FROM TRIANGLES TO MANIFOLDS

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1. Geometry. I believe I am expected to tell you all about geometry; what it is, its developments through the centuries, its current issues and problems, and, if possible, a peep into the future. The first question does not have a clear-cut answer. The meaning of the word *geometry* changes with time and with the speaker. With Euclid, geometry consists of the logical conclusions drawn from a set of axioms. This is clearly not sufficient with the horizons of geometry ever widening. Thus in 1932 the great geometers O. Veblen and J. H. C. Whitehead said, "A branch of mathematics is called geometry, because the name seems good on emotional and traditional grounds to a sufficiently large number of competent people" **[1]**. This opinion was enthusiastically seconded by the great French geometer Elie Cartan **[2]**. Being an analyst himself, the great American mathematician George Birkhoff mentioned a "disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis" **[3]**. Recently my friend André Weil said: "The psychological aspects of true geometric intuition will perhaps never be cleared up. At one time it implied primarily the power of visualization in three-dimensional space. Now that higher-dimensional spaces have mostly driven out the more elementary problems, visualization can at best be partial or symbolic. Some degree of tactile imagination seems also to be involved" **[4]**.

At this point it is perhaps better to let things stand and turn to some concrete topics.

2. Triangles. Among the simplest geometrical figures is the triangle, which has many beautiful properties. For example, it has one and only one inscribed circle and also one and only one circumscribed circle. At the beginning of this century the nine-point circle theorem was known to almost every educated mathematician. But its most intriguing property concerns the sum of its angles. Euclid says that it is equal to 180°, or π by radian measure, and deduces this from a sophisticated axiom, the so-called *parallel axiom*. Efforts to avoid this axiom failed. The result was the discovery of non-Euclidean geometries in which the sum of angles of a triangle is less or greater than π, according as the geometry is hyperbolic or elliptic. The discovery of hyperbolic non-Euclidean geometry, in the eighteenth century by Gauss, John Bolyai, and Lobatchevsky, was one of the most brilliant chapters in human intellectual history.

The generalization of a triangle is an n-gon, a polygon with n sides. By cutting the n-gon into $n-2$ triangles, one sees that the sum of its angles is $(n-2)\pi$. It is better to measure the sum of the exterior angles! The latter is equal to 2π, for all n-gons, including triangles.

3. Curves in the plane; rotation index and regular homotopy. By applying calculus we can consider smooth curves and closed smooth curves in the plane, i.e., curves with a tangent line everywhere and varying continuously. As a point moves along a closed smooth (oriented) curve C once, the lines through a fixed point O and parallel to the tangent lines of C rotate through an angle $2n\pi$ or rotate n times about O. This integer n is called the rotation index of C. (See Fig. 1.) A famous theorem in differential geometry says that if C is a simple curve, i.e., if C does not intersect itself, $n = \pm 1$.

Clearly, there should be a theorem combining the theorem on the sum of exterior angles of an n-gon and the rotation index theorem of a simple closed smooth curve. This is achieved by considering the wider class of simple closed sectionally smooth curves. The rotation index of

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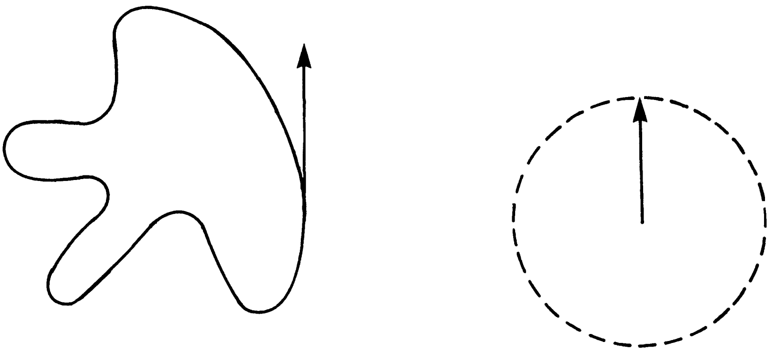


FIG. 1

such a curve can be defined in a natural way by turning the tangent at a corner an amount equal to the exterior angle. (See Fig. 2.) Then the rotation index theorem above remains valid for simple closed sectionally smooth curves. In the particular case of an n-gon formed by straight segments, this reduces to the statement that the sum of its exterior angles is 2π.

This theorem can be further generalized. Instead of simple closed curves we can allow closed curves to intersect themselves. A generic self-intersection can be assigned a sign. Then, if the curve is properly oriented, the rotation index is equal to one plus the algebraic sum of the number of self-intersections. (See Fig. 3.) For example, the figure 8 has the rotation index zero.

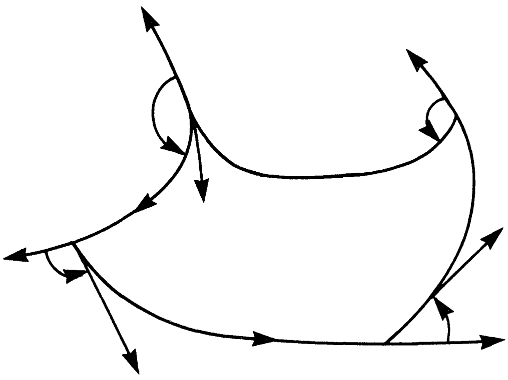


FIG. 2

$n=4$ $n=-2$

FIG. 3

A fundamental notion in geometry, or in mathematics in general, is *deformation* or *homotopy*. Two closed smooth curves are said to be *regularly homotopic* if one can be deformed to the other through a family of closed smooth curves. Since the rotation index is an integer and varies continuously in the family, it must remain a constant; i.e., it keeps the same value when the curve is regularly deformed. A remarkable theorem of Graustein-Whitney says that the converse is true [5]: Two closed smooth curves with the same rotation index are regularly homotopic.

It is a standard practice in mathematics that in order to study closed smooth curves in the plane it is more profitable to look at all curves and to put them into classes, the regular homotopy classes in this case being an example. This may be one of the essential methodological differences between theoretical science and experimental science, where such a procedure is impractical. The Graustein-Whitney theorem says that the only invariant of a regular homotopy class is the rotation index.

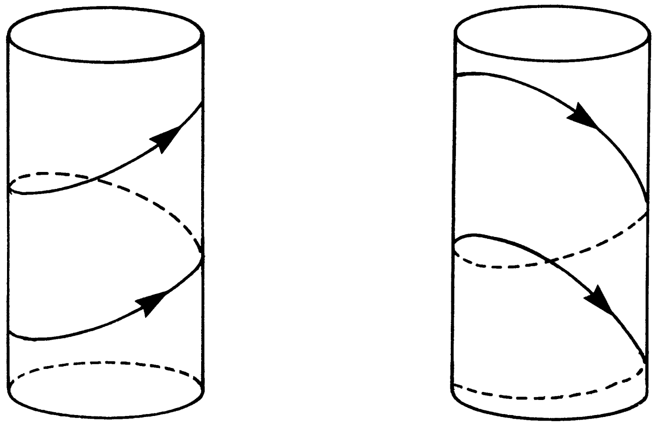


FIG. 4

4. Euclidean three-space. From the plane we pass to the three-dimensional Euclidean space where the geometry is richer and has distinct features. Perhaps the nicest space curve which does not lie in a plane is a circular helix. It has constant curvature and constant torsion and is the only curve admitting ∞^1 rigid motions. There is an essential difference between right-handed and left-handed helices (See Fig. 4), depending on the sign of the torsion; a right-handed helix cannot be congruent to a left-handed one, except by a mirror reflection. Helices play an important role in mechanics. From a geometrical viewpoint it may not be an entire coincidence that the Crick-Watson model of a DNA-molecule is double-helical. A double helix has interesting geometrical properties. In particular, by joining the end points of the helices by segments or arcs, we get two closed curves. In three-dimensional space they have a linking number. (See Fig. 5.)

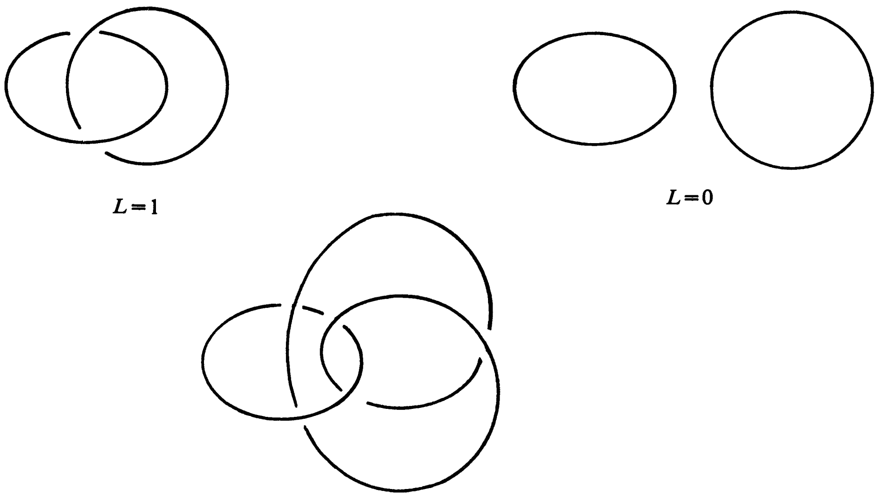


FIG. 5

A recent controversial issue in biochemistry, raised by the mathematicians William Pohl and George Roberts, is whether the chromosomal DNA is double-helical. In fact, if it is, it will have two closed strands with a linking number of the order of 300,000. The molecule is replicated by separation of the strands and formation of the complementary strand of each. With such a large linking number Pohl and Roberts showed that the replication process would have severe mathematical difficulties. Thus the double-helical structure of the DNA molecule, at least for the chromosome, has been questioned [6]. (Added January 26, 1979: A number of recent experiments have shown that some of the mathematical difficulties for the double helical structure of the DNA-molecule can be overcome by enzymatic activities (cf. F. H. C. Crick, Is DNA really a double helix? preprint, 1978).)

The linking number L is determined by the formula of James H. White [7]:

$$T + W = L, \tag{1}$$

where T is the total twist and W the writhing number. The latter can be experimentally measured and changes by the action of an enzyme. This formula is of fundamental importance in molecular biology. Generally DNA molecules are long. In order to store them in limited space, the most economical way is to writhe and coil them. These discussions could indicate the beginning of a stochastic geometry, with the main examples drawn from biology.

In a three-dimensional space surfaces have far more important properties than curves. Gauss's fundamental work elevated differential geometry from a chapter of calculus to an independent discipline. His *Disquisitiones generales circa superficies curvas* (1827) is the birth certificate of differential geometry. The main idea is that a surface has an intrinsic geometry based on the measure of arc length alone. From the element of arc other geometric notions, such as the angle between curves and the area of a piece of surface, can be defined. Plane geometry is thus generalized to any surface Σ based only on the local properties of the element of arc. This localization of geometry is both original and revolutionary. In place of the straight lines are the geodesics, the "shortest" curves between any two points (sufficiently close). More generally, a curve on Σ has a "geodesic curvature" generalizing the curvature of a plane curve and geodesics are the curves whose geodesic curvature vanishes identically.

Let the surface Σ be smooth and oriented. At every point p of Σ there is a unit normal vector $\nu(p)$ which is perpendicular to the tangent plane to Σ at p. (See Fig. 6.) The vector $\nu(p)$ can be viewed as a point of the unit sphere S_0 with center at the origin of the space. By sending p to $\nu(p)$ we get the Gauss mapping

$$g : \Sigma \to S_0. \tag{2}$$

The ratio of the element of the area of S_0 by the element of area of Σ under this mapping is called the Gaussian curvature. Gauss's "remarkable theorem" says that the Gaussian curvature depends only on the intrinsic geometry of Σ. In fact, in a sense it characterizes this geometry. Clearly the Gaussian curvature is zero if Σ is the plane.

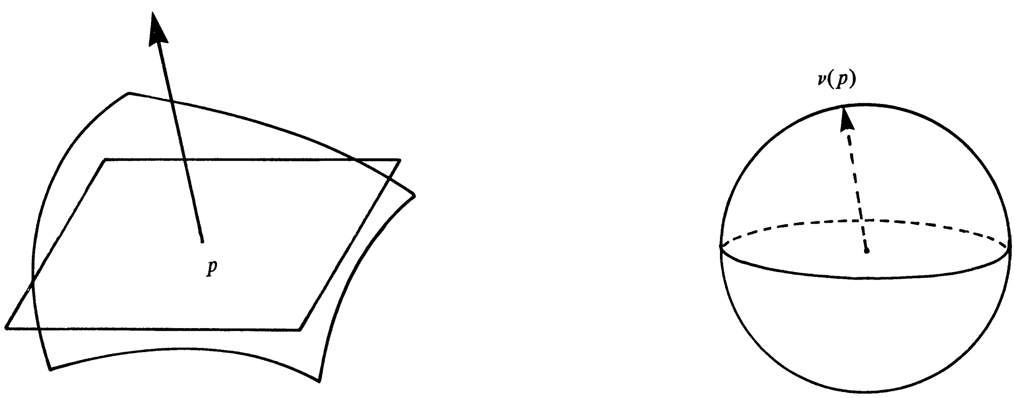


FIG. 6

As in plane geometry we consider on Σ a domain D bounded by one or more sectionally smooth curves. D has an important topological invariant $\chi(D)$, called its Euler characteristic, which is most easily defined as follows: Cut D into polygons in a "proper way" and denote by v, e, and f the number of vertices, edges, and faces, respectively. Then

$$\chi(D) = v - e + f. \tag{3}$$

(Euler's polyhedral theorem was known before Euler, but Euler seems to have been the first one to recognize explicitly the importance of the "alternating sum.")

The Gauss-Bonnet formula in surface theory is

$$\Sigma \text{ ext angles} + \int_{\partial D} \text{geod curv} + \int\int_D \text{Gaussian curv} = 2\pi\chi(D), \tag{4}$$

where ∂D is the boundary of D. For a plane domain the Gaussian curvature is zero. If in addition the domain is simply connected, we have $\chi(D) = 1$. Then this formula reduces to the rotation index theorem discussed in §3. We are indeed a long way from the sum of angles of a triangle.

Generalizing the geometry of closed plane curves we can consider closed oriented surfaces in space. The generalization of the rotation index is the degree of the Gauss mapping g in (2). The precise definition of the degree is sophisticated. Intuitively it is the number of times that the image $g(\Sigma)$ covers S_0, counted with sign. Unlike the plane, where the rotation index can be any integer, the degree d is completely determined by the topology of Σ; it is equal to

$$d = \frac{1}{2}\chi(\Sigma). \tag{5}$$

For the imbedded unit sphere this degree is $+1$ independently of its orientation. A surprising result of S. Smale [**8**] says that the two oppositely oriented unit spheres are indeed regularly homotopic or, more intuitively, that the unit sphere can be turned inside out through a regular homotopy. It is essential that at each stage of the homotopy the surface has a tangent plane everywhere, but is allowed to intersect itself.

5. From coordinate spaces to manifolds. It was Descartes who in the seventeenth century revolutionized geometry by using coordinates. Quoting Hermann Weyl, "The introduction of numbers as coordinates was an act of violence" [**9**]. From now on, paraphrasing Weyl, figure and number, like angel and devil, fight for the soul of every geometer. In the plane the Cartesian coordinates of a point are its distances, with signs, from two fixed perpendicular lines, the coordinate axes. A straight line is the locus of all points whose coordinates x,y satisfy a linear equation

$$ax + by + c = 0. \tag{6}$$

The result is the translation of geometry into algebra.

Once the door was opened for analytic geometry, other coordinate systems came into play. Among them are polar coordinates in the plane and spherical coordinates, cylindrical coordinates in space, and elliptic coordinates in the plane and in space. The latter are adapted to the confocal quadrics and are particularly suited to the study of the ellipsoids, which include our earth.

There is also a need for higher dimensions. For even if we start with a three-dimensional space, the theory of relativity calls for the inclusion of time as a fourth dimension. On a more elementary level, to record the motion of a particle, including its velocity, requires six coordinates (the hodograph). All the continuous functions in one variable form an infinite-dimensional space. Those which are square-integrable form a Hilbert space, which can be coordinatized by an infinite sequence of coordinates. Such a viewpoint, of considering all functions with prescribed properties, is fundamental in mathematics.

From the proliferation of coordinate systems it is natural to have a theory of coordinates.

General coordinates need only the property that they can be identified with points; i.e., there is a one-to-one correspondence between points and their coordinates—their origin and meaning are inessential.

If you find it difficult to accept general coordinates, you will be in good company. It took Einstein seven years to pass from his special relativity in 1908 to his general relativity in 1915. He explained the long delay in the following words: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning" **[10]**.

After being served by coordinates in the study of geometry, we now wish to be free from their bond. This leads to the fundamental notion of a *manifold*. A manifold is described locally by coordinates, but the latter are subject to arbitrary transformations. In other words, it is a space with transient or relative coordinates (principle of relativity). I would compare the concept with the introduction of clothing to human life. It was a historical event of the utmost importance that human beings began to clothe themselves. No less significant was the ability of human beings to change their clothing. If geometry is the human body and coordinates are clothing, then the evolution of geometry has the following comparison.

Synthetic geometry	Naked man
Coordinate geometry	Primitive man
Manifolds	Modern man

A manifold is a sophisticated concept even for mathematicians. For example, a great mathematician such as Jacques Hadamard "felt insuperable difficulty . . . in maintaining more than a rather elementary and superficial knowledge of the theory of Lie groups" **[11]**, a notion based on that of a manifold.

6. Manifolds; local tools. With coordinates practically meaningless there is a need for new tools in studying manifolds. The key word is invariance. Invariants are of two kinds: local and global. The former refer to the behavior under a change of the local coordinates, while the latter are global invariants of the manifold, examples being the topological invariants. Two of the most important local tools are the exterior differential calculus and Ricci's tensor analysis.

An exterior differential form is the integrand of a multiple integral, such as

$$\iint_D Pdydz + Qdzdx + Rdxdy, \tag{7}$$

in (x,y,z)-space, where P, Q, R are functions in x,y,z and D is a two-dimensional domain. It is observed that a change of variables in D (supposed to be oriented) will be taken care of automatically if the multiplication of differentials is anti-symmetric, i.e.,

$$dy \wedge dz = -dz \wedge dy, \text{ etc.,} \tag{8}$$

where the symbol $\wedge$ is used to denote exterior multiplication. It is also more suggestive to introduce the exterior two-form

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \tag{9}$$

and to write the integral (7) as a pairing (D,ω) of the domain D and the form ω.

For if the same is done in n-space, then Stokes's theorem can be written

$$(D,d\omega) = (\partial D,\omega), \tag{10}$$

where D is an r-dimensional domain and ω is an exterior $(r-1)$-form; ∂D is the boundary of D and $d\omega$ is the exterior derivative of ω and is an r-form. Formula (10), the fundamental formula in multi-variable calculus, shows that ∂ and d are adjoint operators. The remarkable fact is that, while the boundary operator ∂ on domains is global, the exterior differentiation operator d on forms is local. This makes d a powerful tool. When applied to a function ($=0$-form) and a

1-form, it gives the gradient and the curl, respectively. All the smooth forms, of all degrees ($\leqslant$ dim of manifold), of a differentiable manifold constitute a ring with the exterior differentiation operator d. Elie Cartan used the exterior differential calculus most efficiently in local problems of differential geometry and partial differential equations. The global theory was founded by G. de Rham, after initial work of Poincaré. This will be discussed in the next section.

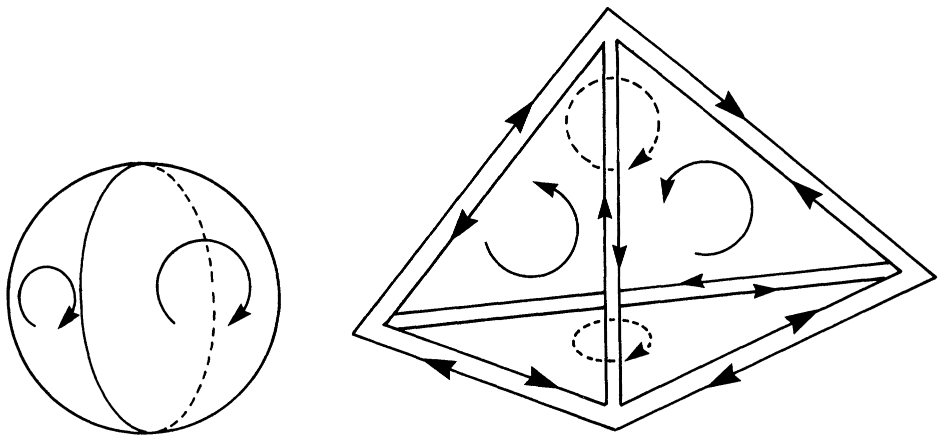
In spite of its importance the exterior differential calculus is inadequate in describing the geometrical and analytical phenomena on a manifold. A broader concept is Ricci's tensor analysis. Tensors are based on the fact that a manifold, being smooth, can be approximated at every point by a linear space, called its tangent space. The tangent space at a point leads to associated tensor spaces. Differentiation of tensor fields needs an additional structure, called an affine connection. If the manifold has a Riemannian or Lorentzian structure, the corresponding Levi-Civita connection will serve the purpose.

7. Homology. Historically a systematic study of the global invariants of a manifold began with combinatorial topology. The idea is to decompose the manifold into cells and see how they fit together. (The decomposition satisfies some mild conditions, which we will not specify.) In particular, if M is a closed manifold of dimension n and α_k denotes the number of k cells of the decomposition, $k=0,1,\ldots,n$, then, as a generalization of (3), the Euler-Poincaré characteristic of M is defined by

$$\chi(M)=\alpha_0-\alpha_1+\cdots+(-1)^n\alpha_n. \tag{11}$$

The basic notion in homology theory is that of a boundary. A chain is a sum of cells with multiplicities. It is called a cycle if it has no boundary, i.e., if its boundary is zero. The boundary of a chain is a cycle (see Fig. 7). The number of linearly independent k-dimensional cycles modulo k-dimensional boundaries is a finite integer b_k, called the kth Betti number. The Euler-Poincaré formula says

$$\chi(M)=b_0-b_1+\cdots+(-1)^n b_n. \tag{12}$$



(a) (b)

FIG. 7

The Betti numbers b_k, and hence $\chi(M)$ itself, are topological invariants of M, that is, they are independent of the decomposition and remain invariant under a topological transformation of M. This and more general statements could be considered the fundamental theorems of combinatorial topology. After the path-breaking works of Poincaré and L. E. J. Brouwer, combinatorial topology blossomed in the U.S. in the 1920's under the leadership of Veblen, Alexander, and Lefschetz.

While this is an effective way in deriving topological invariants, the danger in cutting a manifold is that it might be "killed." Precisely, this means that by using a combinatorial approach we may lose sight of the relations of the topological invariants with local geometrical properties. It turns out that, while homology theory depends on the boundary operator ∂, there is a dual cohomology theory based on the exterior differentiation operator d, the latter being a local operator.

The resulting de Rham cohomology theory can be summarized as follows: The operator d has the fundamental property that, when applied repeatedly it gives the zero form; that is, for any k-form α, the exterior derivative of the $(k+1)$-form $d\alpha$ is zero. This corresponds to the geometrical fact that the boundary of any chain (or domain) has no boundary. (See (10).) A form α is called closed, if $d\alpha=0$. It is called a derived form, if there exists a form β, of degree $k-1$, such that it can be written $\alpha=d\beta$. Thus a derived form is always closed. Two closed forms are called cohomologous if they differ by a derived form. All the closed k-forms which are cohomologous to each other constitute the k-dimensional cohomology class. The remarkable fact is that, while the families of k-forms, closed k-forms, derived k-forms are immensely large, the k-dimensional cohomology classes constitute a finite-dimensional linear space whose dimension is the kth Betti number b_k.

De Rham cohomology is the forerunner of sheaf cohomology, which was founded by J. Leray [**12**] and perfected and applied with great success by H. Cartan and J.-P. Serre.

8. Vector fields and generalizations. On a manifold M it is natural to consider continuous vector fields, i.e., the attachment of a tangent vector to each point, varying in a continuous manner. If the Euler-Poincaré characteristic $\chi(M)$ is not zero, there is at least one point of M at which the vector vanishes. In other words, when the wind blows there is at least one spot on earth with no wind (for the Euler characteristic of the two-dimensional sphere is equal to 2). More precisely, at an isolated zero of a continuous vector field, an integer, called the index, can be defined, which describes to a certain extent the behavior of the vector field at the zero, i.e., whether it is a source, a sink, or otherwise. No matter what the vector field is, so long as it is continuous and has only a finite number of zeros, then the theorem of Poincaré-Hopf says that the sum of its indices at all the zeros is a topological invariant which is precisely $\chi(M)$.

This is a statement on the tangent bundle of M, i.e., the collection of the tangent spaces of M. More generally, a family of vector spaces parametrized by a manifold M and satisfying a local product condition is called a vector bundle over M.

A fundamental question is whether such a bundle is globally a product. The above discussion shows that the tangent bundle is not a product if $\chi(M)\neq 0$; for if it were a product, there would exist a continuous vector field which is nowhere zero. The existence of a space which is locally but not globally a product, such as the tangent bundle of a manifold M with $\chi(M)\neq 0$, is not easy to visualize; geometry thus enters a more sophisticated phase.

To describe the global deviation of a vector bundle from a product space the first invariants are the so-called characteristic cohomology classes. The Euler-Poincaré characteristic is the simplest of the characteristic classes.

The Gauss-Bonnet formula (4) in §4 takes the particularly simple form

$$\int\int K dA = 2\pi\chi(\Sigma) \tag{4a}$$

when the surface Σ has no boundary. In this formula K is the Gaussian curvature and dA is the element of area. Formula (4a) is of paramount importance because it expresses the global invariant $\chi(\Sigma)$ as the integral of a local invariant, which is perhaps the most desirable relationship between local and global properties. This result has a wide generalization.

Let

$$\pi : E \to M \tag{13}$$

be a vector bundle. The generalization of a tangent vector field on M is a section of the bundle, i.e., a smooth mapping $s: M \to E$, such that the composition $\pi \circ s$ is the identity. Since E is only locally a product, the differentiation of s needs an additional structure, usually called a connection. The resulting differentiation, called covariant differentiation, is generally not commutative. The notion of curvature is a measure of the noncommutativity of covariant differentiation. Suitable combinations of the curvature give rise to differential forms which represent characteristic cohomology classes in the sense of the de Rham theory, of which the Gauss-Bonnet formula (4a) is the simplest example [**13**]. I believe that the concepts of vector bundles, connections, and curvature are so fundamental and so simple that they should be included in any introductory course on multivariable calculus.

9. Elliptic differential equations. When M has a Riemannian metric, there is an operator $*$ sending a k-form α to the $(n-k)$-form $*\alpha, n = \dim M$. It corresponds to the geometrical construction of taking the orthogonal complement of a linear subspace of the tangent space. With $*$ and the differential d we introduce the codifferential

$$\delta = (-1)^{nk+n+1} * d * \tag{14}$$

and the Laplacian

$$\Delta = d\delta + \delta d. \tag{15}$$

Then the operator δ sends a k-form to a $(k-1)$-form and Δ sends a k-form to a k-form. A form α satisfying

$$\Delta\alpha = 0 \tag{16}$$

is called *harmonic*. A harmonic form of degree 0 is a harmonic function in the usual sense.

The equation (16) is an elliptic partial differential equation of the second order. If M is closed, all its solutions form a finite dimensional vector space. By a classical theorem of Hodge this dimension is exactly the kth Betti number b_k. It follows by (12) that the Euler characteristic can be written

$$\chi(M) = d_e - d_0, \tag{17}$$

where d_e (respectively, d_0) is the dimension of the space of harmonic forms of even (respectively odd) degree. The exterior derivative d is itself an elliptic operator and (17) can be regarded as expressing $\chi(M)$ as the index of an elliptic operator. The latter is, for any linear elliptic operator, equal to the dimension of the space of solutions minus the dimension of the space of solutions of the adjoint operator.

The expression of the index of an elliptic operator as the integral of a local invariant culminates in the Atiyah-Singer index theorem. It includes as special cases many famous theorems, such as the Hodge signature theorem, the Hirzebruch signature theorem, and the Riemann-Roch theorem for complex manifolds. An important by-product of this study is the recognition of the need to consider pseudo-differential operators on manifolds, which are more general than differential operators.

Elliptic differential equations and systems are closely enmeshed with geometry. The Cauchy-Riemann differential equations, in one or more complex variables, are at the foundation of complex geometry. Minimal varieties are solutions of the Euler-Lagrange equations of the variational problem minimizing the area. These equations are quasi-linear. Perhaps the "most" non-linear equations are the Monge-Ampère equations, which are of importance in several geometrical problems. Great progress has been made in these areas in recent years [**14**]. With this heavy intrusion of analysis George Birkhoff's remark quoted above sounds even more disturbing. However, while analysis maps a whole mine, geometry looks out for the beautiful stones. Geometry is based on the principle that not all structures are equal and not all equations are equal.

10. Euler characteristic as a source of global invariants. To summarize, the Euler characteristic is the source and common cause of a large number of geometrical disciplines. I will illustrate this relationship by a diagram. (See Fig. 8.)

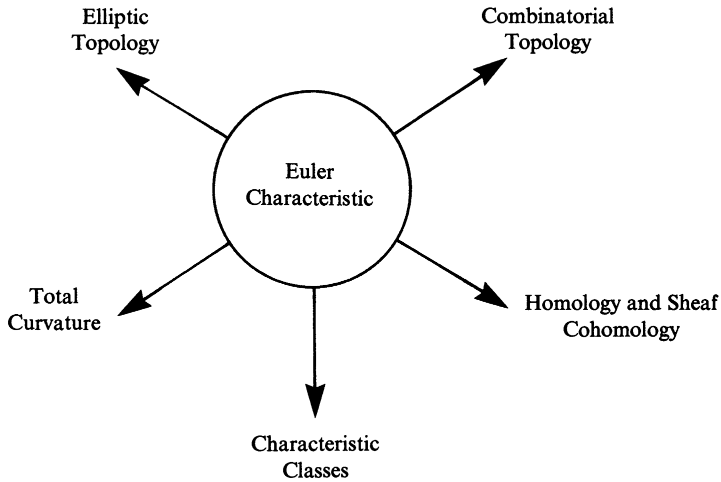


FIG. 8

11. Gauge field theory. At the beginning of this century differential geometry got the spotlight through Einstein's theory of relativity. Einstein's idea was to interpret physical phenomena as geometrical phenomena and to construct a space which would fit the physical world. It was a gigantic task and it is not clear whether he said the last word on a unified field theory of gravitational and electromagnetic fields. The introduction of vector bundles described above, and particularly the connections in them with their characteristic classes and their relations to curvature, widened the horizon of geometry. The case of a line bundle (i.e., when the fiber is a complex line) furnishes the mathematical basis of Weyl's gauge theory of an electromagnetic field. The Yang-Mills theory, based on an understanding of the isotopic spin, is the first example of a nonabelian gauge theory. Its geometrical foundation is a complex plane bundle with a unitary connection. Attempts to unify all field theories, including strong and weak interactions, have recently focused on a gauge theory, i.e., a geometrical model based on bundles and connections. It is with great satisfaction to see geometry and physics united again.

Bundles, connections, cohomology, characteristic classes are sophisticated concepts which crystallized after long years of search and experimentation in geometry. The physicist C. N. Yang wrote [**15**]: "That nonabelian gauge fields are conceptually identical to ideas in the beautiful theory of fiber bundles, developed by mathematicians without reference to the physical world, was a great marvel to me." In 1975 he mentioned to me: "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere." This puzzling is mutual. In fact, referring to the role of mathematics in physics, Eugene Wigner spoke about the unreasonable effectiveness of mathematics [**16**]. If one has to find a reason, it might be expressed in the vague term "unity of science." Fundamental concepts are always rare.

12. Concluding remarks. Modern differential geometry is a young subject. Not counting the strong impetus it received from relativity and topology, its developments have been continuous.

I am glad that we do not know what it is and, unlike many other mathematical disciplines, I hope it will not be axiomatized. With its contact with other domains in and outside of mathematics and with its spirit of relating the local and the global, it will remain a fertile area for years to come.

It may be interesting to characterize a period of mathematics by the number of variables in the functions or the dimension of the spaces it deals with. In this sense nineteenth century mathematics is one-dimensional and twentieth century mathematics is n-dimensional. It is because of the multi-variables that algebra acquires paramount importance. So far most of the global results on manifolds are concerned with even-dimensional ones. In particular, all complex algebraic varieties are of even real dimension. Odd-dimensional manifolds are still very mysterious. I venture to hope that they will receive more attention and substantial clarification in the twenty-first century. Recent works on hyperbolic 3-manifolds by W. Thurston [**17**] and on closed minimal surfaces in a 3-manifold by S. T. Yau, W. Meeks, and R. Schoen have thrown considerable light on 3-manifolds and their geometry. Perhaps the problem of problems in geometry is still the so-called Poincaré conjecture which says that a closed simply connected 3-dimensional manifold is homeomorphic to the 3-sphere. Topological and algebraic methods have so far not led to a clarification of this problem. It is conceivable that tools in geometry and analysis will be found useful.

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