Journal of Scientific Computing, Vol. 17, Nos. 1–4, December 2002 (© 2002)

A Particle Method for the KdV Equation

Alina Chertock[1] and Doron Levy[2]

Received August, 29, 2001; accepted (in revised form) October 22, 2001

We extend the dispersion-velocity particle method that we recently introduced to advection models in which the velocity does not depend linearly on the solution or its derivatives. An example is the Korteweg de Vries (KdV) equation for which we derive a particle method and demonstrate numerically how it captures soliton–soliton interactions.

KEY WORDS: Particle methods; dispersion-velocity method; KdV equation.

1. INTRODUCTION

Particle methods have been used in recent years for approximating solutions for a variety of partial differential equations (PDEs). In these methods, the initial data are represented as a collection of particles, located at points x_i and carrying masses w_i. At later times, the locations of the particles and/or their weights are allowed to change. The solution is then found by following the time evolution of the locations and of the weights of the particles. Due to the Lagrangian nature of the method, small scales that might develop in the solution can be easily captured with a small number of particles.

In a recent work [2] we have introduced the *dispersion-velocity particle method* for approximating solutions of linear and nonlinear dispersive equations. This was the first time that particle methods were used for approximating this type of equations. Our method was based on the *diffusion-velocity particle method* [6] for approximating solutions of parabolic equations. In the *diffusion-velocity* method, one defines a convective field associated with the heat operator which then allows the particles to convect in a standard way. For example, the one-dimensional heat equation, $u_t = u_{xx}$, is rewritten as $u_t + (a(u)\, u)_x = 0$, where the velocity $a(u)$ is given by $-u_x/u$.

[1] Department of Mathematics, University of California and Lawrence Berkeley National Laboratory, Berkeley, California 94720. E-mail: alina@math.lbl.gov
[2] Department of Mathematics, Stanford University, Stanford, California 94305-2125. E-mail: dlevy@math.stanford.edu

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Particles carrying fixed masses will then be convected with speed $a(u)$. Issues of existence and uniqueness of solutions to the diffusion velocity transport equations were investigated in [11, 12]. Convergence results for a porous-media equation were recently obtained in [13].

We note in passing that there exist other methods for treating diffusion terms with particles, such as the random vortex method introduced by Chorin in [3]. The theory of particle methods as well as their applications to various fields are reviewed in [4, 5, 8, 15, 16].

In [2] we have developed a particle method for linear and non-linear dispersive equations. In the nonlinear setup we were interested in approximating solutions to equations which generate compactly supported solutions with non-smooth fronts. The prototype of such equations is the $K(m, n)$ equation $u_t + (u^m)_x + (u^n)_{xxx} = 0$, $m > 0, 1 < n \leqslant 3$, introduced by Rosenau and Hyman in [17]. The fundamental solutions of the $K(m, n)$ equation are compactly supported solitons, the so-called *compactons*. The nonlinear compacton equation we considered in [2] was of the form $u_t + (u^2)_x + (u(u)_{xx})_x = 0$. Since it is already written in an advective form, the velocity $a = u + u_{xx}$ depends linearly on u and its derivatives.

In the present work the goal is to show that our method can be applied to nonlinear dispersive equations that when written in a convective form, do not have a linear dependence on the solution and its derivatives. In this context, our model problem is the KdV equation introduced by Korteweg and de Vries in [10],

$$u_t + 3(u^2)_x + u_{xxx} = 0$$

This equation which was developed for modeling shallow water waves, has been found relevant in other physical models such as, e.g., ion acoustic waves in a plasma [9] and acoustic waves in an anharmonic crystal [18]. For a comprehensive overview of the analysis and applications of the KdV equation we refer the reader to [7, 9] and the references therein.

The structure of the paper is as follows: we start in Section 2 by introducing the dispersion-velocity method for the KdV equation. A short time existence and uniqueness Theorem for a solution of the resulting dispersion-velocity transport equation is stated in Theorem 2.1. We then demonstrate in Section 3 the implementation of our method in several test cases: a single translating soliton, a two-soliton problem, and a soliton–soliton interaction.

2. THE DISPERSION-VELOCITY METHOD

In this section we present the *dispersion-velocity* method for approximating solutions of the KdV equation. The *dispersion-velocity* method is based on the *diffusion-velocity* method introduced by Degond and Mustieles in [6]. Our model problem is the KdV equation

$$u_t + 3(u^2)_x + u_{xxx} = 0 \tag{2.1}$$

which we augment with the initial data $u(x, t=0) = u_0(x)$. Boundary conditions will be specified below. We rewrite (2.1) as a convection equation

$$u_t + (a(x, t)\, u)_x = 0 \tag{2.2}$$

where the coefficient $a(x, t)$ in (2.2) is

$$a(x, t) = 3u(x, t) + \frac{u_{xx}(x, t)}{u(x, t)} \tag{2.3}$$

A "standard" particle method for approximating solutions for (2.3) when $a(x, t)$ is known is based on introducing a distribution of the form

$$u_N(x, t) = \sum_{i=1}^{N} w_i \delta(x - x_i(t))$$

where the initial data is approximated by

$$u_N(x, 0) = \sum_{i=1}^{N} w_i \delta(x - x_i(0)) \simeq u_0(x)$$

Here $x_i(t)$ is the characteristic curve associated with $a(x, t)$, which starts at the point x_i^0, i.e.,

$$\begin{cases} \dfrac{dx_i}{dt} = a(x_i(t), t), \\ x_i(0) = x_i^0 \end{cases} \tag{2.4}$$

According to (2.3), $a(x, t)$ depends on u and on its second derivative, u_{xx}, and, therefore, it can not be considered as a given function. Moreover, since the product of δ functions is not well defined, the standard particle method has to be modified. Consequently, we introduce a smoothed approximation, $u_N^\epsilon(x, t)$,

$$u_N^\epsilon(x, t) = (u_N * \zeta_\epsilon)(x, t) = \sum_{i=1}^{N} w_i \zeta_\epsilon(x - x_i(t)) \tag{2.5}$$

The function $\zeta_\epsilon(x)$ (which is also called the "cutoff function") is taken as a smooth approximation of the δ function and satisfies

$$\zeta_\epsilon(x) = \frac{1}{\epsilon} \zeta\left(\frac{x}{\epsilon}\right), \qquad \text{and} \qquad \int \zeta(x)\, dx = 1 \tag{2.6}$$

Given an appropriate smoothing function $\zeta_\epsilon(x)$, we can approximate $a(x, t)$ in (2.3) by

$$a_\zeta(x, t) = 3(u * \zeta_\epsilon) + \frac{u * \zeta''_\epsilon}{u * \zeta_\epsilon} \tag{2.7}$$

resulting with the *dispersion-velocity transport equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(a_\zeta u) = 0, \\ u(x, t=0) = u_0(x) \end{cases} \tag{2.8}$$

The *dispersion-velocity method* is then obtained by considering a particle approximation as a distribution of the form (2.5), where $x_i(t)$ are the solutions of

$$\begin{cases} \dfrac{dx_i}{dt} = 3u^\epsilon_N(x_i, t) + \dfrac{(u^\epsilon_N(x_i, t))''}{u^\epsilon_N(x_i, t)} = 3 \sum_{j=1}^{N} w_j \zeta_\epsilon(x_i - x_j) + \dfrac{\sum_{j=1}^N w_j \zeta_\epsilon''(x_i - x_j)}{\sum_{j=1}^N w_j \zeta_\epsilon(x_i - x_j)}, \\ x_i(0) = x_i^0 \end{cases} \tag{2.9}$$

Local existence and uniqueness of a solution for the system of ODEs, (2.9), can be obtained from standard ODE theorems if the initial data is smooth. For non smooth initial data a theorem similar to Theorem 2.1 in [2] reads:

Theorem 2.1 (Local Existence and Uniqueness). Assume $\zeta \in C^4(\mathbb{R})$, $u_0 \in W^{1,\infty}(\mathbb{R})$, and that there exist constants $\alpha, \beta > 0$ such that $\alpha \leqslant u_0 \leqslant \beta$. Then there exists T_0 such that (2.8) has a unique solution in $W^{1,\infty}(\mathbb{R} \times (0, T_0))$.

The proof of Theorem 2.1 is similar to the proof of Theorem 2.1 in [2]. We would like to remark that the regularity of the solution is the same regularity of the initial data. Also, a similar Theorem can be formulated for periodic boundary conditions. Finally, we would like to stress that Theorem 2.1 does not imply the stability or the convergence of the numerical scheme, (2.9), as the existence time vanishes with ϵ.

Remarks

1. In order to approximate the initial data, we would like to choose constants $\{w_i\}$ such that $u_N(x, 0) = \sum_i w_i \delta(x - x_i(0))$ approximates $u_0(x)$. This is done in the sense of measures on $\mathbb{R}$. Given a test function $\phi \in C^0_0(\mathbb{R})$, the inner product

$$(u_0(\cdot), \phi(\cdot)) = \int_{\mathbb{R}} u_0(x)\, \phi(x)\, dx$$

should then be approximated by

$$(u_N(\cdot), \phi(\cdot)) = \sum_i w_i \phi(x_i)$$

The constants $\{w_i\}$ can then be determined by solving the standard numerical quadrature problem

$$\int u_0(x)\, \phi(x)\, dx \approx \sum_i w_i \phi(x_i)$$

One way of solving the last equation can be, e.g., to cover $\mathbb{R}$ with a uniform mesh of spacing $h > 0$ and set

$$w_i = h u_0(x_i)$$

2. Clearly, the accuracy of the dispersion-velocity method will depend on the choice of the cutoff function $\zeta_\epsilon(x)$ and on its width ϵ. For a discussion on the role of the cutoff function we refer the reader to our previous work [2] and the references therein.
3. Since we are dealing with dispersive equations, we do not expect any bounds on the distance between particles (both lower and upper bounds). The natural way to overcome this difficulty is to redistribute the particles in fixed times, which sould be selected in such a way as to prevent the particles from spreading too far from each other. It is well known in particle applications that redistribution of the particles might be crucial for a successful implementation of the method, e.g., see [1, 14]. Without redistribution one might fail to capture the long time behavior of the solution.

3. NUMERICAL SIMULATIONS

In this section we present several examples in which we test our particle method for Eq. (2.1). The kernel we used in all the examples is in the form

$$\xi(x) = \frac{1}{\sqrt{\pi}} \left(\frac{3}{2} - x^2 \right) e^{-x^2} \tag{3.1}$$

The time integration was done using a standard fourth-order Runge–Kutta method with a fixed time step that was chosen small enough to ensure the local stability of the Runge–Kutta method. For simplicity periodic boundary conditions were used in all simulations.

We start by considering the KdV equation (2.1) subject to the initial data

$$u(x, 0) = 0.5 \operatorname{sech}^2(0.5x) \tag{3.2}$$

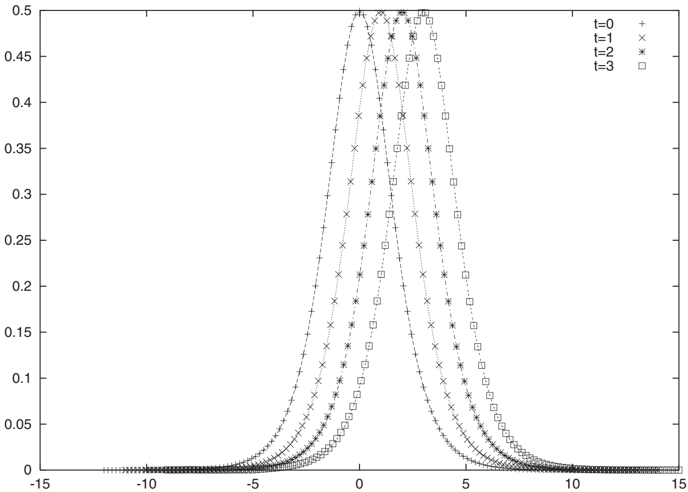

Fig. 1. The solution of (2.1) with initial data (3.2) on $[-12, 12]$. $N=128$, $\epsilon=\sqrt{h}$. The points represent the location of the particles. The solid lines represent exact solution (3.3).

in the interval $[-12, 12]$. In this case, the solution is a fundamental soliton, which is a traveling wave of the form

$$u(x, t) = 0.5\,\mathrm{sech}^2(0.5(x-t)) \tag{3.3}$$

The exact and numerical solutions at times $t=1, 2, 3$ are presented in Fig. 1. The number of particles was taken as $N=128$ and they were

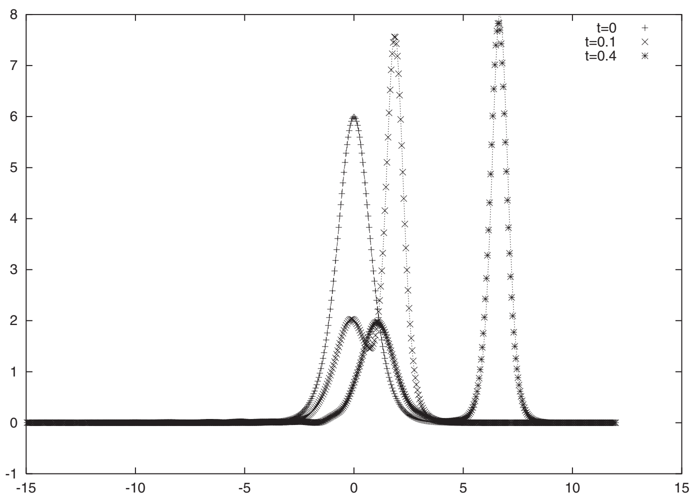

Fig. 2. The solution of (2.1) with initial data (3.4) on $[-15, 12]$. $N=500$, $\epsilon=\sqrt{h}$. The points represent the location of the particles. The solid lines represent exact solution (3.5).

equally spaced at time $t=0$, with spacing $h=24/N$. The width of the kernel was set as $\epsilon=\sqrt{h}$.

In the second example we present a two-soliton problem. Here we solve (2.1), subject to the initial data

$$u(x,0)=6\,\mathrm{sech}^2\,x \tag{3.4}$$

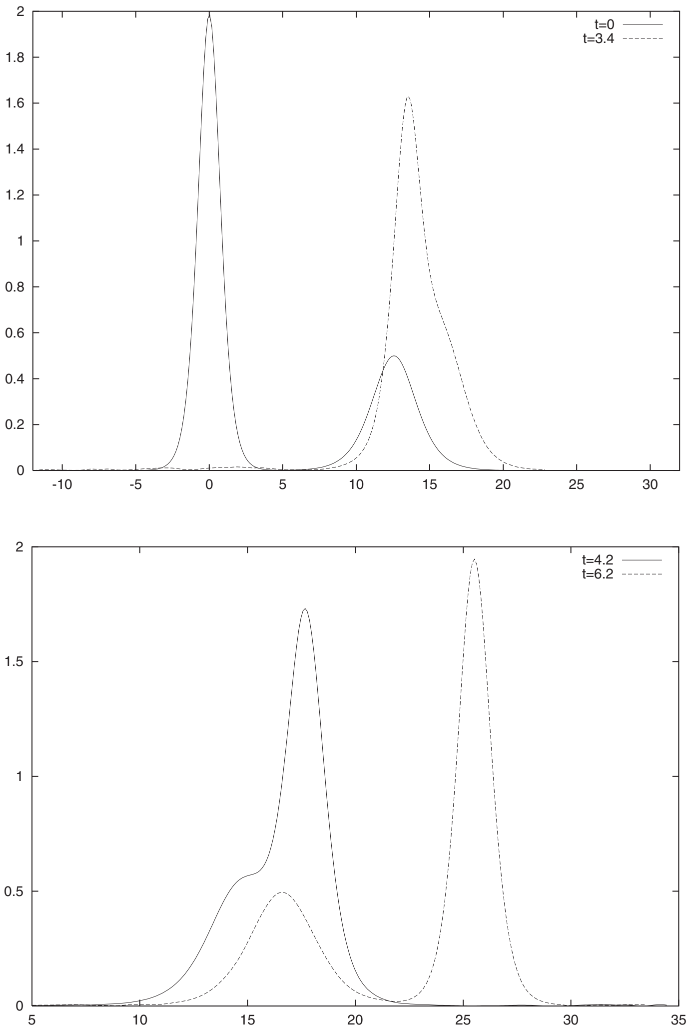

Fig. 3. The solution of (2.1) with initial data (3.6) on $[-3\pi, 10\pi]$. $N=400$, $\epsilon=\sqrt{h}$.

For such initial data, the solution can be expressed as (see [7])

$$u(x, t) = 12\,\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{(3\cosh(x-28t)+\cosh(3x-36t))^2} \tag{3.5}$$

In Fig. 2 we plot the exact and the numerical solutions at $=0.1, 0.4$. The number of particles was taken $N=500$. The width of the kernel was taken as $\epsilon=\sqrt{h}$, where h is the initial spacing between particles. As expected, two solitons split from the initial data.

Finally, we compute the double soliton collision. Here the initial condition is taken as a sum of two solitons:

$$u(x, 0) = 2\,\mathrm{sech}^2(x)+0.5\,\mathrm{sech}^2(0.5(x-4\pi)) \tag{3.6}$$

in the interval $[-3\pi, 10\pi]$. The results are presented in Fig. 3. Once again, the width of the kernel was taken as $\epsilon=\sqrt{h}$, where $h=13\pi/N$ and $N=400$. As one can see, the higher soliton (to the left) that travels with a higher velocity ($\lambda=2$), passes through the lower soliton which travels slower ($\lambda=1$) after going through a nonlinear interaction. Evidently, the particles are capable of capturing the non-linear interaction. The solitons emerge from the interaction in the canonical soliton shape. It is important to mention, however, that redistribution of particles was essential in this case; without such a process the soliton–soliton interaction can not be captured. Redistribution was applied in fixed time intervals of $\Delta t=0.4$ using third-order splines.

ACKNOWLEDGMENTS

We would like to thank Prof. A. Chorin for his valuable comments. The work of A.C. was supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research the U.S. Department of Energy, under Contract DE-AC03-76-SF00098, and in part by the National Science Foundation under Grant DMS98-14631.

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