, we calculate the corresponding integral for each subinterval and consider the sum of the integrals.

(5) For the special case $n = 2$, we must not confuse the integral with integrals of the type

$$\int_\phi f(z) \, dz$$

which arise frequently in complex analysis.

EXAMPLE 9.1.1. Suppose that $\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$ and $f(x, y, z) = x + y + z$. Then

$$\int_\phi f \, ds = \int_0^{2\pi} f(\cos t, \sin t, t) \|(-\sin t, \cos t, 1)\| \, dt = \int_0^{2\pi} (\cos t + \sin t + t)\sqrt{2} \, dt = 2\pi^2\sqrt{2}.$$

EXAMPLE 9.1.2. Suppose that $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$ and $f(x, y) = \sqrt{2y}$. Then

$$\int_\phi f \, ds = \int_0^{2\pi} f(t - \sin t, 1 - \cos t) \|(1 - \cos t, \sin t)\| \, dt = \int_0^{2\pi} (2 - 2\cos t)^{1/2} (2 - 2\cos t)^{1/2} \, dt = 4\pi.$$

EXAMPLE 9.1.3. Suppose that $\phi : [0, \pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t)$ and $f(x, y) = 2 + 8y^2$. Then

$$\begin{aligned}\int_\phi f \, ds &= \int_0^{\pi} f(\cos^3 t, \sin^3 t) \|(-3\cos^2 t \sin t, 3\sin^2 t \cos t)\| \, dt = \int_0^{\pi} (2 + 8\sin^6 t) 3|\sin t \cos t| \, dt \\ &= \int_0^{\pi/2} (6 + 24\sin^6 t) \sin t \cos t \, dt - \int_{\pi/2}^{\pi} (6 + 24\sin^6 t) \sin t \cos t \, dt = 12.\end{aligned}$$

EXAMPLE 9.1.4. The three distinct paths

$$\begin{aligned}&\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t) \\ &\psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \\ &\eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2)\end{aligned}$$

satisfy $\phi([0, 2\pi]) = \psi([0, 1]) = \eta([0, 1]) = C$, the unit circle in $\mathbb{R}^2$. Note also that the path ϕ follows C in a clockwise direction, while the paths ψ and η follow C in an anticlockwise direction. Now consider the function $f(x, y) = 1 + x + y$. Then

$$\begin{aligned}\int_\phi f \, ds &= \int_0^{2\pi} f(\cos t, -\sin t) \|(-\sin t, -\cos t)\| \, dt = \int_0^{2\pi} (1 + \cos t - \sin t) \, dt = 2\pi, \\ \int_\psi f \, ds &= \int_0^{1} f(\cos 2\pi t, \sin 2\pi t) \|(-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)\| \, dt = \int_0^{1} (1 + \cos 2\pi t + \sin 2\pi t) 2\pi \, dt = 2\pi, \\ \int_\eta f \, ds &= \int_0^{1} f(\cos 2\pi t^2, \sin 2\pi t^2) \|(-4\pi t \sin 2\pi t^2, 4\pi t \cos 2\pi t^2)\| \, dt = \int_0^{1} (1 + \cos 2\pi t^2 + \sin 2\pi t^2) 4\pi t \, dt \\ &= 2\pi.\end{aligned}$$

Note that all three integrals have the same value. We shall show in Section 9.3 that this is not just a coincidence.

9.2. Line Integrals

Suppose that the path

$$\phi : [A, B] \to \mathbb{R}^n : t \mapsto \phi(t) = (x_1(t), \ldots, x_n(t))$$

is continuously differentiable. For any vector field $F(x_1, \ldots, x_n)$ such that the composition function

$$F \circ \phi : [A, B] \to \mathbb{R}^n : t \mapsto F(x_1(t), \ldots, x_n(t))$$

is continuous, we define

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi F(x_1, \ldots, x_n) \cdot \mathrm{d}\mathbf{s} = \int_A^B F(\phi(t)) \cdot \phi'(t) \, \mathrm{d}t.$$

Remarks. (1) We are mainly interested in the special cases $n = 2$ and $n = 3$, and write respectively

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi F(x, y) \cdot \mathrm{d}\mathbf{s} \qquad \text{and} \qquad \int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi F(x, y, z) \cdot \mathrm{d}\mathbf{s}.$$

Writing $F = (F_1, F_2)$ and $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$ in the case $n = 2$ and $F = (F_1, F_2, F_3)$ and $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in the case $n = 3$, we have respectively

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi (F_1, F_2) \cdot (\mathrm{d}x, \mathrm{d}y) = \int_\phi (F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y) = \int_A^B \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} \right) \mathrm{d}t$$

and

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi (F_1, F_2, F_3) \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \int_\phi (F_1 \, \mathrm{d}x + F_2 \, \mathrm{d}y + F_3 \, \mathrm{d}z) = \int_A^B \left(F_1 \frac{\mathrm{d}x}{\mathrm{d}t} + F_2 \frac{\mathrm{d}y}{\mathrm{d}t} + F_3 \frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t.$$

(2) Note that F has only to be defined on the image curve $C = \phi([A, B])$ of the path ϕ for our definition to make sense. The continuity of the composition function $F \circ \phi$ on the closed interval $[A, B]$ ensures the existence of the integral.

(3) Sometimes, ϕ may only be piecewise continuously differentiable. As in the last section, we can calculate the corresponding integral for each subinterval in a dissection of the interval $[A, B]$ and consider the sum of the integrals.

(4) Note that if $\phi'(t) \neq \mathbf{0}$ for any $t \in [A, B]$, then

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_A^B \left(F(\phi(t)) \cdot \frac{\phi'(t)}{\|\phi'(t)\|} \right) \|\phi'(t)\| \, \mathrm{d}t = \int_A^B f(\phi(t)) \|\phi'(t)\| \, \mathrm{d}t,$$

where

$$f(\phi(t)) = F(\phi(t)) \cdot \frac{\phi'(t)}{\|\phi'(t)\|}.$$

Here $\phi'(t)/\|\phi'(t)\|$ denotes the unit tangent vector along the path ϕ. The integral now becomes one of the type discussed in the last section.

(5) Suppose that F is a force field; for example, gravitational field or magnetic field. Suppose also that a particle is moving along a path ϕ. At any time t, the force on the particle will be given by $F(\phi(t))$. On the other hand, a small displacement in the time interval $[t, t + \mathrm{d}t]$ can be described by the velocity differential $\mathrm{d}\mathbf{s} = \phi'(t) \, \mathrm{d}t$. It follows that the scalar product $F(\phi(t)) \cdot \phi'(t) \, dt$ denotes the work done in the time interval $[t, t + \mathrm{d}t]$. Hence the integral describes the total work done.

EXAMPLE 9.2.1. Suppose that $\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$ and $F(x, y, z) = (x, y, z)$. Then

$$\begin{aligned}\int_\phi F \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} F(\cos t, \sin t, t) \cdot (-\sin t, \cos t, 1)\,\mathrm{d}t \\ &= \int_0^{2\pi} (\cos t, \sin t, t) \cdot (-\sin t, \cos t, 1)\,\mathrm{d}t \\ &= \int_0^{2\pi} t\,\mathrm{d}t = 2\pi^2.\end{aligned}$$

EXAMPLE 9.2.2. Suppose that $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$ and $F(x, y) = (y, -x)$. Then

$$\begin{aligned}\int_\phi F \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} F(t - \sin t, 1 - \cos t) \cdot (1 - \cos t, \sin t)\,\mathrm{d}t \\ &= \int_0^{2\pi} (1 - \cos t, \sin t - t) \cdot (1 - \cos t, \sin t)\,\mathrm{d}t \\ &= \int_0^{2\pi} (2 - 2\cos t - t\sin t)\,\mathrm{d}t = 6\pi.\end{aligned}$$

EXAMPLE 9.2.3. Suppose that $\phi : [0, \pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t)$ and $F(x, y) = (-y, x)$. Then

$$\begin{aligned}\int_\phi F \cdot \mathrm{d}\mathbf{s} &= \int_0^{\pi} F(\cos^3 t, \sin^3 t) \cdot (-3\cos^2 t \sin t, 3\sin^2 t \cos t)\,\mathrm{d}t \\ &= \int_0^{\pi} (-\sin^3 t, \cos^3 t) \cdot (-3\cos^2 t \sin t, 3\sin^2 t \cos t)\,\mathrm{d}t \\ &= \int_0^{\pi} 3\sin^2 t \cos^2 t\,\mathrm{d}t = \frac{3}{4}\int_0^{\pi} \sin^2 2t\,\mathrm{d}t = \frac{3}{8}\int_0^{\pi} (1 - \cos 4t)\,\mathrm{d}t = \frac{3\pi}{8}.\end{aligned}$$

EXAMPLE 9.2.4. The three distinct paths

$$\begin{aligned}&\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t) \\ &\psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \\ &\eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2)\end{aligned}$$

satisfy $\phi([0, 2\pi]) = \psi([0, 1]) = \eta([0, 1]) = C$, the unit circle in $\mathbb{R}^2$. Note also that the path ϕ follows C in a clockwise direction, while the paths ψ and η follow C in an anticlockwise direction. Now consider the vector field $F(x, y) = (-y, x)$. Then

$$\begin{aligned}\int_\phi F \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} F(\cos t, -\sin t) \cdot (-\sin t, -\cos t)\,\mathrm{d}t \\ &= \int_0^{2\pi} (\sin t, \cos t) \cdot (-\sin t, -\cos t)\,\mathrm{d}t = \int_0^{2\pi} (-1)\,\mathrm{d}t = -2\pi, \\ \int_\psi F \cdot \mathrm{d}\mathbf{s} &= \int_0^{1} F(\cos 2\pi t, \sin 2\pi t) \cdot (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)\,\mathrm{d}t \\ &= \int_0^{1} (-\sin 2\pi t, \cos 2\pi t) \cdot (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)\,\mathrm{d}t = \int_0^1 2\pi\,\mathrm{d}t = 2\pi, \\ \int_\eta F \cdot \mathrm{d}\mathbf{s} &= \int_0^{1} F(\cos 2\pi t^2, \sin 2\pi t^2) \cdot (-4\pi t \sin 2\pi t^2, 4\pi t \cos 2\pi t^2)\,\mathrm{d}t \\ &= \int_0^{1} (-\sin 2\pi t^2, \cos 2\pi t^2) \cdot (-4\pi t \sin 2\pi t^2, 4\pi t \cos 2\pi t^2)\,\mathrm{d}t = \int_0^1 4\pi t\,\mathrm{d}t = 2\pi.\end{aligned}$$

Note that

$$-\int_\phi F\cdot d\mathbf{s}=\int_\psi F\cdot d\mathbf{s}=\int_\eta F\cdot d\mathbf{s},$$

where ψ and η follow the unit circle C in the same direction while ϕ follows the unit circle C in the opposite direction. We shall show in Section 9.3 that this is not just a coincidence.

EXAMPLE 9.2.5. The three distinct paths

$$\begin{aligned}&\phi:[0,1]\to\mathbb{R}^2:t\mapsto(t,t)\\&\psi:[0,1]\to\mathbb{R}^2:t\mapsto(t,t^2)\\&\eta:[0,\pi/2]\to\mathbb{R}^2:t\mapsto(1-\cos t,\sin t)\end{aligned}$$

all have the same initial point $(0,0)$ and the same terminal point $(1,1)$. The curve $\phi([0,1])$ is part of the straight line $y=x$, the curve $\psi([0,1])$ is part of the parabola $y=x^2$, while the curve $\eta([0,\pi/2])$ is part of the circle $(x-1)^2+y^2=1$. Hence the three paths have different curves. Consider now the vector field $F(x,y)=(y,x)$. Then

$$\begin{aligned}\int_\phi F\cdot d\mathbf{s}&=\int_0^1 F(t,t)\cdot(1,1)\,dt=\int_0^1(t,t)\cdot(1,1)\,dt=\int_0^1 2t\,dt=1,\\\int_\psi F\cdot d\mathbf{s}&=\int_0^1 F(t,t^2)\cdot(1,2t)\,dt=\int_0^1(t^2,t)\cdot(1,2t)\,dt=\int_0^1 3t^2\,dt=1,\\\int_\eta F\cdot d\mathbf{s}&=\int_0^{\pi/2}F(1-\cos t,\sin t)\cdot(\sin t,\cos t)\,dt=\int_0^{\pi/2}(\sin t,1-\cos t)\cdot(\sin t,\cos t)\,dt\\&=\int_0^{\pi/2}(\cos t+\sin^2 t-\cos^2 t)\,dt=\int_0^{\pi/2}(\cos t-\cos 2t)\,dt=1.\end{aligned}$$

Next, note that $F=\nabla f$, where $f(x,y)=xy$. Hence in particular, we have

$$\int_\phi F\cdot d\mathbf{s}=\int_\phi\nabla f\cdot d\mathbf{s},\qquad\int_\psi F\cdot d\mathbf{s}=\int_\psi\nabla f\cdot d\mathbf{s},\qquad\int_\eta F\cdot d\mathbf{s}=\int_\eta\nabla f\cdot d\mathbf{s}.$$

Observe now that $f(1,1)-f(0,0)=1$, so is it a coincidence that

$$\int_\phi\nabla f\cdot d\mathbf{s}=\int_\psi\nabla f\cdot d\mathbf{s}=\int_\eta\nabla f\cdot d\mathbf{s}=f(1,1)-f(0,0),\tag{1}$$

so that the integrals depend only on the endpoints of the paths? On the other hand, note that F is the total derivative of f, so (1) is really just a statement like the Fundamental theorem of calculus.

Let us investigate this problem in general. Suppose that F is a gradient vector field in $\mathbb{R}^n$, so that there exists a continuously differentiable function $f(x_1,\ldots,x_n)$ such that $F=\nabla f$. Suppose that $\phi:[A,B]\to\mathbb{R}^n$ is a continuously differentiable path. Consider the composition function

$$g=f\circ\phi:[A,B]\to\mathbb{R}.$$

By the Chain rule, we have

$$g'(t)=\begin{pmatrix}\dfrac{\partial f}{\partial x_1}(\phi(t)) & \ldots & \dfrac{\partial f}{\partial x_n}(\phi(t))\end{pmatrix}\begin{pmatrix}\phi_1'(t)\\\vdots\\\phi_n'(t)\end{pmatrix},$$

where the right hand side is the matrix product of the total derivatives $(\mathbf{D}f)(\phi(t))$ and $(\mathbf{D}\phi)(t)$. It follows that

$$g'(t) = (\nabla f)(\phi(t)) \cdot \phi'(t) = F(\phi(t)) \cdot \phi'(t),$$

and so

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_A^B F(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t = \int_A^B g'(t)\,\mathrm{d}t = g(B) - g(A) = f(\phi(B)) - f(\phi(A))$$

by the Fundamental theorem of calculus applied to the function g. We have proved the following result.

THEOREM 9A. *Suppose that $F = \nabla f$ is a gradient vector field in $\mathbb{R}^n$. Then for any continuously differentiable path $\phi : [A, B] \to \mathbb{R}^n$ such that the composition function $F \circ \phi : [A, B] \to \mathbb{R}^n$ is continuous, we have*

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = f(\phi(B)) - f(\phi(A)).$$

9.3. Equivalent Paths

We return to the questions posed by Examples 9.1.4 and 9.2.4.

DEFINITION. Suppose that $\phi : [A_1, B_1] \to \mathbb{R}^n$ and $\psi : [A_2, B_2] \to \mathbb{R}^n$ are two continuously differentiable paths. Then we say that ϕ and ψ are equivalent if there exists a continuously differentiable and strictly monotonic function $h : [A_1, B_1] \to [A_2, B_2]$ such that $h([A_1, B_1]) = [A_2, B_2]$ and $\phi = \psi \circ h$. In this case, we say that the function h defines a change of parameter. Furthermore, we say that the change of parameter is orientation preserving if h is strictly increasing and orientation reversing if h is strictly decreasing.

REMARKS. (1) It is easy to see that if two paths are equivalent, then they have the same curve. If the change of parameter is orientation preserving, then the curve is followed in the same direction. If the change of parameter is orientation reversing, then the curve is followed in different directions.

(2) Note that the change of parameter is orientation preserving if and only if $h'(t) \geq 0$ for every $t \in [A_1, B_1]$, and orientation reversing if and only if $h'(t) \leq 0$ for every $t \in [A_1, B_1]$.

(3) Since $h : [A_1, B_1] \to [A_2, B_2]$ is strictly monotonic and onto, it follows that it has an inverse function $h^{-1} : [A_2, B_2] \to [A_1, B_1]$. Clearly $\psi = \phi \circ h^{-1}$. Furthermore, the inverse function is also continuously differentiable.

EXAMPLE 9.3.1. Recall the three distinct paths

$$\begin{aligned} &\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t) \\ &\psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \\ &\eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2) \end{aligned}$$

considered in Examples 9.1.4 and 9.2.4. Let us examine first of all ψ and η. The function

$$h_1 : [0, 1] \to [0, 1] : t \to \sqrt{t}$$

is strictly increasing, and defines an orientation preserving change of parameter with $\psi = \eta \circ h_1$. Note that the inverse function

$$h_1^{-1} : [0, 1] \to [0, 1] : t \to t^2$$

is also strictly increasing, and $\eta = \psi \circ h_1^{-1}$. Clearly ψ and η follow the unit circle in the same direction. Consider next ϕ and ψ. The function

$$h_2 : [0, 2\pi] \to [0, 1] : t \to 1 - \frac{t}{2\pi}$$

is strictly decreasing, and defines an orientation reversing change of parameter with $\phi = \psi \circ h_2$. Note that the inverse function

$$h_2^{-1} : [0, 1] \to [0, 2\pi] : t \to 2\pi - 2\pi t$$

is also strictly decreasing, and $\psi = \phi \circ h_2^{-1}$. Clearly ϕ and ψ follow the unit circle in opposite directions.

THEOREM 9B. *Suppose that $\phi : [A_1, B_1] \to \mathbb{R}^n$ and $\psi : [A_2, B_2] \to \mathbb{R}^n$ are two equivalent continuously differentiable paths. Then for any real valued function $f(x_1, \ldots, x_n)$ such that the composition functions $f \circ \phi : [A_1, B_1] \to \mathbb{R}$ and $f \circ \psi : [A_2, B_2] \to \mathbb{R}$ are continuous, we have*

$$\int_\phi f \, \mathrm{d}s = \int_\psi f \, \mathrm{d}s.$$

Proof. Since ϕ and ψ are equivalent, there exists $h : [A_1, B_1] \to [A_2, B_2]$ such that $\phi = \psi \circ h$. It follows from the Chain rule that $\phi'(t) = \psi'(h(t))h'(t)$, and so

$$\int_\phi f \, \mathrm{d}s = \int_{A_1}^{B_1} f(\phi(t))\|\phi'(t)\| \, \mathrm{d}t = \int_{A_1}^{B_1} f(\psi(h(t)))\|\psi'(h(t))h'(t)\| \, \mathrm{d}t.$$

In the orientation preserving case, we have $h'(t) \geq 0$ always, and so, with a change of variables $u = h(t)$, we have

$$\int_\phi f \, \mathrm{d}s = \int_{A_1}^{B_1} f(\psi(h(t)))\|\psi'(h(t))\|h'(t) \, \mathrm{d}t = \int_{A_2}^{B_2} f(\psi(u))\|\psi'(u)\| \, \mathrm{d}u = \int_\psi f \, \mathrm{d}s.$$

In the orientation reversing case, we have $h'(t) \leq 0$ always, and so, with a change of variables $u = h(t)$, we have

$$\begin{aligned} \int_\phi f \, \mathrm{d}s &= -\int_{A_1}^{B_1} f(\psi(h(t)))\|\psi'(h(t))\|h'(t) \, \mathrm{d}t = -\int_{B_2}^{A_2} f(\psi(u))\|\psi'(u)\| \, \mathrm{d}u \\ &= \int_{A_2}^{B_2} f(\psi(u))\|\psi'(u)\| \, \mathrm{d}u = \int_\psi f \, \mathrm{d}s. \end{aligned}$$

This completes the proof. ◯

THEOREM 9C. *Suppose that $\phi : [A_1, B_1] \to \mathbb{R}^n$ and $\psi : [A_2, B_2] \to \mathbb{R}^n$ are two equivalent continuously differentiable paths. Then for any vector field $F(x_1, \ldots, x_n)$ such that the composition functions $F \circ \phi : [A_1, B_1] \to \mathbb{R}^n$ and $F \circ \psi : [A_2, B_2] \to \mathbb{R}^n$ are continuous, we have*

$$\int_\phi F \cdot \mathbf{ds} = \pm \int_\psi F \cdot \mathbf{ds},$$

where the equality holds with the $+$ sign if the change of parameter is orientation preserving and with the $-$ sign if the change of parameter is orientation reversing.

Proof. Since ϕ and ψ are equivalent, there exists $h : [A_1, B_1] \to [A_2, B_2]$ such that $\phi = \psi \circ h$. It follows from the Chain rule that $\phi'(t) = \psi'(h(t))h'(t)$, and so

$$\int_\phi F \cdot \mathbf{ds} = \int_{A_1}^{B_1} F(\phi(t)) \cdot \phi'(t) \, \mathrm{d}t = \int_{A_1}^{B_1} F(\psi(h(t))) \cdot \psi'(h(t))h'(t) \, \mathrm{d}t.$$

With a change of variables $u = h(t)$, we have, in the orientation preserving case,

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u)\,\mathrm{d}u = \int_\psi F \cdot \mathrm{d}\mathbf{s},$$

and in the orientation reversing case,

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_{B_2}^{A_2} F(\psi(u)) \cdot \psi'(u)\,\mathrm{d}u = -\int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u)\,\mathrm{d}u = -\int_\psi F \cdot \mathrm{d}\mathbf{s}.$$

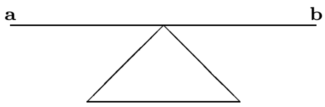
This completes the proof. ○

Remark. Theorems 9B and 9C have natural extensions to the case when the paths are piecewise continuously differentiable. In this case, one can clearly break the paths into continuously differentiable pieces and apply Theorems 9B and 9C to each piece.

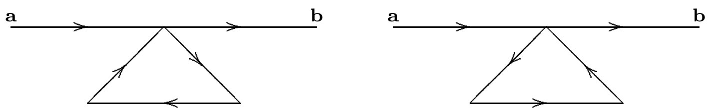
9.4. Simple Curves

Theorems 9B and 9C demonstrate that integrals over differentiable paths depend only on the (oriented) curves of these paths. It therefore seems natural to try to express the theory in terms of these curves instead of the paths. The purpose of this section is to consider this problem. Before we start, we examine the example below which suggests that some care is required.

Example 9.4.1. Consider the curve below with endpoints indicated.



Clearly it is not enough to say that a path has initial point **a** and terminal point **b**, since any two paths that trace the curve in the two different ways indicated below are clearly not equivalent.



To temporarily avoid situations like this, we make the following definition.

Definition. By a simple curve C in $\mathbb{R}^n$, we mean the image $C = \phi([A,B])$ of a piecewise continuously differentiable path $\phi : [A,B] \to \mathbb{R}^n$ with the property that $\phi(t_1) \neq \phi(t_2)$ whenever $A \le t_1 < t_2 \le B$, with the possible exception that $\phi(A) = \phi(B)$ may hold. A simple curve together with a direction is called an oriented simple curve. The function ϕ is called a parametrization of the oriented simple curve C, and the parametrization is said to be orientation preserving if ϕ follows the direction of C, and orientation reversing if ϕ follows the opposite direction of C.

Suppose that C is an oriented simple curve in $\mathbb{R}^n$. For any real valued function $f(x_1,\ldots,x_n)$ continuous on C, we can define

$$\int_C f\,\mathrm{d}s = \int_\phi f\,\mathrm{d}s,$$

where ϕ is any parametrization of C. For any vector field $F(x_1,\ldots,x_n)$ continuous on C, we can define

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_\phi F \cdot \mathrm{d}\mathbf{s},$$

where ϕ is any orientation preserving parametrization of C. The integrals

$$\int_C f\,\mathrm{d}s \qquad \text{and} \qquad \int_C F\cdot\mathrm{d}\mathbf{s}$$

are well defined in view of Theorems 9B and 9C respectively.

REMARKS. (1) Suppose that the oriented simple curve C^- is obtained from the oriented simple curve C by taking the opposite orientation. Then

$$\int_C f\,\mathrm{d}s = \int_{C^-} f\,\mathrm{d}s \qquad \text{and} \qquad \int_C F\cdot\mathrm{d}\mathbf{s} = -\int_{C^-} F\cdot\mathrm{d}\mathbf{s}.$$

(2) The theory can be extended to curves that are not simple, provided that we indicate very carefully how these curves are to be followed, and take note where some parts may be followed more than once. In particular, it is often convenient to break up an oriented curve into several components, each of which is simple. For example, if $C = C_1 + \ldots + C_k$, where the sum denotes that the oriented curve C is obtained by following the oriented (simple) curves $C_1, \ldots, C_k$ one after another, then we have

$$\int_C f\,\mathrm{d}s = \sum_{i=1}^{k}\int_{C_i} f\,\mathrm{d}s \qquad \text{and} \qquad \int_C F\cdot\mathrm{d}\mathbf{s} = \sum_{i=1}^{k}\int_{C_i} F\cdot\mathrm{d}\mathbf{s}.$$

In this case, each of $C_1, \ldots, C_k$ can be parametrized separately.

EXAMPLE 9.4.2. Let $F(x,y) = (3xy, -y^2)$, and let C denote the path of the parabola $y = 2x^2$ from $(1,2)$ to $(0,0)$. Clearly $\phi : [0,1] \to \mathbb{R}^2 : t \mapsto (t, 2t^2)$ is an orientation preserving parametrization of C^-, and so

$$\begin{aligned}\int_{C^-} F\cdot\mathrm{d}\mathbf{s} = \int_\phi F\cdot\mathrm{d}\mathbf{s} &= \int_0^1 F(\phi(t))\cdot\phi'(t)\,\mathrm{d}t = \int_0^1 F(t,2t^2)\cdot(1,4t)\,\mathrm{d}t \\ &= \int_0^1 (6t^3, -4t^4)\cdot(1,4t)\,\mathrm{d}t = \int_0^1 (6t^3 - 16t^5)\,\mathrm{d}t = -\frac{7}{6}.\end{aligned}$$

Hence

$$\int_C F\cdot\mathrm{d}\mathbf{s} = \frac{7}{6}.$$

EXAMPLE 9.4.3. Let $F(x,y,z) = (2x - y + z, x + y - z^2, 3x - 2y + 4z)$, and let C denote the circle on the xy-plane with centre at the origin and radius 3, followed in the anticlockwise direction on the xy-plane. Clearly $\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (3\cos t, 3\sin t, 0)$ is an orientation preserving parametrization of C, and so

$$\begin{aligned}\int_C F\cdot\mathrm{d}\mathbf{s} = \int_\phi F\cdot\mathrm{d}\mathbf{s} &= \int_0^{2\pi} F(\phi(t))\cdot\phi'(t)\,\mathrm{d}t = \int_0^{2\pi} F(3\cos t, 3\sin t, 0)\cdot(-3\sin t, 3\cos t, 0)\,\mathrm{d}t \\ &= \int_0^{2\pi} (6\cos t - 3\sin t, 3\cos t + 3\sin t, 9\cos t - 6\sin t)\cdot(-3\sin t, 3\cos t, 0)\,\mathrm{d}t \\ &= \int_0^{2\pi} (9 - 9\sin t\cos t)\,\mathrm{d}t = 18\pi.\end{aligned}$$

EXAMPLE 9.4.4. Let $F(x,y,z) = (3x^2+6y, -14yz, 20xz^2)$, and let C denote a succession of the straight line segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. Let C_1 denote the straight line segment from $(0,0,0)$ to $(1,0,0)$, C_2 denote the straight line segment from $(1,0,0)$ to $(1,1,0)$, and C_3 denote the straight line segment from $(1,1,0)$ to $(1,1,1)$. Clearly

$$\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (t,0,0), \qquad \psi : [0,1] \to \mathbb{R}^3 : t \mapsto (1,t,0), \qquad \eta : [0,1] \to \mathbb{R}^3 : t \mapsto (1,1,t)$$

are orientation preserving parametrization of C_1, C_2, C_3 respectively. Hence

$$\begin{aligned}
\int_{C_1} F \cdot \mathrm{d}\mathbf{s} &= \int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_0^1 F(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t = \int_0^1 F(t,0,0) \cdot (1,0,0)\,\mathrm{d}t \\
&= \int_0^1 (3t^2,0,0) \cdot (1,0,0)\,\mathrm{d}t = \int_0^1 3t^2\,\mathrm{d}t = 1, \\
\int_{C_2} F \cdot \mathrm{d}\mathbf{s} &= \int_\psi F \cdot \mathrm{d}\mathbf{s} = \int_0^1 F(\psi(t)) \cdot \psi'(t)\,\mathrm{d}t = \int_0^1 F(1,t,0) \cdot (0,1,0)\,\mathrm{d}t \\
&= \int_0^1 (3+6t,0,0) \cdot (0,1,0)\,\mathrm{d}t = \int_0^1 0\,\mathrm{d}t = 0, \\
\int_{C_3} F \cdot \mathrm{d}\mathbf{s} &= \int_\eta F \cdot \mathrm{d}\mathbf{s} = \int_0^1 F(\eta(t)) \cdot \eta'(t)\,\mathrm{d}t = \int_0^1 F(1,1,t) \cdot (0,0,1)\,\mathrm{d}t \\
&= \int_0^1 (9,-14t,20t^2) \cdot (0,0,1)\,\mathrm{d}t = \int_0^1 20t^2\,\mathrm{d}t = \frac{20}{3},
\end{aligned}$$

and so

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_{C_1} F \cdot \mathrm{d}\mathbf{s} + \int_{C_2} F \cdot \mathrm{d}\mathbf{s} + \int_{C_3} F \cdot \mathrm{d}\mathbf{s} = \frac{23}{3}.$$

Next, let $f(x,y,z) = x^2+y^2+z^2$. Then

$$\begin{aligned}
\int_{C_1} f\,\mathrm{d}s &= \int_\phi f\,\mathrm{d}s = \int_0^1 f(\phi(t))\|\phi'(t)\|\,\mathrm{d}t = \int_0^1 f(t,0,0)\|(1,0,0)\|\,\mathrm{d}t = \int_0^1 t^2\,\mathrm{d}t = \frac{1}{3}, \\
\int_{C_2} f\,\mathrm{d}s &= \int_\psi f\,\mathrm{d}s = \int_0^1 f(\psi(t))\|\psi'(t)\|\,\mathrm{d}t = \int_0^1 f(1,t,0)\|(0,1,0)\|\,\mathrm{d}t = \int_0^1 (1+t^2)\,\mathrm{d}t = \frac{4}{3}, \\
\int_{C_3} f\,\mathrm{d}s &= \int_\eta f\,\mathrm{d}s = \int_0^1 f(\eta(t))\|\eta'(t)\|\,\mathrm{d}t = \int_0^1 f(1,1,t)\|(0,0,1)\|\,\mathrm{d}t = \int_0^1 (2+t^2)\,\mathrm{d}t = \frac{7}{3},
\end{aligned}$$

and so

$$\int_C f\,\mathrm{d}s = \int_{C_1} f\,\mathrm{d}s + \int_{C_2} f\,\mathrm{d}s + \int_{C_3} f\,\mathrm{d}s = 4.$$

PROBLEMS FOR CHAPTER 9

1. For each of the following, evaluate the integral $\displaystyle\int_\phi f\,\mathrm{d}s$:
   a) $f(x,y,z) = \dfrac{1}{y^3}$; $\phi : [1,\mathrm{e}] \to \mathbb{R}^3 : t \mapsto (\log t, t, 2)$
   b) $f(x,y,z) = \cos z$; $\phi : [0,2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$
   c) $f(x,y,z) = x\cos z$; $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (t, t^2, 0)$
   d) $f(x,y,z) = yz$; $\phi : [0,3] \to \mathbb{R}^3 : t \mapsto (t, 2t, 3t)$

e) $f(x,y,z) = z$; $\phi : [0,2\pi] \to \mathbb{R}^3 : t \mapsto (t\cos t, t\sin t, t)$

f) $f(x,y,z) = \dfrac{x+y}{y+z}$; $\phi : [1,2] \to \mathbb{R}^3 : t \mapsto (3t, 2t^{3/2}, 3t)$

2. For each of the following, evaluate the integral $\displaystyle\int_\phi F \cdot d\mathbf{s}$:

a) $F(x,y,z) = (y,z,x)$; $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (t, t+1, t-1)$
b) $F(x,y,z) = (3xy, -5z, 10x)$; $\phi : [1,2] \to \mathbb{R}^3 : t \mapsto (t^2+1, 2t^2, t^3)$
c) $F(x,y,z) = (x,y,z)$; $\phi : [0,2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, 0)$
d) $F(x,y,z) = (y, 2x, y)$; $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (t, t^2, t^3)$
e) $F(x,y,z) = (x^2+y^2, z, xy)$; $\phi : [0,\pi] \to \mathbb{R}^3 : t \mapsto (\sin t, \cos t, t)$

3. For each of the following, evaluate the integral $\displaystyle\int_C F \cdot d\mathbf{s}$:

a) $F(x,y,z) = (yz, zx, xy)$; C is the straight line segment from $(1,0,0)$ to $(0,1,0)$ followed by the straight line segment from $(0,1,0)$ to $(0,0,1)$.
b) $F(x,y,z) = (x^2, -xy, 1)$; C is the parabola $z = x^2$ on the plane $y = 0$ from $(-1,0,1)$ to $(1,0,1)$.
c) $F(x,y,z) = (x,y,z)$; C is the parabola $y = x^2$ on the plane $z = 0$ from $(-1,1,0)$ to $(2,4,0)$.

4. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (3x^2, 2xz - y, z)$.

a) Evaluate the integral $\displaystyle\int_\phi F \cdot d\mathbf{s}$ over each of the paths ϕ below:
  (i) $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (2t, t, 3t)$
  (ii) $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (2t, t^2, 3t^3)$
  (iii) $\phi : [0,1] \to \mathbb{R}^3 : t \mapsto (2t^2, t, 4t^2 - t)$
b) Determine $\operatorname{curl} F$.
c) Is F a gradient vector field? Justify your assertion in two different ways, once using part (a), and once using part (b).

5. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (y^2z^3\cos x - 4x^3z, 2yz^3\sin x, 3y^2z^2\sin x - x^4)$.
a) Show that $\operatorname{curl} F = \mathbf{0}$ everywhere in $\mathbb{R}^3$.
b) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$ everywhere in $\mathbb{R}^3$.

6. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (yz, xz + z^2, xy + 2yz)$.

a) Let $\phi : [0,2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, 1)$. Show that $\displaystyle\int_\phi F \cdot d\mathbf{s} = 0$.
b) Show that $\operatorname{curl} F = \mathbf{0}$ everywhere in $\mathbb{R}^3$.
c) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$ everywhere in $\mathbb{R}^3$.
d) Explain your result in part (a) in terms of part (c), quoting any results that you need.
e) Let $\psi : [0,\pi/2] \to \mathbb{R}^3 : t \mapsto (t\sin^7 t, t^2\cos^5 t, \cos^4 t + \sin^4 t)$. Evaluate the integral $\displaystyle\int_\psi F \cdot d\mathbf{s}$.

7. Let $F(x,y,z) = (z^3 + 2xy, x^2, 3xz^2)$, and consider the integral $\displaystyle\int_C F \cdot d\mathbf{s}$.

a) Evaluate the integral when C is the square path with vertices $(\pm1, \pm1, 0)$ and followed in the anticlockwise direction on the xy-plane with initial point $(1,1,0)$.
b) Find a real valued function $f(x,y,z)$ such that $F = \nabla f$.
c) Explain your answer in part (a) in terms of Theorem 9A.
d) Evaluate the integral when C is the straight line segment from $(1,-2,1)$ to $(3,1,4)$.
e) What is the value of the integral when C is any curve from $(1,-2,1)$ to $(3,1,4)$?

8. Evaluate the integral $\displaystyle\int_C F \cdot d\mathbf{s}$ when $F(x,y,z) = (2xyz, x^2z, x^2y)$ and C is any simple curve from $(1,1,1)$ to $(1,2,4)$.

9. It is known that $\nabla f = F$, where $F(x,y,z) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2})$ for every $(x,y,z) \in \mathbb{R}^3$. If $f(0,0,0) = 3$, what is $f(1,1,2)$?

10. A path $\phi : [\theta_1, \theta_2] \to \mathbb{R}^2$ on the xy-plane is given in polar coordinates by $r = r(\theta)$, where $r(\theta)$ is continuously differentiable in the interval $[\theta_1, \theta_2]$.
 a) Determine $\phi(\theta)$ and $\phi'(\theta)$ for any $\theta \in [\theta_1, \theta_2]$.
 b) Write the integral $\displaystyle\int_\phi f\,\mathrm{d}s$ as an integral over θ.
 c) Find the arc length of the path $r = 1 + \cos\theta$ where $\theta \in [0, 2\pi]$.

11. Suppose that $\phi : [A, B] \to \mathbb{R}^n$ is a continuously differentiable path with arc length ℓ. Suppose further that the vector field F satisfies $\|F(\mathbf{x})\| \le M$ for every $\mathbf{x} \in \phi([A, B])$. Show that

$$\left| \int_\phi F \cdot \mathrm{d}\mathbf{s} \right| \le M\ell.$$

12. Suppose that $\phi : [A, B] \to \mathbb{R}^3$ is a continuously differentiable path. Prove the following:
 a) If $F(\phi(t))$ is perpendicular to $\phi'(t)$ for every $t \in [A, B]$, then $\displaystyle\int_\phi F \cdot \mathrm{d}\mathbf{s} = 0$.
 b) If $F(\phi(t))$ is is in the same direction as $\phi'(t)$ for every $t \in [A, B]$, then $\displaystyle\int_\phi F \cdot \mathrm{d}\mathbf{s} = \int_\phi \|F\|\,\mathrm{d}s$.
 c) Discuss the case when $F(\phi(t))$ is is in the opposite direction to $\phi'(t)$ for every $t \in [A, B]$.

13. Suppose that F is a vector field in $\mathbb{R}^3$ such that $\operatorname{curl} F = \mathbf{0}$ everywhere in $\mathbb{R}^3$. Follow the steps below to show that there exists a real valued function $f(x,y,z)$ such that $\nabla f = F$:
 a) Write $F = (F_1, F_2, F_3)$. Show that

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

 b) For any $(x,y,z) \in \mathbb{R}^3$, let C denote a succession of straight line segments from $(0,0,0)$ to $(x,0,0)$ to $(x,y,0)$ to (x,y,z). Show that

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_0^x F_1(t,0,0)\,\mathrm{d}t + \int_0^y F_2(x,t,0)\,\mathrm{d}t + \int_0^z F_3(x,y,t)\,\mathrm{d}t.$$

 c) Let $f(x,y,z) = \displaystyle\int_C F \cdot \mathrm{d}\mathbf{s}$. Show that $\dfrac{\partial f}{\partial z} = F_3$.
 d) Use $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}$ to show that $\dfrac{\partial f}{\partial y} = F_2$.
 e) Use $\dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$ and $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ to show that $\dfrac{\partial f}{\partial x} = F_1$.

14. Consider the vector field

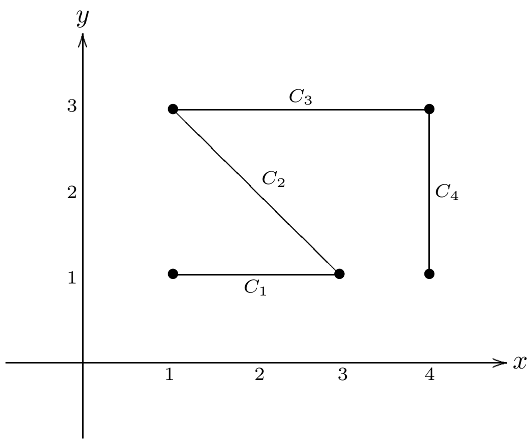
$$F : A \to \mathbb{R}^3 : (x,y,z) \mapsto \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right),$$

where $A = \mathbb{R}^3 \setminus \{(0,0,0)\}$. Suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points in A. Let C be any simple curve from (x_1, y_1, z_1) to (x_2, y_2, z_2).
 a) Show that there exists a function $f : A \to \mathbb{R}$ such that $\nabla f = F$ everywhere in A.
 b) Without using the function f in part (a) explicitly, use Question 9 to show that the integral

$$\int_C F \cdot \mathrm{d}\mathbf{s}$$

 depends only on the real numbers $R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $R_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.
 c) Use Theorem 9A and the function f in part (a) to draw the same conclusion as in part (b).

15. Let $F(x,y) = (x^2y, y^2)$, and consider the oriented curve $C = C_1 + C_2 + C_3 + C_4$ from $(1,1)$ to $(4,1)$ shown in the picture below.



a) Find a parametrization ϕ_i of the oriented line segment C_i for each $i = 1, 2, 3, 4$.

b) Evaluate the integral $\int_{C_i} F \cdot d\mathbf{s}$ for each $i = 1, 2, 3, 4$, and find the integral $\int_C F \cdot d\mathbf{s}$.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 10

PARAMETRIZED SURFACES

10.1. Introduction

In this chapter, we discuss parametrized surfaces. Recall that a path is essentially a parametrization of a curve. In a similar way, a surface can be parametrized. Whereas a curve can be parametrized by the use of a single real parameter, a surface can be parametrized by the use of two real parameters. We shall be concerned only with surfaces in $\mathbb{R}^3$. Before we give any formal definition, let us consider two examples.

EXAMPLE 10.1.1. Consider a function $f : [A, B] \times [C, D] \to \mathbb{R}$. Then the graph

$$\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D] \text{ and } z = f(x, y)\}$$

is a surface in $\mathbb{R}^3$. Each point (x, y, z) on this surface is determined precisely by the values of the variables x and y.

EXAMPLE 10.1.2. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, with radius 1 and centre $(0, 0, 0)$. Using spherical coordinates, we can write

$$x = \sin\phi\cos\theta, \qquad y = \sin\phi\sin\theta, \qquad z = \cos\phi,$$

where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Each point (x, y, z) on the sphere is determined precisely by the values of the variables ϕ and θ.

DEFINITION. By a parametrized surface in $\mathbb{R}^3$, we mean a function of the type

$$\Phi : R \to \mathbb{R}^3,$$

where $R \subseteq \mathbb{R}^2$ is a domain. The range

$$\Phi(R) = \{\Phi(u, v) : (u, v) \in R\} \subseteq \mathbb{R}^3$$

of the function Φ is called a surface.

REMARKS. (1) For every $(u,v) \in R$, we can write $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, with components $x(u,v), y(u,v), z(u,v) \in \mathbb{R}$.

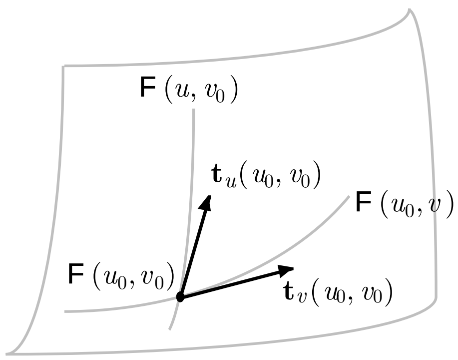
(2) We can think of the function Φ as twisting and bending the region R to give a surface $S = \Phi(R)$. The position of a point $(x(u,v), y(u,v), z(u,v))$ on S is determined by the values of the parameters u and v.

(3) Often, we refer to the parametrized surface $\Phi(u,v)$ without specifying the domain of definition of the function Φ. This is a convenient abuse of rigour.

DEFINITION. We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is continuously differentiable if the function Φ is differentiable and the partial derivatives are continuous.

Suppose that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is differentiable at a point $(u_0, v_0) \in R$. Keeping the first parameter u fixed at u_0, we consider the function $v \mapsto \Phi(u_0, v)$ in a neighbourhood of v_0. The image of this function is a curve on the surface $S = \Phi(R)$, and a tangent vector to this curve at the point $\Phi(u_0, v_0)$ is given by

$$\mathbf{t}_v(u_0,v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right).$$



On the other hand, keeping the second parameter v fixed at v_0, we consider the function $u \mapsto \Phi(u, v_0)$ in a neighbourhood of u_0. The image of this function is a curve on the surface $S = \Phi(R)$, and a tangent vector to this curve at the point $\Phi(u_0, v_0)$ is given by

$$\mathbf{t}_u(u_0,v_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right).$$

It follows that $\mathbf{t}_u(u_0,v_0)$ and $\mathbf{t}_v(u_0,v_0)$ are two vectors tangent to the surface $S = \Phi(R)$ at the point $\Phi(u_0,v_0)$. Unless they are parallel or opposite, these two vectors determine the tangent plane to the surface $S = \Phi(R)$ at the point $\Phi(u_0,v_0)$. In this case, the vector $\mathbf{t}_u(u_0,v_0) \times \mathbf{t}_v(u_0,v_0)$ is a vector normal to the surface $S = \Phi(R)$ at the point $\Phi(u_0,v_0)$.

DEFINITION. We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is smooth at a point $\Phi(u_0,v_0)$ if

$$\mathbf{t}_u(u_0,v_0) \times \mathbf{t}_v(u_0,v_0) \neq \mathbf{0}.$$

We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is smooth if it is smooth at every point $\Phi(u_0,v_0)$ where $(u_0,v_0) \in R$; in other words, if $\mathbf{t}_u \times \mathbf{t}_v \neq \mathbf{0}$ in R.

EXAMPLE 10.1.3. For the parametrized cone

$$\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, u),$$

we have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 1) \qquad \text{and} \qquad \mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u\sin v, u\cos v, 0).$$

Hence

$$\mathbf{t}_u \times \mathbf{t}_v = (\cos v, \sin v, 1) \times (-u\sin v, u\cos v, 0) = (-u\cos v, -u\sin v, u).$$

It follows that the parametrized cone is smooth everywhere except at $(0,0,0)$.

EXAMPLE 10.1.4. For the parametrized sphere

$$\Phi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin u\cos v, \sin u\sin v, \cos u),$$

we have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos u\cos v, \cos u\sin v, -\sin u)$$

and

$$\mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-\sin u\sin v, \sin u\cos v, 0).$$

Hence

$$\begin{aligned}\mathbf{t}_u \times \mathbf{t}_v &= (\cos u\cos v, \cos u\sin v, -\sin u) \times (-\sin u\sin v, \sin u\cos v, 0) \\ &= (\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u).\end{aligned}$$

It follows that the parametrized sphere is smooth everywhere except at $(0,0,\pm 1)$. A similar argument shows that the parametrized sphere $\Psi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\cos u, \sin u\cos v, \sin u\sin v)$ is smooth everywhere except at $(\pm 1,0,0)$. Note that both Φ and Ψ are parametrizations of the same unit sphere $x^2+y^2+z^2=1$.

EXAMPLE 10.1.5. For the parametrized surface $\Phi : [-1,1] \times [-1,1] \to \mathbb{R}^3 : (u,v) \mapsto (u,v,0)$, we have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (1,0,0) \qquad \text{and} \qquad \mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0,1,0).$$

Hence

$$\mathbf{t}_u \times \mathbf{t}_v = (1,0,0) \times (0,1,0) = (0,0,1).$$

It follows that the parametrized surface is smooth everywhere. Note that $\Phi([-1,1]\times[-1,1])$ is the square with vertices $(\pm 1,\pm 1,0)$. For the parametrized surface $\Psi : [-1,1] \times [-1,1] \to \mathbb{R}^3 : (u,v) \mapsto (u^3,v^3,0)$, we have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (3u^2,0,0) \qquad \text{and} \qquad \mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0,3v^2,0).$$

Hence

$$\mathbf{t}_u \times \mathbf{t}_v = (3u^2,0,0) \times (0,3v^2,0) = (0,0,9u^2v^2).$$

It follows that the parametrized surface is smooth everywhere except at points $\Psi(u,v)$ where $u=0$ or $v=0$. Note that $\Psi([-1,1]\times[-1,1])$ is also the square with vertices $(\pm 1,\pm 1,0)$.

REMARKS. (1) Examples 10.1.4 and 10.1.5 show that smoothness depends on the parametrization and not just the surface. Indeed, we say that a surface S is smooth if there exists a parametrization $\Phi : R \to \mathbb{R}^3$ which is smooth and such that $\Phi(R) = S$.

(2) Suppose that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is smooth at a point $(x_0, y_0, z_0) = \Phi(u_0, v_0)$. Then $\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v$, evaluated at (u_0, v_0), is a normal vector to the surface $S = \Phi(R)$ at (x_0, y_0, z_0). It follows that the equation of the tangent plane to S at (x_0, y_0, z_0) is given by $(x-x_0, y-y_0, z-z_0)\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is evaluated at (u_0, v_0).

EXAMPLE 10.1.6. Suppose that $f : [A, B] \times [C, D] \to \mathbb{R}$ is a differentiable function. Then its graph

$$\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D] \text{ and } z = f(x, y)\}$$

is the range of the function $\Phi : [A, B] \times [C, D] \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v))$. We have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \left(1, 0, \frac{\partial f}{\partial u}\right) \qquad \text{and} \qquad \mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(0, 1, \frac{\partial f}{\partial v}\right).$$

Hence

$$\mathbf{t}_u \times \mathbf{t}_v = \left(1, 0, \frac{\partial f}{\partial u}\right) \times \left(0, 1, \frac{\partial f}{\partial v}\right) = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) \neq \mathbf{0}.$$

It follows that the surface is smooth.

EXAMPLE 10.1.7. The hyperboloid of one sheet is given by the equation $x^2 + y^2 - z^2 = 1$. We can write

$$x = r\cos u \qquad \text{and} \qquad y = r\sin u,$$

so that $r^2 - z^2 = 1$. We can then write

$$r = \cosh v \qquad \text{and} \qquad z = \sinh v.$$

Hence we have the parametrization

$$x = \cos u \cosh v, \qquad y = \sin u \cosh v, \qquad z = \sinh v.$$

Consider now the function $\Phi(u, v) = (\cos u \cosh v, \sin u \cosh v, \sinh v)$. We have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (-\sin u \cosh v, \cos u \cosh v, 0)$$

and

$$\mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (\cos u \sinh v, \sin u \sinh v, \cosh v).$$

Hence

$$\begin{aligned} \mathbf{t}_u \times \mathbf{t}_v &= (-\sin u \cosh v, \cos u \cosh v, 0) \times (\cos u \sinh v, \sin u \sinh v, \cosh v) \\ &= (\cos u \cosh^2 v, \sin u \cosh^2 v, -\cosh v \sinh v) \neq \mathbf{0}. \end{aligned}$$

It follows that this surface is smooth.

10.2. Surface Area

For the remainder of these notes, we restrict our attention to piecewise smooth surfaces. These are finite unions of the ranges of parametrized surfaces of the type $\Phi_i : R_i \to \mathbb{R}^3$, where R_i is an elementary region in $\mathbb{R}^2$, Φ_i is continuously differentiable and one-to-one, except possibly on the boundary of R_i, and $S_i = \Phi(R_i)$ is smooth, except possibly at a finite number of points.

For a discussion of elementary regions in $\mathbb{R}^2$, the reader is referred to Section 5.4.

Definition. Suppose that $\Phi : R \to \mathbb{R}^3$ is a continuously differentiable parametrized surface. Then the quantity

$$\mathcal{A} = \iint_R \|\mathbf{t}_u \times \mathbf{t}_v\| \, \mathrm{d}u\mathrm{d}v$$

is called the surface area of the parametrized surface Φ.

Remarks. (1) Note that

$$\begin{aligned}
\mathbf{t}_u \times \mathbf{t}_v &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \\
&= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) \\
&= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right),
\end{aligned}$$

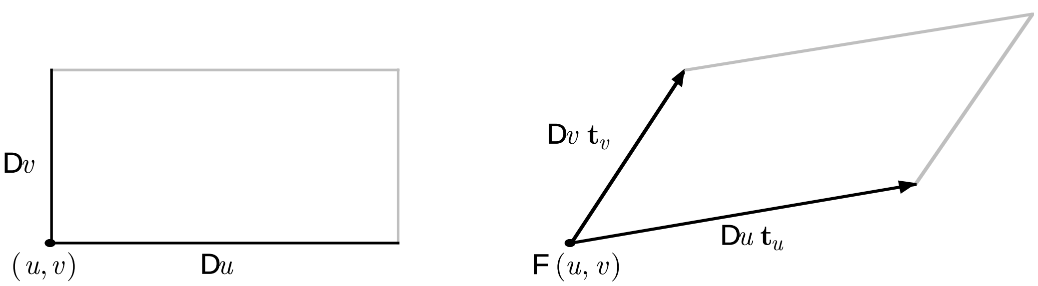
and so

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{1/2} = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{1/2}.$$

Hence

$$\mathcal{A} = \iint_R \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{1/2} \mathrm{d}u\mathrm{d}v.$$

(2) To justify the definition, consider a small rectangle in R with bottom left vertex (u, v) and top right vertex $(u + \Delta u, v + \Delta v)$.



The image under Φ of this rectangle can be approximated by a parallelogram in $\mathbb{R}^3$, with area

$$\|\Delta u \mathbf{t}_u \times \Delta v \mathbf{t}_v\| = \|\mathbf{t}_u \times \mathbf{t}_v\| \, \Delta u \Delta v.$$

Example 10.2.1. For the parametrized cone

$$\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, u),$$

we have

$$\mathbf{t}_u \times \mathbf{t}_v = (-u\cos v, -u\sin v, u),$$

so that

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \big(u^2\cos^2 v + u^2\sin^2 v + u^2\big)^{1/2} = \sqrt{2}u.$$

Alternatively, we have

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 \right)^{1/2} = \big(u^2 + u^2\sin^2 v + u^2\cos^2 v\big)^{1/2} = \sqrt{2}u.$$

Hence the surface area is given by

$$\mathcal{A} = \int_0^{2\pi} \left(\int_0^1 \sqrt{2}u \, \mathrm{d}u \right) \mathrm{d}v = \pi\sqrt{2}.$$

Example 10.2.2. For the parametrized sphere

$$\Phi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin u\cos v, \sin u\sin v, \cos u),$$

we have

$$\mathbf{t}_u \times \mathbf{t}_v = (\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u),$$

so that

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \big(\sin^4 u\cos^2 v + \sin^4 u\sin^2 v + \cos^2 u\sin^2 u\big)^{1/2} = |\sin u|.$$

Alternatively, we have

$$\begin{aligned}\|\mathbf{t}_u \times \mathbf{t}_v\| &= \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 \right)^{1/2} \\ &= \big(\cos^2 u\sin^2 u + \sin^4 u\sin^2 v + \sin^4 u\cos^2 v\big)^{1/2} = |\sin u|.\end{aligned}$$

Hence the surface area is given by

$$\mathcal{A} = \int_0^{2\pi} \left(\int_0^{\pi} |\sin u| \, \mathrm{d}u \right) \mathrm{d}v = 2\int_0^{2\pi} \left(\int_0^{\pi/2} \sin u \, \mathrm{d}u \right) \mathrm{d}v = 4\pi.$$

Example 10.2.3. For the helicoid

$$\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, v),$$

we have

$$\mathbf{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 0) \qquad \text{and} \qquad \mathbf{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u\sin v, u\cos v, 1).$$

Hence

$$\mathbf{t}_u \times \mathbf{t}_v = (\cos v, \sin v, 0) \times (-u \sin v, u \cos v, 1) = (\sin v, -\cos v, u),$$

so that

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = (\sin^2 v + \cos^2 v + u^2)^{1/2} = (1 + u^2)^{1/2}.$$

Alternatively, we have

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \left(\left| \frac{\partial(x,y)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(x,z)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(y,z)}{\partial(u,v)} \right|^2 \right)^{1/2} = (u^2 + \cos^2 v + \sin^2 v)^{1/2} = (u^2+1)^{1/2}.$$

Hence the surface area is given by

$$\mathcal{A} = \int_0^{2\pi} \left(\int_0^1 (1+u^2)^{1/2} \, \mathrm{d}u \right) \mathrm{d}v = \pi(\sqrt{2} + \log(1+\sqrt{2})).$$

Example 10.2.4. Suppose that $f : R \to \mathbb{R}$ is a continuously differentiable function, where $R \subseteq \mathbb{R}^2$ is an elementary region. Then its graph

$$\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } z = f(x,y)\}$$

is the range of the function $\Phi : R \to \mathbb{R}^3 : (u,v) \mapsto (u, v, f(u,v))$, and

$$\mathbf{t}_u \times \mathbf{t}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right).$$

Hence the surface area of the graph is

$$\mathcal{A} = \iint_R \left(\left(\frac{\partial f}{\partial u} \right)^2 + \left(\frac{\partial f}{\partial v} \right)^2 + 1 \right)^{1/2} \mathrm{d}u\mathrm{d}v.$$

Problems for Chapter 10

1. For each of the following parametrized surfaces, find a unit vector normal to the surface at a point $\Phi(u,v)$:
   a) $\Phi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (3 \sin u \cos v, 2 \sin u \sin v, \cos u)$
   b) $\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin v, u, \cos v)$
   c) $\Phi : [-\pi,\pi] \times [-\pi,\pi] \to \mathbb{R}^3 : (u,v) \mapsto ((2-\cos u)\sin v, (2-\cos u)\cos v, \sin u)$
   d) $\Phi : [0,1] \times [0,1] \to \mathbb{R}^3 : (u,v) \mapsto (u,v,v)$

2. For each of the following parametrized surfaces, find the equation of the tangent plane, if it exists, at the point given, and determine also whether the surface is smooth:
   a) $\Phi : [-1,1] \times [0,2] \to \mathbb{R}^3 : (u,v) \mapsto (u^2+v, v^2, 2u)$; at $\Phi(0,1)$
   b) $\Phi : [-1,1] \times [0,2] \to \mathbb{R}^3 : (u,v) \mapsto (u^2-v^2, u+v, u^2+4v)$; at $\Phi(0,1)$
   c) $\Phi : [0,2] \times [-\pi,\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u^2 \cos v, u^2 \sin v, u)$; at $\Phi(1,0)$

3. Consider the hyperboloid $x^2 - y^2 + z^2 = 9$.
   a) Find a parametrization of the hyperboloid.
   b) Use your parametrization in part (a) to find a unit normal to the surface.
   c) Find the equation of the tangent plane to the surface at a point $(x_0, 0, z_0)$, where $x_0^2 + z_0^2 = 9$.
   d) Show that the lines $(x_0, 0, z_0) + \lambda(-z_0, 3, x_0)$ and $(x_0, 0, z_0) + \lambda(z_0, 3, -x_0)$ are part of the surface as well as the tangent plane in part (c).

4. Consider a parametrized surface $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, smooth at a point $\Phi(u_0, v_0)$.
   a) Show that the linear approximation
   $$(u, v) \mapsto \Phi(u_0, v_0) + (\mathbf{D}\Phi)(u_0, v_0) \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}$$
   represents a plane through the point $\Phi(u_0, v_0)$.
   b) Show that the plane in part (a) is the tangent plane to the surface at $\Phi(u_0, v_0)$.

5. Consider the paraboloid parametrized by $\Phi : [0, 2] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, u^2)$.
   a) Find an equation in x, y, z describing the surface.
   b) Find a unit vector orthogonal to the surface at $\Phi(u, v)$.
   c) Find the surface area.

6. Let $R \subseteq \mathbb{R}^2$ be the unit disc with centre $(0, 0)$ and radius 1. Find the area of the parametrized surface $\Phi : R \to \mathbb{R}^3 : (u, v) \mapsto (u - v, u + v, uv)$.

7. Find the area of the parametrized surface $\Phi : [0, 1] \times [0, 2\pi] : (u, v) \mapsto (u \cos v, 2u \cos v, u)$.

8. Find a parametrization of the ellipsoid
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$
   and express the surface area of the (parametrized) ellipsoid as an integral.

9. Suppose that $a > 0$ is fixed. By using a suitable parametrization, find the surface area of the part of the cylinder $x^2 + z^2 = a^2$ that is inside the cylinder $x^2 + y^2 = 2ay$ and also in the first octant $x, y, z \geq 0$.

10. Suppose that $g : [A, B] \to \mathbb{R}$ is a continuous function such that $g(x) \geq 0$ for every $x \in [A, B]$. Now rotate the graph $\{(x, g(x)) : x \in [A, B]\}$ about the x-axis. Follow the steps below to calculate the surface area generated by this rotation.
    a) Convince yourself that the function $\Phi : [A, B] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (u, g(u) \cos v, g(u) \sin v)$ is a parametrization of the surface of revolution.
    b) Show that the surface area of Φ is given by $\mathcal{A} = 2\pi \displaystyle\int_A^B g(u)(1 + |g'(u)|^2)^{1/2} \, \mathrm{d}u$.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 11

INTEGRALS OVER SURFACES

11.1. Integrals of Scalar Functions over Parametrized Surfaces

Suppose that the parametrized surface

$$\Phi : R \to \mathbb{R}^3 : (u,v) \mapsto \Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

is continuously differentiable. For any real valued function $f(x,y,z)$ such that the composition function

$$f \circ \Phi : R \to \mathbb{R} : (u,v) \mapsto f(x(u,v), y(u,v), z(u,v))$$

is continuous, we define

$$\int_\Phi f \,\mathrm{d}S = \int_\Phi f(x,y,z)\,\mathrm{d}S = \iint_R f(\Phi(u,v))\|\mathbf{t}_u \times \mathbf{t}_v\|\,\mathrm{d}u\mathrm{d}v.$$

Remarks. (1) Here $\mathrm{d}S = \|\mathbf{t}_u \times \mathbf{t}_v\|\,\mathrm{d}u\mathrm{d}v$ can be considered to be the surface area differential of the parametrized surface Φ.

(2) Clearly

$$\int_\Phi f\,\mathrm{d}S = \iint_R f(x(u,v), y(u,v), z(u,v)) \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 \right)^{1/2} \mathrm{d}u\mathrm{d}v.$$

(3) Suppose that $f = 1$ identically. Then the integral simply represents the surface area of Φ.

(4) Note that f has only to be defined on the image surface $S = \Phi(R)$ of the parametrized surface Φ for our definition to make sense. The continuity of the composition function $f \circ \Phi$ on the elementary region R ensures the existence of the integral.

(5) Sometimes, Φ may only be piecewise continuously differentiable; in other words, there exists a partition of the region R into a finite union of elementary regions R_i, where $i = 1, \ldots, k$, such that Φ is continuously differentiable in R_i for each $i = 1, \ldots, k$. In this case, we define

$$\int_\Phi f \, \mathrm{d}S = \sum_{i=1}^{k} \int_{\Phi_i} f(\Phi(u,v)) \|\mathbf{t}_u \times \mathbf{t}_v\| \, \mathrm{d}u\mathrm{d}v,$$

where $\Phi_i : R_i \to \mathbb{R}^3 : (u,v) \mapsto \Phi(u,v)$. In other words, we calculate the corresponding integral for each subregion and consider the sum of the integrals.

Example 11.1.1. For the parametrized cone $\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, u)$, and for the function $f(x,y,z) = x^2 + y^2 + z^2 + y$, we have, writing $R = [0,1] \times [0,2\pi]$ and noting Example 10.2.1,

$$\begin{aligned}
\int_\Phi f \, \mathrm{d}S &= \iint_R f(u\cos v, u\sin v, u) \|(-u\cos v, -u\sin v, u)\| \, \mathrm{d}u\mathrm{d}v \\
&= \iint_R (u^2\cos^2 v + u^2\sin^2 v + u^2 + u\sin v)\sqrt{2}u \, \mathrm{d}u\mathrm{d}v \\
&= \iint_R (2u^2 + u\sin v)\sqrt{2}u \, \mathrm{d}u\mathrm{d}v = 2\sqrt{2}\iint_R u^3 \, \mathrm{d}u\mathrm{d}v + \sqrt{2}\iint_R u^2 \sin v \, \mathrm{d}u\mathrm{d}v \\
&= 2\sqrt{2}\left(\int_0^1 u^3 \, \mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) + \sqrt{2}\left(\int_0^1 u^2 \, \mathrm{d}u\right)\left(\int_0^{2\pi} \sin v \, \mathrm{d}v\right) = \pi\sqrt{2}.
\end{aligned}$$

Example 11.1.2. For the helicoid $\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, v)$, and for the function $f(x,y,z) = (1+x^2+y^2)^{1/2} + z$, we have, writing $R = [0,1] \times [0,2\pi]$ and noting Example 10.2.3,

$$\begin{aligned}
\int_\Phi f \, \mathrm{d}S &= \iint_R f(u\cos v, u\sin v, v) \|(\sin v, -\cos v, u)\| \, \mathrm{d}u\mathrm{d}v \\
&= \iint_R ((1+u^2)^{1/2} + v)(1+u^2)^{1/2} \, \mathrm{d}u\mathrm{d}v = \iint_R (1+u^2) \, \mathrm{d}u\mathrm{d}v + \iint_R (1+u^2)^{1/2} v \, \mathrm{d}u\mathrm{d}v \\
&= \left(\int_0^1 (1+u^2) \, \mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) + \left(\int_0^1 (1+u^2)^{1/2} \, \mathrm{d}u\right)\left(\int_0^{2\pi} v \, \mathrm{d}v\right) \\
&= \frac{8\pi}{3} + \pi^2(\sqrt{2} + \log(1+\sqrt{2})).
\end{aligned}$$

Example 11.1.3. The three distinct parametrized surfaces

$$\begin{aligned}
\Phi &: [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u) \\
\Psi &: [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin u \sin v, \sin u \cos v, \cos u) \\
\Lambda &: [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\cos u, \sin u \sin v, \sin u \cos v)
\end{aligned}$$

satisfy $\Phi([0,\pi] \times [0,2\pi]) = \Psi([0,\pi] \times [0,2\pi]) = \Lambda([0,\pi] \times [0,2\pi]) = S$, the unit sphere in $\mathbb{R}^3$. We have shown earlier that for the parametrized surface Φ, we have

$$\mathbf{t}_u \times \mathbf{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u).$$

Similarly, it can be shown that for the paremetrized surfaces Ψ and Λ, we have respectively

$$\mathbf{t}_u \times \mathbf{t}_v = (-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u)$$

and

$$\mathbf{t}_u \times \mathbf{t}_v = (-\cos u \sin u, -\sin^2 u \sin v, -\sin^2 u \cos v).$$

Now consider the function $f(x, y, z) = z^2$. We have, writing $R = [0, 1] \times [0, 2\pi]$ and noting Example 10.2.2,

$$\begin{aligned}\int_\Phi f \,\mathrm{d}S &= \iint_R f(\sin u \cos v, \sin u \sin v, \cos u)\|(\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)\| \,\mathrm{d}u\mathrm{d}v \\ &= \iint_R \cos^2 u |\sin u| \,\mathrm{d}u\mathrm{d}v = \left(\int_0^\pi \cos^2 u \sin u \,\mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) = \frac{4\pi}{3},\end{aligned}$$

as well as

$$\begin{aligned}\int_\Psi f \,\mathrm{d}S &= \iint_R f(\sin u \sin v, \sin u \cos v, \cos u)\|(-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u)\| \,\mathrm{d}u\mathrm{d}v \\ &= \iint_R \cos^2 u |\sin u| \,\mathrm{d}u\mathrm{d}v = \left(\int_0^\pi \cos^2 u \sin u \,\mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) = \frac{4\pi}{3},\end{aligned}$$

and

$$\begin{aligned}\int_\Lambda f \,\mathrm{d}S &= \iint_R f(\cos u, \sin u \sin v, \sin u \cos v)\|(-\cos u \sin u, -\sin^2 u \sin v, -\sin^2 u \cos v)\| \,\mathrm{d}u\mathrm{d}v \\ &= \iint_R \sin^2 u \cos^2 v |\sin u| \,\mathrm{d}u\mathrm{d}v = \left(\int_0^\pi \sin^3 u \,\mathrm{d}u\right)\left(\int_0^{2\pi} \cos^2 v \,\mathrm{d}v\right) = \frac{4\pi}{3}.\end{aligned}$$

Note that the three integrals have the same value. We shall discuss this in greater detail in Section 11.3.

11.2. Surface Integrals

Suppose that the parametrized surface

$$\Phi : R \to \mathbb{R}^3 : (u, v) \mapsto \Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

is continuously differentiable. For any vector field $F(x, y, z)$ such that the composition function

$$F \circ \Phi : R \to \mathbb{R}^3 : (u, v) \mapsto F(x(u, v), y(u, v), z(u, v))$$

is continuous, we define

$$\int_\Phi F \cdot \mathrm{d}\mathbf{S} = \int_\Phi F(x, y, z) \cdot \mathrm{d}\mathbf{S} = \iint_R F(\Phi(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v) \,\mathrm{d}u\mathrm{d}v.$$

Remarks. (1) Here $\mathrm{d}\mathbf{S} = (\mathbf{t}_u \times \mathbf{t}_v)\,\mathrm{d}u\mathrm{d}v$ is the parametrized surface analogue of the velocity differential $\mathrm{d}\mathbf{s} = \phi'(t)\,\mathrm{d}t$ of a path.

(2) Clearly

$$\int_\Phi F \cdot \mathrm{d}\mathbf{S} = \iint_R F(x(u, v), y(u, v), z(u, v)) \cdot \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)}\right) \mathrm{d}u\mathrm{d}v.$$

(3) Note that F has only to be defined on the image surface $S = \Phi(R)$ of the parametrized surface Φ for our definition to make sense. The continuity of the composition function $F \circ \Phi$ on the elementary region R ensures the existence of the integral.

(4) Sometimes, Φ may only be piecewise continuously differentiable. As in the last section, we can calculate the corresponding integral for each subregion in a partition of the region R and consider the sum of the integrals.

(5) Note that if $\mathbf{t}_u \times \mathbf{t}_v \neq \mathbf{0}$ for any $(u,v) \in R$, then

$$\int_\Phi F \cdot \mathrm{d}\mathbf{S} = \iint_R \left(F(\Phi(u,v)) \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|} \right) \|\mathbf{t}_u \times \mathbf{t}_v\| \,\mathrm{d}u\mathrm{d}v = \iint_R f(\Phi(u,v)) \|\mathbf{t}_u \times \mathbf{t}_v\| \,\mathrm{d}u\mathrm{d}v,$$

where

$$f(\Phi(u,v)) = F(\Phi(u,v)) \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}.$$

Here

$$\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}$$

denotes a unit vector normal to the parametrized surface Φ. Then

$$\int_\Phi F \cdot \mathrm{d}\mathbf{S} = \int_\Phi F \cdot \mathbf{n} \,\mathrm{d}S,$$

and the integral now becomes one of the type discussed in the last section.

(6) Suppose that F is an electric field and $S = \Phi(R)$ is a closed surface like a sphere. Then the integral denotes the flux of F over S. Gauss's law states that this is equal to the net charge enclosed by the surface S.

Example 11.2.1. For the parametrized cone $\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, u)$, and for the vector field $F(x,y,z) = (-x,y,z)$, we have, writing $R = [0,1] \times [0,2\pi]$ and noting Example 10.2.1,

$$\begin{aligned}
\int_\Phi F \cdot \mathrm{d}\mathbf{S} &= \iint_R F(u\cos v, u\sin v, u) \cdot (-u\cos v, -u\sin v, u) \,\mathrm{d}u\mathrm{d}v \\
&= \iint_R (-u\cos v, u\sin v, u) \cdot (-u\cos v, -u\sin v, u) \,\mathrm{d}u\mathrm{d}v \\
&= \iint_R (u^2\cos^2 v - u^2\sin^2 v + u^2) \,\mathrm{d}u\mathrm{d}v = \iint_R u^2(1+\cos 2v) \,\mathrm{d}u\mathrm{d}v \\
&= \left(\int_0^1 u^2 \,\mathrm{d}u \right) \left(\int_0^{2\pi} (1+\cos 2v) \,\mathrm{d}v \right) = \frac{2\pi}{3}.
\end{aligned}$$

Example 11.2.2. For the helicoid $\Phi : [0,1] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (u\cos v, u\sin v, v)$, and for the vector field $F(x,y,z) = (x,y,z)$, we have, writing $R = [0,1] \times [0,2\pi]$ and noting Example 10.2.3,

$$\begin{aligned}
\int_\Phi F \cdot \mathrm{d}\mathbf{S} &= \iint_R F(u\cos v, u\sin v, v) \cdot (\sin v, -\cos v, u) \,\mathrm{d}u\mathrm{d}v \\
&= \iint_R (u\cos v, u\sin v, v) \cdot (\sin v, -\cos v, u) \,\mathrm{d}u\mathrm{d}v \\
&= \iint_R uv \,\mathrm{d}u\mathrm{d}v = \left(\int_0^1 u \,\mathrm{d}u \right) \left(\int_0^{2\pi} v \,\mathrm{d}v \right) = \pi^2.
\end{aligned}$$

EXAMPLE 11.2.3. The three distinct parametrized surfaces

$$\begin{aligned}
\Phi &: [0,\pi]\times[0,2\pi]\to\mathbb{R}^3 : (u,v)\mapsto(\sin u\cos v,\sin u\sin v,\cos u)\\
\Psi &: [0,\pi]\times[0,2\pi]\to\mathbb{R}^3 : (u,v)\mapsto(\sin u\sin v,\sin u\cos v,\cos u)\\
\Lambda &: [0,\pi]\times[0,2\pi]\to\mathbb{R}^3 : (u,v)\mapsto(\cos u,\sin u\sin v,\sin u\cos v)
\end{aligned}$$

satisfy $\Phi([0,\pi]\times[0,2\pi])=\Psi([0,\pi]\times[0,2\pi])=\Lambda([0,\pi]\times[0,2\pi])=S$, the unit sphere in $\mathbb{R}^3$. We have respectively

$$\begin{aligned}
\mathbf{t}_u\times\mathbf{t}_v&=(\sin^2u\cos v,\sin^2u\sin v,\cos u\sin u),\\
\mathbf{t}_u\times\mathbf{t}_v&=(-\sin^2u\sin v,-\sin^2u\cos v,-\cos u\sin u),\\
\mathbf{t}_u\times\mathbf{t}_v&=(-\cos u\sin u,-\sin^2u\sin v,-\sin^2u\cos v).
\end{aligned}$$

Now consider the vector field $F(x,y,z)=(xz,yz,z)$. We have, writing $R=[0,1]\times[0,2\pi]$,

$$\begin{aligned}
\int_\Phi F\cdot\mathrm{d}\mathbf{S}&=\iint_R F(\sin u\cos v,\sin u\sin v,\cos u)\cdot(\sin^2u\cos v,\sin^2u\sin v,\cos u\sin u)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(\sin u\cos v\cos u,\sin u\sin v\cos u,\cos u)\cdot(\sin^2u\cos v,\sin^2u\sin v,\cos u\sin u)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(\sin^3u\cos u+\cos^2u\sin u)\,\mathrm{d}u\mathrm{d}v=\left(\int_0^\pi(\sin^3u\cos u+\cos^2u\sin u)\,\mathrm{d}u\right)\left(\int_0^{2\pi}\mathrm{d}v\right)=\frac{4\pi}{3},
\end{aligned}$$

as well as

$$\begin{aligned}
\int_\Psi F\cdot\mathrm{d}\mathbf{S}&=\iint_R F(\sin u\sin v,\sin u\cos v,\cos u)\cdot(-\sin^2u\sin v,-\sin^2u\cos v,-\cos u\sin u)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(\sin u\sin v\cos u,\sin u\cos v\cos u,\cos u)\cdot(-\sin^2u\sin v,-\sin^2u\cos v,-\cos u\sin u)\,\mathrm{d}u\mathrm{d}v\\
&=-\iint_R(\sin^3u\cos u+\cos^2u\sin u)\,\mathrm{d}u\mathrm{d}v=-\frac{4\pi}{3},
\end{aligned}$$

and

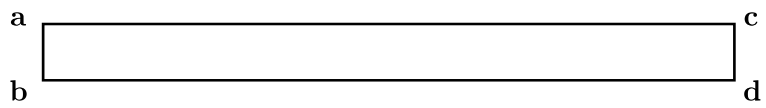
$$\begin{aligned}
\int_\Lambda F\cdot\mathrm{d}\mathbf{S}&=\iint_R F(\cos u,\sin u\sin v,\sin u\cos v)\cdot(-\cos u\sin u,-\sin^2u\sin v,-\sin^2u\cos v)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(\cos u\sin u\cos v,\sin^2u\sin v\cos v,\sin u\cos v)\cdot(-\cos u\sin u,-\sin^2u\sin v,-\sin^2u\cos v)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(-\cos^2u\sin^2u\cos v-\sin^4u\sin^2v\cos v-\sin^3u\cos^2v)\,\mathrm{d}u\mathrm{d}v\\
&=-\left(\int_0^\pi\cos^2u\sin^2u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\cos v\,\mathrm{d}v\right)-\left(\int_0^\pi\sin^4u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\sin^2v\cos v\,\mathrm{d}v\right)\\
&\quad-\left(\int_0^\pi\sin^3u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\cos^2v\,\mathrm{d}v\right)\\
&=-\frac{4\pi}{3}.
\end{aligned}$$

Note that the three integrals differ only in sign. We shall discuss this in greater detail in Section 11.3.

11.3. Equivalent Parametrized Surfaces

Just as most curves have two endpoints, many surfaces have two sides. For our discussion here, we shall ignore surfaces like the Möbius strip, and consider only those surfaces in $\mathbb{R}^3$ which have two sides.

Remark. The Möbius strip has only one side. To construct it, take a long rectangular strip of paper as shown below.



Hold the edge $\overline{\mathbf{ab}}$ stationary and give the edge $\overline{\mathbf{cd}}$ a 180° twist. Now join the edges $\overline{\mathbf{ab}}$ and $\overline{\mathbf{cd}}$ so that $\mathbf{a}$ coincides with $\mathbf{d}$ and $\mathbf{b}$ coincides with $\mathbf{c}$, and admire your artwork.

Let us return to Examples 11.1.3 and 11.2.3. Here the unit sphere S is the range of the parametrized surfaces. The surface of the unit sphere clearly has two sides, the inside and the outside. For the parametrized surface Φ, we have

$$\mathbf{t}_u \times \mathbf{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u) = (\sin u)\Phi(u,v).$$

Note that $\sin u \geq 0$, and so the vector $\mathbf{t}_u \times \mathbf{t}_v$ at $\Phi(u,v)$ points away from the origin. For the parametrized surface Ψ, we have

$$\mathbf{t}_u \times \mathbf{t}_v = (-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u) = -(\sin u)\Psi(u,v).$$

Hence the vector $\mathbf{t}_u \times \mathbf{t}_v$ at $\Psi(u,v)$ points towards the origin.

Definition. Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_2 \to \mathbb{R}^3$ are continuously differentiable parametrized surfaces. Then we say that Φ and Ψ are equivalent if there exists a piecewise continuously differentiable function $h : R_1 \to R_2$ satisfying the following conditions:
(ES1) $h : R_1 \to R_2$ is essentially one-to-one and onto.
(ES2) $\Phi = \Psi \circ h$.
(ES3) Writing $(s,t) = h(u,v)$, we have either

(1) $$\frac{\partial(s,t)}{\partial(u,v)} \geq 0 \qquad \text{for every } (u,v) \in R_1,$$

or

(2) $$\frac{\partial(s,t)}{\partial(u,v)} \leq 0 \qquad \text{for every } (u,v) \in R_1.$$

In this case, we say that h defines a change of paremeters. Furthermore, we say that the change of parameters is orientation preserving if (1) holds and orientation reversing if (2) holds.

Remark. The condition (ES1) is essential for integration of double integrals by change of variables. For more details, see Chapter 6. The condition (ES3) will be clear from the sketched proofs of Theorems 11A and 11B later in this section.

Example 11.3.1. The parametrized surfaces

$$\Phi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (u,v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$$
$$\Psi : [0,\pi] \times [0,2\pi] \to \mathbb{R}^3 : (s,t) \mapsto (\sin s \sin t, \sin s \cos t, \cos s)$$

are equivalent. To see this, consider the function $h : [0,\pi] \times [0,2\pi] \to [0,\pi] \times [0,2\pi] : (u,v) \mapsto (s,t)$, where $s = u$ and

$$t = \begin{cases} \frac{\pi}{2} - v & \text{if } v \leq \frac{\pi}{2}, \\ \frac{5\pi}{2} - v & \text{if } v > \frac{\pi}{2}. \end{cases}$$

Clearly h is essentially one-to-one and onto. Furthermore,

$$\frac{\partial(s,t)}{\partial(u,v)} = \det\begin{pmatrix} \dfrac{\partial s}{\partial u} & \dfrac{\partial s}{\partial v} \\ \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial v} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1,$$

so that the change of parameters is orientation reversing.

THEOREM 11A. *Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_2 \to \mathbb{R}^3$ are two equivalent smooth continuously differentiable parametrized surfaces. Then for any real valued function $f(x,y,z)$ such that the composition functions $f \circ \Phi : R_1 \to \mathbb{R}$ and $f \circ \Psi : R_2 \to \mathbb{R}$ are continuous, we have*

$$\int_\Phi f\,\mathrm{d}S = \int_\Psi f\,\mathrm{d}S.$$

SKETCH OF PROOF. Since Φ and Ψ are equivalent, there exists $h : R_1 \to R_2$ such that $\Phi = \Psi \circ h$. Now

$$\int_\Phi f\,\mathrm{d}S = \iint_{R_1} f(\Phi(u,v)) \left\| \left(\frac{\partial(\Phi_2,\Phi_3)}{\partial(u,v)}, \frac{\partial(\Phi_3,\Phi_1)}{\partial(u,v)}, \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} \right) \right\| \,\mathrm{d}u\mathrm{d}v.$$

By the Chain rule and writing $(s,t) = h(u,v)$, we have

$$\frac{\partial(\Phi_i,\Phi_j)}{\partial(u,v)} = \frac{\partial(\Psi_i,\Psi_j)}{\partial(s,t)} \frac{\partial(s,t)}{\partial(u,v)}.$$

It follows that

$$\begin{aligned} \int_\Phi f\,\mathrm{d}S &= \iint_{R_1} f(\Psi(s,t)) \left\| \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \right\| \left| \frac{\partial(s,t)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v \\ &= \iint_{R_2} f(\Psi(s,t)) \left\| \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \right\| \mathrm{d}s\mathrm{d}t = \int_\Psi f\,\mathrm{d}S. \end{aligned}$$

This completes the proof. ◯

THEOREM 11B. *Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_2 \to \mathbb{R}^3$ are two equivalent smooth continuously differentiable parametrized surfaces. Then for any vector field $F(x,y,z)$ such that the composition functions $F \circ \Phi : R_1 \to \mathbb{R}^3$ and $F \circ \Psi : R_2 \to \mathbb{R}^3$ are continuous, we have*

$$\int_\Phi F \cdot \mathrm{d}\mathbf{S} = \pm \int_\Psi F \cdot \mathrm{d}\mathbf{S},$$

where the equality holds with the $+$ sign if the change of parameters is orientation preserving and with the $-$ sign if the change of parameters is orientation reversing.

SKETCH OF PROOF. Since Φ and Ψ are equivalent, there exists $h : R_1 \to R_2$ such that $\Phi = \Psi \circ h$. Now

$$\begin{aligned} \int_\Phi F \cdot \mathrm{d}\mathbf{S} &= \iint_{R_1} F(\Phi(u,v)) \cdot \left(\frac{\partial(\Phi_2,\Phi_3)}{\partial(u,v)}, \frac{\partial(\Phi_3,\Phi_1)}{\partial(u,v)}, \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} \right) \mathrm{d}u\mathrm{d}v \\ &= \iint_{R_1} F(\Psi(s,t)) \cdot \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \frac{\partial(s,t)}{\partial(u,v)} \mathrm{d}u\mathrm{d}v \\ &= \pm \iint_{R_1} F(\Psi(s,t)) \cdot \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \left| \frac{\partial(s,t)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v \\ &= \pm \iint_{R_2} F(\Psi(s,t)) \cdot \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \mathrm{d}s\mathrm{d}t = \pm \int_\Psi F \cdot \mathrm{d}\mathbf{S}. \end{aligned}$$

This completes the proof. ◯

REMARK. Theorems 11A and 11B have natural extensions to the case when the parametrized surfaces are piecewise continuously differentiable. In this case, one can partition the parametrized surfaces into continuously differentiable pieces and apply Theorems 11A and 11B to each piece.

11.4. Parametrization of Surfaces

As discussed at the beginning of the last section, we shall restrict our attention to surfaces in $\mathbb{R}^3$ which have two sides and are smooth, except possibly at a finite number of points. Our first task is to define an orientation for such surfaces.

Suppose that $\mathbf{x}$ is a point on a smooth surface S. Then if $\mathbf{n}$ is a unit vector normal to the surface S at $\mathbf{x}$, then $-\mathbf{n}$ is also a unit vector normal to the surface S at $\mathbf{x}$, but in the opposite direction. We now need to make a choice as to which side of the surface we consider to be the positive side and which side we consider to be the negative side. Having made such a choice, we now take unit normal vectors $\mathbf{n}$ to be those that point from the negative side of the surface towards the positive side. In this case, we say that S is an oriented surface.

EXAMPLE 11.4.1. Suppose that S is the unit sphere in $\mathbb{R}^3$, and we choose the outside surface to be the positive side. Then unit normal vectors point away from the origin.

EXAMPLE 11.4.2. Suppose that S is the xy-plane in $\mathbb{R}^3$, and we choose the bottom surface to be the positive side. Then unit normal vectors are of the form $(0, 0, -1)$.

Recall now that for any smooth parametrized surface $\Phi : R \to \mathbb{R}^3$, the unit vector

$$\frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|}$$

is normal to the surface at the point $\Phi(u, v)$.

DEFINITION. Suppose that S is an oriented surface in $\mathbb{R}^3$. Then a piecewise continuously differentiable function $\Phi : R \to \mathbb{R}^3$ such that $\Phi(R) = S$ is called a parametrization of S. We say that the parametrization Φ is orientation preserving if

$$\frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|} = \mathbf{n}$$

at every point $\Phi(u, v)$ which is smooth, and orientation reversing if

$$\frac{\mathbf{t}_u \times \mathbf{t}_v}{\|\mathbf{t}_u \times \mathbf{t}_v\|} = -\mathbf{n}$$

at every point $\Phi(u, v)$ which is smooth.

Suppose that S is an oriented surface in $\mathbb{R}^3$. For any real valued function $f(x, y, z)$ continuous on S, we can define

$$\int_S f \, \mathrm{d}S = \int_\Phi f \, \mathrm{d}S,$$

where Φ is any parametrization of S. For any vector field $F(x, y, z)$ continuous on S, we can define

$$\int_S F \cdot \mathrm{d}\mathbf{S} = \int_\Phi F \cdot \mathrm{d}\mathbf{S},$$

where Φ is any orientation preserving parametrization of S.

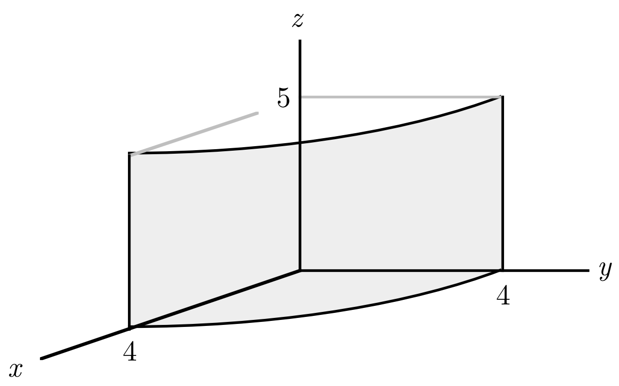
EXAMPLE 11.4.3. Suppose that S denotes the unit sphere $x^2+y^2+z^2=1$. Let $f(x,y,z)=x+y+z$, and consider the integral

$$\int_S f\,\mathrm{d}S.$$

Now $\Phi:[0,\pi]\times[0,2\pi]\to\mathbb{R}^3:(u,v)\mapsto(\sin u\cos v,\sin u\sin v,\cos u)$ is a parametrization of S. We have, writing writing $R=[0,1]\times[0,2\pi]$,

$$\begin{aligned}
\int_S f\,\mathrm{d}S=\int_\Phi f\,\mathrm{d}S&=\iint_R f(\sin u\cos v,\sin u\sin v,\cos u)\|(\sin^2 u\cos v,\sin^2 u\sin v,\cos u\sin u)\|\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(\sin u\cos v+\sin u\sin v+\cos u)|\sin u|\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R\sin u(\cos v+\sin v)|\sin u|\,\mathrm{d}u\mathrm{d}v+\iint_R\cos u|\sin u|\,\mathrm{d}u\mathrm{d}v\\
&=\left(\int_0^\pi\sin^2 u\,\mathrm{d}u\right)\left(\int_0^{2\pi}(\cos v+\sin v)\,\mathrm{d}v\right)+\left(\int_0^\pi\sin u\cos u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\mathrm{d}v\right)=0.
\end{aligned}$$

EXAMPLE 11.4.4. Let S denote the part of the cylinder $x^2+y^2=16$ in the first octant between $z=0$ and $z=5$, with normal vector away from the z-axis.



Note that S can be parametrized by

$$\Phi:R\to\mathbb{R}^3:(u,v)\mapsto(4\sin u,4\cos u,v),$$

where $R=[0,\pi/2]\times[0,5]$. Let $F(x,y,z)=(z,x,-3y^2z)$, and consider the integral

$$\int_S F\cdot\mathrm{d}\mathbf{S}.$$

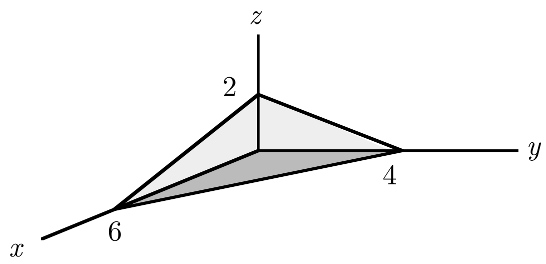
Note that

$$\mathbf{t}_u\times\mathbf{t}_v=\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right)\times\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right)=(4\cos u,-4\sin u,0)\times(0,0,1)=(-4\sin u,-4\cos u,0),$$

so Φ is an orientation reversing parametrization of S. We have

$$\begin{aligned}
\int_S F\cdot\mathrm{d}\mathbf{S}=-\int_\Phi F\cdot\mathrm{d}\mathbf{S}&=-\iint_R F(4\sin u,4\cos u,v)\cdot(-4\sin u,-4\cos u,0)\,\mathrm{d}u\mathrm{d}v\\
&=-\iint_R(v,4\sin u,-48v\cos^2 u)\cdot(-4\sin u,-4\cos u,0)\,\mathrm{d}u\mathrm{d}v\\
&=\iint_R(4v\sin u+16\sin u\cos u)\,\mathrm{d}u\mathrm{d}v=4\iint_R v\sin u\,\mathrm{d}u\mathrm{d}v+16\iint_R\sin u\cos u\,\mathrm{d}u\mathrm{d}v\\
&=4\left(\int_0^{\pi/2}\sin u\,\mathrm{d}u\right)\left(\int_0^5 v\,\mathrm{d}v\right)+16\left(\int_0^{\pi/2}\sin u\cos u\,\mathrm{d}u\right)\left(\int_0^5\mathrm{d}v\right)=90.
\end{aligned}$$

Example 11.4.5. Let S denote the part of the plane $2x+3y+6z=12$ in the first octant, with normal vector away from the origin.



Note that S is a triangle with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$, and can be parametrized by

$$\Phi : R \to \mathbb{R}^3 : (u,v) \mapsto \left(u, v, \frac{12-2u-3v}{6}\right),$$

where R is the triangular region

$$R = \{(u,v) \in \mathbb{R}^2 : u,v \geq 0 \text{ and } 2u+3v \leq 12\}.$$

Let $F(x,y,z) = (18z, -12, 3y)$, and consider the integral

$$\int_S F \cdot \mathrm{d}\mathbf{S}.$$

Note that

$$\mathbf{t}_u \times \mathbf{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(1, 0, -\frac{1}{3}\right) \times \left(0, 1, -\frac{1}{2}\right) = \left(\frac{1}{3}, \frac{1}{2}, 1\right),$$

so Φ is an orientation preserving parametrization of S. We have

$$\begin{aligned}
\int_S F \cdot \mathrm{d}\mathbf{S} = \int_\Phi F \cdot \mathrm{d}\mathbf{S} &= \iint_R F\left(u, v, \frac{12-2u-3v}{6}\right) \cdot \left(\frac{1}{3}, \frac{1}{2}, 1\right) \mathrm{d}u\mathrm{d}v \\
&= \iint_R (36-6u-9v, -12, 3v) \cdot \left(\frac{1}{3}, \frac{1}{2}, 1\right) \mathrm{d}u\mathrm{d}v = \iint_R (6-2u)\,\mathrm{d}u\mathrm{d}v \\
&= \int_0^4 \left(\int_0^{6-\frac{3}{2}v} (6-2u)\,\mathrm{d}u\right) \mathrm{d}v = \int_0^4 \left(9v - \frac{9}{4}v^2\right) \mathrm{d}v = 24.
\end{aligned}$$

Example 11.4.6. Let S denote the surface of the cube with vertices $(\pm1, \pm1, \pm1)$, with outward normal vector. Let $F(x,y,z) = (x,y,z)$, and consider the integral

$$\int_S F \cdot \mathrm{d}\mathbf{S}.$$

To evaluate this integral, consider first of all the face S_1 with vertices $(\pm1, \pm1, 1)$. The function

$$\Phi : [-1,1] \times [-1,1] \to \mathbb{R}^3 : (u,v) \mapsto (u,v,1)$$

is a parametrization of S_1. Note that

$$\mathbf{t}_u \times \mathbf{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (1,0,0) \times (0,1,0) = (0,0,1),$$

so Φ is an orientation preserving parametrization of S_1. We have, writing $R = [-1,1] \times [-1,1]$,

$$\int_{S_1} F \cdot \mathrm{d}\mathbf{S} = \int_{\Phi} F \cdot \mathrm{d}\mathbf{S} = \iint_R F(u,v,1) \cdot (0,0,1)\,\mathrm{d}u\mathrm{d}v$$
$$= \iint_R (u,v,1) \cdot (0,0,1)\,\mathrm{d}u\mathrm{d}v = \iint_R \mathrm{d}u\mathrm{d}v = 4.$$

Consider next the face S_2 with vertices $(\pm 1, \pm 1, -1)$. The function

$$\Psi : [-1,1] \times [-1,1] \to \mathbb{R}^3 : (u,v) \mapsto (u,v,-1)$$

is a parametrization of S_2. Note that

$$\mathbf{t}_u \times \mathbf{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (1,0,0) \times (0,1,0) = (0,0,1),$$

so Ψ is an orientation reversing parametrization of S_2. We have, writing $R = [-1,1] \times [-1,1]$,

$$\int_{S_2} F \cdot \mathrm{d}\mathbf{S} = -\int_{\Psi} F \cdot \mathrm{d}\mathbf{S} = -\iint_R F(u,v,-1) \cdot (0,0,1)\,\mathrm{d}u\mathrm{d}v$$
$$= -\iint_R (u,v,-1) \cdot (0,0,1)\,\mathrm{d}u\mathrm{d}v = \iint_R \mathrm{d}u\mathrm{d}v = 4.$$

Hence

$$\int_{S_1} F \cdot \mathrm{d}\mathbf{S} + \int_{S_2} F \cdot \mathrm{d}\mathbf{S} = 8.$$

It follows from symmetry arguments that

$$\int_S F \cdot \mathrm{d}\mathbf{S} = 24.$$

Problems for Chapter 11

1. Evaluate the integral $\int_{\Phi} (x^2 + y^2 + z^2)\,\mathrm{d}S$, where $\Phi : [0,1] \times [0,1] \to \mathbb{R}^3 : (u,v) \mapsto (u+v, u, v)$.

2 Find a parametrization Φ of the upper half sphere $x^2 + y^2 + z^2 = 1$ and $z \geq 0$, and evaluate the following integrals:

   a) $\int_{\Phi} z\,\mathrm{d}S$

   b) $\int_{\Phi} (x + 3y^5, y + 10xz, z - xy) \cdot \mathrm{d}\mathbf{S}$

3. Evaluate each of the following integrals by first finding a suitable parametrization of the set given:

   a) $\int_{\Phi} (x^2 + y - 4, 3xy, 2xz + z^2) \cdot \mathrm{d}\mathbf{S}$; parametrization Φ of the sphere $x^2 + y^2 + z^2 = 16$

   b) $\int_{\Phi} (3xy^2, 3x^2y, z^3) \cdot \mathrm{d}\mathbf{S}$; parametrization Φ of the unit sphere $x^2 + y^2 + z^2 = 1$

   c) $\int_{\Phi} (1, 1, z(x^2 + y^2)^2) \cdot \mathrm{d}\mathbf{S}$; parametrization Φ of the cylinder $x^2 + y^2 = 1$ and $0 \leq z \leq 1$

4. For each of the following, evaluate the integral $\displaystyle\int_S f\,\mathrm{d}S$:
   a) $f(x,y,z) = xyz$ and S is the triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,1,1)$.
   b) $f(x,y,z) = z$ and S is the surface $z = x^2 + y^2$ with $x^2 + y^2 \le 1$.
   c) $f(x,y,z) = x^2$ and S is the part of the plane $x = z$ inside the cylinder $x^2 + y^2 = 1$.
   d) $f(x,y,z) = xyz$ and S is the rectangle with vertices $(1,0,1)$ $(2,0,0)$, $(1,1,1)$ and $(2,1,0)$.
   e) $f(x,y,z) = x + y$ and S is the unit sphere $x^2 + y^2 + z^2 = 1$.
   f) $f(x,y,z) = x^2 + y^2 + z^2$ and S is the part of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = x + 3$.
   g) $f(x,y,z) = x^2 + y^2$ and S is the part of the paraboloid $z = 2 - (x^2 + y^2)$ above the xy-plane.
   h) $f(x,y,z) = x^2 + y^2$ and S is the cone $z^2 = 3(x^2 + y^2)$ bounded by the planes $z = 0$ and $z = 3$.

5. Suppose that S is the sphere with radius R and centred at the origin. Explain why

$$\int_S x^2\,\mathrm{d}S = \int_S y^2\,\mathrm{d}S = \int_S z^2\,\mathrm{d}S.$$

   Hence determine the value of the integrals with very little computation.

6. For each of the following, evaluate the integral $\displaystyle\int_S F \cdot \mathrm{d}\mathbf{S}$:
   a) $F(x,y,z) = (x,y,z)$ and S is the the upper half sphere $x^2+y^2+z^2 = 1$ and $z \ge 0$, with normal vector pointing away from the origin.
   b) $F(x,y,z) = (x^2,y^2,z^2)$ and S is the part of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$, with normal vector pointing out of the cone.
   c) $F(x,y,z) = (x,y,-y)$ and S is the cylinder $x^2 + y^2 = 1$ and $0 \le z \le 1$, with normal vector pointing out of the cylinder.
   d) $F(x,y,z) = (xy,-x^2,x+z)$ and S is the part of the plane $2x + 2y + z = 6$ in the first octant, with normal vector pointing away from the origin.
   e) $F(x,y,z) = (2x - z, x^2y, -xz^2)$ and S is the surface of the cube $[0,1]^3$, with normal vector pointing out of the cube.
   f) $F(x,y,z) = (xz^2, x^2y - z^3, 2xy + y^2z)$ and S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$, with normal vector pointing out of the region.
   g) $F(x,y,z) = (z^2 - x, -xy, 3z)$ and S is the surface of the region bounded by $z = 4 - y^2$, $x = 0$, $x = 3$ and $z = 0$, with normal vector pointing out of the region.
   h) $F(x,y,z) = (2x + 3z, -xz - y, y^2 + 2z)$ and S is the surface of the sphere with centre $(3,-1,2)$ and radius 3, with normal vector pointing out of the region.
   i) $F(x,y,z) = (4xz, -y^2, yz)$ and S is the surface of the cube $[0,1]^3$, with normal vector pointing out of the cube.
   j) $F(x,y,z) = (2xy, yz^2, xz)$ and S is the surface of the parallelepiped $[0,2] \times [0,1] \times [0,3]$, with normal vector pointing out of the region.

MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 12

INTEGRATION THEOREMS

12.1. Green's Theorem

Recall from Section 5.4 that a region of the type

$$R = \{(x,y) \in \mathbb{R}^2 : x \in [A_1,B_1] \text{ and } \phi_1(x) \le y \le \phi_2(x)\}, \tag{1}$$

where the functions $\phi_1 : [A_1,B_1] \to \mathbb{R}$ and $\phi_2 : [A_1,B_1] \to \mathbb{R}$ are continuous in the interval $[A_1,B_1]$ and where $\phi_1(x) \le \phi_2(x)$ for every $x \in [A_1,B_1]$, is called an elementary region of type 1. A region of the type

$$R = \{(x,y) \in \mathbb{R}^2 : y \in [A_2,B_2] \text{ and } \psi_1(y) \le x \le \psi_2(y)\}, \tag{2}$$

where the functions $\psi_1 : [A_2,B_2] \to \mathbb{R}$ and $\psi_2 : [A_2,B_2] \to \mathbb{R}$ are continuous in the interval $[A_2,B_2]$ and where $\psi_1(y) \le \psi_2(y)$ for every $y \in [A_2,B_2]$, is called an elementary region of type 2. Furthermore, an elementary region of type 3 is one which is of both type 1 and type 2; in other words, one that can be described by both (1) and (2).

Green's theorem relates a line integral along a simple closed curve C in $\mathbb{R}^2$ to a double integral over the region R enclosed by the curve. We say that C has positive orientation if the region R is on the left when we follow the curve C, and has negative orientation otherwise. For example, a circle followed in the anticlockwise direction has positive orientation with respect to the region it encloses.

THEOREM 12A. (GREEN'S THEOREM) *Suppose that $R \subseteq \mathbb{R}^2$ is an elementary region of type 3, with boundary curve C followed with positive orientation. Suppose further that the functions $P : R \to \mathbb{R}$ and $Q : R \to \mathbb{R}$ are both continuously differentiable. Then*

$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y. \tag{3}$$

REMARKS. (1) Consider the vector field $F = (P, Q)$ in $\mathbb{R}^2$. Then (3) can be written as

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

(2) Consider a vector field $F(x, y, z) = (P(x, y), Q(x, y), 0)$ in $\mathbb{R}^3$, and imagine the region R to be a surface S on the xy-plane, with boundary curve C. We have

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_C (P, Q, 0) \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y). \tag{4}$$

On the other hand, we can parametrize the surface S by the function $\Phi : R \to \mathbb{R}^3 : (x, y) \mapsto (x, y, 0)$. Then $\mathbf{t}_x = (1, 0, 0)$ and $\mathbf{t}_y = (0, 1, 0)$, so that $\mathbf{t}_x \times \mathbf{t}_y = (0, 0, 1)$. Hence

$$\int_\Phi (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} = \iint_R \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot (0, 0, 1)\,\mathrm{d}x\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y. \tag{5}$$

If we take the oriented surface S to have normal in the positive z-direction, then

$$\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} = \int_\Phi (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S}. \tag{6}$$

Combining (3)–(6), we conclude that

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S}.$$

This is known as Stokes's theorem. We shall study this in Section 12.2.

(3) Replacing Q by P and replacing P by $-Q$ in (3), we obtain

$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \int_C (P\,\mathrm{d}y - Q\,\mathrm{d}x). \tag{7}$$

Consider now a vector field $F = (P, Q)$ in $\mathbb{R}^2$. Then

$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \iint_R (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y. \tag{8}$$

Next, suppose that ϕ is an orientation preserving parametrization of C. Then a tangent vector at a point $(x(t), y(t))$ is given by $(x'(t), y'(t))$. Rotating this vector in the clockwise direction by an angle $\pi/2$ gives an outward normal vector to C at the point $(x(t), y(t))$. This outward normal vector is $(y'(t), -x'(t))$, with unit vector

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\|(x'(t), y'(t))\|}.$$

It follows that

$$\int_C (P\,\mathrm{d}y - Q\,\mathrm{d}x) = \int_C F \cdot \mathbf{n}\,\mathrm{d}s. \tag{9}$$

Combining (7)–(9), we obtain

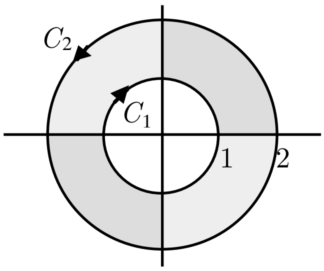
$$\iint_R (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y = \int_C F \cdot \mathbf{n}\,\mathrm{d}s.$$

This is the 2-dimensional version of Gauss's divergence theorem which we shall study in Section 12.3.

(4) Green's theorem can be extended to regions R which are finite unions of essentially disjoint elementary regions of type 3. For example, consider the annulus

$$R = \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}.$$

We can cut R into four subregions of type 3 by the lines $x = 0$ and $y = 0$.



The boundary curve is now the union of the two circles

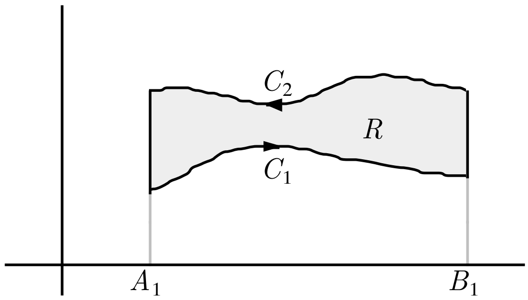
$$C_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \qquad \text{and} \qquad C_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\},$$

with C_1 followed in the clockwise direction and C_2 followed in the anticlockwise direction.

Proof of Theorem 12A. Consider first of all the integral

$$\int_C P \, \mathrm{d}x. \tag{10}$$

Since R is an elementary region of type 3, it is an elementary region of type 1, and so can be described in the form (1).



The boundary curve C of this region can be split into four parts. There are two straight line segments, from $(A_1, \phi_2(A_1))$ to $(A_1, \phi_1(A_1))$ and from $(B_1, \phi_1(B_1))$ to $(B_1, \phi_2(B_1))$. There are also two curves

$$C_1 = \{(x, \phi_1(x)) : x \in [A_1, B_1]\} \qquad \text{and} \qquad C_2 = \{(x, \phi_2(x)) : x \in [A_1, B_1]\},$$

followed from $(A_1, \phi_1(A_1))$ to $(B_1, \phi_1(B_1))$ and from $(B_1, \phi_2(B_1))$ to $(A_1, \phi_2(A_1))$ respectively. The contribution from the two straight line segments to the integral (10) is zero, since $\mathrm{d}x = 0$ on these two line segments. It follows that

$$\begin{aligned}\int_C P \, \mathrm{d}x = \int_{C_1} P \, \mathrm{d}x + \int_{C_2} P \, \mathrm{d}x &= \int_{A_1}^{B_1} P(x, \phi_1(x)) \, \mathrm{d}x - \int_{A_1}^{B_1} P(x, \phi_2(x)) \, \mathrm{d}x \\ &= -\int_{A_1}^{B_1} (P(x, \phi_2(x)) - P(x, \phi_1(x))) \, \mathrm{d}x.\end{aligned}$$

On the other hand, it follows from Fubini's theorem that

$$\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y = \int_{A_1}^{B_1} \left(\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,\mathrm{d}y \right) \mathrm{d}x = \int_{A_1}^{B_1} (P(x,\phi_2(x)) - P(x,\phi_1(x)))\,\mathrm{d}x,$$

by the Fundamental theorem of calculus. Hence

$$\int_C P\,\mathrm{d}x = -\iint_R \frac{\partial P}{\partial y}\,\mathrm{d}x\mathrm{d}y. \tag{11}$$

Similarly, it can be proved that

$$\int_C Q\,\mathrm{d}y = \iint_R \frac{\partial Q}{\partial x}\,\mathrm{d}x\mathrm{d}y. \tag{12}$$

(3) now follows on combining (11) and (12). ◯

EXAMPLE 12.1.1. Consider the special case when $P(x,y) = -y/2$ and $Q(x,y) = x/2$. Then (3) becomes

$$\frac{1}{2}\int_C (x\,\mathrm{d}y - y\,\mathrm{d}x) = \iint_R \mathrm{d}x\mathrm{d}y.$$

This is equal to the area of R. Suppose now that R is the region bounded by the hypocycloid C of four cusps, given by the equation $x^{2/3} + y^{2/3} = 1$ and parametrized by

$$\phi : [0,2\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t).$$

Then $\mathrm{d}x = -3\cos^2 t \sin t\,\mathrm{d}t$ and $\mathrm{d}y = 3\sin^2 t \cos t\,\mathrm{d}t$. Hence the area of the region bounded by the hypocycloid is given by

$$\frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^4 t + 3\cos^2 t\sin^4 t)\,\mathrm{d}t = \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{3\pi}{8}.$$

EXAMPLE 12.1.2. Let $P(x,y) = x^2 y\cos x + 2xy\sin x - y^2\mathrm{e}^x$ and $Q(x,y) = x^2\sin x - 2y\mathrm{e}^x$. Then

$$\frac{\partial Q}{\partial x} = x^2\cos x + 2x\sin x - 2y\mathrm{e}^x = \frac{\partial P}{\partial y}.$$

It follows from Green's theorem that

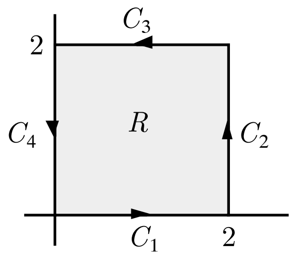
$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = 0 \tag{13}$$

for the boundary curve C of any elementary region of type 3. Note that (13) holds if C is the boundary curve of any elementary region of type 3 in which the equality

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

holds.

Example 12.1.3. Let $P(x,y)=x^2-xy^3$ and $Q(x,y)=y^2-2xy$, and let R denote the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.



The boundary curve is then $C=C_1\cup C_2\cup C_3\cup C_4$, where C_1,C_2,C_3,C_4 are the four sides of R followed in the anticlockwise direction with initial point $(0,0)$, and can be parametrized respectively by

$$\begin{aligned}
\phi_1 &: [0,2]\to\mathbb{R}^2 : t\mapsto (t,0),\\
\phi_2 &: [0,2]\to\mathbb{R}^2 : t\mapsto (2,t),\\
\phi_3 &: [0,2]\to\mathbb{R}^2 : t\mapsto (2-t,2),\\
\phi_4 &: [0,2]\to\mathbb{R}^2 : t\mapsto (0,2-t).
\end{aligned}$$

We have

$$\begin{aligned}
\int_{C_1}(P\,\mathrm{d}x+Q\,\mathrm{d}y) &= \int_{\phi_1}(P,Q)\cdot\mathrm{d}\mathbf{s} = \int_0^2 (P(t,0),Q(t,0))\cdot\phi_1'(t)\,\mathrm{d}t\\
&= \int_0^2 (t^2,0)\cdot(1,0)\,\mathrm{d}t = \frac{8}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_{C_2}(P\,\mathrm{d}x+Q\,\mathrm{d}y) &= \int_{\phi_2}(P,Q)\cdot\mathrm{d}\mathbf{s} = \int_0^2 (P(2,t),Q(2,t))\cdot\phi_2'(t)\,\mathrm{d}t\\
&= \int_0^2 (4-2t^3,t^2-4t)\cdot(0,1)\,\mathrm{d}t = -\frac{16}{3},
\end{aligned}$$

as well as

$$\begin{aligned}
\int_{C_3}(P\,\mathrm{d}x+Q\,\mathrm{d}y) &= \int_{\phi_3}(P,Q)\cdot\mathrm{d}\mathbf{s} = \int_0^2 (P(2-t,2),Q(2-t,2))\cdot\phi_3'(t)\,\mathrm{d}t\\
&= \int_0^2 ((2-t)^2-8(2-t),4-4(2-t))\cdot(-1,0)\,\mathrm{d}t = \frac{40}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_{C_4}(P\,\mathrm{d}x+Q\,\mathrm{d}y) &= \int_{\phi_4}(P,Q)\cdot\mathrm{d}\mathbf{s} = \int_0^2 (P(0,2-t),Q(0,2-t))\cdot\phi_4'(t)\,\mathrm{d}t\\
&= \int_0^2 (0,(2-t)^2)\cdot(0,-1)\,\mathrm{d}t = -\frac{8}{3}.
\end{aligned}$$

Hence

$$\int_C(P\,\mathrm{d}x+Q\,\mathrm{d}y) = \int_{C_1}(P\,\mathrm{d}x+Q\,\mathrm{d}y)+\int_{C_2}(P\,\mathrm{d}x+Q\,\mathrm{d}y)+\int_{C_3}(P\,\mathrm{d}x+Q\,\mathrm{d}y)+\int_{C_4}(P\,\mathrm{d}x+Q\,\mathrm{d}y) = 8.$$

This calculation can be somewhat simplified by noting that $\mathrm{d}x = 0$ on C_2 and C_4, while $\mathrm{d}y = 0$ on C_1 and C_3. Hence

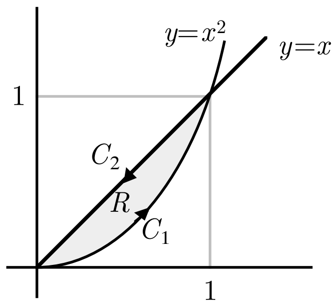
$$\begin{aligned}\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) &= \int_0^2 P(x,0)\,\mathrm{d}x + \int_0^2 Q(2,y)\,\mathrm{d}y - \int_0^2 P(x,2)\,\mathrm{d}x - \int_0^2 Q(0,y)\,\mathrm{d}y \\ &= \int_0^2 x^2\,\mathrm{d}x + \int_0^2 (y^2-4y)\,\mathrm{d}y - \int_0^2 (x^2-8x)\,\mathrm{d}x - \int_0^2 y^2\,\mathrm{d}y = 8.\end{aligned}$$

On the other hand, we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y. = \iint_R (3xy^2-2y)\,\mathrm{d}x\mathrm{d}y = \int_0^2 \left(\int_0^2 (3xy^2-2y)\,\mathrm{d}x\right) \mathrm{d}y = \int_0^2 (6y^2-4y)\,\mathrm{d}y = 8.$$

This verifies Green's theorem.

EXAMPLE 12.1.4. Let $P(x,y) = xy + y^2$ and $Q(x,y) = x^2$, and let R denote the region bounded by the line $y = x$ and the parabola $y = x^2$.



The boundary curve is then $C = C_1 \cup C_2$, where C_1 is the part of the parabola from $(0,0)$ to $(1,1)$ and C_2 is the part of the line from $(1,1)$ and $(0,0)$. The curves C_1 and C_2 can be parametrized respectively by

$$\phi_1 : [0,1] \to \mathbb{R}^2 : t \mapsto (t,t^2) \qquad \text{and} \qquad \phi_2 : [0,1] \to \mathbb{R}^2 : t \mapsto (1-t,1-t).$$

We have

$$\begin{aligned}\int_{C_1} (P\,\mathrm{d}x + Q\,\mathrm{d}y) &= \int_{\phi_1} (P,Q)\cdot \mathrm{d}\mathbf{s} = \int_0^1 (P(t,t^2),Q(t,t^2))\cdot \phi_1'(t)\,\mathrm{d}t \\ &= \int_0^1 (t^3+t^4,t^2)\cdot(1,2t)\,\mathrm{d}t = \frac{19}{20}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} (P\,\mathrm{d}x + Q\,\mathrm{d}y) &= \int_{\phi_2} (P,Q)\cdot \mathrm{d}\mathbf{s} = \int_0^1 (P(1-t,1-t),Q(1-t,1-t))\cdot \phi_2'(t)\,\mathrm{d}t \\ &= \int_0^1 (2(1-t)^2,(1-t)^2)\cdot(-1,-1)\,\mathrm{d}t = -1.\end{aligned}$$

Hence

$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_{C_1} (P\,\mathrm{d}x + Q\,\mathrm{d}y) + \int_{C_2} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = -\frac{1}{20}.$$

On the other hand, we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y. = \iint_R (x-2y)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_{x^2}^x (x-2y)\,\mathrm{d}y\right) \mathrm{d}x = \int_0^1 (x^4 - x^3)\,\mathrm{d}y = -\frac{1}{20}.$$

This verifies Green's theorem.

12.2. Stokes's Theorem

Stokes's theorem relates a line integral along a simple closed curve C in $\mathbb{R}^3$ to a surface integral over a surface S with boundary curve C. A special case of it is Green's theorem discussed in the last section.

Clearly any relationship between the line integral and the surface integral requires a convention concerning the orientation of the curve C with respect to the orientation of the surface S. We use the right hand thumb rule: Extend the thumb on our right hand and close the fingers. If the thumb points in the direction of the chosen normal of S, then the curve C is said to have positive orientation if it follows the direction of the fingers. In other words, if we follow the curve C in positive orientation, then the surface S is on the left.

THEOREM 12B. (STOKES'S THEOREM) *Suppose that $S \subseteq \mathbb{R}^3$ is an oriented surface, defined by an orientation preserving parametrization $\Phi : R \to \mathbb{R}^3$ for some elementary region $R \subseteq \mathbb{R}^2$ of type 3, and with boundary curve C followed with positive orientation. Suppose further that the vector field F is continuously differentiable in S. Then*

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S}.$$

We shall not give a rigorous proof here. Instead, we only very roughly give an outline of the main ideas, and show that the result may be deduced from Green's theorem. In the sketch below, we often make extra assumptions which are not normally necessary.

HEURISTICS OF THEOREM 12B. Write $F = (F_1, F_2, F_3)$. Then

$$\int_C F \cdot \mathrm{d}\mathbf{s} = \int_C (F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z) \tag{14}$$

and

$$\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} = \int_S (\operatorname{curl}(F_1, 0, 0)) \cdot \mathrm{d}\mathbf{S} + \int_S (\operatorname{curl}(0, F_2, 0)) \cdot \mathrm{d}\mathbf{S} + \int_S (\operatorname{curl}(0, 0, F_3)) \cdot \mathrm{d}\mathbf{S}. \tag{15}$$

Suppose that

$$\Phi : R \to \mathbb{R}^3 : (u, v) \mapsto (x, y, z).$$

Let C' denote the boundary curve of R, and consider the integral

$$\int_C F_1\,\mathrm{d}x.$$

Since

$$\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v,$$

it follows from Green's theorem that

$$\int_C F_1\,\mathrm{d}x = \int_{C'} \left(F_1 \frac{\partial x}{\partial u}\,\mathrm{d}u + F_1 \frac{\partial x}{\partial v}\,\mathrm{d}v \right) = \iint_R \left(\frac{\partial}{\partial u}\left(F_1 \frac{\partial x}{\partial v} \right) - \frac{\partial}{\partial v}\left(F_1 \frac{\partial x}{\partial u} \right) \right) \mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} + F_1 \frac{\partial^2 x}{\partial u \partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} - F_1 \frac{\partial^2 x}{\partial v \partial u} \right) \mathrm{d}u\mathrm{d}v = \iint_R \left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} \right) \mathrm{d}u\mathrm{d}v.$$

Next, note that

$$\frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} \qquad \text{and} \qquad \frac{\partial F_1}{\partial v} = \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}.$$

Hence

(16)
$$\int_C F_1\,\mathrm{d}x = \iint_R \left(\left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} - \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} \right) \mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left(\left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} - \left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} \right) \mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left(\frac{\partial F_1}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \right) - \frac{\partial F_1}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \right) \mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)} \right) \mathrm{d}u\mathrm{d}v.$$

On the other hand,

(17)
$$\int_S (\mathrm{curl}(F_1,0,0)) \cdot \mathrm{d}\mathbf{S} = \iint_R \left(0, \frac{\partial F_1}{\partial z}, -\frac{\partial F_1}{\partial y} \right) \cdot \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) \mathrm{d}u\mathrm{d}v$$
$$= \iint_R \left(\frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)} \right) \mathrm{d}u\mathrm{d}v.$$

Combining (16) and (17), we obtain

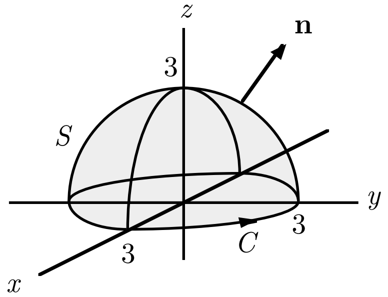
(18)
$$\int_C F_1\,\mathrm{d}x = \int_S (\mathrm{curl}(F_1,0,0)) \cdot \mathrm{d}\mathbf{S}.$$

Similarly, we have

(19)
$$\int_C F_2\,\mathrm{d}y = \int_S (\mathrm{curl}(0,F_2,0)) \cdot \mathrm{d}\mathbf{S} \qquad \text{and} \qquad \int_C F_3\,\mathrm{d}z = \int_S (\mathrm{curl}(0,0,F_3)) \cdot \mathrm{d}\mathbf{S}.$$

The result follows on combining (14), (15), (18) and (19). ○

EXAMPLE 12.2.1. Let S denote the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$, with outward pointing normal.



Then the boundary curve C is given by $x^2+y^2=9$, followed in the anticlockwise direction. Consider the vector field $F(x,y,z)=(2y,3x,-z^2)$. Let us first of all evaluate the integral

$$\int_C F\cdot \mathrm{d}\mathbf{s}.$$

By using the orientation preserving parametrization

$$\phi:[0,2\pi]\to\mathbb{R}^3 : t\mapsto(3\cos t,3\sin t,0),$$

we have

$$\begin{aligned}\int_C F\cdot\mathrm{d}\mathbf{s}&=\int_0^{2\pi}F(\phi(t))\cdot\phi'(t)\,\mathrm{d}t=\int_0^{2\pi}F(3\cos t,3\sin t,0)\cdot(-3\sin t,3\cos t,0)\,\mathrm{d}t\\&=\int_0^{2\pi}(6\sin t,9\cos t,0)\cdot(-3\sin t,3\cos t,0)\,\mathrm{d}t=\int_0^{2\pi}(27\cos^2 t-18\sin^2 t)\,\mathrm{d}t=9\pi.\end{aligned}$$

Next, let us evaluate the integral

$$\int_S(\operatorname{curl}F)\cdot\mathrm{d}\mathbf{S}.$$

Consider the parametrization

$$\Phi:R\to\mathbb{R}^3:(u,v)\mapsto(3\sin u\cos v,3\sin u\sin v,3\cos u),$$

where $R=[0,\pi/2]\times[0,2\pi]$. We have

$$\mathbf{t}_u\times\mathbf{t}_v=(9\sin^2 u\cos v,9\sin^2 u\sin v,9\cos u\sin u),$$

so that Φ is an orientation preserving parametrization of S. It is easy to see that

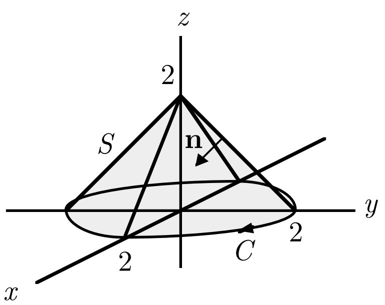
$$\operatorname{curl}F=(0,0,1),$$

so

$$\begin{aligned}\int_S(\operatorname{curl}F)\cdot\mathrm{d}\mathbf{S}&=\iint_R(0,0,1)\cdot(9\sin^2 u\cos v,9\sin^2 u\sin v,9\cos u\sin u)\,\mathrm{d}u\mathrm{d}v\\&=\iint_R 9\cos u\sin u\,\mathrm{d}u\mathrm{d}v=9\left(\int_0^{\pi/2}\cos u\sin u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\mathrm{d}v\right)=9\pi.\end{aligned}$$

This verifies Stokes's theorem.

EXAMPLE 12.2.2. Let S denote the surface of the cone $z=2-\sqrt{x^2+y^2}$ above the xy-plane, with inward pointing normal.



Then the boundary curve C is given by $x^2 + y^2 = 4$, followed in the clockwise direction. Consider the vector field $F(x, y, z) = (x - z, x^3 + yz, -3xy^2)$. Let us first of all evaluate the integral

$$\int_C F \cdot \mathrm{d}\mathbf{s}.$$

By using the orientation reversing parametrization

$$\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (2\cos t, 2\sin t, 0),$$

we have

$$\begin{aligned}\int_C F \cdot \mathrm{d}\mathbf{s} &= -\int_0^{2\pi} F(\phi(t)) \cdot \phi'(t)\,\mathrm{d}t = -\int_0^{2\pi} F(2\cos t, 2\sin t, 0) \cdot (-2\sin t, 2\cos t, 0)\,\mathrm{d}t \\ &= -\int_0^{2\pi} (2\cos t, 8\cos^3 t, -24\cos t\sin^2 t) \cdot (-2\sin t, 2\cos t, 0)\,\mathrm{d}t \\ &= \int_0^{2\pi} (4\cos t\sin t - 16\cos^4 t)\,\mathrm{d}t = -12\pi.\end{aligned}$$

Next, let us evaluate the integral

$$\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S}.$$

Consider the parametrization

$$\Phi : R \to \mathbb{R}^3 : (u, v) \mapsto (u\cos v, u\sin v, 2 - u),$$

where $R = [0, 2] \times [0, 2\pi]$. We have

$$\mathbf{t}_u \times \mathbf{t}_v = (\cos v, \sin v, -1) \times (-u\sin v, u\cos v, 0) = (u\cos v, u\sin v, u),$$

so that Φ is an orientation reversing parametrization of S. Since

$$\operatorname{curl} F = (-6xy - y, -1 + 3y^2, 3x^2),$$

it follows that

$$\begin{aligned}\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} &= -\iint_R (-6u^2\cos v\sin v - u\sin v, -1 + 3u^2\sin^2 v, 3u^2\cos^2 v) \cdot (u\cos v, u\sin v, u)\,\mathrm{d}u\mathrm{d}v \\ &= \iint_R (6u^3\cos^2 v\sin v + u^2\sin v\cos v + u\sin v - 3u^3\sin^3 v - 3u^3\cos^2 v)\,\mathrm{d}u\mathrm{d}v \\ &= \int_0^2 \left(\int_0^{2\pi} (6u^3\cos^2 v\sin v + u^2\sin v\cos v + u\sin v - 3u^3\sin^3 v - 3u^3\cos^2 v)\,\mathrm{d}v \right) \mathrm{d}u \\ &= -3\int_0^2 \left(\int_0^{2\pi} u^3\cos^2 v\,\mathrm{d}v \right) \mathrm{d}u = -12\pi.\end{aligned}$$

This verifies Stokes's theorem.

Suppose that $F = \nabla f$ is a gradient vector field in $\mathbb{R}^3$. Then it follows from Theorem 9A that for any continuously differentiable path $\phi : [A, B] \to \mathbb{R}^3$ such that the composition function $F \circ \phi : [A, B] \to \mathbb{R}^3$ is continuous, we have

$$\int_\phi F \cdot \mathrm{d}\mathbf{s} = f(\phi(B)) - f(\phi(A)).$$

In other words, the value of the integral depends only on the endpoints of the path ϕ.

With the help of Stokes's theorem, we can characterize gradient vector fields.

THEOREM 12C. *Suppose that $F:\mathbb{R}^3\to\mathbb{R}^3$ is a continuously differentiable vector field. Then the following statements are equivalent:*
(a) For any oriented simple closed curve C, we have

$$\int_C F\cdot \mathrm{d}\mathbf{s}=0.$$

(b) For any two oriented simple curves C_1 and C_2 with the same initial point and the same terminal point, we have

$$\int_{C_1} F\cdot \mathrm{d}\mathbf{s}=\int_{C_2} F\cdot \mathrm{d}\mathbf{s}.$$

(c) There exists a function $f:\mathbb{R}^3\to\mathbb{R}$ such that $F=\nabla f$ everywhere in $\mathbb{R}^3$.
(d) We have $\operatorname{curl} F=\mathbf{0}$ everywhere in $\mathbb{R}^3$.

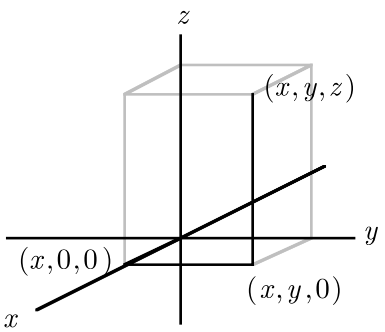
Sketch of Proof. We shall show that (a)⇒(b)⇒(c)⇒(d)⇒(a). To show that (a)⇒(b), let C be the curve C_1 followed by C_2. Then C is closed. If C is simple, then

$$\int_{C_1} F\cdot \mathrm{d}\mathbf{s}-\int_{C_2} F\cdot \mathrm{d}\mathbf{s}=\int_C F\cdot \mathrm{d}\mathbf{s}=0.$$

If C is not simple, then an elaboration of this argument will give the same result. To show that (b)⇒(c), let C be any oriented simple curve with initial point $(0,0,0)$ and terminal point (x,y,z), and write

$$f(x,y,z)=\int_C F\cdot \mathrm{d}\mathbf{s}.$$

Since (b) holds, $F(x,y,z)$ is independent of the choice of C. In particular, we can take C to be the line segment from $(0,0,0)$ to $(x,0,0)$, followed by the line segment from $(x,0,0)$ to $(x,y,0)$, followed by the line segment from $(x,y,0)$ to (x,y,z).



Assume first of all that x,y,z are all positive. Then the three line segments can be parametrized respectively by

$$\phi_1:[0,x]\to\mathbb{R}^3:t\mapsto(t,0,0),$$
$$\phi_2:[0,y]\to\mathbb{R}^3:t\mapsto(x,t,0),$$
$$\phi_3:[0,z]\to\mathbb{R}^3:t\mapsto(x,y,t),$$

so that writing $F=(F_1,F_2,F_3)$, we have

$$f(x,y,z)=\int_0^x F_1(t,0,0)\,\mathrm{d}t+\int_0^y F_2(x,t,0)\,\mathrm{d}t+\int_0^z F_3(x,y,t)\,\mathrm{d}t.$$

With a little modification in argument, this last formula can be shown to hold even if x, y, z are not all positive. By the Fundamental theorem of calculus, we clearly have

$$\frac{\partial f}{\partial z} = F_3.$$

By using different paths, it can be shown that

$$\frac{\partial f}{\partial x} = F_1 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = F_2,$$

so that $\nabla f = F$. That (c)⇒(d) is proved in Theorem 8G. Finally, to prove that (d)⇒(a), we simply apply Stokes's theorem with any surface S whose boundary is C. ◯

REMARKS. (1) In the statement of Theorem 12C, it is possible to assume that the vector field F is continuously differentiable in $\mathbb{R}^3$, except possibly at a finite number of points. The proof only needs minor modification.

(2) There is a 2-dimensional version of Theorem 12C. Recall that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

is the scalar curl of a vector field $F = (P, Q)$ in $\mathbb{R}^2$. Thus there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $F = \nabla f$ everywhere in $\mathbb{R}^2$ if and only if

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

everywhere in $\mathbb{R}^2$. Here Green's theorem plays the role of Stokes's theorem in establishing the result. However, we cannot allow exceptions to the condition that F is continuously differentiable in $\mathbb{R}^2$.

(3) Theorem 12C is in some sense the converse of Theorem 8G. Recall now Theorem 8F, that for any twice continuously differentiable vector field F in $\mathbb{R}^3$, we have $\text{div}(\text{curl}\, F) = 0$. One can prove that if G is a continuously differentiable vector field in $\mathbb{R}^3$ with $\text{div}\, G = 0$, then there exists a vector field F in $\mathbb{R}^3$ such that $G = \text{curl}\, F$.

12.3. Gauss's Theorem

Gauss's theorem relates a surface integral over a closed surface S in $\mathbb{R}^3$ to a volume integral over a region V with boundary surface S.

We shall be concerned with regions in $\mathbb{R}^3$ of the type

$$V = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R \text{ and } \phi_1(x, y) \le z \le \phi_2(x, y)\}, \tag{20}$$

where R is an elementary region in $\mathbb{R}^2$, and where the functions $\phi_1 : R \to \mathbb{R}^2$ and $\phi_2 : R \to \mathbb{R}^2$ are continuous, with $\phi_1(x, y) \le \phi_2(x, y)$ for every $(x, y) \in R$. There are two other types, one with y bounded between continuous functions of (x, z) in an elementary region, the other with x bounded between continuous functions of (y, z) in an elementary region. A region in $\mathbb{R}^3$ which can be simultaneously described in all these three ways is called a symmetric elementary region in $\mathbb{R}^3$. Clearly we can evaluate triple integrals of continuous functions over such regions; see Section 5.7.

THEOREM 12D. (GAUSS'S THEOREM) *Suppose that $V \subseteq \mathbb{R}^3$ is a symmetric elementary region, with boundary surface S oriented with outward normal. Suppose further that the vector field F is continuously differentiable on V. Then*

$$\int_S F \cdot \mathrm{d}\mathbf{S} = \iiint_V (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

REMARKS. (1) Sometimes, we write

$$\int_S F \cdot \mathrm{d}\mathbf{S} = \int_V (\operatorname{div} F)\,\mathrm{d}V.$$

(2) Gauss's theorem is in fact valid for any region V which can be expressed as a union of finitely many essentially disjoint symmetric elementary regions.

(3) We shall see that the proof of Gauss's theorem is very similar to that of Green's theorem.

SKETCH OF PROOF OF THEOREM 12D. Write $F = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\int_S F \cdot \mathrm{d}\mathbf{S} &= \int_S ((F_1,0,0) + (0,F_2,0) + (0,0,F_3)) \cdot \mathrm{d}\mathbf{S} \\ &= \int_S (F_1,0,0)\cdot \mathrm{d}\mathbf{S} + \int_S (0,F_2,0)\cdot \mathrm{d}\mathbf{S} + \int_S (0,0,F_3)\cdot \mathrm{d}\mathbf{S}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\iiint_V (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iiint_V \frac{\partial F_1}{\partial x}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_V \frac{\partial F_2}{\partial y}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint_V \frac{\partial F_3}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned} \tag{22}$$

We shall show first of all that

$$\int_S (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \iiint_V \frac{\partial F_3}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{23}$$

Since V is a symmetric elementary region, it can be described in the form (20), so that

$$\begin{aligned}\iiint_V \frac{\partial F_3}{\partial z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iint_R \left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} \frac{\partial F_3}{\partial z}\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y \\ &= \iint_R (F_3(x,y,\phi_2(x,y)) - F_3(x,y,\phi_1(x,y)))\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{24}$$

On the other hand, the boundary surface S can be partitioned into six surfaces, with bottom surface

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } z = \phi_1(x,y)\},$$

top surface

$$S_2 = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R \text{ and } z = \phi_2(x,y)\},$$

and four side surfaces S_3, S_4, S_5, S_6 corresponding to the four edges of the elementary region R. The normal vectors to the surfaces S_3, S_4, S_5, S_6 are all horizontal, with no component in the z-direction. Hence

$$\int_{S_3} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \int_{S_4} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \int_{S_5} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \int_{S_6} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = 0,$$

and so

$$\int_S (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \int_{S_1} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} + \int_{S_2} (0,0,F_3)\cdot \mathrm{d}\mathbf{S}. \tag{25}$$

The surface S_1 can be parametrized by

$$\Phi : R \to \mathbb{R}^3 : (x,y) \mapsto (x,y,\phi_1(x,y)),$$

with normal vector

$$\mathbf{t}_x \times \mathbf{t}_y = \left(1,0,\frac{\partial \phi_1}{\partial x}\right) \times \left(0,1,\frac{\partial \phi_1}{\partial y}\right) = \left(-\frac{\partial \phi_1}{\partial x}, -\frac{\partial \phi_1}{\partial y}, 1\right).$$

Hence Φ is an orientation reversing parametrization of S_1, and so

$$\int_{S_1} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = -\iint_R (0,0,F_3)\cdot \left(-\frac{\partial \phi_1}{\partial x}, -\frac{\partial \phi_1}{\partial y}, 1\right) \mathrm{d}x\mathrm{d}y = -\iint_R F_3(x,y,\phi_1(x,y))\,\mathrm{d}x\mathrm{d}y. \tag{26}$$

The surface S_2 can be parametrized by

$$\Psi : R \to \mathbb{R}^3 : (x,y) \mapsto (x,y,\phi_2(x,y)),$$

with normal vector

$$\mathbf{t}_x \times \mathbf{t}_y = \left(1,0,\frac{\partial \phi_2}{\partial x}\right) \times \left(0,1,\frac{\partial \phi_2}{\partial y}\right) = \left(-\frac{\partial \phi_2}{\partial x}, -\frac{\partial \phi_2}{\partial y}, 1\right).$$

Hence Ψ is an orientation preserving parametrization of S_2, and so

$$\int_{S_2} (0,0,F_3)\cdot \mathrm{d}\mathbf{S} = \iint_R (0,0,F_3)\cdot \left(-\frac{\partial \phi_2}{\partial x}, -\frac{\partial \phi_2}{\partial y}, 1\right) \mathrm{d}x\mathrm{d}y = \iint_R F_3(x,y,\phi_2(x,y))\,\mathrm{d}x\mathrm{d}y. \tag{27}$$

(23) now follows on combining (24)–(27). Similarly, we have

$$\int_S (F_1,0,0)\cdot \mathrm{d}\mathbf{S} = \iiint_V \frac{\partial F_1}{\partial x}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad \text{and} \qquad \int_S (0,F_2,0)\cdot \mathrm{d}\mathbf{S} = \iiint_V \frac{\partial F_2}{\partial y}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{28}$$

The result now follows on combining (21)–(23) and (28). ○

Example 12.3.1. Let V denote the unit ball $x^2+y^2+z^2 \le 1$. Then the boundary surface S is given by $x^2+y^2+z^2=1$. Consider the vector field $F(x,y,z)=(2x,y^2,z^2)$. Let us first of all calculate the integral

$$\int_S F\cdot \mathrm{d}\mathbf{S}.$$

The surface S can be parametrized by

$$\Phi : R \to \mathbb{R}^3 : (u,v) = (\sin u\cos v, \sin u \sin v, \cos u),$$

where $R=[0,\pi]\times[0,2\pi]$, and where

$$\mathbf{t}_u \times \mathbf{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u) = (\sin u)\Phi(u,v).$$

This is an orientation preserving parametrization, and so

$$\begin{aligned}
\int_S F\cdot \mathrm{d}\mathbf{S} &= \iint_R F(\sin u\cos v, \sin u\sin v, \cos u)\cdot(\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u)\,\mathrm{d}u\mathrm{d}v \\
&= \iint_R (2\sin u\cos v, \sin^2 u\sin^2 v, \cos^2 u)\cdot(\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u)\,\mathrm{d}u\mathrm{d}v \\
&= \iint_R (2\sin^3 u\cos^2 v + \sin^4 u\sin^3 v + \sin u\cos^3 u)\,\mathrm{d}u\mathrm{d}v \\
&= 2\left(\int_0^\pi \sin^3 u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\cos^2 v\,\mathrm{d}v\right) + \left(\int_0^\pi \sin^4 u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\sin^3 v\,\mathrm{d}v\right) \\
&\quad + \left(\int_0^\pi \sin u\cos^3 u\,\mathrm{d}u\right)\left(\int_0^{2\pi}\mathrm{d}v\right) \\
&= \frac{8\pi}{3}.
\end{aligned}$$

Next, note that

$$\iiint_V (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_V (2+2y+2z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\iiint_V (1+y+z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We can write

$$V = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1 \text{ and } -\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}\},$$

so that

$$\iiint_V z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{x^2+y^2\le 1}\left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y = 0.$$

Similarly

$$\iiint_V y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0.$$

Hence

$$\iiint_V (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\iiint_V \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{8\pi}{3},$$

since the volume of the unit sphere is equal to $4\pi/3$. This verifies Gauss's theorem.

EXAMPLE 12.3.2. Let V be the cube with vertices $(\pm1,\pm1,\pm1)$, with boundary surface S. Consider the vector field $F(x,y,z)=(x,y,z)$. We have shown in Example 11.4.6 that

$$\int_S F\cdot\mathrm{d}\mathbf{S} = 24.$$

Now

$$\iiint_V (\operatorname{div} F)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\iiint_V \mathrm{d}x\mathrm{d}y\mathrm{d}z = 24.$$

This verifies Gauss's theorem. In fact, we can generalize this observation. Suppose that S is the boundary surface of any region V in $\mathbb{R}^3$ for which Gauss's theorem holds. Then

$$\int_S \mathbf{r}\cdot\mathrm{d}\mathbf{S} = 3\iiint_V \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where $\mathbf{r}=(x,y,z)$ denotes any point on S.

We conclude this chapter by proving the following famous result.

THEOREM 12E. (GAUSS'S LAW) *Suppose that $V \subseteq \mathbb{R}^3$ is a symmetric elementary region, with boundary surface S oriented with outward normal. Suppose further that $(0,0,0) \notin S$. Then*

$$\int_S \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} = \begin{cases} 4\pi & \text{if } (0,0,0) \in V, \\ 0 & \text{if } (0,0,0) \notin V, \end{cases}$$

where $\mathbf{r} = (x,y,z)$ denotes any point on S, and $r = \|\mathbf{r}\| = (x^2+y^2+z^2)^{1/2}$.

SKETCH OF PROOF. Suppose first of all that $(0,0,0) \notin V$. Then the vector field

$$\frac{\mathbf{r}}{r^3}$$

is continuously differentiable on V, and so it follows from Gauss's theorem that

$$\int_S \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} = \iiint_V \operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

It is easy to check that

$$\operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) = 0$$

whenever $r \neq 0$. The desired conclusion therefore holds in this case. Suppose now that $(0,0,0) \in V$. Since $(0,0,0) \notin S$, it follows that there exists $\epsilon > 0$ such that the open ball $B(\epsilon)$, with centre $(0,0,0)$ and radius ϵ, satisfies $B(\epsilon) \subseteq V$. Now let $\Omega = V \setminus B(\epsilon)$, the region V with the open ball $B(\epsilon)$ removed. Clearly this region has boundary surface $S \cup T$, where T is the boundary surface of $B(\epsilon)$ with normal pointing towards $(0,0,0)$. Applying Gauss's theorem to this region Ω (note that Ω is not an elementary region), we have

$$\int_S \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} + \int_T \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} = \iiint_\Omega \operatorname{div}\left(\frac{\mathbf{r}}{r^3}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0,$$

so that

$$\int_S \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} = -\int_T \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S}.$$

The boundary surface T can be parametrized by

$$\Phi : R \to \mathbb{R}^3 : (u,v) \mapsto \epsilon(\sin u \cos v, \sin u \sin v, \cos u),$$

where $R = [0,\pi] \times [0,2\pi]$, with

$$\mathbf{t}_u \times \mathbf{t}_v = \epsilon^2(\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u).$$

This is an orientation reversing parametrization, and so

$$\begin{aligned} -\int_T \frac{\mathbf{r}}{r^3} \cdot \mathrm{d}\mathbf{S} &= \iint_R \frac{\epsilon(\sin u \cos v, \sin u \sin v, \cos u)}{\epsilon^3} \cdot \epsilon^2(\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)\,\mathrm{d}u\mathrm{d}v \\ &= \iint_R (\sin u \cos v, \sin u \sin v, \cos u) \cdot (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)\,\mathrm{d}u\mathrm{d}v \\ &= \iint_R \sin u\,\mathrm{d}u\mathrm{d}v = \left(\int_0^\pi \sin u\,\mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) = 4\pi. \end{aligned}$$

This gives the desired conclusion. ◯

Problems for Chapter 12

1. In each of the following cases, verify Green's theorem:
   a) $P(x,y) = xy^2$, $Q(x,y) = -yx^2$ and R is the disc with centre $(0,0)$ and radius 1.
   b) $P(x,y) = x + y$, $Q(x,y) = y$ and C is the circle with centre $(0,0)$ and radius 1.
   c) $P(x,y) = y - \sin x$, $Q(x,y) = \cos x$ and R is the triangular region with vertices $(0,0)$, $(\pi/2, 0)$ and $(\pi/2, 1)$.
   d) $P(x,y) = 2x^3 - y^3$, $Q(x,y) = x^3 + y^3$ and R is the disc with centre $(0,0)$ and radius 1.
   e) $P(x,y) = x^3 - x^2y$, $Q(x,y) = xy^2$ and R is the region bounded by the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

2. Use Green's theorem to find the area enclosed by each of the following:
   a) The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
   b) The cycloid parametrized by $\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$.

3. Use Green's theorem to evaluate the integral $\displaystyle\int_C (y\,\mathrm{d}x - x\,\mathrm{d}y)$, where C is the boundary of the square with vertices $(\pm 1, \pm 1)$.

4. Let

$$P(x,y) = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad Q(x,y) = \frac{x}{x^2 + y^2},$$

and let R be the disc in $\mathbb{R}^2$ with centre $(0,0)$ and radius 1, with boundary C followed in the anticlockwise direction. Evaluate the integrals

$$\int_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) \qquad \text{and} \qquad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y,$$

and explain why they have different values.

5. Consider the paraboloid $S = \{(x,y,z) \in \mathbb{R}^3 : 2z = x^2 + y^2 \text{ and } z \le 2\}$, with boundary curve C. Consider also the vector field $F(x,y,z) = (3y, -xz, yz^2)$.
   a) Show that $\operatorname{curl} F = (z^2 + x, 0, -z - 3)$.
   b) Let $R = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \le 4\}$. Show that

$$\Phi : R \to \mathbb{R}^3 : (u,v) \mapsto \left(u, v, \frac{u^2 + v^2}{2} \right)$$

   is a parametrization of S, and that

$$\int_\Phi (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} = -\iint_R \left(u \left(\frac{u^2 + v^2}{2} \right)^2 + u^2 + \frac{u^2 + v^2}{2} + 3 \right) \mathrm{d}u\mathrm{d}v.$$

   Using the substitution $u = r\cos\theta$ and $v = r\sin\theta$, show that the integral above has value -20π.
   c) Suppose that the surface S is oriented with normal vector pointing towards the positive z-axis. Determine whether the parametrization Φ is orientation preserving or orientation reversing, and write down the value of the integral

$$\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S}.$$

   d) Find a suitable orientation of the path C which you must specify carefully, and verify Stokes's theorem.

6. In each of the following cases, verify Stokes's theorem:
   a) $F(x,y,z) = (x^2, 2xy + x, z)$ and S is the disc $x^2 + y^2 \le 1$ on the plane $z = 0$.
   b) $F(x,y,z) = (2x - y, -yz^2, -y^2 z)$ and S is the surface of the part of the sphere $x^2 + y^2 + z^2 = 1$ with $z \ge 0$.
   c) $F(x,y,z) = (y - z + 2, yz + 4, -xz)$ and S is the surface of the cube $[0,2]^3$ excluding the part contained in the xy-plane.
   d) $F(x,y,z) = (xz, -y, x^2 y)$ and S is the surface of the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + 2z = 8$ excluding the part contained in the xz-plane.

7. Suppose that $F(x,y,z) = (y + z, -xz, y^2)$. Consider the region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $y = 2$, $z = 0$ and $2x + z = 6$. Verify Stokes's theorem where S is the surface of the region but excluding the part contained in
   a) the plane $z = 0$;
   b) the plane $y = 2$;
   c) the plane $2x + z = 6$.

8. In each of the following cases, use Stokes's theorem to evaluate the integral $\displaystyle\int_S (\mathrm{curl}\, F) \cdot \mathrm{d}\mathbf{S}$:
   a) $F(x,y,z) = (x^2 + y - 4, 3xy, 2xz + z^2)$ and S is the part of the sphere $x^2 + y^2 + z^2 = 16$ above the xy-plane.
   b) $F(x,y,z) = (x^2 + y - 4, 3xy, 2xz + z^2)$ and S is the part of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane..
   c) $F(x,y,z) = (2yz, 2 - x - 3y, x^2 + z)$ and S is the surface of the intersection of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in the first octant $x, y, z \ge 0$.

9. Suppose that $F(x,y,z) = (2xy + 3, x^2 - 4z, -4y)$.
   a) Show that $\mathrm{curl}\, F = \mathbf{0}$ everywhere in $\mathbb{R}^3$.
   b) Choose any curve C from $(3,-1,2)$ to $(2,1,-1)$, and evaluate the integral $\displaystyle\int_C F \cdot \mathrm{d}\mathbf{s}$
   c) Suppose that C' is a curve from $(3,-1,2)$ to $(2,1,-1)$ different from your choice in part (b). Comment on the value of the integral $\displaystyle\int_{C'} F \cdot \mathrm{d}\mathbf{s}$, and justify your assertion by quoting the relevant results.

10. Consider two surfaces S_1 and S_2 with the same boundary. Describe with sketches how S_1 and S_2 must be oriented to ensure that

$$\int_{S_1} (\mathrm{curl}\, F) \cdot \mathrm{d}\mathbf{S} = \int_{S_2} (\mathrm{curl}\, F) \cdot \mathrm{d}\mathbf{S}.$$

11. Suppose that $F(x,y,z) = (2xz^3 + 6y, 6x - 2yz, 3x^2 z^2 - y^2)$.
    a) Show that $\mathrm{curl}\, F = \mathbf{0}$ in $\mathbb{R}^3$.
    b) Evaluate the integral $\displaystyle\int_C F \cdot \mathrm{d}\mathbf{s}$, where C is any path from $(1,-1,1)$ to $(2,1,-1)$.

12. Suppose that $b > a > 0$, and that C is any path joining any point on the sphere $x^2 + y^2 + z^2 = a^2$ to any point on the sphere $x^2 + y^2 + z^2 = b^2$. Suppose also that $F(x,y,z) = 5r^3 \mathbf{r}$, where $\mathbf{r} = (x,y,z)$ and $r = \|\mathbf{r}\| = (x^2 + y^2 + z^2)^{1/2}$. Use Theorem 12C to show that

$$\int_C F \cdot \mathrm{d}\mathbf{s} = b^5 - a^5.$$

13. Suppose that $F(x,y,z) = (4x, -2y^2, z^2)$. Suppose further that V is the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$, and that S is the boundary surface of V oriented with outward normal. Verify that

$$\int_S F \cdot \mathrm{d}S = \iiint_V (\mathrm{div}\, F)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

14. In each of the following cases, verify Gauss's theorem:
   a) $F(x, y, z) = (x, y, z)$ and V is the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$.
   b) $F(x, y, z) = (2x - z, x^2y, -xz^2)$ and V is the unit cube $[0, 1]^3$.
   c) $F(x, y, z) = (2xy + z, y^2, -x - 3y)$ and V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 6$.
   d) $F(x, y, z) = (2x^2y, -y^2, 4xz^2)$ and V is the region in the first octant bounded by the plane $x = 2$ and the cylinder $y^2 + z^2 = 9$.

15. Consider Question 6, parts (g), (h) and (i), in Problems for Chapter 11. Verify Gauss's theorem in each case.

16. Suppose also that V is a symmetric elementary region in $\mathbb{R}^3$, with boundary surface S oriented with outward normal. Suppose also that $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\| = (x^2 + y^2 + z^2)^{1/2}$. Show that

$$\int_S \frac{\mathbf{r}}{r^2} \cdot \mathrm{d}\mathbf{S} = \iiint_V \frac{1}{r^2} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

17. Suppose that $\mathbf{n}$ is the outward unit normal at any point on the surface S of a region V. Prove that the surface area of S is equal to

$$\iiint_V (\operatorname{div} \mathbf{n}) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

18. Suppose that S is the boundary of a region in $\mathbb{R}^3$. It is known that

$$\int_S (\operatorname{curl} F) \cdot \mathrm{d}\mathbf{S} = 0.$$

   a) Explain this result in terms of Stokes's theorem.
   b) Explain this result in terms of Gauss's theorem.

19. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are twice continuously differentiable functions. Suppose also that V is a symmetric elementary region in $\mathbb{R}^3$, with boundary surface S oriented with outward normal.
   a) Prove Green's first identity, that

$$\int_S (f \nabla g) \cdot \mathrm{d}\mathbf{S} = \iiint_V (f \nabla^2 g + (\nabla f) \cdot (\nabla g)) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

   b) Deduce Green's second identity, that

$$\int_S (f \nabla g - g \nabla f) \cdot \mathrm{d}\mathbf{S} = \iiint_V (f \nabla^2 g - g \nabla^2 f) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$