Galilean-invariant Nonlinear PDEs and their Exact Solutions

Roman M. CHERNIHA

Institute of Mathematics of the National Academy of Sciences of Ukraina,
Tereshchenkivs'ka Street 3, Kyiv 4, Ukraina

Abstract

All systems of (n+1)-dimensional quasilinear evolutional second- order equations invariant under the chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$ are described. The obtained results are illustrated by examples of nonlinear Schrödinger equations.

1   Introduction

The $(n + 1)$-dimensional diffusion (heat) system of equations

$$\begin{aligned} \lambda_1 U_t &= \Delta U, \\ \lambda_2 V_t &= \Delta V \end{aligned} \tag{1}$$

(where $U = U(t, x), V = V(t, x)$ are unknown differentiable real functions, $U_t = \dfrac{\partial U}{\partial t}$, $V_t = \dfrac{\partial V}{\partial t}$, $x = (x_1, \ldots x_n), \lambda_1, \lambda_2 \in \mathbf{R}$) is known to be invariant under the generalized Galilei algebra $AG_2(1.n)$ [1, 2]

$$P_t = \partial_t, \quad P_a = \partial_a, \tag{2a}$$

$$Q_\lambda, G_a = tP_a - \frac{x_a}{2} Q_\lambda, J_{ab} = x_a P_b - x_b P_a, \tag{2b}$$

$$D = 2tP_t + x_a P_a + I_\alpha, \tag{2c}$$

$$\Pi = t^2 P_t + tx_a P_a - \frac{1}{4} |x|^2 Q_\lambda + tI_\alpha, \alpha_k = -n/2. \tag{2d}$$

In relations (2) and always hereinafter $Q_\lambda = \lambda_1 U \partial_U + \lambda_2 V \partial_V$, $I_\alpha = \alpha_1 U \partial_U + \alpha_2 V \partial_V$, $\partial_U \equiv \dfrac{\partial}{\partial U}$, $\partial_V \equiv \dfrac{\partial}{\partial V}$, $\partial_t \equiv \dfrac{\partial}{\partial t}$, $\partial_a \equiv \dfrac{\partial}{\partial x_a}$, $\alpha_k \in \mathbf{R}$, $k = 1, 2$ and summation is assumed from 1 to n over repeated indices.

The algebra produced by operators (2a)–(2b) is called the Galilei algebra $AG(1.n)$, and its extension by using the operator (2c) will be refered to as $AG_1(1.n)$ [1, 2].

It is clear that in the case $V(t, x) = 0$ from the system (1) we obtain the single heat equation

$$\lambda_1 U_t = \Delta U \qquad (1^1)$$

which is invariant under the $AG_2(1.n)$ algebra (2) with $\lambda_2 = \alpha_2 = 0$. In [3] it was proved that a standard generalization of Eq.(equation) (1^1) of the form

$$\lambda_1 U_t = \nabla(D(U)\nabla U) + B(U)$$

is not invariant under the Galilei algebra for all functions $D(U) \neq c_1$, $B(U) \neq c_2$, c_1 and c_2 are constants. Moreover, in paper [3] we have constructed all quasilinear generalizations of the heat equation that are invariant with respect to subalgebras of the generalized Galilei algebra $AG_2(1.n)$. In particular we have found the following nonlinear equation

$$\lambda_1 U_t = \Delta U + \lambda_0 t^{-2} U \left(\frac{U}{E(t,x)}\right)^{\beta},$$

where λ_0, β are arbitrary constants and E is the fundamental solution of the heat Eq.(1^1), which is invariant with respect to the algebra with the basic operators G_a, J_{ab}, D and Π.

Now consider a system of quasilinear generalizations of diffusion Eqs. (1) of the form

$$\begin{aligned} \lambda_1 U_t &= A_{ab}U_{ab} + C_{ab}V_{ab} + B_1, \\ \lambda_1 V_t &= D_{ab}U_{ab} + E_{ab}V_{ab} + B_2 \end{aligned} \qquad (3)$$

$A_{ab}, C_{ab}, D_{ab}, E_{ab}, B_1, B_2$ being arbitrary real or complex differentiable functions of $2n+2$ variables $U, V, U_1, \ldots U_n, V_1, \ldots V_n$. The indices $a = 1, \ldots n$ and $b = 1, \ldots n$ of the functions U and V denote differentiating with respect to x_a and x_b.

System (3) generalizes practically all known nonlinear systems of first- and second-order evolutional equations, describing various processes in physics, chemistry, biology: heat-and-mass transfer, filtration of two-phase liquid,diffusion at chemical reactions, etc. (see, for example, [4–7]).

In the case of complex $U = \overset{*}{V}$, $A_{ab} = \overset{*}{E}_{ab}$, $C_{ab} = \overset{*}{D}_{ab}$, $B_1 = \overset{*}{B}_2 = B$, $\lambda_1 = \overset{*}{\lambda}_2 = i$, system (3) is transformed into a pair of complex conjugate equations. We treat them as a class of nonlinear generalizations of Schrödinger equations, namely:

$$iU_t = A_{ab}U_{ab} + \overset{*}{D}_{ab}\overset{*}{U}_{ab} + B, \qquad (4a)$$

$$-i\overset{*}{U}_t = \overset{*}{A}_{ab}\overset{*}{U}_{ab} + D_{ab}U_{ab} + \overset{*}{B} \qquad (4b)$$

(hereinafter complex conjugate Eqs.(4b) are omitted).

For $A_{ab} = D_{ab} = D_{aa} = 0$, $a \neq b$, $A_{aa} = -1$, Eq.(4a) is obviously transformed into a Schrödinger equation with the nonlinear potential B:

$$iU_t + \Delta U = B. \qquad (4^1)$$

By choice of the corresponding potential $B = B(U, \overset{*}{U}, U_1 \ldots U_n, \overset{*}{U}_1, \ldots \overset{*}{U}_n)$, a great variety of Schrödinger equation generalizations known from the literature can be obtained.

In the case of zero potential B a classical Schrödinger equation is obtained

$$iU_t + \Delta U = 0 \qquad (5)$$

invariant under the $AG_2(1.n)$ algebra with the basic operators (2) , where

$$Q_\lambda = -i(U\partial_U - \overset{*}{U}\partial_{\overset{*}{U}}), \quad I_\alpha = \alpha(U\partial_U + \overset{*}{U}\partial_{\overset{*}{U}}) \tag{6}$$

In series of our papers [1, 2, 4, 8] written in collaboration with Professor W.Fushchych, all systems of evolutional equations of the form (3) invariant under the chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$ are described. The obtained results are illustrated by examples of nonlinear Schrödinger equations (NSE). In particular we have obtained the NSE

$$iU_t + U_{xx} + \lambda_1 U|U|^4 + \lambda_2 U|U||U|_x = 0 \tag{7}$$

that is invariant under the generalized Galilei algebra. In [9] all nonequivalent Lie ansätze and wide classes of exact solutions to NSE (7) were constructed. Note that in the case $\lambda_1 = 1, \lambda_2 = 4$ Eq.(7) is called the Eckhaus equation and can be linearized by the integral substitution [10].

2   Description of systems (3) with Galilean symmetry

The algebra of symmetries for the system of Eqs.(1) contains the Galilei operators G_a, $a = 1, \ldots n$, being a mathematical expression of the Galilei relativistic principle for Eqs.(1). The Galilei operators are also known [3] to be closely related with the fundamental solution of the diffusion equation. We recall that if some system of PDEs is invariant with respect to the Galilei algebra or its extention, then it gives a wide range of possibilities for construction of multiparametric families of exact solutions [1,9,11]. Moveover, the Galilei operators and projective operator (2d) generate nontrivial formulae of multiplication of solutions. These formulae can be used to convert stationary (time-independent) into non-stationary (time-dependent) ones with a different structure.

In view of this, it seems reasonable to search for Galilean-invariant nonlinear generalizations of system (1) in the class of system (3).

Theorem 1. *The system of nonlinear Eqs.(3) is invariant under the Galilei algebra in the representation (2a),(2b) if and only if it has the form:*

$$\lambda_1 U_t = \Delta U + U[A_1\Delta lnU + C_1\Delta lnV + B_1] +$$

$$+ U[A_2\omega_a\omega_b(lnU)_{ab} + C_2\omega_a\omega_b(lnV)_{ab}], \tag{8}$$

$$\lambda_2 V_t = \Delta V + V[D_1\Delta lnU + E_1\Delta lnV + B_2] +$$

$$+ V[D_2\omega_a\omega_b(lnU)_{ab} + E_2\omega_a\omega_b(lnV)_{ab}]$$

where $(lnU)_{ab} \equiv \dfrac{\partial^2 lnU}{\partial x_a \partial x_b}$, $(lnV)_{ab} \equiv \dfrac{\partial^2 lnV}{\partial x_a \partial x_b}$, $\Delta lnU \equiv (lnU)_{11} + \ldots + (lnU)_{nn}$, $\Delta lnV \equiv (lnV)_{11} + \ldots + (lnV)_{nn}$, $\omega = U^{\lambda_2}V^{-\lambda_1}$, $\omega_a = \dfrac{\partial\omega}{\partial x_a} \equiv (\lambda_2 U_a/U - \lambda_1 V_a/V)\omega$ *and* $A_k, B_k, C_k, D_k, E_k, k = 1, 2$ *are arbitrary functions of absolute invariants of the* $AG(1.n)$ *algebra* ω *and* $\theta = \omega_a\omega_a$.

The proof of this and the following theorems is based on the classical Lie scheme, which is realized in [2, 3] for obtaining the Galilei invariant equations. The detailed cumbersome calculations are omitted.

Note that in the case where $\lambda_1 = 0$, i.e., the first equation of system (3) being elliptical, the absolute invariants of the Galilei algebra are considerably simpler, namely: $\omega = U, \theta = U_a U_a$.

In case of systems of the form (3), being $AG_1(1.n)-$ and $AG_2(1.n)$- invariant, the structure of such systems essentially depends on the determinant $\delta = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{vmatrix}$. It is clear that the unit operators I_α and Q_λ are linearly dependent only in the case , where the determinant $\delta = 0$. As a result, we obtain two different essentially (i.e. nonequivalent) representations of the algebras $AG_1(1.n)$ and $AG_2(1.n)$ for $\delta = 0$ and $\delta \neq 0$, in contrast to the case of a single diffusion equation .

We omit theorem that describes $AG_1(1.n)$-invariant systems (see [4,8]).

The $AG_2(1.n)$-invariant systems are described by the following

Theorem 2. *The nonlinear system of Eqs.(3) is invariant with respect to algebra $AG_2(1.n)$ with basis operators (2) iff it has the form:*

1. In the case when $\delta \neq 0$

$$\lambda_1 U_t = \hat{\alpha}_1 \Delta U + U A(\hat{\theta})(\lambda_2 \Delta ln U - \lambda_1 \Delta ln V) + U \omega^{-2/\delta} B_1(\hat{\theta}) +$$

$$+ (1 - \hat{\alpha}_1) U_a U_a / U + U \omega^{2/\delta - 2} \omega_a \omega_b [\lambda_2 (ln U)_{ab} - \lambda_1 (ln V)_{ab}] C(\hat{\theta}), \qquad (9)$$

$$\lambda_2 V_t = \hat{\alpha}_2 \Delta V + V D(\hat{\theta})(\lambda_2 \Delta ln U - \lambda_1 \Delta ln V) + V \omega^{-2/\delta} B_2(\hat{\theta}) +$$

$$+ (1 - \hat{\alpha}_2) V_a V_a / V + V \omega^{2/\delta - 2} \omega_a \omega_b [\lambda_2 (ln U)_{ab} - \lambda_1 (ln V)_{ab}] E(\hat{\theta}).$$

2. In the case when $\delta = 0$

$$\lambda_1 U_t = \hat{\alpha}_1 \Delta U + U A(\omega)(\lambda_2 \Delta ln U - \lambda_1 \Delta ln V) + U \omega_a \omega_a B_1(\omega) +$$

$$+ (1 - \hat{\alpha}_1) U_a U_a / U + U (\omega_{a_1} \omega_{a_1})^{-1} \omega_a \omega_b [\lambda_2 (ln U)_{ab} - \lambda_1 (ln V)_{ab}] C(\omega), \qquad (10)$$

$$\lambda_2 V_t = \hat{\alpha}_2 \Delta V + V D(\omega)(\lambda_2 \Delta ln U - \lambda_1 \Delta ln V) + V \omega_a \omega_a B_2(\omega) +$$

$$+ (1 - \hat{\alpha}_2) V_a V_a / V + V (\omega_{a_1} \omega_{a_1})^{-1} \omega_a \omega_b [\lambda_2 (ln U)_{ab} - \lambda_1 (ln V)_{ab}] E(\omega),$$

A, B_1, B_2, C, D, E *being arbitrary functions, and* $\omega = U^{\lambda_2} V^{-\lambda_1}, \hat{\theta} = \omega_a \omega_a \omega^{2/\delta - 2}$.

Note that the systems of reaction-diffusion equations

$$\begin{aligned} \lambda_1 U_t &= \Delta U + f(U, V), \\ \lambda_2 V_t &= \Delta V + g(U, V), \end{aligned} \qquad (11)$$

which are intensively studied recently (see, e.g., [6, 7]), are particular case of the system (3). So, as follows from Theorem 2, the nonlinear systems (11) preserv $AG_2(1.n)$-symmetry of the linear system of Eqs.(1) if and only if they have the form

$$\begin{aligned} \lambda_1 U_t &= \Delta U + \beta_1 U^{1 + \lambda_2 \gamma} V^{-\lambda_1 \gamma}, \\ \lambda_2 V_t &= \Delta V + \beta_2 V^{1 - \lambda_1 \gamma} U^{\lambda_2 \gamma}, \end{aligned} \qquad (12)$$

where $\gamma = 4/(n(\lambda_2 - \lambda_1)), \lambda_2 \neq \lambda_1, \beta_k \in \mathbf{R}$.

In the case, where the first of Eqs.(3) degenerates into an elliptical one ($\lambda_1 = 0$),the $AG_2(1.n)$-invariant systems was also constructed. Note that in paper [8] integration of two-dimensional systems of Eqs.(9), (10) was reduced in this case to integration of the linear heat equation with a source.

3 Galilean-invariant nonlinear generalizations of the Schrödinger equation

As is noted above, a class of nonlinear generalizations of the Schrödinger Eq.(4) is a specific case of evolutional equations systems (3). This enables one to describe on the basis of Theorems 1 and 2 all quasilinear generalizations of the Schrödinger Eq.(5), which are invariant with respect to the Galilei and generalized Galilei algebras.

Note that the algebra $AG_2(1.n)$ in the case of Schrödinger equations is called the Schrödinger algebra.

Corollary 1. In the class of nonlinear equations of the form (4) the algebra AG(1.n) (2a),(2b) with $Q_\lambda = -i(U\partial_U - \overset{*}{U} \partial_{\overset{*}{U}})$ is admitted only for equations given by

$$iU_t + \Delta U = U[A_1\Delta lnU + A_2\Delta ln \overset{*}{U} + B] +$$

$$+U[A_3|U|_a|U|_b(lnU)_{ab} + A_4|U|_a|U|_b(ln \overset{*}{U})_{ab}], \tag{13}$$

where $A_j, j = 1, 2, 3, 4$ and B are arbitrary complex functions of two arguments $|U|$ and $|U|_a|U|_a$; $|U|^2 = U\overset{*}{U}$, $|U|_a = \dfrac{\partial|U|}{\partial x_a}$.

In the case $A_j = 0$, the class of Eqs.(13) is reduced to the equation

$$iU_t + \Delta U = UB(|U|, |U|_a|U|_a) \tag{14}$$

obtained in [1], whose specific case is a Schrödinger equation with the power nonlinearity $U|U|^\beta$, $\beta = $ const.

Corollary 2. Within the class of nonlinear equations of the form (4), the algebra $AG_2(1.n)$ (2), (6) for $\alpha = -n/2$ of the linear Schrödinger equation (5) is conserved only for equations given by

$$iU_t + \Delta U = UE_1\Delta ln|U| + U|U|^{4/n}B +$$

$$+U|U|^{-4/n-2}E_2|U|_a|U|_b(ln|U|)_{ab}. \tag{15}$$

In Eq.(15), E_1, E_2 and B are arbitrary complex functions of the argument $|U|^{-4/n-2} \times |U|_a|U|_a$, which is an absolute invariant of the generalized Galilei algebra $AG_2(1.n)$.

In the case $E_1 = E_2 = 0$, from the class of Eqs.(15) the following equation

$$iU_t + \Delta U = U|U|^{4/n}B \tag{16}$$

is obtained, which had been obtained in [1, 2]. Note that for $B = c = $ const, Eq. (16) is transformed into an equation with a fixed power nonlinearity, studied in a series of papers ([12, 13] for $n = 1$, [15] for $n = 2$, [1, 2, 11] for $n = 3$). In [1, 2] multiparametric families of invariant solutions of Eq.(16) of the form

$$iU_t + \Delta U = cU\frac{|U|_a|U|_a}{|U|^2} \tag{17}$$

and

$$iU_t + \Delta U = cU|U|^{4/3} \tag{18}$$

are constructed and systematized.

Being written in the case of one spatial variable (n=1), the class of Eqs.(15) after simple transformations is given by

$$iU_t + U_{xx} = U E_1 (ln|U|)_{xx} + U|U|^4 B, \quad U = U(t,x), x = x_1 \tag{19}$$

E_1 and B being arbitrary complex functions of the argument $|U|^{-3}|U|_x$.

Obviously, a specific case of Eq.(21) is given by Eq. (7) that was studied in detail in [9] for arbitrary constant values of λ_1 and λ_2. A multidimensional generalization of Eq.(7), posessing the $AG_2(1.n)$ symmetry, can be proposed as

$$iU_t + \Delta U + c_1 U|U|^{4/n} + c_2 U|U|^{-1+2/n}(|U|_a|U|_a)^{1/2} = 0. \tag{20}$$

4 Ansätze and formulae of multiplication of exact solutions of NSE (7)

In this section we consider NSE (7), namely:

$$iU_t + U_{xx} + \lambda_1 \mid U \mid^4 U + \lambda_2 \mid U \mid\mid U \mid_x U = 0,$$

where $U_t = \dfrac{\partial U}{\partial t}$, $U_{xx} = \dfrac{\partial^2 U}{\partial x^2}$, $\lambda_k = a_k + ib_k$, $a_k, b_k \in \mathbf{R}$, $k = 1, 2$.

For $\lambda_2 = 0, \lambda_1 = a_1$, NSE (7) is transformed into a Schrödinger equation with the power nonlinearity without derivatives

$$iU_t + U_{xx} + a_1 \mid U \mid^4 U = 0 \tag{21}$$

which was studied in [12, 13].

Contrary to a well-known NSE with a cubic nonlinearity which is integrated by the inverse scattering problem method [14], Eq.(21) cannot thus be integrated.

At $\lambda_1 = 0, \lambda_2 = a_2$, NSE (7) is transformed into a Davey-Stewartson-type equation for the case of a single spatial variable [16]

$$iU_t + U_{xx} + a_2 \mid U \mid\mid U \mid_x U = 0. \tag{22}$$

Finally for $16\lambda_1 =\mid \lambda_2 \mid^2, \mid \lambda_2 \mid^2 =\mid a_2 + ib_2 \mid^2 \equiv a_2^2 + b_2^2$, NSE (7) was studied in [10,17], where it was shown to be reduced to the linear Schrödinger Eq.(5) by the integral substitution. Note that it cannot be linearized for arbitrary λ_1, λ_2 [17].

As is shown in the section 3, the class of $AG_2(1.1)$-invariant Eqs.(15) contains NSE (7). In this section we will construct ansätze for this equation over all nonequivalent subalgebras of the generalized Galilei algebra $AG_2(1.1)$. We will suggest also the formulae for multiplication of exact solutions of NSE (7) into multiparametric families of ones [9].

In [18] systems of all nonequivalent (nonconjugate) subalgebras of the Galilei algebra and its generalizations are constructed. The complete set of nonconjugate subalgebras of the $AG_2(1.1)$ algebra with basic operators (2) for $\lambda_1 = \overset{*}{\lambda_2} = -i, n = 1, \alpha = -1/2$ is obtained as

$$X_1 = P_x, \quad X_2 = Q_\lambda, \quad X_3 = P_t - \alpha Q_\lambda$$
$$X_4 = P_t \mp G_x, \quad X_5 = D + \alpha Q_\lambda \tag{23}$$

$$X_6 = P_t + \Pi - \alpha Q_\lambda, \quad \alpha \in \mathbf{R}.$$

By solving corresponding Lagrange equations for each of the operators (23), the following ansätze for the function U are obtained

$$X_1: \quad U = \varphi(t) \tag{24a}$$

$$X_2: \quad U = \gamma \exp(i\varphi(t,x)), \tag{24b}$$

$$X_3: \quad U = \varphi(x) \exp(-i\alpha t) \tag{24c}$$

$$X_4: \quad U = \exp[\mp\frac{it}{2}(x + \frac{t^2}{3})]\varphi(\omega), \quad \omega = t^2 \pm 2x \tag{24d}$$

$$X_5: \quad U = t^{-(1+i\alpha)/4}\varphi(\omega), \quad \omega = \frac{x}{t^{1/2}} \tag{24e}$$

$$X_6: \quad U = (t^2 + 1)^{-1/4} \exp\left[\frac{i}{4}\left(\frac{tx^2}{1+t^2} + 2\alpha \arctan t\right)\right]\varphi(\omega), \quad \omega = x(1+t^2)^{-1/2} \tag{24f}$$

where α, γ are arbitrary real parameters, in (24b) φ is a real function.

After putting anzätze (24) into NSE (7), the following reduction equations are obtained (in ansatz (24d) for convenience, the upper signs are fixed):

$$i\frac{d\varphi}{dt} + \lambda_1 \mid \varphi \mid^4 \varphi = 0, \tag{25a}$$

$$i\left(i\frac{\partial\varphi}{\partial t} + \frac{\partial^2\varphi}{\partial x^2}\right) - \left(\frac{\partial\varphi}{\partial x}\right)^2 + \lambda_1\gamma^4 = 0, \tag{25b}$$

$$\frac{d^2\varphi}{dx^2} + \alpha\varphi + \left(\lambda_1 \mid \varphi \mid^4 + \lambda_2 \mid \varphi \mid \frac{d \mid \varphi \mid}{dx}\right)\varphi = 0, \tag{25c}$$

$$4\frac{d^2\varphi}{d\omega^2} + \frac{1}{4}\omega\varphi + \left(\lambda_1 \mid \varphi \mid^4 + 2\lambda_2 \mid \varphi \mid \frac{d \mid \varphi \mid}{d\omega}\right)\varphi = 0, \tag{25d}$$

$$\frac{d^2\varphi}{d\omega^2} - \frac{i}{2}\omega\frac{d\varphi}{d\omega} + \frac{\alpha - i}{4}\varphi + \left(\lambda_1 \mid \varphi \mid^4 + \lambda_2 \mid \varphi \mid \frac{d \mid \varphi \mid}{d\omega}\right)\varphi = 0, \tag{25e}$$

$$\frac{d^2\varphi}{d\omega^2} - \frac{1}{4}(2\alpha + \omega^2)\varphi + \left(\lambda_1 \mid \varphi \mid^4 + \lambda_2 \mid \varphi \mid \frac{d \mid \varphi \mid}{d\omega}\right)\varphi = 0. \tag{25f}$$

All Eqs.(25) are ordinary differential equations (ODE), except for equations (25b), and from their solutions by means of Eqs.(24), the exact solutions of NSE (7) are readily obtained. Eq. (25b) obtained by reduction of NSE (7) by unit operator $X_2 = Q_\lambda$ is easily integrated.

All the solutons that can be obtained by means of the $AG_2(1.1)$-symmetry are limited to NSE (7) invariant solutions obtained by means of Eqs.(24)–(25). This is due to the fact any other invariant solution can be obtained by applying corresponding finite transformations generated by operators (2) to Eq.(24)-type solutions. Successive application of finite transformations, generated by basic operators (2) for $\lambda_1 = \overset{*}{\lambda_2} = -i, n = 1, \alpha = -1/2$, to an arbitrary fixed solution V(t,x) of NSE (7) gives the following formula of its multiplica-

tion into a six-parameter family of solutions (for more detail see [1, 2])

$$U(t,x) = \Big(\frac{m}{d_0 - pm^2 t}\Big)^{1/2} \exp\Big[i\gamma + i\frac{pm^2 x^2 + 2m(\varepsilon + pd)x + m^2\varepsilon^2 t + b_0}{4(pm^2 t - d_0)}\Big] \times$$

$$V\Big(\frac{m^2 t + d_1}{d_0 - pm^2 t},\ \frac{mx + m^2\varepsilon t + \varepsilon d_1 + d}{d_0 - pm^2 t}\Big), \tag{26}$$

where $d_0 = 1 - pd_1, b_0 = pd^2 + 2\varepsilon d + \varepsilon^2 d_1$ and $\gamma, p, m, \varepsilon, d_1, d$ being arbitrary parameters.

The formula

$$U(t,x) = \exp\Big[-\frac{i\varepsilon}{2}\Big(x + \frac{\varepsilon t}{2}\Big)\Big]V(t, x + \varepsilon t) \tag{27}$$

is a specific case of formula (26) for $\gamma = p = d_1 = d = 0, m = 1$, and formula

$$U = \frac{1}{\sqrt{1 - pt}} \exp\Big[-\frac{ipx^2}{4(1 - pt)}\Big]V\Big(\frac{t}{1 - pt}, \frac{x}{1 - pt}\Big) \tag{28}$$

for $\gamma = \varepsilon = d_1 = d = 0, m = 1$. Formulae (27), (28) generated by Galilean and projective transformations enable nonstationary solutions of NSE (7) to be obtained from the $U = V(x)$ stationary solutions.

If in formula (26) for $\varepsilon = \gamma = d = 0, d_1 = \frac{1}{p}$ a limiting transition for $p \to \infty, m \to 0, pm \to -1$ is made, formula for obtaining new solutions is obtained

$$U(t,x) = \frac{1}{\sqrt{t}} \exp\Big(\frac{ix^2}{4t}\Big)V\Big(-\frac{1}{t}, \frac{x}{t}\Big), \tag{29}$$

which is well-known for the linear Schrödinger and heat equations. Moreover, it is also valid for any other nonlinear equations invariant with respect to P_t, D, Π operators .

It should be noted that application of formulae (29) and (26) for $p = \varepsilon = d_1 = d = 0$ to an evident generalization of ansatz (24d)

$$U(t,x) = \exp\Big[-\frac{it}{2}\Big(x + \frac{t^2}{3} + 2\alpha\Big)\Big]\varphi(\omega),\ \ \omega = t^2 + 2x$$

reduces it to the form

$$U(t,x) = \frac{1}{\sqrt{t}} \exp\Big[\frac{i}{12t^3}(3x^2 t^2 + 6m^3 xt + 2m^6 +$$

$$+12\alpha m^2 t^2 + 12\gamma t^3)\Big]\varphi\Big(\frac{2xt + m^3}{t^2}\Big), \tag{30}$$

m, α, γ being arbitrary parameters. Ansatz (30) is quoted in Ref.17 as an example of non-Lie ansatz, though as shown above, it is the Lie ansatz.

Moreover, in the new paper [19] the formula (29) is called one to represent a discrete symmetry. But, as you can see, this formula can be obtained also from a Lie symmetry. Note that multidimensional generalization of this formula was constructed in our paper [1].

5 Construction of exact solutions of NSE (7)

Exact solutions of NSE (7) will be obtained from solutions of reduction Eqs.(25) with subsequent application of ansätze (24). Here I give only examples (in detail see [9]).

Example 1. The NSE (7) for $\alpha < 0, \lambda_1 = a_1 + ib_1, \lambda_2 = a_2 + ib_2, a_2 = b_1 = 0, 16a_1 > b_2^2$ has the following exact solution

$$U = \frac{A_-}{(\cosh 2\sqrt{-\alpha}(x + \varepsilon t))^{\frac{1}{2}}} \exp\Big[-\frac{i\varepsilon}{2}\Big(x + \Big(2\alpha + \frac{\varepsilon}{2}\Big)t + \frac{8b_2}{\varepsilon(-\alpha)^{1/2}A_-^2} \times \tag{31}$$

$$\times \arctan(\exp 2\sqrt{-\alpha}(x + \varepsilon t))\Big)\Big],$$

where $A_- = \Big(\dfrac{-a_1 + \frac{b_2^2}{16}}{3\alpha}\Big)^{1/4}$, α, ε are arbitrary real parameters.

Evidently, for $b_2 = 0$ the solution of Eq.(21) from the solution (31) follows

$$U = \frac{A_-}{(\cosh 2\sqrt{-\alpha}(x + \varepsilon t))^{\frac{1}{2}}} \exp\Big[-\frac{i\varepsilon}{2}\Big(x + \Big(2\alpha + \frac{\varepsilon}{2}\Big)t\Big)\Big] \tag{32}$$

which is called soliton-like similarly to the known Zakharov-Shabat solution [14] for the NSE with the nonlinearity $U \mid U \mid^2$.

Since the function *arctan* is limited from below and from above, the solution (31) of the NSE (7) will be have similarly to solution (32) for $\mid x + \varepsilon t \mid \to \infty$.

Example 2. The NSE (7) for $\lambda_1 = a_1, \lambda_2 = a_2 + ib_2, 16a_1 = b_2^2, a_2 \neq 0$ has the following exact solution

$$U = R^{-1}(x + c_0) \exp\Big[-\frac{ib_2}{4}\int(R^{-1}(x + c_0))^2 dx\Big], \tag{33}$$

where R^{-1} is an inverse function to R . The function R satisfies the transcendental relation

$$x + c_0 = \frac{1}{2a_2 d^2} \ln\Big|\frac{(\rho - d)^3}{\rho^3 - d^3}\Big| + \frac{\sqrt{3}}{a_2 d^2} \arctan\Big(\frac{2\rho + d}{-\sqrt{3}d}\Big) \equiv R(\rho)$$

where c_0, d are arbitrary real parameters.

The exact solution (33) posessing a discontinuity at the point $x = \dfrac{\sqrt{3}\pi}{2a_2 d^2} - c_0$ (i.e., a resonance point is present, when $U \to \infty$) and at infinity $(x \to \infty)$ is limited.

Note that by applying any of formulae (26)–(29), solution (33) is transformed into a non-stationary solution of NSE (7).

Example 3. The NSE (7) for $b_1 = 0, a_2 < 0, 16a_1 = b_2^2$ has the following exact solution

$$U = \frac{2x + t^2}{\sqrt{-8a_2}} \exp\Big[-\frac{i}{2}\Big(tx + \frac{t^3}{3} + \frac{b_2}{96a_2}(2x + t^2)^3\Big)\Big].$$

The constructed exact solutions of the NSE (7) exist only at $16\lambda_1 \neq |\lambda_2|^2$, i.e., in the case, where a nonlocal linearization of this equation being lacking.

We have constructed also new exact solutions of the NSE (7) in the case $16\lambda_1 = |\lambda_2|^2$.

Example 4. The NSE (7) for $\lambda_1 = a_1, \lambda_2 = ib_2, 16a_1 = b_2^2$ has the following exact solution

$$U = \sqrt{\frac{x}{t}} Z_{\frac{1}{4}}\Big(-\frac{x^2}{2t}\Big) \exp\Big[\frac{i}{4}\Big(\frac{x^2}{2t} - b_2 \int \omega Z_{\frac{1}{4}}^2\Big(-\frac{\omega^2}{2}\Big) d\omega\Big)\Big],$$

where $\omega = \dfrac{x}{\sqrt{t}}$ and $Z_{\frac{1}{4}}(\cdot)$ is an arbitrary cylindrical function.

References

[1] Fushchych W.I., Cherniha R.M., *Ukrainian Math. J.*, 1989, V.41, N 10, 1161-1167; V.41, N 12 1456–1463.

[2] Fushchych W.I., Cherniha R.M., On Exact Solutions of Two Multidimensional Nonlinear Schrödinger-type Equations, 1986 (Kyiv, Preprint: Institute of Mathematics, Ukrainian Acad.Sci. N 86.85).

[3] Fushchych W.I., Cherniha R.M., *J. Phys. A*, 1985, V.18, 3491–3503.

[4] Fushchych W.I., Cherniha R.M., *Dopovidi Akad. Nauk Ukrainy* (Proc. Ukr. Acad. Sci.), 1994, N 3, 31–38.

[5] Wilhelmsson H., *Ukrain. Fiz. Zhurn.* (Ukrainian Phys. J.), 1993, V.38, N 1 44–53; *Physica Scripta*, 1992, V.46, 177–181.

[6] Fife P.C., Mathematical Aspects of Reacting and Diffusing Systems, Springer, Berlin, 1979.

[7] Murray J.D., Mathematical Biology, Springer, Berlin, 1989.

[8] Fushchych W.I., Cherniha R.M., *Dopovidi Akad. Nauk Ukrainy* (Proc. Ukr. Acad.Sci.), 1993, N 8, 44–51.

[9] Cherniha R.M., *Ukrain. Fiz. Zhurn.* (Ukrainian Phys.J.), 1995, V.40, N 4, 376–384.

[10] Calogero F., De Lillo S., *Inverse Problems*, 1988, V.4, L33–L37.

[11] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics, Dordrecht, Kluwer Academic Publishers, 1993.

[12] Clarkson P.A., Cosgrove C.M., *J. Phys. A*, 1987, V.20, 2003–2024.

[13] Gagnon L., Winternitz P., *J. Phys. A*, 1988, V.21, 1493–1511.

[14] Zakharov V.E., Shabat A.B., *Zhurn. Eksper. Teor. Fiz.* (JETP), 1971, V.61, N 1, 118–134.

[15] Gagnon L., *J. Phys. A*, 1982, V.25, 2649–2667.

[16] Davey A., Stewartson K., *Proc. Roy. Soc. A (London)*, 1974, V.338, 101.

[17] Clarkson P.A., *Nonlinearity*, 1992, V.5, 453–472.

[18] Olver P.J., Applications of Lie Groups to Differential Equations, Springer-Verlag, 1986.

[19] Qiong-gui Lin, *J. Phys. A*, 1995, V.28, 231–251.