The "ghost" symmetry of the BKP hierarchy

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In this paper, we systematically develop the "ghost" symmetry of the Kadomtsev-Petviashvili sub-hierarchy of B type (BKP) through its actions on the Lax operator L, the eigenfunctions, and the τ function. In this process, the spectral representation of the eigenfunctions and a new potential are introduced by using squared eigenfunction potential of the BKP hierarchy. Moreover, the bilinear identity of the constrained BKP hierarchy and Adler–Shiota–van Moerbeke formula of the BKP hierarchy are rederived compactly by means of the spectral representation and "ghost" symmetry. © *2010 American Institute of Physics.* [doi:10.1063/1.3397943]

I. INTRODUCTION

Symmetry[1] plays an important role in the study of the integrable system. Many crucial properties of the integrable system, such as the Noether conserved laws, Hamiltonian structure, Darboux transformation, and reduction, are closely connected with symmetries. There are several kinds of symmetry of the integrable system. For instance, in the well-known Kadomtsev-Petviashvili (KP) theory, there is an important symmetry called "ghost" symmetry.[2] By identifying the "ghost" symmetry with the kth time flow, the constrained KP (cKP) hierarchy[3–11] can be easily defined. In this paper, we shall focus on the study of the "ghost" symmetry of the Kadomtsev-Petviashvili sub-hierarchy of B type (BKP) hierarchy.

The "ghost" symmetry was first introduced by Oevel[2] in studying the solutions of the cKP hierarchy. Then it was extensively studied in Refs. 5 and 12–16. In the KP hierarchy, the "ghost" symmetry is closely related with a squared eigenfunction potential (SEP), which is associated with a pair of arbitrary eigenfunction $\Phi(t)$ and adjoint eigenfunction $\Psi(t)$ by means of following definition:[2]

$$\frac{\partial}{\partial t_n} S(\Phi(t), \Psi(t)) = \mathrm{Res}(\partial^{-1} \Psi M_n \Phi \partial^{-1}). \tag{1}$$

Here $M_n = L_+^n$ and L is a Lax operator of the KP hierarchy. [In this paper, we use the notations $(\Sigma_i a_i \partial^i)_+ = \Sigma_{i \geq 0} a_i \partial^i$, $(\Sigma_i a_i \partial^i)_- = \Sigma_{i<0} a_i \partial^i$, $(\Sigma_i a_i \partial^i)_{[k]} = a_k$, $\mathrm{Res}(\Sigma_i a_i \partial^i) = a_{-1}$, and $(\Sigma_i a_i \partial^i)^* = \Sigma_i (-\partial)^i a_i$.] The predecessor of SEP was, in fact, the Cauchy–Baker–Akhiezer kernel introduced in Ref. 17, which is an important object for a study of vector fields action on Riemann surfaces and Virasoro action on tau functions. In Ref. 15, Aratyn *et al.* gave a systematic study for the SEP and the "ghost" symmetry in KP case. By using SEP as a basic building block in the definition of the KP hierarchy, they established a new way to reformulate the theory of the KP hierarchy called SEP method. The crucial fact of the SEP method is that there exists a spectral representation for any eigenfunction of the KP hierarchy with SEP as an spectral density. They also showed that the "ghost" symmetry,[5,13] which is generated by SEP, has close relation with the additional symmetries of the KP hierarchy.[18–24] In fact, SEP can be regarded as a generating function for the

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additional symmetries of the KP hierarchy when both eigenfunction $\Phi(t)$ and adjoint eigenfunction $\Psi(t)$ defining the SEP are Baker–Akhiezer (BA) functions.

In present work, we would like to consider the "ghost" symmetry for the BKP hierarchy. Here BKP hierarchy[25] is an important reduction in the ordinary KP hierarchy under the constraints on the Lax operator $L^*=-\partial L\partial^{-1}$. In contrast with the KP hierarchy, the SEP of the BKP hierarchy cannot generate directly a symmetry flow due to the BKP constraints $L^*=-\partial L\partial^{-1}$.[16] Thus, we have to find a new potential, which is used to generate the "ghost" symmetry of the BKP hierarchy and is expected to be expressed by SEP. So this new potential is called SEP of B-type (BSEP). Fortunately, as we shall show, the BSEP was first introduced by Loris[16] in the study of symmetry reduction in the BKP hierarchy.

Similar to the case of the KP hierarchy,[15] before giving the "ghost" symmetry of the BKP hierarchy, we need to study SEP first. Starting from the BKP bilinear identity, we shall show that there is also a spectral representations for the eigenfunctions of the BKP hierarchy, i.e., any eigenfunction of BKP hierarchy can be represented as a spectral integral over BA wave function with a spectral density expressed in terms of SEP. Then according to the differential Fay identity of the BKP hierarchy, we get the expression of the basic SEP (the one whose defining eigenfunctions are BA functions). Thus we can give the general expressions of SEP for the BKP hierarchy with the spectral representation. We then point out the importance of the spectral representations by showing that it can, in fact, provide another definition of the BKP hierarchy. In other words, we get an equivalent formulation of BKP hierarchy. We also call it SEP method for the BKP hierarchy.

Next, after BSEP is systematically studied, we define the "ghost" symmetry flows ∂_α for the BKP hierarchy by means of its action on the Lax operator L and the dressing operator W. Furthermore, actions of ∂_α on the eigenfunction Φ and τ function are given by BSEP.

At last, we consider applications for above theory. We shall first derive the bilinear identities for the cBKP hierarchy[16,26,27] with the SEP method. Moreover, then by letting eigenfunctions in the BSEP be BA functions, we get the relation between the "ghost" symmetry and the additional symmetry: in this case, the BSEP becomes a generating function for the additional symmetries of the BKP hierarchy. With the help of this fact, we shall give a simple and straightforward proof for the Adler–Shiota–van Moerbeke formula of the BKP hierarchy.[28–31]

This paper is organized in the following way. In Sec. II, some basic facts about the BKP hierarchy are reviewed. Then, SEP for the BKP hierarchy is studied in detail in Sec. III. After some interesting properties of the BSEP studied in Sec. IV, the "ghost" symmetry for BKP is showed in Sec. V. At last, we devote Sec. VI to two applications on the spectral representation and the "ghost" symmetry.

II. BKP HIERARCHY

Here, we shall review some basic facts about the BKP hierarchy.[25] The BKP hierarchy can be defined in Lax form as

$$\partial_{2n+1}L=[B_{2n+1},L],\quad B_{2n+1}=(L^{2n+1})_+,\quad n=0,1,2,\ldots, \tag{2}$$

where the Lax operator is given by

$$L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\cdots, \tag{3}$$

with coefficient functions u_i depending on the time variables $t=(t_1=x,t_3,t_5,\ldots)$ and satisfies the BKP constraint,

$$L^*=-\partial L\partial^{-1}. \tag{4}$$

It can be shown[25] that the constraint (4) is equivalent to the condition $(B_{2n+1})_{[0]}=0$.

The Lax equation (2) is equivalent to the compatibility condition of the linear system

$$L(\psi_{BA}(t,\lambda))=\lambda\psi_{BA}(t,\lambda),\quad \partial_{2n+1}\psi_{BA}(t,\lambda)=B_{2n+1}(\psi_{BA}(t,\lambda)), \tag{5}$$

[for a differential operator A and a function f, $A(f)$ denotes the action of A on f], where $\psi_{BA}(t,\lambda)$ is called BA wave function. The whole hierarchy can be expressed in terms of a dressing operator W, so that

$$L=W\,\partial\,W^{-1},\quad W=1+\sum_{j=1}^{\infty}w_j\partial^{-j},$$

and the Lax equation is equivalent to the Sato's equation,

$$\partial_{2n+1}W=-(L^{2n+1})_{-}W, \tag{6}$$

with constraint

$$W^{*}\,\partial\,W=\partial. \tag{7}$$

Let the solutions of the linear system (5) be the form

$$\psi_{BA}(t,\lambda)=W(e^{\xi(t,\lambda)})=w(t,\lambda)e^{\xi(t,\lambda)}, \tag{8}$$

where $\xi(t,\lambda)=\Sigma_{i=0}^{\infty}t_{2i+1}\lambda^{2i+1}$ and $w(t,\lambda)=1+w_1/\lambda+w_2/\lambda^2+\cdots$. Then $\psi_{BA}(t,z)$ is a wave function of the BKP hierarchy if and only if it satisfies the bilinear identity,[25]

$$\int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)=1,\quad \forall\, t,t', \tag{9}$$

where $\int d\lambda\equiv\oint_{\infty}d\lambda/2\pi i=\mathrm{Res}_{\lambda=\infty}$ and $t=(t_1=x,t_3,t_5,\cdots)$.

In the BKP hierarchy, if Φ (or Ψ) satisfies

$$\partial_{2n+1}\Phi=B_{2n+1}(\Phi)(\text{or }\ \partial_{2n+1}\Psi=-B_{2n+1}^{*}(\Psi)),\quad n=0,1,2,\ldots, \tag{10}$$

we shall call Φ (or Ψ) eigenfunction (or adjoint eigenfunction) of the BKP hierarchy. Obviously, $\psi_{BA}(t,\lambda)$ is also an eigenfunction. The relation between the eigenfunctions and adjoint eigenfunctions can be seen from the fact $B_{2n+1}^{*}\partial=-\partial B_{2n+1}$. This fact implies that any eigenfunction Φ gives rise to an adjoint eigenfunction $\Psi=\Phi_x$. In particular, we have $\psi_{BA}^{*}(t,\lambda)=-\lambda^{-1}\psi_{BA}(t,-\lambda)_x$, where $\psi_{BA}^{*}(t,\lambda)\equiv W^{*-1}(e^{-\xi(t,\lambda)})$. Moreover, from the bilinear identity (9), solutions of the BKP hierarchy can be characterized by a single function $\tau(t)$ called τ-function such that[25]

$$w(t,\lambda)=\frac{\tau(t-2[\lambda^{-1}])}{\tau(t)}, \tag{11}$$

where $[\lambda^{-1}]=(\lambda^{-1},\frac{1}{3}\lambda^{-3},\ldots)$. This implies that all dynamical variables $\{u_i\}$ in the Lax operator L can be expressed by τ-function, which is an essential character of the KP and BKP hierarchies. Moreover, another important property of τ function of the BKP is the following Fay-like identity.

Proposition 1: (Reference 30) (Fay identity) The tau function of the BKP hierarchy satisfies

$$\sum_{(s_1,s_2,s_3)}\frac{(s_1-s_0)(s_1+s_2)(s_1+s_3)}{(s_1+s_0)(s_1-s_2)(s_1-s_3)}\tau(t+2[s_2]+2[s_3])\tau(t+2[s_0]+2[s_1])$$
$$+\frac{(s_0-s_1)(s_0-s_2)(s_0-s_3)}{(s_0+s_1)(s_0+s_2)(s_0+s_3)}\tau(t+2[s_0]+2[s_1]+2[s_2]+2[s_3])\tau(t)=0, \tag{12}$$

where (s_1,s_2,s_3) stands for cyclic permutations of s_1, s_2, and s_3.

Proposition 2: (Reference 30) (Differential Fay identity) For the BKP hierarchy,

$$\left(\frac{1}{s_2^2}-\frac{1}{s_1^2}\right)\{\tau(t+2[s_1])\tau(t+2[s_2])-\tau(t+2[s_1]+2[s_2])\tau(t)\}=\left(\frac{1}{s_2}+\frac{1}{s_1}\right)\{\partial\tau(t+2[s_2])\tau(t+2[s_1])$$
$$-\partial\tau(t+2[s_1])\tau(t+2[s_2])\}+\left(\frac{1}{s_2}-\frac{1}{s_1}\right)\{\tau(t+2[s_1]+2[s_2])\,\partial\,\tau(t)-\partial\tau(t+2[s_1]+2[s_2])\tau(t)\}. \tag{13}$$

Note that these identities are indeed different from the counterpart of the KP hierarchy because of the BKP constraint (4). In the next context, we shall show it is for the same reason that the SEP of the BKP hierarchy cannot generate directly the symmetry flow.

III. SEP FOR THE BKP HIERARCHY

As mentioned in Sec. I, we hope to get a new potential BSEP from the SEP of the BKP hierarchy. So we shall study some interesting properties of the SEP of the BKP hierarchy in this section. For any pair of (adjoint) eigenfunctions $\Phi(t),\Psi(t)$, there exists a function $S(\Phi(t),\Psi(t))$ called SEP, determined by the following equations:

$$\frac{\partial}{\partial t_{2n+1}}S(\Phi(t),\Psi(t))=\mathrm{Res}(\partial^{-1}\Psi(L^{2n+1})_+\Phi\partial^{-1}),\quad n=0,1,2,3,\ldots. \tag{14}$$

In particular, for $n=0$, we have

$$\partial_x S(\Phi(t),\Psi(t))=\Phi(t)\Psi(t). \tag{15}$$

One can see that this definition is the same as the one[2] in the KP hierarchy except even number flows are frozen. There are two properties of SEP for the BKP hierarchy.

Lemma 3: If $\Phi(t)$ and $\Psi(t)$ are BKP eigenfunction and adjoint eigenfunction respectively, then one has the following relation:

$$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)=e^{-\xi(t,\lambda)}(\Phi(t)+\mathcal{O}(\lambda^{-1})), \tag{16}$$

$$S(\psi_{BA}(t,\lambda),\Psi(t))=e^{\xi(t,\lambda)}(\Psi(t)\lambda^{-1}+\mathcal{O}(\lambda^{-2})). \tag{17}$$

Proof: We only prove the first identity since the proof of the second one is similar. Because $\psi_{BA}(t,-\lambda)_x=e^{-\xi(t,\lambda)}(-\lambda+\mathcal{O}(1))$ and

$$\int e^{-x\lambda}\Phi(t)dx=-\int\lambda^{-1}\Phi(t)de^{-x\lambda}=-\lambda^{-1}e^{-x\lambda}\Phi(t)+\lambda^{-1}\int e^{-x\lambda}\Phi_x(t)dx$$
$$=\cdots=e^{-x\lambda}(-\lambda^{-1}\Phi(t)+\mathcal{O}(\lambda^{-2})),$$

we find

$$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)=\int\Phi(t)\psi_{BA}(t,-\lambda)_x dx=\int\Phi(t)e^{-\xi(t,\lambda)}(-\lambda+\mathcal{O}(1))dx$$
$$=-\lambda\int\Phi(t)e^{-\xi(t,\lambda)}dx+e^{-\xi(t,\lambda)}\mathcal{O}(\lambda^{-1})$$
$$=-\lambda e^{-\xi(t,\lambda)}(-\lambda^{-1}\Phi(t)+\mathcal{O}(\lambda^{-2}))+e^{-\xi(t,\lambda)}\mathcal{O}(\lambda^{-1})$$
$$=e^{-\xi(t,\lambda)}(\Phi(t)+\mathcal{O}(\lambda^{-1})).$$

■

Lemma 4: If Φ_1 and Φ_2 are two eigenfunctions of the BKP hierarchy, then

$$\Phi_1\Phi_2=S(\Phi_1,\Phi_{2x})+S(\Phi_2,\Phi_{1x}). \tag{18}$$

Proof: Some useful formulas below are needed in the proof.

$$\mathrm{Res}(A) = -\mathrm{Res}(A^*), \tag{19}$$

$$a_x = \partial a - a\,\partial\,, \tag{20}$$

$$\mathrm{Res}(A\partial^{-1}) = A_{[0]}, \tag{21}$$

where A is a pseudodifferential operator, and a is a function,

$$\begin{aligned}
\partial_{t_{2n+1}} S(\Phi_2, \Phi_{1x}) &= Res(\partial^{-1}\Phi_{1x}B_{2n+1}\Phi_2\partial^{-1}) = -Res(\partial^{-1}\Phi_2 B^*_{2n+1}\Phi_{1x}\partial^{-1}) \quad using\ (19)\\
&= Res(\partial^{-1}\Phi_2\,\partial\,B_{2n+1}\partial^{-1}\Phi_{1x}\partial^{-1}) \quad using\ B^*_{2n+1} = -\partial B_{2n+1}\partial^{-1}\\
&= Res(\partial^{-1}\Phi_2\,\partial\,B_{2n+1}\Phi_1\partial^{-1}) - Res(\partial^{-1}\Phi_2\,\partial\,B_{2n+1}\partial^{-1}\Phi_1)\ using\ (20)\\
&= Res(\Phi_2 B_{2n+1}\Phi_1\partial^{-1}) - Res(\partial^{-1}\Phi_{2x}B_{2n+1}\Phi_1\partial^{-1})\\
&\quad + Res(\partial^{-1}\Phi_2 B^*_{2n+1}\Phi_1)\ using\ (20)\ and\ B^*_{2n+1} = -\partial B_{2n+1}\partial^{-1}\\
&= \Phi_2 Res(B_{2n+1}\Phi_1\partial^{-1}) - Res(\partial^{-1}\Phi_{2x}B_{2n+1}\Phi_1\partial^{-1}) + Res(B_{2n+1}\Phi_2\partial^{-1})\Phi_1\ using\ (19)\\
&= \Phi_2 B_{2n+1}(\Phi_1) - Res(\partial^{-1}\Phi_{2x}B_{2n+1}\Phi_1\partial^{-1}) + B_{2n+1}(\Phi_2)\Phi_1\ using\ (21)\\
&= \Phi_2(\partial_{t_{2n+1}}\Phi_1) - \partial_{t_{2n+1}}S(\Phi_1, \Phi_{2x}) + (\partial_{t_{2n+1}}\Phi_2)\Phi_1 = \partial_{t_{2n+1}}(\Phi_1\Phi_2) - \partial_{t_{2n+1}}S(\Phi_1, \Phi_{2x}).
\end{aligned}$$

■

Proposition 5: (Spectral representation) If $\Phi(t)$ *is an eigenfunction of the BKP hierarchy, then*

$$\Phi(t) = \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)S(\Phi(t'), \psi_{BA}(t', -\lambda)_{x'}), \tag{22}$$

where the time t' *is taken at some arbitrary fixed value. In other words,* $\Phi(t)$ *owns a spectral representation in the form of*

$$\Phi(t) = \int d\lambda\lambda^{-1}\varphi(\lambda)\psi_{BA}(t,\lambda), \tag{23}$$

with spectral densities given by SEP, that is, $\varphi(\lambda) = S(\Phi(t'), \psi_{BA}(t', -\lambda)_{x'})$.

Proof: Denote the right hand side of (22) by $I(t,t')$. Then by the BKP bilinear identity (9), one finds that $\partial_{t'_m} I(t,t') = 0$. Hence $I(t,t') = f(t)$. By considering (16), we have

$$I(t, t' = t) = \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)e^{-\xi(t,\lambda)}(\Phi(t) + \mathcal{O}(\lambda^{-1})) = \Phi(t).$$

■

Remark 1: Here we only give the spectral representation for eigenfunctions. As for the adjoint eigenfunctions, the spectral representation can be derived similarly by considering (9) and (17), that is,

$$\Psi(t) = \int d\lambda\psi^*_{BA}(t,\lambda)S(\psi_{BA}(t',\lambda), \Psi(t')). \tag{24}$$

However, because of the relation between the eigenfunctions and adjoint eigenfunctions, we must show that our spectral representations for BKP hierarchy are compatible.

In fact, any adjoint eigenfunction Ψ for BKP can be written as the derivative of an eigenfunction Φ, that is, $\Psi = \Phi_x$. So with the help of (22), (9), and (18) and $\psi^*_{BA}(t,\lambda) = -\lambda^{-1}\psi_{BA}(t, -\lambda)_x$, then

$$\Phi_x(t) = \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)_x S(\Phi(t'),\psi_{BA}(t',-\lambda)_{x'}) = \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)_x\psi_{BA}(t',-\lambda)\Phi(t')$$
$$- \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)_x S(\psi_{BA}(t',-\lambda),\Phi(t')_{x'})$$
$$= -\int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)_x S(\psi_{BA}(t',-\lambda),\Phi(t')_{x'})$$
$$= -\int d\lambda\lambda^{-1}\psi_{BA}(t,-\lambda)_x S(\psi_{BA}(t',\lambda),\Phi(t')_{x'}) \ \ letting \ \ \lambda\to-\lambda$$
$$= \int d\lambda\psi^*_{BA}(t,\lambda)S(\psi_{BA}(t',\lambda),\Phi(t')_{x'}) = \int d\lambda\psi^*_{BA}(t,\lambda)S(\psi_{BA}(t',\lambda),\Psi(t')) = \Psi(t).$$

So our representation is consistent with $\Psi=\Phi_x$, which shows it is necessary to only study the spectral representation of the eigenfunctions for the BKP hierarchy.

Remark 2: Since $\psi^*_{BA}(t,\lambda)=-\lambda^{-1}\psi_{BA}(t,-\lambda)_x$, so we can rewrite (22) as

$$\Phi(t) = -\int d\lambda\psi_{BA}(t,\lambda)S(\Phi(t'),\psi^*_{BA}(t',\lambda)). \tag{25}$$

Our results (24) and (25) can be regarded as a natural reduction from corresponding ones[15] of the KP hierarchy by considering BKP constraints $L^*=-\partial L\partial^{-1}$ and $\Psi=\Phi_x$.

Remark 3: In particular,

$$\psi_{BA}(t,\mu) = \int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)S(\psi_{BA}(t',\mu),\psi_{BA}(t',-\lambda)_{x'}) \tag{26}$$

is given from (22) by setting $\Phi(t)=\psi_{BA}(t,\mu)$.

Now we shall use the above obtained spectral representation to get general expressions of SEP. Before this we will use the differential Fay identity (13) to get $S(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x)$, which is a basic and useful SEP of the BKP hierarchy. According to Proposition 2, set $s_1=\lambda^{-1}$ and $s_2=-\mu^{-1}$, we can find

$$\partial_x\left(\frac{\tau(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\tau(t)}\right) = \frac{\partial_x\tau(t+2[\lambda^{-1}]-2[\mu^{-1}])\tau(t)-\tau(t+2[\lambda^{-1}]-2[\mu^{-1}])\partial_x\tau(t)}{\tau^2(t)}$$
$$= \frac{(-\mu+\lambda)}{(-\mu-\lambda)}\frac{(\partial_x\tau(t-2[\mu^{-1}])\tau(t+2[\lambda^{-1}])-\tau(t-2[\mu^{-1}])\partial_x\tau(t+2[\lambda^{-1}]))}{\tau^2(t)}$$
$$- \frac{(-\mu+\lambda)(\tau(t-2[\mu^{-1}])\tau(t+2[\lambda^{-1}])-\tau(t-2[\lambda^{-1}]-2[\mu^{-1}])\tau(t))}{\tau^2(t)}.$$

Taking into account of the following identity:

$$\partial_x\left(\frac{\tau(t-2[\mu^{-1}])}{\tau(t)}\right)\tau(t+2[\lambda^{-1}]) - \tau(t-2[\mu^{-1}])\partial_x\left(\frac{\tau(t+2[\lambda^{-1}])}{\tau(t)}\right)$$
$$= \frac{\partial_x\tau(t-2[\mu^{-1}])\tau(t+2[\lambda^{-1}])-\tau(t-2[\mu^{-1}])\partial_x\tau(t+2[\lambda^{-1}])}{\tau^2(t)},$$

then,

$$\partial_x\left(\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}\right)$$
$$=\frac{\mu-\lambda}{\mu+\lambda}\left(\partial_x\left(\frac{\pi(t-2[\mu^{-1}])}{\pi(t)}\right)\pi(t+2[\lambda^{-1}])-\pi(t-2[\mu^{-1}])\partial_x\left(\frac{\pi(t+2[\lambda^{-1}])}{\pi(t)}\right)\right)$$
$$+(\mu-\lambda)\left(\frac{\pi(t-2[\mu^{-1}])\pi(t+2[\lambda^{-1}])}{\pi^2(t)}-\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}\right). \tag{27}$$

Furthermore, in order to get $S(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x)$, by using (8) and (11), we first calculate

$$\partial_x(\psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],-\lambda))=\partial_x\left(\frac{\lambda+\mu}{\mu-\lambda}e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}\right)$$
$$=(\lambda+\mu)e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}-\frac{\lambda+\mu}{\lambda-\mu}e^{\xi(t,\mu)-\xi(t,\lambda)}\partial_x\left(\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}\right).$$

Note

$$e^{\xi(-2[\mu^{-1}],-\lambda)}=e^{2(\lambda/\mu+(\lambda/\mu)^3+(\lambda/\mu)^5+\cdots)}=e^{\ln(1+\lambda/\mu)-\ln(1-\lambda/\mu)}=\frac{\mu+\lambda}{\mu-\lambda} \tag{28}$$

is used in the first equality above. Taking (27) into the last term of above formula, then

$$\partial_x(\psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],-\lambda))$$
$$=(\lambda+\mu)e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}-\frac{\lambda+\mu}{\lambda-\mu}e^{\xi(t,\mu)-\xi(t,\lambda)}\left(\frac{\mu-\lambda}{\mu+\lambda}\left\{\partial_x\left(\frac{\pi(t+2[\lambda^{-1}])}{\pi(t)}\right)\pi(t\right.\right.$$
$$\left.+2[\lambda^{-1}])-\pi(t-2[\mu^{-1}])\partial_x\left(\frac{\pi(t+2[\lambda^{-1}]}{\pi(t)}\right)\right\}+(\mu-\lambda)\left(\frac{\pi(t-2[\mu^{-1}])\pi(t+2[\lambda^{-1}])}{\pi^2(t)}\right.$$
$$\left.\left.-\frac{\pi(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\pi(t)}\right)\right)$$
$$=e^{\xi(t,\mu)-\xi(t,\lambda)}\left(\partial_x\left(\frac{\pi(t-2[\mu^{-1}])}{\pi(t)}\right)\pi(t+2[\lambda^{-1}])-\pi(t-2[\mu^{-1}])\partial_x\left(\frac{\pi(t+2[\lambda^{-1}])}{\pi(t)}\right)\right.$$
$$\left.+(\mu+\lambda)\frac{\pi(t-2[\mu^{-1}])\pi(t+2[\lambda^{-1}])}{\pi^2(t)}\right)$$
$$=\partial_x\psi_{BA}(t,\mu)\psi_{BA}(t,-\lambda)-\psi_{BA}(t,\mu)\partial_x\psi_{BA}(t,-\lambda)$$
$$=\partial_x(\psi_{BA}(t,\mu)\psi_{BA}(t,-\lambda))-2\psi_{BA}(t,\mu)\partial_x\psi_{BA}(t,-\lambda).$$

Note the first term cancels the fourth term of the first equality above. So we have

$$\psi_{BA}(t,\mu)\psi_{BA}(t,-\lambda)_x=\tfrac{1}{2}\partial_x\{(\psi_{BA}(t,-\lambda)-\psi_{BA}(t-2[\mu^{-1}],-\lambda))\psi_{BA}(t,\mu)\},$$

which implies

$$S(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x)=\tfrac{1}{2}(\psi_{BA}(t,-\lambda)-\psi_{BA}(t-2[\mu^{-1}],-\lambda))\psi_{BA}(t,\mu). \tag{29}$$

Next, we shall give the expression of another basic SEP-$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)$. According to the spectral representation of $\Phi(t)$ in (23), then

$$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)=S\left(\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x\right)$$

$$=\int d\mu\mu^{-1}\varphi(\mu)S(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x)$$

$$=\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t,\mu)\frac{1}{2}(\psi_{BA}(t,-\lambda)-\psi_{BA}(t-2[\mu^{-1}],-\lambda)) \text{ using (29)}$$

$$=\frac{1}{2}\psi_{BA}(t,-\lambda)\Phi(t)-\frac{1}{2}\underline{\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t-2[\mu^{-1}],-\lambda)\psi_{BA}(t,\mu)}.$$

Thus we only need to compute the underlied part above. To this end, with the help of $e^{\xi(-2[\mu^{-1}],-\lambda)}=(\mu+\lambda)/(\mu-\lambda)$ in (28), we first calculate

$$\psi_{BA}(t-2[\mu^{-1}],-\lambda)\psi_{BA}(t,\mu)=(\lambda+\mu)\frac{1}{\mu}\frac{1}{1-\frac{\lambda}{\mu}}e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\tau(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\tau(t)}$$

$$=(\lambda+\mu)\left[\delta(\lambda,\mu)-\frac{1}{\lambda}\frac{1}{1-\frac{\mu}{\lambda}}\right]e^{\xi(t,\mu)-\xi(t,\lambda)}\frac{\tau(t+2[\lambda^{-1}]-2[\mu^{-1}])}{\tau(t)}$$

$$=-\psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda)+(\lambda+\mu)\delta(\lambda,\mu). \tag{30}$$

Here, the delta function is defined as

$$\delta(\lambda,\mu)=\frac{1}{\mu}\sum_{n=-\infty}^{\infty}\left(\frac{\mu}{\lambda}\right)^n=\frac{1}{\lambda}\frac{1}{1-\frac{\mu}{\lambda}}+\frac{1}{\mu}\frac{1}{1-\frac{\lambda}{\mu}}, \tag{31}$$

and the following property of delta function is used: given a function $f(z)=\Sigma_{i=-\infty}^{\infty}a_iz^i$,

$$f(z)\delta(\lambda,z)=f(\lambda)\delta(\lambda,z)$$

as is seen from $z^i\Sigma_n(z/\lambda)^n=\lambda^i\Sigma_n(z/\lambda)^{n+i}$. Thus taking (30) back into the underlined part above, then

$$\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}\ (t-2[\mu^{-1}],-\lambda))\psi_{BA}(t,\mu)$$

$$=-\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda)+\int d\mu\mu^{-1}\varphi(\mu)(\lambda+\mu)\delta(\lambda,\mu)$$

$$=-\Phi(t+2[\lambda^{-1}])\psi_{BA}(t,-\lambda)+\text{the term independent of } t.$$

So we get

$$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)=\tfrac{1}{2}\psi_{BA}(t,-\lambda)(\Phi(t+2[\lambda^{-1}])+\Phi(t)), \tag{32}$$

since the definition of SEP up to the term independent of t.

Similarly, we can get the expressions of $S(\psi_{BA}(t,\lambda),\Phi_x(t))$ and $S(\Phi_1(t),\Phi_{2x}(t))$ by considering $\Phi_x(t)=\int d\lambda\lambda^{-1}\varphi(\lambda)\psi_{BA}(t,\lambda)_x$, see Appendices A and B. Thus we have the following corollary.

Corollary 6: If $\Phi(t),\Phi_1(t),\Phi_2(t)$ are eigenfunctions of the BKP hierarchy, then

$$S(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)_x)=\tfrac{1}{2}(\psi_{BA}(t,-\lambda)-\psi_{BA}(t-2[\mu^{-1}],-\lambda))\psi_{BA}(t,\mu), \tag{33}$$

$$S(\Phi(t),\psi_{BA}(t,-\lambda)_x)=\tfrac{1}{2}\psi_{BA}(t,-\lambda)(\Phi(t+2[\lambda^{-1}])+\Phi(t)), \tag{34}$$

$$S(\psi_{BA}(t,\lambda),\Phi_x(t))=\tfrac{1}{2}\psi_{BA}(t,\lambda)(\Phi(t)-\Phi(t-2[\lambda^{-1}])), \tag{35}$$

$$S(\Phi_1(t),\Phi_{2x}(t))=\int\int d\lambda d\mu\lambda^{-1}\mu^{-1}\varphi_1(\mu)\varphi_2(\lambda)S(\psi_{BA}(t,\mu),\psi_{BA}(t,\lambda)_x). \tag{36}$$

Remark 4: Note that (35) is also derived by Loirs[16] by a different method.

Remark 5: According to Proposition 5 and Corollary 6, we have

$$\Phi(t)=\int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)\left[\frac{1}{2}\Phi(t'+2[\lambda^{-1}])+\frac{1}{2}\Phi(t')\right]. \tag{37}$$

In fact, the inverse of Proposition 5 is also correct and it provides another formulation of the BKP hierarchy.

Proposition 7: Given a function $\psi(t,\lambda)$ *which has the form* $\psi(t,\lambda)=e^{\xi(t,\lambda)}\sum_{j=0}^{\infty}\omega_j(t)\lambda^{-j}$ *with* $\omega_0=1$ *and* $\xi(t,\lambda)$ *as in (8), where multitime* $t=(t_1=x,t_3,\ldots)$ *and* λ *is the spectral parameter, let us assume that* $\psi(t,\lambda)$ *has the following spectral representation:*

$$\psi(t,\mu)=\int d\lambda\lambda^{-1}\psi(t,\lambda)S(t';\lambda,\mu) \tag{38}$$

for two arbitrary multitimes t *and* t' *, where the function* $S(t;\lambda,\mu)$ *is defined such that* $(\partial/\partial t_1)S(t;\lambda,\mu)=\psi(t,\mu)\psi(t,-\lambda)_x$ *. Then, (38) is equivalent to the Hirota bilinear identity (9), so in this way* $\psi(t,\lambda)$ *becomes BA functions of the associated BKP hierarchy.*

Proof: The proof for one side of the equivalence that Hirota bilinear identity (9) imply the spectral representation (38) is contained in the proof of (26). So we only need to show that (38) implies (9). To the end, by differentiating both side of (38) with respect to t_1', then,

$$0=\partial\psi(t,\lambda)/\partial t_1'=\psi(t',\mu)\int d\lambda\lambda^{-1}\psi(t,\lambda)\psi(t',-\lambda)_{x'}.$$

So

$$\int d\lambda\lambda^{-1}\psi(t,\lambda)\psi(t',-\lambda)\equiv C.$$

By letting $t'=t$, and considering $\psi(t,\lambda)\psi(t,-\lambda)=1+\mathcal{O}(\lambda^{-1})$, we have $C=1$. Thus $\psi(t,\lambda)$ satisfies $\int d\lambda\lambda^{-1}\psi(t,\lambda)\psi(t',-\lambda)=1$, i.e., the Hiorta biliner equations of the BKP hierarchy. ■

By now, we have established the SEP method for the BKP hierarchy, which provides another formulation of the BKP hierarchy.

IV. BSEP

Based on the useful properties of the SEP given Sec. III, we are now in a position to discuss a new potential Ω-BSEP,[1,6] which will be used to generate the "ghost" flow of the BKP hierarchy in Sec. V. We first provide three expressions of Ω for different eigenfunctions, and then give their identities.

BSEP is also defined as a function of a pair of BKP eigenfunctions Φ_1 and Φ_2,

$$\Omega(\Phi_1,\Phi_2)=S(\Phi_2,\Phi_{1x})-S(\Phi_1,\Phi_{2x}). \tag{39}$$

The definition of BSEP can be up to a constant of integration. It is obvious that $\Omega(\Phi_1,\Phi_2)=-\Omega(\Phi_2,\Phi_1)$ and that $\Omega(\Phi,1)=\Phi$ (since 1 is an eigenfunction). So according to (29) and (30), we have

$$\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)) = -\psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda) + \tfrac{1}{2}(\lambda+\mu)\delta(\lambda,\mu)$$
$$= \psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],-\lambda) - \tfrac{1}{2}(\lambda+\mu)\delta(\lambda,\mu). \tag{40}$$

Remark 6: As the definition of BSEP can be up to the term independent of t, we can omit the terms independent of t in (40). That is,

$$\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)) = -\psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda). \tag{41}$$

We would like to mention there is another expression for $\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda))$, i.e.,

$$\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda)) = -\frac{\mathcal{X}(\lambda,\mu)\tau(t)}{\tau(t)}. \tag{42}$$

Here the vertex operator[25] is defined as follows:

$$\mathcal{X}(\lambda,\mu) \equiv :e^{\theta(\lambda)}::e^{-\theta(\mu)}: = e^{\xi(t+2[\lambda^{-1}],\mu)-\xi(t,\lambda)}e^{\sum_1^\infty 2/(2l-1)(\lambda^{-(2l-1)}-\mu^{-(2l-1)})\partial/(\partial t_{2l-1})}$$
$$= -e^{\xi(t,\mu)-\xi(t-2[\mu^{-1}],\lambda)}e^{\sum_1^\infty 2/(2l-1)(\lambda^{-(2l-1)}-\mu^{-(2l-1)})\partial/(\partial t_{2l-1})} + (\lambda+\mu)\delta(\lambda,\mu) \tag{43}$$

where

$$\theta(\lambda) \equiv -\sum_{l=1}^{\infty}\lambda^{2l-1}t_{2l-1} + \sum_{l=1}^{\infty}\frac{1}{2l-1}\lambda^{-(2l-1)}\frac{\partial}{\partial t_{2l-1}} \tag{44}$$

the columns $:\cdots:$ indicate Wick Normal ordering with respect to the creation/annihilation "modes" t_l and $\partial/\partial t_l$, respectively. Thus according to the definition of the Vertex operator (43) and the wave functions (8) and (11), we can easily get

$$\frac{\mathcal{X}(\lambda,\mu)\tau(t)}{\tau(t)} = \psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda) = -\psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],-\lambda) + (\lambda+\mu)\delta(\lambda,\mu). \tag{45}$$

So (42) is true.

As for $\Omega(\Phi(t),\psi_{BA}(t,-\lambda))$, according to the definition of Ω, two identities (34) and (35), then $\Omega(\Phi(t),\psi_{BA}(t,-\lambda))$ can expressed by the form of

$$\Omega(\Phi(t),\psi_{BA}(t,-\lambda)) = S(\psi_{BA}(t,-\lambda),\Phi_x(t)) - S(\Phi(t),\psi_{BA}(t,-\lambda)_x)$$
$$= \tfrac{1}{2}(\Phi(t)-\Phi(t+2[\lambda^{-1}]))\psi_{BA}(t,-\lambda) - \tfrac{1}{2}\psi_{BA}(t,-\lambda)[\Phi(t+2[\lambda^{-1}])+\Phi(t)]$$
$$= -\psi_{BA}(t,-\lambda)\Phi(t+2[\lambda^{-1}]). \tag{46}$$

Note that (46) implies (41) in Remark 6 as we expected.

Remark 7: In fact, with the help of spectral representation (23) for the BKP hierarchy and the expression for $\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda))$ (41), $\Omega(\Phi(t),\psi_{BA}(t,-\lambda))$ is derived alternatively as

$$\Omega(\Phi(t),\psi_{BA}(t,-\lambda)) = \int d\mu\mu^{-1}\varphi(\mu)\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,-\lambda))$$
$$= -\int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t+2[\lambda^{-1}],\mu)\psi_{BA}(t,-\lambda)$$
$$= -\Phi(t+2[\lambda^{-1}])\psi_{BA}(t,-\lambda).$$

We further show a more general Ω of eigenfunctions Φ_1 and Φ_2,

$$\Omega(\Phi_1,\Phi_2)=\int\int d\lambda d\mu\lambda^{-1}\mu^{-1}\varphi_1(\mu)\varphi_2(\lambda)\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,\lambda)). \tag{47}$$

Next, we would like to show three identities on Ω of the BKP hierarchy.

Lemma 8: For the BKP *hierarchy,*

$$\Delta_z(\psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],\lambda))=\psi_{BA}(t,\lambda)\psi_{BA}(t-2[z^{-1}],\mu)-\psi_{BA}(t,\mu)\psi_{BA}(t-2[z^{-1}],\lambda), \tag{48}$$

where $\Delta_z=e^{-\sum_{l=1}^{\infty}[2/(2l-1)]z^{-(2l-1)}[\partial/(\partial t_{2l-1})]}-1$ *is a shift-difference operator.*

Proof: First of all, we move all terms in right side hand of (48) to the left, take $\psi_{BA}(t,\lambda)$ in (8) and (11) into it, then

$$\begin{aligned}
&(48)\text{holds}\Leftrightarrow \psi_{BA}(t-2[z^{-1}],\mu)\psi_{BA}(t-2[\mu^{-1}]-2[z^{-1}],\lambda)-\psi_{BA}(t,\mu)\psi_{BA}(t-2[\mu^{-1}],\lambda)\\
&\quad+\psi_{BA}(t,\mu)\psi_{BA}(t-2[z^{-1}],\lambda)-\psi_{BA}(t,\lambda)\psi_{BA}(t-2[z^{-1}],\mu)=0\\
&\quad\Leftrightarrow\frac{(z-\mu)(z-\lambda)(\mu-\lambda)}{(z+\mu)(z+\lambda)(\mu+\lambda)}\frac{\tau(t-2[\mu^{-1}]-2[z^{-1}]-2[\lambda^{-1}])}{\tau(t-2[z^{-1}])}-\frac{\mu-\lambda}{\mu+\lambda}\frac{\tau(t-2[\mu^{-1}]-2[\lambda^{-1}])}{\tau(t)}\\
&\quad+\frac{z-\lambda}{z+\lambda}\frac{\tau(t-2[\mu^{-1}])}{\tau(t)}\frac{\tau(t-2[z^{-1}]-2[\lambda^{-1}])}{\tau(t-2[z^{-1}])}-\frac{z-\mu}{z+\mu}\frac{\tau(t-2[\lambda^{-1}])}{\tau(t)}\frac{\tau(t-2[z^{-1}]-2[\mu^{-1}])}{\tau(t-2[z^{-1}])}\\
&=0\ \text{using(28), removing}\ e^{\xi(t,\lambda)+\xi(t,\mu)}\\
&\quad\Leftrightarrow\frac{(z-\mu)(z-\lambda)(\mu-\lambda)}{(z+\mu)(z+\lambda)(\mu+\lambda)}\tau(t-2[\mu^{-1}]-2[z^{-1}]-2[\lambda^{-1}])\tau(t)-\frac{\mu-\lambda}{\mu+\lambda}\tau(t-2[\mu^{-1}]\\
&\quad-2[\lambda^{-1}])\tau(t-2[z^{-1}])+\frac{z-\lambda}{z+\lambda}\tau(t-2[\mu^{-1}])\tau(t-2[z^{-1}]-2[\lambda^{-1}])\\
&\quad-\frac{z-\mu}{z+\mu}\tau(t-2[\lambda^{-1}])\tau(t-2[z^{-1}]-2[\mu^{-1}])=0\ \text{multiplying}\ \tau(t)\tau(t-2[z^{-1}])\\
&\quad\Leftrightarrow\frac{(\mu-z)(\mu-\lambda)}{(\mu+z)(\mu+\lambda)}\tau(t)\tau(t-2[\mu^{-1}]-2[z^{-1}]-2[\lambda^{-1}])+\frac{(z+\lambda)(z-\mu)}{(z-\lambda)(z+\mu)}\\
&\qquad\times\tau(t-2[z^{-1}]-2[\mu^{-1}])\tau(t-2[\lambda^{-1}])\\
&\quad+\frac{(\lambda-\mu)(\lambda+z)}{(\lambda+\mu)(\lambda-z)}\tau(t-2[\mu^{-1}]-2[\lambda^{-1}])\tau(t-2[z^{-1}])\\
&\quad-\tau(t-2[\mu^{-1}])\tau(t-2[z^{-1}]-2[\lambda^{-1}])=0,\quad\text{multiplying}\frac{\lambda+z}{\lambda-z}.
\end{aligned}$$

For convenience, denote the left hand side of above equality by C. Second, we shall prove indeed $C=0$ from the Fay identity (12) of the BKP hierarchy, thus (48) is proven. To this end, by letting $s_0=0$ in Fay identity (12), then

$$\begin{aligned}
&\frac{(s_1+s_2)(s_1+s_3)}{(s_1-s_2)(s_1-s_3)}\tau(t+2[s_2]+2[s_3])\tau(t+2[s_1])+\frac{(s_2+s_3)(s_2+s_1)}{(s_2-s_3)(s_2-s_1)}\tau(t+2[s_3]+2[s_1])\tau(t+2[s_2])\\
&\quad+\frac{(s_3+s_1)(s_3+s_2)}{(s_3-s_1)(s_3-s_2)}\tau(t+2[s_1]+2[s_2])\tau(t+2[s_3])-\tau(t+2[s_1]+2[s_2]+2[s_3])\tau(t)=0.
\end{aligned}$$

Then, after shifting $t\mapsto t-2[s_2]-2[s_3]$ and letting $[s_1]\mapsto-[s_1]$ in above equation, it becomes

$$\frac{(s_1-s_2)(s_1-s_3)}{(s_1+s_2)(s_1+s_3)}\tau(t)\tau(t-2[s_1]-2[s_2]-2[s_3])+\frac{(s_2+s_3)(s_2-s_1)}{(s_2-s_3)(s_2+s_1)}\tau(t-2[s_2]-2[s_1])\tau(t-2[s_3])$$
$$+\frac{(s_3-s_1)(s_3+s_2)}{(s_3+s_1)(s_3-s_2)}\tau(t-2[s_1]-2[s_3])\tau(t-2[s_2])-\tau(t-2[s_1])\tau(t-2[s_2]-2[s_3])=0.$$

At last, setting $s_1=\mu^{-1}$, $s_2=z^{-1}$, $s_3=\lambda^{-1}$, we have

$$\frac{(\mu-z)(\mu-\lambda)}{(\mu+z)(\mu+\lambda)}\tau(t)\tau(t-2[\mu^{-1}]-2[z^{-1}]-2[\lambda^{-1}])+\frac{(z+\lambda)(z-\mu)}{(z-\lambda)(z+\mu)}\tau(t-2[z^{-1}]-2[\mu^{-1}])\tau(t$$
$$-2[\lambda^{-1}])+\frac{(\lambda-\mu)(\lambda+z)}{(\lambda+\mu)(\lambda-z)}\tau(t-2[\mu^{-1}]-2[\lambda^{-1}])\tau(t-2[z^{-1}])-\tau(t-2[\mu^{-1}])\tau(t-2[z^{-1}]$$
$$-2[\lambda^{-1}])=0,$$

i.e., $C=0$, as we claimed before. This is the end of the proof. ■

After the preparation above, now we can give two important identities of the BSEP below.

Proposition 9: Under shift of the times t of the BKP *hierarchy,* BSEP *obeys*

$$\Omega(\Phi_1(t-2[z^{-1}]),\Phi_2(t-2[z^{-1}]))-\Omega(\Phi_1(t),\Phi_2(t))=\Phi_1(t-2[z^{-1}])\Phi_2(t)-\Phi_1(t)\Phi_2(t-2[z^{-1}]), \tag{49}$$

$$\Omega(\Phi_1(t+2[z^{-1}]),\Phi_2(t+2[z^{-1}]))-\Omega(\Phi_1(t),\Phi_2(t))=\Phi_1(t+2[z^{-1}])\Phi_2(t)-\Phi_1(t)\Phi_2(t+2[z^{-1}]). \tag{50}$$

Proof: By a straightforward calculation, then

$$\begin{aligned}\Delta_z\Omega(\Phi_1,\Phi_2)&=\int\int d\lambda d\mu\lambda^{-1}\mu^{-1}\varphi_1(\mu)\varphi_2(\lambda)\Delta_z\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,\lambda))\ \text{using(47)}\\&=\int\int d\lambda d\mu\lambda^{-1}\mu^{-1}\varphi_1(\mu)\varphi_2(\lambda)(\psi_{BA}(t,\lambda)\psi_{BA}(t-2[z^{-1}],\mu)\\&\quad-\psi_{BA}(t,\mu)\psi_{BA}(t-2[z^{-1}],\lambda)),\ \text{using(41) and (48)}\\&=\Phi_1(t-2[z^{-1}])\Phi_2(t)-\Phi_1(t)\Phi_2(t-2[z^{-1}])\quad\text{using (23)}.\end{aligned}$$

So (49) is proven. By shift $t\mapsto t+2[z^{-1}]$, (50) is derived from (49). ■

Remark 8: In fact, these identities above have been given in Loris' paper,[16] but here we give another proof and our proof is much easier.

V. "GHOST" SYMMETRY

After the preparation above, now we can define the "ghost" symmetry flows generated by the BSEP through its action on the Lax operator. We shall further show its actions on the dressing operator, eigenfunction $\Phi(t)$, and τ function.

Given a set of eigenfunctions $\Phi_{1a},\Phi_{2a},a\in\{\alpha\}$, the "ghost" symmetry of the BKP hierarchy is defined in the following way:

$$\partial_\alpha L\equiv\left[\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\partial^{-1}\Phi_{2a,x}),L\right],\quad \partial_\alpha W\equiv\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\partial^{-1}\Phi_{2a,x})W. \tag{51}$$

Next, we need to show that the definition above is consistent with the BKP constraint (4) and ∂_α commutes with $\partial_{t_{2n+1}}$. In other words, ∂_α is indeed a kind of symmetry flow of the BKP hierarchy.

For simplicity in the next context, we introduce an operator $A=\Sigma_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\partial^{-1}\Phi_{2a,x})$.

Proposition 10: ∂_α is consistent with the BKP *constraint (4), i.e.,* $(\partial_\alpha L^*)\partial+\partial(\partial_\alpha L)=0$.

Proof: According to the definition of A, and using a identity $\partial^{-1}f=f\partial^{-1}-\partial^{-1}f_x\partial^{-1}$, we have

$$\begin{aligned}A^*\partial+\partial A=&\sum_{a\in\{\alpha\}}(\Phi_{2a,x}\partial^{-1}\Phi_{1a}\partial-\Phi_{1a,x}\partial^{-1}\Phi_{2a}\partial+\partial\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\partial\Phi_{1a}\partial^{-1}\Phi_{2a,x})=\sum_{a\in\{\alpha\}}(\Phi_{2a,x}\Phi_{1a}\\&-\Phi_{2a,x}\partial^{-1}\Phi_{1a,x}-\Phi_{1a,x}\Phi_{2a}+\Phi_{1a,x}\partial^{-1}\Phi_{2a,x}+\Phi_{2a}\Phi_{1a,x}+\Phi_{2a,x}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\Phi_{2a,x}\\&-\Phi_{1a,x}\partial^{-1}\Phi_{2a,x})=0.\end{aligned}$$

Furthermore, using the definition of ∂_α, a simple computation leads to

$$\begin{aligned}(\partial_\alpha L^*)\partial+\partial(\partial_\alpha L)=&[A,L]^*\partial+\partial[A,L]=-[A^*,L^*]\partial+\partial[A,L]=-\partial L\partial^{-1}A^*\partial+A^*\partial L\partial^{-1}\partial+\partial[A,L]\\=&-\partial(L\partial^{-1}A^*\partial-\partial^{-1}A^*\partial L)+\partial[A,L]=\partial[\partial^{-1}A^*\partial,L]+\partial[A,L]=\partial[\partial^{-1}A^*\partial+A,L]\\=&0\end{aligned}$$

because of the above identity on A. This means $\partial_\alpha L^*$ is consistent with BKP constraint (4). ■

Proposition 11: ∂_α commutes with $\partial_{t_{2n+1}}$.

Proof: We first claim the following equations

$$\partial_\alpha B_{2n+1}-\partial_{t_{2n+1}}A=[A,B_{2n+1}] \tag{52}$$

hold for A and ∂_α, which will be proven latter. With the help of above equation, a simple calculation infers

$$\begin{aligned}[\partial_{t_{2n+1}},\partial_\alpha]L=&\partial_{t_{2n+1}}([A,L])-\partial_\alpha([B_{2n+1},L])=[\partial_{t_{2n+1}}A,L]+[A,[B_{2n+1},L]]-[\partial_\alpha B_{2n+1},L]\\&-[B_{2n+1},[A,L]]=[\partial_{t_{2n+1}}A-\partial_\alpha B_{2n+1}+[A,B_{2n+1}],L]=0,\end{aligned}$$

which shows ∂_α commutes with $\partial_{t_{2n+1}}$. Therefore, the remaining part of the proof is to show our claimed statement (52). First of all, the definition of the "ghost" flows $\partial_\alpha L=[A,L]$ implies obviously $\partial_\alpha L^{2n+1}=[A,L^{2n+1}]$. Thus, we have

$$\partial_\alpha B_{2n+1}=([A,L^{2n+1}])_+=([A,B_{2n+1}])_+. \tag{53}$$

Second, the derivative of A with respect to t_{2n+1} is given by

$$\begin{aligned}\partial_{t_{2n+1}}A=&\sum_{a\in\{\alpha\}}((\partial_{t_{2n+1}}\Phi_{2a})\partial^{-1}\Phi_{1a,x}-(\partial_{t_{2n+1}}\Phi_{1a})\partial^{-1}\Phi_{2a,x})+\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}(\partial_{t_{2n+1}}\Phi_{1a,x})\\&-\Phi_{1a}\partial^{-1}(\partial_{t_{2n+1}}\Phi_{2a,x})).\end{aligned}$$

Taking (10) into it, then

$$\begin{aligned}\partial_{t_{2n+1}}A=&\sum_{a\in\{\alpha\}}(B_{2n+1}(\Phi_{2a})\partial^{-1}\Phi_{1a,x}-B_{2n+1}(\Phi_{1a})\partial^{-1}\Phi_{2a,x})-\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}B^*_{2n+1}(\Phi_{1a,x})\\&-\Phi_{1a}\partial^{-1}B^*_{2n+1}(\Phi_{2a,x})).\end{aligned}$$

Note $\Phi_{1a,x}$ and $\Phi_{2a,x}$ are two adjoint eigenfunctions. Furthermore,

$$\begin{aligned}\partial_{t_{2n+1}}A=&\left(B_{2n+1}\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\partial^{-1}\Phi_{2a,x})\right)_-\left(\sum_{a\in\{\alpha\}}(\Phi_{2a}\partial^{-1}\Phi_{1a,x}-\Phi_{1a}\partial^{-1}\ \Phi_{2a,x}))B_{2n+1}\right)_-\\=&(B_{2n+1}A)_--(AB_{2n+1})_-.\end{aligned}$$

Here the relation $(F_+\partial^{-1})_-=F_{[0]}\partial^{-1}$ (F is a pseudodifferential operator) is used. With the relation $(\partial^{-1}F_+)_-=\partial^{-1}(F^*)_{[0]}$ (F is a pseudodifferential operator.) Thus, we have

$$\partial_{t_{2n+1}} A = -([A, B_{2n+1}])_-, \tag{54}$$

At last, according to (53) and (54), (52) is obtained. ■

Next, let us see the action of the above "ghost" flows on the eigenfunctions Φ.

Proposition 12: The "ghost" symmetry is the compatible condition of the linear problems,

$$\partial_{t_{2n+1}} \Phi = B_{2n+1}(\Phi), \tag{55}$$

$$\partial_\alpha \Phi = \frac{1}{2} \sum_{a \in \{\alpha\}} (\Phi_{2a} \Omega(\Phi_{1a}, \Phi) - \Phi_{1a} \Omega(\Phi_{2a}, \Phi)). \tag{56}$$

Proof: The main idea of the proof is to use (52), which is equivalent to ghost symmetry flow (51), to get the commutativity of the flows $\partial_\alpha \partial_{t_{2n+1}} \Phi = \partial_{t_{2n+1}} \partial_\alpha \Phi$. So, according to (18), we can rewrite (56) into

$$\begin{aligned} \partial_\alpha \Phi &= \frac{1}{2} \sum_{a \in \{\alpha\}} \Phi_{2a}(S(\Phi, \Phi_{1ax}) - S(\Phi_{1a}, \Phi_x)) - (1 \leftrightarrow 2) = \sum_{a \in \{\alpha\}} \left(\Phi_{2a} S(\Phi, \Phi_{1ax}) - \frac{1}{2} \Phi_{1a} \Phi_{2a} \Phi \right) \\ &\quad - (1 \leftrightarrow 2) = \sum_{a \in \{\alpha\}} (\Phi_{2a} S(\Phi, \Phi_{1ax}) - \Phi_{1a} S(\Phi, \Phi_{2ax})), \end{aligned} \tag{57}$$

and then

$$\begin{aligned} \partial_{t_{2n+1}}(\partial_\alpha \Phi) &= \sum_{a \in \{\alpha\}} \{(\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1ax}) - (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2ax}) + \Phi_{2a} \operatorname{Res}(\partial^{-1} \Phi_{1ax} B_{2n+1} \Phi \partial^{-1}) \\ &\quad - \Phi_{1a} \operatorname{Res}(\partial^{-1} \Phi_{2ax} B_{2n+1} \Phi \partial^{-1})\} = \sum_{a \in \{\alpha\}} \{(\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1ax}) - (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2ax})\} \\ &\quad + \operatorname{Res}(A B_{2n+1} \Phi \partial^{-1}). \end{aligned} \tag{58}$$

By a tedious but straightforward calculation, we have (see Appendix C)

$$\operatorname{Res}(A B_{2n+1} \Phi \partial^{-1}) = \partial_\alpha(\partial_{t_{2n+1}} \Phi) - \sum_{a \in \{\alpha\}} ((\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1a,x}) - (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2a,x})). \tag{59}$$

Thus by substituting (59) and (58), we get

$$\partial_\alpha(\partial_{t_{2n+1}} \Phi) = \partial_{t_{2n+1}}(\partial_\alpha \Phi).$$

■

We now consider the commutativity of two "ghost" symmetries generated by different pairs of eigenfunctions $\{\Phi_{1a}, \Phi_{2a}\}_{a \in \{\alpha\}}$ and $\{\Phi_{1b}, \Phi_{2b}\}_{b \in \{\beta\}}$, and their corresponding flows are $\partial_\alpha L = [A, L]$ and $\partial_\beta L = [A', L]$. Here $A = \Sigma_{a \in \{\alpha\}} (\Phi_{2a} \partial^{-1} \Phi_{1a,x} - \Phi_{1a} \partial^{-1} \Phi_{2a,x})$ as before, $A' = \Sigma_{b \in \{\beta\}} \Phi_{2b} (\partial^{-1} \Phi_{1b,x} - \Phi_{1b} \partial^{-1} \Phi_{2b,x})$.

Proposition 13: If two "ghost" symmetry flows ∂_α and ∂_β are generated by A and A′ above, then $[\partial_\alpha, \partial_\beta] = 0$.

Proof: By using the relation

$$f_1 \partial^{-1} g_1 f_2 \partial^{-1} g_2 = f_1 S(f_2, g_1) \partial^{-1} g_2 - f_1 \partial^{-1} S(f_2, g_1) g_2, \tag{60}$$

then,

$$
\begin{aligned}
AA' &= \sum_{a,b} (\Phi_{2a}\partial^{-1}\Phi_{1a,x} - \Phi_{1a}\partial^{-1}\Phi_{2a,x})(\Phi_{2b}\partial^{-1}\Phi_{1b,x} - \Phi_{1b}\partial^{-1}\Phi_{2b,x}) = \sum_{a,b} (\Phi_{2a}\partial^{-1}\Phi_{1a,x}\Phi_{2b}\partial^{-1}\Phi_{1b,x} \\
&\quad - \Phi_{2a}\partial^{-1}\Phi_{1a,x}\Phi_{1b}\partial^{-1}\Phi_{2b,x} - \Phi_{1a}\partial^{-1}\Phi_{2a,x}\Phi_{2b}\partial^{-1}\Phi_{1b,x} + \Phi_{1a}\partial^{-1}\Phi_{2a,x}\Phi_{1b}\partial^{-1}\Phi_{2b,x}) \\
&= \sum_{a,b} (\Phi_{2a}S(\Phi_{2b},\Phi_{1a,x})\partial^{-1}\Phi_{1b,x} - \Phi_{2a}\partial^{-1}S(\Phi_{2b},\Phi_{1a,x})\Phi_{1b,x} - \Phi_{2a}S(\Phi_{1b},\Phi_{1a,x})\partial^{-1}\Phi_{2b,x} \\
&\quad + \Phi_{2a}\partial^{-1}S(\Phi_{1b},\Phi_{1a,x})\Phi_{2b,x} - \Phi_{1a}S(\Phi_{2b},\Phi_{2a,x})\partial^{-1}\Phi_{1b,x} + \Phi_{1a}\partial^{-1}S(\Phi_{2b},\Phi_{2a,x})\Phi_{1b,x} \\
&\quad + \Phi_{1a}S(\Phi_{1b},\Phi_{2a,x})\partial^{-1}\Phi_{2b,x} - \Phi_{1a}\partial^{-1}S(\Phi_{1b},\Phi_{2a,x})\Phi_{2b,x}).
\end{aligned}
$$

Collecting terms in AA' according to $\partial^{-1}\Phi_{1b,x}$, $\partial^{-1}\Phi_{2b,x}$, $\Phi_{2a}\partial^{-1}$, and $\Phi_{1a}\partial^{-1}$ in order, then using (57), we have

$$
\begin{aligned}
AA' &= \sum_{a,b} (\Phi_{2a}S(\Phi_{2b},\Phi_{1a,x}) - \Phi_{1a}S\ (\Phi_{2b},\Phi_{2a,x}))\partial^{-1}\Phi_{1b,x} - (\Phi_{2a}S(\Phi_{1b},\Phi_{1a,x}) \\
&\quad - \Phi_{1a}S(\Phi_{1b},\Phi_{2a,x}))\partial^{-1}\Phi_{2b,x} + \Phi_{2a}\partial^{-1}(S(\Phi_{1b},\Phi_{1a,x})\Phi_{2b,x} - S(\Phi_{2b},\Phi_{1a,x})\Phi_{1b,x}) \\
&\quad + \Phi_{1a}\partial^{-1}(S(\Phi_{2b},\Phi_{2a,x})\Phi_{1b,x} - S(\Phi_{1b},\Phi_{2a,x})\Phi_{2b,x})) \\
&= \sum_{b} ((\partial_\alpha\Phi_{2b})\partial^{-1}\Phi_{1b,x} - (\partial_\alpha\Phi_{1b})\partial^{-1}\Phi_{2b,x}) + \sum_{a} (-\Phi_{2a}\partial^{-1}(\partial_\beta\Phi_{1a,x}) + \Phi_{1a}\partial^{-1}(\partial_\beta\Phi_{2a,x})).
\end{aligned}
$$

So

$$
\begin{aligned}
[A,A'] &= \sum_{b} \{(\partial_\alpha\Phi_{2b})\partial^{-1}\Phi_{1b,x} - (\partial_\alpha\Phi_{1b})\partial^{-1}\Phi_{2b,x}\} + \sum_{a} \{-\Phi_{2a}\partial^{-1}(\partial_\beta\Phi_{1a,x}) + \Phi_{1a}\partial^{-1}(\partial_\beta\Phi_{2a,x})\} \\
&\quad + \sum_{a} \{-(\partial_\beta\Phi_{2a})\partial^{-1}\Phi_{1a,x} + (\partial_\beta\Phi_{1a})\partial^{-1}\Phi_{2a,x}\} + \sum_{b} \{\Phi_{2b}\partial^{-1}(\partial_\alpha\Phi_{1b,x}) - \Phi_{1b}\partial^{-1}(\partial_\alpha\Phi_{2b,x})\} \\
&= \sum_{a} \{-\Phi_{2a}\partial^{-1}(\partial_\beta\Phi_{1a,x}) + \Phi_{1a}\partial^{-1}(\partial_\beta\Phi_{2a,x}) - (\partial_\beta\Phi_{2a})\partial^{-1}\Phi_{1a,x} + (\partial_\beta\Phi_{1a})\partial^{-1}\Phi_{2a,x}\} \\
&\quad + \sum_{b} \{\Phi_{2b}\partial^{-1}(\partial_\alpha\Phi_{1b,x}) - \Phi_{1b}\partial^{-1}(\partial_\alpha\Phi_{2b,x}) + (\partial_\alpha\Phi_{2b})\partial^{-1}\Phi_{1b,x} - (\partial_\alpha\Phi_{1b})\ \partial^{-1}\Phi_{2b,x})\} \\
&= -\partial_\beta A + \partial_\alpha A'.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
[\partial_\alpha,\partial_\beta]L &= \partial_\alpha[A',L] - \partial_\beta[A,L] = [\partial_\alpha A' - \partial_\beta A,L] + [A',[A,L]] - [A,[A',L]] \\
&= [\partial_\alpha A' - \partial_\beta A + [A',A],L] = 0.
\end{aligned}
$$

■

At last, let us see the action of "ghost" flow on the τ function.
Proposition 14:

$$
\partial_\alpha\tau(t) = \frac{1}{2}\sum_{a\in\{\alpha\}} \Omega(\Phi_{2a}(t),\Phi_{1a}(t))\tau(t). \tag{61}
$$

Proof: Since $\psi_{BA}(t,\lambda)$ is also an eigenfunction, so (56) implies

$$\partial_\alpha \psi_{BA}(t,\lambda) = \frac{1}{2}\sum_{a\in\{\alpha\}} [\Phi_{2a}(t)\Omega(\Phi_{1a}(t),\psi_{BA}(t,\lambda)) - \Phi_{1a}(t)\Omega(\Phi_{2a}(t),\psi_{BA}(t,\lambda))]$$

$$= \frac{1}{2}\sum_{a\in\{\alpha\}} [-\Phi_{2a}(t)\Phi_{1a}(t-2[\lambda^{-1}]) + \Phi_{1a}(t)\Phi_{2a}(t-2[\lambda^{-1}])]\psi_{BA}(t,\lambda) \text{ using (46)}$$

$$= \frac{1}{2}\sum_{a\in\{\alpha\}} \Delta_\lambda \Omega(\Phi_{2a}(t),\Phi_{1a}(t))\psi_{BA}(t,\lambda) \text{ using (49)}$$

So we have $\partial_\alpha \tau(t) = \frac{1}{2}\Sigma_{a\in\{\alpha\}}\Omega(\Phi_{2a}(t),\Phi_{1a}(t))\tau(t)$ using the expression of $\psi_{BA}(t,\lambda)$ in (8) and (11). ■

Remark 9: So starting from the "ghost" symmetry, we find that $C\tau + \frac{1}{2}\Sigma_{a\in\{\alpha\}}\Omega(\Phi_{2a}(t),\Phi_{1a}(t))\tau(t)$ is a new τ function of the BKP hierarchy. This transformation is also given by Loris[16] started from bilinear identity.

Remark 10: The symmetry reduction in BKP hierarchy, which is now called constrained BKP (cBKP) hierarchy,[16] is just to identify ∂_α with $-\partial_{t_{2n+1}}$, i.e.,

$$(L^{2n+1})_- = \sum_{a\in\{\alpha\}} \Phi_{2a}\partial^{-1}\Phi_{1a,x} - \Phi_{1a}\partial^{-1}\Phi_{2a,x} \tag{62}$$

or

$$\partial_{t_{2n+1}}\tau(t) = -\frac{1}{2}\sum_{a\in\{\alpha\}} \Omega(\Phi_{2a}(t),\Phi_{1a}(t))\tau(t) = \frac{1}{2}\sum_{a\in\{\alpha\}} \Omega(\Phi_{1a}(t),\Phi_{2a}(t))\tau(t). \tag{63}$$

Note if set $(L^{2n+1})_- = A$ as (62), then $\partial_{t_{2n+1}}L = [B_{2n+1},L] = [-L_-^{2n+1},L] = -\partial_\alpha L$. So $\partial_\alpha = -\partial_{t_{2n+1}}$.

To conclude this section, we would like to stress that the "ghost" symmetry (51) of the BKP hierarchy is indeed different from the counterpart in the KP hierarchy. This difference is due to the BKP constraint (4). Moreover, the BSEP provides a convenient tool to show it.

VI. APPLICATIONS

In this section, we shall show two applications for previous results.

First, let us derive a bilinear identity for the cBKP hierarchy (62) through the spectral representation of the BKP hierarchy. Since

$$-\lambda^{2n+1}\psi_{BA}(t,-\lambda) = L^{2n+1}(\psi_{BA}(t,-\lambda)) = (L^{2n+1})_+(\psi_{BA}(t,-\lambda)) + (L^{2n+1})_-(\psi_{BA}(t,-\lambda))$$

$$= \partial_{t_{2n+1}}\psi_{BA}(t,-\lambda) + \partial_\alpha\psi_{BA}(t,-\lambda) = \partial_{t_{2n+1}}\psi_{BA}(t,\lambda)$$

$$+ \frac{1}{2}\sum_{a\in\{\alpha\}} [\Phi_{2a}(t)\Omega(\Phi_{1a}(t),\Psi_{BA}(t,-\lambda))$$

$$- \Phi_{1a}(t)\Omega(\Phi_{2a}(t),\Psi_{BA}(t,-\lambda))] \text{ using (56)}$$

$$= \partial_{t_{2n+1}}\psi_{BA}(t,-\lambda) + \frac{1}{2}\sum_{a\in\{\alpha\}} [-\Phi_{2a}(t)\Phi_{1a}(t+2[\lambda^{-1}])$$

$$+ \Phi_{1a}(t)\Phi_{2a}(t+2[\lambda^{-1}])]\Psi_{BA}(t,-\lambda) \text{ using (46)}. \tag{64}$$

So according to (64) and the bilinear identity of the BKP hierarchy, we have

$$\int d\lambda\lambda^{2n}\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)=\sum_{a\in\{\alpha\}}\frac{1}{2}\int d\lambda\lambda^{-1}[\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)\Phi_{2a}(t')\Phi_{1a}(t'+2[\lambda^{-1}])$$

$$-\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)\Phi_{1a}(t')\Phi_{2a}(t'+2[\lambda^{-1}])]=\sum_{a\in\{\alpha\}}\left[\Phi_{2a}(t')\int d\lambda\lambda^{-1}\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)\right.$$

$$\left.\times\left[\frac{1}{2}\Phi_{1a}(t'+2[\lambda^{-1}])+\frac{1}{2}\Phi_{1a}(t')\right]-(1\leftrightarrow 2)\right.$$

$$=\sum_{a\in\{\alpha\}}[\Phi_{2a}(t')\Phi_{1a}(t)-\Phi_{1a}(t')\Phi_{2a}(t)]\ \text{using}\ (37).$$

Thus we get the following.

Proposition 15: For the constrained BKP hierarchies (62),the bilinear identity can be written as

$$\int d\lambda\lambda^{2n}\psi_{BA}(t,\lambda)\psi_{BA}(t',-\lambda)=\sum_{a\in\{\alpha\}}[\Phi_{2a}(t')\Phi_{1a}(t)-\Phi_{1a}(t')\Phi_{2a}(t)]. \tag{65}$$

Remark 11: The bilinear identity of the cBKP is the same with Loris' paper.[16]

Next, we will study the relation between the "ghost" symmetry and the additional symmetry. By using the "ghost" symmetry of the BKP hierarchy, we shall give a simple proof of the Adler–Shiota–van Moerbeke formula[28–30] of the BKP which provides the connection between the form of additional symmetries of the BKP hierarchy acting on BA functions and Sato Backlund symmetry acting on the tau functions of the BKP hierarchy. To this end, let $Y(\lambda,\mu)\equiv\psi_{BA}(t,-\lambda)\partial^{-1}\psi_{BA}(t,\mu)_x-\psi_{BA}(t,\mu)\partial^{-1}\psi_{BA}(t,-\lambda)_x$ be pseudodifferential operator inducing a special "ghost" symmetry flow $\partial_{(\lambda,\mu)}W\equiv Y(\lambda,\mu)W$ according to (51). In this case, the "ghost" symmetry flow is generated by an infinite combination of additional symmetries.[30,31] Then, $\partial_{(\lambda,\mu)}W\equiv Y(\lambda,\mu)W$ infers its actions on wave function,

$$\partial_{(\lambda,\mu)}(\psi_{BA}(t,z))=Y(\lambda,\mu)(\psi_{BA}(t,z)).$$

Taking (56) into it, we have

$$Y(\lambda,\mu)(\psi_{BA}(t,z))=\tfrac{1}{2}(\psi_{BA}(t,-\lambda)\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,z))-\psi_{BA}(t,\mu)\Omega(\psi_{BA}(t,-\lambda),\psi_{BA}(t,z))). \tag{66}$$

Further, according to (8) and (11), the action of the vertex operator $\mathcal{X}(\lambda,\mu)$ on the BA function $\psi_{BA}(t,z)$ is as follows:

$$\mathcal{X}(\lambda,\mu)\psi_{BA}(t,z)=\psi_{BA}(t,z)\Delta_z\frac{\mathcal{X}(\lambda,\mu)\tau(t)}{\tau(t)}. \tag{67}$$

Now, the above results allow us to establish the connection between $\mathcal{X}$ and Y.

Proposition 16:

$$\mathcal{X}(\lambda,\mu)\psi_{BA}(t,z)=2Y(\lambda,\mu)\psi_{BA}(t,z). \tag{68}$$

Proof:

$$\mathcal{X}(\lambda,\mu)\psi_{BA}(t,z)=\psi_{BA}(t,z)\Delta_z\frac{\mathcal{X}(\lambda,\mu)\tau(t)}{\tau(t)}=-\psi_{BA}(t,z)\psi_{BA}(t,-\lambda)\psi_{BA}(t-2[z^{-1}],\mu)$$

$$+\psi_{BA}(t,z)\psi_{BA}(t,\mu)\psi_{BA}(t-2[z^{-1}],-\lambda)\text{using(45)and(48)}$$

$$=\psi_{BA}(t,-\lambda)\Omega(\psi_{BA}(t,\mu),\psi_{BA}(t,z))-\psi_{BA}(t,\mu)\Omega(\psi_{BA}(t,-\lambda),\psi_{BA}(t,z))\text{using(41)}$$

$$= 2Y(\lambda,\mu)(\psi_{BA}(t,z)) \text{ using}(66).$$

■

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APPENDIX A: PROOF OF (35)

$$S(\psi_{BA}(t,\lambda),\Phi_x(t)) = S\left(\psi_{BA}(t,\lambda), \int d\mu\mu^{-1}\varphi(\mu)\psi_{BA}(t,\mu)_x\right) \text{ using } (23)$$

$$= \int d\mu\mu^{-1}\varphi(\mu)S(\psi_{BA}(t,\lambda),\psi_{BA}(t,\mu)_x) = \int d\mu\mu^{-1}\varphi(\mu)\frac{1}{2}(\psi_{BA}(t,\mu)$$

$$- \psi_{BA}(t-2[\lambda^{-1}],\mu))\psi_{BA}(t,\lambda) \text{ using } (33)$$

$$= \frac{1}{2}\psi_{BA}(t,\lambda)(\Phi(t)-\Phi(t-2[\lambda^{-1}])) \text{ using } (23).$$

APPENDIX B: PROOF OF (36)

$$S(\Phi_1(t),\Phi_{2x}(t)) = S\left(\int d\mu\mu^{-1}\varphi_1(\mu)\psi_{BA}(t,\mu), \int d\lambda\lambda^{-1}\varphi_2(\lambda)\psi_{BA}(t,\lambda)_x\right) \text{ using } (23)$$

$$= \int\int d\lambda d\mu\lambda^{-1}\mu^{-1}\varphi_1(\mu)\varphi_2(\lambda)S(\psi_{BA}(t,\mu),\psi_{BA}(t,\lambda)_x).$$

APPENDIX C: PROOF OF (59)

According to (52),

$$\begin{aligned}
&\mathrm{Res}(AB_{2n+1}\Phi\partial^{-1}) \\
&= \mathrm{Res}(\partial_\alpha B_{2n+1}\Phi\partial^{-1}) - \mathrm{Res}(\partial_{t_{2n+1}}A\Phi\partial^{-1}) + \mathrm{Res}(B_{2n+1}A\Phi\partial^{-1}) \\
&= (\partial_\alpha B_{2n+1})(\Phi) + \mathrm{Res}(B_{2n+1}A\Phi\partial^{-1}) \text{ using } (21) \\
&= (\partial_\alpha B_{2n+1})(\Phi) + \sum_{a\in\{\alpha\}} \mathrm{Res}(B_{2n+1}(\Phi_{2a}\partial^{-1}\Phi_{1a,x} - \Phi_{1a}\partial^{-1}\Phi_{2a,x})\Phi\partial^{-1}) \\
&= (\partial_\alpha B_{2n+1})(\Phi) + \sum_{a\in\{\alpha\}} (\mathrm{Res}(B_{2n+1}\Phi_{2a}\partial^{-1}(\partial S(\Phi,\Phi_{1a,x}) - S(\Phi,\Phi_{1a,x})\partial)\partial^{-1}) \\
&- (1\leftrightarrow 2)) \text{ using } (20) \\
&= (\partial_\alpha B_{2n+1})(\Phi) + \sum_{a\in\{\alpha\}} (\mathrm{Res}(B_{2n+1}\Phi_{2a}S(\Phi,\Phi_{1a,x})\partial^{-1}) \\
&- \mathrm{Res}(B_{2n+1}\Phi_{2a}\partial^{-1}S(\Phi,\Phi_{1a,x})) - (1\leftrightarrow 2)) \\
&= (\partial_\alpha B_{2n+1})(\Phi) + \sum_{a\in\{\alpha\}} (B_{2n+1}(\Phi_{2a}S(\Phi,\Phi_{1a,x})) - B_{2n+1}(\Phi_{2a})S(\Phi,\Phi_{1a,x}) - (1\leftrightarrow 2)) \text{ using } (21)
\end{aligned}$$

$$= (\partial_\alpha B_{2n+1})(\Phi) + B_{2n+1} \sum_{a\in\{\alpha\}} (\Phi_{2a} S(\Phi, \Phi_{1a,x}) - \Phi_{1a} S(\Phi, \Phi_{2a,x}))$$

$$- \sum_{a\in\{\alpha\}} ((\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1a,x}) - (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2a,x}))$$

$$= (\partial_\alpha B_{2n+1})(\Phi) + B_{2n+1}(\partial_\alpha \Phi) - \sum_{a\in\{\alpha\}} ((\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1a,x})$$

$$- (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2a,x})) \text{ using } (57)$$

$$= \partial_\alpha (\partial_{t_{2n+1}} \Phi) - \sum_{a\in\{\alpha\}} ((\partial_{t_{2n+1}} \Phi_{2a}) S(\Phi, \Phi_{1a,x}) - (\partial_{t_{2n+1}} \Phi_{1a}) S(\Phi, \Phi_{2a,x})).$$

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