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CHARACTERISTIC CLASSES OF HERMITIAN MANIFOLDS

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INTRODUCTION

In recent years the works of Stiefel,[1] Whitney,[2] Pontrjagin,[3] Steenrod,[4] Feldbau,[5] Ehresmann,[6] etc. have added considerably to our knowledge of the topology of manifolds with a differentiable structure, by introducing the notion of so-called fibre bundles. The topological invariants thus introduced on a manifold, called the characteristic cohomology classes, are to a certain extent susceptible of characterization, at least in the case of Riemannian manifolds,[7] by means of the local geometry. Of these characterizations the generalized Gauss-Bonnet formula of Allendoerfer-Weil[8] is probably the most notable example.

In the works quoted above, special emphasis has been laid on the sphere bundles, because they are the fibre bundles which arise naturally from manifolds with a differentiable structure. Of equal importance are the manifolds with a complex analytic structure which play an important rôle in the theory of analytic functions of several complex variables and in algebraic geometry. The present paper will be devoted to a study of the fibre bundles of the complex tangent vectors of complex manifolds and their characteristic classes in the sense of Pontrjagin. It will be shown that there are certain basic classes from which all the other characteristic classes can be obtained by operations of the cohomology ring. These basic classes are then identified with the classes obtained by generalizing Stiefel-Whitney's classes to complex vectors. In the sense of de Rham the cohomology classes can be expressed by exact exterior differential forms which are everywhere regular on the (real) manifold. It is then shown that, in case the manifold carries an Hermitian metric, these differential forms can be constructed from the metric in a simple way. This means that the characteristic classes are completely determined by the local structure of the Hermitian metric. This result also includes the formula of Allendoerfer-Weil and can be regarded as a generalization of that formula.

Concerning the relations between the characteristic classes of a complex manifold and an Hermitian metric defined on it, the problem is completely solved by the above results. It is to be remarked that corresponding questions for Rie-

[1] STIEFEL, [24]. The number in the bracket refers to the bibliography at the end of the paper.
[2] WHITNEY, [29], [30].
[3] PONTRJAGIN, [18], [19].
[4] STEENROD, [21], [22].
[5] FELDBAU, [12].
[6] EHRESMANN, [10].
[7] CHERN, [5], [6], [7].
[8] ALLENDOERFER-WEIL, [2].

mannian manifolds remain open. Roughly speaking, the difficulty in the real case lies in the existence of finite homotopy groups of certain real manifolds, namely the manifolds formed by the ordered sets of linearly independent vectors of a finite-dimensional vector space.

The paper is divided into five chapters. In Chapter I we consider the fibre bundles which include the bundles of tangent complex vectors of a complex manifold and which are called complex sphere bundles. To a given base space a complex sphere bundle can be defined by a continuous mapping of the base space into a complex Grassmann manifold and it is shown that this is the most general way of generating a complex sphere bundle. We take the Grassmann manifold to be that in a complex vector space of sufficiently high dimension and define a characteristic cohomology class in the base space to be the inverse image under this mapping of a cohomology class of the Grassmann manifold. We are therefore led to the study of the cocycles or cycles on a complex Grassmann manifold, a problem treated exhaustively by Ehresmann.[9] A close examination of Ehresmann's results is therefore made in Chapter II, in the light of the problems which concern us here. In fact, we are only interested in the cocycles of the Grassmann manifold which are of dimension not greater than the dimension of the base space. If the Grassmann manifold is that of the linear spaces of n (complex) dimensions in a linear vector space of $n + N$ dimensions, there are on it n basic cocycles such that all other cocycles of dimension $\leqq 2n$ can be obtained from them by operations of the cohomology ring. The cycles corresponding to these cocycles are determined and geometrically interpreted. In Chapter III we identify the images of these cocycles in the base space with the cocycles obtained by generalizing the Stiefel-Whitney invariants to complex vectors. A new definition of these cocycles is given, which is important for applications to differential geometry in the large. Chapter IV is devoted to the study of a complex manifold with an Hermitian metric. It is proved that the n basic cocycles in question can be characterized in a simple way in terms of differential forms constructed from the Hermitian metric. These results are then applied in Chapter V to the complex projective space with the elliptic Hermitian metric. Classical formulas of Cartan[10] and Wirtinger[11] are derived from our formulas as particular cases.

Chapter I

Complex Sphere Bundles and Their Imbedding

1. The complex sphere

Various definitions have been given of a fibre bundle. For definiteness we shall adopt the one of Steenrod[12] and follow his terminology. We are, however, going to restrict the kind of fibre bundles under consideration.

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[9] Ehresmann, [8].
[10] Cartan, [3].
[11] Wirtinger, [31].
[12] Steenrod, [22].

Let $E(n; C)$ be a complex vector space of n dimensions,[13] whose vectors will be denoted by small German letters. In $E(n; C)$ suppose a positive definite Hermitian form be given, which, in terms of a suitable base, has the expression

$$(1) \qquad \mathfrak{z}\bar{\mathfrak{z}} = \sum_{i=1}^{n} z^i \bar{z}^i,$$

where z^i are the components of the vector $\mathfrak{z}$ in terms of the base and the bar denotes the operation of taking the complex conjugate. A vector $\mathfrak{z}$ such that $\mathfrak{z}\bar{\mathfrak{z}} = 1$ is called a unit vector. The group of linear transformations

$$(2) \qquad \mathfrak{z}^{*i} = \sum_{j=1}^{n} a_j^i \mathfrak{z}^j, \qquad i = 1, \cdots, n,$$

which leaves the form (1) unaltered is the unitary group and will be denoted by $U(n; C)$. We shall call the complex sphere $S(n; C)$ the manifold of all the unit vectors of $E(n; C)$. It is homeomorphic to the real sphere of topological dimension $2n - 1$. The letter C in these notations will be dropped, when there is no danger of confusion.

In this paper we shall be concerned with fibre bundles such that the fibres are homeomorphic to the complex sphere $S(n)$ and that the group in each fibre is the unitary group $U(n)$. Such a fibre bundle is called a *complex sphere bundle.*

The most important complex sphere bundle is obtained from the consideration of the complex tangent vectors of a complex manifold $M(n)$ of complex dimension n and topological dimension $2n$. By a complex manifold $M(n)$ we shall mean a connected Hausdorff space which satisfies the following conditions:

1) It is covered by a finite or denumerable set of neighborhoods each of which is homeomorphic to the interior of the polycylinder

$$|z^i| < 1, \qquad i = 1, \cdots, n,$$

in the space of n complex variables, so that z^i can be taken as local coordinates of $M(n)$.

2) In a region in which two local coordinate systems z^i and z^{*i} overlap the coordinates of the same point are connected by the relations

$$(3) \qquad z^{*i} = f^i(z^1, \cdots, z^n),$$

where f^i are analytic functions.

It follows from this definition that the notions of $M(n)$ which are expressed in terms of local coordinates but which remain invariant under the transformations (3) have an intrinsic meaning in $M(n)$. This is in particular true of a *tangent vector* at a point, which we define in the usual way as an object which has n com-

[13] Throughout this paper we shall mean by dimension the complex dimension. The dimension of a manifold in the sense of topology will be called the topological dimension, which is twice the complex dimension. The dimensions of simplexes, chains, cycles, homology groups, etc., are understood in the sense of topology, so long as there is no danger of confusion.

ponents Z^i in each local coordinate system and whose components Z^i, Z^{*i} in two local coordinate systems z^i, z^{*i} are transformed according to the equations

$$Z^{*i} = \sum_{k=1}^{n} \frac{\partial z^{*i}}{\partial z^k} Z^k. \tag{4}$$

It is clear that the space of the tangent vectors at a point is homeomorphic to $E(n)$. We consider the non-zero tangent vectors and call two such vectors equivalent if their components Z^i, W^i with respect to the same local coordinate system satisfy the conditions

$$W^i = \rho Z^i,$$

where ρ is a positive real quantity. This relation remains unchanged under transformation of local coordinates. Also it is an equivalence relation in the sense of algebra, being reflexive, symmetric, and transitive. Hence the non-zero tangent vectors can be divided by means of this equivalence relation into mutually disjoint classes. We call such a class of non-zero tangent vectors a direction. With a natural topology the space of directions at a point is homeomorphic to the complex sphere $S(n)$. Furthermore, by using the so-called unitarian trick in group theory, it is easy to verify that the manifold of all directions at the points of $M(n)$ is a complex sphere bundle with $M(n)$ as the base space It will be called the tangent bundle of $M(n)$.

Although all the results in this chapter will be formulated for general complex sphere bundles, it is the particular case of the tangent bundle of a complex manifold that justifies the study of complex sphere bundles.

2. The Grassmann manifold and the imbedding theorems

Consider the space $E(n + N;C)$ and the linear subspaces of $E(n + N;C)$ of dimension n. The manifold of all such linear subspaces is called a Grassmann manifold and will be denoted by $H(n, N;C)$ or simply $H(n, N)$. It is of dimension nN. The unit vectors of the complex sphere $S(n + N)$ in $E(n + N)$, which belong to a linear subspace $E(n)$ of dimension n, constitute a complex sphere $S(n)$, to be denoted by $S(n + N) \cap E(n)$.

Now let B be a finite polyhedron in the sense of combinatorial topology and let f be a continuous mapping of B into $H(n, N)$. From the mapping f we can define a complex sphere bundle $\mathfrak{F}$ with B as base space as follows: $\mathfrak{F}$ consists of the points $(b, \mathfrak{v})$ of the topological product $B \times S(n + N)$ such that $\mathfrak{v} \in f(b) \cap S(n + N)$, and the projection π of $\mathfrak{F}$ onto B is defined by $\pi(b, \mathfrak{v}) = b$. It is easy to verify that $\mathfrak{F}$ is a complex sphere bundle over B, which we shall call the *induced bundle* over B.

The importance of the notion of complex sphere bundles induced by the mapping of the base space into a Grassmann manifold is justified by the following theorems:

THEOREM 1. *To every bundle $\mathfrak{F}$ of complex spheres $S(n)$ over a finite polyhedron B of topological dimension d there exists a continuous mapping f of B into $H(n, N)$ with $N \geq d/2$, such that $\mathfrak{F}$ is equivalent to the bundle induced by f.*

THEOREM 2. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two bundles of complex spheres $S(n)$ over a finite polyhedron B of topological dimension d induced by the mappings f_1, f_2 respectively of B into $H(n, N)$, $N \geqq d/2$. The bundles $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent when and only when the mappings f_1 and f_2 are homotopic.*

Similar theorems for real sphere bundles are known.[14] It follows from these theorems that to a class of equivalent bundles of complex spheres $S(n)$ over a finite polyhedron B of topological dimension d corresponds a class of homotopic mappings of B into the Grassmann manifold $H(n, N)$, where N is an integer satisfying $2N \geqq d$. This class of mappings induces a homomorphism h of the cohomology groups of dimension $\leqq d$ of $H(n, N)$ into the cohomology groups of the same dimension of B. A cohomology class of B which is the image under h of a cohomology class of $H(n, N)$ is called a *characteristic cohomology class* or simply a *characteristic class* and each of its cocycles is called a *characteristic cocycle.*

3. Proofs of Theorems 1 and 2

The proofs of the Theorems 1 and 2 do not differ essentially from the real case. We shall therefore restrict ourselves to a brief description of the general procedure. We need the following two lemmas:

LEMMA 1. (Covering homotopy theorem)[15] *Let $\mathfrak{F}$ be a fibre space over a base space B, which is a compact metric space. Let S be a compact topological space and let I be the unit interval. Suppose a mapping $h(S \times I) \subset \mathfrak{F}$ be given having the property: There exists a mapping $H(S \times 0) \subset \mathfrak{F}$ such that*

$$h(p \times 0) = \pi H(p \times 0), \qquad p \in S,$$

where π is the projection of $\mathfrak{F}$ into B. Then there exists a mapping $H(S \times I) \subset \mathfrak{F}$ such that

$$h(p \times t) = \pi H(p \times t), \qquad p \in S, \qquad 0 \leqq t \leqq 1.$$

From Lemma 1 follows the lemma:

LEMMA 2 (Feldbau).[16] *Let $\mathfrak{F}$ be a fibre bundle over a compact metric base space B. If B is contractible to a point, then $\mathfrak{F}$ is equivalent to the topological product of B and one of its fibres F.*

To prove Theorem 1 we are going to define the mapping f whose existence was asserted by the theorem. We take a simplicial decomposition of B which is so fine that each simplex lies in a neighborhood, and denote by σ_i^k, $i = 1, \cdots, \alpha_k$, $k = 0, 1, \cdots, d$, its simplexes. We denote as usual by π the projection of $\mathfrak{F}$ onto B. Our purpose is to define a mapping $f(B) \subset H(n, H)$ and a mapping $f^*(\mathfrak{F}) \subset B \times S(n + N)$ such that

[14] WHITNEY, [29]; STEENROD, [22].

[15] HUREWICZ-STEENROD, [15]. The theorem is given in various papers, in slightly different versions.

[16] FELDBAU, [12].

(5) $$f^*(p) \in \pi(p) \times \{f(\pi(p)) \cap S(n + N)\}$$

and that for a fixed $\pi(p)$ the mapping $f^*(p)$ is a homeomorphism preserving the scalar product. The definition of these mappings is given by induction on the dimension of the simplexes of B. The images $f(\sigma_i^0) \in H(n, H)$, $i = 1, \cdots, \sigma_0$, are defined in an arbitrary way and it is clear how $f^*(p)$ can be defined for all $p \in \mathfrak{F}$ such that $\pi(p) = \sigma_i^o$, $i = 1, \cdots, \alpha_0$. We suppose the mappings be defined over the $(k - 1)$-dimensional skeleton of B and consider any simplex σ^k of dimension k. We take a neighborhood U which contains σ^k and decompose the set $\pi^{-1}(U)$ into a topological product of U and a complex sphere $S_0(n)$. Then we can define n mappings $\varphi_i(\sigma^k) \subset \mathfrak{F}$, $i = 1, \cdots, n$, such that: 1) $\pi\varphi_i(p) = p$, $p \in \sigma^k$; 2) $\varphi_i(p)$, $\varphi_j(p)$, $i \neq j$, are orthogonal vectors on the complex sphere $\pi^{-1}(p)$. We proceed to define by induction $f^*(\varphi_i(p)) = p \times q_i$, which will satisfy the condition that q_i, q_j for $i \neq j$ are orthogonal vectors of $S(n + N)$. By hypothesis, $f^*(\varphi_1(p))$ is defined for all $p \in \partial\sigma^k$.[17] Since $\partial\sigma^k$ is topologically a sphere of topological dimension $k - 1 \leqq d - 1 \leqq 2N - 1 < 2(n + N) - 1$, which is the topological dimension of the complex sphere $S(n + N)$, and since $\pi(p)$, $p \in \varphi_1(\sigma^k)$, is the cell σ^k, it follows that $f^*(\varphi_1(p))$, $p \in \partial\sigma^k$, is contractible in $B \times S(n + N)$. This means that there is a continuous mapping $g(\partial\sigma^k \times t) \subset B \times S(n + N)$, $0 \leqq t \leqq 1$, such that the following conditions are satisfied: 1) $g(\partial\sigma^k \times 0)$ is a point; 2) $g(\partial\sigma^k \times 1)$ is identical with $f^*(\varphi_1(p))$. On the other hand, we can introduce in σ^k the "polar coordinates" ρ, p, where $0 \leqq \rho \leqq 1$ and $p \in \partial\sigma^k$. For a point of σ^k having the coordinates ρ, p we define

$$f^*(\varphi_1(\rho, p)) = g(p \times \rho).$$

Suppose now that

$$f^*(\varphi_1(p)) = p \times q_1, \cdots, f^*(\varphi_{i-1}(p)) = p \times q_{i-1}, \qquad p \in \sigma^k,$$

are defined, such that q_k, q_j, $k, j = 1, \cdots, i - 1$, $k \neq j$, are orthogonal vectors of $S(n + N)$. To define $f^*(\varphi_i(p))$ we consider on $S(n + N)$ the complex spheres $S(n + N - i + 1)$ whose vectors are orthogonal to $q_1, \cdots, q_{i-1}$. These complex spheres $S(n + N - i + 1)$, depending on p, constitute a complex sphere bundle over the simplex σ^k. By Lemma 2, it is a topological product of σ^k and a complex sphere $S_0(n + N - i + 1)$. By induction hypothesis, the boundary $\partial\sigma^k$ is mapped into $S_0(n + N - i + 1)$, by means of the vectors $q_i \in S(n + N - i + 1)(p)$. Since the topological dimension $k - 1$ of $\partial\sigma^k$ is smaller than the topological dimension $2(n + N - i) + 1$ of $S_0(n + N - i + 1)$, the map is contractible and the mapping of $\partial\sigma^k$ can be extended continuously throughout σ^k. It follows that a mapping $h(\sigma^k) \subset S_0(n + N - i + 1)$ and hence a mapping $h_1(\sigma^k) \subset S(n + N)$ can be defined such that

$$h_1(p) \in S(n + N - i + 1)(p), \qquad p \in \sigma^k.$$

We then define $f^*(\varphi_i(p)) = p \times h_1(p) = p \times q_i$, $p \in \sigma^k$. Clearly the vector q_i is orthogonal to $q_1, \cdots, q_{i-1}$.

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[17] We shall make use of the notation $\partial\sigma^k$ to denote both the combinatorial and the set-theoretical boundary of the simplex σ^k, as the meaning will be clear by context.

To complete the induction on the dimension k let $p^* \epsilon \mathfrak{F}$ such that $\pi(p^*) = p \epsilon \sigma^k$. Then p^* has n components $u_1, \cdots, u_n$ with respect to $\varphi_1(p), \cdots, \varphi_n(p)$. We define $f(p)$ to be the linear space of n dimensions of $E(n + N)$ which contains the complex sphere determined by $q_1, \cdots, q_n$ and

$$f^*(p^*) = p \times q,$$

where q belongs to $f(p) \cap S(n + N)$ and has the components $u_1, \cdots, u_n$ with respect to $q_1, \cdots, q_n$.

Thus our induction is complete and it is easily seen that the mappings f and f^* fulfill our desired conditions. It is also clear that the complex sphere bundle induced by the mapping $f(B) \subset H(n, N)$ is equivalent to $\mathfrak{F}$. This proves our Theorem 1.

Concerning Theorem 2 it is not difficult to prove that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent if f_1 and f_2 are homotopic. The converse is proved by defining a mapping $f(B \times I) \subset H(n, H)$, with $f(B \times 0)$ and $f(B \times 1)$ coinciding with the given mappings f_1 and f_2 respectively. Because of the equivalence of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ a complex sphere bundle can be defined over $B \times I$ in an obvious way. The rest of the argument consists of defining the mapping $f(B \times I)$ by an extension process analogous to the proof of Theorem 1. We shall omit the details here.

Chapter II

Study of the Cocycles on a Complex Grassmann Manifold

1. Summary of some known results

Let $H(n, N)$ be the Grassmann manifold of n-dimensional linear subspaces in $E(n, N)$. Our main purpose in this chapter is to give a homology base for the cocycles of dimension $\leqq 2n$ of $H(n, N)$. It is to be remarked that, $H(n, N)$ being a manifold of topological dimension $2nN$, there corresponds to each cycle of dimension s a cocycle of dimension $2nN - s$, and vice versa.

There are two different ways to describe the cocycles of $H(n, N)$, which are both useful to our purpose.

To explain the first method let $0 \leqq \varphi(i) \leqq N$, $1 \leqq i \leqq n$, be a non-decreasing integral-valued function. Let L_i, $1 \leqq i \leqq n$, be a linear vector space of dimension $i + \varphi(i)$ in $E(n + N)$, such that

$$L_1 \subset L_2 \subset \cdots \subset L_n.$$

Let $Z(\varphi(i))$ be the set of all n-dimensional linear spaces $X(n)$ such that

$$\dim\ (X(n) \cap L_i) \geqq i, \qquad i = 1, \cdots, n,$$

where the notation in the parenthesis denotes the linear space common to $X(n)$ and L_i. $Z(\varphi(i))$ is called a Schubert variety in algebraic geometry. It is a pseudo-manifold of dimension $s = \sum_{i=1}^{n} \varphi(i)$ and carries an integral cycle of dimension $2s$ of $H(n, N)$. Concerning the significance of the Schubert varieties for the topology of Grassmann manifolds the following theorem was proved by Ehresmann:[18]

[18] Ehresmann, [8], p. 418.

THEOREM 3. *The Grassmann manifold $H(n, N)$ has no torsion coefficients and has all its Betti numbers of odd dimension equal to zero. Its Betti number of dimension 2s is equal to the number of distinct non-decreasing integral-valued functions $\varphi(i)$, $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} \varphi(i) = s$. The integral cycles carried by the corresponding Schubert varieties $Z(\varphi(I))$ constitute a homology base for the Betti group of dimension 2s.*

The second method to describe the cocycles of $H(n, N)$ is by means of differential forms. Let $X(n) \in H(n, H)$ and let $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ be n vectors in $X(n)$ such that

$$\mathfrak{e}_i \cdot \bar{\mathfrak{e}}_j = \delta_{ij}, \qquad 1 \leqq i, j \leqq n.$$

To these n vectors we add N further vectors $\mathfrak{e}_{n+1}, \cdots, \mathfrak{e}_{n+N}$ satisfying the conditions

$$\mathfrak{e}_A \cdot \bar{\mathfrak{e}}_B = \delta_{AB}, \qquad 1 \leqq A, B \leqq n + N. \tag{6}$$

When there is a differentiable family of the vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$, we put

$$\theta_{AB} = d\mathfrak{e}_A \cdot \bar{\mathfrak{e}}_B, \tag{7}$$

which are linear differential forms satisfying the conditions

$$\theta_{AB} + \bar{\theta}_{BA} = 0. \tag{8}$$

Among θ_{AB} the forms

$$\theta_{ir}, \qquad \bar{\theta}_{ir}, \qquad 1 \leqq i \leqq n, \qquad n + 1 \leqq r \leqq n + N,$$

constitute a set of $2nN$ linearly independent forms at each point of $H(n, N)$. Let Θ be a form in θ_{ir}, $\bar{\theta}_{ir}$ with constant coefficients. Θ is called an invariant form if it remains unchanged under the groups of transformations

$$\theta^*_{ir} = \sum_{j=1}^{n} a_{ij} \theta_{jr}, \tag{9}$$

$$\theta^*_{ir} = \sum_{s=n+1}^{n+N} b_{rs} \theta_{is}, \tag{10}$$

where (a_{ij}), (b_{rs}) are arbitrary unitary matrices. It is called exact, if $d\Theta = 0$. It is well-known that on a differentiable manifold of class two a cocycle with rational coefficients can be expressed by an exact differential form, and conversely. For convenience we shall therefore call an exact differential form a cocycle. Then we have the following theorem of E. Cartan:[19]

THEOREM 4. *Every invariant form of $H(n, N)$ is exact. The Betti number of dimension 2s of $H(n, N)$ is equal to the number of linearly independent (with constant coefficients) invariant differential forms of degree 2s. The set of these forms constitutes a cohomology base of dimension 2s.*

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[19] CARTAN, [3]; EHRESMANN, [8], p. 409

2. The basic forms

Let r be an integer between 1 and n. For reasons which will be clear later we shall be particularly interested in the n cycles Z_r, $r = 1, \cdots, n$, carried by the Schubert varieties defined by the functions

$$(11)\qquad \begin{aligned} \varphi_r(i) &= N - 1, \qquad i = 1, \cdots, n - r + 1, \\ \varphi_r(i) &= N, \qquad\quad i = n - r + 2, \cdots, n. \end{aligned}$$

The cycle Z_r is of dimension $2(Nn - n + r - 1)$. We shall find the invariant differential form which gives the cocycle of dimension $2(n - r + 1)$ corresponding to Z_r.

For this purpose we put

$$(12)\qquad \Theta_{ij} = \sum_{s=n+1}^{n+N} \theta_{is}\theta_{sj}, \qquad 1 \leqq i, j \leqq n,$$

and

$$(13)\qquad \Phi_r = \frac{1}{(2\pi\sqrt{-1})^{n-r+1}(n - r + 1)!} \sum \delta(i_1 \cdots i_{n-r+1}; j_1 \cdots j_{n-r+1}) \Theta_{i_1j_1} \cdots \Theta_{i_{n-r+1}j_{n-r+1}},$$

where $\delta(i_1 \cdots i_{n-r+1}; j_1 \cdots j_{n-r+1})$ is zero except when $j_1, \cdots, j_{n-r+1}$ form a permutation of $i_1, \cdots, i_{n-r+1}$, in which case it is $+1$ or -1 according as the permutation is even or odd, and where the summation is extended over all indices $i_1, \cdots, i_{n-r+1}$ from 1 to n. It is easy to verify that Φ_r is an invariant form on $H(n, N)$. Our problem is then solved by the following theorem:

THEOREM 5. *The invariant differential form Φ_r defines a cocycle of dimension $2(n - r + 1)$ on $H(n, N)$, which corresponds to the cycle Z_r in the sense that, for any cycle ζ of dimension $2(n - r + 1)$, the relation*

$$(14)\qquad KI(\zeta_1, Z_r) = \int_\zeta \Phi_r$$

holds,[20] *whenever both sides are defined.*

To prove this theorem we notice that both sides of the equation (14) are linear in ζ, so that it is sufficient to prove (14) for the cycles of a homology base of dimension $2(n - r + 1)$. By Theorem 3, these are the cycles carried by the Schubert varieties $Z(\varphi(i))$ such that

$$\sum_{i=1}^{n} \varphi(i) = n - r + 1.$$

Since the function $\varphi(i)$ is non-decreasing, we must have

$$(15)\qquad \varphi(1) = \cdots = \varphi(r - 1) = 0.$$

[20] The notation KI, due to Lefschetz, means the Kronecker index (or the intersection number).

Let ζ_k, $k = 1, \cdots, m$, be the cycles of a homology base of dimension $2(n - r + 1)$ defined in this way, and let ζ_1 be the cycle defined by

(16) $\varphi(1) = \cdots = \varphi(r-1) = 0, \qquad \varphi(r) = \cdots = \varphi(n) = 1.$

Then for the cycles ζ_k, $k \neq 1$, we must also have

$$\varphi(r) = 0.$$

It is therefore sufficient to prove that

(17) $$KI(\zeta_k, Z_r) = \int_{\zeta_k} \Phi_r, \qquad k = 1, \cdots, m,$$

for the cycles ζ_k which are well chosen so that both sides of (17) are defined.

Let us first assume that $k \neq 1$. By definition, there exists a fixed linear vector space $L(r)$ of dimension r such that any $X(n)$ of the Schubert variety carrying ζ_k satisfies the condition

$$\dim (X(n) \cup L(r)) \geqq r,$$

which means that

$$X(n) \supset L(r).$$

On the other hand, any $X(n)$ of the Schubert variety carrying Z_r has the property that it intersects a fixed linear space $L(N + n - r)$ of dimension $N + n - r$ in a linear space of dimension $\geqq n - r + 1$. We take in $E(n + N)$ a frame $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$ and let $L(r)$ and $L(N + n - r)$ be spanned by the vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_r$ and $\mathfrak{e}_{r+1}, \cdots, \mathfrak{e}_{n+N}$ respectively. There is clearly no $X(n)$ which contains $\mathfrak{e}_1, \cdots, \mathfrak{e}_r$ and has in common with $L(N + n - r)$ a linear space of dimension $n - r + 1$, which means that

$$KI(Z_r, \zeta_k) = 0, \qquad k \neq 1.$$

To show that the integral on the right-hand side of (17) is also zero, we choose the frame $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_r$ belong to $L(r)$ and $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ belong to $X(n)$. It is obviously possible to choose $\mathfrak{e}_1, \cdots, \mathfrak{e}_r$ to be fixed. Under this choice we have

$$\Theta_{ij} = 0, \qquad 1 \leqq i, j \leqq r,$$

and hence

$$\Phi_r = 0.$$

Thus the relations (17) are proved for $k \neq 1$.

To define ζ_1 we take two linear vector spaces $L(r - 1)$, $L(n + 1)$ of dimensions $r - 1$, $n + 1$ respectively, such that $L(r - 1) \subset L(n + 1)$. An element $X(n)$ of ζ_1 is then defined by the conditions

$$L(r - 1) \subset X(n) \subset L(n + 1).$$

Let $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$ be a frame in $E(n + N)$. We suppose the linear vector spaces in question to be so chosen that

$$\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1} \subset L(r-1),$$
$$\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+1} \subset L(n+1),$$
$$\mathfrak{e}_r, \cdots, \mathfrak{e}_{n-1}, \mathfrak{e}_{n-1}, \cdots, \mathfrak{e}_{n+N} \subset L(n+N-r).$$

By this choice $L(n+1)$ and $L(n+N-r)$ have in common a linear vector space of dimension $n-r+1$, namely the one spanned by $\mathfrak{e}_r, \cdots, \mathfrak{e}_{n-1}, \mathfrak{e}_{n+1}$. It follows that Z_r and ζ_1 have in common only one $X(n)$, which is spanned by $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n-1}, \mathfrak{e}_{n+1}$. This intersection is to be counted simply, and we have[21]

$$KI(Z_r, \zeta_1) = 1.$$

It now only remains to evaluate the integral in the right-hand side of (17) for $k = 1$. The linear vector spaces $L(r-1)$ and $L(n+1)$ being fixed, we choose a fixed frame $\mathfrak{a}_1, \cdots, \mathfrak{a}_{n+N}$ in $E(n+N)$ such that $\mathfrak{a}_1, \cdots, \mathfrak{a}_{r-1}$ belong to $L(r-1)$ and $\mathfrak{a}_1, \cdots, \mathfrak{a}_{n+1}$ to $L(n+1)$. If $X(n) \in \zeta_1$, we choose the frame $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$, whose first n vectors belong to $X(n)$ such that

$$(18)\qquad \begin{aligned} \mathfrak{e}_A &= \mathfrak{a}_A, \qquad A = 1, \cdots, r-1, n+2, \cdots, n+N, \\ \mathfrak{e}_B &= \sum_C u_{BC}\mathfrak{a}_C, \qquad B, C = r, \cdots, n+1, \end{aligned}$$

where u_{BC} are the elements of a unitary matrix. It follows that

$$(19)\qquad \begin{aligned} \theta_{ki} &= d\mathfrak{e}_k \cdot \bar{\mathfrak{e}}_i = 0, \\ \theta_{ik} &= -\bar{\theta}_{ki} = 0, \end{aligned} \qquad k = n+2, \cdots, n+N, \quad i = 1, \cdots, n+N,$$

and hence that

$$\Theta_{ij} = \theta_{in+1}\theta_{n+1j}, \qquad 1 \leqq i, j \leqq n.$$

For simplicity we shall write $\mathfrak{e}$ for $\mathfrak{e}_{n+1}$, u_B for $u_{n+1,B}$, and θ_i for $\theta_{n+1,i}$. We remark that $X(n)$ is completely determined by the vector $\mathfrak{e}$, whose components with respect to $\mathfrak{a}_r, \cdots, \mathfrak{a}_{n+1}$ are $u_r, \cdots, u_{n+1}$. Our purpose therefore is to transform the form Φ_r into an exterior differential form in $u_r, \cdots, u_{n+1}$, from which the integration can be carried out. With this purpose in mind, we have

$$(20)\qquad \begin{aligned} \theta_i &= d\mathfrak{e} \cdot \bar{\mathfrak{e}}_i = \left(\sum_{B=r}^{n+1} du_B\, \mathfrak{a}_B\right)\left(\sum_{C=r}^{n+1} \bar{u}_{iC}\, \bar{\mathfrak{a}}_C\right) \\ &= \sum_{B=r}^{n+1} du_B\, \bar{u}_{iB}, \qquad i = r, \cdots, n, \\ \theta_i &= 0, \qquad i = 1, \cdots, r-1, \end{aligned}$$

and

$$\Theta_{ij} = -\bar{\theta}_i\theta_j, \qquad i, j = r, \cdots, n,$$

all other Θ_{ij} being zero. It follows that Φ_r is equal to

[21] Cf. EHRESMANN, [8], p. 421.

$$(21)\quad \frac{(2\pi\sqrt{-1})^{r-n-1}}{(n-r+1)!}\cdot\sum \delta(i_1\cdots i_{n-r+1}\,;j_1\cdots j_{n-r+1})\theta_{i_1}\bar{\theta}_{j_1}\cdots\theta_{i_{n-r+1}}\bar{\theta}_{j_{n-r+1}}$$

$$= (-1)^{\frac{1}{2}(n-r)(n-r+1)}\frac{(n-r+1)}{(2\pi\sqrt{-1})^{n-r+1}}\theta_r\cdots\theta_n\bar{\theta}_r\cdots\bar{\theta}_n\,.$$

From now on we shall agree on the following ranges of indices:

$$\gamma \leqq \alpha,\beta \leqq n,\qquad \gamma \leqq i,j \leqq n+1.$$

From (20) and

$$\theta_{n+1} = \sum_j du_j\bar{u}_j$$

we get, by solving for du_i,

$$(22)\qquad du_i = \sum_\alpha u_{\alpha i}\theta_\alpha + u_i\theta_{n+1}\,.$$

We notice that

$$(23)\qquad \theta_{n+1} + \bar{\theta}_{n+1} = 0.$$

Consider now the form

$$(24)\quad \Psi_r = \sum_{k=0}^{n-r+1} du_r\cdots du_{r+k-1}\,du_{r+k-1}\cdots du_{n+1}\,d\bar{u}_r\cdots d\bar{u}_{r+k-1}\,d\bar{u}_{r+k+1}\cdots d\bar{u}_{n+1}\,,$$

which we shall prove to differ from Φ_r by a numerical factor. In fact, it is easy to verify, by means of the fact that the matrix (u_{ij}) is unitary, that

$$\Psi_r = \theta_r\cdots\theta_n\bar{\theta}_r\cdots\bar{\theta}_n\,,$$

and hence that

$$(25)\qquad \Phi_r = (-1)^{\frac{1}{2}(n-r)(n-r+1)}\frac{(n-r+1)!}{(2\pi\sqrt{-1})^{n-r+1}}\Psi_r\,.$$

To integrate Φ_r or Ψ_r over ζ_1 let us notice that $X(n)$ will remain unchanged if $\mathfrak{e}$ is replaced by the vector $e^{\sqrt{-1}\rho}\,\mathfrak{e}$, ρ being real. We can therefore normalize the coordinates $u_r, \cdots, u_{r+1}$ of $X(n)$ by assuming that u_{n+1} is real and positive. Then we have

$$\Psi_r = du_r\cdots du_n\,d\bar{u}_r\cdots d\bar{u}_n\,,$$

which is to be integrated over the domain

$$u_{n+1}^2 + u_r\bar{u}_r + \cdots + u_n\bar{u}_n = 1,\qquad u_{n+1} > 0.$$

This integration is then easily achieved. In fact, we put

$$u_r = v_r + \sqrt{-1}\,w_r\,,$$

$$\bar{u}_r = v_r - \sqrt{-1}\,w_r\,.$$

Then

$$\Psi_r = (-1)^{\frac{1}{2}(n-r)(n-r+1)}(-2\sqrt{-1})^{n-r+1}dv_r\,dw_r\cdots dv_n\,dw_n\,,$$

and the integral is over the domain D:

$$u_{n+1}^2 + v_r^2 + w_r^2 + \cdots + v_n^2 + w_n^2 = 1, \qquad u_{n+1} > 0.$$

But the integral

$$\int_D dw_r\, dv_r \cdots dw_n\, dv_n$$

is the volume of the domain bounded by the unit hypersphere of dimension $2n - 2r + 1$, which is equal to $\pi^{n-r+1}/(n - r + 1)!$. Hence we have

$$\int_{\mathfrak{z}_1} \Psi_r = (-1)^{\frac{1}{2}(n-r)(n-r+1)} \frac{(2\pi\sqrt{-1})^{n-r+1}}{(n - r + 1)!},$$

and finally,

$$\int_{\mathfrak{z}_1} \Phi_r = 1.$$

Our Theorem 5 is therefore completely proved.

3. The basis theorem

The importance of the invariant differential forms Φ_r, $1 \leqq r \leqq n$, lies in a theorem we proceed to prove, which asserts that every invariant differential form of degree $\leqq 2n$ in $H(n, N)$ is a polynomial in Φ_r with constant coefficients. The exact statement of our theorem is as follows:

THEOREM 6. *Every invariant differential form of degree $\leqq 2n$ in $H(n, N)$ is a polynomial in Φ_r, $1 \leqq r \leqq n$, with constant coefficients. If the form defines an integral cocycle on $H(n, N)$, the coefficients are integers.*

The theorem follows easily from the so-called first main theorem on vector invariants for the unitary group, which we state as follows:

LEMMA 3. *Let $\mathfrak{v}_1, \cdots, \mathfrak{v}_m$ be a set of vectors in $E(n)$ under transformations of the unitary group $U(n)$. Every integral rational invariant in the components of $\mathfrak{v}_k$, $1 \leqq k \leqq m$, is an integral rational function of the scalar products $\mathfrak{v}_i\bar{\mathfrak{v}}_k$, $1 \leqq i, k \leqq m$.*

It is known that[22] under the unimodular unitary group such an invariant is an integral rational function of the scalar products and of determinants of the form $[\mathfrak{v}_1 \cdots \mathfrak{v}_n]$ or $[\bar{\mathfrak{v}}_1 \cdots \bar{\mathfrak{v}}_n]$. But under a general unitary transformation of determinant $e^{\sqrt{-1}\alpha}$ the determinants $[\mathfrak{v}_1 \cdots \mathfrak{v}_n]$ and $[\bar{\mathfrak{v}}_1 \cdots \bar{\mathfrak{v}}_n]$ will be multiplied by $e^{\sqrt{-1}\alpha}$ and $e^{-\sqrt{-1}\alpha}$ respectively. It follows that an invariant will involve the determinants only in products of the form $[\mathfrak{v}_1 \cdots \mathfrak{v}_n]\cdot[\bar{\mathfrak{v}}_1' \cdots \bar{\mathfrak{v}}_n']$, which can however be expressed as a determinant of scalar products:

$$[\mathfrak{v}_1 \cdots \mathfrak{v}_n][\bar{\mathfrak{v}}_1' \cdots \bar{\mathfrak{v}}_n'] = \begin{vmatrix} \mathfrak{v}_1\bar{\mathfrak{v}}_1' & \cdots & \mathfrak{v}_1\bar{\mathfrak{v}}_n' \\ \cdots & \cdots & \cdots \\ \mathfrak{v}_n\bar{\mathfrak{v}}_1' & \cdots & \mathfrak{v}_n\bar{\mathfrak{v}}_n' \end{vmatrix}.$$

Thus the lemma is proved.

[22] WEYL, [28], p. 45.

To prove Theorem 6 let Ψ be an invariant differential form of degree $2s \leqq 2n$ in $H(n, N)$, which is therefore an exterior form in θ_{ir}, $\bar{\theta}_{ir}$, $1 \leqq i \leqq n$, $n + 1 \leqq r \leqq n + N$, with constant coefficients. The form Ψ being in particular invariant under the transformation $\theta^*_{ir} = e^{\sqrt{-1}\alpha}\theta_{ir}$, it follows that Ψ, when reduced to its lowest terms, will contain in each term exactly s factors each of θ_{ir} and $\bar{\theta}_{ir}$. Let us fix our attention for the moment to the group (10). We take from Ψ all the terms of the form

$$\text{const. } \theta_{i_1 r_1} \cdots \theta_{i_s r_s} \bar{\theta}_{j_1 t_1} \cdots \bar{\theta}_{j_s t_s},$$

with a fixed set of the indices $i_1, \cdots, i_s, j_1, \cdots, j_s$, and call their sum Ψ_1. Since the indices $i_1, \cdots, i_s, j_1, \cdots, j_s$ are now fixed, we shall drop them for simplicity.

Now it is well-known that there is an isomorphism between the ring of exterior forms and the ring of multilinear forms with alternating coefficients. To Ψ_1 corresponds, in the complex vector space of N dimensions, an alternating multilinear form of degree $2s$. Since Ψ_1 is invariant under the unitary group (10), the same is true of its corresponding alternating multilinear form. By our Lemma the latter is an integral rational function of the scalar products. It follows by the isomorphism that Ψ_1 can be expressed as a polynomial in sums of the form $\sum_r \theta_{ir}\bar{\theta}_{jr} = -\Theta_{ij}$. Consequently, Ψ is a polynomial in Θ_{ij}, $1 \leqq i, j \leqq n$, with constant coefficients.

Let us now put

$$(26) \qquad P_r = \sum \Theta_{i_1 i_2}\Theta_{i_2 i_3} \cdots \Theta_{i_{n-r+1} i_1}, \qquad r = 1, \cdots, n.$$

By the same argument as above, we can prove that Ψ, being also invariant under the group (9), is a polynomial in P_r, $1 \leqq r \leqq n$, with constant coefficients. On the other hand, it is easy to show, by induction on r, that P_r is a polynomial in $\Phi_1, \cdots, \Phi_r$, with constant coefficients. Hence the first part of our theorem is proved.

To prove the second part of the theorem consider the products of the form

$$(27) \qquad \Phi_n^{\lambda_1} \cdots \Phi_1^{\lambda_n},$$

such that

$$(27a) \qquad \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n \doteq s.$$

These forms constitute a basis for all invariant differential forms of degree $2s \leqq 2n$ on $H(n, N)$. Since $s \leqq n$, their number is equal to the number of partitions of s as a sum of integral summands. By Theorem 3 this is equal to the Betti number of dimension $2s$ of $H(n, N)$. It follows that the products in (27) are linearly independent, and that every invariant differential form of degree $2s$ of $H(n, N)$ representing an integral cocycle is equal to a linear combination of the products (27) with integral coefficients.

Chapter III

The Basic Characteristic Classes on a Complex Manifold

1. A second definition of the basic characteristic classes

Let M be a complex manifold of dimension n. We consider the complex sphere bundle defined from the tangent vectors of M and imbed it, according to Theorem 1, in a Grassmann manifold $H(n, N)$, $N \geqq n$, by means of a mapping of M into $H(n, N)$. It follows from Theorem 2 that the inverse image of a cohomology class of dimension $\leqq 2n$ of $H(n, N)$ induced by this mapping is an invariant of M (or rather of the analytic structure of M), which we have called a characteristic cohomology class of M. From Theorem 6 we see that of all the characteristic cohomology classes of M those which are inverse images of the cohomology classes of $H(n, N)$ containing the cocycles Φ_r, $1 \leqq r \leqq n$, play a particularly important rôle, because all the others can be obtained from them by operations of the cohomology ring. We therefore call these n classes the basic characteristic classes, the inverse image of the class containing Φ_r being the r^{th} basic class.

Our first aim is to identify these basic classes with the classes obtained by generalizing to complex manifolds the well-known procedure of Stiefel-Whitney.[23]

In order to understand the situation we recall briefly the results of Stiefel-Whitney for real sphere bundles, emphasizing the differences between the real and complex cases. From a bundle of real spheres of dimension $n - 1$ over a polyhedron as base space Stiefel and Whitney considered the fibre bundle over the same base space whose fibre at each point is the manifold $V(n, r)$ of $r(1 \leqq r \leqq n)$ linearly independent points of the real sphere at this point. It was proved that all homology groups of dimension $<n - r$ of $V(n, r)$ vanish and that the homology group $H^{n-r}(V(n, r))$ of dimension $n - r$ of $V(n, r)$ is the free cyclic group or the cyclic group of order 2 according as the following condition is satisfied or not: $n - r$ is even or $r = 1$. To define a generator of $H^{n-r}(V(n, r))$ we take an ordered set of $r - 1$ mutually perpendicular unit vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$ in the Euclidean space E^n of dimension n which contains the sphere of dimension $n - 1$. The unit vector $\mathfrak{e}_r$ of E^n perpendicular to $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$ describes a sphere of dimension $n - r$, which, when oriented, defines a cycle ζ_0^{n-r} belonging to one of the generating homology classes of $H^{n-r}(V(n, r))$. We then take a simplicial decomposition K of the base space such that each simplex of K belongs to a neighborhood. Let the simplexes of K be oriented. It is possible to define a continuous mapping of the $(n - r)$-dimensional skeleton of K into the fibre bundle such that each point is mapped into a point belonging to the fibre over it. Let σ^{n-r+1} be a simplex of dimension $n - r + 1$ of K. In a neighborhood containing σ^{n-r+1} the fibre bundle can be resolved into a topological product and can therefore be mapped into one fibre. Since the mapping is defined over the boundary $\partial\sigma^{n-r+1}$, we get a mapping of

[23] Stiefel, [24]; Whitney, [29].

a sphere of dimension $n - r$ into a fibre and hence an element h of $H^{n-r}(V(n, r))$. The delicate point is to get from this element h an integer or a residue class mod. 2. This is possible if a generating element of $H^{n-r}(V(n, r))$ is defined. When $H^{n-r}(V(n, r))$ is cyclic of order two, it has only one generating element, so that no further assumption is necessary. When $H^{n-r}(V(n, r))$ is a free cyclic group, we assume that a continuous field of generating elements of $H^{n-r}(V(n, r))$ can be defined over the whole M, which is possible if M is orientable. The element h is then equal to the generator of $H^{n-r}(V(n, r))$ so defined, multiplied by an integer or a residue class mod. 2. Taking this integer or the residue class mod. 2 as the value of a cochain for the simplex σ^{n-r+1}, we get an integral cochain or a cochain mod. 2. It was proved that the cochain is a cocycle and that its cohomology class is independent of the choice of the mapping from which it is defined. This cohomology class is the class of Stiefel-Whitney. It is to be remarked that the definition can be given under more general conditions, but we shall be satisfied with the above résume.

The situation is simpler in the case of complex sphere bundles. From a bundle of complex spheres $S(n)$ we consider the fibre bundle $\mathfrak{F}^{(r)}$ over the same base space whose fibre at each point is the manifold $U(n, r)$ of $r(1 \leqq r \leqq n)$ linearly independent vectors in $E(n)$. It can be proved that[24] all homology groups of dimension $<2n - 2r + 1$ of $U(n, r)$ vanish and that the homology group $H^{2n-2r+1}(U(n, r))$ of dimension $2n - 2r + 1$ is a free cyclic group. To define a generator of $H^{2n-2r+1}(U(n, r))$ we take in $E(n)$ an ordered set of $r - 1$ mutually perpendicular unit vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$. The unit vector $\mathfrak{e}_r$ in $E(n)$ perpendicular to $e_1, \cdots, e_{r-1}$ describes a complex sphere in the $E(n - r + 1)$ perpendicular to $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$. The complex sphere $S(n - r + 1)$ in $E(n - r + 1)$ is topologically a real sphere of topological dimension $2n - 2r + 1$. Its two orientations define two cycles belonging respectively to the two generating classes of $H^{2n-2r+1}(U(n, r))$. The cycle carried by the oriented real sphere $S(n - r + 1)$ is completely determined by the orientation of $E(n - r + 1)$ considered as a real Euclidean space of topological dimension $2(n - r + 1)$. This orientation is independent of the order of the vectors $\mathfrak{e}_r, \cdots, \mathfrak{e}_n$ in $E(n - r + 1)$. It follows that the fibre bundle $\mathfrak{F}^{(r)}$ is orientable in the sense of Steenrod, which means that there is an isomorphism in the large between the $(2n - 2r + 1)$-dimensional homology groups of the fibres of $\mathfrak{F}^{(r)}$, or that a continuous field of generating elements of $H^{2n-2r+1}(U(n, r))$ can be defined over the whole manifold. The fibre bundle $\mathfrak{F}^{(r)}$ has two opposite orientations and we shall from now on make a definite choice of one of them. Using this "oriented" fibre bundle, we shall be allowed to replace an element of $H^{2n-2r+1}(U(n, r))$ at a point by the integer which, when multiplied by the generating element at this point, is equal to the element in question.

With these explanations understood, we have:

[24] Ehresmann, [9]. This fact is easily proved by making use of the covering homotopy theorem, the complex case being even simpler than the real case.

THEOREM 7. *The r^{th} basic characteristic class of a complex manifold M of dimension n can be defined as follows: Take a simplicial decomposition of M each of whose simplexes belongs to a neighborhood, and define over its skeleton of dimension $2n - 2r + 1$ a continuous field of ordered sets of r linearly independent complex tangent vectors. To each simplex of dimension $2n - 2r + 2$ take a point in its interior and consider the manifold of the ordered sets of r linearly independent complex tangent vectors at that point. The field on the boundary of the simplex defines a mapping of the boundary into this manifold and hence an element of its $(2n - 2r + 1)$-dimensional homotopy or homology group, which is free cyclic. Attach the corresponding integer to the simplex. The cochain so defined is a cocycle and belongs to the r^{th} basic class.*

To prove this theorem we take on $H(n, N)$ a definite Schubert variety V defined by the function in (11), which carries a cycle Z, dual to the cocycle Φ_r. The Schubert variety being an algebraic variety on the algebraic variety $H(n, N)$, it follows from the triangulation theorem of algebraic varieties[25] that $H(n, N)$ can be covered by a complex L such that V is a subcomplex. Let L^* be the dual cellular subdivision of L and let f be a mapping of M into L^*, which, according to Theorem 1, induces over M a complex sphere bundle equivalent to the tangent bundle of M. By the theorem on the simplicial approximation of mappings there exists a subdivision of M and a simplicial mapping f_1 of the subdivision into L^* such that f and f_1 are homotopic. By Theorem 2 the complex sphere bundle over M induced by f_1 is equivalent to the tangent bundle of M. For simplicity of notation we can therefore assume f to be a simplicial mapping of M into L^*.

Let K be the skeleton of M of dimension $2(n - r + 1)$. $f(K)$ is a subcomplex of L^* of dimension $\leqq 2(n - r + 1)$. The r^{th} basic cocycle of M is by definition a linear function of the integral chains of dimension $2(n - r + 1)$ of K such that its value at a simplex σ of K is the intersection number of $f(\sigma)$ and V.

To give a description of the Schubert variety V we take in $E(n + N)$ a linear subspace $L(n + N - r)$ of dimension $n + N - r$, and let $L(r)$ be the linear subspace of dimension r which is totally perpendicular to $L(n + N - r)$. Then V consists of all $X(n)$ of $H(n, N)$ satisfying the condition

$$\dim(X(n) \cap L(n + N - r)) \geqq n - r + 1.$$

We take in $L(r)$ r mutually perpendicular unit vectors $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$. To each $X(n)$ not belonging to V the projection of $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ on $X(n)$ will give r vectors which are linearly independent, and the construction fails exactly for the $X(n)$ belonging to V.

With these preparations consider a simplex σ of K. If the intersection number $KI(f(\sigma), V) = 0$, the above construction will give a continuous field of r linearly independent complex vectors at each point of $f(\sigma)$ and hence also at each point of σ, which shows that the integer attached to σ according to the statement of the theorem is also zero. It remains therefore to consider the case

[25] VAN DER WAERDEN, [26].

that $KI(f(\sigma), V) = \epsilon \neq 0$. In this case $f(\sigma)$ is of dimension $2(n - r + 1)$. Since $f(\sigma)$ and V belong to dual subdivisions, we have $\epsilon = +1$. Let $X_0(n)$ be the linear space of $H(n, N)$ common to V and $f(\sigma)$. The orthogonal projection of $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ defines on each $X(n) \neq X_0(n)$ of $f(\sigma)$ r linearly independent complex vectors, which, by the resolution of all the $X(n)$ belonging to $f(\sigma)$ into a topological product, are mapped into the manifold μ of all the ordered sets of r linearly independent vectors in $X_0(n)$. Our purpose is to prove that by means of the sets of vectors on the boundary $\delta f(\sigma)$, $\delta f(\sigma)$ is mapped into a cycle belonging to the generator of the $(2n - 2r + 1)st$ homology group of μ. For simplicity of language let us call index the integer m obtained by mapping the field on the boundary $\delta f(\sigma)$ into a fibre $U(n, r)$, the image cycle being in the homology class equal to m times the generator of $H^{2n-2r+1}(U(n, r))$.

Suppose first that a continuous field of ordered sets of n linearly independent vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ is defined throughout $f(\sigma)$ and let r linearly independent vectors $\mathfrak{f}_1, \cdots, \mathfrak{f}_r$ be defined over $\delta f(\sigma)$ such that

$$\mathfrak{f}_i = \sum_{k=1}^{n} \mathfrak{f}_{ik} \mathfrak{e}_k, \qquad 1 \leqq i \leqq r.$$

Regarding $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ as fixed, these equations also define a mapping of $\partial f(\sigma)$ into μ. We assert that their indices are equal. In fact, the existence of the field $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ throughout $f(\sigma)$ provides exactly a deformation of the field $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ over $\partial f(\sigma)$ into vectors $\mathfrak{e}_1', \cdots, \mathfrak{e}_n'$ which are constant.

We assume that $X_0(n)$ has the property that the orthogonal projections of $a_1, \cdots, a_{r-1}$ onto it are linearly independent, which is possible, after applying a small deformation if necessary. Since $f(\sigma)$ is a simplex, we can define over $f(\sigma)$ a continuous field of n linearly independent vectors, such that in every $X(n)$ the first $r - 1$ of these vectors are the orthogonal projections in $X(n)$ of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{r-1}$ respectively. This continuous field is then deformed into a continuous field $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ over $f(\sigma)$ such that in each $X(n)$ the vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ constitute a frame (that is, are mutually perpendicular unit vectors). It is well-known that the deformation can be so chosen that during the deformation the vector subspace determined by the first s vectors $(1 \leqq s \leqq n)$ remains fixed. With these deformations performed, we proceed to study the orthogonal projection $\mathfrak{a}_r^*$ of $\mathfrak{a}_r$ in $X(n)$.

The index of the field of orthogonal projections of $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ on $\partial f(\sigma)$ is equal to the index of the field $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}, \mathfrak{a}_r^*$ on $\partial f(\sigma)$ and is also equal to the index of the same field, when $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$ are considered as constant vectors. We also remark that the vector $\mathfrak{a}_r^*$ is linearly independent of $\mathfrak{e}_1, \cdots, \mathfrak{e}_{r-1}$ at every point $\neq X_0(n)$ of $f(\sigma)$.

To show that the index in question is 1 we choose the continuous field of vectors $\mathfrak{e}_{n+1}, \cdots, \mathfrak{e}_{n+N}$ over $f(\sigma)$ such that $\mathfrak{e}_A$, $1 \leqq A \leqq n + N$, is a frame in $E(n + N)$. Then we have

$$\mathfrak{a}_r = \sum_{A=1}^{n+N} u_A \mathfrak{e}_A, \qquad \sum_{A=1}^{n+N} u_A \bar{u}_A = 1,$$

and

$$\mathfrak{a}_r^* = \sum_{i=1}^{n} u_{ri}\, \mathfrak{e}_i \, .$$

According to our previous remark we can regard the vectors $\mathfrak{e}_A$, $1 \leqq A \leqq n + N$, as fixed and consider the mapping of $f(\sigma) - X_0(n)$ into $S(n - r)$ defined by the vector whose components with respect to a fixed frame are $u_{r,r+1}, \cdots, u_{rn}$. Thus we see that the index is 1, and Theorem 7 is proved.

It is also possible to introduce from a bundle of complex spheres $S(n)$ the fibre bundles $\mathfrak{F}^{(r)*}$ over the same base space whose fibre at each point is the manifold $U^*(n, r)$ of all ordered sets of $r(1 \leqq r \leqq n)$ mutually perpendicular vectors of $S(n)$. The manifold $U^*(n, r)$ is an absolute retract of $U(n, r)$. Theorem 7 still holds, if we replace everywhere the phrase "ordered sets of r linearly independent complex vectors" by "ordered sets of r mutually perpendicular vectors of $S(n)$", and, naturally also the manifold $U(n, r)$ by $U^*(n, r)$.

2. A third definition of the basic characteristic classes

We suppose in this section that the base space M, which is a complex manifold, is compact. From the tangent bundle $\mathfrak{F}$ over M we construct the fibre bundle $\mathfrak{F}^{(r)*}$ $(1 \leqq r \leqq n)$ as explained at the end of the last section. Then the following theorem gives a third definition of the r^{th} basic characteristic class:

THEOREM 8. *The r^{th} basic characteristic class of M is the cohomology class of M, each of whose cocycles γ has the following property: Under the projection of $\mathfrak{F}^{(r)*}$ into M, γ is mapped into a cocycle γ^*. There exists in $\mathfrak{F}^{(r)*}$ a cochain β^*, such that $\delta\beta^* = \gamma^*$ and that β^* reduces to a fundamental cocycle on each fibre of $\mathfrak{F}^{(r)*}$.*

The last statement in the theorem needs some explanation. Let P be a polyhedron and $Q \subset P$ a closed subpolyhedron of P. If γ is a cochain in P, its *reduced cochain* on Q is the cochain γ' such that $\gamma' \cdot \sigma = \epsilon\, \gamma \cdot \sigma$, where $\epsilon = 1$ or 0 according as the simplex σ belongs to Q or not. Moreover, the integral cohomology group of dimension $2n - 2r + 1$ of a fibre being free cyclic, a *fundamental cocycle* on the fibre is a cocycle of dimension $2n - 2r + 1$ which belongs to a generator of the cohomology group. It is also understood that the cycles and cocycles are defined in terms of simplicial decompositions of M and $\mathfrak{F}^{(r)*}$. To define the inverse mapping of the cocycles of M into the cocycles of $\mathfrak{F}^{(r)*}$ induced by the projection π of $\mathfrak{F}^{(r)*}$ into M, we therefore take a simplicial approximation π' of π. Let σ^* be a simplex of dimension $2n - 2r + 2$ of $\mathfrak{F}^{(r)*}$. Then we define

$$\gamma^* \cdot \sigma^* = \gamma \cdot \pi'(\sigma^*),$$

if $\pi'(\sigma^*)$ is of dimension $2n - 2r + 2$ and $\gamma^* \cdot \sigma^* = 0$ if $\pi'(\sigma^*)$ is of lower dimension. The cocycle γ^* depends on π', but its cohomology class is independent of it. The theorem asserts that any such cocycle γ^* reduces to a fundamental cocycle on a fibre.

In order to prove Theorem 8 we need the following lemma:

LEMMA 4. *With the notations of Theorem 8 let U be a neighborhood of M and let $\pi^{-1}(U)$ be its complete inverse image in $\mathfrak{F}^{(r)*}$. Let K be a finite complex and L its skeleton of dimension $\leqq 2n - 2r$. If f and g are two continuous mappings into $\pi^{-1}(U)$ of K and L respectively, there is a continuous mapping f^* of K into $\pi^{-1}(U)$ which is homotopic to f and coincides with g on L.*

We denote by I the unit segment $0 \leqq t \leqq 1$ and consider the topological product $K \times I$. To prove Lemma 4 is to define a continuous mapping $f(K \times I) \subset \pi^{-1}(U)$, with $f(K \times 0)$ and $f(L \times 1)$ given. For this purpose we decompose K simplicially and arrange the simplexes of the decomposition in a sequence that every simplex is preceded by its faces. The mapping is then defined by successive extensions over the prisms constructed on the simplexes of the sequence. We resolve $\pi^{-1}(U)$ into the topological product of U and a fixed fibre F_0 and denote by λ the projection of $\pi^{-1}(U)$ onto F_0. Let σ^0 be a vertex of K. Then $f(\sigma^0 \times 0)$ and $f(\sigma^0 \times 1)$ are both defined in $\mathfrak{F}^{(r)*}$, and can be joined by a segment, on which the prism on σ^0 is mapped. Using mathematical induction we suppose f be defined over all prisms on simplexes preceding σ^m of the sequence, and consider σ^m, $m \leqq 2n - 2r$. By hypothesis, the mapping is defined over $\partial(\sigma^m \times I)$, which is topologically a sphere of topological dimension m. The mapping can be extended over $\sigma^m \times I$, if and only if $f(\partial(\sigma^m \times I))$ is homotopic to zero in $\mathfrak{F}^{(r)*}$, that is, by the covering homotopy theorem, if and only if $\lambda f(\partial(\sigma^m \times I))$ is homotopic to zero in F_0. The latter is the case, because the m^{th} homotopy group of F_0 is zero. It follows that f is defined for the subcomplex $L \times I + K \times 0$ of the prism $K \times I$. By a well-known elementary geometric construction[26] f is then extended over $K \times I$. Thus the lemma is proved.

We proceed to prove Theorem 8. Let γ be a cocycle of M belonging to the r^{th} basic class defined by the construction of Theorem 7, with the bundle $\mathfrak{F}^{(r)}$ replaced by $\mathfrak{F}^{(r)*}$. To explain Theorem 7 for this case, we take a simplicial decomposition of M which is so fine that each simplex belongs to a neighborhood of M. Let $K^{2n-2r+1} = K$ be the $(2n - 2r + 1)$-dimensional skeleton of the simplicial decomposition. There exists a continuous mapping Ψ of K into $\mathfrak{F}^{(r)*}$ such that $\pi\Psi(p) = p$ for every $p \in K$. Let σ be a simplex of dimension $2n - 2r + 2$. The mapping Ψ defines a mapping of the boundary $\partial\sigma$ of σ into $\mathfrak{F}^{(r)*}$ and the mapping $\lambda\Psi$ defines a mapping of $\partial\sigma$ into F_0, and hence a cycle of dimension $2n - 2r + 1$ of F_0. Let this cycle be homologous to a multiple $\gamma(\sigma)$ of the generating cycle of dimension $2n - 2r + 1$ of F_0. According to Theorem 7 the cocycle $\gamma(\psi)$ defined by assigning the integer $\gamma(\sigma)$ to σ belongs to the r^{th} basic class. Moreover, it was also proved[27] that to a given cocycle γ of the r^{th} basic class there exists a mapping ψ such that $\gamma(\psi) = \gamma$. We suppose ψ to be so chosen.

Let π' be a simplicial approximation of π and let γ^* be the inverse image of γ under π' as defined above. We shall show that γ^* is the coboundary of an

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[26] ALEXANDROFF-HOPF, [1], p. 501.

[27] STEENROD, [21], p. 124.

integral cochain β^* of dimension $2n - 2r + 1$ of $\mathfrak{F}^{(r)*}$. To define β^* let τ^* be a simplex of dimension $2n - 2r + 1$ of $\mathfrak{F}^{(r)*}$, and let $\tau = \pi'(\tau^*)$. The discussion will be divided into two cases, according as τ is of dimension equal to or less than $2n - 2r + 1$.

Suppose τ be of dimension $2n - 2r + 1$. In this case π' establishes a simplicial and therefore topological mapping between τ and τ^*. By Lemma 4 there exists a mapping $\pi''(\tau) \subset \mathfrak{F}^{(r)*}$, which is homotopic to $\pi'^{-1}(\tau)$ and coincides with Ψ on the boundary $\partial\tau$. We then take an oriented sphere of topological dimension $2n - 2r + 1$ and denote by H_1, H_2 its two hemispheres. We map H_1 and H_2 into τ by the mappings h_1 and h_2 of the degrees -1 and $+1$ respectively, such that h_1 and h_2 are identical on the "equator" $H_1 \cap H_2$. A mapping f of the sphere $H_1 + H_2$ into $\mathfrak{F}^{(r)*}$ is then defined by the conditions

$$f(p) = \pi''h,\ (p), \qquad p \in H_1,$$
$$f(p) = \psi h_2(p), \qquad p \in H_2.$$

This mapping f is by construction continuous. Taking its projection λf on F_0, we get an element of the $(2n - 2r + 1)$-dimensional homotopy group of the fibre and hence an integer, on account of the orientability of the bundle $\mathfrak{F}^{(r)*}$. This integer we define to be $\beta^* \cdot \tau^*$. It is to be remarked that $\beta^* \cdot \tau^*$ in general depends on the deformation which carries π'^{-1} to π'', but only one such deformation will be utilized, and $\beta^* \cdot \tau^*$ is thus well defined.

Next let τ be of dimension $< 2n - 2r + 1$. We suppose without loss of generality that $p_0 = \pi(F_0) \in \tau$. Let any simplex τ' of dimension $2n - 2r + 1$ be mapped into τ^* by a non-degenerate orientation-preserving simplicial mapping. By Lemma 4 this mapping is homotopic to a mapping $\pi'''(\tau') \subset \mathfrak{F}^{(r)*}$ such that $\pi'''(\partial\tau') = q_0$, where q_0 is a point of F_0. We identify all the points on $\partial\tau'$, thus getting an oriented sphere of topological dimension $2n - 2r + 1$, which is mapped into F_0 by the mapping $\lambda\pi'''(\tau') \subset F_0$. This mapping defines an element of the $(2n - 2r + 1)$-dimensional homotopy group and hence an integer, which is defined to be $\beta^* \cdot \tau^*$.

It remains to show that the coboundary of the cochain β^* so defined is equal to γ^*. For this purpose let σ^* be a simplex of $\mathfrak{F}^{(r)*}$ of dimension $2n - 2r + 2$. It is sufficient to verify that $\gamma^* \cdot \sigma^* = \delta\beta^* \cdot \sigma^* = \beta^* \cdot \partial\sigma^*$.

Suppose first that $\sigma = \pi'(\sigma^*)$ is of dimension $2n - 2r + 2$. The mapping π'' is defined for each simplex of $\partial\sigma$ and hence for $\partial\sigma$ itself, because it coincides with ψ on the $(2n - 2r)$-dimensional skeleton K^{2n-2r}. We therefore have two mappings, $\lambda\psi$ and $\lambda\pi''$ respectively, of $\partial\sigma$ into F_0 such that they are identical on K^{2n-2r}. Our fibre F_0 being $(2n - 2r)$-simple, this is a situation discussed by Eilenberg,[28] who introduced several cochains, denoted in his notation by $c(\lambda\psi)$, $c(\lambda\pi'')$, $d(\lambda\psi, \lambda\pi'')$ respectively. In our notation they are given by

$$c(\lambda\psi) \cdot \sigma = \gamma^* \cdot \sigma^*,$$
$$c(\lambda\pi'') = 0,$$
$$d(\lambda\psi, \lambda\pi'') \cdot \partial\sigma = \beta^* \cdot \partial\sigma^*,$$

[28] Eilenberg, [11], pp. 235–237.

where the second relation follows from the fact that $\lambda\pi''$ is defined for the simplex σ bounded by $\partial\sigma$. From a theorem of Eilenberg we have

$$\delta d(\lambda\psi, \lambda\pi'') = c(\lambda\psi) - c(\lambda\pi''),$$

or

$$\gamma^* \cdot \sigma^* = \beta^* \cdot \partial\sigma^*,$$

which is to be proved.

Next suppose $\sigma = \pi'(\sigma^*)$ be of dimension $< 2n - 2r + 2$. If each simplex of $\partial\sigma^*$ is mapped by π' into a simplex of dimension $< 2n - 2r + 1$, $\beta^* \cdot \partial\sigma^*$ is clearly zero. The other possibility is that σ is of dimension $2n - 2r + 1$ and that exactly two of the simplexes of $\partial\sigma^*$, say τ_1^* and τ_2^*, are mapped by π' into σ. If τ_1^* and τ_2^* are coherently oriented with the boundary $\partial\sigma^*$, it follows by definition that $\beta^* \cdot (\tau_1^* + \tau_2^*) = 0$. On the other hand, it is not difficult to see that $\beta^* \cdot (\partial\sigma^* - \tau_1^* - \tau_2^*) = 0$. Hence we have $\beta^* \cdot \partial\sigma^* = 0$. We have thus proved that γ^* is the coboundary of a cochain β^*.

To see that β^* reduces on a fibre F_0, we suppose the simplicial decomposition of $\mathfrak{F}^{(r)*}$ so made that F_0 is a subcomplex. Every simplex of F_0 is mapped by π' into a point, so that we have $\gamma^* \cdot \sigma^* = 0$ for every simplex σ^* of dimension $2n - 2r + 2$ of F_0, which shows that β^* reduces to a cocycle on F_0. To show that β^* is the fundamental cocycle on F_0, it is sufficient to show that one is the value of its product with a cycle belonging to the generating homology class of dimension $2n - 2r + 1$ of F_0. For this purpose we take an oriented sphere of topological dimension $2n - 2r + 1$ and map it simplicially into F_0 such that the map belongs to the generator of the $(2n - 2r + 1)$-dimensional homotopy group of F_0. The image of this map defines a cycle of F_0 belonging to the generating homology class of the $(2n - 2r + 1)$-dimensional homology group, and its product with β^* is 1. It follows that β^* reduces to a fundamental cocycle on F_0.

Our Theorem 8 is now completely proved.

3. In terms of differential forms

Consider again the Grassmann manifold $H(n, N)$. We take as point of a new space a linear space $E(n)$ and r vectors $\mathfrak{e}_{n-r+1}, \cdots, \mathfrak{e}_n$ belonging to $E(n)$ such that $\mathfrak{e}_i\bar{\mathfrak{e}}_j = \delta_{ij}, n - r + 1 \leqq i, j \leqq n$. This space, to be denoted by $R(r, n, N)$, is clearly a fibre bundle over $H(n, N)$, the projection of a point of $R(r, n, N)$ being the corresponding $E(n)$ and each fibre being homeomorphic to $U^*(n, r)$. This fibre bundle $R(r, n, N)$ is transformed transitively by the unitary group in the space $E(n + N)$. Let W_r be the r^{th} basic class of $H(n, N)$ and $\gamma \epsilon W_r$ be one of its cocycles. γ is mapped by the inverse mapping induced by any simplicial approximation of π into a cocycle having the properties asserted by Theorem 8. In particular, we can take for γ the cocycle defined by the differential form Φ_r. The inverse image of Φ_r in $R(r, n, N)$ under π is a differential form which for convenience we denote by Φ_r. The form Φ_r then defines a cocycle γ^* in $R(r, n, N)$.

From Theorem 8 it follows that[29] there exists a cochain β^* in $R(r, n, N)$ such that $\delta\beta^* = \gamma^*$ and such that β^* reduces to a fundamental cocycle on a fibre.

In order to define β^* in $R(r, n, N)$ by means of a suitably chosen differential form, we shall make use of the following lemma proved by de Rham:[30]

LEMMA 5. *Let M be a compact differentiable manifold of class $\geqq 2$ and let K be a simplicial decomposition of M, whose simplexes are σ_i^p, $i = 1, \cdots, \alpha_p$, $p = 0, 1, \cdots, n$, and whose incidence relations are*

$$\partial \sigma_i^p = \sum_{j=1}^{\alpha_{p-1}} \eta_{ij}^{(p)} \sigma_j^{p-1}$$

Then there exists a set of differential forms

$$\varphi_i^{(p)}, \qquad i = 1, \cdots, \alpha_p, \qquad p = 0, 1, \cdots, n,$$

such that the following conditions are satisfied:

$$1) \qquad \int_{\sigma_j(p)} \varphi_i^{(p)} = \delta_{ij},$$

$$2) \qquad d\varphi_i^{(p)} = \sum \eta_{ji}^{(p+1)} \varphi_j^{(p+1)}.$$

We apply this lemma to the manifold $R(r, n, N)$, and write for simplicity $p = 2n - 2r + 1$. The cochain β^* is defined by definition by a system of equations

$$\beta^* \cdot \sigma_i^p = \lambda_i, \qquad i = 1, \cdots, \alpha_p .$$

Let

$$\omega = \sum_{i=1}^{\alpha_p} \lambda_i \varphi_i^{(p)}.$$

Then we have

$$\int_{\sigma_i^p} \omega = \lambda_i$$

which shows that the differential form ω defines the cochain β^*. By construction we have

$$d\omega = \Phi_r .$$

Now the unitary group $U(n + N)$ in $E(n + N)$ transforms transitively the manifold $R(r, n, N)$. Let s be a transformation of $U(n + N)$. If θ is a differen-

[29] We have tacitly assumed at this point of our discussion that the Theorem 8, proved by a combinatorial construction for the simplicial approximations of the projection π, holds for π itself, when the cocycles are expressed by means of differential forms. It is, however, possible to avoid this assumption by observing that the cochain β^* exists in $R(r, n, N)$ such that $\delta\beta^* = \gamma^*$. That β^* reduces to a fundamental cocycle on a fibre then follows from the very definition of the characteristic cocycle on $H(n, N)$.

[30] DE RAHM, [20], p. 178.

tial form in $R(r, n, N)$, we shall denote by $s\theta$ its transform by the transformation **s.** We also use the notation $\theta \sim 0$ to denote that θ is derived.

LEMMA 6. *Let β^* be a cochain of dimension $2n - 2r + 1$, whose coboundary is γ^*. Let ω be a differential form which defines β^*. Then $s\omega - \omega \sim 0$.*

First of all, the differential form $s\omega - \omega$ is exact, since we have

$$d(s\omega - \omega) = s\Phi_r - \Phi_r = 0.$$

Let ζ be a cycle of dimension $2n - 2r + 1$ of $R(r, n, N)$. It is sufficient to prove that

$$\int_\zeta s\omega = \int_\zeta \omega.$$

The cycle $s\omega - \omega$ being homologous to zero, let Z^* be the chain it bounds. Then we have

$$\int_\zeta s\omega - \omega = \int_{s\zeta - \zeta} \omega = \int_{\partial Z^*} \omega = \int_{Z^*} \Phi_r .$$

Let Z be the projection in the base space $H(n, N)$ of the chain Z^* in $R(r, n, N)$. The boundary of Z is the projection of $s\zeta - \zeta$. Since the Betti group of dimension $2n - 2r + 1$ of $H(n, N)$ is zero, the projection of ζ bounds in $H(n, N)$ a chain which we shall call Z_1. Then the projection of $s\zeta$ bounds the chain sZ_1, and we have

$$Z \sim sZ_1 - Z_1 .$$

It follows that

$$\int_{Z^*} \Phi_r = \int_Z \Phi_r = \int_{Z_1} s\Phi_r - \int_{Z_1} \Phi_r = 0.$$

Thus Lemma 6 is proved.

THEOREM 9. *Under the projection of $R(r, n, N)$ into $H(n, N)$ the differential form Φ_r is mapped by the inverse mapping into $R(r, n, N)$. There exists a differential form π which is invariant under transformations of the unitary group $U(n + N)$ operating in $R(r, n, N)$ and whose exterior derivative $d\pi$ is equal to Φ_r.*

By Lemma 5 we can construct the differential form ω in $R(r, n, N)$ such that

$$d\omega = \Phi_r .$$

For such a differential form ω it follows from Lemma 6 that

$$\delta\omega - \omega \sim 0.$$

Let dv be the invariant volume element of $U(n + N)$ such that the integral of dv over $U(n + N)$ is equal to 1. We put

$$\Pi = \int_{U(n+N)} s\omega \, dv.$$

Then Π is invariant under $U(n+N)$ and we have

$$\lambda\Pi = \int_{U(n+N)} s \cdot d\omega \cdot dv = \Phi_r \int_{U(n+N)} dv = \Phi_r ,$$

which proves our theorem.

To a point of $R(r, n, N)$ we now attach the frames $\mathfrak{e}_1, \cdots, \mathfrak{e}_{n+N}$ in $E(n+N)$ such that $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$ determine the $E(n)$ and that $\mathfrak{e}_{n-r+1}, \cdots, \mathfrak{e}_n$ are the vectors in question. For clearness let us agree in the remainder of this section on the following ranges of indices:

$$1 \leqq \alpha, \beta, \gamma \leqq n - r, \qquad n - r + 1 \leqq A, B, C \leqq n,$$
$$n + 1 \leqq i, j, k \leqq n + N.$$

In a neighborhood of $R(r, n, N)$ we can choose a differentiable family of such frames, one attached at each point of the neighborhood. By means of the family of frames the forms $\theta_{A\alpha}$, $\bar{\theta}_{A\alpha}$, θ_{Ai}, $\bar{\theta}_{Ai}$, θ_{AB} can be constructed according to the equations (7). They constitute a set of linearly independent linear differential forms at each point of $R(r, n, N)$. Our form π, whose existence was asserted by Theorem 9 and which is invariant under $U(n+N)$, is necessarily a polynomial (in the sense of Grassmann algebra) in the forms of this set, with constant coefficients. On the other hand, the form Π, being itself in $R(r, n, N)$, must be invariant under the transformation

$$\theta^*_{Ai} = \sum_j a_{ij} \theta_{Aj}$$

where a_{ij} are the elements of a unitary matrix. We put

$$\Theta_{AB} = \sum_i \theta_{Ai} \theta_{iB}$$

It follows from the first main theorem on vector invariants of the unitary group that Π is a polynomial in $\theta_{A\alpha}$, $\bar{\theta}_{A\alpha}$, θ_{AB}, Θ_{AB}, with constant coefficients. Moreover, on a fibre, that is, omitting all terms in Θ_{AB}, Π becomes a fundamental cocycle.

All these results can be summarized in the following theorem:

THEOREM 10. *There exists in $R(r, n, N)$ a polynomial Π in $\theta_{A\alpha}$, $\bar{\theta}_{A\alpha}$, θ_{AB}, Θ_{AB}, with constant coefficients, such that $d\Pi = \Phi_r$. When all terms involving Θ_{AB} in Π are omitted, the form defines a fundamental cocycle on a fibre.*

CHAPTER IV

HERMITIAN MANIFOLDS

1. Fundamental formulas of Hermitian Geometry

Let M be a compact complex manifold. M is called an Hermitian manifold, if an intrinsic Hermitian differential form is given throughout the manifold. In each local coordinate system z^i the Hermitian differential form is defined by

$$(28) \qquad ds^2 = \sum_{i,j=1}^{n} g_{ij}(z, \bar{z})(dz^i d\bar{z}^j), \qquad \bar{g}_{ij} = g_{ji},$$

where, as well as in later formulas, we insert a parenthesis to designate that the multiplication of the differential forms in question is ordinary multiplication. We shall agree, unless otherwise stated, that the indices i, j, k take the values 1 to n.

Our main result in this chapter is to establish that the n basic classes which arise from the analytic structure of a complex manifold, are completely determined by the Hermitian metric, if the manifold in question is an Hermitian manifold. In particular, as we shall see later, the theorem for the class W_1 reduces to the formula of Allendoerfer-Weil, if we interpret the Hermitian metric as a Riemannian metric for the real manifold of $2n$ topological dimensions.

We begin by establishing the fundamental formulas for local Hermitian Geometry.

For this purpose we determine in a neighborhood of M n linear differential forms φ_i in the local coordinates z^i such that

$$(29) \qquad ds^2 = \sum_{i=1}^{n} (\varphi_i \bar{\varphi}_i).$$

The forms φ_i are determined up to a unitary transformation:

$$(30) \qquad \omega_i = \sum_{i=1}^{n} u_{ij}\varphi_j, \qquad i = 1 \cdots, n,$$

where u_{ij} are the elements of a unitary matrix

$$U = (n_{ij}),$$

and which we take to be independent variables. Let $\mathfrak{e}_i$ be the dual base corresponding to ω_k, so that

$$\omega_i(\mathfrak{e}_k) = \delta_{ik}.$$

From the Hermitian differential form a scalar product of the contravariant vectors can be defined, and we have

$$\mathfrak{e}_i \cdot \bar{\mathfrak{e}}_k = \delta_{ik}.$$

We shall call a frame the figure formed by a point P and n such vectors $\mathfrak{e}_i$. With a natural topology the set of frames constitutes a fibre bundle over M.

The forms ω_i are intrinsically defined in the fibre bundle. By actual calculation in terms of a local coordinate system, we find that their exterior derivatives are of the form

$$d\omega_i = \sum_j \omega_j \omega_{ji} + \sum_{j,k} a_{ijk}\omega_j\omega_k + \sum_{j,k} b_{ijk}\omega_j\bar{\omega}_k, \qquad a_{ijk} + a_{ijk} = 0,$$

where

$$\omega_{ij} = \sum_k \bar{u}_{ik} d\bar{u}_{jk},$$

so that

$$\cdot\ \omega_{ij} + \omega_{ji} = 0.$$

But the forms ω_{ij} are defined up to the transformation

$$\omega_{ij} \rightarrow \omega_{ij} + \sum \lambda_{ijk}\omega_k - \sum \bar{\lambda}_{jik}\bar{\omega}_k ,$$

where the quantities λ_{ijk} are arbitrary. It is easy to show that there exists one, and only one, set of forms ω_{ij}, such that the following equations are satisfied:

$$(31) \quad d\omega_i = \sum_j \omega_j \omega_{ji} + \sum_{jik} A_{ijk}\omega_j\omega_k, \qquad A_{ijk} + A_{ikj} = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0,$$

From the uniqueness of this set of forms follows the fact that they are intrinsically defined in the fibre bundle. The forms ω_i, ω_{ij} constitute therefore a set of linearly independent linear differential forms in the fibre bundle.

From equations (31) it is possible to draw all the consequences of local Hermitian Geometry. In fact, we put

$$(32) \qquad \Omega_i = \sum_{j,k} A_{ijk}\omega_j\omega_k .$$

Exterior differentiation of the first set of equations in (31) will give

$$d\Omega_i - \sum_k \omega_k \Omega_{ki} + \sum_k \Omega_k \omega_{ki} = 0,$$

where we have put

$$(33) \qquad \Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik}\omega_{kj} .$$

We remark that $d\Omega_i + \sum_k \Omega_k\omega_{ki}$ is of the form $\sum_{j,k} \psi_{ijk}\omega_j\omega_k$. It follows that

$$\omega_1 \cdots \omega_n \Omega_{ij} = 0$$

and that we can put

$$\Omega_{ik} = \sum \chi_{ikj}\omega_j .$$

On the other hand, we have

$$\Omega_{ik} + \bar{\Omega}_{ki} = 0$$

or

$$\sum_j \chi_{ikj}\omega_j + \sum_j \bar{\chi}_{kij}\bar{\omega}_j = 0,$$

which shows that χ_{ikj} is a linear combination of ω_k, $\bar{\omega}_k$:

$$\chi_{ikj} = \sum_{l=1}^{n} a_{ikjl}\omega_l + \sum_{l=1}^{n} b_{ijkl}\bar{\omega}_l .$$

Substituting this expression of χ_{ikj} into the last equation, we see immediately that a_{ikjl} must be symmetric in the last two indices j, l. It follows that Ω_{ik} is of the form

$$\Omega_{ik} = \sum b_{ikjl}\bar{\omega}_l\omega_j .$$

The equations for the exterior derivatives $d\omega_i$, $d\omega_{sj}$ and the equations obtained therefrom by exterior differentiation we shall call the fundamental equations of local Hermitian Geometry. These equations will now be summarized as follows:

$$\begin{aligned}
&d\omega_i = \sum_j \omega_j\omega_{ji} + \Omega_i ,\\
&d\omega_{ij} = \sum_k \omega_{ik}\omega_{kj} + \Omega_{ij} ,\\
&d\Omega_i + \sum_j \Omega_j\omega_{ji} - \sum_j \omega_j\Omega_{ji} = 0 ,\\
&d\Omega_{ij} + \sum_k \Omega_{ik}\omega_{kj} - \sum_k \omega_{ik}\Omega_{kj} = 0 ,\\
&\Omega_i = \sum A_{ijk}\omega_j\omega_k , \qquad A_{ijk} + A_{ikj} = 0 ,\\
&\Omega_{ij} = \sum_{l,m=1}^{n} R_{ij,lm}\bar{\omega}_l\omega_m , \qquad R_{ij,lm} = \bar{R}_{ji,ml} ,\\
&\omega_{ij} + \bar{\omega}_{ji} = 0 , \qquad \Omega_{ij} + \bar{\Omega}_{ji} = 0 .
\end{aligned} \tag{34}$$

In a well-known way the forms ω_i, ω_{ij} can be interpreted as defining an infinitesimal displacement, by means of the equations

$$\begin{aligned}
dp &= \sum_i \omega_i \mathfrak{e}_i ,\\
d\mathfrak{e}_i &= \sum_j \omega_{ij}\mathfrak{e}_j .
\end{aligned} \tag{35}$$

Of importance are the Hermitian metrics satisfying the condition

$$\Omega_i = 0, \tag{36}$$

which will be called Hermitian metrics without torsion. An Hermitian metric without torsion can be characterized by the condition

$$d(\sum_i \omega_i\bar{\omega}_i) = 0 , \tag{37}$$

and was studied by E. Kähler.[31] Kähler proved that in this case there exists locally a function $F(z^i, \bar{z}^i)$ such that the metric can be written in the form

$$ds^2 = \sum_{i,k} \frac{\partial^2 F}{\partial z^i\, \partial \bar{z}^k} (dz^i\, d\bar{z}^k). \tag{38}$$

Hermitian metrics without torsion play an important rôle in the theory of automorphic functions of several complex variables.

[31] KÄHLER, [16].

2. Formulas for the basic characteristic classes

As defined in Chapter III there are on M n basic characteristic classes, the r^{th} one $(1 \leqq r \leqq n)$ being of dimension $2(n - r + 1)$. We shall show that, if M is an Hermitian manifold, these classes are defined by the local properties of the Hermitian metric. For this purpose we put

$$\Psi_r = \frac{1}{(2\pi\sqrt{-1})^{n-r+1}(n-r+1)!} \sum \delta(i_1 \cdots i_{n-r+1}; j_1 \cdots j_{n-r+1})\Omega_{i_1 j_1} \cdots \Omega_{i_{n-r+1} j_{n-r+1}}, \tag{39}$$

where Ω_{ij} are the forms defined in (34) and where the meaning of the summation has been explained before. Then we have the following theorem, which is the main result of this paper:

THEOREM 11. *The form Ψ_r defined by* (39) *is the form corresponding to the r^{th} basic characteristic class W_r in the sense that the product of any homology class ζ of dimension $2(n - r + 1)$ with W_r is equal to the integral of Ψ_r over ζ:*

$$\zeta \cdot W_r = \int_\zeta \Psi_r . \tag{40}$$

We first establish the following lemma:

LEMMA 7. *Let Δ be the differential form Π in Theorem* 10, *with every θ and Θ replaced by the corresponding ω and Ω with the same indices. Then $d\Delta = \Psi_r$.*

We observe that the equations for $d\omega_{ij}$, $d\Omega_{ij}$ are exactly of the same form as the equations for $d\theta_{ij}$, $d\Theta_{ij}$, the only difference being that Θ_{ij} is given by the equation (12). It follows that

$$d\Delta - \Psi_r \equiv 0 \ \text{mod.}\ \Omega_{ij} - \sum_{B=n+1}^{n+N} \theta_{iB}\,\theta_{Bj}.$$

By mathematical induction on the degree of $d\Delta - \Psi_r$ it is easy to show that then

$$d\Delta - \Psi_r = 0.$$

To prove Theorem 11 we make use of the definition of W_r given in Theorem 7, with $\mathfrak{F}^{(r)}$ replaced by $\mathfrak{F}^{(r)*}$. A sufficiently fine simplicial decomposition K of M is taken and a continuous mapping ψ of its $(2n - 2r + 1)$-dimensional skeleton into $\mathfrak{F}^{(r)*}$ is defined, such that the image of every point belongs to the fibre over it. Let $\sigma_A^{2n-2r+2}$ or σ_A, $1 \leqq A \leqq \alpha_{2n-2r+2}$, be the simplexes of dimension $2n - 2r + 2$ of K. By the construction of Theorem 7, an integer $\gamma(\sigma_A)$ is defined for every σ_A, and the corresponding cocycle γ belongs to W_r. It is sufficient to prove that

$$\zeta \cdot \gamma = \int_\zeta \Psi_r . \tag{40a}$$

Both sides of the equation (40a) being linear in ζ, equation (40a) will follow from the relation

$$\gamma(\sigma_A) = \int_{\sigma_A} \Psi_r .$$

But, by Lemma 7,

$$\int_{\sigma_A} \Psi_r = \int_{\psi(\sigma_A)} \Psi_r = \int_{\psi(\sigma_A)} d\Delta = \int_{\partial\psi(\sigma)_A} \Delta.$$

The image $\psi(\sigma_A)$ has a singular point and we see that the last integral is precisely the definition of $\gamma(\sigma_A)$ given in integral form. Hence Theorem 11 is proved.

3. The case $r = 1$ and the formula of Allendoerfer-Weil

In the case $r = 1$ we can take for the cycle ζ in (40) one of the fundamental cycles of the manifold M. Then we have $\zeta \cdot W_1 = \chi$, the Euler-Poincaré characteristic of M. On the other hand, the manifold M can be considered as a real differentiable manifold and the Hermitian metric can be used to define a Riemannian metric in the real manifold. It is to be expected that the formula (40) will then reduce to the formula of Allendoerfer-Weil. We shall show that this is actually the case, if the Hermitian metric is without torsion.

To study the Hermitian metric as a Riemannian metric, we decompose each of the forms ω_i, ω_{ij}, Ω_{ij} into its real and imaginary parts, writing

$$\begin{aligned} \omega_i &= \theta_i + \sqrt{-1}\psi_i, \\ \omega_{ij} &= \theta_{ij} + \sqrt{-1}\psi_{ij}, \\ \Omega_{ij} &= \Theta_{ij} + \sqrt{-1}\Psi_{ij}. \end{aligned} \tag{41}$$

From the last two equations of (34) we have

$$\begin{aligned} \theta_{ij} + \theta_{ji} &= 0, \qquad \psi_{ij} - \psi_{ji} = 0, \\ \Theta_{ij} + \Theta_{ji} &= 0, \qquad \Psi_{ij} - \Psi_{ji} = 0. \end{aligned} \tag{41a}$$

The Hermitian metric can then be written as a Riemannian metric in the form:

$$ds^2 = \sum_i \{(\theta_i)^2 + (\psi_i)^2\}.$$

We get, moreover, by separating the real and imaginary parts of the equations for $d\omega_i$, $d\omega_{ij}$, the following equations:

$$\begin{aligned} d\theta_i &= \sum_j \theta_j\theta_{ji} - \sum_j \psi_j\psi_{ji}, \\ d\psi_i &= \sum_j \theta_j\psi_{ji} + \sum_j \psi_j\theta_{ji}, \end{aligned} \tag{42}$$

and

$$\begin{aligned} d\theta_{ij} &= \sum_k \theta_{ik}\theta_{kj} - \sum_k \psi_{ik}\psi_{kj} + \Theta_{ij}, \\ d\psi_{ij} &= \sum_k \theta_{ik}\psi_{kj} + \sum_k \psi_{ik}\theta_{kj} + \Psi_{ij}. \end{aligned} \tag{43}$$

It follows that, for the Riemannian Geometry of $2n$ dimensions thus obtained, the curvature forms can be conveniently described by the matrix

$$(44) \qquad \begin{pmatrix} \Theta_{ij} & -\Psi_{ij} \\ \Psi_{ij} & \Theta_{ij} \end{pmatrix}$$

or simply by

$$\begin{pmatrix} \Theta & -\Psi \\ \Psi & \Theta \end{pmatrix}.$$

It only remains to compare the integrand of the Allendoerfer-Weil formula calculated from this matrix of curvature forms with the expression Ψ_1 defined in (39).

Let Ω denote the integrand of the Allendoerfer-Weil formula. We observe that $(-2\pi)^n\Omega$ obeys the same expansion rule as a so-called Pfaffian function of the $2n^{\text{th}}$ order,[32] which is an integral rational function in a number of independent variables, whose square is equal to the value of a skew-symmetric determinant of order $2n$. It follows that

$$(2\pi)^{2n}\Omega^2 = \begin{vmatrix} \Theta & -\Psi \\ \Psi & \Theta \end{vmatrix}.$$

On the other hand, we have

$$\Psi_1 = \frac{1}{(2\pi\sqrt{-1})^n}\,|\Omega_{ij}| = \frac{1}{(2\pi\sqrt{-1})^n}\,|\Theta + \sqrt{-1}\Psi|,$$

from which we get, after some reduction,

$$(-1)^n(2\pi)^{2n}\Psi_1^2 = \begin{vmatrix} \Theta + \sqrt{-1}\Psi & 0 \\ 0 & \Theta + \sqrt{-1}\Psi \end{vmatrix}$$

$$= \begin{vmatrix} \Theta + \sqrt{-1}\Psi & 0 \\ \Theta + \sqrt{-1}\Psi & -\Theta + \sqrt{-1}\Psi \end{vmatrix} = (-1)^n \begin{vmatrix} \Theta & -\Psi \\ \Psi & \Theta \end{vmatrix}.$$

Hence we have

$$\Omega^2 = \Psi_1^2,$$

and finally

$$\Omega = \Psi_1,$$

by comparing the coefficients of one of the terms in both sides.

Chapter V

Applications to Elliptic Hermitian Geometry

1. Preliminaries

We are going to make use of the above results to derive some consequences for elliptic Hermitian Geometry.

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[32] Pascal, [17], pp. 60–64.

The local Hermitian Geometry whose fundamental equations are given by (34) is called elliptic Hermitian, if we have

$$(45)\qquad \begin{aligned} \Omega_{ii} &= \omega_i\bar{\omega}_i + \sum_k \omega_k\bar{\omega}_k , \\ \Omega_{ij} &= \omega_j\bar{\omega}_i , \qquad i \neq j. \end{aligned}$$

This definition owes its origin to the type of geometry studied by G. Fubini,[33] E. Study,[34] and E. Cartan,[35] which we shall prove to satisfy the conditions (45).

The elliptic Hermitian Geometry of Fubini-Study is defined as follows: We consider the complex projective space of n dimensions P_n, with the homogeneous coordinates z^α, where, as well as throughout this section, we shall use the following ranges of our indices:

$$0 \leqq \alpha, \beta, \gamma \leqq n, \qquad 1 \leqq i, j, k \leqq n.$$

In P_n let a positive definite Hermitian form be given:

$$(46)\qquad (z\bar{z}) = \sum_\alpha z^\alpha \bar{z}^{-\alpha},$$

which will serve to define the scalar product of two vectors in the affine space A_{n+1} of $n+1$ dimensions, with the (non-homogeneous) coordinates z^α. We normalize the coordinates z^α in P_n, such that

$$(47)\qquad (z\bar{z}) = 1.$$

An Hermitian metric in P_n is then defined by the Hermitian form

$$(48)\qquad ds^2 = (dz\, d\bar{z}) - (z\, d\bar{z})\cdot(\bar{z}\, dz).$$

The group of linear transformations in A_{n+1} which leaves the form (46) invariant is the unitary group $U(n+1)$. We take in A_{n+1} $n+1$ vectors A_k such that

$$(49)\qquad A_\alpha \bar{A}_\beta = \delta_{\alpha\beta} .$$

For a differentiable family of such sets of vectors we have

$$(50)\qquad dA_\alpha = \sum_\beta \theta_{\alpha\beta} A_\beta ,$$

where

$$(51)\qquad \theta_{\alpha\beta} + \bar{\theta}_{\beta\alpha} = 0,$$

and where the forms $\theta_{\alpha\beta}$ satisfy the equations of structure:

$$(52)\qquad d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma}\theta_{\gamma\beta} .$$

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[33] FUBINI, [13].
[34] STUDY, [25].
[35] CARTAN, [4].

Let B_α be fixed vectors in A_{n+1}, satisfying the equations

$$B_\alpha \bar{B}_\beta = \delta_{\alpha\beta} .$$

The coordinates of a vector A_0 with respect to B_α are defined by the equation

$$A_0 = \sum_\alpha z^\alpha B_\alpha ,$$

from which we find

$$(dA_0\, d\bar{A}_0) - (A_0\, d\bar{A}_0)(\bar{A}_0\, dA_0) = (dz\, d\bar{z}) - (z\, d\bar{z})(\bar{z}\, dz).$$

It follows that, if we regard A_0 as defining the points in P_n, the Hermitian form in (48) can be written as

$$ds^2 = (dA_0\, d\bar{A}_0) - (A_0\, d\bar{A}_0)(\bar{A}_0\, dA_0),$$

and, by (50), as

$$ds^2 = \sum_i (\theta_{0i} \bar{\theta}_{0i}). \tag{53}$$

This proves in particular that the Hermitian form in (48) is positive definite.

To calculate the curvature of the metric (53) we shall make use of the equations (52). Using the notations of local Hermitian Geometry in the last Chapter, we put

$$\omega_i = \theta_{0i} .$$

Then, by (52), we get

$$d\omega_i = d\theta_{0i} = \sum_{j \neq i} \omega_j \theta_{ji} + \omega_i(\theta_{ii} - \theta_{00}).$$

From the uniqueness of the set of forms ω_{ij} satisfying the first and the ninth equations of (34) it follows that

$$\omega_{ij} = \theta_{ij} , \qquad i \neq j,$$

$$\omega_{ii} = \theta_{ii} - \theta_{00} .$$

We find then

$$d\omega_{ij} = \sum_k \omega_{ik}\omega_{kj} + \theta_{i0}\theta_{0j} , \qquad i \neq j,$$

$$d\omega_{ii} = \sum_k \omega_{ik}\omega_{ki} + \theta_{i0}\theta_{0i} + \sum_k \theta_{k0}\theta_{0k} ,$$

and therefore

$$\Omega_{ij} = -\bar{\omega}_i \omega_j , \qquad i \neq j,$$

$$\Omega_{ii} = -\bar{\omega}_i \omega_i - \sum_k \bar{\omega}_k \omega_k ,$$

which are exactly the equations (45). Thus we have proved that it is possible to define in the complex projective space an elliptic Hermitian Geometry.

2. Formulas of Cartan and Wirtinger

When the Hermitian manifold is locally elliptic, it is possible to calculate the forms Ψ_r in (39) more explicitly. In fact, we are going to prove the following theorem:

THEOREM 12. *In a locally elliptic Hermitian manifold let Λ be the exterior differential form corresponding to the Hermitian differential form, that is, $\Lambda = \sum_i \omega_i \bar{\omega}_i$ in case the given Hermitian form is $ds^2 = \sum_i (\omega_i \bar{\omega}$ Then we have*

$$\Psi_r = \frac{1}{(2\pi\sqrt{-1})^{n-r+1}} \binom{n+1}{r} \Lambda^{n-r+1}, \qquad 1 \leqq r \leqq n. \tag{54}$$

It is clear that the construction of Λ from ds^2 is independent of the choice of the base linear differential forms ω_i in terms of which ds^2 is expressed.

The theorem is proved by induction on $n - r$. If $n - r = 0$, that is, $r = n$, we have

$$\Psi_n = \frac{1}{2\pi\sqrt{-1}} \sum_i \Lambda_{ii} = \frac{n+1}{2\pi\sqrt{-1}} \Lambda.$$

Suppose the formula (54) be true for $r + 1, \cdots, n$. We have

$$\Psi_r = \frac{1}{(2\pi\sqrt{-1})^{n-r+1}(n-r+1)!} \{\sum \delta(i_1 \cdots i_{n-r}; j_1 \cdots j_{n-r}) \cdot\Omega_{i_1 j_1} \cdots \Omega_{i_{n-r} j_{n-r}} \sum_k \Omega_{kk} - (n-r) \sum \delta(i_1 \cdots i_{n-r}; j_1 \cdots j_{n-r}) \cdot\Omega_{i_1 j_1} \cdots \Omega_{i_{n-r-1} j_{n-r-1}} \cdot \sum_k \Omega_{i_{n-r} k} \Omega_{k j_{n-r}}\}.$$

Consider the second sum inside the braces. If $i_{n-r} \neq j_{n-r}$, we can replace the sum $\sum_k \Omega_{i_{n-r}k}\Omega_{kj_{n-r}}$ by $\omega_{j_{n-r}}\bar{\omega}_{i_{n-r}}\Lambda = \Omega_{i_{n-r}j_{n-r}}\Lambda$. If $i_{n-r} = j_{n-r}$, the sum can be replaced by $(\omega_{i_{n-r}}\bar{\omega}_{i_{n-r}} + \Lambda)\Lambda = \Omega_{i_{n-r}j_{n-r}}\Lambda$. By our induction hypothesis we get easily the desired formula (54).

As an application let us determine the n basic classes of the complex projective space of n dimensions. The result is given by the following theorem:

THEOREM 13. *The r^{th} basic characteristic class ($1 \leqq r \leqq n$) of a complex projective space of n dimensions is dual to the homology class containing the cycle carried by a linear subspace of dimension $r - 1$ multiplied by $\binom{n+1}{r}$.*

To prove this theorem we consider the affine space A_{n+1} of dimension $n + 1$ with the coordinates z^α such that the projective space P_n under consideration is the hyperplane at infinity with which A_{n+1} is made into a projective space of $n + 1$ dimensions. As before, z^α are homogeneous coordinates in P_n. Suppose that we have defined at each point $\mathfrak{z}$ of A_{n+1} different from the origin r vectors $\mathfrak{v}_i$, $1 \leqq i \leqq r$, whose components $v_i^0, \cdots, v_i^n$ are linear forms in the coordinates

z^α of $\mathfrak{z}$. Then all the points on the line joining the origin 0 to $\mathfrak{z}$ are projected from 0 into the same point p of P_n and the vectors $\mathfrak{v}_i$, $1 \leqq i \leqq r$, at the points of $0\mathfrak{z}$ are projected from 0 into the same vector of P_n, which we attach to p. It is easy to verify that the r vectors thus defined at each point of P_n are linearly independent when and only when the $r + 1$ vectors $\mathfrak{z}, \mathfrak{v}_1, \cdots, \mathfrak{v}_r$ in A_{n+1} are linearly independent.

We take

$$v_i^\alpha = a_i^\alpha z^\alpha, \qquad 1 \leqq i \leqq r,$$

where a_i^α are constants. The a_i^α can be so chosen that none of the determinants of order $r + 1$ of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ a_1^0 & \cdots & a_1^n \\ \cdots & \cdots & \cdots \\ a_r^0 & \cdots & a_r^n \end{pmatrix}$$

will vanish. It then follows that the vectors $\mathfrak{z}, \mathfrak{v}_1, \cdots, \mathfrak{v}_r$ will be linearly dependent when and only when all products

$$z^{\alpha_0} \cdots z^{\alpha_r} = 0, \qquad 0 \leqq \alpha_0, \cdots, \alpha_r \leqq n,$$

the indices $\alpha_0, \cdots, \alpha_r$ being distinct from each other. This is possible when and only when $n + 1 - r$ of the coordinates z^α will vanish. In other words, for this particular field of r vectors the points of P_n at which the r vectors are linearly dependent are the linear spaces of dimension $r - 1$ defined by setting $n + 1 - r$ of the homogeneous coordinates of P_n to zero. The number of such linear spaces is $\binom{n+1}{r}$ and each of them is to be counted simply. Hence our theorem is proved.

THEOREM 14. (E. Cartan).[36] *Let M be a closed submanifold of topological dimension $2n - 2r + 2$ of the complex projective space of n dimensions P_n in which an elliptic Hermitian Geometry is defined. Let m be the number of points of intersection of M with a generic linear subspace of dimension $r - 1$ of P_n. Then*

$$(55) \qquad m = \frac{1}{(2\pi\Gamma - 1^{n-r+1})} \int_m \Lambda^{n-r+1}$$

In particular, if M is an algebraic variety of dimension $n - r + 1$ in P_n, m is its order.

This theorem is an immediate consequence of the Theorems 11, 12, and 13.

Related to these discussions is also a formula due to W. Wirtinger. In Hermitian Geometry, as in Riemannian Geometry, it is common to define an element of volume of topological dimension $2p$ by means of the equation

$$(56) \qquad \Delta_{2p} = \pm \sum_{(i_1 \cdots i_p)} \omega_{i_1} \cdots \omega_{i_p} \bar{\omega}_{i_1} \cdots \bar{\omega}_{i_p},$$

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[36] CARTAN, [3], p. 206.

the summation being over all the combinations of $i_1, \cdots, i_p$ from 1 to n. Up to a sign Λ_{2p} is equal to $\frac{1}{p!}\Lambda^p$. We define

$$\Delta_{2p} = \frac{1}{p!}\Lambda^p. \tag{57}$$

From (55) follows the theorem:

THEOREM 15. (W. Wirtinger).[37] *In the complex projective space of n-dimensions with the elliptic Hermitian metric let V_{2p} be a p-dimensional algebraic variety of order m and volume V. Then*

$$V = \frac{(2\pi)^p}{p!}m. \tag{58}$$

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