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Lie and Non-Lie Symmetries of Nonlinear Diffusion Equations with Convection Term

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Abstract

Lie and conditional symmetries of nonlinear diffusion equations with convection term are described. Examples of new ansätze and exact solutions are presented.

1. Introduction

In the present paper, we consider nonlinear diffusion equations with convection term of the form

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U), \tag{1}$$

where $U = U(t, x)$ is the unknown function and $A(U)$, $B(U)$, $C(U)$ are arbitrary smooth functions. The indices t and x denote differentiating with respect to these variables. Equation (1) generalizes a great number of the well-known nonlinear second-order evolution equations, describing various processes in physics, chemistry, biology (in paper [1], one can find a wide list of references).

In the case $A = 1$, $B = C = 0$, the classical heat equation

$$U_t = U_{xx} \tag{2}$$

follows from equation (1). S. Lie [2] was the first to calculate the maximal invariance algebra (i.e., the Lie symmetry) of the linear heat equation (2). The algebra found is the generalized Galilei algebra $AG_2(1.1)$ that generates the six-parameter group of time- and space translations and Galilei, scale, and projective transformations (for details, see, e.g., [3, 4]).

In the case $B = C = 0$, the standard nonlinear heat equation

$$U_t = [A(U)U_x]_x \tag{3}$$

follows from equation (1). Lie symmetries of equation (3) were completely described by Ovsyannikov [5]. It has been shown that equation (3) is invariant under a non-trivial algebra of Lie symmetries only in the cases $A = \lambda_0 \exp(mU)$ and $A = \lambda_0(U + \alpha_0)^k$, where $\lambda_0, m, k, \alpha_0$ are arbitrary constants.

Finally, in the case $B(U) = 0$, we obtain the known nonlinear heat equation with a source

$$U_t = [A(U)U_x]_x + C(U). \tag{4}$$

The Lie symmetry of equation (4) was completely described in [6].

An approach to finding a non-classical symmetry (conditional symmetry) of the linear heat equation (2) was suggested in [7] (see also [8]). The conditional symmetry of nonlinear heat equation (4) was studied in [9, 10].

In the present paper (section 2), the Lie symmetries of equation (1) are *completely* described. In particular, such operators of invariance are found that are absent for equations (3) and (4).

In section 3, new operators of conditional symmetry of equations (1) and (4) are constructed. With the help of these operators, new ansätze and exact solutions for some equations of the form (1) are found.

2. Lie symmetries of the nonlinear equation (1)

It is easily shown that equation (1) is reduced by the local substitution (see, e.g., [11])

$$U \to U^* = \int A(U)dU \equiv A_0(U) \tag{5}$$

to the form

$$U^*_{xx} = F_0(U^*)U^*_t + F_1(U^*)U^*_x + F_2(U^*), \tag{6}$$

where

$$F_0 = \frac{1}{A(U)}\Big|_{U=A_0^{-1}(U^*)}, \quad F_1 = -\frac{B(U)}{A(U)}\Big|_{U=A_0^{-1}(U^*)}, \quad F_2 = -C(U)\Big|_{U=A_0^{-1}(U^*)} \tag{7}$$

and A_0^{-1} is the inverse function to $A_0(U)$. Hereinafter, the sign $*$ is omitted, i.e., the equation

$$U_{xx} = F_0(U)U_t + F_1(U)U_x + F_2(U) \tag{8}$$

is considered.

Now let us formulate theorems which give complete information on local symmetry properties of equation (8). Note that we do not consider the cases in which $F_1(U) = 0$ since these cases have been studied in [6].

It is clear that equation (8) is invariant with respect to the trivial algebra

$$P_t = \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x} \tag{9}$$

for arbitrary functions $F_0(U), F_1(U), F_2(U)$. Hereinafter, operators (9) are not listed since they are common for all cases.

Theorem 1. *The maximal algebra of invariance of equation (8) in the case $F_0 = 1$ is the Lie algebra with basic operators (9) and*

a) $D_1 = 2mtP_t + mxP_x - UP_U$, *if* $F_1 = \lambda_1 U^m$, $F_2 = \lambda_2 U^{2m+1}$;

$$b)\quad \mathcal{G}_1 = \exp(-\lambda_2 t)\left(P_x - \frac{\lambda_2}{\lambda_1}P_U\right), \quad \text{if} \quad F_1 = \lambda_1 U, \quad F_2 = \lambda_2 U, \quad \lambda_2 \neq 0;$$

$$c)\quad G_1 = tP_x + \frac{1}{\lambda_1}UP_U, \quad D_2 = 2tP_t + xP_x - UP_U,$$
$$\Pi_1 = t^2P_t + txP_x + \left(\frac{x}{\lambda_1} - tU\right)P_U \quad \text{if} \quad F_1 = \lambda_1 U, \quad F_2 = 0;$$

$$d)\quad G_1 = tP_x + \frac{1}{\lambda_1}UP_U, \quad \text{if} \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U;$$

$$e)\quad \mathcal{G}_2 = \exp(-\lambda_3 t)\left(P_x - \frac{\lambda_3}{\lambda_1}UP_U\right),$$
$$\text{if} \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U + \lambda_3 U \ln U, \quad \lambda_3 \neq 0;$$

$$f)\quad Y = \exp\left[\left(\frac{\lambda_1^2}{4} - \lambda_3\right)t + \frac{\lambda_1}{2}x\right]UP_U,$$
$$\text{if} \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U + \lambda_3 U \ln U - \frac{\lambda_1^2}{4}U\ln^2 U,$$

where $\lambda_1 \neq 0, \lambda_2, \lambda_3$ *and* $m \neq 0$ *are arbitrary constants,* $P_U = \dfrac{\partial}{\partial U}$.

Theorem 2. *The maximal algebra of invariance of equation (8) in the case* $F_0 = \exp(mU)$ *is the Lie algebra with basic operators (9) and*

$$D = (2n - m)tP_t + nxP_x - P_U$$

if $F_1 = \lambda_1 \exp(nU)$, $F_2 = \lambda_2 \exp(2nU)$, *where* $\lambda_1 \neq 0, \lambda_2, m$ *and* $n \neq 0$ *are arbitrary constants.*

Theorem 3. *The maximal algebra of invariance of equation (8) in the case* $F_0 = U^k$, $k \neq 0$ *is the Lie algebra with basic operators (9) and*

$$a)\quad D_1 = (2m - k)tP_t + mxP_x - UP_U,$$
$$\text{if} \quad F_1 = \lambda_1 U^m, \quad F_2 = \lambda_2 U^{2m+1}, \quad m \neq 0, m \neq k, m \neq k/2;$$

$$b)\quad T = \exp(-\lambda_3 kt)(P_t - \lambda_3 UP_U), \quad \text{if} \quad F_1 = \alpha_1, \quad F_2 = \lambda_2 U + \lambda_3 U^{k+1};$$

$$c)\quad X = \exp(-\alpha_1 kx)(P_x + 2\alpha_1 UP_U),$$
$$\text{if} \quad F_1 = \lambda_1 U^{k/2} + (k+4)\alpha_1, \quad F_2 = \lambda_2 U^{k+1} - 2\alpha_1\lambda_1 U^{k/2+1} + \lambda_4 U, \quad k \neq -4;$$

$$d)\quad D_2 = ktP_t + UP_U, \quad X, \quad \text{if} \quad F_1 = \alpha_1(k+4), \quad F_2 = \lambda_4 U, \quad k \neq -4;$$

$$e)\quad T,\ X, \quad \text{if} \quad F_1 = \alpha_1(k+4), \quad F_2 = \lambda_4 U + \lambda_3 U^{k+1}, \quad k \neq -4,$$

where $\alpha_1 \neq 0, \lambda_1 \neq 0, \lambda_2,$ *and* $\lambda_3 \neq 0$ *are arbitrary constants* $\lambda_4 = -2\alpha_1^2(k+2)$.

The proofs of Theorems 1, 2 and 3 are based on the classical Lie scheme (see, e.g., [12, 13]) and here they are omitted. Note that these proofs are non-trivial because equation (8) contains three arbitrary functions $F_0(U), F_1(U), F_2(U)$ (for details see our recently published paper [14]).

Using Theorems 1–3 and [6], one can show that some nonliner convection equations of the form (8) contain the operators G_1, Y and X that are not invariance operators for any nonlinear equation of the form (4).

In particular, it follows from Theorem 1 that the nonlinear convection equation

$$U_t = U_{xx} - \lambda_1(\ln U)U_x - \lambda_2 U$$

is invariant under the Galilei algebra with basic operators P_t, P_x and G_1, in which the unit operator is absent. Note that the Burgers equation is also invariant under the Galilei algebra (see Theorem 1, case (c)), which does not contain a unit operator. All second-order equations, which are invariant with respect to the similar representation of the Galilei algebra, were described in [3].

On the other hand, *all nonlinear* equations of the form (4) do not have the Galilei symmetry [3]. The Galilei algebra of the linear heat equation contains the unit operator $I = U\partial_U$ and is essentially different from that of the Burgers equation. Nonlinear equations and systems of equations, preserving the Galilei algebra of the linear heat equation, have been described in [4], [15], [16].

3. Conditional symmetries of the nonlinear equation (1)

In this section, we study the Q-conditional symmetry (see the definition of conditional symmetry in [13]) of the nonlinear equation (8) if $F_1(U) \neq 0$.

Theorem 4. *Equation (8) is Q-conditional invariant under the operator*

$$Q = \xi^0(t,x,U)P_t + \xi^1(t,x,U)P_x + \eta(t,x,U)P_U$$

if the functions ξ^0, ξ^1, η satisfy the following equations:

case 1.

$$\begin{cases} \xi^0 = 1, \quad \xi^1_{UU} = 0, \quad \eta_{UU} = 2\xi^1_U(F_1 - \xi^1 F_0) + 2\xi^1_{xU}, \\ \eta(F_1 - \xi^1 F_0)_U - (\xi^1_t + 2\xi^1\xi^1_x - 3\xi^1_U\eta)F_0 + \xi^1_x F_1 + 3\xi^1_U F_2 - 2\eta_{xU} + \xi^1_{xx} = 0, \\ \eta(\eta F_0 + F_2)_U + (2\xi^1_x - \eta_U)(\eta F_0 + F_2) + \eta_t F_0 + \eta_x F_1 - \eta_{xx} = 0; \end{cases} \tag{10}$$

case 2.

$$\begin{cases} \xi^0 = 0, \quad \xi^1 = 1, \quad \eta(\eta_x + \eta\eta_U - \eta F_1 - F_2)\dot{F}_0 = \\ = (\eta_{xx} + 2\eta\eta_{xU} + \eta^2\eta_{UU} - \eta^2\dot{F}_1 - \eta_x F_1 - \eta\dot{F}_2 + \eta_U F_2)F_0 + \eta_t F_0^2. \end{cases} \tag{11}$$

The dot above F_0, F_1, F_2 denotes differentiating with respect to the variable U.

One can prove this theorem using [13], §5.7 .

The systems of the nonlinear equations (10) and (11) are very complicated and we did not construct their general solutions. A partial solution of equations (10) has the form

$$\xi^1 = U + \lambda_4, \quad \eta = \mathcal{P}_3(U),$$

$$F_1 = (U + \lambda_4)F_0 + 3\lambda_3 U + \lambda_2, \quad F_2 = -\mathcal{P}_3(U)(F_0 + \lambda_3),$$

where $\mathcal{P}_3(U) = \lambda_0 + \lambda_1 U + \lambda_2 U^2 + \lambda_3 U^3$, $\lambda_\mu \in \mathbf{R}$, $\mu = 0, \dots, 4$, $F_0(U)$ is an arbitrary smooth function.

A partial solution of equation (11) is the following:

$$\eta = \frac{1}{t} H(U), \quad F_0 = \lambda_0 \dot{H}, \quad F_1 = \lambda_1 \dot{H} + \lambda_0 \dot{H} \int \frac{dU}{H(U)}, \quad F_2 = \lambda_2 H \dot{H},$$

where $H = H(U)$ is an arbitrary smooth function. So, we obtain the following result: the equation

$$U_{xx} = F_0(U)[U_t + (U + \lambda_4) U_x - \mathcal{P}_3(U)] + (3\lambda_3 U + \lambda_2) U_x - \lambda_3 \mathcal{P}_3(U) \tag{12}$$

is a Q-conditional invariant under the operator

$$Q = P_t + (U + \lambda_4) P_x + \mathcal{P}_3(U) P_U \tag{13}$$

and the equation

$$U_{xx} = \dot{H}(U) \left[\lambda_0 U_t + \left(\lambda_1 + \lambda_0 \int \frac{dU}{H(U)} \right) U_x + \lambda_2 H(U) \right] \tag{14}$$

is a Q-conditional invariant under the Galilei operator

$$G = t P_x + H(U) P_U. \tag{15}$$

Using operators (13) and (15), we find the ansätze

$$\int \frac{U + \lambda_4}{\mathcal{P}_3(U)} dU - x = \varphi(\omega), \quad \omega = \int \frac{dU}{\mathcal{P}_3(U)} - t; \tag{16}$$

$$\int \frac{dU}{H(U)} = \varphi(t) + \frac{x}{t}. \tag{17}$$

After the substitution of ansätze (16) and (17) into (12) and (14), some ODEs are obtained that can be solved. Having solutions of these ODEs and using ansätze (16) and (17), we obtain the exact solution

$$\int \frac{dU}{\mathcal{P}_3(U)} - \int \frac{d\tau}{\mathcal{P}_3(\tau)} = t, \qquad \int \frac{U + \lambda_4}{\mathcal{P}_3(U)} dU - \int \frac{\tau + \lambda_4}{\mathcal{P}_3(\tau)} d\tau = x \tag{18}$$

of equation (12) in the parametrical form, and the solution

$$\int \frac{dU}{H(U)} = \frac{x}{t} + \frac{1}{\lambda_0} \left(\frac{1}{t} \ln t - \lambda_1 - \frac{1}{2} \lambda_2 t \right) \tag{19}$$

of equation (14).

Finally, note that operators of conditional symmerty give a possibility to construct ansätze and solutions of PDEs that can not be obtained by the Lie method. A construction of new ansätze and solutions of other types will be considered in the next paper.

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