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GEOMETRY OF A QUADRATIC DIFFERENTIAL FORM*

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1. The geometry of a quadratic differential form was founded by Gauss and Riemann, when it was shown that euclidean geometry can be given a significant generalization by basing all metrical notions on a positive definite quadratic differential form

$$\sum_{i,k} g_{ik} dx^i\, dx^k. \tag{1}$$

Relativity theory calls for the study of a differential form which is non-degenerate but of signature $(-+\cdots+)$. Such a data will be called a Lorentzian structure. Although there is yet no geometrical or physical justification, it is a natural mathematical problem to study a non-degenerate quadratic differential form with arbitrary signature.

Modern mathematics requires us to say at the outset in what space the quadratic differential form is defined and the study takes place. Such spaces are called differentiable manifolds, which are, roughly speaking, spaces where differentiation makes sense and where tensor fields can be defined. The prime example is the n-dimensional number space and a general differentiable manifold behaves locally like it. From a given differentiable manifold further manifolds are constructed by taking the submanifolds or the quotient manifolds. A compact manifold is one which can be covered by a finite number of coordinate patches. A manifold with a non-degenerate (resp. positive definite, resp. of signature $(- + \cdots +)$) quadratic differential form defined everywhere is called pseudo-riemannian (resp. riemannian, resp. Lorentzian).

Does such a pseudo-riemannian structure exist on a differentiable manifold? If the form is positive definite, this is so, because the set of all possible positive semi-definite quadratic differential forms has the convexity property. For a non-degenerate form of signature (p, q) (p positive squares and q negative squares in the normal form) the existence of such a form on a manifold is equivalent to the existence of a continuous field of p-dimensional subspaces of the tangent spaces.[1] If the manifold is compact and p is odd, a necessary condition is that the Euler-Poincaré characteristic of the manifold should be zero [2], [3]. A complete topological characterization of such manifolds is not known. It follows, however, from the above

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[1] Cf., for example, [1], p. 207.

result that a compact 4-manifold has a Lorentzian structure if and only if its Euler-Poincaré characteristic is zero.[2]

2. Once a pseudo-riemannian structure is given on a manifold, its local properties are expressed by the Levi-Civita connection it determines. The latter allows us to transport tangent vectors parallelly along curves and leads to the curvature tensor R_{ijkl}. From the curvature tensor one defines, for any two-dimensional subspace E of the tangent space spanned by the vectors X^i, Y^j, the sectional or riemannian curvature

$$(2) \quad R(E) = -\sum R_{ijkl}X^iY^jX_kY^l / \sum (g_{ik}g_{jl} - g_{il}g_{jk})X^iY^jX_kY^l.$$

Do these local properties have implications on the global properties of the manifold? The answer is yes, and its precise statements contain some of the most beautiful discoveries in modern mathematics. To give an example we take a compact oriented four-dimensional pseudo-riemannian manifold. Instead of the curvature tensor it is convenient to consider the tensorial differential forms

$$(3) \qquad \Omega_{ik} = \tfrac{1}{2}\sum R_{ikjl}dx^j \wedge dx^l.$$

Then the four-form

$$(4) \qquad \Pi = \frac{1}{24\pi^2}\sum g^{ik}\, g^{jl}\, \Omega_{ij}\, \Omega_{kl}$$

is a multiple of the volume element by a scalar invariant P. *Its integral over the manifold is a topological invariant, called the index or signature of the manifold.* The latter is defined as follows: Let $\gamma_1, \cdots, \gamma_b$ be a basis of the second homology group of the manifold. Let $KI(\gamma_i, \gamma_j)$, $1 \leqq i, j \leqq b$, be the intersection number of γ_i, γ_j. It is symmetric in its two arguments and can be considered as the coefficients of a quadratic form. The signature of the latter (i.e., the number of its positive squares minus that of its negative squares in its diagonalized normal form) is called the signature of the manifold.

Thus the theorem relates in a precise way a topological invariant depending on the multiplicative homology structure of the manifold with an expression in the curvature tensor of a pseudo-riemannian structure. It is among the most difficult theorems in mathematics. Its proof requires the

[2] The question whether there exists on a compact oriented 4-manifold a continuous field of oriented two-dimensional subspaces of the tangent spaces was completely answered by F. Hirzebruch and H. Hopf, cf. [4]. The corresponding problem for the unoriented two-dimensional sub-spaces, which is equivalent to the existence or non-existence of a quadratic differential form of signature (2, 2), seems to be more difficult and is to my knowledge unsolved.

cohomology ring theory, characteristic classes, Thom's cobordism theory, etc.[3] Could the scalar invariant P, homogeneous and of the second degree in R_{ijkl}, have some use in physics?

Encouraged by this result, one would like to construct further four-fold integrals which, when integrated over the manifold, will give topological invariants. I only know one other instance. This is the Gauss-Bonnet formula for a riemannian manifold. It says that the integral of the four-form

$$\Gamma = \frac{1}{8\pi^2 \mid \det g_{ik} \mid^{\frac{1}{2}}} (\Omega_{12} \wedge \Omega_{34} + \Omega_{13} \wedge \Omega_{42} + \Omega_{14} \wedge \Omega_{23}) \tag{5}$$

is equal to the Euler-Poincaré characteristic of the manifold [8]. The form in (5) is defined for any oriented pseudo-riemannian structure, but I do not know whether the statement remains true.[4]

These results have generalizations to higher dimensions. At the root of them is de Rham's theorem. We say that an exterior differential form of degree r is closed if its exterior derivative is zero and that it is a derived form if it is the exterior derivative of a form of degree $r - 1$. If the manifold is compact, de Rham's theorem says that the quotient space of the space of all closed forms of degree r divided by the space of all derived forms of degree r is isomorphic to the r-dimensional cohomology group with real coefficients. The generalizations of the above theorems consist in identifying characteristic cohomology classes of the manifold with differential forms constructed from the curvature tensor of a pseudo-riemannian structure.[5]

3. The simplest pseudo-riemannian structure is when the sectional curvature (2) is constant for all two-dimensional subspaces E. A manifold with such a structure is said to be of constant curvature. This obviously imposes strong restrictions on the manifold. In the riemannian case the study of such manifolds is called the Clifford-Klein space problem. A Lorentzian manifold of constant curvature is also called a universe with the

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[3] A brief outline of a proof of this theorem is as follows:

(a) By a theorem of Thom and Hirzebruch the signature of a compact oriented 4-manifold is equal to its Pontrjagin number divided by three;

(b) Pontrjagin numbers can be expressed as integrals of expressions formed from the components of the curvature tensor of a riemannian metric;

(c) a theorem of A. Weil says that the integral has the same value when the riemannian structure is replaced by a pseudo-riemannian one.

Cf. [5], p. 85; [6]; and [7].

[4] This question has since been answered. With a suitable modification the Gauss-Bonnet formula is true for any compact oriented even-dimensional pseudo-riemannian manifold. Cf. a forthcoming paper by the author in Summa Brasiliensis Mathematicae.

[5] Cf. [5], [6], and [7].

locally perfect cosmological principle. The questions on pseudo-riemannian manifolds of constant curvature have been studied only recently by E. Calabi and L. Markus [11] and J. A. Wolf [9], [10].

We begin by giving some simple examples of such manifolds. The n-dimensional number space $R_q^{\ n}$ with the coordinates $(x_1, \cdots, x_n)$ and the quadratic differential form

$$-dx_1^{\ 2} - \cdots - dx_q^{\ 2} + dx_{q+1}^2 + \cdots + dx_n^{\ 2} \tag{6}$$

is an example of such a manifold with sectional curvature zero. To get examples of manifolds with non-zero constant sectional curvature consider in R^{n+1} the quadric $Q_q^{\ n}$ defined by the equation

$$-x_1^{\ 2} - \cdots - x_q^{\ 2} + x_{q+1}^2 + \cdots + x_{n+1}^2 = 1. \tag{7}$$

Topologically $Q_q^{\ n}$ is homeomorphic to $R^q \times S^{n-q}$, where S^{n-q} denotes the sphere of $n - q$ dimensions. In R^{n+1} take the form

$$\Psi_q = -dx_1^{\ 2} - \cdots - dx_q^{\ 2} + dx_{q+1}^2 + \cdots + dx_{n+1}^2 . \tag{8}$$

The quadric $Q_q^{\ n}$ with the differential form induced by Ψ_q we denote by $S_q^{\ n}$ and the quadric $Q_q^{\ n}$ with the differential form induced by $-\Psi_q$ we denote by H_{n-q}^n. Then $S_q^{\ n}$ and $H_q^{\ n}$ are manifolds of dimension n with a non-degenerate quadratic differential form of signature $(n - q, q)$, whose sectional curvatures are respectively the constants 1 and -1. They are not necessarily connected nor simply connected, and we denote the corresponding connected and simply connected manifolds by $\tilde{S}_q^{\ n}$ and $\tilde{H}_q^{\ n}$ respectively. In particular, $S_1^{\ n}$ and $H_1^{\ n}$ are Lorentzian manifolds of curvatures 1 and -1 respectively. $S_1^{\ n}$ is also called a de Sitter space.

These spaces play a rôle which is more than just examples. We say that a manifold is complete if the geodesics can be extended to arbitrary values of the affine parameter. Then the following theorem was proved by Wolf: *A complete connected pseudo-riemannian manifold of constant sectional curvature* 1, 0, or -1 *is covered respectively by* $\tilde{S}_q^{\ n}$, $R_q^{\ n}$ $\tilde{H}_q^{\ n}$.

This reduces the study of the pseudo-riemannian manifolds of constant curvature to that of discontinuous groups of isometries without fixed points in the spaces $\tilde{S}_q^{\ n}$, $R_q^{\ n}$, $\tilde{H}_q^{\ n}$. Those of particular interest are the ones which are at the same time homogeneous spaces, i.e., spaces on which the group of isometries acts transitively. A homogeneous Lorentzian manifold of constant curvature is also called a universe with the globally perfect cosmological principle. The connected homogeneous pseudo-riemannian manifolds of non-zero constant curvature have been completely classified by J. Wolf. In particular, a connected homogeneous riemannian manifold of constant negative curvature is isometric to the hyperbolic space $H_0^{\ n}$. Those of constant positive curvature are also relatively few and they include the quotient spaces of a sphere S^{4k+3} of dimension $4k + 3$ by the polyhedral groups. The case of zero curvature is more complicated, of which only partial results are known.

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