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Quantum Integrable Systems

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Dedicated to our mothers

Rama Roy Chowdhury
and
Ashima Ghose Choudhury

Contents

Preface

Integrable systems have opened up new horizons in the field of classical physics over the past few decades. The concepts of solitons and solitary waves have provided a foundation for both analytical studies as well as practical applications. Consequently, research in nonlinear systems has gained immense popularity and has led to forays into newer areas. A particularly rich field has been the subatomic world, where quantized counterparts of the classical treatments are applicable. However, a point of caution must be mentioned; whereas the usual techniques of quantization rely heavily on the notions of linearity and superposition principles, such niceities are not present in nonlinear systems. Therefore, new approaches have been developed to obtain a "proper" quantization procedure for such systems.

In this volume, our objective is to clarify some of the developments in the domain of quantum integrable systems, keeping in mind the corresponding classical notions. The latter have been included to produce a self-contained book as far as possible, although some background reading is always useful for a better understanding of the material.

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Chapter 1

Nonlinear Systems and Classical IST

1.1 Introduction

Nonlinear integrable systems constitute one of the most fascinating discoveries of applied mathematics and theoretical physics. The subject developed rapidly because of its wide applicabilty in a wide range of physical situations. It began with the study of shallow water waves in fluid mechanics, and is now popular with both physicsts and mathematicians. Initial studies were confined to the classical aspects of nonlinear partial differential equations, which were integrable. Later it was observed that their applicability could be extended to explain several phenomena in particle physics, condensed matter and even laser physics. Hence it was considered necessary to develop a quantum mechanical counterpart of the classical inverse scattering transform formalism.

In the meantime, exhaustive studies of classical soliton theory had led to the identification of several basic features of the theory such as the Hamilton structure, existence of an infinite number of conservation laws, bi-Hamiltonian structure, action-angle variables and so on. Consequently, soliton theory may be considered a full fledged classical field theory. In this volume we will discuss primarily the developements that took place in the quantum mechanical domain of integrable systems. But prior to that we will acquaint the readers with the basic formalism of classical integrable systems, which is necessary to understand the salient features of the quantum theory.

To begin with let us discuss some of the fundamental aspects of classical nonlinear integrable systems. In this chapter we will discuss the fundamental aspects of classical nonlinear integrable systems.

1.2 Definition of Integrability

Before proceeding to the theoretical framework, one should bear in mind that even today there is no foolproof definition of a completely integrable system. There are a number of different approaches to defining integrability.

The classical method is that of Painlevé, who studied nonlinear dynamical systems long ago in an attempt to deduce a sufficient number of first integrals, so that the task of solving a nonlinear problem is reduced to that of a quadrature. But even in the case of the simple pendulum, the inadequacy of this approach is evident as the motion, when reduced to that of a quadrature, is defined by an integral of the form,

$$t - t_0 = \int_{u_0}^{u} \frac{dy}{\sqrt{(1-y^2)(1-K^2y^2)}}, \tag{1.2.1}$$

where K is a constant. However this elliptic integral does not provide the result, i.e., the position as a function of time in general.

In this context it is interesting to recall a famous comment made by Painlevé about the "double interest" of a differential equation. He considered them either as the source for defining new functions or as a class of equations to be integrated with the existing functions at our disposal. The study of integrability is essentially a study of the singularity structure of the solution of differential equations.

There exists a deep difference between the singularities of solutions of linear and nonlinear equations. While the singularities of a linear equation control that of its solutions, the same does not hold for nonlinear equations. Linear equations have a fixed set of singularities for its solution, for nonlinear equations one may find solutions having singularities not present in the original equation. Such singularities are termed *movable singularities*. The complete integrability of a nonlinear ordinary differential equation implies the nonexistence of movable singularities. This is to be expected since the integration of an ordinary differential equation is essentially an acquisition of global knowledge of its general solution and not just a local knowledge as ensured by the existence of Cauchy's theorem. Thus the most demanding definition for integrability is the single valuedness of its general solution, so that the same may be consistent with any kind of initial condition. In short we may say that the Painlevé criterion for complete integrability of

a nonlinear ordinary differential equation is the uniformizability of its general solution. The Painlevé criterion may also be stated as "the absence of movable critical points in the general solution".

An alternative approach to the concept of complete integrability utilizes the existence of Lax pairs and the notion of the existence of an infinite number of conservation laws. This is true both for partial and ordinary differential equations, although in the previous paragraph we restricted ourselves to ordinary differential equations. The idea of extending the Painlevé criterion to partial differential equations was due to Ablowitz, Ramani and Segur who conjectured that all possible reductions of a partial differential equation to an ordinary differential equation will be completely integrable if the original partial differential equation is completely integrable. Subsequently the case of nonlinear partial differential equations was analyzed by John Weiss [1], who developed an extension of the original approach of Painlevé.

In the Lax pair method one usually asserts that a system of partial differential equations is completely integrable if it can be deduced as the consistency condition of two linear problems.

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \tag{1.2.2}$$

where Ψ is a n component vector and U, V are matrices which belong to either $SL(n), GL(n)$ or any other $n-$dimensional Lie algebra. Several authors have shown how to prove the existence of an infinite number of conservation laws from (1.2.2); and the fact that they are in involution. The matrix functions U and V depend on the nonlinear field variable and their derivatives. The basic idea of the *inverse scattering transform* (IST) is to determine the nonlinear field variables from a study of the linear auxiliary equations (1.2.2). At present there are several alternate routes to this destination: the Riemann-Hilbert approach or in more than one dimension the so-called $\bar{\partial}$ bar problem. On the other hand, for a discrete nonlinear integrable system, the Lax pair is written as

$$\Psi_{n+1} = L_n\Psi_n, \qquad \Psi_{nt} = L_n\Psi_n, \tag{1.2.3}$$

and the corresponding nonlinear set becomes

$$\frac{\partial L_n}{\partial t} = [L_n, V_n]. \tag{1.2.4}$$

All the techniques of the IST formalism are equally applicable for this case. But a second look at (1.2.3) reveals its striking resemblence to

our well-known Hamilton's equation in classical mechanics, with the difference that the right-hand side is a commutator and not a Poisson bracket.

1.3 Lax Pair Technique

The technique of generating a nonlinear equation from a pair of linear problems is usually referred to as the *Lax pair* technique. Initially, there was no well-defined procedure for obtaining a Lax pair for a given nonlinear partial differential equation. However a novel approach was suggested in the seminal papers of Ablowitz, Kaup, Newell, Segur [2] and Zakharov and Shabat [3]. They showed that if one begins with two linear problems,

$$\Psi_x = U\Psi \qquad U = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \tag{1.3.1}$$

and

$$\Psi_t = V\Psi, \qquad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{1.3.2}$$

then by demanding consistency of (1.3.1) and (1.3.2) one can generate several types of nonlinear integrable systems by assuming that A, B, C are appropriate analytic functions of λ. Almost simultaneously a breakthrough was made by Wahlquist and Estabrook [4], who showed that by using Cartan calculus of differential forms and prolongation techniques, it is possible to find the Lax pair of a given nonlinear problem. It is perhaps the only direct method of finding the Lax pair that otherwise had to be "pulled out of the hat" many times. The nonlinear fields occur as coefficients in the linear Lax equations, which for the famous KdV case turns out to the familiar stationary Schrödinger equation and which were referred to as "potentials" or "pseudo potentials". The basic idea of inverse scattering is to obtain detailed information about these potentials (i.e., nonlinear fields) from the data on the wave function, which is a eigenfunction of the Lax equation. The usual approach of quantum mechanics is based on the idea of scattering in a potential as the distance goes to positive or negative infinity. The other methods like Riemann-Hilbert or dressing operator approach do not rely on the

concept of "scattering", and thus have wider applicability. Besides in more than $1+1$ dimension we now have the well formulated $\bar{\partial}$-problem [5].

1.4 Inverse Scattering Transform

The subject of IST developed because of its importance in the domain of atomic and particle physics. It was the endeavour of researchers worldwide to reconstruct the potential responsible for atomic and subatomic phenomena from the phase shifts of different partial waves in scattering experiments. Similar approaches were also adopted in the classical domain; its application to the field of oil prospecting highlights its immense utility. Here a wave with known characteristics is sent through the earth's crust and the reflected wave is monitored. From the change in the nature of this wave it is possible to predict the density at a particular depth, providing thereby an indication of the possible existence of petroleum. Though the two situations differ considerably, the basic philosophy remains the same. Theoretically there are some differences in the treatment of the classical and quantum cases. In the classical situation, one studies the form of the "potential" or the nonlinear field, which may either be a single soliton state or even multisoliton states. In the quantum case we generally deal with the nature of the excitation spectrum for a particular "potential", which is expressed through the Bethe ansatz equations. Recently there have been attempts to construct the operator corresponding to quantum mechanical nonlinear fields. These topics will be discussed in some detail in the next few chapters . At this point, we should recognize some of the shortcomings of the classical IST. Since in this formulation we always speak of $x \longrightarrow \pm\infty$ (x being the coordinate), it is essential that care is taken to apply this method to bounded systems. In this respect Zakharov's seminal work, based on Riemann-Hilbert transforms, should be mentioned. In this case the scattering behaviour is replaced by the "analyticity property in the complex eigenvalue plane."

So far we have discussed classical systems in the $1+1$ dimensional system. In the classical case of the $(2+1)$ dimension Ablowitz et al. formulated the $\bar{\partial}$-bar problem, which elegantly solves the inverse problem in the $(2+1)$ dimension. In the classical case we have the famous

Gelfand-Levitan-Marchenko (GLM) equation that is the cornerstone in the developement of the subject. Now-a-days we do have a quantum version of the GLM equation, which again is due to the efforts of the Russian school. Incidentally, it may be mentioned that a peculiar difficulty exists with the *quantum inverse scattering method* (QISM). While classically it is easier to deal with continuous systems, in the corresponding quantum case such systems invariably suffer from problems of regularization. In the latter case it is obviously easier to deal with discrete systems. However, attempts have also been made to discretize the continuous Lax equation for constructing the monodromy matrix. On the other hand, two excellent papers by Sklyanin [6], which may be interpreted as a new attempt to understand the quantum inverse scattering method in the continuous case, lead to the fact that the Riemann-Hilbert technique is an extremely potent tool for solving these problems. In these papers Sklyanin proposed a rigorous formulation of the quantum Sine-Gordon and the nonlinear Schrödinger problem.

The classical inverse scattering problem was initiated as a result of the remarkable observation of P.D. Lax that the KdV equation for shallow water waves [7],

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.4.1}$$

could be obtained as the consistency condition of the following linear equations:

$$-\Psi_{xx} + \lambda^2\Psi = u\Psi, \tag{1.4.2}$$

$$\Psi_t = A(u)\Psi_x + B(u)\Psi. \tag{1.4.3}$$

It will be noticed that (1.4.2) is the familiar Schrödinger equation with potential $u(x,t)$. Now Lax showed that if the scattering data for the stationary Schrödinger equation is known, then it is possible to reconstruct the potential $u(x,t)$, by making use of (1.4.3). This remarkable feature allows for the solution of the nonlinear evolution equation, by means of analysing the two auxillary linear problems given by (1.4.2 and 1.4.3). The entire problem can be formulated by demanding that the potential u goes to zero as $x \to \pm\infty$, i.e., the solitary character of the potential. One then defines the Jost functions $\Psi, \bar{\Psi}, \Phi, \bar{\Phi}$ as solutions of (1.4.2), which are connected by means of the transmission and reflection coefficients as follows:

$$\Psi = a\Phi + b\bar{\Phi},$$

$$\bar{\Psi} = \bar{a}\bar{\Phi} + \bar{b}\Phi. \tag{1.4.4}$$

One can then show that the classical GLM equation may be written as

$$K(x,y) + G(x,y) + \int_x^\infty G(x,z)K(z,y)dz = 0, \tag{1.4.5}$$

where $G(x,y)$ is the kernel of the equation and is given by

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\lambda)}{a(\lambda)} e^{i\lambda(x+y)} d\lambda. \tag{1.4.6}$$

Here b/a is the reflection coefficient. It can further be shown that the nonlinear variable $u(x)$ is connected to $K(x,y)$ in (1.4.5) by

$$u(x) = -2\frac{dK(x,x)}{dx}. \tag{1.4.7}$$

Then the time dependence of $u(x,t)$ can be ascertained from the equation (1.4.3). Thus the procedure for solving the intial value problem can be outlined as follows. One specifies some initial profile of the nonlinear wave $u(x,t)_{t=0} = u(x,0)$, and uses (1.4.2) to solve the direct scattering problem; that is one determines the position and number of bound states λ_j, together with their normalization constants C_j for the corresponding eigenfunctions and also the continuum contribution in the form of the quantity $b(\lambda)/a(\lambda)$ called the reflection coefficient. Then (1.4.3) is used to determine the time evolution of this set known as the scattering data; i.e., $\lambda_j(t), C_j(t)$ and $b(\lambda,t)/a(\lambda,t)$. These are then used in the construction of the kernel $G(x,y)$. Actually the discrete part of the spectrum is exactly solvable. In (1.4.6) if we consider only the zeros of $a(\lambda)$, then we find that

$$G(x,y) = \sum_i C_i e^{-\lambda_i(x+y)}.$$

Inserting this in (1.4.5) then leads to a degeneration of the integral equation into a set of coupled linear equations, once we write

$$K(x,y) = \sum_i f_i(x) e^{-\lambda_i y},$$

which can then be solved for the f_i's. Finally we can determine the nonlinear variable $u(x)$ from (1.4.7). In this procedure one can always substitute the time variation of the parameters beforehand, and get the

full-time dependent multisoliton solution.

It should be mentioned that essentially the same procedure is followed in case of the $n \times n$ matrix Lax equation, where one gets a coupled set of more than one GLM equation. We shall discuss a specific case in Appendix D to illustrate the procedure in the case of a matrix Lax equation.

The entire scheme of classical inverse scattering problem is best summarized by the schematic diagram shown in Figure (1.4.1).

It may be mentioned that the same can also be derived without using the terminology of scattering theory. This is the method of dressing operators formulated by Zakharov [8] and others. In this formalism one starts with a trivial solution of the nonlinear system for which the Lax operators become constant coefficient differential operators, say L_0, M_0.The dressing operator approach then demands the factorization of L in terms of Volterra operator. The main objective is to find operators $\hat{K}^+$ and $\hat{K}^-$ so that the 'dressed' and 'undressed' operators $\hat{L}$ and $\hat{L}_0$ are related by

$$\hat{L} = (I + \hat{K}^+)\hat{L}_0(I + \hat{K}^+)^{-1}, \tag{1.4.8}$$

In general $\hat{L}$ consists of two parts, a differential operator and an integral Volterra operator. The operators $\hat{K}^+$ and $\hat{K}^-$ have the effect of making

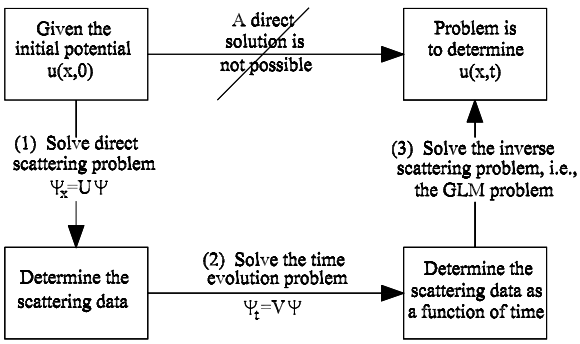

FIGURE 1.4.1: Schematic diagram of the inverse scattering method.

zero the contribution from this integral operator. Suppose we have a Fredholm operator $\hat{F}$,

$$\hat{F}\Psi = \int_{-\infty}^{\infty} F(z,z')\Psi(z')dz', \tag{1.4.9}$$

and we factorize it as follows:

$$I + \hat{F} = (I + \hat{K}^{+})^{-1}(I + \hat{K}^{-}), \tag{1.4.10}$$

where

$$\hat{K}^{-}\Psi = \int_{-\infty}^{z} K^{-}(z,z')\Psi(z')dz', \tag{1.4.11}$$

$\hat{K}^{\pm}$ are the Volterra factors of the operator F. In the inverse scattering we are interested in those F that commute with differential operators. A simple case is

$$[\hat{F}, \hat{M}_0] = 0 \tag{1.4.12}$$

with $\hat{M}_0 = \alpha\frac{\partial}{\partial x} + \hat{L}_0$. Then (1.4.12) leads to a linear differential equation for F. On the other hand the same treatment for the time part,

$$\hat{N}_0 = \beta\frac{\partial}{\partial t} + \hat{V}_0, \tag{1.4.13}$$

leads to

$$[\hat{F}, \hat{N}_0] = 0, \tag{1.4.14}$$

which is again a linear differential equation giving the time dependence of F. Hence the kernel function F of the operator $\hat{F}$ is completely determined. The most interesting part of the procedure is that if the dressed form of the operator $\hat{L}_0$ is $\hat{L}$ and that of $\hat{N}_0$ is $\hat{N}$ then we get

$$\alpha\frac{\partial\hat{N}}{\partial x} - \beta\frac{\partial\hat{L}}{\partial t} = [\hat{L}, \hat{N}], \tag{1.4.15}$$

which is the required Lax equation generating the nonlinear integrable system! On the other hand one also gets equations connecting the nonlinear fields with the kernel $K(x,x)$, which can always be explicitly obtained from the integral Gelfand-Levitan-Marchenko equation once the function F is known. Consequently, it is in principle possible to obtain all the results of the classical IST by the dressing operator approach. The soliton class can also be generalized as we have not imposed any condition such as $x \longrightarrow \pm\infty$.

It should be mentioned that there is another method, namely the Riemann-Hilbert transform, which, too, does not utilize the boundary conditions at infinite distance. Though a discussion of these techniques is outside the scope of this book, we have mentioned them in order to give readers an overall flavour of just how well formulated the classical theory of inverse scattering problem is. Though it should in principle be possible to consider the quantum version of these techniques, only the first approach using the idea of scattering has been properly translated into a quantum version, and many aspects of the classical IST are yet to be properly formulated and analyzed in the quantum context.

1.5 Hamiltonian Structure

Since quantization of a system requires full information about the canonical Poisson bracket relations between its variables, the Hamiltonian structure of nonlinear equations will be of utmost importance in our future discussions. In this context one should remember that nonlinear integrable systems possess infinite numbers of conserved quantities, of which any one can serve as the Hamiltonian. At present there are several methods for the derivation of the Hamiltonian structure of a nonlinear partial differential equation. Mention should be made of the work of F. Magri [9], who first showed that these integrable systems are bi-Hamiltonian in character, that is they may be generated from two Hamiltonians H_1 and H_2 and two symplectic operators θ_1 and θ_2 i.e.,

$$q_t = \theta_1 \nabla H_1 = \theta_2 \nabla H_2. \tag{1.5.1}$$

For quantization, we require the explicit forms of θ_1 and θ_2 as well as H_1, H_2 for the particular equation under consideration. For example, in case of the KdV equation we have,

$$q_t = \frac{\partial}{\partial x}\frac{\delta H_1}{\delta q}, \qquad H_1 = -u_{xx} + 3u^2 \tag{1.5.2}$$

and

$$q_t = (-\partial_x^3 + 4u\partial_x + 2\partial_x u)\frac{\delta H_2}{\delta q}, \qquad H_2 = \frac{1}{2}u^2 \tag{1.5.3}$$

while for the nonlinear Schrödinger equation we have,

$$\begin{pmatrix} q_t \\ q_t^* \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla H_1, \tag{1.5.4}$$

with $H_1 = \nabla\psi\nabla\psi^* + \lambda \mid \psi \mid^4$. Let us now point out the basic difference between (1.5.3) and (1.5.4). While the symplectic operator in (1.5.3) is a differential operator, in (1.5.4) it is a constant. The former is usually referred to as the "nonultralocal" case, while the latter is said to be "ultralocal". As of now, the quantum inverse problem has been rigorously formulated only in the case of ultralocal systems and the nonultralocal cases need to be studied further.

A particularly useful method for obtaining the Hamiltonian structure is that of Drinfeld-Sokolov [10]. Besides, small amplitude expansion [11] and the trace identity method [12] have also been employed for this purpose. Once the symplectic structure is explicitly known, an important element that may be constructed from it is the classical r-matrix, a notion introduced by the Leningrad school, which has found universal acceptance. The Poisson racket between the entries of the Lax operators $L(x,\lambda)$ and $L(y,\mu)$ is denoted as $\{L(x,\lambda) \overset{\otimes}{,} L(y,\mu)\}$ and the fundamental equation determining the classical r-matrix is

$$\{L(x,\lambda) \overset{\otimes}{,} L(y,\mu)\} = [r(\lambda,\mu), L(x,\lambda)\otimes I + I\otimes L(y,\mu)]\delta(x-y). \tag{1.5.5}$$

The delta function on the right-hand side shows that this relation is valid for "ultralocal" systems. The quantum counterpart of (1.5.5) plays a vital role in the formulation of the quantum inverse scattering problem. In case of nonultralocal systems, a more generalized version of such an equation can be constructed. However, quantization of such systems is plagued with several problems and is not well formulated. On the other hand, it is possible to generate a hierarchy of integrable equations if the space part of the Lax equation and the corresponding $r(\lambda,\mu)$ matrix is known [13]. This clearly indicates the important role played by the classical r matrix in case of classical integrable systems.

In the case of nonultralocal systems, quantization is a nontrivial task, but one method that has proved to be quite useful is that of operator product expansions (OPE), a technique used frequently in quantum field theory, together with that of the coordinate *Bethe ansatz*, which will be discussed in the next chapter.

There is another approach: an equivalence class of operators used in the case of KdV equation by Fuchssteiner and Roy Chowdhury [14]. This complements the techniques of quantum inverse scattering method, operator product expansions, etc. It should be emphasised that while it is not possible to translate all aspects of the classical theory to their corresponding quantum versions, the last technique allows for the construction of the quantum recursion operator, which is essential for proving complete quantum integrability, even though the latter is not a well-defined concept.

Trace identity is another elegant method for deducing the symplectic form in which one assumes that the space part of the Lax equation for a particular nonlinear problem is

$$\Psi_x = U(\lambda)\Psi, \tag{1.5.6}$$

and that the nth flow is given by

$$U_t - V_x^{(n)} = [V^{(n)}, U]. \tag{1.5.7}$$

The time part of the Lax pair is taken as

$$\Psi_t = V^{(n)}\Psi, \tag{1.5.8}$$

and the matrix $V^{(n)}$ is determined through

$$V_x^{(n)} = [U, V^{(n)}]. \tag{1.5.9}$$

If $V^{(n)}$ is a solution of this equation, then

$$\bar{V} = \lambda^{\mu} V(n) \tag{1.5.10}$$

is also a solution. The trace identity technique can then be used to show that

$$(\frac{\delta}{\delta u_i})tr(\bar{V}\frac{\partial U}{\partial \lambda}) = \frac{\partial}{\partial \lambda}tr(\bar{V}\frac{\partial U}{\partial u_i}), \tag{1.5.11}$$

where $u_i (i = 1, 2, ...p)$ are the nonlinear field variables on which U depends. Here $\frac{\delta}{\delta u_i}$ stands for the variational derivatives given by

$$\frac{\delta}{\delta u_i} = \sum_{j=0}(-\partial)^j \frac{\partial}{\partial u_i^j}; \partial = \frac{\partial}{\partial x}; u_i^j = \partial^j u_i. \tag{1.5.12}$$

The equation (1.5.12) actually expresses the variational derivatives of the conserved densities in terms of the coefficients of $V^{(n)}$, which in turn are connected to the time evolution in a particular flow.

In the foregoing discussion we have given an overview of the various aspects of classical nonlinear integrable systems. The classical and quantum treatments differ in many respects. Some properties of nonlinear integrable systems are easily interpretable in the quantum language while others remain confined to their classical domain. An important example is the existence of an infinite number of conserved quantities. Whereas quantum mechanically it is easy to prove that this existence is associated with the factorization of the scattering matrx, yet for a long time, a suitable interpretation of this fact was obscure in the classical domain. On the other hand though the GLM equation played a key role in classical inverse scattering transform, the corresponding equation for the quantum inverse scattering method was derived albeit much later. Thus it is fascinating to compare the developement of classical inverse and quantum inverse scattering transforms step by step, and we hope that the reader will appreciate in this process the role of quantum mechanics in nonlinear systems. Of course in these discussions we have not brought up the notion of the possibility of a quantum mechanical Lax operator. It has been shown from the Yang-Baxter equation that such an operator does exist. Several such ideas and the various intricacies of the quantization of nonlinear partial differential equations will be analysed in the subsequent chapters as we go through the sequential developement of the subject.

Chapter 2

Coordinate Bethe Ansatz

2.1 Introduction

In the last chapter, we presented a general overview of classical integrable systems and their essential features. Although it has not been possible to construct a quantum mechanical counterpart of every classical technique, remarkable progress has been made in the developement of quantum integrable systems. Regarding quantization of nonlinear equations, the simplest approach is to consider the nonlinear field occurring in a nonlinear equation as an operator and assume the equation to be derivable from a suitable Hamiltonian with a well-defined commutation rule. Indeed this was the essential approach adopted by Bethe in his treatment of the many-body bosonic system with a δ-function potential [15]. This system is today better known as the δ-function Bose gas. Later, it was observed that such a system is equivalent to the nonlinear Schrödinger equation. We start therefore with a very brief introduction to the Bose gas, adopting the approach as outlined in the celebrated article by Fowler [16].

In this original study, the many-particle system was modeled as a collection of n perfectly elastic billiard balls of equal mass, constrained to move along a line. Consequently in each collision the incoming and the outgoing momenta are coincident, and as the system evolves with time, the momenta of the n particles $k_1, k_2, ..., k_n$ are preserved. The Schrödinger equation for such a system is

$$\left(-\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i,j} \delta(x_i - x_j) \right) \Psi = E\Psi. \tag{2.1.1}$$

Now for $n = 2$, this is equivalent to the quantum mechanical problem of a single particle interacting in a δ-function potential. The associated

boundary condition is

$$\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\Psi(x_1=x_2^+)-\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\Psi(x_1=x_2^-) = 2c\Psi(x_1=x_2), \tag{2.1.2}$$

where $x_i^{\pm}$ represents the limiting values of the coordinate from larger to smaller values. Assuming the basic form of the wave function for the two particle system to be

$$\Psi(x_1,x_2)=\exp\{i(k_1x_1+k_2x_2)\}+e^{i\theta_{12}}\exp\{i(k_2x_1+k_1x_2)\} \tag{2.1.3}$$

for the region $x_1 < x_2$, one can show that the same wave function for the region $x_1 > x_2$ may be obtained by the symmetry transformation $x_1 \to x_2$. In that case the boundary condition yields

$$e^{i\theta_{12}}=-\frac{c-i(k_1-k_2)}{c+i(k_1-k_2)}. \tag{2.1.4}$$

For the N particle system the generalization is fairly obvious:

$$\Psi(x_1,x_2...x_n)=\sum_P a(P)\exp\{i\sum_{j=1}^{n}k_{P_j}x_j\},\quad x_1<x_2<....<x_n, \tag{2.1.5}$$

where the sum over P is a sum over all permutations of $1,2,...n$, and the function $a(P)$ stands for factors of the form $\exp(i\theta_{12})$ and their products. This wave function follows bosonic symmetry. The total energy and momentum of the system is given by

$$E=\sum_{i=1}^{n}k_i^2, \qquad k=\sum_{i=1}^{n}k_i. \tag{2.1.6}$$

Since the wave function is of the plane-wave type, hence the relation (2.1.5) prompts us to interpret the system to be quasi-free. However in pratice these momenta are not observable owing to large phase shifts at the points of collision resulting in the k_i's getting smeared out. As of now we have not defined the system completely, since we are yet to impose a suitable boundary condition. The latter will lead to a restriction of the values of the momenta k_i. Thus while all possible wave functions of the form (2.1.5) are admissible, all values of the momenta k_i are not permissible. One should also note that all the momenta k_i should be different; for if we set $k_1 = k_2$ we get $\exp(i\theta_{12}) = -1$

and $\Psi(x_1, x_2)$ becomes identically zero. This causes the particles to behave as fermions. There is a boson-fermion correspondence in two dimensions that is manifested in the equivalence between the Thirring model and the Sine-Gordon model. If we impose the periodic boundary conditions for free particles we get

$$\exp(ik_iL) = 1, \qquad k_iL = 0, 2\pi, ..., \tag{2.1.7}$$

assuming that the total length occupied by the particles is L.

In case of the Bethe ansatz wave function, there are two mechanisms for phase change. The kinetic phase change, which causes $\exp(ik_ix)$ to change between collisions, and the phase shift *at* collisions. Hence in this case (2.1.7) is modified to

$$k_jL + \sum_{l\neq j} \theta_{jl} = 2\pi I_j, \tag{2.1.8}$$

where

$$\exp(i\theta_{jl}) = -\frac{c - i(k_j - k_l)}{c + i(k_j - k_l)} \tag{2.1.9}$$

and the I_j's are integers, which are nothing but the quantum numbers. The set (2.1.8) forms a large number of coupled equations that are usually difficult to solve. If on the other hand we let $L \to \infty$, it can be shown [17] that the k_j's are closely spaced, so that upon writing (2.1.8) for k_{j+1} we have

$$k_{j+1}L + \sum_{l\neq j} \theta j + 1, l(k_{j+1}, k_l) = 2\pi I_{j+1}. \tag{2.1.10}$$

Subtracting (2.1.8) and (2.1.10) we get

$$\begin{aligned}(k_{j+1} - k_j)L &= \sum\{\theta_{j+1,l} - \theta_{j,l}\} + 2\pi, \\ &\approx -2c(k_{j+1}-k_j)\sum_l \frac{1}{c^2 + (k_j - k_l)^2} + 2\pi. \end{aligned} \tag{2.1.11}$$

Defining $f(k)$ by

$$k_{j+1} - k_j = \frac{1}{Lf(k_j)}, \tag{2.1.12}$$

so that $Lf(k)dk$ is the number of k's in the range $(k, k + dk)$ we have upon proceeding to the limit $L \to \infty$,

$$2f(k) - 1 = 2c\int_{-k}^{k} \frac{f(p)}{c^2 + (p - k)^2} dp, \tag{2.1.13}$$

and

$$L\int_{-k}^{k} f(p)dp = n, \tag{2.1.14}$$

which may be solved by iteration. The ground state energy is therefore given by

$$E = L\int_{-k}^{k} p^2 f(p)dp. \tag{2.1.15}$$

We shall not elaborate further on this issue, though it remains at the heart of any discussion of the coordinate Bethe ansatz. Analysis of more general situations can be found in the original articles by B. Sutherland [18] and C. N. Yang [19].

2.2 Nonlinear Systems and the CBA

The above discussion does not make it clear how a nonlinear integrable system is related to the technique of coordinate Bethe ansatz, but for the fact that the nonlinear Schrödinger equation is equivalent to the δ-function Bose gas. To illustrate the connection it is helpful to take a concrete example. We make use of the extended derivative nonlinear Schrödinger equation, which is written as [20]

$$iq_{0,t} = -q_{0xx} + i\epsilon_0 q_1^2 q_{0x}^\star + 2iV_0 q_{1x} - \epsilon_1 |q_1|^2 V_0 q_1 - V_1 q_0, \tag{2.2.1}$$

$$iq_{1,t} = -q_{1xx} - i\epsilon_1 q_1^2 q_{1x}^\star - V_1 q_1 + 2V_0 q_0. \tag{2.2.2}$$

The system is completely integrable in the classical sense and possesses a Lax pair. It is known that the lowest Hamiltonian is given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx \mathcal{H}(x),$$

$$\mathcal{H}(x) = \Big(\epsilon_0 q_{1xx} q_0^* + \epsilon q_{0xx} q_1^* + \frac{3}{5} i\epsilon_1\epsilon_0 q_{1x} q_1 q_1^* q_0^* - \frac{3}{5} i\epsilon_1^2 q_0 q_1 q_{1x}^* q_1^* + V_0 V_1\Big), \tag{2.2.3}$$

where

$$V_0 = \epsilon_1 (q_0 q_1^*) + \epsilon_0 q_1 q_0^*,$$

$$V_1 = \frac{1}{2}\epsilon_1^2|q_1|^4 - 2\epsilon_0|q_0|^2.$$

The canonical Poisson brackets are

$$\{\epsilon_0 q_0^*(x), q_1(y)\} = \delta(x-y), \tag{2.2.4}$$

$$\{\epsilon_1 q_0(x), q_1^*(y)\} = -\delta(x-y), \tag{2.2.5}$$

where ϵ_1, ϵ_0 are ± 1. To construct the corresponding quantum mechanical system, we consider q_0, q_1 as operators and convert the Poisson bracket relations (2.2.4 and 2.2.5) to commutators, assuming that a vacuum state exists and is given by

$$q_1^*|0\rangle = q_0^*|0\rangle = 0. \tag{2.2.6}$$

Here a general state vector can contain excitations of both q_1 type and q_0 type and is therefore written as $|N, L, M\rangle$, where N is the number of excitations of the q_1 type, L is the mixed state containing both q_0 and q_1, while M denotes the excitations of q_0 type. Let us now assume that the simplest one-particle state can be written as

$$|1,1\rangle = \int dx f_1(x) q_1(x)|0\rangle + \int dx f_2(x) q_0(x)|0\rangle, \tag{2.2.7}$$

where f_1, f_0 are the respective wave functions. If we demand

$$H|1,1\rangle = E|1,1\rangle, \tag{2.2.8}$$

then we get

$$\frac{1}{2}\frac{\partial^2 f_1}{\partial x^2} = E f_1, \qquad \frac{1}{2}\frac{\partial^2 f_2}{\partial x^2} = E f_2.$$

For the two-particle state the state vector is written in the form,

$$\begin{aligned} |2,11,2\rangle = &\int\int dx_1 dx_2 g_1(x_1,x_2) q_1(x) q_1(x_2)|0\rangle + \\ &\int\int dx_1 dx_2 g_2(x_1,x_2) q_1(x) q_0(x_2)|0\rangle \\ &+ \int\int dx_1 dx_2 g_3(x_1,x_2) q_0(x_1) q_0(x_2)|0\rangle. \end{aligned} \tag{2.2.9}$$

Then the condition

$$H|2,11,2\rangle = E|2,11,2\rangle,$$

leads to the following equations:

$$\frac{1}{2}\left(\frac{\partial^2 g_2}{\partial x_1^2}+\frac{\partial^2 g_2}{\partial x_2^2}\right)-\frac{3i}{10}\left(\frac{\partial g_3}{\partial x_1}+\frac{\partial g_3}{\partial x_2}\right)\delta(x_1-x_2)-$$

$$-4g_1(x_1,x_2)\delta(x_1-x_2)=Eg_2(x_1,x_2), \tag{2.2.10}$$

$$\frac{1}{2}\left(\frac{\partial^2 g_1}{\partial x_1^2}+\frac{\partial^2 g_1}{\partial x_2^2}\right)-\frac{3i}{10}\left(\frac{\partial g_2}{\partial x_1}+\frac{\partial g_2}{\partial x_2}\right)\delta(x_1-x_2)=Eg_1(x_1,x_2), \tag{2.2.11}$$

$$\frac{1}{2}\left(\frac{\partial^2 g_3}{\partial x_1^2}+\frac{\partial^2 g_3}{\partial x_2^2}\right)-2g_2(x_1,x_2)\delta(x_1-x_2)=Eg_3(x_1,x_2). \tag{2.2.12}$$

In a similar way the corresponding three-particle case can be expressed as follows:

$$|3,12,21,3\rangle=\int\int dx_1dx_2dx_3h_1(x_1,x_2,x_3)q_1(x_1)q_1(x_2)q_1(x_3)|0\rangle$$

$$+\int\int dx_1dx_2dx_3h_2(x_1,x_2,x_3)q_0(x_1)q_1(x_2)q_1(x_3)|0\rangle$$

$$+\int\int dx_1dx_2dx_3h_3(x_1,x_2,x_3)q_0(x_1)q_0(x_2)q_1(x_3)|0\rangle$$

$$+\int\int dx_1dx_2dx_3h_4(x_1,x_2,x_3)q_0(x_1)q_0(x_2)q_0(x_3)|0\rangle. \tag{2.2.13}$$

The corresponding equation for h_i $(i=1,...4)$ turns out to be

$$\frac{1}{2}\nabla^2 h_1+\frac{3i}{10}\left(\Sigma_{12}h_2\delta(x_2-x_1)+\Sigma_{13}h_2\delta(x_3-x_1)\right)+$$

$$+h_3\delta(x_1-x_3)\delta(x_2-x_3)=Eh_1, \tag{2.2.14}$$

$$\frac{1}{2}\nabla^2 h_2+\frac{3i}{5}\left(\Sigma_{23}h_3\delta(x_2-x_3)+\Sigma_{21}h_3\delta(x_1-x_2)\right)+$$

$$+3h_4\delta(x_2-x_3)\delta(x_1-x_3)-4h_1\left(\delta(x_1-x_2)+\delta(x_2-x_3)\right)=Eh_2, \tag{2.2.15}$$

$$\frac{1}{2}\nabla^2 h_3-\frac{3i}{5}\left(\Sigma_{31}h_4\delta(x_1-x_3)+\Sigma_{32}h_4\delta(x_2-x_3)\right)+$$

$$-4h_2\left(\delta(x_1-x_2)+\delta(x_2-x_3)\right)=Eh_3, \tag{2.2.16}$$

$$\frac{1}{2}\nabla^2 h_4-2h_3\left(\delta(x_1-x_3)+\delta(x_2-x_3)\right)=Eh_4, \tag{2.2.17}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \qquad \Sigma_{ij} = \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}.$$

This time the equations for the wave functions are not as simple as in case of the δ-function Bose gas. One can actually write down the equations for the N particle state, but their explicit form is rather complicated. To solve the set (2.2.10–2.2.12) we set

$$g_i(x_1, x_2) = A_i e^{i(k_1 x_1 + k_2 x_2)}\theta(x_1 - x_2) + B_i e^{i(k_1 x_2 + k_2 x_1)}\theta(x_2 - x_1), \tag{2.2.18}$$

which immediately leads to

$$E = -\frac{1}{2}(k_1^2 + k_2^2), \tag{2.2.19}$$

$$\frac{i}{2}(k_1 - k_2)(A_1 - B_1) = \frac{3i}{5}\left[\frac{i}{2}(k_1 + k_2)(A_3 + B_3)\right] - 2(A_2 + B_2),$$

$$\frac{i}{2}(k_1 - k_2)(A_2 - B_2) = \frac{3}{20}(k_1 + k_2)(A_1 + B_1),$$

$$\frac{i}{2}(k_1 - k_2)(A_3 - B_3) = A_1 + B_1. \tag{2.2.20}$$

Using the usual normalization conditions on g_i we obtain

$$\frac{A_1}{B_1} = \frac{k_1(10 + 6i) - k_2(10 - 6i) - 40i}{k_1(10 - 6i) - k_2(10 + 6i) + 40i}, \tag{2.2.21}$$

$$\frac{A_2}{B_2} = \frac{k_1(10 - 3i) - k_2(10 + 3i)}{k_1(10 + 3i) - k_2(10 - 3i)}, \tag{2.2.22}$$

$$\frac{A_3}{B_3} = \frac{k_1 - k_2 - 2i}{k_1 - k_2 + 2i}. \tag{2.2.23}$$

A similar analysis holds for the case of three-particle states, however, the number of equations are very large. The solution involves 22 equations in all, and as the symmetry of the δ-function Bose gas is no longer valid, we do not have a compact expression for either the set of equations or its solutions. On the other hand, the present set of equations does have an important implication. Under the reduction condition $q_1 = \pm q_0^\star$ the set (2.2.1 and 2.2.2) reduces to the derivative nonlinear Schrödinger equation:

$$iu_t + u_{xx} + \epsilon_1(|u|^2 u)_x = 0. \tag{2.2.24}$$

This equation plays an important role in various situations in plasma physics [21] and elsewhere [22]. However, its most important property in the present context is that it represents an example of a nonultralocal integrable system. This means that its Hamiltonian structure is governed by a symplectic operator involving the derivative of the δ-function. Written explicitly we have

$$H = \int dx \left[-\frac{1}{2}(uu_x^* - u_x u^*) + \epsilon_1 |u(x)|^4 \right], \tag{2.2.25}$$

while the basic Poisson bracket is

$$\{u(x), u^*(y)\} = \frac{\partial}{\partial x}\delta(x-y). \tag{2.2.26}$$

Nonultralocal systems are in general rather difficult to tackle within the framework of algebraic Bethe ansatz. Though at present some solutions have been found [23], a rigorous and completely satisfactory formalism is yet to emerge. However, we can proceed with the coordinate Bethe ansatz trick for solving the quantum mechanical problem if we agree to set aside subtle questions regarding normalization and a proper Hilbert space, etc.

To proceed we shall write the one-, two- and three-particle states as follows:

$$\begin{aligned} |1\rangle &= \int dx_1 f_1(x_1) u(x_1)|0\rangle, \\ |2\rangle &= \int\int dx_1 dx_2 f_2(x_1, x_2) u(x_1) u(x_2)|0\rangle, \\ |3\rangle &= \int\int\int dx_1 dx_2 dx_3 f_3(x_1, x_2, x_3) u(x_1) u(x_2) u(x_3)|0\rangle. \end{aligned} \tag{2.2.27}$$

For the one-particle state we have

$$E = -k^2, \qquad \frac{\partial^2 f_1}{\partial x_1^2} = E f_1. \tag{2.2.28}$$

The two-particle case leads to

$$\nabla_2^2 f_2 + 2\epsilon_1 \frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_2}\delta(x_1 - x_2) = E f_2(x_1, x_2), \tag{2.2.29}$$

with the boundary condition given by

$$\left(\frac{\partial f_1}{\partial x_{12}}\right)_+ - \left(\frac{\partial f_1}{\partial x_{12}}\right)_- + 2\epsilon_1 \left(\frac{\partial^2 f_1}{\partial x_{12}^2}\right)_{x_{12}=0} = 0, \tag{2.2.30}$$

where $x_{12} = x_1 - x_2$. The solution for $f_2(x_1, x_2)$ is

$$f_2(x_1, x_2) = \left[a_2 e^{-ik_2(x_1+x_2)} + b_2 e^{ik_2(x_1+x_2)}\right] \times$$
$$\times \left[(1+F)e^{ik_1(x_1-x_2)}\theta(x_1 - x_2)\right.$$
$$\left. +\{e^{ik_1(x_1-x_2)} + Be^{-ik_1(x_1-x_2)}\}\theta(x_1 - x_2)\right], \tag{2.2.31}$$

where

$$B = F = -\frac{i\epsilon_1 k_1}{1 + i\epsilon_1 k_1}.$$

In the case of the three-particle system, the corresponding set of equations is

$$\nabla_3^2 f_3 + 2\epsilon_1 \left[\frac{\partial f_3}{\partial x_1}\frac{\partial}{\partial x_2}\delta(x_2 - x_1) + P\right] = E f_3, \tag{2.2.32}$$

with P denoting terms obtained by permutation of $(1, 2, 3)$. Note that in this manner it is not difficult to formulate the n-particle problem. Moreover, the solution may be obtained by following the previous strategy. Hence, even though the commutation relation is not canonical, it is in principle possible to solve the quantum mechanical problem. Formulation of the n-particle case essentially represents a combinatorial problem.

Let us now return to the Hamiltonian in (2.2.25). The equation for the N-particle system is

$$\left[i\sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\delta'(x_i - x_j)\right]\phi(x_1, ...x_N) = E\phi(x_1, ...x_N). \tag{2.2.33}$$

Considering for simplicity the case $N = 2$, we write the two-particle wave function as

$$\phi(x_1, x_2) = \alpha(12, 12)e^{i(k_1x_1+k_2x_2)} + \alpha(12, 21)e^{i(k_2x_1+k_1x_2)} \qquad x_1 < x_2,$$
$$\phi(x_1, x_2) = \alpha(21, 12)e^{i(k_1x_1+k_2x_2)} + \alpha(21, 21)e^{i(k_2x_1+k_1x_2)} \qquad x_1 > x_2, \tag{2.2.34}$$

with the boundary condition expressed in the following way:

$$\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right)_{x_i>x_j} - \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right)_{x_i<x_j} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{x_i=x_j}. \tag{2.2.35}$$

Applying this to (2.2.34) we get

$$\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix}=\begin{pmatrix}1+\frac{ik_1k_2}{k_1-k_2} & \frac{ik_1k_2}{k_1-k_2}\\ -\frac{ik_1k_2}{k_1-k_2} & 1-\frac{ik_1k_2}{k_1-k_2}\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix}. \tag{2.2.36}$$

Next let us consider the three-particle state with $(x_1 < x_2 < x_3)$,

$$\begin{aligned}\phi(x_1,x_2,x_3)=\alpha(123,123)e^{i(k_1x_1+k_2x_2+k_3x_3)}+\\ +\alpha(123,213)e^{i(k_2x_1+k_1x_2+k_3x_3)}\\ +\alpha(123,231)e^{i(k_2x_1+k_3x_2+k_1x_3)}+\alpha(123,321)e^{i(k_3x_1+k_2x_2+k_1x_3)}\\ +\alpha(123,312)e^{i(k_3x_1+k_1x_2+k_2x_3)}+\alpha(123,132)e^{i(k_1x_1+k_3x_2+k_2x_3)}.\end{aligned} \tag{2.2.37}$$

The basic assumption here is that the wave function in the other sectors $(x_2 < x_1 < x_3)$ is obtained by a permutation of the coordinates x_1, x_2, x_3, which was essentially the approach used by Yang and Yang [15]. If we now impose the boundary condition on $\phi(x_1, x_2, x_3)$, then we can relate the coefficients $\alpha(ijk, lmn)$ in different sectors. Here we have followed the convention that in the notation for α, the first three indices denote the sector $x_i < x_j < x_k$, while the second set denote the momenta k_l, k_m, k_n associated with the positions. Organizing these coefficients as a vector,

$$\vec{\alpha}(123,123)=[\alpha^{123}_{123},\alpha^{213}_{123},\alpha^{231}_{123},\alpha^{312}_{123},\alpha^{321}_{123},\alpha^{132}_{123}]^t, \tag{2.2.38}$$

we arrive at the following conditions:

$$\begin{aligned}\vec{\alpha}(123,123)_i&=(N_1)_{ij}\vec{\alpha}(213,213)_j,\\ \vec{\alpha}(213,213)_i&=(N_2)_{ij}\vec{\alpha}(231,231)_j,\\ \vec{\alpha}(231,231)_i&=(N_3)_{ij}\vec{\alpha}(321,321)_j,\end{aligned} \tag{2.2.39}$$

where N_1, N_2 and N_3 are 6×6 matrices. We also have

$$\begin{aligned}\vec{\alpha}(123,123)_a&=(M_1)_{ab}\vec{\alpha}(132,132)_b,\\ \vec{\alpha}(132,132)_a&=(M_2)_{ab}\vec{\alpha}(312,312)_b,\\ \vec{\alpha}(312,312)_a&=(M_3)_{ab}\vec{\alpha}(321,321)_b.\end{aligned} \tag{2.2.40}$$

The relations (2.2.39) and (2.2.40) imply that we should have

$$M_1M_2M_3=N_1N_2N_3, \tag{2.2.41}$$

which is nothing but the Yang-Baxter equation. We can quote a couple of these matrices [24]:

$$N_1 = \begin{pmatrix} 1+A_{12} & A_{12} & 0 & 0 & 0 & 0 \\ A_{21} & 1+A_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+A_{23} & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & 1+A_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+A_{31} & A_{31} \\ 0 & 0 & 0 & 0 & A_{31} & 1+A_{13} \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1+A_{13} & 0 & 0 & A_{13} & 0 & 0 \\ 0 & 1+A_{23} & 0 & 0 & A_{23} & 0 \\ 0 & 0 & 1+A_{21} & 0 & 0 & A_{21} \\ A_{31} & 0 & 0 & 1+A_{31} & 0 & 0 \\ 0 & A_{32} & 0 & 0 & 1+A_{32} & 0 \\ 0 & 0 & A_{12} & 0 & 0 & 1+A_{12} \end{pmatrix}, \quad (2.2.42)$$

with similar expressions for other N_i and M_i. The identity can be verified by direct multiplication using

$$(1+A_{12})A_{13}(1+A_{21}) + A_{12}A_{23}A_{12} = A_{12}A_{23}A_{13} + A_{13},$$

with A_{ij} standing for

$$A_{ij} = \frac{k_i k_j}{k_i - k_j}.$$

Imposition of the periodicity condition now leads to

$$e^{ik_i L} = \prod_{j \neq i} \left(\frac{1 - A_{ij}}{1 + A_{ij}} \right).$$

This form of the result suggests that we can construct some matrix analogous to the quantum R matrix if we set

$$R(x) = a(x)P + b(x)I, \quad (2.2.43)$$

with

$$a = \frac{1}{1 + A_{ij}}, \qquad b = \frac{A_{ij}}{1 + A_{ij}}.$$

Here I is a 4×4 unit matrix and

$$P = \frac{1}{2} \begin{pmatrix} 1+\sigma_3 & \sigma_1 - i\sigma_2 \\ \sigma_1 + i\sigma_2 & 1-\sigma_3 \end{pmatrix},$$

so that

$$R(k_i,k_j)=R_{ij}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{1+A_{ij}} & \frac{A_{ij}}{1+A_{ij}} & 0\\ 0 & \frac{A_{ij}}{1+A_{ij}} & \frac{1}{1+A_{ij}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Furthermore, it can be easily checked that

$$(I\otimes R_{32})(R_{31}\otimes I)(I\otimes R_{21})=(R_{21}\otimes I)(I\otimes R_{31})(R_{32}\otimes I). \quad (2.2.44)$$

Relations of this type will be the main topic of discussions in the following chapter, where we will explore their implications and uses. Here, it is sufficient to note that such relations have come out from even a nonultralocal theory.

2.3 Fermionic System

In this section we shall discuss the applicability of the coordinate Bethe ansatz in the case of fermionic systems. An important example of such a system is provided by the Luttinger model [25]. We write the two-component fermionic field operator as $\chi(x,t)=(\chi_1(x,t),\chi_2(x,t))$. The Hamiltonian of the system is

$$H=\int_{-\infty}^{\infty}\mathcal{H}dx,$$

$$\mathcal{H}=iv(\chi_1^\dagger\partial_x\chi_1-\chi_2^\dagger\partial_x\chi_2)+2g\int dy\chi_1^\dagger(x)\chi_2^\dagger(y)V(x-y)\chi_2(y)\chi_1(x), \quad (2.3.1)$$

where v,g are real constants. The potential $V(x)$ is assumed to have the following properties:

$$V(x)=V(-x),\qquad \int_{-\infty}^{\infty}dx'V(x')=\text{finite}. \quad (2.3.2)$$

The field operators χ satisfy equal time anticommutation relations:

$$[\chi_k(x),\chi_m^\dagger(y)]_+=\delta_{km}\delta(x-y),\ [\chi_k(x),\chi_m(y)]_+=[\chi_k^\dagger(x),\chi_m^\dagger(y)]_+=0, \quad (2.3.3)$$

here $(k, m = 1, 2)$. The Hamilton's equations for this model are given by

$$i\partial_t \chi_j(x,t) = [\chi_j(x,t), H] \tag{2.3.4}$$

and are as follows:

$$\partial_t \chi_1 - v\partial_x \chi_1 = -2ig \int dy V(x-y)\chi_2^\dagger(y)\chi_2(y)\chi_1(x), \tag{2.3.5}$$

$$\partial_t \chi_+ v\partial_x \chi_2 = -2ig \int dy V(x-y)\chi_1^\dagger(y)\chi_1(y)\chi_2(x). \tag{2.3.6}$$

Instead of continuing with the Hamiltonian (2.3.1), Komori et al. [25] constructed a transformation that converted (2.3.1) into a quadratic Hamiltonian. The transformation used was

$$\psi_1(x,t) = \exp\left(-i\frac{g}{v}\phi_2(x,t)\right)\chi_1(x,t), \tag{2.3.7}$$

$$\psi_2(x,t) = \exp\left(i\frac{g}{v}\phi_1(x,t)\right)\chi_2(x,t), \tag{2.3.8}$$

where

$$\phi_j(x,t) = \int dx' W(x-x')\chi_j^\dagger(x',t)\chi_j(x',t) \qquad j = 1,2;$$

$$W(x) = \int_{-\infty}^{x} dx' V(x'). \tag{2.3.9}$$

Under the above transformation, the new equations of motion become

$$\partial_t \psi_1 - v\partial_x \psi_1 = 0, \tag{2.3.10}$$

$$\partial_t \psi_2 + v\partial_x \psi_2 = 0. \tag{2.3.11}$$

The corresponding Hamiltonian is

$$H' = \int dx \;\; iv(\psi_1^\dagger \partial_x \psi_1 - \psi_2^\dagger \partial_x \psi_2). \tag{2.3.12}$$

Let us denote by $|n, m\rangle$ the state with n, ψ_1 particles and m, ψ_2 particles. The $|1, 1\rangle$ state is then expressible in the form,

$$|1,1\rangle = \int\int dx_1 dx_2 \;\; \Psi(x_1, x_2)\psi_1^\dagger(x_1)\psi_2^\dagger(x_2)|0\rangle, \tag{2.3.13}$$

with the vacuum defined by

$$\chi_j(x)|0\rangle = 0, \qquad \psi_j(x)|0\rangle = 0, \quad j = 1, 2. \tag{2.3.14}$$

Let us now consider the eigenvalue equation

$$H'|1,1\rangle = E_{1,1}|1,1\rangle, \tag{2.3.15}$$

i.e.,

$$iv\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\Psi(x_1,x_2) = E_{1,1}\psi(x_1,x_2). \tag{2.3.16}$$

The solution of (2.3.16) is

$$\Psi(x_1,x_2) = A\exp[i(k_1x_1+k_2x_2)], \qquad E_{1,1} = v(k_2-k_1). \tag{2.3.17}$$

By using the inverse transformation we get back

$$|1,1\rangle = \int\int dx_1dx_2\Psi(x_1,x_2)\exp\left(\frac{ig}{v}W(x_1-x_2)\right)r\chi_1^\dagger(x_1)\chi_2^\dagger(x_2)|0\rangle, \tag{2.3.18}$$

so that the Bethe wave function is of the form,

$$\hat{\psi}(x_1,x_2) = \exp[i(k_1x_1+k_2x_2)]\exp\left(\frac{ig}{v}W(x_1-x_2)\right). \tag{2.3.19}$$

The construction of the general state $|n,m\rangle, n+m=N$ can be done as follows. We start with

$$|n,m\rangle = \int ... \int \prod_{k=1}^{n+m} dx_k\Psi(x_1..x_n,x_{n+1},..x_{n+m})\times$$

$$\times\prod_{j=1}^{n}\psi_1^\dagger(x_j)\prod_{i=1}^{m}\psi_2^\dagger(x_{n+i})|0\rangle \tag{2.3.20}$$

and setting

$$H'|n,m\rangle = E_{n,m}|n,m\rangle,$$

it is seen that $\Psi(x_1....x_{n+m})$ satisfies the equation,

$$iv\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j} - \sum_{i=1}^{m}\frac{\partial}{\partial x_{n+i}}\right)\psi(x_1,...x_{n+m}) = E_{n,m}\Psi(x_1,...x_{n+m}). \tag{2.3.21}$$

The solution of (2.3.21) is given by

$$\Psi(x_1,...x_{n+m}) = \exp\left(i\sum_{j=1}^{n+m}k_jx_j\right), \tag{2.3.22}$$

with

$$E_{n,m} = v\left(\sum_{j=1}^{n}(-k_j) + \sum_{i=1}^{m} k_{n+i}\right), \tag{2.3.23}$$

so upon transforming back we obtain

$$\hat{\Psi}(x_1, ..x_{n+m}) = \exp\left(i\sum_{j=1}^{n+m} k_j x_j\right) \prod_{1\le j\le n, 1\le i\le m} \exp\left(\frac{ig}{v}W(x_j - x_{n+i})\right). \tag{2.3.24}$$

It will be noticed that in this case the computations were greatly simplified owing to the transformations (2.3.7 and 2.3.8) which converted the Hamiltonian (2.3.1) to a quadratic form. But even without this trick, it is possible to obtain the manifold of solutions for the fermionic system. A generalized fermionic model was studied by Dutyshev in [26]. The original approach uses the Lagrangian,

$$L = i\psi\gamma_\mu\partial_\mu\psi - m\bar{\psi}\psi + L_I, \tag{2.3.25}$$

where L_I is the interaction part given by

$$L_I = -\frac{1}{2}g_0(\bar{\psi}\gamma_\mu\psi)^2 - \frac{1}{2}g_a(\bar{\psi}\gamma_\mu\tau^a\psi)^2. \tag{2.3.26}$$

Here, ψ is a two-component Dirac spinor while τ^a are the Pauli matrices with γ_μ being Dirac matrices. The spinors obey anticommutation rules:

$$[\psi_i^\alpha(x), \psi_j^\beta(y)]_+ = \delta_{ij}\delta^{\alpha\beta}\delta(x-y), \tag{2.3.27}$$

and the Hamiltonian is given by

$$H = \int dx\{-i\psi_i^{\dagger\alpha}\sigma_{ij}^\tau\partial_i\psi_j^\alpha + m_0\psi_i^{\dagger\alpha}\sigma_{ij}\psi_j^\alpha\} + H_I, \tag{2.3.28}$$

where σ_{ij} are Pauli matrices, τ stands for the transpose and H_I is given by

$$H_I = \int \frac{1}{4}\psi_{i_1}^{\dagger\alpha_1}\psi_{j_1}^{\dagger\beta_1}(g_a\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2} + g_a\tau^a_{\alpha_1\alpha_2}\tau^a_{\beta_1\beta_2})\times$$

$$\times(\delta_{i_1i_2}\delta_{j_1j_2} - \sigma^\tau_{i_1i_2}\sigma^\tau_{j_1j_2})\psi_{i_2}^{\alpha_2}\psi_{j_2}^{\beta_2}.$$

If $|0\rangle$ is the vacuum state so that

$$\psi|0\rangle = 0 \quad \text{and} \qquad H|0\rangle = 0, \tag{2.3.29}$$

then an arbitrary eigenstate may be written as

$$|N\rangle = \int dx_1 ... dx_N \Phi^{i_1 ... i_N}_{\alpha_1 ... \alpha_N} \psi^{\dagger \alpha_1}_{i_1} ... \psi^{\dagger \alpha_N}_{i_N} |0\rangle, \tag{2.3.30}$$

which we want to be a solution of

$$H \Phi^{i_1 ... i_N}_{\alpha_1 ... \alpha_N} = E_N \Phi^{i_1 .... i_N}_{\alpha_1 ... \alpha_N}. \tag{2.3.31}$$

Let us first consider the one-particle sector. The wave function satisfies

$$\left(-i\sigma^z \frac{\partial}{\partial x} + m_0 \sigma^x \right) \phi^i_a(x) = E_1 \phi^i_a(x), \tag{2.3.32}$$

with a solution

$$\phi_a(x) = A_a U(\theta) \exp(ikx), \tag{2.3.33}$$

where

$$E_1 = m_0 \cosh\theta, \qquad k = m_0 \sinh\theta,$$

k being the wave vector and θ the rapidity. The function $U(\theta)$ is the Dirac spinor given by

$$U(\theta) = (2\cosh\theta)^{-\frac{1}{2}} \begin{pmatrix} e^{\theta/2} \\ e^{-\theta/2} \end{pmatrix}. \tag{2.3.34}$$

Now the two-particle state can be easily constructed if one recalls that the total energy $E_2 = m_0 \cosh\theta_1 + m_0 \cosh\theta_2$ and the total momenta $K_2 = k_1 + k_2$ are conserved. Explicitly, the two-particle wave function is given by

$$\begin{aligned} \Phi^{(12)}_{\alpha_1 \alpha_2} &= A^{12}_{\alpha_1 \alpha_2} U_1(\theta_1) U_2(\theta_2) e^{i(k_1 x_1 + k_2 x_2)} \\ &\quad - A^{21}_{\alpha_1 \alpha_2} U_1(\theta_2) U_2(\theta_1) e^{i(k_2 x_1 + k_1 x_2)}, \end{aligned} \tag{2.3.35}$$

along with

$$\begin{aligned} \Phi^{(21)}_{\alpha_1 \alpha_2} &= A^{21}_{\alpha_1 \alpha_2} U_1(\theta_1) U_2(\theta_2) e^{i(k_1 x_1 + k_2 x_2)} \\ &\quad - A^{12}_{\alpha_1 \alpha_2} U_1(\theta_2) U_2(\theta_1) e^{i(k_2 x_1 + k_1 x_2)}. \end{aligned} \tag{2.3.36}$$

If we now consider linear combinations of these wave functions:

$$G_0 = \phi_{12} - \phi_{21}, \qquad G_1 = \phi_{11} + \phi_{22},$$

$$G_2 = \phi_{11} - \phi_{22}, \qquad G_3 = \phi_{12} + \phi_{21},$$

they are seen to satisfy

$$-i\partial_x G_\mu^{+-} + i\partial_+ G_\mu^{+-} + m_0(G_\mu^{++} + G_\mu^{--}) + 2f_\mu G_\mu^{+-}\delta(x_1 - x_2) = E_2 G_\mu^{+-}. \tag{2.3.37}$$

Here $\mu = 0, 1, 2, 3$ and $(+-)$ refers to spinor components, the constants f_0, f_1, f_2, f_3 being defined as follows:

$$f_0 = \frac{1}{2}(g_0 - g_3 - g_1 - g_2), \qquad f_1 = \frac{1}{2}(g_0 + g_1 + g_2 + g_3),$$

$$f_2 = \frac{1}{2}(g_0 + g_3 - g_1 + g_2), \qquad f_3 = \frac{1}{2}(g_0 - g_3 + g_1 + g_2).$$

As before one should have appropriate boundary conditions at $x_1 = x_2$, viz:

$$G_\mu^{+-}(x_1 < x_2)|_{x_1=x_2} = e^{2if_\mu} G_\mu^{+-}(x_2 < x_1)|_{x_1=x_2},$$

$$G_\mu^{-+}(x_1 < x_2)|_{x_1=x_2} = e^{-2if_\mu} G_\mu^{-+}(x_2 > x_1)|_{x_1=x_2}, \tag{2.3.38}$$

while the other G_μ's are continuous. The system of equations so obtained is closed, with the periodicity condition of the wave function given by

$$\phi(....x_n + L....) = \phi(....x_n....). \tag{2.3.39}$$

Equations (2.3.37 and 2.3.38) immediately yield

$$B_\mu^{12} = \gamma_\mu(\theta_{12}) B_\mu^{21}, \tag{2.3.40}$$

where $\theta_{12} = \frac{1}{2}(\theta_1 - \theta_2)$, with

$$\gamma_a(\theta) = \frac{\coth\theta + i\lambda_a}{\coth\theta - i\lambda_a}, \qquad a = 1, 2, 3,$$

$$\lambda_\mu = \tan f_\mu, \qquad \gamma_0(\theta) = -\frac{1 + i\lambda_0 \coth\theta}{1 - i\lambda_0 \coth\theta};$$

where

$$B_{0,3} = A_{12} \mp A_{21}, \qquad B_{1,2} = A_{11} \pm A_{22}.$$

Since the B's are linear combinations of A's one can at once solve and get

$$A_{\alpha_1\alpha_2}^{12} = K_{12}(\theta_{12}) A_{\alpha_2\alpha_1}^{21}, \tag{2.3.41}$$

where

$$K_{nm}(\theta) = v^\mu(\theta)\tau_n^\mu \tau_m^\mu, \tag{2.3.42}$$

with

$$v^0 = \frac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3 - \gamma_0), \qquad v^1 = \frac{1}{4}(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_0),$$

$$v^2 = \frac{1}{4}(-\gamma_1 + \gamma_2 + \gamma_3 - \gamma_0), \qquad v^3 = \frac{1}{4}(\gamma_1 + \gamma_2 - \gamma_3 - \gamma_0),$$

the matrix K_{12} is the scattering matrix. For $n > 2$ the relation is

$$A^{....q_n q_{n+1}....}_{....\alpha_n \alpha_{n+1}...} = S_{n,n+1} A^{...q_{n+1} q_n....}_{....\alpha_{n+1} \alpha_n....}. \tag{2.3.43}$$

In the sequel we will prove that the S matrix satisfies the Yang-Baxter equation. Until now we have not used any periodic boundary condition. If we consider the wave function and use periodicity, along with the definition of the S matrix in the N-particle case, then it leads to

$$T_n \Omega = e^{ik_n L} \Omega, \tag{2.3.44}$$

with

$$T_n = S_{n,n+1}....S_{nN} S_{n1}....S_{nn-1}.$$

This equation can be solved by taking recourse to a special trick. We set

$$L(v) = -\prod_{n=1}^{N} S_{0n}\left(\frac{v - \sigma_n}{2}\right), \tag{2.3.45}$$

$$S_{0n}\left(\frac{v - \sigma_n}{2}\right) = \omega^{\mu}\left(\frac{v - \sigma_n}{2}\right) \tau_0^{\mu} \tau_n^{\mu},$$

the operator $L(v)$ acts in a space of 2^{N+1} dimension with an additional particle numbered 0. Let us denote the trace of this operator over the indices of the additional particle by $T(v)$:

$$T(v) = tr_0 L(v), \tag{2.3.46}$$

this $T(v)$ is actually Baxter's transfer matrix [27]. The solution will lead to equations that determine the momenta k_i. One then utilizes the fact that $T(u), T(v)$ commute, i.e.,

$$[T(u), T(v)] = 0 \tag{2.3.47}$$

and represents the operator $L(v)$ in the following form:

$$L(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} = -\prod_{n=1}^{N} \begin{pmatrix} \omega_n^0 + \omega_n^3 \tau_n^3 & \omega_n^1 \tau_n^1 - i\omega_n^2 \tau_n^2 \\ \omega_n^1 \tau_n^1 + i\omega_n^2 \tau_n^2 & \omega_n^0 - \omega_n^3 \tau_n^3 \end{pmatrix}, \tag{2.3.48}$$

where

$$\omega_n^\mu = \omega^\mu \left(\frac{v - \sigma_n}{2} \right).$$

It is then easy to show that $L(v)$ satisfies the relation,

$$R\left(\frac{v-u}{2}\right)(L(v) \otimes L(u)) = (L(u) \otimes L(v))R\left(\frac{v-u}{2}\right), \qquad (2.3.49)$$

which is a fundamental relation of the Yang-Baxter algebra, with R representing the quantum R matrix.

We will further discuss the Yang-Baxter equation and associated algebra in the subsequent chapters, and hence do not pursue the matter further here. Instead, in the next section we shall discuss another method for obtaining the eigenmomenta equation. We remind the reader that we have not used the periodicity condition yet and hence we did not talk about the equation determining the eigenmomenta k_i's.

2.4 Boundary Condition in the Bethe Ansatz

We have already shown in section (2.2) how periodic boundary conditions lead to a set of equations determining the eigenmomenta k_j. However, physical situations in which the boundary conditions are not periodic can often be conceived. Such nonperiodic boundary conditions were considered by several authors, notably Gaudin [28] for the δ-function Bose gas and also by Woynarovich [29]. In this section we take up this interesting issue and discuss how the coordinate Bethe ansatz may be applied in such cases.

For this purpose we once again consider the Hamiltonian for the δ-function Bose gas:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j), \qquad 0 < x_i < L. \qquad (2.4.1)$$

Recall the following:
(i) An elementary solution of the system governed by (2.4.1) is a continuous symmetric function of the coordinates $(x_1, ..., x_N)$.
(ii) Bethe ansatz for the eigenvalue problem associated with H above

yields a solution that is parametrized by the set $\{k\} = (k_1, ...k_N)$ and is given by

$$\Psi_{\{k\}}(x) = \sum_P a(P) \exp\left(i \sum_{i=1}^{N} k_{P_i} x_i\right), \tag{2.4.2}$$

in the domain $D, x_1 < x_2 < ... < x_N$, with the sum over all permutations P of order N over $i = 1$ to $i = N$.
(iii) The coefficients $a(P)$ are given by

$$a(P) = \prod_{i<j} \left(1 + \frac{ic}{k_{P_i} - k_{P_j}}\right). \tag{2.4.3}$$

(iv) The corresponding energy eigenvalue is

$$E_{\{k\}} = \sum_{i=1}^{N} k_i^2. \tag{2.4.4}$$

The usual periodicity condition is of the form

$$\Psi(x_1 = 0, x_2, ...x_N) = \Psi(x_2, x_3, ...x_N, x_1 = L). \tag{2.4.5}$$

Imposition of this condition on $\Psi_{\{k\}}$ leads to

$$a(PC) \exp(ik_{P_1} L) = a(P), \tag{2.4.6}$$

with C being the cyclic permutation of $(1, 2...N)$. From (2.4.2) and (2.4.5) we get

$$k_i L = 2\pi n_i + \sum_j \psi_{ij}, \tag{2.4.7}$$

with ψ_{ij} being the phases and

$$\tan\frac{\psi_{ij}}{2} = \frac{c}{k_i - k_j}, \tag{2.4.8}$$

which is just the eigenvalue equation of the algebraic Bethe ansatz (to be discussed in Chapter 5). That the integers n_i are unique was proved by Yang and Yang [15]. Here we follow Gaudin's argument [28] based on the continuity of c. For $n_i \neq n_j$ we choose the sheet of ψ_{ij}, which in the neighbourhood of $c = 0$ behaves like

$$\psi_{ij} \propto \frac{c}{n_i - n_j} \frac{L}{2\pi}. \tag{2.4.9}$$

Let us consider the ground state of total momentum zero so that $n_i = 0$ $\forall i$. In the vicinity of $c = 0$, we look for a solution of the form

$$k_i \approx \left(\frac{2c}{L}\right)^{\frac{1}{2}} q_i + \mathcal{O}(c),$$

q_i being distinct, so that we get

$$\psi_{ij} \approx (2cL)^{\frac{1}{2}}(q_i - q_j) + \mathcal{O}(c). \tag{2.4.10}$$

Equation (2.4.7) then leads to the fact that the q_i's must satisfy the relation,

$$q_i = \sum_{i \neq j} \frac{1}{q_i - q_j}. \tag{2.4.11}$$

This gives us a precise idea of the distribution of the pseudomomenta k in the limit $c \to 0$. Equation (2.4.11) immediately implies that the q_i's are the zeros of Hermite polynomials of degree N, and therefore satisfy

$$H''(q) - 2qH'(q) + 2NH(q) = 0. \tag{2.4.12}$$

The density of the zeros of $H(q)$ is given by

$$\rho(q) = \frac{1}{\pi}(2N - q^2)^{\frac{1}{2}}, \tag{2.4.13}$$

leading to the following density for k:

$$\rho(k) = \frac{L}{2\pi c}(4c\rho - k^2)^{\frac{1}{2}}, \tag{2.4.14}$$

with $\rho = \frac{N}{L}$ as $c \to 0$.

2.4.1 Semi-infinite axis

We shall next consider the same problem formulated on a semi-infinite axis $(0, L)$. In this case the boundary conditions for the bosonic wave function $\psi(x)$ are

$$\psi(x_1 = 0, x_2 ... x_N) = 0, \tag{2.4.15}$$

$$\psi(x_1, ...x_N = L) = 0, \tag{2.4.16}$$

in the region $0 \leq x_1 \leq x_2 ... \leq x_n < L$. The idea here is to construct elementary solutions of the Schrödinger equation on the semi-infinite axis $x_i > 0$, which must satisfy (2.4.15) at $x_1 = 0$. Here one can effectively utilize the optical analogy given by McGuire [30]. The idea is to construct the wave function, by superposition of all elementary wave functions $\Psi_{\{k\}}$ obtained by reflection at the wall $x = 0$. Such an elementary solution is written as $\Psi_{\{|k_i|\}}(x)$, and is associated with a set of N distinct positive numbers $|k_i|$. If k's are real, then one can choose

$$0 < |k_i| < |k_2| < ... < |k_N|,$$

so that we can define the 2^N sets,

$$\{k\} = \{k_1, k_2, ...k_N\}, \qquad k_i = \epsilon_i |k_i|, \quad \epsilon = \pm 1.$$

All the states $\Psi_{\{k\}}$ have the same energy and we look for a solution of the form,

$$\Psi_{\{|k_i|\}}(x) = \sum_{\epsilon_1 \epsilon_2 ... \epsilon_N} A(\epsilon_1 .... \epsilon_N) \Psi_{\{k\}}(x). \tag{2.4.17}$$

Then the condition (2.4.15) yields

$$\sum_{\epsilon} A\{\epsilon\} \sum_{P} \prod_{i<j} \left(1 + \frac{ic}{k_{P_i} - k_{P_j}}\right) \exp\left(i \sum_{i=2}^{N} k_{P_i} x_i\right) = 0, \tag{2.4.18}$$

leading thereby to 2^{N-1} relations of the form,

$$A(\epsilon_1 ... \epsilon_{P_1} ... \epsilon_N) \prod_{\beta \neq P_1} \left(1 + \frac{ic}{k_{P_1} - k_\beta}\right) +$$

$$+ A(\epsilon_1 ..., -\epsilon_{P_1} ... \epsilon_N) \prod_{\beta \neq P_1} \left(1 + \frac{ic}{-k_{P_1} - k_\beta}\right) = 0, \tag{2.4.19}$$

which must be true for any $\{k\}$ and P. It is therefore sufficient to choose

$$A(\epsilon_1 ... \epsilon_N) = \prod_{i<j} \left(1 - \frac{ic}{k_i + k_j}\right) \epsilon_1 ... \epsilon_N (k_i + k_j). \tag{2.4.20}$$

Thus, we obtain the desired solution,

$$\Psi_{\{|k_i|\}} = \sum_{\epsilon,P} \epsilon_1 ... \epsilon_N \prod_{i<j} \left(1 - \frac{ic}{k_i + k_j}\right) \left(1 + \frac{ic}{k_{P_i} - k_{P_j}}\right) \times$$
$$\times e^{i(k_{P_1} x_1 + ... + k_{P_N} x_N)} \tag{2.4.21}$$

with $k_i = \epsilon_i |k_i|$ and $k_{P_i} = \epsilon_{P_i} |k_{P_i}|$.
After the determination of the wave function that satisfies the condition (2.4.15), let us now impose (2.4.16) on (2.4.21); this yields

$$\sum_{\epsilon,P} \left[\prod_{i<j} \left(1 - \frac{ic}{k_{P_i} + k_{P_j}}\right) \left(1 + \frac{ic}{k_{P_i} - k_{P_j}}\right) \right] e^{ik_{P_N} L} = 0, \tag{2.4.22}$$

for all P and $\{\epsilon\}$. Setting $P_N = \alpha$ we obtain

$$e^{2ik_\alpha L} = \prod_{\beta \neq \alpha} \frac{k_\beta - k_\alpha - ic}{k_\beta - k_\alpha + ic} . \frac{k_\beta + k_\alpha + ic}{k_\beta + k_\alpha - ic}, \tag{2.4.23}$$

which must be satisfied by all possible α and possible signs of k. Equation (2.4.23) can also be recast as

$$e^{2ik_\alpha L} = \prod_{\beta \neq \alpha} \frac{(k_\alpha + ic)^2 - k_\beta^2}{(-k_\alpha + ic)^2 - k_\beta^2}, \tag{2.4.24}$$

leading to

$$2k_\alpha L = m_\alpha + \sum_{\beta \neq \alpha} \left\{ \tan^{-1} \frac{c}{k_\alpha - k_\beta} + \tan^{-1} \frac{c}{k_\alpha + k_\beta} \right\}, \tag{2.4.25}$$

which is the equation for the eigenmomenta under the modified boundary conditions (2.4.15 and 2.4.16).

2.5 Heisenberg Spin Chain

The Heisenberg spin chain model constitutes one of the classic integrable models and has been investigated in great detail. It was originally solved by Onsager by evaluating the partition function [31]. Later, an application of the coordinate Bethe ansatz revealed its rich underlying mathematical structure. Among one-dimensional models, it is

universally recognized as the most important discrete integrable system. The model is defined by the Hamiltonian,

$$H = \sum_{i=1}^{N-1} \left(\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + P \left(S_i^z S_{i+1}^z - 1/4 \right) \right) + \\ + \left(S_N^x S_1^x + S_N^y S_1^y \right) + P \left(S_N^z S_1^z - 1/4 \right), \tag{2.5.1}$$

where S_i^x, S_i^y, S_i^z are the components of the spin operators at the ith lattice site. The periodic boundary condition implies $S_{N+1} = S_1$ and this is represented by the last two terms in the Hamiltonian. Introducing the operators,

$$\sigma_j^{\pm} = S_j^x \pm i S_j^y, \qquad \sigma_j^z = 2 S_j^z, \tag{2.5.2}$$

we have

$$H = H_{kin} + H_{int},$$

where

$$H_{kin} = \sum_{i=1}^{N} \frac{1}{2} \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_{i+1}^+ \sigma_i^- \right), \tag{2.5.3}$$

$$H_{int} = \sum_{i=1}^{N} \frac{P}{4} \left(\sigma_i^z \sigma_{i+1}^z - 1 \right). \tag{2.5.4}$$

The Bethe ansatz is actually a way of constructing the many-particle wave function, which diagonalizes the Hamiltonian. If there are no particles, all the spins point up, and the chain is in the ferromagnetic state ($S^z = N/2$) $|F\rangle$, with zero energy. From this bare vacuum, particles can be created by acting on it with the σ_n^- operators. The one-particle eigenstate is of a plain wave type ($S^z = N/2 - 1$),

$$|\phi^{(1)}\rangle = \sum_{n=1}^{N} a(n) \sigma_n^- |F\rangle, \ \text{ with } \ a_n = e^{ikn}, \tag{2.5.5}$$

and obeys the equation

$$H|\phi^{(1)}\rangle = E^{(1)} |\phi^{(1)}\rangle. \tag{2.5.6}$$

When expressed in component form it reads

$$\langle F|\sigma_n^+ H|\phi^{(1)}\rangle = E^{(1)} \langle F|\sigma_n^+|\phi^{(1)}\rangle, \tag{2.5.7}$$

leading to

$$1/2\{a(n-1)+a(n+1)\}-\rho a(n)=E^{(1)}a(n),\quad 1<n<N,$$

$$1/2\{a(N)+a(2)\}-\rho a(1)=E^{(1)}a(1),$$

$$1/2\{a(N-1)+a(1)\}-\rho a(N)=E^{(1)}a(N). \tag{2.5.8}$$

Substitution of $a(n)$ gives

$$E^{(1)}(k)=(\cos k-\rho), \tag{2.5.9}$$

provided

$$e^{ikN}=1\ \ or\quad k=\frac{2\pi}{N}\lambda;\quad \lambda=1,...N. \tag{2.5.10}$$

The momentum of the state is

$$\rho^{(1)}(k)=k=\frac{2\pi}{N}\lambda. \tag{2.5.11}$$

Next we look at the two-particle state with ($S^z=N/2-2$):

$$|\phi^{(2)}\rangle=\sum_{n_1<n_2}a(n_1,n_2)\sigma^-_{n_1}\sigma^-_{n_2}|F\rangle, \tag{2.5.12}$$

with

$$a(n_1,n_2)=a_{12}e^{i(k_1n_1+k_2n_2)}+a_{21}e^{i(k_2n_1+k_1n_2)}$$

$$=\exp[i(k_1n_1+k_2n_2+\psi_{12}/2)]+\exp[i(k_2n_1+k_1n_2-\psi_{21}/2)]. \tag{2.5.13}$$

For convenience we choose

$$\psi_{21}=-\psi_{12}=\psi(k_1,k_2),\quad \psi(k_2,k_1)=-\psi(k_1,k_2);$$

then the eigenvalue equation is

$$\langle F|\sigma^+_{n_1}\sigma^+_{n_2}H|\phi^{(2)}\rangle=E^{(2)}\langle F|\sigma^+_{n_1}\sigma^+_{n_2}|\phi^{(2)}\rangle, \tag{2.5.14}$$

which leads to

$$1/2\{a(n_1-1,n_2)+a(n_1+1,n_2)+a(n_1,n_2-1)+a(n_1,n_2+1)\}-$$

$$-2\rho a(n_1,n_2)=E^{(2)}a(n_1,n_2),$$

$$1/2\{a(n_1-1,n_1+1)+a(n_1,n_1+2)\}-\rho a(n_1,n_1+1)=E^{(2)}a(n_1,n_1+1),$$

$$1/2\{a(n_2,N)+a(2,n_2)+a(1,n_2-1)+a(1,n_2+1)\}-$$

$$-2\rho a(1,n_2) = E^{(2)} a(1,n_2). \tag{2.5.15}$$

The first equation holds when both particles are apart from each other and end points. The second is valid when they occupy neighbouring sites. Using the form of $a(n_1,n_2)$ as given in (2.5.13) we get

$$E^{(2)}(k_1,k_2) = (\cos k_1 - \rho) + (\cos k_2 - \rho) = E^{(1)}(k_1) + E^{(1)}(k_2). \tag{2.5.16}$$

Using this and the expression for $a(n_1,n_2)$, together with (2.5.15), we get an equation that determines the phase ψ_{12} as a function of k_1, k_2. The simplest way to do this is to observe that

$$1/2\{a(n_1,n_2) + a(n_1+1,n_1+1)\} - \rho a(n_1,n_1+1) = 0, \tag{2.5.17}$$

is equivalent to (2.5.8) if $E^{(2)} = E^{(1)}(k_1) + E^{(1)}(k_2)$. The last equation can be written as

$$\left(e^{i\psi_{12}/2} + e^{-i\psi_{12}/2}\right) + e^{i(k_1+k_2)}\left(e^{i\psi_{12}/2} + e^{-i\psi_{12}/2}\right) -$$

$$-\rho\left(e^{i(k_2+\psi_{12}/2)} + e^{i(k_2+\psi_{12}/2)}\right) = 0, \tag{2.5.18}$$

leading to

$$\cot(\frac{\psi_{12}}{2}) = -\rho\frac{\cot(k_1/2) - \cot(k_2/2)}{(1-\rho)\cot(k_1/2)\cot(k_2/2) - (1+\rho)}. \tag{2.5.19}$$

Finally one can check that (2.5.18) is equivalent to

$$a(n_2,N) = a(0,n_2), \tag{2.5.20}$$

that is,

$$e^{i(k_1n_2+k_2N+\psi_{12}/2)} + e^{i(k_2n_2+k_1N-\psi_{12}/2)} = e^{i(k_2n_2+\psi_{12}/2)} + e^{i(k_1n_2-\psi_{12}/2)}. \tag{2.5.21}$$

This will be satisfied for all n_2 if

$$e^{i(k_1N-\psi_{12})} = 1, \qquad e^{i(k_1n_2-\psi_{12})} = 1; \tag{2.5.22}$$

implying thereby that

$$Nk_1 = 2\pi\lambda_1 + \psi_{12},$$

$$Nk_2 = 2\pi\lambda_2 - \psi_{12}, \tag{2.5.23}$$

λ_{12} being integers. The total momentum is

$$\rho^{(2)}(k_1, k_2) = k_1 + k_2 = \frac{2\pi}{N}(\lambda_1 + \lambda_2).$$

In view of the above, we can write the most general eigenstate of r particles ($S^z = N/2 - r$) as follows:

$$|\phi^{(r)}\rangle = \sum_{n_\alpha < n_{\alpha+1}} a(n_1, n_2, ...n_r) \prod_{\alpha=1}^{r} \sigma^-_{n_\alpha} |F\rangle, \tag{2.5.24}$$

with the coefficients $a(n_1, ...n_r)$ being

$$a(n_1, ...n_r) = \sum_P \exp\left[i \sum_\alpha k_{P_\alpha} n_\alpha + i/2 \sum_{\alpha<\beta} \psi_{P_\alpha P_\beta} \right], \tag{2.5.25}$$

where P is the permutation of the indices $1, 2, ..., r$. The corresponding energy is given by

$$E^{(r)}(k_1, ...k_r) = \sum_\alpha^r (\cos k_\alpha - \rho) = \sum_\alpha^r E^{(1)}(k_\alpha), \tag{2.5.26}$$

provided

$$\cot(\psi_{\alpha\beta}/2) = -\rho \frac{\cot(k_\alpha/2) - \cot(k_\beta/2)}{(1-\rho)\cot(k_\alpha/2)\cot(k_\beta/2) - (1-\rho)}, \tag{2.5.27}$$

and

$$N k_\alpha = 2\pi\lambda_\alpha + \sum_{\beta\neq\alpha} \psi_{\alpha\beta} \qquad (\alpha = 1, 2...r). \tag{2.5.28}$$

The momentum of the state is

$$P^{(r)}(k_1, ...k_r) = \sum_\alpha^r k_\alpha = \frac{2\pi}{N} \sum_\alpha^r \lambda_\alpha. \tag{2.5.29}$$

The calculation is identical to that in case of the single- and two-particle cases:

$$\langle F| \prod_{\alpha=1}^{r} \sigma^+_{n_\alpha} H |\phi^{(r)}\rangle = E^{(r)} \langle F| \prod_{\alpha=1}^{r} \sigma^+_{n_\alpha} |\phi^{(r)}\rangle. \tag{2.5.30}$$

Written explicitly, the above condition gives the following:

$$1/2\{.... + a(...., n-1, n+1, N = 2, ...) + a(...., n, n+1, n+3, ...) + ...\} -$$

$$-(r-2)\rho a(....,n+1,n+2,....) = E^{(r)}a(....,n,n+1,n+2,....),$$
$$1/2\{a(....,n,n,n+2,...0 + a(...,n+1,n=1,n+2,....)\}-$$
$$-\rho a(...,n,n+1,n+2,....) = 0,$$
$$1/2\{a(...,n,n+1,n+1,...) + a(...,n+n+2,n=2,...)\}-$$
$$-\rho a(....,n,n+1,n+2,...) = 0. \tag{2.5.31}$$

The configuration of particles in this case is more complicated than in the two-particle case. Besides, there are other complications associated with the Bethe ansatz for the spin chain, which we do not go into here, but refer the reader to the following reference [32].

2.6 Spin of the Bethe Ansatz State

The solutions obtained in the previous situations are all eigenstates of S^z. The isotropic case corresponds to $\rho = 1$. In this case the Hamiltonian commutes with all three components of the total spin, and it is expected that the eigenstate will also be eigenstates of the total spin S^2, which we write as

$$S^2 = \sigma^-\sigma^+ + S^z(S^z+1), \tag{2.6.1}$$

with $\sigma^\pm = \sum_{i=1}^N \sigma_i^\pm$. The above statement holds if

$$\sigma^+|\phi^{(r)}\rangle = 0. \tag{2.6.2}$$

For $r = 1$, the left-hand side of (2.5.5) is formally a $S^z = N/2$ state that is either zero or a multiple of $|F\rangle$. Actually,

$$\langle F|\sigma^+|\phi^{(1)}\rangle = \sum_{m=1}^{N} a(m) = \sum_{m=1}^{N} e^{ikm}, \tag{2.6.3}$$

which is zero due to the fact that $k = \frac{2\pi}{N}\lambda$. So these are eigenstates of S^2. For $r = 2$ the left-hand side of (2.5.12) is formally an $S^z = N/2-1$ state; thus we have

$$\langle F|\sigma_n^+\sigma^+|\phi^{(2)}\rangle = \sum_{m=1}^{n-1} a(m,n) + \sum_{m=n+1}^{N} a(n,m) = 0, \tag{2.6.4}$$

for $1 \leq n \leq N$. Substituting the expression for $a(n_1, n_2)$ we get

$$\sum_{m=1}^{n-1} a(m,n) + \sum_{m=n+1}^{N} a(m,n)$$

$$= \sum_{m=1}^{n-1} \left(e^{i(k_1 m + k_2 n + \psi_{12}/2)} + e^{i(k_2 m + k_1 n - \psi_{12}/2)} \right)$$

$$+ \sum_{m=n+1}^{N} \left(e^{i(k_1 n + k_2 m + \psi_{12}/2)} + e^{i(k_2 n + k_1 m - \psi_{12}/2)} \right) \tag{2.6.5}$$

A simple calculation then yields

$$= \frac{e^{ik_1} - e^{ik_1 n}}{1 - e^{ik_1}} e^{i(k_2 n + \psi_{12}/2)} + \frac{e^{ik_2} - e^{ik_2 n}}{1 - e^{ik_2}} e^{i(k_1 n - \psi_{12}/2)}$$

$$+ e^{ik_1 n} \frac{e^{ik_2(n+1)} - e^{ik_2(N+1)}}{1 - e^{ik_2}} e^{i\psi_{12}/2} + e^{ik_2 n} \frac{e^{ik_1(n+1)} - e^{ik_1(N+1)}}{1 - e^{ik_1}} e^{-i\psi_{12}/2}. \tag{2.6.6}$$

This can be reduced to

$$e^{i(k_1+k_2)n} \left[\frac{e^{i(k_1 - \psi_{12}/2)} - e^{i\psi_{12}/2}}{1 - e^{ik_1}} + \frac{e^{i(k_2 + \psi_{12}/2)} - e^{-i\psi_{12}/2}}{1 - e^{ik_2}} \right] = 0, \tag{2.6.7}$$

since for $\rho = 1$

$$\cot(\psi_{12}/2) = \frac{1}{2}(\cot(k_1/2) - \cot(k_2/2)). \tag{2.6.8}$$

The generalization of such computations for the case $r > 2$ is straightforward but laborious. However, the above ananlysis shows that in the isotropic case, the Bethe states are eigenvectors of the total spin operator.

2.7 Other Integrable Models

In the preceding sections we have described how the coordinate Bethe ansatz may be applied to specific models like the Heisenberg spin chain, a Fermionic system and the extended derivative nonlinear Schrödinger

equation. We have also described the procedure for incorporating non-periodic boundary conditions and have dwelt on application of this technique for ultralocal and nonultralocal cases. The Heisenberg spin chain and the nonlinear Schrödinger equation are two major models in which the applicability of the coordinate Bethe ansatz was tested in the truest sense. Therafter several other integrable models in two dimensions appeared, such as the three-wave interaction, supersymmetric nonlinear integrable equations, etc., where an application of this technique proved to be immensely succesful, thus revealing the enormous power of the coordinate Bethe ansatz.

2.7.1 Supersymmetric NLS equation

In this section we shall consider one such system, namely the supersymmetric nonlinear Schrödinger equation, which is written as a coupled system [33]:

$$iq_t = -q_{xx} + 2\alpha_1 q^\dagger qq + \alpha_2 \psi^\dagger \psi q - i(\alpha_2)^{\frac{1}{2}} \psi\psi_x,$$

$$i\psi_t = -2\psi_{xx} + \alpha_2 q^\dagger q\psi + i(\alpha_2)^{\frac{1}{2}}(2q\partial_x\psi^\dagger + \psi^\dagger q_x), \tag{2.7.1}$$

with q being a bosonic field and ψ a fermionic field. The Hamiltonian of the system is

$$H = \int dx\{-q^\dagger q_{xx} - 2\psi^\dagger\psi_{xx} + \alpha_1 q^\dagger \dagger qq + \alpha_2 q^\dagger q_x \psi^\dagger\psi$$

$$-i(\alpha_2)^{\frac{1}{2}}(q\psi^\dagger\psi_x^\dagger + q^\dagger\psi\psi_x)\}; \tag{2.7.2}$$

while the equations of motion are given as usual by

$$\frac{\partial\psi}{\partial t} = [H, \psi] \qquad \frac{\partial q}{\partial t} = [H, q]\,. \tag{2.7.3}$$

Note that the commutation and anticommutation relations satisfied by the fields are as follows:

$$\{\psi(x), \psi(y)\} = \{\psi^\dagger(x), \psi^\dagger(y)\} = 0,$$

$$\{\psi(x), \psi^\dagger(y)\} = \delta(x-y), \qquad \left[q(x), q^\dagger(y)\right] = \delta(x-y). \tag{2.7.4}$$

The basic problem is again to search for solutions of

$$H|\phi\rangle = E|\phi\rangle, \tag{2.7.5}$$

where $|\phi\rangle = |m, n\rangle$, with m denoting here the number of bosons and n the number of fermions. To illustrate the calculational procedure, let us consider the state $|1, 2\rangle$ where

$$|1,2\rangle = \int dx_1 f(x_1) q^\dagger(x_1)|0\rangle + \int\int dx_1 dx_2 g(x_1, x_2)\psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle. \tag{2.7.6}$$

The equations for f and g turn out to be as follows:

$$-\frac{\partial^2 f}{\partial x_1^2} = Ef(x_1),$$

$$i(\alpha_2)^{\frac{1}{2}} f(x_1)\delta'(x_2 - x_1) - 2\left[\frac{\partial^2 g(x_1, x_2)}{\partial x_1^2} + \frac{\partial^2 g(x_1, x_2)}{\partial x_2^2}\right] = Eg(x_1, x_2). \tag{2.7.7}$$

The solutions of these can be obtained by the usual procedure and finally we obtain:

$$f(x_1) = a_1 e^{ip_1 x_1} + a_2 e^{-ip_1 x_1}, \qquad E = p_1^2, \tag{2.7.8}$$

$$g(x_1, x_2) = g_1(x_1, x_2)\theta(x_1 - x_2) + g_2(x_1, x_2)\theta(x_2 - x_1), \tag{2.7.9}$$

where

$$\begin{aligned} g_1(x_1, x_2) &= \pi a_1 \sqrt{\alpha_2}\left(i \mp \frac{1}{\sqrt{3}}\right) \exp\left[\left(-\frac{\sqrt{3} \pm i}{2}\right)\sqrt{E}x_1 + \left(\frac{\sqrt{3} \pm i}{2}\right)\sqrt{E}x_2\right] \\ &+ \pi a_2 \sqrt{\alpha_2}\left(i \pm \frac{1}{\sqrt{3}}\right) \exp\left[\left(-\frac{\sqrt{3} \pm i}{2}\right)\sqrt{E}x_1 + \left(\frac{\sqrt{3} \mp i}{2}\right)\sqrt{E}x_2\right], \end{aligned}$$

and

$$\begin{aligned} g_2(x_1, x_2) &= \pi a_1 \sqrt{\alpha_2} \exp\left[\left(\frac{\sqrt{3} \pm i}{2}\right)\sqrt{E}x_1 + \left(-\frac{\sqrt{3} \mp i}{2}\right)\sqrt{E}x_2\right] \\ &+ \pi a_2 \sqrt{\alpha_2} \exp\left[\left(\frac{\sqrt{3} \mp i}{2}\right)\sqrt{E}x_1 + \left(-\frac{\sqrt{3} \pm i}{2}\right)\sqrt{E}x_2\right]. \end{aligned} \tag{2.7.10}$$

The same procedure may be followed if we assume that $E < 0$. We can now consider various types of states, such as

$$|S_1\rangle = \int\int dx_1 dx_2 g(x_1, x_2) q^\dagger(x_1)\psi^\dagger(x_2)|0\rangle$$

$$+\int\int dx_1dx_2dx_3\psi^\dagger(x_1)\psi^\dagger(x_2)\psi^\dagger(x_3)h(x_1,x_2,x_3)|0\rangle. \tag{2.7.11}$$

The eigenvalue problem

$$H|S_1\rangle = E|S_1\rangle \tag{2.7.12}$$

then leads to

$$-\left(\frac{\partial^2 g}{\partial x_1^2}+2\frac{\partial^2 g}{\partial x_2^2}\right)+\alpha_2 g(x_1,x_2)\delta(x_1-x_2)=Eg(x_1,x_2),$$

$$-i\sqrt{\alpha_2}g(x_1,x_2)\delta'(x_3-x_1)-2\left(\frac{\partial^2 h}{\partial x_1^2}+\frac{\partial^2 h}{\partial x_2^2}+\frac{\partial^2 h}{\partial x_3^2}\right)=Eh(x_1,x_2,x_3). \tag{2.7.13}$$

The solutions can always be obtained if we consider the regions $x_1 < x_2$ and $x_1 > x_2$ separately.

We can pass over to multiparticle states and consider

$$|S_2\rangle = \int\int dx_1dx_2g(x_1,x_2)q^\dagger(x_1)q^\dagger(x_2)|0\rangle+$$

$$\int\int\int dx_1dx_2dx_3h(x_1,x_2,x_3)q^\dagger(x_1)\psi^\dagger(x_2)\psi^\dagger(x_3)|0\rangle+$$

$$\int\int\int\int dx_1dx_2dx_3dx_4K(x_1,x_2,x_3,x_4)\psi^\dagger(x_1)\psi^\dagger(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)|0\rangle. \tag{2.7.14}$$

The differential equations for g, h, K are now more complicated, as follows:

$$-\left(\frac{\partial^2 g}{\partial x_1^2}+\frac{\partial^2 g}{\partial x_2^2}\right)+2\alpha_1 h(x_1,x_2)\delta(x_1-x_2)=Eg(x_1,x_2),$$

$$-2i\sqrt{\alpha_2}[g(x_1,x_2)+g(x_2,x_1)]\delta'(x_3-x_1)-\left(\frac{\partial^2 h}{\partial x_1^2}+\frac{\partial^2 h}{\partial x_2^2}+\frac{\partial^2 h}{\partial x_3^2}\right)$$

$$+\alpha_2 h(x_1,x_2,x_3)[\delta(x_1-x_2)+\delta(x_1-x_3)]=Eh(x_1,x_2,x_3),$$

$$-i\sqrt{\alpha_2}h(x_1,x_2,x_3)\delta'(x_4-x_1)-2\left(\frac{\partial^2 K}{\partial x_1^2}+\frac{\partial^2 K}{\partial x_2^2}+\frac{\partial^2 K}{\partial x_3^2}+\frac{\partial^2 K}{\partial x_4^2}\right)$$

$$=EK(x_1,x_2,x_3,x_4). \tag{2.7.15}$$

After a rather lengthy but straightforward calculation, we can obtain the solutions for the above equations in the following form:

$$g(x_1,x_2)=(1+F_1)\exp[i(k_2+Q_2)x_1+i(Q_2-K_2)x_2]\theta(x_1-x_2)$$

$$+\exp[i(k_2+Q_2)x_1+i(Q_2-k_2)x_2]\theta(x_2-x_1)$$
$$-B_1\exp[i(Q_2-k_2)x_1+i(Q_2+k_2)x_2]\theta(x_2-x_1), \tag{2.7.16}$$
$$\begin{aligned}h(x_1,x_2,x_3) = (h_{1b}+h_{2b})\theta(x_1-x_2)\theta(x_2-x_3)\theta(x_1-x_3)+\\ +(h_{1b}+h_{2b}+h_{3a})\theta(x_1-x_2)\theta(x_3-x_2)\theta(x_1-x_3)\\ +(h_{1b}+h_{2a}+h_{3a})\theta(x_1-x_2)\theta(x_3-x_2)\theta(x_3-x_1)\\ +(h_{1a}+h_{2b}+h_{3b})\theta(x_2-x_1)\theta(x_2-x_3)\theta(x_1-x_3)\\ +(h_{1a}+h_{2a}+h_{3b})\theta(x_2-x_1)\theta(x_2-x_3)\theta(x_3-x_1)\\ +(h_{1a}+h_{2a})\theta(x_2-x_1)\theta(x_3-x_2)\theta(x_3-x_1),\end{aligned} \tag{2.7.17}$$

where $h_{1a}, h_{1b}, h_{2a}, h_{2b}$ are combinations of plane waves of the following form:

$$h_{1a} = \exp i\left[(k_3+Q_3)x_1+(-k_3+Q_3/2)x_2+Q_4x_3\right]$$
$$-B_2\exp i\left[(-k_3+Q_3)x_1+(k_3+Q_3/2)x_2+q_4x_3\right], \tag{2.7.18}$$
$$h_{1b} = \exp i\left[(k_3+Q_3)x_1+(-k_3+Q_3/2)x_2+Q_4x_3\right](1+F_2), \tag{2.7.19}$$

with similar expressions for the others. In the present case the structure of the Bethe ansatz wave function is more complicated than in the earlier ones. Consequently the imposition of the periodicity condition results in a complicated equation for the eigenmomenta.

2.7.2 Three-wave interaction problem

We next consider the three-wave interaction problem, which was solved in [34]. The quantum three-wave interaction model in one dimension is governed by the Hamiltonian:

$$H = \int dx[-\sum_{j=1}^{3} ic_jQ_j^\star Q_{jx}+g\{Q_2^\star(x)Q_3(x)Q_1(x)+Q_1^\star(x)Q_3^\star(x)Q_2(x)\}]. \tag{2.7.20}$$

Here $Q_j^\star, Q_j$ are the creation and annihilation operators of the particular wave for $j=1,2,3$. For the purely bosonic case,

$$[Q_j(x,t),Q_k^\star(y,t)] = \delta_{jk}\delta(x-y). \tag{2.7.21}$$

The equations of motion are given by

$$\frac{\partial Q_j(x,t)}{\partial t} = i[H,Q_j(x,t)]. \tag{2.7.22}$$

From these, using the commutation rules, we get

$$\frac{\partial Q_1}{\partial t} + c_1 \frac{\partial Q_1}{\partial x} = -igQ_3^\star Q_2,$$
$$\frac{\partial Q_2}{\partial t} + c_2 \frac{\partial Q_2}{\partial x} = -igQ_3 Q_1,$$
$$\frac{\partial Q_3}{\partial t} + c_3 \frac{\partial Q_3}{\partial x} = -igQ_1^\star Q_2. \tag{2.7.23}$$

For setting up the Bethe ansatz one assumes that

$$Q_j(x,t)|0\rangle = 0, \qquad j = 1,2,3. \tag{2.7.24}$$

The different multiparticle states are constructed as follows:

$$|\lambda_1, ...\lambda_N\rangle = \int dx_1...dx_N \theta(x_1 > ..... > x_N) \exp(i\sum_{i=1}^{N} p_i x_i)\times$$
$$\times Q_1^\star(x_1).....Q_1^\star(x_N)|0\rangle, \tag{2.7.25}$$
$$|\mu_1, ...\mu_N\rangle = \int dx_1..dx_N \theta(x_1 > ..... > x_N) \exp(i\sum_{i=1}^{N} q_i x_i)\times$$
$$\times Q_3^\star(x_1).....Q_3^\star(x_N)|0\rangle, \tag{2.7.26}$$
$$|\lambda_1 + \mu_1, ...\lambda_N + \mu_N\rangle = \int dx_1...dx_N \theta(x_1 > ..... > x_N)\times$$
$$\times \exp(i\sum_{i=1}^{N}(p_i + q_i)x_i) Q_2^\star(x_1).....Q_2^\star(x_N)|0\rangle, \tag{2.7.27}$$

with $\theta(x_1 > x_2... > x_N) = \theta(x_1 - x_2)\theta(x_2 - x_3)...\theta(x_{N-1} - x_N)$ where

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0 \end{cases}$$

and

$$p_j = (c_2 - c_3)\lambda_j, \qquad q_j = (c_1 - c_2)\mu_j.$$

Similarly one can think of states with more than one kind of particle, for example, the state,

$$|\lambda_1, \lambda_2 + \mu_1, \mu_2\rangle = \int\int dx_1 dx_2 dx_3 \theta(x_1 > x_2 > x_3)\times$$

$$\exp\{i[p_1x_1 + (p_2 + q_1)x_2 + q_3x_3]\}Q_1^\star(x_1)Q_2^\star(x_2)Q_3^\star(x_3)|0\rangle. \quad (2.7.28)$$

The Hamiltonian commutes with the number operators,

$$\hat{M} = \int dx(Q_1^\star Q_1 + Q_2^\star Q_2), \qquad \hat{N} = \int dx(Q_2^\star Q_2 + Q_3^\star Q_3).$$

The entire analysis can be repeated as before and it is possible to derive the detailed structure of the energy and momenta eigenvalues for a particular state. The details are very elaborate and we refer the reader to the original articles [35].

In this chapter we have discussed the original method for analysis of a nonlinear quantum mechanical problem. It involves essentially treating the nonlinear equation as an operator equation, and writing a suitable Hamiltonian from which the equation is derivable, through use of appropriate commutation relations. The next step is to set up the eigenvalue problem $H\psi = E\psi$, which leads to the generalized form of the Bethe equations. These are then explicitly solved, whereupon imposition of periodic boundary conditions leads to the equation for the eigenmomenta. In this respect a remarkable result is that of the derivative nonlinear Schrödinger problem. Even in cases where fermions are involved there is essentially no difficulty in applying this method. Of course, in each case, solving the eigenmomenta equation poses a tricky problem, and the usual procedure is to convert it to an integral equation. The solution of the latter constitutes a separate problem in itself. The efficacy of the coordinate Bethe ansatz can be ascertained by looking at its applications to systems other than the δ-function Bose gas or the NLS equation.

However, it is necessary to also comment on the limitations of the method described in this chapter. Notably all the models treated so far are in one space and one temporal dimension. A higher dimensional analog of the Bethe ansatz is yet to emerge. Attempts have also been made to look into the application of the Bethe ansatz by employing the perturbative techniques of quantum field theory. In this respect the works of Thacker et al. are worth mentioning [36]. A similar but more rigorous approach was adopted by Gutkhin [37], in which the concept of intertwining operators were considered. These are operators that yield the solution of interacting systems, from that of the free one. However, till now it has been tested only in the case of the Heisenberg spin chain [38] and the δ-function Bose gas, where the solution is already possible by the the standard Bethe ansatz. Actually, the construction of

intertwining operators is a nontrivial task. It is important to note that although the equations studied are all completely integrable, yet there is no need to invoke the Lax equation. In the following chapters we will show how the Lax equation and its associated inverse scattering can be used and subsequently quantized to set up another variant of the Bethe ansatz, to elegantly solve for the spectrum of the nonlinear quantum mechanical problem. The method, which is algebraic in character, is known as the algebraic Bethe ansatz.

Chapter 3

Yang-Baxter Equation

3.1 Introduction

In the previous chapter we introduced the coordinate Bethe ansatz, which is a fundamental tool for the exact solution of a large number of quantum mechanical many-body problems. We also introduced the notion of the quantum R matrix, a central object in the modern approach to studies of quantum integrable systems deriving in particular a basic equation satisfied by it, namely the Yang-Baxter equation (YBE).

In this chapter we will analyze the Yang-Baxter equation in greater detail, discussing its origin and structure, the nature of its solutions, their properties and also its connection with exactly solvable models of statistical mechanics and the quantum inverse problem in general.

3.2 General Description

The Yang-Baxter equation is a matrix equation, defined in the tensor product of three complex vector spaces $V_1 \otimes V_2 \otimes V_3$ and given by [39]:

$$R^{V_1V_2}(u-v)R^{V_1V_3}(u)R^{V_2V_3}(v) = R^{V_2V_3}(v)R^{V_1V_3}(u)R^{V_1V_2}(u-v). \quad (3.2.1)$$

Each matrix R is assumed to act only on two spaces of the direct product as indicated by the superscripts and as an identity in the remaining space. We introduce the Chevalley basis e_{ij}, which represents a matrix of appropriate dimensions, with unity as the only nonvanishing entry at the intersection of the ith row and the jth column so that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. One can then write

$$R^{V_1V_2}(u) = \left[R^{V_1V_2}(u)\right]_{kl}^{ij} e_{ij}^{V_1} \otimes e_{kl}^{V_2} \otimes I,$$

$$R^{V_1V_3}(v) = \Big[R^{V_1V_3}(v)\Big]^{pq}_{rs}\, e^{V_1}_{pq} \otimes I \otimes e^{V_3}_{rs},$$

$$R^{V_2V_3}(w) = \Big[R^{V_2V_3}(w)\Big]^{ab}_{cd}\, I \otimes e^{V_2}_{ab} \otimes e^{V_3}_{cd}, \tag{3.2.2}$$

so that the Yang-Baxter equation in terms of its elements is as follows:

$$\sum_{\alpha\beta\gamma} \Big[R^{V_1V_2}(u-v)\Big]^{a\alpha}_{c\beta} \Big[R^{V_1V_3}(u)\Big]^{\alpha b}_{e\gamma} \Big[R^{V_2V_3}(v)\Big]^{\beta d}_{\gamma f}$$

$$= \sum_{\alpha\beta\gamma} \Big[R^{V_2V_3}(v)\Big]^{c\alpha}_{e\beta} \Big[R^{V_1V_3}(u)\Big]^{a\gamma}_{\beta f} \Big[R^{V_1V_2}(u-v)\Big]^{\gamma b}_{\alpha d}. \tag{3.2.3}$$

For the time being, let us assume that the V_i's $(i = 1, 2, 3)$ are identical vector spaces of dimension, say N. It is pertinent to mention here that although we shall be following the above notation for the Yang-Baxter equation, there exists in the literature an alternative notation, which, too, is in wide use. Instead of R as introduced above, one defines a matrix $\hat{R}$ by a left multiplication of R by the permutation operator $\mathcal{P}$:

$$\hat{R}(u) = \mathcal{P}R(u) \quad \text{where} \quad \mathcal{P}^{ij}_{kl} = \delta_{il}\delta_{jk}. \tag{3.2.4}$$

In terms of $\hat{R}(u)$, the Yang-Baxter equation assumes the form,

$$(I\otimes\hat{R}(u-v))(\hat{R}(u)\otimes I)(I\otimes\hat{R}(v)) = (\hat{R}(v)\otimes I)(I\otimes\hat{R}(u))(\hat{R}(u-v)\otimes I). \tag{3.2.5}$$

The complex variable u occurring in the argument of the R matrix is called the spectral parameter. However, in general a solution of the Yang-Baxter equation, to be called a *Yang-Baxter sheaf* may depend on additional parameters, which are called *connection constants*. In our discussion, we will consider R to be a function of a single connection constant, besides the spectral parameter, for the sake of simplicity. Consequently, one has not an isolated sheaf but instead a family of sheaves $R(u, \eta)$, depending on the connection constant η. At this point it is imperative to introduce a few definitions. We say that the Yang-Baxter sheaf $R(u, \eta)$ is

(i) **Regular** if the following condition holds: $R(u = 0, \eta) = \mathcal{P}$. (3.2.6)

(ii) **Quasi-classical** if there exists $\eta = \eta_c$ so that $R(u, \eta_c) = I$, (3.2.7)

where I is the identity operator in $V \otimes V$. Furthermore, there exist the following symmetry transformations.

(iii) **Homothetic**: It is obvious from the Yang-Baxter equation that if $R(u)$ is a solution, then $R'(u) = f(u)R(u)$, where $f(u)$ is an arbitrary scalar function, will also be a solution. One then says that the sheaves $R(u)$ and $R'(u)$ are homothetic.
(iv) If T is some nondegenerate operator in V, then the sheaf

$$R'(u) = (T \otimes T)R(u)(T \otimes T)^{-1} \text{ is a solution of the YBE.} \quad (3.2.8)$$

Two homothetic sheaves that are connected by the above similarity transformation are said to be equivalent.
Furthermore, if in the space V the representation $T(g)$ of some group G acts, then one says that the sheaf $R(u)$ is invariant with respect to the representation $T(g)$, if for any $g \in G$ the following holds:

$$R(u)(T(g) \otimes T(g)) = (T(g) \otimes T(g))R(u). \quad (3.2.9)$$

Product of sheaves:
Let $R^{(1)}(u)$ and $R^{(2)}(u)$ be two solutions of the Yang-Baxter equation of dimensions N_1 and N_2, respectively. The tensor product of the sheaves, $R^{(1)}(u)$ and $R^{(2)}(u)$, means the sheaf $(R^{(1)} \otimes R^{(2)})(u)$ of dimension $N_1 \times N_2$ defined by

$$(R^{(1)} \otimes R^{(2)})(u) = R^{(1)}(u) \otimes R^{(2)}(u). \quad (3.2.10)$$

Direct sum:
The direct sum of the sheaves $R^{(1)}(u)$ and $R^{(2)}(u)$ is the sheaf $(R^{(1)}(u)+ R^{(2)}(u))$ of dimension $N_1 + N_2$ defined as follows. Let $e_\alpha \otimes e_\beta$ denote basis vectors so that

$$R(u)(e_\gamma \otimes e_\delta) = (e_\alpha \otimes e_\beta)R^{\alpha\beta}_{\gamma\delta}(u).$$

The operator $(R^{(1)}(u) + R^{(2)}(u))$ acts on the basis vectors of the form,

$$e^{(i)}_{\alpha_i} \otimes e^{(k)}_{\alpha_k} \ \ (i,k = 1,2;\ \ \alpha_i = 1,2...N_i;\ \ e^{(i)}_{\alpha_i} \in V_i),$$

of the space $(V_1 + V_2) \otimes (V_1 + V_2)$ by

$$(R^{(1)}(u) + R^{(2)}(u))(e^{(1)}_{\alpha_1} \otimes e^{(1)}_{\beta_1}) = R^{(1)}(u)(e^{(1)}_{\alpha_1} \otimes e^{(1)}_{\beta_1}),$$

$$(R^{(1)}(u) + R^{(2)}(u))(e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\beta_2}) = (e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\beta_2}),$$

$$(R^{(1)}(u) + R^{(2)}(u))(e^{(2)}_{\alpha_2} \otimes e^{(1)}_{\beta_1}) = (e^{(2)}_{\alpha_2} \otimes e^{(1)}_{\beta_1}),$$

$$(R^{(1)}(u) + R^{(2)}(u))(e^{(2)}_{\alpha_2} \otimes e^{(2)}_{\beta_2}) = R^{(2)}(u)(e^{(2)}_{\alpha_2} \otimes e^{(2)}_{\beta_2}). \qquad (3.2.11)$$

It is seen by direct verification that both $(R^{(1)} \otimes R^{(2)})(u)$ and $(R^{(1)} + R^{(2)})(u)$ satisfy the Yang-Baxter equation (3.2.1). It should be mentioned that, while the tensor product of Yang-Baxter sheaves preserves the regular nature of families of sheaves, the operation of addition preserves only their quasi-classicalism. Moreover, from (3.2.11) it is evident that the direct sum of two sheaves is never a regular sheaf. If the space V admits a decomposition into a direct product of two subspaces V_1 and V_2, so that the action of the operator $R(u)$ on a basis vector $e^{(i)}_{\alpha_i} \otimes e^{(k)}_{\beta_k}$ has the property,

$$R(u)(e^{(i)}_{\alpha_i} \otimes e^{(k)}_{\beta_k}) \in V_i \otimes V_k, \qquad i, k = 1, 2;$$

then we say that the sheaf $R(u)$ is *reducible*. A reducible sheaf is therefore always the direct sum of two sheaves. If such a decomposition does not exist, then the sheaf is said to be *irreducible*. For a reducible sheaf $R(u)$ the operators $R^{(1)}(u)$ and $R^{(2)}(u)$ acting in the spaces $V_1 \otimes V_1$ and $V_2 \otimes V_2$, respectively, according to the formula,

$$R^{(i)}(e^{(i)}_{\alpha_i} \otimes e^{(i)}_{\beta_i}) = R(u)(e^{(i)}_{\alpha_i} \otimes e^{(i)}_{\beta_i}), \quad i = 1, 2, \qquad (3.2.12)$$

will also be Yang-Baxter sheaves. We shall now discuss the Yang-Baxter equation and its generalizations to include quadratic algebras, the so-called *Yang-Baxter algebras*. The same has been extensively used for analysis of quantum groups in recent times. One can identify four major areas where the Yang-Baxter equation and the Yang-Baxter algebras have appeared:

(i) One-dimensional quantum chains, for example, the Heisenberg spin chain, Toda lattice, etc.

(ii) Factorizable scattering in the $1 + 1$ dimension.

(iii) Statistical lattice models in two dimensions.

(iv) Braid groups.

Let us confine ourselves, for the time being, to quantum models on a lattice. An important feature of any regular Yang-Baxter sheaf is that one can associate it to a quantum completely integrable system with locally commuting integrals of motion. To understand how this is accomplished, consider a regular Yang-Baxter sheaf $R(u)$ of dimension N. The state space $\mathcal{A}$ of the required quantum system is assumed to be given by

$$\mathcal{A} = V_1 \otimes V_2 \otimes ... \otimes V_M, \qquad (V_n = \mathcal{C}^n; n = 1, 2, ...M),$$

where M is an arbitrary natural number ≥ 2. In this space $\mathcal{A}$ is defined a Hamiltonian H, where

$$H = \sum_{n=1}^{M-1} \mathcal{H}_{n+1,n} + \mathcal{H}_{1,M}, \tag{3.2.13}$$

with the local Hamiltonian density $\mathcal{H}_{n+1,n}$ given by

$$\mathcal{H}_{n+1,n} = \left(\frac{d}{du} R_{n+1,n}(u)|_{u=0} \right) \mathcal{P}_{n+1,n}. \tag{3.2.14}$$

The operator $R_{n+1,n}(u)$ acts in the space $V_{n+1} \otimes V_n$ in the corresponding Yang-Baxter sheaf $R(u)$ and on the remaining components of $\mathcal{A}$ as an identity. Similar remarks hold for the permutation operator $\mathcal{P}_{n+1,n}$. One can then view the quantum system, as a chain of M "atoms" each of which has N quantum states and interacts only with its nearest neighbours. To construct a sequence of operators commuting with H as defined in (3.2.14), we consider an extended space $\hat{\mathcal{A}} = Q \otimes Q' \otimes \mathcal{A}$, where the two auxillary spaces Q and Q' are isomorphic with $\mathcal{C}^N$. We define the transition operator $T_1^M(u)$ by

$$T_1^M(u) = L_M(u) L_{M-1}(u) .... L_1(u) = \prod_{n=1}^{\overleftarrow{M}} L_n(u), \tag{3.2.15}$$

where $L_n(u) \equiv R_{qn}(u)$, in which the index n is related to the space V_n, and q to the space Q. Replacing Q by Q' one can similarly define the quantities $T_1'^M(u)$ and $L_n'(u)$. With this notation one can rewrite the Yang-Baxter equation (3.2.1) in the following manner [40]:

$$R_{qq'}(u-v) L_n(u) L_n'(v) = L_n'(v) L_n(u) R_{qq'}(u-v). \tag{3.2.16}$$

We shall now present an important theorem in this context.
Theorem: If the matrix $R_{qq'}(u-v)$ intertwines the operators $L_n(u)$ and $L_n'(v)$ as given in (3.2.16), then it also intertwines the transition matrix $T_1^M(u)$, so that

$$R_{qq'}(u-v) T_1^M(u) T_1'^M(v) = T_1'^M(v) T_1^M(u) R_{qq'}(u-v). \tag{3.2.17}$$

Proof: Let us assume that the theorem is true for $M = j$ so that

$$R_{qq'}(u-v) T_1^{j+1}(u) T_1'^{j+1}(v) = R_{qq'}(u-v) L_{j+1}(u) L_{j+1}'(v) \times$$

$$\times R^{-1}_{qq'} R_{qq'}(u-v) T_1^j(u) T_1{}' j(v) R^{-1}_{qq'} R_{qq'}(u-v),$$

$$= L'_{j+1}(v) L_{j+1}(u) T_1'^j(v) T_1^j(u) R_{qq'}(u-v);$$

We have used (3.2.16):

$$= T_1'^{j+1}(v) T_1^{j+1}(u) R_{qq'}(u-v).$$

Thus the theorem is true for $M = j+1$ if it is true for $M = j$. Hence the proof.
Now, the generating function $t(u)$ of the integrals of motion of the quantum system under consideration is defined as the trace of the transition operator $T_1^M(u)$ taken with respect to the auxiliary space Q i.e.,

$$t(u) = \mathrm{tr}_q T_1^M(u). \tag{3.2.18}$$

It follows from (3.2.17), upon taking trace over both Q and Q' (and using the cyclic property of trace operation), that the generating function $t(u)$ forms a commuting set of operators in $\mathcal{A}$:

$$[t(u), t(v)] = 0, \quad \forall\ u, v \quad (u \neq v). \tag{3.2.19}$$

The generating functional of the local integrals of motion for the Hamiltonian H is often given by $\ln(t^{-1}(0)t(u))$ [41]:

$$\mathcal{I}_n = \frac{d^n}{du^n} \ln(t^{-1}(0)t(u))|_{u=0}. \tag{3.2.20}$$

Locality in the present context means that the operator $\mathcal{I}_n$ may be represented as a sum of operators, each of which acts nontrivially at no more than $n+1$ neighbouring nodes of the lattice.

It is also important to dwell on the aspect of completeness of the systems of integrals $\mathcal{I}_n$ in the space $\mathcal{A}$. While this is an open problem in many respects, it has been proved in case of the Heisenberg ferromagnet that these integrals do in fact form a complete set [42]. It is plausible to expect that for $N = 2$ at least these integrals of motion form a complete set. It should be noted that the Yang-Baxter equation, in addition to allowing for the construction of the integrals of motion of a quantum model on a lattice, also allows us to find the eigenfunctions of the Hamiltonian H and to determine the spectrum of H. A systematic procedure for determining these is through the algebraic Bethe ansatz, which will be described in Chapter 5.

As mentioned earlier, the operator R, besides being a function of the spectral parameter u, may depend on additional variables. However for the sake of brevity we assume there is only a single parameter η. This variable takes the same value for all R matrices, while the spectral parameter u takes different but relevant values for different R matrices. The physical meanings of the parameters u and η would depend on the explicit model under investigation. However, in certain cases it is possible to perform the passage to the limit for η, as a result of which one gets a continuous completely integrable model of quantum field theory on a line. The most notable examples of the performance of such a limiting procedure is the XYZ model from which one gets the nonlinear Schrödinger equation [43].

3.3 Factorized Scattering

To discuss the Yang-Baxter equation in the context of factorized scattering, we note that the process is characterized by the following general features.
(i) There is no particle production; the number of particles in the incoming state is the same as those in the outgoing state. Furthermore, the set of outgoing momenta is the same as the set of incoming momenta. However, the particles may exchange momenta in the course of collisions.
(ii) The M-particle S matrix can be represented as a product of $M(M-1)/2$ two-particle S matrices, so that mathematically the process of M-particle scattering is reduced to a sequence of two-particle collisions.

In two dimensions, there are several quantum field theoretic models that exhibit these features, thus allowing for explicit realizations of the factorized S matrix. In such systems, the essential feature is the existence of a two-particle scattering matrix, as all the amplitudes can be expressed as a product of these basic building blocks. Essentially, the elastic nature of the collisions and factorizability arise due to the existence of additional conserved charges, apart from the usual ones like energy, momentum, electric charge, etc. These may either be local or nonlocal. In most cases of interest, there exist an infinite number of "charges" for the integrable field theories. To understand how the factorizability of the S matrix leads to the Yang-Baxter equation, let

us consider the case of $M = 3$ body scattering [44].

Let the initial three-particle state (labelled by the momenta) be specified by $p_1 > p_2 > p_3$. The incoming and/or outgoing particles are in one of the N different internal states designated by i and j, $1 \leq i, j \leq N$. It is possible to parametrize the energy-momenta of the relativistic particles by their rapidity θ, in terms of which the momenta $p_a = m(\cosh\theta_a, \sinh\theta_a)$, in two dimensions. We assume the particles to be of equal mass and use θ_a to denote the rapidity of the ath particle ($a = 1, 2, 3$). Owing to the elastic nature of the scattering, while the internal state indices i and j may undergo changes in the course of collisions, the rapidities do not change. Moreover, due to Lorentz invariance, the S matrix R_{ab} $(a, b = 1, 2, 3)$ being an $N^2 \times N^2$ matrix of the internal indices, is a function only of the difference of the rapidities $\theta_a - \theta_b$. Since in the final state, there are also three particles with the same set of momenta $\{p_1, p_2, p_3\}$, there are three possible collisions, which are depicted in Figure (3.3.1).

In Figure (3.3.1a) and Figure (3.3.1c) the collision proceeds two at a time while in Figure (3.3.1b) there is an intrinsic three-body scattering. According to the particle-displacement argument [45], because of the existence of conserved charges, say $Q_n (n = 1, 2...)$, one can act on the entire initial state by $\exp(i\xi\hat{Q}_N)$, which causes a displacement of each particle by amounts proportional to their momentum. By allowing the

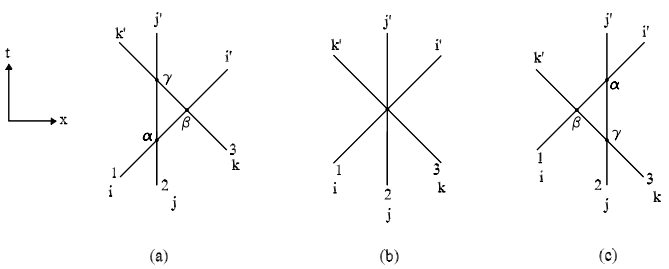

FIGURE 3.3.1: Schematic diagram of possible three-particle collisions.

real parameter ξ to become arbitrarily large one can cause an arbitrary change in the relative position of any two particles. Thus by displacing the line labelled 1 parallel to itself, relative to 2 and 3, one can go from Figure (3.3.1a) to Figure (3.3.1c) continuously. The consequences of the particle displacement argument are far reaching. As Q_N is a conserved charge, it commutes with the Hamiltonian and thus leaves the S matrix invariant. Consequently, the amplitude for all the three processes depicted in Figure (3.3.1a) to Figure (3.3.1c) are the same. Moreover, the processes in Figure (3.3.1a) to Figure (3.3.1c) can be written in terms of the two-body scattering matrices R_{ab}, and since the amplitudes are the same, one has

$$\begin{aligned} R_{123}(\theta_1, \theta_2, \theta_3) &= R_{12}(\theta_1 - \theta_2) R_{13}(\theta_1 - \theta_3) R_{23}(\theta_2 - \theta_3) \\ &= R_{23}(\theta_2 - \theta_3) R_{13}(\theta_1 - \theta_3) R_{12}(\theta_1 - \theta_2). \end{aligned} \tag{3.3.1}$$

Thus, in addition to the factorization of the three-body scattering matrix, it is seen that the individual two-body scattering matrices obey a cubic identity. This cubic identity is of fundamental importance, for it has been shown that even for higher many-body amplitudes the same cubic identity provides the sufficiency condition for factorized scattering [46]. Equation (3.3.1) essentially represents the Yang-Baxter equation in the context of factorized scattering. In the case of relativistic models, such as the Sine-Gordon model, and nonlinear σ-model, Karowski et al. [47] and Zamolodchikov and Zamolodchikov [48] have obtained exact expressions for the S matrices, by solving functional equations derived from certain general considerations like unitarity, crossing symmetry and factorization. We shall however not go into the details of these calculations, but refer the reader to the original articles.

As the majority of completely integrable models are strictly nonrelativistic in nature, it would be appropriate to discuss a nonrelativistic theory of such factorized S matrices. The following discussion is based on the work of Sogo et al. [49]. It may be mentioned here that space-time symmetries impose the following requirements on the S matrix:
(a) Time-reversal invariance $S(\theta) = S^T(\theta)$.
(b) Parity-inversion invariance $S(\theta) = PS(\theta)P$.
(c) In the absence of sources or sinks of particles, unitarity condition requires $S(\theta)S^\dagger(\theta) = I$.
(d) Real analyticity $S^\star(\theta) = S(-\theta)$.
(e) Crossing invariance $S^{cd}_{ab}(\theta) = S^{cb}_{ad}(i\pi - \theta)$.

(f) Crossing symmetry $S^{t_1}(\theta) = (I \otimes W)S(-\theta - \eta)(I \otimes W^{-1})$, where t_1 denotes transposition in the first horizontal space, and W is a constant matrix while η is a model-dependent parameter.

A nonrelativistic theory of factorized S matrix may be constructed by dispensing with the crossing relations and replacing the rapidity variable θ by the velocity p/m. Furthermore, the imposition of Z_4-symmetry, which is by far the most general form of symmetry, may be shown to lead — in the nonrelativistic case — to the nonlinear Schrödinger model, i.e., the δ-function Bose gas.

The S matrix is basically the transformation operator connecting the incoming and outgoing states, that is,

$$|\text{in}\rangle = \hat{S}|\text{out}\rangle. \tag{3.3.2}$$

The $|\text{in}\rangle(|\text{out}\rangle)$ state is one in which all particles are arranged spatially in order of increasing (deceasing) momenta. Consider a pair of particles $A, \bar{A}$, which may either be a particle-antiparticle pair or a spin up-spin down pair, and assume that they are connected by an operator $\hat{C}$ in such a way that

$$\hat{C}|A\rangle = |\bar{A}\rangle, \qquad \hat{C}|\bar{A}\rangle = |A\rangle. \tag{3.3.3}$$

For a two-particle basis state, we may consider the following: $|AA\rangle$, $|\bar{A}\bar{A}\rangle$, $|A\bar{A}\rangle$ and $|\bar{A}A\rangle$. Consequently, the two-body S matrix elements are given by

$$\begin{pmatrix} |A(\theta_1)A(\theta_2)\rangle \\ |\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle \\ |A(\theta_1)\bar{A}(\theta_2)\rangle \\ |\bar{A}(\theta_1)A(\theta_2)\rangle \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix} \begin{pmatrix} |\bar{A}(\theta_2)A(\theta_1)\rangle \\ |\bar{A}(\theta_2)\bar{A}(\theta_1)\rangle \\ |A(\theta_2)\bar{A}(\theta_1)\rangle \\ |\bar{A}(\theta_2)A(\theta_1)\rangle \end{pmatrix}. \tag{3.3.4}$$

The general Z_N operator is defined by

$$Z_N|A\rangle = e^{i\phi}|A\rangle, \qquad Z_N|\bar{A}\rangle = e^{-i\phi}|\bar{A}\rangle, \tag{3.3.5}$$

where $\phi = 2\pi/N$. Invariance, under $\hat{C}$ and Z_N symmetry, requires that

$$[\hat{C}, \hat{S}] = 0, \tag{3.3.6}$$

$$[Z_N, \hat{S}] = 0. \tag{3.3.7}$$

In addition the C invariance implies that

$$S_{ij} = S_{ji}, \qquad i, j = 1, 2, 3, 4, \tag{3.3.8}$$

while Z_N symmetry gives rise to the following conditions:

$$\langle A\bar{A}|\hat{S}|AA\rangle \sin\phi = 0,\ \ \langle \bar{A}A|\hat{S}|AA\rangle \sin\phi = 0,\ \ \langle \bar{A}\bar{A}|\hat{S}|AA\rangle \sin(2\phi) = 0. \tag{3.3.9}$$

Hence, for $N = 4$, we have $\phi = \frac{\pi}{2}$ and as a result the two-body S matrix assumes the following form:

$$\hat{S}^{(2)}(\theta) = \begin{pmatrix} S & S_a & 0 & 0 \\ S_a & S & 0 & 0 \\ 0 & 0 & S_r & s_t \\ 0 & 0 & S_t & S_r \end{pmatrix} \text{ for } Z_4.$$

Applying the same procedure to the three-particle S matrix, one can show that

$$\hat{S}^{(3)}(\theta) = \begin{pmatrix} \tilde{S}^{(3)} & 0 \\ 0 & \tilde{S}^{(3)} \end{pmatrix},$$

where the following basis may be chosen, $|AAA\rangle, |A\bar{A}\bar{A}\rangle, |\bar{A}A\bar{A}\rangle, |\bar{A}\bar{A}A\rangle, |\bar{A}\bar{A}\bar{A}\rangle, |\bar{A}AA\rangle, |A\bar{A}A\rangle$ and $|AA\bar{A}\rangle$ in this order; $\tilde{S}^{(3)}$ being a 4×4 matrix. As the unitarity condition requires

$$\hat{S}^{(2)}(\theta)\hat{S}^{(2)}(-\theta) = I, \tag{3.3.10}$$

it gives rise to the following:

$$\begin{aligned} S(\theta)S(-\theta) + S_a(\theta)S_a(-\theta) &= 1, \\ S(\theta)S_a(-\theta) + S_a(\theta)S(-\theta) &= 0, \\ S_r(\theta)S_r(-\theta) + S_t(\theta)S_t(-\theta) &= 1, \\ S_r(\theta)S_t(-\theta) + S_t(\theta)S_r(-\theta) &= 0. \end{aligned} \tag{3.3.11}$$

In addition, one has for systems with an infinite number of integrals of motion, the factorization equation,

$$\begin{aligned} &\hat{S}^{(3)}((1)23, \theta_{23})\hat{S}^{(3)}(13(2), \theta_{13})\hat{S}^{(3)}((3)12, \theta_{12}) \\ &= \hat{S}^{(3)}(12(3), \theta_{12})\hat{S}^{(3)}((2)13, \theta_{13})\hat{S}^{(3)}(23(1), \theta_{23}), \end{aligned} \tag{3.3.12}$$

where $\theta_{ij} = \theta_i - \theta_j$. Here

$$\hat{S}^{(3)}(ij(k),\theta) = \begin{pmatrix} \tilde{S}^{(3)}(ij(k),\theta) & 0 \\ 0 & \tilde{S}^{(3)}(ij(k),\theta) \end{pmatrix}, \tag{3.3.13}$$

$$\text{and} \quad \hat{S}^{(3)}((k)ij,\theta) = \begin{pmatrix} \tilde{S}^{(3)}((k)ij,\theta) & 0 \\ 0 & \tilde{S}^{(3)}((k)ij,\theta) \end{pmatrix}, \tag{3.3.14}$$

$$\text{with} \qquad \tilde{S}^{(3)}(ij(k);\theta) = \begin{pmatrix} S & 0 & 0 & S_a \\ 0 & S_r & S_t & 0 \\ 0 & S_t & S_r & 0 \\ S_a & 0 & 0 & S \end{pmatrix}, \tag{3.3.15}$$

$$\text{and} \qquad \tilde{S}^{(3)}((k)ij;\theta) = \begin{pmatrix} S & S_a & 0 & 0 \\ S_a & S & 0 & 0 \\ 0 & 0 & S_r & S_t \\ 0 & 0 & S_t & S_r \end{pmatrix}. \tag{3.3.16}$$

$\hat{S}^{(3)}(ij(k);\theta)$, actually means that particles ij collide, while k remains to the right of the collision, and $\hat{S}^{(3)}((k)ij;\theta)$ is similar but with k remaining to the left. From the preceding cubic identity (3.3.12) we obtain the following relations:

$$\begin{aligned} SS_aS_r + S_aS_tS_t &= S_aSS + SS_rS_a, \\ SS_aS_t + S_aS_tS_r &= S_rS_tS_a + S_tS_aS, \\ S_rSS_r + S_tS_rS_t &= S_aSS_a + SS_rS, \\ S_rSS_t + S_tS_rS_r &= S_rS_tS + S_tS_aS_a, \end{aligned} \tag{3.3.17}$$

where the arguments of the S's are implied to be θ_{12}, θ_{13} and θ_{23} in that order, or for greater convenience $\theta, \theta+\theta', \theta'$ after denoting that $\theta = \theta_{12}$ and $\theta' = \theta_{23}$.

If, instead of $|A\rangle$ and $|\bar{A}\rangle$, one employs the doublet

$$\begin{aligned} |A\rangle &= (|A_1\rangle + i|A_2\rangle)/\sqrt{2} \\ |\bar{A}\rangle &= (|A_1\rangle - i|A_2\rangle)/\sqrt{2} \end{aligned} \tag{3.3.18}$$

and takes as basis $|A_1A_1\rangle, |A_2A_2\rangle, |A_1A_2\rangle$ and $|A_2A_1\rangle$, then it leads to the following form of $\hat{S}^{(2)}(\theta)$:

$$\hat{S}^{(2)}(\theta) = \begin{pmatrix} \sigma & \sigma_a & 0 & 0 \\ \sigma_a & \sigma & 0 & 0 \\ 0 & 0 & \sigma_r & \sigma_t \\ 0 & 0 & \sigma_t & \sigma_r \end{pmatrix},$$

where

$$\begin{aligned} 2\sigma &= S + S_a + S_t + S_r, \\ 2\sigma_a &= S_t + S_r - S - S_a, \\ 2\sigma_r &= S - S_a + S_r - S_t, \\ 2\sigma_t &= S - S_a + S_t - S_r. \end{aligned} \tag{3.3.19}$$

The problem now is to solve the functional equations (3.3.11) and (3.3.17). For the nonrelativistic case one need not consider the crossing relations for the nonrelativistic case. Instead, from (3.3.17) we have

$$\begin{aligned} S(\theta) &= \frac{\mathrm{sn}(\mu\theta + 2\xi, l)}{\mathrm{sn}(2\xi, l)} S_r(\theta), \\ S_t(\theta) &= -\frac{\mathrm{sn}(\mu\theta, l)}{\mathrm{sn}(2\xi, l)} S_r(\theta), \\ S_a(\theta) &= -l\mathrm{sn}(\mu\theta, l)\mathrm{sn}(\mu\theta + 2\xi, l) S_r(\theta), \end{aligned} \tag{3.3.20}$$

where l is the modulus of Jacobi's elliptic function $\mathrm{sn}(x, l)$, while μ, ξ are arbitrary constants. Substitution in the unitary relations gives

$$S_r(\theta)S_r(-\theta) = \frac{\mathrm{sn}^2(2\xi, l)}{\mathrm{sn}^2(2\xi, -l) - \mathrm{sn}^2(\mu\theta, l)}. \tag{3.3.21}$$

This equation can be uniquely solved, except for the well-known CDD ambiguity [50]. Setting $f(\theta) = \ln S_r(\theta)$, we can rewrite it as

$$f(\theta) + f(-\theta) = \sum_{n=1}^{\infty} \frac{16q^n}{n(1-q^{2n})} \sin^2\left[\frac{\pi n(2\xi + ik')}{2k}\right] \sin^2\left[\frac{\pi n\mu\theta}{2k}\right], \tag{3.3.22}$$

where k and k' are the complete elliptical integrals of the first kind with moduli l and $l' = \sqrt{1-l^2}$, respectively, and q is defined by

$$q = \exp(-\pi k'/k). \tag{3.3.23}$$

Equation (3.3.22) may be solved under the condition that $S_r(\theta)$ is meromorphic, to obtain

$$f(\theta) = 8\sum_{n=1}^{\infty} \frac{\sin^2\left[\frac{\pi n(2\xi+ik')}{2k}\right]}{n\sinh\left[\frac{\pi nk'}{k}\right]} f_n(\theta), \tag{3.3.24}$$

with

$$f_n(\theta) = \frac{\sin\left[\frac{\pi n\mu\theta}{2k}\right]\sin\left[\frac{\pi n(2\xi+\mu\theta)}{2k}\right]}{2\cos\left[\frac{\pi n\xi}{k}\right]}. \tag{3.3.25}$$

Thus the function $S_r(\theta)$ is given by

$$S_r(\theta) = \exp\left[4\sum_{n=1}^{\infty} \frac{\sin^2\left[\frac{\pi n(2\xi+ik')}{2k}\right]\sin\left[\frac{\pi n\mu\theta}{2k}\right]\sin\left[\frac{\pi n(2\xi+\mu\theta)}{2k}\right]}{n\sinh\left[\frac{\pi nk'}{k}\right]\cos\left[\frac{\pi n\xi}{k}\right]}\right]. \tag{3.3.26}$$

Having determined $S_r(\theta)$, the remaining elements of the S matrix can be obtained from (3.3.20), thus completing the determination of the nonrelativistic factorized S matrix with Z_4 symmetry. Note that the constants μ and ξ are independent.

It is interesting to consider certain special cases. For example, $l = 0$ corresponds to $S_a = 0$ and is interesting, as the functional equations (3.3.20) now reduce to sinusoidal functions given by

$$S(\theta) = \frac{\sin(\mu\theta + 2\xi)}{\sin 2\xi} S_r(\theta),$$

$$S_t(\theta) = -\frac{\sin(\mu\theta)}{\sin 2\xi} S_r(\theta), \tag{3.3.27}$$

$$\text{and } S_r(\theta)S_r(-\theta) + S_t(\theta)S_t(-\theta) = 1. \tag{3.3.28}$$

Substituting $S_t(\theta)$ into the last equation gives

$$S_r(\theta)S_r(-\theta) = \frac{\sin^2 2\xi}{\sin(2\xi+\mu\theta)\sin(2\xi-\mu\theta)}. \tag{3.3.29}$$

Assuming $S_r(\theta)$ to be meromorphic, and considering the gamma function formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

leads to the following solution for $S_r(\theta)$:

$$S_r(\theta) = -\frac{\sin 2\xi}{\sin \mu\theta} \frac{\Gamma(-\frac{2\xi+\mu\theta}{\pi})\Gamma(1+\frac{2\xi-\mu\theta}{\pi})}{\Gamma(-\frac{\mu\theta}{\pi})\Gamma(1-\frac{\mu\theta}{\pi})}. \tag{3.3.30}$$

Hence from (3.3.27), one finds

$$S_t(\theta) = \frac{\Gamma(-\frac{2\xi+\mu\theta}{\pi})\Gamma(1+\frac{2\xi-\mu\theta}{\pi})}{\Gamma(-\frac{\mu\theta}{\pi})\Gamma(1-\frac{\mu\theta}{\pi})}, \tag{3.3.31}$$

$$\text{and } S(\theta) = -\frac{\sin(\mu\theta+2\xi)}{\sin 2\xi} \frac{\Gamma(-\frac{2\xi+\mu\theta}{\pi})\Gamma(1+\frac{2\xi-\mu\theta}{\pi})}{\Gamma(-\frac{\mu\theta}{\pi})\Gamma(1-\frac{\mu\theta}{\pi})}. \tag{3.3.32}$$

It should be pointed out however that there still exists CDD ambiguity in the solution (3.3.30) of the unitary equation (3.3.29).

For subsequent analysis, it is convenient to recast $S_r(\theta)$ and $S(\theta)$, given by (3.3.30) and (3.3.32), respectively, in the following form:

$$S_r(\theta) = \frac{\Gamma(\frac{\mu\theta}{\pi})\Gamma(1+\frac{2\xi-\mu\theta}{\pi})\Gamma(-\frac{2\xi+\mu\theta}{\pi})}{\Gamma(-\frac{\mu\theta}{\pi})\Gamma(-\frac{2\xi}{\pi})\Gamma(1+\frac{2\xi}{\pi})}, \tag{3.3.33}$$

$$S(\theta) = \frac{\Gamma(\frac{\mu\theta}{\pi})\Gamma(1+\frac{2\xi-\mu\theta}{\pi})}{\Gamma(-\frac{\mu\theta}{\pi})\Gamma(1+\frac{2\xi+\mu\theta}{\pi})}. \tag{3.3.34}$$

Zamolodchikov and Zamolodchikov were the first to note the connection between the above S matrices and those of the Calogero system [51]. The latter is in general a completely integrable many-body system, described by the Wierstrass $\wp$ function. However, there is a special case of this system consisting of two species A and $\bar{A}$, with a Hamiltonian given by

$$H = -\frac{\hbar^2}{2m}\sum_A \frac{\partial^2}{\partial x_i^2} + \Sigma U_{AA}(x_i - x_j) - \frac{\hbar^2}{2m}\sum_{\bar{A}} \frac{\partial^2}{\partial y_i^2}$$

$$+\Sigma U_{\bar{A}\bar{A}}(y_i - y_j) + \Sigma U_{A\bar{A}}(x_i - y_j), \tag{3.3.35}$$

where

$$U_{AA}(x) = U_{\bar{A}\bar{A}}(x) = \frac{U_0}{\sinh^2(ax)}, \quad U_{A\bar{A}}(x) = -\frac{U_0}{\cosh^2(ax)}, \tag{3.3.36}$$

with U_0 and a being positive constants. Owing to the complete integrability of the model, the total S matrix can be determined by computing the two-body S matrix. Furthermore, the consequent two-body problem with interaction potential given by (3.3.36) may be solved analytically. In terms of relative coordinates, the problem involves solving the Schrödinger equation:

$$\left(-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2}+\frac{U_0}{\cosh^2 ax}\right)\Psi(x)=E\Psi(x). \tag{3.3.37}$$

Employing the transformation,

$$\xi=\tanh(ax),\ \ \epsilon=-i\frac{\sqrt{mE}}{\hbar a},\ \ s=\frac{1}{2}\left(\sqrt{1-\frac{4mU_0}{\hbar^2a^2}}-1\right), \tag{3.3.38}$$

allows (3.3.37) to be written as

$$\frac{d}{d\xi}(1-\xi^2)\frac{d\Psi}{d\xi}+\left\{s(s+1)-\frac{\epsilon^2}{1-\xi^2}\right\}\Psi=0. \tag{3.3.39}$$

Using

$$\Psi(x)=(1-\xi^2)^\nu w(\xi),\qquad \nu=\epsilon/2, \tag{3.3.40}$$

(3.3.39) reduces to

$$(1-\xi^2)\frac{d^2w}{d\xi^2}-2\xi(2\nu+1)\frac{dw}{d\xi}+(s-\epsilon)(s+\epsilon+1)w=0, \tag{3.3.41}$$

and upon making the identifications,

$$u=(1-\xi)/2,\ \ \alpha=\epsilon-s,\ \ \beta=\epsilon+s+1,\ \ \gamma=\epsilon+1, \tag{3.3.42}$$

it reduces to the familiar hypergeometric equation:

$$u(1-u)\frac{d^2w}{du^2}+\{\gamma-(\alpha+\beta+1)u\}\frac{dw}{du}-\alpha\beta w=0. \tag{3.3.43}$$

The scattering wave solution is therefore given by

$$\Psi(x)=(1-\xi^2)^\nu F(\alpha,\beta,\gamma;(1-\xi)/2), \tag{3.3.44}$$

and has the asymptotic nature:

$$\Psi(x)\sim 4^\nu\exp(-2\alpha\nu x)\quad\text{ as }\quad x\to+\infty, \tag{3.3.45}$$

$$\Psi(x) \sim 4^{\nu}\left[\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}e^{-2\alpha\nu x}+\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}e^{2\alpha\nu x}\right] \tag{3.3.46}$$

as $x \to -\infty$. Here we have used the connection formula for hypergeometric functions:

$$F(\alpha,\beta,\gamma;u)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-u)$$

$$+(1-u)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-u). \tag{3.3.47}$$

The S matrices are then given by

$$S_r=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(\alpha+\beta-\gamma)}, \tag{3.3.48}$$

$$S_t=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}. \tag{3.3.49}$$

Comparing these expressions with (3.3.33) and (3.3.34), it is found that both these S matrices are identical for

$$2\mu=i\frac{m\pi}{\hbar a}, \qquad 2\xi=\pi s. \tag{3.3.50}$$

An interesting result that emerges from this:

$$\int_{-\infty}^{\infty}\frac{U_0}{\cosh^2(ax)}dx=\frac{2U_0}{a};$$

when the limit $U_0, a \to +\infty$ with $U_0/a=\kappa$ (fixed) is taken, then the expressions for the S matrices reduce to

$$S_r(p)=\frac{-i\kappa/\hbar}{\frac{p}{2m}+i\kappa/\hbar}, \tag{3.3.51}$$

$$S_t(p)=\frac{\frac{p}{2m}}{\frac{p}{2m}+i\kappa/\hbar}, \quad S(p)=1, \tag{3.3.52}$$

using (3.3.50) and determining that $p=2\sqrt{mE}$ and $\Gamma(z)\sim\frac{1}{z}$ for $z\sim 0$. These expressions for S_r and S_t are similar to those obtained by Yang when considering the δ-function fermi gas, with the Hamiltonian

$$H=-\frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial x_i^2}-\frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial y_i^2}+2\kappa\sum_i\delta(x_i-y_i) \tag{3.3.53}$$

where $x_i(y_i)$'s are the coordinates of spin-up (down) particles [52]. In this context one regards $A(\bar{A})$ as a spin-up (down) state of fermions of spin 1/2.

The second interesting limit arises when the case $S_a = 0$ and $\sigma_a = 0$ is considered, which forbids the processes $AA \longleftrightarrow \bar{A}\bar{A}$ and $A_1A_1 \longleftrightarrow A_2A_2$, in which case $S = S_t + S_r$, as a result of (3.3.48 and 3.3.49). This case is realized when one takes the limit, $\mu \to 0, 2\xi \to -\pi$, where $(\pi + 2\xi)/\mu \equiv -2i\kappa/\hbar$ is fixed. The resulting S matrix is now

$$S(p) = \frac{\frac{p}{2m} - \frac{i\kappa}{\hbar}}{\frac{p}{2m} + \frac{i\kappa}{\hbar}}, \tag{3.3.54}$$

which matches the result of Lieb and Liniger for the δ-function Bose gas with the Hamiltonian [32]:

$$H = -\frac{\hbar^2}{2m}\sum_i \frac{\partial^2}{\partial x_i^2} + 2\kappa \sum_i \delta(x_i - x_j). \tag{3.3.55}$$

Thus, a nonrelativistic theory of factorized S matrices is not only possible, but also it reproduces certain well-known results, of different many-particle systems under various limiting conditions.

3.4 Baxter's Star-Triangle Relation

Baxter's star-triangle relation marks one of the most outstanding theoretical achievements in the analysis of exactly solvable lattice spin systems. The simplest way to understand this particular relation is to consider a relativistic scattering theory with n kinds of particles, all having identical masses in $1+1$ dimension. The two-particle scattering matrix is given by $S^{\beta_1\beta_2}_{\alpha_1\alpha_2}(\theta_{12})$, where α_1 and α_2 denote the incoming particles and β_1 and β_2 the outgoing ones. This is shown in Figure (3.4.1). As the particles are identical and of equal mass, the energy-momentum conservation relation and a pair of rapidities θ_1 and θ_2 remain unaltered during the scattering process. Furthermore, Lorentz-invariance requires that the S matrix be a function only of the relative rapidity $\theta_1 - \theta_2 = \theta_{12}$. A crucial observation here is if the system admits an infinite set of conserved quantities, then any set of individual

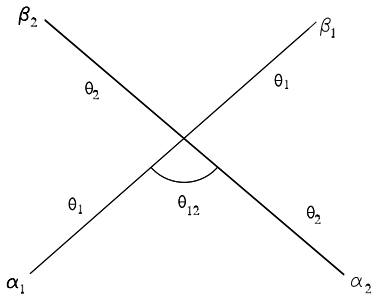

FIGURE 3.4.1: Schematic diagram of the two-particle scattering matrix.

rapidities will be conserved in a multiparticle collision. As a consequence, the resulting amplitude of the S matrix is independent of the sequence of two-particle collisions. This means that the three-particle S matrix element constructed from two-particle amplitudes in different ways must be equal [53]. Two such possibilities are depicted in the Figure (3.4.2). The equalities of the depicted amplitudes are known as the factorization equations and may be expressed as follows:

$$S^{\gamma_1\gamma_2}_{\alpha_1\alpha_2}(\theta_{12})S^{\beta_1\gamma_3}_{\gamma_1\alpha_3}(\theta_{13})S^{\beta_2\beta_3}_{\gamma_2\gamma_3}(\theta_{23}) = S^{\gamma_2\gamma_3}_{\alpha_2\alpha_3}(\theta_{23})S^{\gamma_1\beta_3}_{\alpha_1\gamma_3}(\theta_{13})S^{\beta_1\beta_2}_{\gamma_1\gamma_2}(\theta_{12}), \tag{3.4.1}$$

$$\theta_{mn} = \theta_m - \theta_n, \qquad 0 \leq \alpha_i, \beta_i, \gamma_i < n, \;\; i = 1, 2, 3,$$

where we assume summation over repeated indices. A number of observations follow from (3.4.1):

(i) When the relative rapidities vanish, then there is no scattering. This gives us a natural initial condition,

$$S^{\beta_1\beta_2}_{\alpha_1\alpha_2}(0) = \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}. \tag{3.4.2}$$

This initial condition is compatible with the factorization equation provided:

(ii) $$S^{\gamma_1\gamma_2}_{\alpha_1\alpha_2}(\theta)S^{\beta_2\beta_1}_{\gamma_2\gamma_1}(-\theta) = \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}F(\theta), \tag{3.4.3}$$

which is known as the unitarity condition.

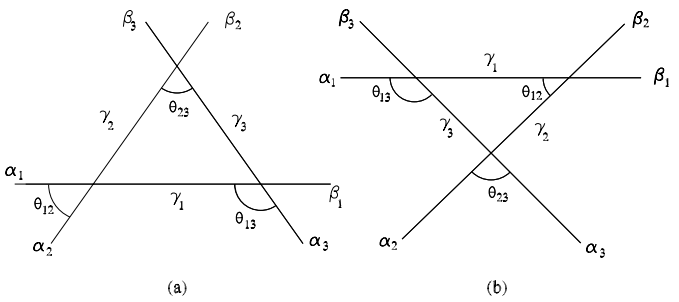

FIGURE 3.4.2: Diagram depicting the equivalence of S matrix element constructed from two-particle amplitudes.

(iii) Furthermore, (3.4.1) admits the following symmetry condition, known as crossing symmetry:

$$S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta) = S_{\alpha_2\beta_1^\star}^{\beta_2\alpha_1^\star}(i\pi - \theta). \tag{3.4.4}$$

We also have

(iv) PT-invariance: $$S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta) = S_{\beta_2\beta_1}^{\alpha_2\alpha_1}(\theta), \tag{3.4.5}$$

(v) CP-invariance: $$S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta) = S_{\alpha_2^\star\alpha_1^\star}^{\beta_2^\star\beta_1^\star}(\theta), \tag{3.4.6}$$

where the $\star$ denotes charge conjugation. Now the type of particle involved in the scattering process is typically specified by its isotopic charge, that is, conserved modulo n. In the case of two-particle scattering it means that

$$\alpha_1 + \alpha_2 = \beta_1 + \beta_2 \pmod{n}, \tag{3.4.7}$$

while charge conjugation stands for a reversal of its sign,

$$\alpha^\star = -\alpha \mod n. \tag{3.4.8}$$

The preceding discussion allows for a smooth transition to analysis of two-dimensional spin statistical systems. For this, it is necessary to assume that the rapidities are purely imaginary. The latter condition

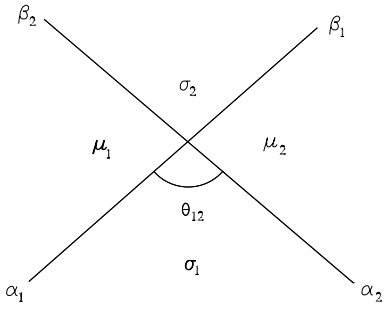

FIGURE 3.4.3: The relative rapidity corresponds to the angle θ_{12}.

allows for the relative rapidity to be given the geometrical meaning of the angle depicted in Figure (3.4.3). Consider now a lattice formed by the vertices shown in Figure (3.4.3), the plaquettes of the latter being occupied by spin variables σ and μ in a way to satisfy the conditions:

$$\alpha_1 = \sigma_1 - \mu_1, \;\; \alpha_2 = \mu_2 - \sigma_1, \;\; \beta_1 = \mu_2 - \sigma_2, \beta_2 = \sigma_2 - \mu_1. \qquad (3.4.9)$$

It is implied that the addition of indices is modulo n. The connection to statistical spin systems is brought about by regarding the S matrix element, as a statistical weight of a corresponding spin configuration, with identification

$$S^{\beta_1\beta_2}_{\alpha_1\alpha_2}(\theta) = R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(-i\theta), \qquad (3.4.10)$$

and depends on the relative orientation of spins. The PT-invariance condition now takes the form

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\sigma_2\sigma_1}_{\mu_1\mu_2}(\theta), \qquad (3.4.11)$$

and is therefore equivalent to symmetry upon interchanging $\sigma_1 \leftrightarrow \sigma_2$. CP-invariance, on the other hand, translates to a symmetry upon interchange of $\mu_1 \leftrightarrow \mu_2$ so that

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\mu_2\mu_1}_{\sigma_1\sigma_2}(\theta). \qquad (3.4.12)$$

Finally crossing symmetry now assumes the form

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\mu_1\mu_2}_{\sigma_1\sigma_2}(\pi - \theta). \qquad (3.4.13)$$

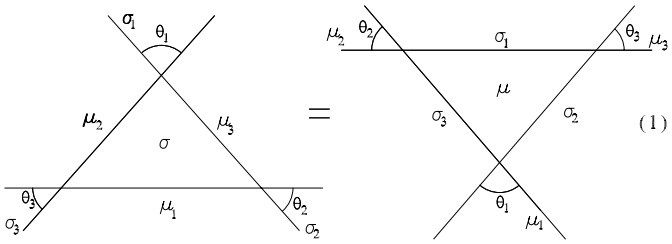

FIGURE 3.4.4: Graphical representation of Equation (3.4.14).

In view of (3.4.10), the factorization equation in terms of the spin variables becomes

$$\sum_{\sigma=0}^{n-1} R_{\mu_2\mu_3}^{\sigma_1\sigma_3}(\theta_1) R_{\mu_3\mu_1}^{\sigma_2\sigma}(\theta_2) R_{\mu_3\mu_2}^{\sigma_3\sigma}(\theta_3) = \sum_{\mu=0}^{n-1} R_{\sigma_2\sigma_3}^{\mu_1\mu}(\theta_1) R_{\sigma_3\sigma_1}^{\mu_2\mu}(\theta_2) R_{\sigma_1\sigma_2}^{\mu_3\mu}(\theta_3), \tag{3.4.14}$$

$$\theta_1 + \theta_2 + \theta_3 = \pi. \tag{3.4.15}$$

Equation (3.4.14) may be graphically represented as in Figure (3.4.4). When (3.4.14) is satisfied, Baxter showed that the partition function of the spin model remained invariant under parallel translations of the direct lines forming the lattice, thus depending only on the angles between them. This property is called Z-invariance [54].

From the standard initial conditions (3.4.2) and using (3.4.9) and (3.4.13), it is possible to derive the boundary conditions on $R(\theta)$, which are as follows:

$$R_{\mu_1\mu_2}^{\sigma_1\sigma_2}(0) = \delta_{\sigma_1,\sigma_2}, \quad R_{\mu_1\mu_2}^{\sigma_1\sigma_2}(\pi) = \delta_{\mu_1,\mu_2}. \tag{3.4.16}$$

The analog of the unitary condition (3.4.3) may now be obtained from (3.4.14), if one of the arguments θ_i is set equal to π and (3.4.15) is used. This leads to the condition,

$$\sum_{\sigma=0}^{n-1} R_{\mu_1\mu_2}^{\sigma_1\sigma}(\theta) R_{\mu_1\mu_2}^{\sigma_2\sigma}(-\theta) = \delta_{\sigma_1,\sigma_2} F(\theta). \tag{3.4.17}$$

As (3.4.17) holds for arbitrary μ_1 and μ_2, it is obvious that $F(\theta)$ must be an even function. It will be noticed that the spin models under consideration are actually built out of two kinds of spin variables σ and μ, which do not interact. The absence of any interaction between the σ and μ means that the statistical weight $R^{\sigma_1\sigma_2}_{\mu_1\mu_2}$ can be factorized:

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = K_{\sigma_1\sigma_2}(\theta)K_{\mu_1\mu_2}(\pi-\theta), \tag{3.4.18}$$

while the crossing condition $R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\pi-\theta)$, implies

$$K_{\sigma_1\sigma_2}(\theta) = K_{\sigma_1\sigma_2}(\pi-\theta), \tag{3.4.19}$$

so that the K's obey crossing symmetry. Again the restriction

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\sigma_1\sigma_2}_{\mu_2\mu_1}(\theta),$$

implies that

$$K_{\mu_1\mu_2}(\pi-\theta) = K_{\mu_2\mu_1}(\pi-\theta), \tag{3.4.20}$$

meaning that $K(\theta)$ is symmetric in its indices. Furthermore,

$$R^{\sigma_1\sigma_2}_{\mu_1\mu_2}(\theta) = R^{\sigma_2\sigma_1}_{\mu_1\mu_2}(\theta)$$

implies that

$$K_{\sigma_1\sigma_2}(\theta) = K_{\sigma_2\sigma_1}(\theta). \tag{3.4.21}$$

Substituting (3.4.18) into (3.4.14) we get

$$\sum_{\sigma=0}^{n-1} K_{\sigma_1\sigma}(\theta_1)K_{\sigma_2\sigma}(\theta_2)K_{\sigma_3\sigma}(\theta_3) = \lambda(\theta_1,\theta_2,\theta_3)K_{\sigma_2\sigma_3}(\pi-\theta_1)\times$$

$$\times K_{\sigma_3\sigma_1}(\pi-\theta)K_{\sigma_1\sigma_2}(\pi-\theta), \tag{3.4.22}$$

where $\theta_1+\theta_2+\theta_3=\pi$ and $\lambda(\theta_1,\theta_2,\theta_3)$ is some function, symmetric in its arguments. Equation (3.4.21) admits the graphical representation shown in Figure (3.4.5) in which $K_{\sigma_1\sigma_2}(\theta)$ is denoted by a line connecting points σ_1 and σ_2, and is called the star-triangle relation (STR). The standard boundary conditions on the statistical weights $R^{\sigma_1\sigma_2}_{\mu_1\mu_2}$, as given in (3.4.16), lead to the following boundary conditions for K's:

$$K_{\sigma_1\sigma_2}(0) = \nu\delta_{\sigma_1,\sigma_2}, \quad K_{\sigma_1\sigma_2}(\pi) = \nu^{-1}. \tag{3.4.23}$$

Setting one of the arguments θ_i in (3.4.21) equal to π and using (3.4.22), we get the following unitarity conditions:

$$\sum_{\sigma} K_{\sigma_1\sigma}(\theta)K_{\sigma_2\sigma}(-\theta) = f(\theta)\delta_{\sigma_1\sigma_2}, \tag{3.4.24}$$

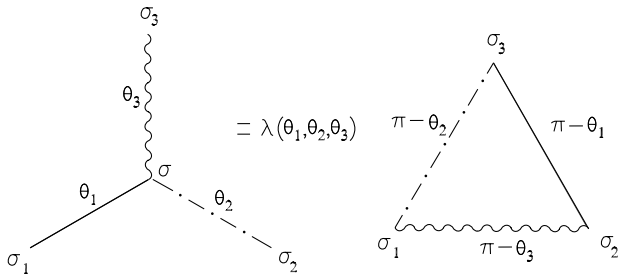

FIGURE 3.4.5: Diagrammatic representation of the star-triangle relation.

$$K_{\sigma_1\sigma_2}(\pi-\theta)K_{\sigma_1\sigma_2}(\pi+\theta)=g(\theta), \tag{3.4.25}$$

where $f(\theta)$ and $g(\theta)$ are some even functions. It might be mentioned here that any two solutions of the star-triangle relation differing by an arbitrary factor $\rho(\theta)$ are equivalent, as it involves a renormalization of the factors $\lambda(\theta_1,\theta_2,\theta_3)$ and $f(\theta)$ and $g(\theta)$ in the respective equations.

Using (3.4.1) and (3.4.14), Baxter showed that the transfer matrices of Z-invariant models formed parametric commutative families. This is also true for spin models obeying the star-triangle relation. We refer the reader to the original work of Baxter [55] for further details.

3.5 Vertex Models

A particularly rich field in which the Yang-Baxter equation has fundamental importance is in the case of two-dimensional vertex models [65, 66, 67]. It can be viewed as giving a sufficient condition for the commutability of the transfer matrices in statistical mechanics. This property enables us to evaluate exactly physical quantities such as the free energy and the one-point function. It is in this sense that such models are termed as exactly solvable. The commutability of the transfer matrices allows us to study the exactly solvable models in (1 + 1)-

dimensional quantum theory and two-dimensional classical statistical mechanics. Moreover, as two-dimensional vertex models appear in diverse physical situations, such as ferroelectric systems, spin models, crystal models like ice and KDP, etc., they constitute a vital part of any theoretical study of real-life physical systems.

Vertex models in statistical physics consist basically of a two-dimensional square lattice of M rows and N columns, where each link can take a different state. Horizontal and vertical links may be in local states belonging to two different vector spaces $\mathcal{A}$ and $\mathcal{V}$. One then assigns the Boltzmann weight (vertex weight) to the configuration of state variables. To get a more concrete idea, let us consider a collection of "atoms" located at the vertices of the two-dimensional lattice and assume that each atom interacts only with its nearest neighbour. In addition, we assume that the interaction energy depends on the "state" of the bonds joining the neighbouring atoms. If the possible states of the bonds are labelled by the elements of a finite set $\{1, 2, ..., n\}$, then the interaction energy will be denoted, say by ϵ_{ij}^{kl}, if the states of the bonds connected to the atoms are as depicted in Figure (3.5.1).

In general ϵ_{ij}^{kl} will depend on certain other parameters such as the external electric or magnetic fields, etc. It will, however, be assumed to be independent of the location of the vertex within the lattice. Knowledge of ϵ_{ij}^{kl} obviously specifies the model. Now, a state of the lattice

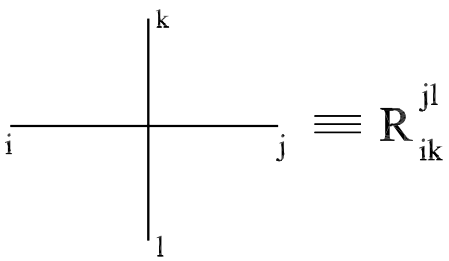

FIGURE 3.5.1: Labelling of the state of bonds.

is given by the assignment of a state to each bond of the lattice, so that the energy of the entire lattice is obtained by summing up ϵ_{ij}^{kl} over all the atoms in the lattice. Obviously for an infinite lattice, such a sum will clearly be divergent and therefore one begins with a lattice of $M \times N$ (finite) rows and columns and finally goes to the limit $M, N \to \infty$. For such finite lattices, it is necessary to impose periodic boundary conditions on the states of the bonds at the edges as shown in Figure (3.5.2).

Now the canonical partition function

$$Z = \sum_{\substack{all \\ configs.}} \exp(-\beta E_{\text{state}}), \tag{3.5.1}$$

where β as usual is proportional to the inverse temperature while

$$\exp(-\beta E_{\text{state}}) = \prod_{\substack{all \\ vertices}} \exp(-\beta \epsilon_{ij}^{kl}). \tag{3.5.2}$$

Hence

$$Z = \sum_{\substack{all \\ configs.}} \prod_{\substack{all \\ vertices}} \exp(-\beta \epsilon_{ij}^{kl}). \tag{3.5.3}$$

The free energy of the system is defined in the thermodynamic limit

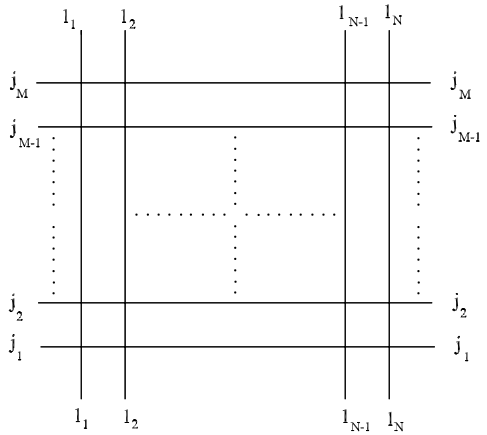

FIGURE 3.5.2: Schematic diagram of a finite $M \times N$ lattice with periodic boundary conditions.

by

$$f = -\lim_{(M,N)\to\infty} \frac{1}{MN} \log Z. \tag{3.5.4}$$

Here $\exp(-\beta\epsilon_{ij}^{kl})$ is called the Boltzmann weight, and we may represent them as the elements of a matrix, i.e., $R_{ik}^{jl}(\theta) = \exp(-\beta\epsilon_{ij}^{kl})$, with θ being some kind of coupling constant, which also depends on the temperature. Now we introduce the monodromy operator $T_{ij}(\theta)$, associated with a horizontal line of the lattice and given by

$$T_{ij}(\theta) = \sum_{a_1,\ldots a_{N-1}=1}^{\dim \mathcal{A}} t_{ia_1}^{(1)}(\theta) \otimes t_{a_1a_2}^{(2)}(\theta) \otimes \ldots \otimes t_{a_{N-1}L}^{(N)}(\theta). \tag{3.5.5}$$

Here each $t_{ij}^{(K)}(\theta)$ acts in the vertical space $\mathcal{V}$ associated to the Kth column of the lattice, and its elements are given by

$$[t_{ij}(\theta)]_{kl} = R_{ik}^{jl}(\theta) = \exp(-\beta\epsilon_{ij}^{kl}).$$

To evaluate the partition function Z one has to sum over all possible states of the bonds in the first row, excluding the free bonds for the time being as shown in Figure (3.5.3).

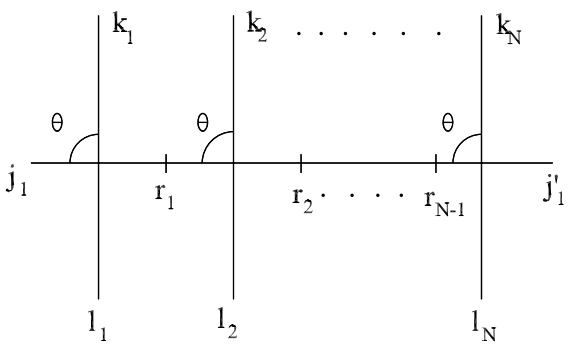

FIGURE 3.5.3: Schematic diagram of the lattice with a single row used for evaluation of the partition function.

Thus we form the monodromy matrix $T(\theta)$, with the elements

$$T^{j_1' l_1 .... l_N}_{j_1 k_1 ... k_N}(\theta) = \sum_{r_1 ... r_{N-1}} R^{r_1 l_1}_{j_1 k_1}(\theta) R^{r_2 l_2}_{r_1 k_2}(\theta) .... R^{j_1' l_N}_{r_{N-1} k_N}(\theta). \tag{3.5.6}$$

To keep track of the indices, it is convenient to introduce an auxiliary N-dimensional vector space $\mathcal{V}$, with basis $\{v_1, ...v_N\}$ and define $R \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$ by

$$R(v_i \otimes v_j) = \sum_{k,l} R^{kl}_{ij} v_k \otimes v_l, \tag{3.5.7}$$

and $T \in \mathrm{End}(\mathcal{A} \otimes \mathcal{V}^{\otimes N})$ by

$$T(v_{j_1} \otimes v_{k_1} \otimes ... \otimes v_{k_N}) = \sum_{j_1, l_1 ... l_N} T^{j_1' l_1 .... l_N}_{j_1 k_1 ... k_N}(v_{j_1'} \otimes v_{l_1} .... \otimes v_{l_N}). \tag{3.5.8}$$

Then the monodromy matrix can be written as

$$T(\theta) = R_{01}(\theta) R_{02}(\theta) .... R_{0N}(\theta), \tag{3.5.9}$$

where R_{ij} here means an R acting in the ith and jth space of the tensor product $\mathcal{A} \otimes \mathcal{V}^{\otimes N}$, the first space being labelled 0 and the rest $1, 2, ..., N$. Recalling that we assumed periodic boundary conditions for the ends, i.e., $j_1 = j_1'$, so that summing over the states of all bonds in the first row gives us nothing but a trace so that we have

$$T^{j_1 l_1 .... l_N}_{j_1 k_1 .... k_N}(\theta) = (\mathrm{tr}_{\mathcal{A}} T(\theta))^{l_1 ..... l_N}_{k_1 .... k_N}. \tag{3.5.10}$$

This is referred to as the *row-to-row transfer matrix* in the literature on statistical mechanics. Consequently, the result of summing the contributions to the partition function Z, over all states of bonds in the first two rows will be

$$(\mathrm{tr}_{\mathcal{A}} T(\theta))^{l_1 ..... l_N}_{k_1 .... k_N} (\mathrm{tr}_{\mathcal{A}} T(\theta))^{k_1 .... k_N}_{m_1 .... m_N} = \left((\mathrm{tr}_{\mathcal{A}} T)^2 \right)^{l_1 ..... l_N}_{m_1 .... m_N}, \tag{3.5.11}$$

due to the summing up over the vertical bonds joining the first two rows. This is depicted in Figure (3.5.4).

Proceeding in this manner, it is obvious that the result of summing over all the vertical bonds together with periodic boundary conditions finally leads to the following expression for the partition function:

$$Z = \mathrm{tr}_{\mathcal{V}^{\otimes N}} \left((\mathrm{tr}_{\mathcal{A}} T(\theta))^M \right) = \mathrm{tr}_{\mathcal{V}^{\otimes N}} \left[\tau(\theta)^M \right], \tag{3.5.12}$$

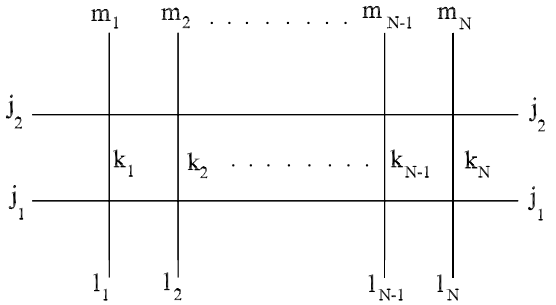

FIGURE 3.5.4: Schematic diagram of the row-to-row transfer matrix for two rows.

where we have denoted the row-to-row transfer matrix by $\tau(\theta)$. Since (3.5.12) gives the partition function Z as a trace over a transfer matrix, its evaluation basically requires determination of the eigenvalues of the operator $\left[\tau^{(N)}(\theta)\right]^M$. Indeed, in the thermodynamic limit, all one requires is a knowledge of the largest eigenvalue $\Lambda^{(N)}_{\max}(\theta)$, to evaluate the free energy since it can be shown that

$$f = -\lim_{N\to\infty} \frac{1}{N} \log \Lambda^{(N)}_{\max}(\theta). \tag{3.5.13}$$

The above considerations are valid irrespective of the integrability of the model. Now, while the notion of complete integrability in Hamiltonian mechanics is a precise notion in view of the Liouville theorem, it does not have a precise meaning in statistical mechanics, where the term *exactly solvable* is more appropriate, provided one can obtain closed-form analytic expressions for the free energy in the thermodynamic limit. We shall see later how the quantum inverse scattering method allows for the exact determination of the eigenvalues of the transfer matrix, by making use of the algebraic Bethe ansatz. In the present context of vertex models, it is appropriate to dwell on the sufficiency condition for exact solvability of the model or integrability generally. This is provided once again by the Yang-Baxter equation which emerges in the following manner. Assuming the existence of a

nonsingular matrix $R(\theta)$, so that

$$R(\theta - \theta')[t(\theta) \otimes t(\theta')] = [t(\theta') \otimes t(\theta)]R(\theta - \theta'), \tag{3.5.14}$$

where $\otimes$ refers to the usual tensor product of matrices acting on h-dimensional horizontal spaces $\mathcal{A}$, with R acting on $\mathcal{A} \otimes \mathcal{A}$, the matrix product being on the vertical spaces $\mathcal{V}$. Elementwise, this equation is written as follows:

$$R^{ef}_{ab}(\theta-\theta')[t_{ec}(\theta)]_{p\gamma}[t_{fd}(\theta')]_{\gamma q} = [t_{ae}(\theta')]_{p\gamma}[t_{bf}(\theta)]_{\gamma q}R^{cd}_{ef}(\theta-\theta'). \tag{3.5.15}$$

If a new vertex configuration is associated to $R^{cd}_{ab}(\theta)$, as shown in Figure (3.5.5). Then equation (3.5.15) can be interpreted in the following graphical manner as shown in Figure (3.5.6).

This graphical description will be particularly helpful when we consider the Yang-Baxter algebra in the presence of finite boundary conditions. Equation (3.5.15), together with its graphical counterpart, shows that one is allowed to shift the wavy lattice line (of type $\mathcal{V}$) through the vertex representing R^{cd}_{ab}, provided we maintain the same relative angles between the different lines. Equation (3.5.14), or its explicit version as given by (3.5.15), represents in a sense a local relation since it is valid for each vertex.

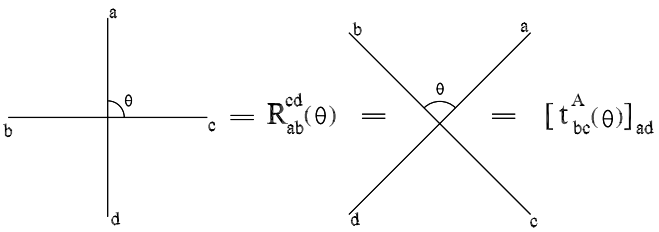

FIGURE 3.5.5: Schematic representation of a vertex configuration.

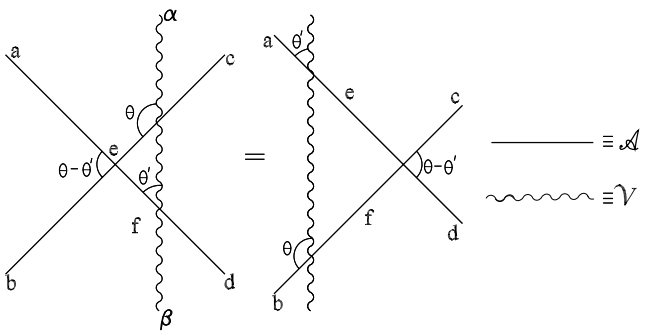

FIGURE 3.5.6: Schematic representation of Equation (3.5.15).

From the definition of $T_{ij}(\theta)$ in (3.5.5), using (3.5.15) one can derive the corresponding relation for $T_{ab}(\theta)$, which reads

$$R(\theta-\theta')[T^{(N)}(\theta)\otimes T^{(N)}(\theta')] = [T^{(N)}\otimes T^{(N)}(\theta)]R(\theta-\theta'), \quad (3.5.16)$$

which is to be interpreted as a global version of (3.5.15). Equation (3.5.16) holds in $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{V}$. Its graphical representation consists of repeated application of the graphical form, depicted in Figure (3.5.7), which is often referred to as the train argument.

From (3.5.16) as $R(\theta-\theta')$ is nonsingular, we have

$$[T^{(N)}(\theta)\otimes T^{(N)}(\theta')] = R^{-1}(\theta-\theta')[T^{(N)}\otimes T^{(N)}(\theta)]R(\theta-\theta').$$

Upon taking trace on $\mathcal{A}\otimes\mathcal{A}$ and using its cyclic property, we arrive at

$$[\tau^{(N)}(\theta), \tau^{(N)}(\theta')] = 0, \quad (3.5.17)$$

with

$$\tau^{(N)}(\theta) \equiv \mathrm{tr}_{\mathcal{A}} T^{(N)}(\theta). \quad (3.5.18)$$

Thus we have a one-parameter family of commuting transfer matrices. It will be realized that by virtue of the relation (3.5.16), the R matrix defines the basic algebraic structure of the theory of integrable vertex models. We shall refer to (3.5.16) as the Yang-Baxter algebra (YBA).

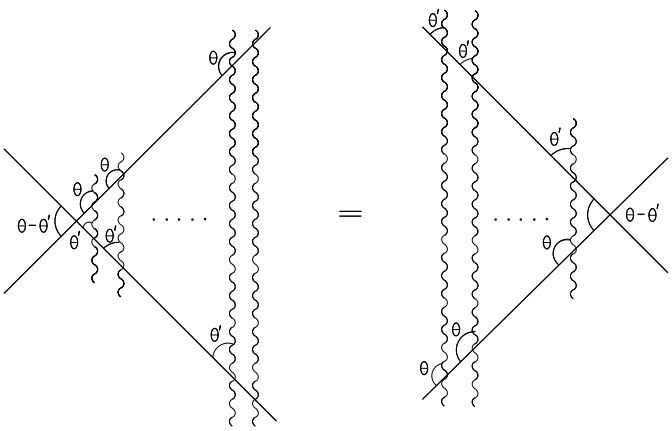

FIGURE 3.5.7: Graphical representation of the Yang-Baxter algebra.

To determine the consistency of this algebra, we may consider the transformation of the product $T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)$ to $T(\theta_3)\otimes T(\theta_2)\otimes T(\theta_1)$, using (3.5.16), in order to establish the Yang-Baxter equation. Firstly, one can convert $T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3)$ to $T(\theta_3)\otimes T(\theta_2)\otimes T(\theta_1)$ as follows:

$$T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3) = (R^{-1}(\theta_1-\theta_2)\otimes I)(I\otimes R^{-1}(\theta_1-\theta_3))$$
$$\times(R^{-1}(\theta_2-\theta_3)\otimes I)[T(\theta_3)\otimes T(\theta_2)\otimes T(\theta_1)]$$
$$\times(R(\theta_2-\theta_3)\otimes I)(I\otimes R(\theta_1-\theta_3))(R(\theta_1-\theta_2)\otimes I).$$

Alternatively the same conversion may be obtained in the following manner:

$$T(\theta_1)\otimes T(\theta_2)\otimes T(\theta_3) = (I\otimes R^{-1}(\theta_2-\theta_3))(R^{-1}(\theta_1-\theta_3)\otimes I)$$
$$\times(I\otimes R^{-1}(\theta_1-\theta_2))\times[T(\theta_3)\otimes T(\theta_2)\otimes T(\theta_1)]$$
$$\times(I\otimes R(\theta_1-\theta_2))(R(\theta_1-\theta_3)\otimes I)(I\otimes R(\theta_2-\theta_3)).$$

The Yang-Baxter equation then emerges as the sufficiency condition of these two equations and is represented by

$$(I\otimes R(\theta_1-\theta_2))(R(\theta_1-\theta_3)\otimes I)(I\otimes R(\theta_2-\theta_3))$$
$$= (R(\theta_2-\theta_3)\otimes I)(I\otimes R(\theta_1-\theta_3))(R(\theta_1-\theta_2)\otimes I). \qquad (3.5.19)$$

3.6 Reflection Equation Algebra

In this section we consider an extension of the Yang-Baxter equation by incorporating an additional element called the reflection matrix. The resulting quadratic algebra may be shown to have new properties with a direct bearing on physical problems. Historically, just as the Yang-Baxter equation arose from factorizable scattering on the entire line, the new *reflection equation algebra* can be shown to emerge from factorizable scattering on a half line. Furthermore, whereas the Yang-Baxter equation gives the sufficiency condition for commutativity of transfer matrices of exactly solvable models in statistical mechanics assuming periodic boundary conditions. The new reflection equation leads to commutability of transfer matrices for lattice models with boundaries, the so-called open chains. This provides motivation for studying the reflection equation and its associated algebra. Apart from open chains, the reflection equation has also appeared in the context of quantum groups and quantum algebras, in noncommutative differential geometry, etc., to name a few branches.

To initiate a discussion on the reflection algebra we consider the Zamolodchikov algebra [56]

$$A_\alpha(u)A_\beta(v) = R^{\alpha\beta}_{\gamma\delta}(u-v)A_\gamma(u)A_\delta(u), \tag{3.6.1}$$

and assume that there exists a matrix $K(u)$, so that

$$A_\alpha(u) = K^\alpha_{\alpha'}(u)A_{\alpha'}(-u). \tag{3.6.2}$$

From (3.6.2), the matrix $K^\alpha_{\alpha'}(u)$ may be interpreted as the amplitude for a particle to be reflected elastically from a wall. This provides a simple visual description of factorized scattering on a half line. Setting $u = 0$ gives

$$K(0) = I. \tag{3.6.3}$$

Using (3.6.2) twice we get $A_\alpha(u) = K^\alpha_{\alpha'}(u)K^{\alpha'}_{\alpha''}(-u)A_{\alpha''}(u)$, implying that

$$K(u)K(-u) = I, \tag{3.6.4}$$

giving us the unitarity condition. Considering monomials of degree two, i.e., objects $A_\alpha(u)A_\beta(v)$, it can be shown that there are two distinct ways of applying the extended Zamoldchikov algebra twice, to obtain an

expression proportional to $A_{\alpha'}(-u)A_{\beta'}(-u)$. Subsequent assumption of linear independence of these monomials leads to the relation [57]

$$R_{12}(u-v)\overset{1}{K}(u)\,\mathcal{P}_{12}R_{12}(u+v)\mathcal{P}_{12}\overset{2}{K}(v)$$
$$=\overset{2}{K}(v)\,R_{12}(u+v)\overset{1}{K}(u)\,\mathcal{P}_{12}R_{12}(u-v)\mathcal{P}_{12}. \tag{3.6.5}$$

Equation (3.6.5) will be referred to as the reflection factorization equation.

Thus, given a solution of $R(u)$ the Yang-Baxter equation (3.6.5) may be solved for $K(u)$. We give a few examples below.

Example 1: The R matrix for the spin $\frac{1}{2}$ $A_1^{(1)}$ matrix is [58]

$$R(u,\eta)=\frac{1}{Z(u,\eta)}\begin{pmatrix}\sinh(u+\eta) & 0 & 0 & 0\\ 0 & \sinh u\sinh\eta & 0 & 0\\ 0 & \sinh\eta\sinh u & 0 & 0\\ 0 & 0 & 0 & \sinh(u+\eta)\end{pmatrix}, \tag{3.6.6}$$

where $Z(u,\eta)=\sqrt{|\sinh(u+\eta)\sinh(-u+\eta)|}$. It is seen to be symmetric, i.e.,

$$\mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12}=R_{12}(u).$$

The solution of $K(u)$ for such an R matrix is

$$K(u,\xi)=\frac{1}{\sqrt{|\sinh(u+\xi)\sinh(-u+\xi)|}}\begin{pmatrix}\sinh(u+\xi) & 0\\ 0 & -\sinh(u-\xi)\end{pmatrix}. \tag{3.6.7}$$

Example 2: In the case of the Fateev-Zamolodchikov R matrix representing spin 1, the $A_1^{(1)}$ matrix [59], the solution of the reflection equation is

$$K(u,\xi)=\rho(u,\xi)\begin{pmatrix}a_{++} & 0 & 0\\ 0 & a_{-+} & 0\\ 0 & 0 & a_{--}\end{pmatrix}, \tag{3.6.8}$$

where

$$a_{++}=\sinh(u+\xi)\sinh(u+\xi-\eta),$$
$$a_{-+}=-\sinh(u-\xi)\sinh(u+\xi-\eta),$$
$$a_{--}=\sinh(u-\xi)\sinh(u-\xi+\eta),$$

and

$$\rho(u,\xi)=\sqrt{\sinh(u+\xi)\sinh(-u+\xi)\sinh(u-\eta+\xi)\sinh(-u-\eta+\eta)},$$

with ξ being an arbitrary parameter. It is convenient at this point to return to a diagrammatic approach in order to understand the reflection equation. For simplicity we will assume that the R matrix, following from the Yang-Baxter equation, satisfies the following conditions:

- Symmetry $$\mathcal{P}_{12}R_{12}\mathcal{P}_{12} = R_{12}, \quad R_{12}^{t_1} = R_{12}^{t_2}, \tag{3.6.9}$$

where t_i denotes a transposition in the ith space.

- Unitarity $$R_{12}(\theta)R_{12}(-\theta) = \rho(\theta). \tag{3.6.10}$$

- Crossing unitarity $$R_{12}^{t_1}(\theta)R_{12}^{t_2}(-\theta - 2i) = \tilde{\rho}(\theta), \tag{3.6.11}$$

where $\rho(\theta)$ and $\tilde{\rho}(\theta)$ are some scalar functions.

Example: The simplest example of a R matrix fulfilling these conditions is that which corresponds to the scattering of two spin $\frac{1}{2}$ particles, and is given by

$$R(\theta) = -i\theta I + \mathcal{P}, \tag{3.6.12}$$

$$(\mathcal{P})_{i_1 i_2}^{j_1 j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1}. \tag{3.6.13}$$

As noted earlier, the scattering process on a half line can be visualized by postulating the existence of a "wall" either on the right or left. On striking the wall, a particle is assumed to be reflected elastically so that its rapidity undergoes a reflection $\theta \longrightarrow -\theta$, without any alteration of its intrinsic quantum numbers; i.e., the indices on the R matrix. The

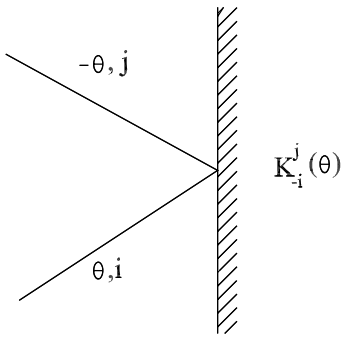

FIGURE 3.6.1: Schematic diagram of a reflection matrix.

amplitude for scattering off the wall is therefore given by the "reflection matrix", say, $K^{j}_{-i}(\theta)$ and is shown in Figure (3.6.1).

Let us recall that the essential feature in the case of factorized scattering within the diagrammatic approach was the fact that one could displace any line parallel to itself through the intersection of the other two lines. Retaining this view would mean, in the case of two-particle scattering from a half line, the equivalence of the two diagrams, shown in Figure (3.6.2).

Associating an R matrix to the intersection of two lines and a K_- matrix to reflection from the wall with time flowing from the bottom to top gives us the following relation:

$$R_{12}(\theta_1-\theta_2)\overset{1}{K}_-(\theta_1)R_{12}(\theta_1+\theta_2)\overset{2}{K}_-(\theta_2)$$

$$=\overset{2}{K}_-(\theta_2)R_{12}(\theta_1+\theta_2)\overset{1}{K}_-(\theta_1)R_{12}(\theta_1-\theta). \tag{3.6.14}$$

For the R matrix given by (3.6.12), this equation admits the following solution for $K_-(\theta)$:

$$K_-(\theta,\xi_-)=\left(I-\frac{i\theta}{\xi_-}\sigma^3\right). \tag{3.6.15}$$

If the "wall" is situated to the left as shown in Figure (3.6.3), then we may associate a matrix K^{j}_{+i} for the elastic reflection.

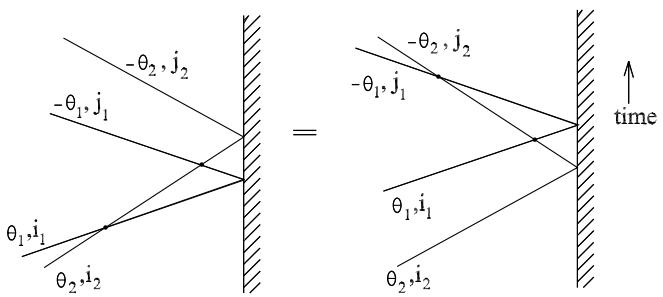

FIGURE 3.6.2: Schematic diagram of Equation (3.6.2).

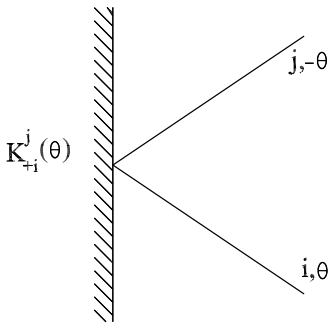

FIGURE 3.6.3: Schematic representation of reflection matrix from a left "wall".

In that case the factorized reflection equation may be depicted as in Figure (3.6.4). This corresponds to the following equation:

$$R_{12}(-\theta_1+\theta_2)\overset{1}{K}{}^{t_1}_{+}\,R_{12}(-\theta_1-\theta_2-2i)\overset{2}{K}{}^{t_2}_{+}(\theta_2)$$

$$=K^{t_2}_{+}(\theta_2)R_{12}(-\theta_1-\theta_2-2i)\overset{1}{K}{}^{t_1}_{+}\,R_{12}(-\theta_1+\theta_2). \tag{3.6.16}$$

With the R matrix given in (3.6.12), equation (3.6.15) admits the following solution for $K_+(\theta)$:

$$K_+(\theta)=K^{t}_{-}(-\theta-i,\xi_+). \tag{3.6.17}$$

To establish a connection between the above discussion and quantum spin chains, it is convenient to once again introduce the notion of an auxiliary vector space of rapidity, say a, to be denoted by straight lines and a set of quantum spaces that will be denoted by wavy lines of zero rapidity. In addition, it is necessary to analytically continue the spectral parameter or rapidity variable θ_i, to purely imaginary values, $\theta = iu$ with $u \in \mathcal{R}$. Remembering that $L_n(u) \equiv R_{an}(u)$, the equivalence of the schematic representation shown in Figure (3.6.5) is obvious. Let $\mathcal{T}_-(u)$ denote this sequential matrix product and define the monodromy matrix by $T(u) \equiv L_n(u).....L_1(u)$. We then have

$$\mathcal{T}(u)=T(u)K_-(u)T^{-1}(-u). \tag{3.6.18}$$

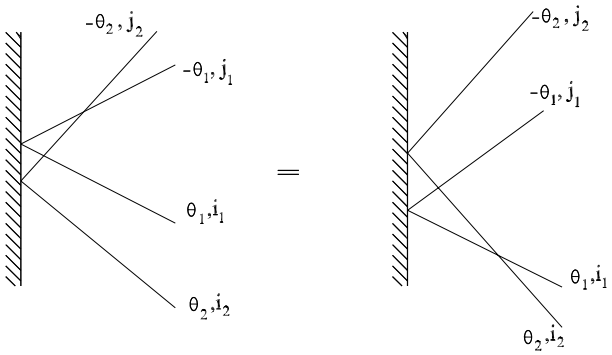

FIGURE 3.6.4: Schematic representation of Equation (3.6.16).

As a result of the Yang-Baxter equation, the monodromy matrix $T(u)$ satisfies the relation

$$R(u_1 - u_2)\overset{1}{T}(u_1)\overset{2}{T}(u_2) = \overset{2}{T}(u_2)\overset{1}{T}(u_1)R(u_1 - u_2), \tag{3.6.19}$$

and may be understood from Figure (3.6.6).

Introduction of a right wall modifies the figures on either side of the equality as represented in Figure (3.6.6) to the Figures (3.6.7) and (3.6.8), respectively.

Their equality establishes the analog of (3.6.19) in the case of open chains, viz,

$$R_{12}(u_1 - u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{12}(u_1 + u_2)\overset{2}{\mathcal{T}}_-(u_2) =$$

$$\overset{2}{\mathcal{T}}_-(u_2)R_{12}(u_1 + u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{12}(u_1 - u_2). \tag{3.6.20}$$

For reflection from a left wall, the corresponding algebra is

$$R_{12}(u_1 - u_2)\overset{1^{t_1}}{\mathcal{T}}_+(u_1)R_{12}(-u_1 - u_2 - 2i)\overset{2^{t_2}}{\mathcal{T}}_+(u_2) =$$

$$\overset{2^{t_2}}{\mathcal{T}}_+(u_2)R_{12}(-u_1 - u_2 - 2i)\overset{1^{t_1}}{\mathcal{T}}_+(u_1)R_{12}(-u_1 + u_2). \tag{3.6.21}$$

The above discussions might tempt one to introduce, in case of open chains with boundary terms at both ends, two walls, one at either end. However, such a consideration does not lead to a natural definition for

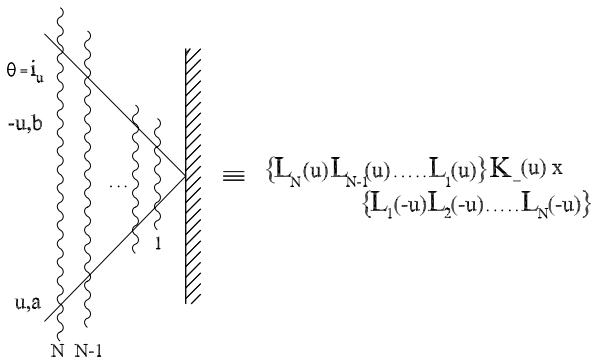

FIGURE 3.6.5: Diagram representing sequential matrix products of $L_n(u)$ together with reflection.

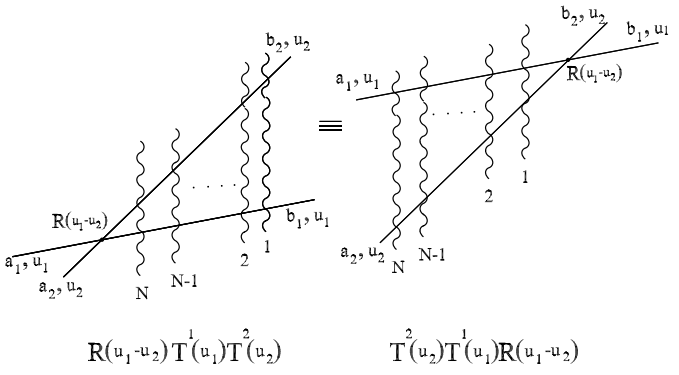

FIGURE 3.6.6: Graphical representation of the Yang-Baxter algebra.

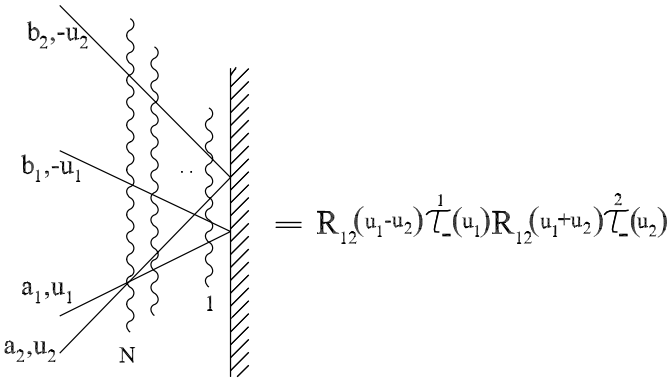

FIGURE 3.6.7: Diagram showing modification of left-hand-side of Figure (3.6.6) in the presence of a right boundary.

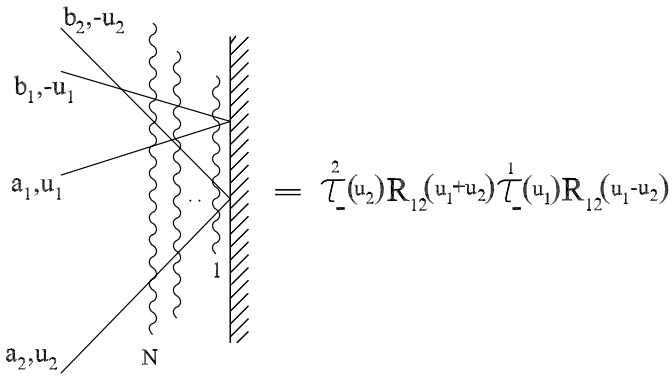

FIGURE 3.6.8: Diagram showing modification of right-hand-side of Figure (3.6.6) in the presence of a right boundary.

the transfer matrix for such chains. The credit for deducing a proper and consistent definition for the transfer matrix in this context goes to Sklyanin, who first demonstrated that the quantity $t(u)$ given by [62],

$$t(u) = \mathrm{tr}_a K_+(u)\mathcal{T}_-(u), \tag{3.6.22}$$

is a transfer matrix, with

$$[t(u), t(v)] = 0 \ \forall \ u, v.$$

It is natural to enquire about the expressions for the Hamiltonian of the system in the presence of such boundaries. This is given by

$$H = \frac{t'(0) - \mathrm{tr}K'_+(0)}{2\mathrm{tr}K_+(0)} = \sum_{n=1}^{N-1} H_{n,n+1} + H_{\mathrm{boundary}}, \tag{3.6.23}$$

where

$$H_{n,n+1} = \mathcal{P}_{n,n+1}\frac{d}{du}R_{n,n+1}(u)|_{u=0} \tag{3.6.24}$$

and the boundary term is given by

$$H_{\mathrm{boundary}} = \frac{1}{2}\overset{1}{K}{}'_-(0) + \frac{\mathrm{tr}_0 \overset{0}{K}{}_+(0)H_{N0}}{\mathrm{tr}K_+(0)}. \tag{3.6.25}$$

The proof of (3.6.23–3.6.25) is based on the following assumptions:

$$(i) \quad L_n(u) = R_{n,0}(u) \quad (ii) \quad R_{m,n}(0) = \mathcal{P}_{mn} \quad (iii) \quad K_-(0) = I.$$

Consider the derivative of $t(u)$ as given in (3.6.22), with respect to u and evaluate it at $u = 0$. Straightforward calculation gives

$$t'(0) = 2\mathrm{tr}[K_+(0)T'(0)T^{-1}(0)] + \mathrm{tr}[K_+(0)T(0)K'_-(0)T^{-1}(0)] + \mathrm{tr}(K'_+(0)), \tag{3.6.26}$$

where the primes denote differentiation with respect to u and $T(0) = \mathcal{P}_{N,0}\ \mathcal{P}_{N-1,0}.....\mathcal{P}_{1,0}$. Consider the last term in (3.6.26) and denote it by, say X_3, i.e.,

$$X_3 = \mathrm{tr}[K_+(0)\mathcal{P}_{N,0}....\mathcal{P}_{1,0}K'_-(0)\mathcal{P}_{1,0}....\mathcal{P}_{N,0}].$$

As $\mathcal{P}_{1,0}K'_-(0) = \mathcal{P}_{1,0}\left(I^{(1)} \otimes \overset{(0)}{K}{}'_-(0)\right)$ hence $\mathcal{P}_{1,0}K'_-(0)\mathcal{P}_{1,0} = \left(\overset{(1)}{K}{}'_-(0) \otimes I^{(0)}\right)\mathcal{P}^2_{1,0} = \overset{1}{K}{}'_-(0)$. Because the trace operation is taken

over the auxiliary space "0", $\overset{1}{K'_-}(0)$ may be taken outside the trace, leading to

$$X_3 = \overset{1}{K'_-}(0)\mathrm{tr}[K_+(0)\mathcal{P}_{N,0}....\mathcal{P}_{2,0}.\mathcal{P}_{2,0}....\mathcal{P}_{N,0}].$$

However, $\mathcal{P}^2_{m,n} = I$, so

$$X_3 = \overset{1}{K'_-}(0)\mathrm{tr}(K_+(0)). \tag{3.6.27}$$

Denoting $X_2 = \mathrm{tr}[K_+(0)T'(0)T^{-1}(0)]$,

$$= \sum_{j=1}^{N} \mathrm{tr}\left[K_+(0)\left(\mathcal{P}_{N,0}....\frac{dR_{j,0}(u)}{du}|_{u=0}\mathcal{P}_{j-1,0}....\mathcal{P}_{1,0}\right)T^{-1}(0)\right],$$

$$= \mathrm{tr}\left[K_+(0)\frac{dR_{N,0}(u)}{du}|_{u=0}(\mathcal{P}_{N-1,0}.....\mathcal{P}_{1,0}) \times (\mathcal{P}_{1,0}....\mathcal{P}_{N,0})\right] +$$

$$+ \sum_{j=1}^{N-1} \mathrm{tr}[K_+(0)\mathcal{P}_{N,0}....Y_{j,0}\mathcal{P}_{j-1,0}....\mathcal{P}_{1,0}T^{-1}(0)],$$

$$= \mathrm{tr}\left[K_+(0)\frac{dR_{N,0}(u)}{du}|_{u=0}\mathcal{P}_{N,0}\right] + \sum_{j=1}^{N-1} M_{j,j+1},$$

where

$$Y_{j,0} \equiv \frac{dR_{j,0}(u)}{du}|_{u=0},$$

$$M_{j,j+1} \equiv \mathrm{tr}\left[K_+(0)\mathcal{P}_{N,0}.....Y_{j,0}\mathcal{P}_{j-1,0}....\mathcal{P}_{1,0}T^{-1}(0)\right],$$

leading to

$$X_2 = \mathrm{tr}\left[K_+(0)\mathcal{P}_{N,0}....Y_{j,o}(\mathcal{P}_{j-1,0}...\mathcal{P}_{1,0})\right.$$
$$\left.\{(\mathcal{P}_{1,0}...\mathcal{P}_{j-1,0})(\mathcal{P}_{j,0}\mathcal{P}_{j+1,0}...\mathcal{P}_{N,0})\}\right].$$

But $\mathcal{P}^2_{k,o} = I$ so

$$M_{j,j+1} = \mathrm{tr}\left[K_+(0)\mathcal{P}_{N,0}...\mathcal{P}_{j+2,0}\mathcal{P}_{j+1,0}Y_{j,0}\mathcal{P}_{j,0}\mathcal{P}_{j+1,0}...\mathcal{P}_{N,0}\right].$$

With the property $\mathcal{P}_{j+1,0}Y_{j,0} = Y_{j,j+1}\mathcal{P}_{j+1,0}$, $Y_{j+1,j}$ can be shifted leftward and taken outside the trace, to finally get $M_{j,j+1} = Y_{j,j+1}\mathcal{P}_{j,j+1}$ $\mathrm{tr}(K_+(0))$ resulting in

$$X_2 = \mathrm{tr}\left[K_+(0)\frac{dR_{N,0}(u)}{du}|_{u=0}\mathcal{P}_{N,0}\right] + \mathrm{tr}(K_+(0))$$

$$\sum_{j=1}^{N-1} \frac{dR_{j,j+1}(u)}{du}|_{u=0}\mathcal{P}_{j,j+1}. \tag{3.6.28}$$

Consequently,

$$t'(0) = \mathrm{tr}(K'_+(0)) + 2\left[\mathrm{tr}K_+(0)\sum_{j=1}^{N-1} h_{j,j+1} + \mathrm{tr}\{\overset{0}{K}_+(0)\, H_{N,0}\}\right]$$

$$+ \overset{1}{K}_-{}'(0)\mathrm{tr}K_+(0),$$

or alternatively,

$$t'(0) = 2\mathrm{tr}K_+(0)\left[\sum_{j=1}^{N-1} h_{j,j+1} + \frac{1}{2}\overset{1}{K}'_-(0) + \frac{\mathrm{tr}_0\{\overset{0}{K}_+(0)H_{N,0}\}}{\mathrm{tr}(K_+(0))}\right]$$

$$+\mathrm{tr}(K'_+(0)). \tag{3.6.29}$$

The last term within square brackets represents the Hamiltonian of the model in the presence of finite boundary conditions, the first term representing the bulk Hamiltonian; the remaining terms represent the contributions to the Hamiltonian from either end.

We now consider a general solution of the reflection matrices for the six-vertex model, depending on four arbitrary parameters. This problem was analysed by H.J. de Vega and A. González Ruiz in [63]. Going back to (3.6.5) and assuming that the R matrix has P, T and crossing symmetry, one can obtain the following:

$$\hat{R}(u-v)[K^-(u)\otimes I]\hat{R}(u+v)[K^-(v)\otimes I]$$

$$= [K^-(v)\otimes I]\hat{R}(u+v)[K^-(u)\otimes I]\hat{R}(u-v), \tag{3.6.30}$$

$$\bar{R}(u-v)[I\otimes K^+(u)]\bar{R}(u+v)[I\otimes K^+(v)]$$

$$= [I\otimes K^+(v)]\bar{R}(u+v)[I\otimes K^+(u)]\bar{R}(u-v), \tag{3.6.31}$$

where $\hat{R}(u)\equiv\mathcal{P}_{12}R_{12}(u)$ and $\bar{R}(u) = R_{12}(u)\mathcal{P}_{12}$. From (3.6.9) however, $\hat{R}(u) = \bar{R}(u) = R(u)$ (say), and satisfies the Yang-Baxter equation:

$$[I\otimes R(u-v')][R(u)\otimes I][I\otimes R(v)] = [R(v)\otimes I][I\otimes R(u)][R(u-v)\otimes I]. \tag{3.6.32}$$

From above, $K^+_{ab}(u)$ and $K^-_{ab}(u)$ describe scattering of the particles by the right and left boundaries, respectively. We follow [63] to discuss the procedure for finding the general solutions of $K^{\pm}_{ab}(u)$, for a given solution of the Yang-Baxter equation, in the case of the six-vertex model. For this model $R(u)$ has the following form:

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh\eta}{\sinh(u+\eta)} & \frac{\sinh u}{\sinh(u+\eta)} & 0 \\ 0 & \frac{\sinh u}{\sinh(u+\eta)} & \frac{\sinh\eta}{\sinh(u+\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.6.33}$$

Consider the following ansatz for $K(u)$:

$$K(u) = \begin{pmatrix} x(u) & y(u) \\ z(u) & t(u) \end{pmatrix}. \tag{3.6.34}$$

Substitution of $R(u)$ and $K(u)$ in (3.6.30) or (3.6.31), since they are equivalent for the six-vertex model above, gives rise to a set of functional equations determining the unknown functions $x(u), y(u), z(u)$ and $t(u)$. The relevant equations are

$$z(u)y(v) = z(v)y(u), \tag{3.6.35}$$

$$\sinh(u-v)[x(u)x(v) - t(u)t(v)] + \sinh(u+v)[x(v) - x(u)t(v)] = 0, \tag{3.6.36}$$

$$y(u)x(v)\sinh 2v = [\sinh(u+v)x(u) + \sinh(u-v)t(u)]y(v). \tag{3.6.37}$$

From (3.6.35) it follows that

$$\frac{y(u)}{z(u)} = \frac{y(v)}{z(v)} = k_1 \ \text{(say)}$$

k_1 being an arbitrary constant. Defining $a(u) = \frac{t(u)}{x(u)}$, enables one to write (3.6.36) as

$$\frac{a(u) - a(v)}{a(u)a(v) - 1} = \frac{\tanh u - \tanh v}{\tanh u + \tanh v}, \tag{3.6.38}$$

which gives rise to the solution

$$a(u) = \frac{t(u)}{x(u)} = \frac{\sinh(\xi - u)}{\sinh(\xi + u)}, \tag{3.6.39}$$

with ξ being an arbitrary constant. Finally, (3.6.37) yields

$$y(u) = \mu \sinh 2u, \tag{3.6.40}$$

with μ being another constant. The remaining equations are identically satisfied, so that the general solution $K(u)$ for the six-vertex model can be represented as

$$K(u,k,\lambda,\mu,\xi) = \begin{pmatrix} k\sinh(\xi+u) & \mu\sinh 2u \\ \lambda\sinh 2u & k\sinh(\xi-u) \end{pmatrix}, \tag{3.6.41}$$

where k, μ, λ and ξ are arbitrary parameters. This solution was also obtained by A.B. Zamolodchikov. Note that the choice $\mu = \lambda = 0$ leads to the singular solution. The Hamiltonian associated with this solution for the boundary matrices is of the form

$$H = A\left\{\sum_{n=1}^{N-1} h_{n,n+1} + \frac{1}{2}\dot{K}_1^-(0) + \frac{\mathrm{tr}_0[K_0^{+t}(-\eta)h_{N0}]}{\mathrm{tr}[K^+(-\eta)]}\right\}. \tag{3.6.42}$$

Here $h_{n,n+1} = \frac{d}{du}R_{n,n+1}(u)|_{u=0}$, gives the two-site bulk Hamiltonian, while $(n, n+1)$ label the lattice site on which the R matrix acts and $A =$ constant. Choosing in this particular case,

$$K^{\pm}(u) = K(u, k_\pm, \lambda_\pm, \mu_\pm, \xi_\pm), \tag{3.6.43}$$

we find that to maintain the bulk XXZ Hamiltonian with the first derivative of the transfer matrix, it is necessary that the value of $K^-(u=0)$ be non-zero. Consequently, we must have $k_- \neq 0$ and without loss of generality set $K^-(0) = 1$, leading to $K_- = \frac{1}{\sinh\xi}$. Similar reasoning dictates that $k_+ \neq 0$. The preceding formula for the Hamiltonian becomes, upon insertion of (3.6.33), (3.6.41) and (3.6.43) into (3.6.42):

$$H = \sum_{n=1}^{N-1}\left[\sigma_n^1\sigma_{n+1}^1 + \sigma_n^2\sigma_{n+1}^2 + \cosh\eta\sigma_n^3\sigma_{n+1}^3\right]$$

$$+ \sinh\eta\left[b_-\sigma_1^3 + c_-\sigma_1^- + d_-\sigma_1^+ - b_+\sigma_N^3 - c_+\sigma_N^- - d_+\sigma_N^+\right]. \tag{3.6.44}$$

Here $(\sigma_k^1, \sigma_k^2, \sigma_k^3)$ or $(\sigma_k^3, \sigma_k^\pm)$ refer to the Pauli matrices at the kth site. In arriving at this form, A has been set equal to $2\sinh\eta$ and terms proportional to the identity have been neglected. Note that the Pauli

matrices at site 1 have been obtained from the term $\frac{1}{2}\dot{K}^-(0)$, the dot referring to differentiation with respect to u, while $\sigma_N^{\pm}, \sigma_N^3$'s have arisen from the $\mathrm{tr}_0[....]$ term. The parameters $b_\pm, c_\pm, d_\pm$ are given as follows:

$$b_\pm = \coth\xi_\pm, c_- = 2\lambda_-, d_- = 2\mu_-, c_+ = \frac{2\lambda_+}{k_+\sinh\xi_+}, d_+ = \frac{2\mu_+}{k_+\sinh\xi_+}.$$

One should compare (3.6.42) with (3.6.25) and note that evaluation of the boundary term at $\theta = -\eta$ in (3.6.42) as opposed to $\theta = 0$ in (3.6.25), has led to the appearance of $\sigma_N^{\pm}$ in the expression for the Hamiltonian (3.6.44).

Chapter 4

Continuous Integrable Systems

4.1 Introduction

In Chapter 3 we discussed the properties of the Yang-Baxter equation and the associated quantum R matrix, which are essential ingredients for setting up the algebraic Bethe ansatz. The latter is undoubtedly one of the most significant contributions of the St. Petersburg school to the theory of quantum inverse scattering method. Subsequently a number of variants of the algebraic Bethe ansatz have been developed to deal with situations where the standard technique fails, namely the functional Bethe ansatz, analytical Bethe ansatz, etc.

The quantum inverse scattering method is, in a sense, tailor made for discrete integrable systems. Initial studies of integrable systems were however confined mostly to continuous systems, and their quantization posed a major problem. With this in mind we analyze certain continuous integrable systems to provide the requisite background for further discussions.

4.2 Quantum Continuous Integrable Systems

Application of the quantum inverse scattering method to continuous integrable systems, presents a number of difficulties. In case of continuous systems the occurrence of divergences is a pertinent issue. In view of this Faddeev and his school [68] initially formulated the quantum inverse problem for continuous systems by making a suitable space discretization of the continuous Lax operator. However, the problem of deriving a suitable Lax operator correct up to all orders of Δ, the cell length of discretization, still remained. There are only a few continuous

models for which such an exact discretization is possible.

To initiate the discussion of continuous integrable systems and their quantization, we may recall that the essential reason for introducing the concept of *normal ordering* in field theory stemmed from the fact that in any quantum field theory, operator products unless properly written are often ill-defined objects. The expressions of the Hamiltonian, momenta, etc., have to be normal ordered to have precise meaning. In the discrete case, owing to analogy with statistical mechanics, one often overlooks these issues in the eagerness to obtain the energy excitations and the Bethe equations for the eigenmomenta together with their solution. However we will illustrate how the usual concepts of quantum field theory may be applied to continuous integrable systems with suitable examples. Our first example is the nonlinear Schrödinger model.

4.2.1 Nonlinear Schrödinger model

We begin by considering a particularly well-known continuous system, the nonlinear Schrödinger (NLS) model whose Hamiltonian is

$$H = \int dx \left(\frac{\partial u^\star}{\partial x} \frac{\partial u}{\partial x} + c u^\star(x)^2 u(x)^2 \right). \tag{4.2.1}$$

The Lax equation for this is

$$\frac{\partial \Psi}{\partial x} = L(\lambda)\Psi, \qquad \frac{\partial \Psi}{\partial t} = M(\lambda)\Psi, \tag{4.2.2}$$

where

$$L(\lambda) = \frac{i}{2}\lambda\sigma_3 + i\sqrt{c}u(x)\sigma_+ - i\sqrt{c}u^\star(x)\sigma_-. \tag{4.2.3}$$

We define the transition matrix $T(x, y; \lambda)$ as the solution of the following equation:

$$\frac{\partial T(x, y; \lambda)}{\partial x} = L(x, \lambda)T(x, y; \lambda), \text{ with } T(x, x; \lambda) = I. \tag{4.2.4}$$

Here $\sigma_i \, (i = 1, 2, 3)$ are the Pauli matrices with $\sigma_\pm = 1/2(\sigma_1 \pm \sigma_2)$. In the quantum mechanical case we assume $u(x), u^\star(x)$ are operators acting on a Fock space and obeying the commutation relations:

$$[u(x), u(y)] = [u^\star(x), u^\star(y)] = 0, \quad [u(x), u^\star(y)] = \delta(x - y). \tag{4.2.5}$$

The equations for the transition matrix are then given by

$$\frac{\partial T(x, y; \lambda)}{\partial x} =: L(x, \lambda)T(x, y; \lambda) :, \tag{4.2.6}$$

$$\frac{\partial T(x,y;\lambda)}{\partial y} = - : T(x,y;\lambda)L(y,\lambda) :, \quad \text{with} \quad T(x,x;\lambda) = I. \quad (4.2.7)$$

Here $: A :$ stands for normal ordering of the expression A, meaning that the creation and annihilation operators, $u^\star$ and u, respectively, occurring in the expression for A are written in such a manner that the annihilation operators always appear to the right of the creation operators. It is customary to convert the forward and backward equations for the monodromy matrix, i.e., (4.2.6 and 4.2.7) respectively, to integral equations and then compute the commutation relations for the field operators $u, u^\star$ with $T(x,y;\lambda)$. The results are as follows:

$$[u(z), T(x,y;\lambda)] = [u^\star(z), T(x,y;\lambda)] = 0 \quad z \notin [y,x],$$

$$[u(x), T(x,y;\lambda)] = -i\frac{\sqrt{c}}{2}\sigma_- T(x,y;\lambda),$$

$$[u(y), T(x,y;\lambda)] = -i\frac{\sqrt{c}}{2}T(x,y;\lambda)\sigma_-,$$

$$[u^\star(x), T(x,y;\lambda)] = -\frac{i\sqrt{c}}{2}\sigma_+ T(x,y;\lambda),$$

$$[u^\star(y), T(x,y;\lambda)] = -\frac{i\sqrt{c}}{2}T(x,y;\lambda)\sigma_+,$$

$$[u(z), T(x,y;\lambda)] = -i\sqrt{c}T(x,z;\lambda)\sigma_- T(z,y;\lambda) \text{ when } y < z < x. \quad (4.2.8)$$

Now consider the quantity $[T(x,y;\lambda) \overset{\otimes}{,} T(x,y;\mu)]$ whose matrix elements are defined by

$$[T(x,y;\lambda) \overset{\otimes}{,} T(x,y;\mu)]_{ij,kl} = [T_{ik}(x,y;\lambda), T_{jl}(x,y;\lambda)]. \quad (4.2.9)$$

It is easy to see that

$$[T(x,y;\lambda) \overset{\otimes}{,} T(x,y;\mu)] = T(x,y;\lambda) \otimes T(x,y;\mu)$$

$$-\mathcal{P}T(x,y;\mu) \otimes T(x,y;\lambda)\mathcal{P}. \quad (4.2.10)$$

Using the commutation relations (4.2.8), we can rewrite (4.2.6) and (4.2.7) as

$$\frac{\partial T(x,y;\lambda)}{\partial x} = \widetilde{L}(x,\lambda)T(x,y;\lambda),$$

$$\frac{\partial T(x,y;\lambda)}{\partial y} = -T(x,y;\lambda)\widetilde{L}(x,\lambda), \quad (4.2.11)$$

where $\widetilde{L}(x,\lambda) = L(x,\lambda)\frac{c}{2}\sigma_+\sigma_-$ and we have dropped the normal ordering symbol. An important relation that will be required in our subsequent analysis is

$$[T(x,y;\lambda) \overset{\otimes}{,} T(x,y;\mu)] = \int_y^x \int_y^x dzdz' T(x,y;\lambda)\otimes T(x,z';\mu)\times$$

$$\times[\widetilde{L}(z,\lambda)\otimes\widetilde{L}(z',\mu)]\mathcal{P}T(z',y;\mu)\otimes T(z,y;\lambda)\mathcal{P}. \tag{4.2.12}$$

Regarding the proof of this relation we consider the $(\alpha\beta,\gamma\delta)$ element of the right-hand side that gives

$$\int_y^x \int_y^x dzdz' T_{\alpha\rho}(x,z;\lambda)T_{\beta\sigma}(x,z';\mu)(\widetilde{L}_{\rho\theta}(z,\lambda)\widetilde{L}_{\sigma\phi}(z',\mu)-$$

$$-\widetilde{L}_{\sigma\phi}(z',\mu)\widetilde{L}_{\rho\theta}(z,\lambda))T_{\phi\delta}(z',y;\mu)T_{\theta\gamma}(z,y;\lambda). \tag{4.2.13}$$

Using the equations for the transition matrix in their component form, viz

$$\widetilde{L}_{\sigma\phi}(z',\mu)T_{\phi\delta}(z',y;\mu) = \frac{\partial}{\partial z'}T_{\sigma\delta}(z',y;\mu),$$

$$-T_{\beta\sigma}(x,z';\mu)\widetilde{L}_{\sigma\phi}(z';\mu) = \frac{\partial}{\partial z'}T_{\beta\phi}(x,z';\mu), \tag{4.2.14}$$

we can write (4.2.13) in the following form:

$$\int_y^x dz T_{\alpha\rho}(x,z;\lambda)\int_y^x \frac{\partial}{\partial z'}\left(T_{\beta\sigma}(x,z';\mu)\widetilde{L}_{\rho\theta}(z,\lambda)T_{\sigma\delta}(z',y;\mu)\right)T_{\theta\gamma}(z,y;\lambda).$$

Integrating with respect to z' and using $T_{\beta\alpha}(x,x;\mu) = \delta_{\beta\alpha}$ we therefore obtain

$$\int_y^x T_{\alpha\rho}(x,z;\lambda)\widetilde{L}_{\rho\theta}(z,\lambda)T_{\beta\delta}(x,y;\mu)T_{\theta\delta}(z,y;\lambda)-$$

$$-T_{\alpha\rho}(x,z;\lambda)T_{\beta\delta}(x,y;\mu)\widetilde{L}_{\rho\theta}(z,\lambda)T_{\theta\gamma}(z,y;\lambda). \tag{4.2.15}$$

Owing to the fact that

$$T_{\alpha\rho}(x,z;\lambda)\widetilde{L}_{\rho\theta}(z;\lambda) = -\frac{\partial}{\partial z}T_{\alpha\theta}(x,z;\lambda),$$

$$\widetilde{L}_{\rho\theta}(z,\lambda)T_{\theta\gamma}(z,y;\lambda) = \frac{\partial}{\partial z}T_{\rho\gamma}(z,y;\lambda), \tag{4.2.16}$$

the second integration can be performed to obtain

$$-T_{\beta\delta}(x,y;\mu)T_{\alpha\gamma}(x,y;\lambda) + T_{\alpha\gamma}(x,y;\lambda)T_{\beta\delta}(x,y;\mu), \tag{4.2.17}$$

which is nothing but the left-hand side of (4.2.12). Making use of the relation

$$[\widetilde{L}(z;\lambda)\otimes\widetilde{L}(z';\mu)] = c[\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+]\delta(z-z'), \tag{4.2.18}$$

in (4.2.12) yields

$$[T(x,y;\lambda)\overset{\otimes}{,}T(x,y;\mu)] = \frac{c}{2}\int_y^x dz T(x,z;\lambda)\otimes T(x,z;\mu)(\sigma_3\otimes I - I\otimes\sigma_3)$$

$$\times T(z,y;\mu)\otimes T(z,y;\lambda)\mathcal{P}. \tag{4.2.19}$$

Since

$$[T(x,z;\mu),T(z,y;\mu)] = 0, \tag{4.2.20}$$

the right-hand side of (4.2.19) is simply

$$\frac{c}{2}\int_y^x dz(T(x,z;\lambda)\sigma_3 T(z,y;\mu))\otimes T(x,z;\mu)T(z,y;\lambda) -$$

$$-T(x,z;\lambda)T(z,y;\mu)\otimes T(x,z;\mu)\sigma_3 T(z,y;\lambda)\mathcal{P}. \tag{4.2.21}$$

One can further prove that

$$2i(\lambda-\mu)^{-1}\frac{\partial}{\partial z}\left(T(x,z;\lambda)T(z,y;\mu)\otimes T(x,z;\mu)T(z,y;\lambda)\right)$$

$$= T(x,z;\lambda)\sigma_3 T(z,y;\mu)\otimes T(x,z;\mu)T(z,y;\lambda) -$$

$$-T(x,z;\lambda)T(z,y;\mu)\otimes T(x,z;\mu)\sigma_3 T(z,y;\lambda), \tag{4.2.22}$$

which, when used in (4.2.21), gives

$$[T(x,y;\lambda)\otimes T(x,y;\mu)]$$

$$= \left(\frac{ic}{\lambda-\mu}\right)[T(x,y;\mu)\otimes T(x,y;\lambda) - T(x,y;\lambda)\otimes T(x,y;\mu)]\mathcal{P}. \tag{4.2.23}$$

Finally using (4.2.10) we can convert (4.2.23) to

$$R(\lambda,\mu)T(x,y;\lambda)\otimes T(x,y;\mu) = T(x,y;\mu)\otimes T(x,y;\mu)R(\lambda,\mu), \tag{4.2.24}$$

where

$$R(\lambda,\mu) = -\frac{ic}{\lambda-\mu-ic}I + \frac{\lambda-\mu}{\lambda-\mu-ic}\mathcal{P}. \tag{4.2.25}$$

Equation (4.2.24) is the fundamental relation for the formulation of the quantum inverse problem. An important observation here is that this equation is actually equivalent to the following differential identity:

$$\frac{\partial}{\partial z}[T(x,z;\mu)\otimes T(x,z;\lambda)R(\lambda,\mu)T(z,y;\lambda)\otimes T(z,y;\mu)]=0. \quad (4.2.26)$$

A simple proof of this assertion follows upon integration over the interval $y \le z \le x$, which leads to the Yang-Baxter equation. Conversely, if the Yang-Baxter equation holds for all x and y, then it certainly holds for $y = z$. Multiplying the Yang-Baxter equation for x, z on the right by $T(z,y;\lambda)\otimes T(z,y;\mu)$, we get

$$\begin{aligned} &R(\lambda,\mu)[\{T(x,z;\lambda)\otimes T(x,z;\mu)\}\{T(z,y\lambda)\otimes T(z,y;\mu\}] \\ &= T(x,z;\mu)\otimes T(x,z;\lambda)R(\lambda,\mu)T(z,y;\lambda)\otimes T(z,y;\mu). \end{aligned}$$

Since $T(x,z;\mu)$ and $T(z,y;\mu)$ depend on field operators on nonoverlapping intervals and as these field operators commute, we find that

$$[T(x,z;\lambda)\otimes T(x,z;\mu)][T(z,y;\lambda)\otimes T(z,y;\mu)] = T(x,y;\lambda)\otimes T(x,y;\mu). \quad (4.2.27)$$

Hence, the left-hand side is independent of z and differentiation with respect to z proves the identity.

The above discussion shows how the usual procedure of normal ordering together with the Lax operator can be used to obtain the Yang-Baxter equation in case of continuous systems. It should be mentioned however that other authors have adopted a slightly different approach and have used the properties of square eigenfunctions to verify the same relation.

We shall now consider the example of the massive Thirring model to further illustrate this procedure, since it involves the use of graded products.

4.2.2 Thirring model

The Thirring model, apart from being an example of an integrable system, is important from a physical point of view, as it describes essentially the dynamics of a two-dimensional fermionic field. There are two different formulations for describing this system, depending on the commuting or anticommuting properties of the field. The transition matrix for the Thirring model is defined as usual to be the solution of

$$\frac{\partial T(x;\lambda)}{\partial x} =: L(x;\lambda)T(x;\lambda) :, \quad (4.2.28)$$

with the condition that $T(x=0;\lambda)=I$. Here we are to follow normal ordering of fermions. The Lax operator is given by

$$L(x;\lambda)=\frac{i}{2}m\sinh\lambda\tau_3+\Sigma(x)+S(x;\lambda), \tag{4.2.29}$$

with

$$\Sigma(x)=\begin{pmatrix}-\phi^+(x)(1-e^{ic\sigma_3})\phi(x) & 0\\ 0 & -\phi^+(x)(1+e^{ic\sigma_3})\phi(x)\end{pmatrix}, \tag{4.2.30}$$

$$\frac{S(x;\lambda)}{i\sqrt{m\sin c}}=\begin{pmatrix}0 & e^{-\lambda/2}\phi_1(x)-e^{\lambda/2}\phi_2(x)\\ -e^{-\lambda/2}\phi_1^\dagger(x)+e^{\lambda/2}\phi_2^\dagger(x) & 0\end{pmatrix}, \tag{4.2.31}$$

where the Lax matrix is a super matrix with its row and column parities $p(1)=0$ and $p(2)=1$, respectively. As discussed previously we can convert (4.2.28) to the following integral equation over the interval $0\le x\le M$:

$$T(x;\lambda)=\exp(\frac{im}{2}x\sinh\lambda\tau_3)+\int_0^x dz\exp\{\frac{im}{2}(x-z)\sinh\lambda\tau_3\}$$
$$\times:(\Sigma(z)+S(z;\lambda))T(z;\lambda):. \tag{4.2.32}$$

From this we can derive the following auxiliary relations:

$$\phi_\nu T(x;\lambda)=\tau_3T(x;\lambda)\tau_3\phi_\nu(x)+\frac{1}{2}:E_\nu(x;\lambda)T(x;\lambda):,$$

$$T(x;\lambda)\phi_\nu^\dagger(x)=\phi_\nu^\dagger(x)\tau_3T(x;\lambda)\tau_3+\frac{1}{2}:F_\nu(x;\lambda)\tau_3T(x;\lambda)\tau_3:, \tag{4.2.33}$$

with

$$E_\nu(x;\lambda)=-\phi_\nu(x)[1-\tau_3e^{ic\nu\tau_3(-1)^{\nu+1}}]+i\sqrt{m\sin c}(-1)^\nu e^{(-1)^\nu\lambda/2}\tau_-$$
$$F_\nu(x;\lambda)=-\phi_\nu^+(x)[1-\tau_3e^{ic\nu\tau_3(-1)^{\nu+1}}]-i\sqrt{m\sin c}(-1)^\nu e^{(-1)^\nu\lambda/2}\tau_+. \tag{4.2.34}$$

Now let us define a tensor product,

$$K(x;\lambda,\mu)=T(x,\lambda)\otimes_s T(x;\mu), \tag{4.2.35}$$

where the tensor product $\otimes_s$ is of the super form defined by

$$(A\otimes_s B)_{ij}^{kl}=-(-1)^{p(j)(p(k)+p(i))}A_{ik}B_{jl}. \tag{4.2.36}$$

By differentiating (4.2.35) and using (4.2.28) and (4.2.29), we get

$$\partial_x K(x;\lambda,\mu) =: M(x;\lambda,\mu)K(x;\lambda,\mu) :, \tag{4.2.37}$$

with

$$M(x;\lambda,\mu) = L(x;\lambda)\otimes_s I + I\otimes_s L(x;\mu) + \sum_\nu (x;\lambda)\otimes_s E_\nu(x;\mu). \tag{4.2.38}$$

One can then show that the following relation holds:

$$R(\lambda-\mu)M(x;\lambda,\mu) = M(x;\mu,\lambda)R(\lambda-\mu), \tag{4.2.39}$$

which means that the R matrix can exchange λ,μ in $L(x;\lambda,\mu)$ in the same manner as in the Yang-Baxter equation. It also implies that this relation also holds good for $K(x;\lambda,\mu)$, which is nothing but the Yang-Baxter equation. The quantity $K(x;\lambda,\mu)$ is similar to the classical square eigenfunctions, which are known to form a complete set, from classical inverse scattering theory. In case of the fermionic model, the transfer matrix $t(\lambda)$ is given by the super trace of $T(\lambda)$, which is $t(\lambda) = T_{11}(\lambda) - T_{22}(\lambda)$. Integrability then implies that

$$[t(\lambda), t(\mu)] = 0. \tag{4.2.40}$$

We have shown how a continuous classical integrable equation may be analysed in the light of the quantization procedure and a reasonbly proper derivation of the Yang-Baxter equation can be obtained. The algebraic Bethe ansatz actually starts from the Yang-Baxter equation after representing the monodromy matrix in the following manner:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{4.2.41}$$

to get the commutation relations of $A(\lambda), B(\lambda)$, etc. These are then utilized to define the Hamiltonian and creation/ annihilation operators for the n excitation Bethe states. In the above example we wanted to show that it is possible to prove the Yang-Baxter equation for the case of a nonlinear integrable system defined in continuous space time. Furthermore, it is in the continuous domain alone that one can understand the problem for nonultralocal systems and the difficulties they pose even for a proof of the Yang-Baxter equation. Moreover, the identification of creation and annihilation operators in the continuous case has great implications on our subsequent discussion.

4.3 Conserved Quantities

Since complete integrability is always associated with the existence of an infinite number of conserved quantities, we will now dwell on the issue of conserved quantities for quantized systems. We shall explain their occurrence with the quantum nonlinear Schrödinger equation as our prototype example. The space part of the Lax operator for this equation is

$$\psi_{1x} + \frac{i\lambda}{2}\psi_1 = i\sqrt{c}u^\star\psi_2,$$

$$\psi_{2x} - \frac{i\lambda}{2}\psi_2 = -i\epsilon\sqrt{c}\psi_1 u. \tag{4.3.1}$$

On the other hand, the corresponding time part is given by

$$\psi_{1t} = c_1\psi_1 - i\sqrt{c}\lambda u^\star\psi_2 + \sqrt{c}u_x^\star\psi_2 + iku^\star\psi_1 u,$$

$$\psi_{2t} = c_2\psi_2 + i\epsilon\sqrt{c}\lambda u\psi_1 + \epsilon\sqrt{c}u_x\psi_1 - iku^\star\psi_2 u, \tag{4.3.2}$$

where $\epsilon = \pm 1, k = \epsilon c$ and $c_1 - c_2 = i\lambda^2$. Here we have followed the convention of Wadati et al. [69].

The asymptotic behaviour of $\Psi = (\psi_1, \psi_2)^t$ is important both in the classical and quantum contexts. It is given by

$$\Psi(x;\lambda) = \begin{pmatrix}\psi_1(x;\lambda)\\ \psi_2(x,\lambda)\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}\exp(-i\lambda x/2), \text{ as } x \to -\infty \tag{4.3.3}$$

and

$$\Psi(x;\lambda) = \begin{pmatrix}\psi_1(x;\lambda)\\ \psi_2(x,\lambda)\end{pmatrix} \to \begin{pmatrix}A(\lambda)\exp(-i\lambda x/2)\\ B(\lambda)\exp(i\lambda x/2)\end{pmatrix}, \text{ as } x \to +\infty. \tag{4.3.4}$$

The differential equations (4.3.1) and the boundary conditions can be combined into the following integral equations:

$$\psi_1(x;\lambda)e^{i\lambda x/2} = 1 + i\sqrt{c}\int dy\theta(x>y)e^{i\lambda y}u^\star(y)\psi_2(y;\lambda)e^{-i\lambda y},$$

$$\psi_2(x;\lambda)e^{-i\lambda x/2} = -i\epsilon\sqrt{c}\int dy\theta(x>y)e^{-i\lambda y}(y)\psi_1(y;\lambda)e^{i\lambda y}u(y). \tag{4.3.5}$$

By iteration we then have

$$\psi_1(x;\lambda)e^{i\lambda x/2} = 1 + \sum_{n=1}^{\infty} \sqrt[n]{c} \int ... \int dx_1...dx_n dy_1....dy_n \times$$

$$\times \theta(x > x_1 > y_1.... > x_n > y_n) e^{i\lambda \sum_{i=1}^{n}(x_i - y_i)}$$

$$\times u^{\star}(x_1)...u^{\star}(x_n)u(y_n)....u(y_1)$$

$$\psi_2(x;\lambda)e^{-i\lambda x/2} = -i\epsilon \sum_{n=1}^{\infty} \sqrt[n+1]{c} \int ... \int dx_1...dx_n dy_1....dy_{n+1} \times$$

$$e^{i\lambda \sum_{i=1}^{n}(x_i - y_i)} \theta(x > x_1 > y_1.... > x_n > y_{n+1}) \times$$

$$u^{\star}(x_1)...u^{\star}(x_n)u(y_{n+1})....u(y_1), \tag{4.3.6}$$

with $\theta(x_1 > x_2 > ... > x_n) = \theta(x_1 - x_2)\theta(x_2 - x_3)....\theta(x_{n-1} - x_n)$. If the asymptotic limits are taken of these expressions, then we find that

$$A(\lambda) = 1 + \sum_{n=1}^{\infty} \sqrt[n]{c} \int ... \int dx_1..dx_n dy_1...dy_n e^{i\lambda \sum_{i=1}^{n}(x_i - y_i)} \times$$

$$\times \theta(x_1 > x_2 > ... > y_1 > ... > y_n) u^{\star}(x_1)....u^{\star}(x_n)u(y_1)...u(y_n),$$

$$B(\lambda) = -i\epsilon \sum_{n=1}^{\infty} \sqrt[n+1]{c} \int ... \int dx_1...dx_n dy_1....dy_{n+1} e^{i\lambda \sum_{i=1}^{n}(x_i - y_i)}$$

$$\times \theta(x > x_1 > y_1.... > x_n > y_{n+1}) u^{\star}(x_1)...u^{\star}(x_n)u(y_{n+1})....u(y_1). \tag{4.3.7}$$

It is interesting to note that the operator $A(\lambda)$ contains equal numbers of creation and annihilation operators. Furthermore, it is known that $A(\lambda)$ is independent of time and generates conserved quantities. On the other hand in the expression for $B(\lambda)$, there is an additional creation operator, and hence the reason for using $B(\lambda)$ to generate a n-excitation state in the case of the algebraic Bethe ansatz. The infinite number of conserved quantities can be obtained by an expansion of $A(\lambda)$ of the form

$$A(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{A_n}{(i\lambda)^n}. \tag{4.3.8}$$

4.4 Nonultralocal Systems and the YBE

Having discussed continuous systems and their quantization in the previous sections, we are now in a position we consider the case of integrable nonultralocal systems. For classical integrable systems, when one evaluates the symplectic operators it often turns out that they contain derivatives of the delta function. These are cases that have no counterparts in usual quantum field theory, and as such they were discarded due to the presence of divergences and for not being renormalizable. However, in two dimensions it is possible to analyse such systems with a certain degree of rigour.

In Chapter 3 we indicated how the scattering matrix may be related to the Yang-Baxter equation. The former is of prime concern in any quantum field theoretical analysis. We shall try to give an idea of the attempts that have been made to encompass different nonultralocal theories within the broader framework of the Yang-Baxter equation. Until now our treatment has centered around describing continuous integrable systems in quantum mechanical terms, using the notions of operators and normal orderings. However, one can always employ the purely classical Lax operators for a derivation of the Yang-Baxter equation. In the following we shall concern ourselves with the initial attempts at understanding nonultralocal systems and will follow mostly Tsyplyev's paper [70].

We once again consider the equation for the transition matrix, i.e.,

$$\frac{\partial}{\partial x}T(x,y;\lambda) = L(x;\lambda)T(x,y;\lambda) \text{ with } T(x,x;\lambda) = I. \tag{4.4.1}$$

In the case of fields that decrease sufficiently rapidly as $|x| \to \infty$, one can define the monodromy matrix on the entire x axis, in the following manner:

$$T(\lambda) = \lim_{\substack{x\to\infty \\ y\to-\infty}} T_+^{-1}(x;\lambda)T(x,y;\lambda)T_-(y;\lambda), \tag{4.4.2}$$

where $T_\pm(x;\lambda)$ denotes the solutions of (4.4.1) for the asymptotic matrix,

$$L_\pm(\lambda) = \lim_{x\to\pm\infty} L(x;\lambda).$$

For the Poisson bracket of the monodromy matrix $T(\lambda)$ we find that

$$\{T(\lambda)\otimes, T(\mu)\} = r_+(\lambda,\mu)T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu)r_-(\lambda,\mu), \tag{4.4.3}$$

where $r_{\pm}(\lambda,\mu)$ are defined by

$$r_{\pm}(\lambda,\mu) = \lim_{x\to\pm\infty} T_{\pm}^{-1}(x;\lambda)\otimes T_{\pm}^{-1}(x;\mu) r(\lambda,\mu) T_{\pm}(x;\lambda)\otimes T_{\pm}(x,\mu). \tag{4.4.4}$$

Defining the Jost functions as

$$\phi_{\pm}(x;\lambda) = \lim_{y\to\pm\infty} T(x,y;\lambda) T_{\pm}(x;\lambda), \tag{4.4.5}$$

the Poisson bracket $\{T(\lambda)\otimes, T(\mu)\}$ may be evaluated from the relation

$$\{T(\lambda)\otimes T(\mu)\} = \int\int_{-\infty}^{\infty} dxdy \phi_{+}^{-1}(x;\lambda)\otimes\phi_{+}^{-1}(y;\mu)\{L(x;\lambda)\otimes L(y;\mu)\}\times$$

$$\times\phi_{-}(x;\lambda)\otimes\phi_{-}(y;\mu). \tag{4.4.6}$$

In the derivation of this relation use has been made of the well-known variational formula,

$$\delta T(\lambda) = \int_{-\infty}^{\infty} dx \phi_{+}^{-1}(x;\lambda)\delta L(x;\lambda)\phi_{-}(x;\lambda).$$

For a nonultralocal system let us assume that the explicit computation of Poisson brackets between elements of the Lax operator can be expressed as

$$\{L(x;\lambda) \overset{\otimes}{,} L(y;\mu)\} = \left(\sum_{i,k=0}^{N} \omega_{ik}\partial_x^i\partial_y^k\right)\delta(x-y), \tag{4.4.7}$$

where ω_{ik} are nonconstant matrices, defined in the tensor product $V\otimes V$, which depend on λ and μ as on the nonlinear field variables. The antisymmtery of the Poisson brackets implies that

$$P\omega_{ij}(\lambda,\mu;x,y)P = -\omega_{ji}(\lambda,\mu;x,y).$$

Using the equation

$$(\partial_x - L(x;\lambda))\phi_{\pm}(x;\lambda) = 0,$$

it is possible to perform one integration in (4.4.6) to get

$$\{T(\lambda)\otimes, T(\mu)\} = \int_{-\infty}^{\infty} dx \phi_{+}^{-1}(x;\lambda)\otimes\phi_{+}^{-1}(x;\mu)$$

$$\times\Omega(\lambda,\mu;x)\phi_{-}(x;\lambda)\otimes\phi_{-}(x;\mu). \tag{4.4.8}$$

The matrix $\Omega(\lambda,\mu;x)$ can be constructed out of ω_{ik} and $L(x;\lambda)$ and is given by

$$\Omega(\lambda,\mu;x)=\sum_{i,k=0}^{N}\sum_{m,n,s=0}^{\substack{l+n=k\\ s+m=i}}P^{ik}_{slmn}D^{sl}_{-}(\lambda,\mu;x)$$

$$\times\left(\partial_x^{i-s-m}\partial_y^{k-i-n}\omega_{ik}(\lambda,\mu;x,y)\right)_{y=x}D^{mn}_{+}(\lambda,\mu;x). \tag{4.4.9}$$

It is important to note that the integrand of (4.4.8) will be the derivative of the following expression:

$$\phi_+^{-1}(x;\lambda)\otimes\phi_+^{-1}(x;\mu)r(\lambda,\mu;x)\phi_-(x;\lambda)\otimes\phi_-(x;\mu)$$

if the matrix $r(\lambda,\mu)$ satisfies the equation

$$\partial_x r(\lambda,\mu;x)+[r(\lambda,\mu;x),L(x;\lambda)\otimes I+I\otimes L(x;\mu)]=\Omega(\lambda,\mu;x,y), \tag{4.4.10}$$

whence one obtains (4.4.3), i.e.,

$$\{T(\lambda)\otimes_{,}T(\mu)\}=r_+(\lambda,\mu)T(\lambda)\otimes T(\mu)-T(\lambda)\otimes T(\mu)r_-(\lambda,\mu),$$

with $r_\pm(\lambda,\mu)$ connected to the classical r matrix, $r(\lambda,\mu)$ by the relation (4.4.4).

Example: Let us consider an equation studied by Wadati et al. [34, 35] namely

$$\partial_t q=-\frac{1}{2}\partial_x^2\{q(1+q^\star q)^{-\frac{1}{2}}\}, \tag{4.4.11}$$

where q is a complex field. The Hamiltonian for this equation is

$$H=\int dx\{(1+q^\star q)^{\frac{1}{2}}-1\}.$$

The complex nonlinear fields obey the Poisson brackets:

$$\{q(x),q^\star(y)\}=\delta''(x-y),\quad \{q(x),q(y)\}=\{q^\star(x),q^\star(y)\}=0. \tag{4.4.12}$$

Here the $'$ denotes differentiation of the delta function with respect to its argument. The Lax matrix associated with this equation is

$$L(x;\lambda)=-i\lambda(\sigma_3+q(x)\sigma_++q^\star(x)\sigma_-). \tag{4.4.13}$$

It is then straightforward to calculate the Poisson brackets of the Lax matrix, and we find that

$$\{L(x;\lambda)\otimes_{,}L(y;\mu)\}=4\omega\delta''(x-y), \tag{4.4.14}$$

where

$$\omega = -\frac{i}{4}\lambda\mu(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+). \tag{4.4.15}$$

Let us now write

$$L_\pm(\lambda, \mu; x) = L(x; \lambda) \otimes I \pm I \otimes L(x; \mu), \tag{4.4.16}$$

then (4.4.10) for the r matrix becomes

$$\partial_x r + [r, L_+] = \partial_x[\omega, L_+] + [L_-, [L_-, \omega]].$$

As we are interested in a local solution of this equation, we demand that

$$\partial_x r = \partial_x[\omega, L_+],$$

which implies that

$$r(\lambda, \mu; x) = [\omega, L_+] + r'(\lambda, \mu).$$

Here $r'(\lambda, \mu)$ satisfies the equation.

$$[r', L_+] = [L_-, [L_-, \omega]] - [L_+, [L_+, \omega]].$$

Hence, by writing $Q(x) = q(x)\sigma_+ + q^\star(x)\sigma_-$, we obtain

$$r(\lambda, \mu; x) = \lambda\mu \left[\frac{\lambda\mu}{\lambda - \mu}\sigma_3 \otimes \sigma_3 + \frac{(\lambda + \mu)^2}{2(\lambda - \mu)}(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)\right]$$

$$+\frac{\lambda\mu}{4}(\lambda\sigma_3 \otimes Q(x) + \mu Q(x) \otimes \sigma_3). \tag{4.4.17}$$

From (4.4.17) we can easily compute $r_\pm$ if we note that the asymptotic solutions $\phi_\pm(x; \lambda)$ can be taken in the form,

$$\phi_\pm(x; \lambda) = \exp(-i\lambda\sigma_3 x).$$

Consequently, we obtain from (4.4.4):

$$r_\pm(\lambda, \mu) = \lambda^2\mu^2 \left[\frac{1}{\lambda - \mu}\sigma_3 \otimes \sigma_3 \pm 2\pi i\delta(\lambda - \mu)(\sigma_+ \otimes \sigma_-\sigma_- \otimes \sigma_+)\right]. \tag{4.4.18}$$

Thus we have obtained a well defined r matrix for the nonultralocal system under discussion. We shall now discuss the quantization of such systems.

4.4.1 Quantization of nonultralocal systems

In order to quantize such systems we use, as a preliminary approximation, the technique of lattice approximation as formulated by Faddeev and others [71]. First of all we note from (4.4.1), that the transition matrix is formally given by

$$T(x,y;\lambda) = X \exp\left(\int_y^x L(z;\lambda)dz\right), \tag{4.4.19}$$

where "X" denotes x ordering, which is basically similar to the notion of time ordering in quantum field theory. We split the interval (x,y) into N intervals of length $\Delta = \frac{1}{N}(x-y)$ and denote by x_n the coordinate of the nth lattice point, with $x_0 = y, x_N = x$. Then the matrix $T(x,y;\lambda)$ can be represented as a product,

$$T(x,y;\lambda) = \prod_n T_n(\lambda) \quad \text{where} \quad T_n(\lambda) = T_n(x_n, x_{n-1};\lambda).$$

Furthermore, we demand that

$$\{T_n(\lambda) \overset{\otimes}{,} T_m(\mu)\} = 0, \text{ for } m \neq n. \tag{4.4.20}$$

Because of this one can consider the formula locally only for the matrices $T_n(\lambda)$ at each lattice site. In the usual ultralocal situation it is possible to restrict oneself to only a first approximation in Δ in the formula for $T_n(\lambda)$, i.e.,

$$T_n(\lambda) \approx 1 + \int_{x_{n-1}}^{x_n} L(z;\lambda)dz,$$

because we consider $\delta(x-y)$ to be of the order of Δ^{-1}. But in the nonultralocal case if in general the Poisson brackets of the elements of $L(x;\lambda)$ involve the nth derivatives of the delta function, then since $\delta^{(n)} \sim \Delta^{-(n+1)}$, one has to modify the above approximation to include higher-order terms so that

$$T_n(\lambda) \approx X \sum_{k=0}^{N+1} \frac{1}{k!} \left(\int_{x_{n-1}}^{x_n} L(z;\lambda)dz\right)^k. \tag{4.4.21}$$

Example:
To illustrate this point we consider the Sine-Gordon equation, which in light-cone coordinates is given by

$$\phi_{xt} + \frac{1}{2}\sin 2\phi = 0.$$

Here $t = (x_0 - x_1)/2$ and $x = (x_0 + x_1)/2$. The Hamiltonian for this equation is

$$H = \int_{-\infty}^{\infty} \sin^2 \phi dx,$$

with the fields obeying the Poisson brackets $\{\phi(x), \phi_y(y)\} = \frac{1}{2}\delta(x-y)$. Now the Lax matrix for the Sine-Gordon equation is given by

$$L(x;\lambda) = i(u(x)\sigma_3 + \lambda\sigma_2),$$

where $u(x) = \phi_x(x)$, and one can easily verify that

$$\{L(x;\lambda) \overset{\otimes}{,} L(y;\mu)\} = -\frac{1}{2}\sigma_3 \otimes \sigma_3 \delta'(x-y).$$

Retaining therefore terms up to order Δ^2, we get for the discretized monodromy matrix the following approximation:

$$T_n(\lambda) \approx \begin{pmatrix} 1 + ip_n - p_n^2/2 & \Delta\lambda(1 - iq_n) \\ -\Delta\lambda(1 + iq_n) & 1 - ip_n - p_n^2/2 \end{pmatrix}, \tag{4.4.22}$$

where

$$p_n = \int_{x_{n-1}}^{x_n} u(x)dx, \qquad q_n = \Delta^{-1} \int_{x_{n-1}}^{x_n} dx_n u(x)(x_n + x_{n-1} - 2x). \tag{4.4.23}$$

The discrete variables (p_n, q_n) satisfy the following canonical Poisson brackets:

$$\{p_n, p_m\} = \{q_n, q_m\} = 0 \quad \text{and} \quad \{q_n, p_m\} = \frac{1}{2}\delta_{nm}.$$

Consequently, the relation (4.4.21) is satisfied up to the required accuracy in Δ if the classical r matrix is given by

$$r(\lambda,\mu) = \left[\frac{\lambda^2 + \mu^2}{4(\lambda^2 - \mu^2)}\sigma_3 \otimes \sigma_3 + \frac{\lambda\mu}{2(\lambda^2 - \mu^2)}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)\right].$$

For calculating the quantum R matrix we are forced to use the monodromy matrix, as there is no formal prescription to compute the quantum counterpart of $T_n(\lambda)$. For this we notice that since the commutator $[\hat{u}(x), \hat{u}(y)] = \hat{u}(x)\hat{u}(y) - \hat{u}(y)\hat{u}(x)$ is equal to $\frac{i\hbar}{2}\delta'(x-y) \sim \Delta^{-2}$, we can ascribe to $\hat{u}(x)$ an order Δ^{-1}. Here $\hat{u}(x)$ is the operator corresponding to the field $u(x)$. It may be commented that the construction of the

quantum version of the matrix $T_n(\lambda)$ is still an open problem. So with these dimensional arguments, it is found that the operators p_n, q_n as defined by (4.4.23) in which $u(x)$ is replaced by $\hat{u}(x)$ have order Δ^{-1}. This makes it necessary to retain all powers of p_n and q_n in the expression for $T_n(\lambda)$ and we may assume that the quantum monodromy matrix is of the following form:

$$T_n(\lambda) = \begin{pmatrix} e^{i\hat{p}_n} & \Delta\lambda e^{-i\hat{q}_n} \\ -\Delta\lambda e^{i\hat{q}_n} & e^{-i\hat{p}_n} \end{pmatrix}. \tag{4.4.24}$$

The commutation relations for the operators $\hat{q}_n, \hat{p}_n$ are now as follows:

$$[\hat{p}_n, \hat{p}_m] = [\hat{q}_n, \hat{q}_m] = 0, \qquad [\hat{q}_n, \hat{p}_m] = \frac{i\hbar}{2}\delta_{nm}.$$

Evaluating $T(x, y; \lambda)$ in operator form and keeping in mind the commutation rules, we find that

$$R(\lambda, \mu) = \left[\frac{\sinh\left(\frac{\alpha-\beta-i\hbar}{2}\right)}{2\sinh\left(\frac{\alpha-\beta}{2}\right)} \sigma_0 \otimes \sigma_0 + \frac{\cosh\left(\frac{\alpha-\beta-i\hbar}{2}\right)}{2\cosh\left(\frac{\alpha-\beta}{2}\right)} \sigma_3 \otimes \sigma_3 \right] +$$

$$+ \sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ \text{ where } \alpha = \ln\lambda, \quad \beta = \ln\mu; \quad \sigma_0 = I.$$

It is evident that as $\hbar \to 0$ we get back the classical r matrix.

This example serves to illustrate the initial attempts at deriving the classical and quantum r matrices in case of nonultralocal systems. It is by no means a rigorous technique, since there are several obvious gaps, and more studies are required to tackle these and other related issues. For example, one still needs to develop a rigorous prescription for the construction of $T_n(\lambda)$, which will be valid for all orders of the lattice spacing Δ. Secondly, the exact form of the quantised L operator is also needed. However, we should mention here an important communication by Sklyanin [72], who showed how one can construct an exact quantum mechanical L operator from the discretization of the space part of the Lax pair, in case of the nonlinear Schrödinger equation. For more details on this method, including its application to the Sine-Gordon equation, we refer to the articles [73, 74].

4.5 Operator Product Expansion and the YBE

For continuous integrable systems there is an alternative procedure for deriving the Yang-Baxter equation, which is fundamental to the study of quantum field theory. It is well known from quantum field theory that the occurrence of ultraviolet diveregences is primarily due to the improper definition of operator products arising in the theory. It was through the remarkable work of K. Wilson [75] that one recognized how to obtain the equations of motion in a divergenceless manner in quantum field theory, through the concept of operator product expansion (OPE). The latter contains all the information of the commutation rules, besides taking care of the singularities arising in the various invariant delta functions.

In this section we show how by the simple use of OPEs, one is led to the Yang-Baxter equation. Moreover, we shall also in this context consider situations where the nonlinear field variables take asymptotically nonzero values. This allows us to derive a more general form of the quantum R matrix, along with the Yang-Baxter equation.

Let us once again consider the nonlinear Schrödinger equation, satisfying the Lax equation,

$$\Phi_x = iL(x;\lambda)\Phi, \tag{4.5.1}$$

with

$$L(x;\lambda) = \begin{pmatrix} i\lambda & \sqrt{\chi}\bar{\psi} \\ \sqrt{\chi}\psi & -i\lambda \end{pmatrix}. \tag{4.5.2}$$

The fields satisfy the Poisson brackets:

$$\{\psi(x), \bar{\psi}(y)\} = i\delta(x-y), \;\; \{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0.$$

As before we define the transition matrix $T(x, y; \lambda)$ as the solution of

$$\frac{\partial T(x,y;\lambda)}{\partial x} = L(x;\lambda)T(x,y;\lambda) \;\text{ with }\; T(x,x;\lambda) = I. \tag{4.5.3}$$

The solution of (4.5.3) is formally written as

$$T(x,y;\lambda) = P\exp\left\{\int_y^x L(\xi;\lambda)d\xi\right\} \tag{4.5.4}$$

where P now denotes an ordered exponential. Since $T(x,y;\lambda)$ is functionally dependent on the field variables $\psi,\bar{\psi}$ through $L(\xi;\lambda)$, the proper approach is to consider the singularities in the product $\psi(x)\bar{\psi}(y)$, which occur in the tensor product $T(x,y;\lambda)\otimes T(x,y;\mu)$. Using (4.5.4) we can write this tensor product as

$$T(x,y;\lambda)\otimes T(x,y;\mu)=\left(P\exp\left\{\int_{x-\Delta}^{x}L(\xi;\lambda)d\xi\right\}\right)\otimes$$

$$\otimes\left(P\exp\left\{\int_{x-\Delta}^{x}L(\zeta;\lambda)d\zeta\right\}\right)T(x-\Delta,y;\lambda)\otimes T(x-\Delta,y;\mu), \quad (4.5.5)$$

for arbitrary but small Δ. Differentiating this with respect to x we get

$$\frac{\partial}{\partial x}\left(T(x,y;\lambda)\otimes T(x,y;\mu)\right)=$$

$$\left[iL(x;\lambda)P\exp\left\{\int_{x-\Delta}^{x}L(\xi;\lambda)d\xi\right\}\otimes P\exp\left\{\int_{x-\Delta}^{x}L(\zeta;\lambda)d\zeta\right\}\right]\times$$

$$\times T(x-\Delta,y;\lambda)\otimes T(x-\Delta,y;\mu)+$$

$$+\left[P\exp\left\{\int_{x-\Delta}^{x}L(\xi;\lambda)d\xi\right\}\otimes iL(x;\mu)P\exp\left\{\int_{x-\Delta}^{x}L(\zeta;\lambda)d\zeta\right\}\right]\times$$

$$\times T(x-\Delta,y;\lambda)\otimes T(x-\Delta,y;\mu). \quad (4.5.6)$$

Next expanding the exponentials as $\Delta\to 0$, from (4.5.6) we obtain

$$\frac{\partial}{\partial x}\left(T(x,y;\lambda)\otimes T(x,y;\mu)\right)=\Gamma(x;\lambda,\mu)\left(T(x,y;\lambda)\otimes T(x,y;\mu)\right), \quad (4.5.7)$$

where $\Gamma(x;\lambda,\mu)$ is given by

$$\Gamma(x;\lambda,\mu)=iL(x;\lambda)\otimes I+iI\otimes L(x;\mu)-$$

$$-\int_{x-\Delta}^{x}L(x;\lambda)\otimes L(\zeta;\mu)d\zeta-\int_{x-\Delta}^{x}L(\xi;\lambda)\otimes L(x;\mu)d\xi+\mathcal{O}(\Delta). \quad (4.5.8)$$

Let us now write the Lax matrix in terms of the $su(2)$ generators, viz

$$L(x;\lambda)=t_i(\lambda)L_i(x),$$

where

$$t_1(\lambda)=\sqrt{\chi}\sigma_+,\quad t_2(\lambda)=\sqrt{\chi}\sigma_-,\quad t_3(\lambda)=\frac{\lambda\sigma_3}{2i}\quad \text{and}$$

$$L_1(x) = \bar{\psi}(x), \;\; L_2(x) = \psi(x), \;\; L_3(x) = 1.$$

Since in the tensor product $L(x;\lambda) \otimes L(\zeta;\mu)$, the operator product $L_i(x)L_j(\zeta)$ is governed by the commutator

$$[L_i(x), L_j(\zeta)],$$

therefore from the singularity as $\Delta \to 0$, we get

$$\int_{x-\Delta}^{x} L(x;\lambda) \otimes L(\zeta;\mu)d\zeta = t_i(\lambda) \otimes t_j(\mu) \int_{x-\Delta}^{x} L_i(x)L_j(\zeta)d\zeta$$

$$\approx t_i(\lambda) \otimes t_j(\mu) \int_{x-\Delta}^{x} [L_i(x), L_j(\zeta)]d\zeta \quad \text{as } \Delta \to 0. \tag{4.5.9}$$

Using the procedure outlined in (4.5.9), one can work out the rest of the operator products occurring in (4.5.8) to finally obtain the following expression for $\Gamma(x,\lambda,\mu)$, viz

$$\Gamma(x;\lambda,\mu) = i\sqrt{\chi}\bar{\psi}(x)(\sigma_+ \otimes I + I \otimes \sigma_+) + i\sqrt{\chi}\psi(x)(\sigma_- \otimes I + I \otimes \sigma_-) +$$

$$+\frac{1}{2i}(\lambda\sigma_3 \otimes I + \mu I \otimes \sigma_3) + 2\chi\hbar(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+). \tag{4.5.10}$$

The quantum R matrix can now be obtained from the fact that it intertwines $T(x,y;\lambda) \otimes T(x,y;\mu)$, i.e.,

$$R(\lambda,\mu)\,(T(x,y;\lambda) \otimes T(x,y;\mu)) = (T(x,y;\mu) \otimes T(x,y;\lambda))\,R(\lambda,\mu),$$

where

$$R(\lambda,\mu)\Gamma(x;\lambda,\mu) = \Gamma(x;\lambda,\mu)R(\lambda,\mu). \tag{4.5.11}$$

Assuming a general structure for the R matrix of the form

$$R(\lambda,\mu) = I \otimes I + R_1(\sigma_3 \otimes \sigma_3) + R_2(\sigma_2 \otimes \sigma_2) + R_3(\sigma_1 \otimes \sigma_1),$$

we find that

$$R(\lambda,\mu) = \begin{pmatrix} 1+p & 0 & 0 & 0 \\ 0 & 1-p & 2p & 0 \\ 0 & 2p & 1-p & 0 \\ 0 & 0 & 0 & 1+p \end{pmatrix}, \tag{4.5.12}$$

where $$p = \left(1 + \frac{8i\hbar\chi}{\lambda-\mu}\right)^{-1}.$$

As of now we have not used the asymptotic behaviour of the nonlinear fields. Let us therefore suppose that as $x \to \pm\infty$, $\psi(x)$ does not tend to zero but satisfies instead

$$\lim_{x\to\pm\infty} \psi(x) = \rho e^{i\phi_\pm} \quad \text{with } \phi_+ - \phi_- = \theta. \tag{4.5.13}$$

Such a situation corresponds to the case of finite density. For convenience we set $\phi_- = 0$ and $\phi_+ = \theta$, so that

$$\lim_{x\to-\infty} L(x;\lambda) = \frac{1}{2}\begin{pmatrix} -i\lambda & 2\sqrt{\chi}\rho \\ 2\sqrt{\chi}\rho & i\lambda \end{pmatrix} = L_-(\lambda). \tag{4.5.14}$$

Let $E_\rho^-(x;\lambda)$ denote the matrix solution of

$$\frac{d}{dx}E_\rho^-(x;\lambda) = L_-(\lambda)E_\rho^-(x;\lambda). \tag{4.5.15}$$

One can easily verify that

$$\begin{aligned} E_\rho^-(x;\lambda) &= \begin{pmatrix} 1 & i(k-\lambda)/\omega \\ i(\lambda-k)/\omega & 1 \end{pmatrix} \exp(ikx\sigma_3/2) \\ &= A_-(\omega,\lambda)\exp(ikx\sigma_3/2), \end{aligned} \tag{4.5.16}$$

with

$$A_-(\omega,\lambda) = \begin{pmatrix} 1 & i(k-\lambda)/\omega \\ i(\lambda-k)/\omega & 1 \end{pmatrix},$$

$$k(\lambda) = \sqrt{\lambda^2-\omega^2}, \qquad \omega = 2\sqrt{\chi}\rho.$$

A similar consideration for $x \to +\infty$, yields the solution

$$E_\rho^+(x;\lambda) = A_+(\omega,\lambda)\exp(-ikx\sigma_3/2), \tag{4.5.17}$$

with

$$A_+(\omega,\lambda) = \begin{pmatrix} 1 & \frac{-i(\lambda-k)}{\omega}e^{i\theta} \\ \frac{i(\lambda-k)}{\omega}e^{i\theta} & 1 \end{pmatrix}.$$

Consequently, the limiting form of $\Gamma(x;\lambda,\mu)$ is given by

$$\begin{aligned} \Gamma_0(\lambda,\mu) &= i\sqrt{\chi}\rho\left[(\sigma_1\otimes I + I\otimes\sigma_1)\cos\theta + (\sigma_2\otimes I + I\otimes\sigma_2)\sin\theta\right] + \\ &+ i\sqrt{\chi}\rho\left[i\hbar\chi(\sigma_2\otimes\sigma_1 - \sigma_1\otimes\sigma_2) + \frac{1}{2i}(\lambda\sigma_3\otimes I + \mu I\otimes\sigma_3)\right]. \end{aligned} \tag{4.5.18}$$

We clearly have from (4.5.11),

$$R(\lambda,\mu)\Gamma_0(\lambda,\mu) = \Gamma_0(\lambda,\mu)R(\lambda,\mu),$$

so that

$$\exp(-i\Gamma_0(\lambda,\mu))R(\lambda,\mu) = R(\lambda,\mu)\exp(-i\Gamma_0(\lambda,\mu)).$$

Now the scattering data is defined by

$$\lim_{\substack{y\to-\infty\\ x\to+\infty}} T(x,y;\lambda) = E^+_\rho(x,;\lambda)T(\lambda)(E^-_\rho(x;\lambda))^{-1}. \tag{4.5.19}$$

Upon passing over to these limits in the intertwining relation we get

$$R_1(x;\lambda,\mu)T(\lambda)\otimes T(\mu) = T(\mu)\otimes T(\lambda)R_2(x;\lambda,\mu), \tag{4.5.20}$$

where

$$R_1(x;\lambda,\mu) = Q_1R(\lambda,\mu)Q_2, \qquad R_2(x;\lambda,\mu) = S_1R(\lambda,\mu)S_2$$

and

$$\begin{aligned} Q_1 &= \exp(ik(\mu)x\sigma_3/2)A_+^{-1}(\mu)\otimes\exp(ik(\lambda)x\sigma_3/2)A_+^{-1}(\lambda),\\ Q_2 &= A_+(\lambda)\exp(-ik(\lambda)x\sigma_3/2)\otimes A_+(\mu)\exp(-ik(\mu)x\sigma_3/2),\\ S_1 &= \exp(ik(\mu)y\sigma_3/2)A_-^{-1}(\mu)\otimes\exp(ik(\lambda)y\sigma_3/2)A_-^{-1}(\lambda),\\ S_2 &= A_-(\lambda)\exp(-ik(\lambda)y\sigma_3/2)\otimes A_-(\mu)\exp(-ik(\mu)y\sigma_3/2). \end{aligned} \tag{4.5.21}$$

Finally, we arrive at

$$R_+(\lambda,\mu)T(\lambda)\otimes T(\mu) = T(\mu)\otimes T(\lambda)R_-(\lambda,\mu), \tag{4.5.22}$$

where

$$R_\pm(\lambda,\mu) = \lim_{x\to\pm\infty}\left(e^{ik(\mu)x\sigma_3/2}\otimes e^{ik(\lambda)x\sigma_3/2}\right)(A_\pm^{-1}(\mu)\otimes A_\pm^{-1}(\lambda))$$

$$\times R(\lambda,\mu)(A_\pm(\lambda)\otimes A_\pm(\mu))\lim_{x\to\pm\infty}\left(e^{-ik(\lambda)x\sigma_3/2}\otimes e^{-ik(\mu)x\sigma_3/2}\right). \tag{4.5.23}$$

The above expressions can be explicitly evaluated; for example, one finds that

$$A_-^{-1}(\mu)\sigma_1A_-(\lambda) = \left\{\frac{\omega^2}{2\mu(\mu-k(\mu))} + \frac{\lambda-k(\lambda)}{2\mu}\right\}\sigma_1$$

$$+\frac{i\omega}{2\mu}\left(1+\frac{\lambda-k(\lambda)}{\mu-k(\mu)}\right)\sigma_3,$$

and

$$A_-^{-1}(\mu)\sigma_2 A_-(\lambda)=\frac{\omega^2}{2\mu(\mu-k(\mu))}\sigma_2+\frac{i(\lambda-k(\lambda))}{2\mu}\sigma_3$$

$$+\frac{\omega}{2\mu}\left(\frac{\lambda-k(\lambda)}{\mu-k(\mu)}-1\right)I.$$

4.6 Finite Boundary Conditions

In Chapter 3 we dwelt on the effect of nonperiodic boundary conditions on the Yang-Baxter equation in the context of scattering theory within the framework developed by Cherednik [76]. The most notable outcome of the discussion was that the factorization of the n-particle scattering could be conceived of as a multitude of two-particle scattering amplitudes. However, this did not shed light on the more familiar concept of a boundary condition in the usual sense of classical field theory. Indeed, the notion of boundary conditions is intimately connected to the famous Dirichlet and Neumann problems. A crucial factor in case of integrable systems is that the imposition of boundary conditions should not destroy the integrability of the system. In a series of papers, Sklyanin et al. [78], have investigated the possible forms of nontrivial boundary conditions in relation to the KdV, mKdV equations. An alternative approach to analyse the effect of boundary conditions was given by Saha et al. [77].

In this section we first consider, the effect of different types of boundary conditions on the existence of the infinite conserved quantities, which are a characteristic feature of integrable systems. We will then investigate them in relation to the classical r matrix and Yang-Baxter equation. In discussing these issues we closely follow Sklyanin's formalism and consider the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}+2\bar{\psi}\psi^2. \tag{4.6.1}$$

The Hamiltonian for this equation is

$$H = \int dx \left\{ \frac{\partial \bar{\psi}}{\partial x} \frac{\partial \psi}{\partial x} + \bar{\psi}^2 \psi^2 \right\}, \tag{4.6.2}$$

where the nonlinear fields obey the Poisson brackets:

$$\{\psi(x), \bar{\psi}(y)\} = i\delta(x-y), \;\; \{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0. \tag{4.6.3}$$

It is customary to assume in classical inverse scattering theory that as $x \to \pm\infty$, $\psi(x) \to 0$. That is, one considers usually the entire x-axis and asymptotically decaying nonlinear fields. In the finite case this condition may be generalized to

$$\psi(x_+) = e^{i\theta} \psi(x_-), \tag{4.6.4}$$

where $[x_+, x_-]$ is the portion of the x-axis that is considered; this is referred to as the quasi-periodic situation. By analogy with the Sturm-Liouville theorem, it is natural to consider even more general boundary conditions such as:

$$\frac{\partial \psi}{\partial x}|_{x=x_\pm} + \theta_\pm \psi_\pm = 0, \qquad \psi_\pm = \psi(x_\pm). \tag{4.6.5}$$

The condition $\theta_\pm = 0$ now corresponds to the Neumann condition while the condition $\theta \to \infty$ represents the Dirichlet condition.

Let us first derive the form of the generating function for the infinite number of conserved quantities, when the boundary conditions are not periodic. Obviously, the preservation of the integrals of motion requires that the generating function should also be preserved. In the continuous case the transition matrix is formally given by

$$T(x, y; \lambda) = \overleftarrow{\exp} \left(\int_y^x L(z, \lambda) dz \right), \tag{4.6.6}$$

where $T(x, y; \lambda)$ satisfies the following group properties:

$$\begin{aligned} T(x, z; \lambda) T(z, y; \lambda) &= T(x, y; \lambda), \\ T(x, y; \lambda) &= T^{-1}(y, x; \lambda), \\ \frac{\partial T(x, y; \lambda)}{\partial y} &= -T(x, y; \lambda) L(y, \lambda) \end{aligned} \tag{4.6.7}$$

and

$$\det T(x, y; \lambda) = 1 \qquad \text{since} \quad \text{tr } L(x; \lambda) = 0. \tag{4.6.8}$$

Let γ be a curve with initial point (x_0, t_0) and final point (x, t) in the two-dimensional plane $R^2 : (x, t)$. Parallel transport from (x_0, t_0) to (x, t) along γ is given by

$$\Omega_\gamma = \overleftarrow{\exp} \left(\int_\gamma L dx + M dt \right), \tag{4.6.9}$$

where the integration is understood multiplicatively. Consider a partition of γ into $N - 1$, sections denoted by, say $\gamma_1, \gamma_2, ..., \gamma_N$, so that one can write

$$L_n = 1 + \int_{\gamma_n} (L dx + M dt), \tag{4.6.10}$$

$$\Omega_n = \overleftarrow{\prod_{n=1}^{N}} L_n = L_N L_{N-1} ..... L_1, \tag{4.6.11}$$

with

$$\Omega_\gamma = \lim_{N \to \infty} \Omega_N. \tag{4.6.12}$$

Now the parallel transport of any vector F along γ is given by

$$F_\gamma = \Omega_\gamma F. \tag{4.6.13}$$

We also note that the gauge transformation properties of L, M, Ω are as follows:

$$\begin{aligned} L &\to \frac{\partial G}{\partial x} G^{-1} + G L G^{-1}, \\ M &\to \frac{\partial G}{\partial x} G^{-1} + G M G^{-1}, \\ \Omega_\gamma &\to G(x, t) \Omega_\gamma G^{-1}(x_0, t_0). \end{aligned} \tag{4.6.14}$$

The vanishing of the curvature means that Ω_γ depends only on the initial and final points and not on the path γ itself. Locally, the zero curvature condition then amounts to having $\Omega_\gamma = 1$.

Next let us proceed to the quasi-periodic case. Here the monodromy matrix written as T_L is the matrix of parallel transport along the contour $t = t_0, -L \leq x \leq L$, oriented along the positive x-axis so that we have

$$T_L(\lambda, t_0) = \overleftarrow{\exp} \left(\int_{-L}^{L} L(x, t_0; \lambda) dx \right). \tag{4.6.15}$$

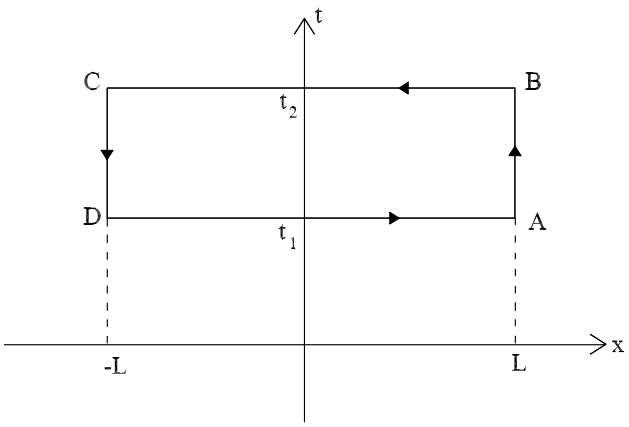

FIGURE 4.6.1: Contour for evaluation of the monodromy matrix.

Next we consider the contour ABCD in the (x,t) plane and we evaluate Ω around it.

Starting from D one finds

$$\begin{aligned}
&(1) \quad D \to A && T_L(\lambda,t_1) = \overleftarrow{\exp}\left(\int_{-L}^{L} L(x,t_1;\lambda)dx\right) = \Omega_{DA},\\
&(2) \quad A \to B && S_+ = \overleftarrow{\exp}\left(\int_{t_1}^{t_2} M(+L,t;\lambda)dx\right) = \Omega_{AB},\\
&(3) \quad B \to C && \widetilde{T}_L(\lambda,t_2) = \overleftarrow{\exp}\left(\int_{-L}^{L} L(x,t_2;\lambda)dx\right) = \Omega_{BC},\\
&(4) \quad C \to D && \widetilde{S}_- = \overleftarrow{\exp}\left(\int_{t_2}^{t_1} M(-L,t;\lambda)dx\right) = \Omega_{CD}.
\end{aligned} \tag{4.6.16}$$

Since $\Omega_\gamma = I$ we get $\Omega_{CD}\Omega_{BC}\Omega_{AB}\Omega_{DA} = I$, i.e.,

$$S_-^{-1}T_L^{-1}(t_2)S_+T_L(t_1) = I. \tag{4.6.17}$$

Under quasi-periodic boundary conditions, we have

$$M(-L,L;\lambda) = Q(\theta)M(L)Q^{-1}(\theta), \quad S_- = Q(\theta)S_+Q^{-1}(\theta), \tag{4.6.18}$$

so that

$$T_L(t_2)Q(\theta) = S_+(T_L(t_1)Q(\theta))S_+^{-1},$$

or

$$\mathrm{tr}\{T_L(t_2)Q(\theta)\} = \mathrm{tr}\{T_L(t_1)Q(\theta)\}. \tag{4.6.19}$$

Thus we see that, tr $\{T_L(t)Q(\theta)\}$ is independent of time and hence is a generator of the conserved quantities in the presence of quasi-periodic boundary conditions. Note that the explicit form of the boundary matrix $Q(\theta)$ depends on the particular form of the temporal part, $M(x,t;\lambda)$, of the Lax pair for the nonlinear equation under investigation.

Let us now consider a more general type of boundary condition [78]. For the nonlinear Schrödinger equation the temporal part of the Lax pair is

$$M(x,t;\lambda) = i\begin{pmatrix} \lambda^2/2 + \bar{\psi}\psi & -i\bar{\psi}_x - \lambda\bar{\psi} \\ -i\psi_x + \lambda\psi & -\lambda^2/2 - \bar{\psi}\psi \end{pmatrix}. \tag{4.6.20}$$

With the boundary condition,

$$\frac{\partial\psi}{\partial x}|_{x=x_\pm} + \theta_\pm\psi(x=x_\pm) = 0, \tag{4.6.21}$$

it is then found that there exists a matrix $K(\lambda)$ so that

$$K_\pm(\lambda)M(x_\pm, \pm\lambda) = M(x_\pm, \mp\lambda)K_\pm(\lambda), \tag{4.6.22}$$

where

$$K_\pm(\lambda) = \begin{pmatrix} \lambda \pm i\theta_\pm & 0 \\ 0 & -\lambda \pm i\theta_\pm \end{pmatrix}. \tag{4.6.23}$$

It can then be shown that

$$S_+(\lambda,t,t_2) = \overleftarrow{\exp}\left[\int_{t_1}^{t_2} K_+^{-1}(\lambda)M(x_+,t;-\lambda)K_+(\lambda)dt\right]. \tag{4.6.24}$$

Upon using the identity

$$\exp(g^{-1}Ag) = g^{-1}\exp(A)g \tag{4.6.25}$$

where g, A are matrices, one can show that

$$S_+(\lambda,t,t_2) = K_+^{-1}(\lambda)S_+(-\lambda,t,t_2)K_+(\lambda). \tag{4.6.26}$$

Similarly it can also be shown that

$$S_-(\lambda,t,t_2) = \overleftarrow{\exp}\left[\int_{t_1}^{t_2} K_-(\lambda)M(x_-,t;-\lambda)K_-^{-1}(\lambda)dt\right],$$

$$= K_-(\lambda)S_-(-\lambda,t,t_2)K_-^{-1}(\lambda). \tag{4.6.27}$$

Since $\Omega_\gamma = I$ we get

$$[K_-(\lambda)S_-^{-1}(-\lambda)K_-^{-1}(\lambda)]T_\pm^{-1}(t_2)[K_+^{-1}(\lambda)S_+(-\lambda)K_+(\lambda)]T_\pm(t_1) = I, \tag{4.6.28}$$

or

$$K_+(\lambda)T_{+-}(t_1)K_-(\lambda)T_\pm^{-1}(-\lambda) =$$
$$S_+^{-1}(-\lambda)\{K_+(\lambda)T_\pm(t_2)K_(\lambda)T_\pm^{-1}(-\lambda)\}T_\pm(t_2,\lambda)S_-(-\lambda)T_\pm^{-1}(t_1,-\lambda). \tag{4.6.29}$$

Since

$$S_-^{-1}(\lambda)T_\pm^{-1}(t_2)S_+(\lambda)T_\pm(t_1) = I, \tag{4.6.30}$$

we find that

$$K_+(\lambda)T_\pm(t_1,\lambda)K_-(\lambda)T_\pm^{-1}(t_1,-\lambda) =$$
$$S_+^{-1}(\lambda)[K_+(\lambda)T_\pm(t_2,\lambda)K_-(\lambda)T_\pm^{-1}(t_2,-\lambda)]S_+(-\lambda), \tag{4.6.31}$$

and finally,

$$\tau(\lambda) = \mathrm{tr}\left[K_+(\lambda)T_\pm(t,\lambda)K_-(\lambda)T_\pm^{-1}(t,-\lambda)\right] \tag{4.6.32}$$

is independent of time and therefore is the generator of the conserved quantities. In this case the contour is obtained by joining the points (x_-,t_1), $(x_+,t_1),(x_+,t_2)$ and (x_-,t_2).

4.7 Modified Classical Yang-Baxter Equation

After ensuring the existence of an infinite number of conservation laws, we now proceed to establish the form of the Yang-Baxter equation (in the classical limiting case) after the imposition of nonperiodic boundary conditions. Our discussion will be based on an important integrable system, known in the literature as the *Liouville-Thirring model*. This model is of additional relevance to our discussions as it is nonultralocal, and we shall describe how finite boundary conditions may be imposed on such a model without destroying its integrability [81].

The classical Liouville-Thirring model is governed by the following coupled equations:

$$\partial^2\Phi = -J^2\exp(\Phi), \qquad i\hat{\partial}\Psi = 4J\Psi\exp(\Phi), \tag{4.7.1}$$

where Φ is a scalar field and Ψ is a two-component fermionic field. We have used the following notation: $\hat{\partial} = \gamma^\mu \partial_\mu, J^\mu = \bar{\Psi}\gamma_\mu\Psi, \Psi = \Psi^\dagger\gamma^0, J^2 = J_\mu J^\mu, \bar{\Psi} = \gamma^0\Psi$. Such a model was first investigated by Pogrebkov et al. in [79]. The space part of the Lax operator for (4.7.1) can be written as follows:

$$L(x;\lambda) = \begin{pmatrix} p(x) & q(x) \\ r(x) & -p(x) \end{pmatrix} - i\lambda\sigma_3, \tag{4.7.2}$$

where

$$\begin{aligned} p(x) &= -\rho_1(x)\alpha_1(x) + \frac{1}{4}(\phi'(x) + \pi(x)), \\ r(x) &= -\rho_2(x)e^{\phi(x)} + \alpha_1'(x) + \rho_1(x)\alpha_1^2(x) - \frac{1}{2}\alpha_1(x)(\phi'(x) + \pi(x)), \\ q(x) &= -\rho_1(x). \end{aligned} \tag{4.7.3}$$

Here the primes denote differentiation with respect to the space variable x, while $\pi(x)$ is the field momentum conjugate to the field ϕ; i.e., $\pi(x) = \partial_t\phi$. In addition the substitution $\Psi_j = \sqrt{\rho_j(t,x)}\exp[(-1)^{j+1}$ $\alpha_j(t,x)]$ has been used. The canonical Poisson brackets are

$$\begin{aligned} \{\pi(x), \phi(y)\} &= \delta(x-y), \\ \{\alpha_j(x), \rho_k(y)\} &= \frac{1}{4}(-1)^{j+1}\delta_{jk}\delta(x-y). \end{aligned} \tag{4.7.4}$$

Using (4.7.3) and (4.7.4) we can calculate the Poisson brackets between the elements of $L(x,\lambda)$:

$$\begin{aligned} \{p(x), q(y)\} &= -\frac{1}{4}q(x)\delta(x-y), \quad \{r(x), p(y)\} = -\frac{1}{4}r(x)\delta(x-y), \\ \{p(x), r(y)\} &= \frac{1}{4}r(x)\delta(x-y), \quad \{q(x), p(y)\} = \frac{1}{4}q(x)\delta(x-y), \\ \{p(x), p(y)\} &= -\frac{1}{8}\delta'(x-y), \\ \{r(x), q(y)\} &= \frac{1}{2}p(x)\delta(x-y) - \frac{1}{4}\delta'(x-y), \\ \{q(x), r(y)\} &= -\frac{1}{2}p(x)\delta(x-y) - \frac{1}{4}\delta'(x-y). \end{aligned} \tag{4.7.5}$$

Now, a simple computation shows that

$$\{L(z,\lambda) \overset{\otimes}{,} L(w,\mu)\} = [s(\lambda,\mu), L(z,\lambda)\otimes I - I\otimes L(w,\mu)]\delta(z-w)$$

$$-[r(\lambda,\mu), L(z,\lambda)\otimes I + I\otimes L(w,\mu)]\delta(z-w) - 2s(\lambda,\mu)\delta'(z-w)$$
$$= A(z,w,\lambda)\delta(z-w) - 2s(\lambda,\mu)\delta'(z-w), \tag{4.7.6}$$

with

$$r(\lambda,\mu) = \frac{1}{8}\frac{\lambda+\mu}{\lambda-\mu}\mathcal{P}, \qquad s(\lambda,\mu) = \frac{1}{16}(2\mathcal{P} - I). \tag{4.7.7}$$

Here $\mathcal{P}$ stands for the permutation matrix. The particular form of the Poisson brackets of the Lax matrix $L(x,\lambda)$ as given by (4.7.6), is known in the literature as the $(r-s)$ Poisson structure and was exhaustively studied by de Vega, Maillet and several other authors [80].

Consider the Poisson brackets between the elements of the transition matrix defined earlier. However, due to the nonultralocal character of the Poisson brackets (4.7.5) one has to be careful about the nature of the end points occurring in the definition of the monodromy matrix ensuring that the end points in the expression for $\{T \otimes, T\}$ should not coincide. One can then show that the Poisson brackets of the transition matrix $T(x,y;\lambda)$ is as follows [79]:

$$\{T(x,y;\lambda) \overset{\otimes}{,} T(u,v,\mu)\} = \int_y^x dz \int_v^u dw \epsilon(x-y)\epsilon(u-v)\chi(z;x,y)\chi(w;u,v)$$
$$\times T(x,z;\lambda)\otimes T(u,w;\mu)\{L(z,\lambda) \overset{\otimes}{,} L(w,\mu)\}T(z,y;\lambda)\otimes T(w,v;\mu), \tag{4.7.8}$$

where

$$\epsilon(x-y) = \begin{cases} 1 & \text{when } x > y \\ 0 & \text{when } x = y \\ -1 & \text{when } x < y \end{cases}$$

and

$$\chi(z;x,y) = \begin{cases} \alpha \text{ for } z = \min(x,y) \\ 1 \ \min(x,y) < z < \max(x,y) \\ \beta \text{ for } z = \max(x,y) \\ 0 \text{ otherwise} \end{cases}. \tag{4.7.9}$$

To compute $\{T \overset{\otimes}{,} T\}$ we calculate first of all $\{T \overset{\otimes}{,} L\}$, which by definition is given by

$$\{T(x,y;\lambda) \overset{\otimes}{,} L(w,\mu)\} = \int_y^x dz \epsilon(x-y)\chi(z;x,y)(T(x,z;\lambda)\otimes I)$$
$$\times\{L(z,\lambda) \overset{\otimes}{,} L(w,\mu)\}(T(z,y;\lambda)\otimes I), \tag{4.7.10}$$
$$= \int_y^x dz \epsilon(x-y)\chi(z;x,y)(T(x,z;\lambda)\otimes I)$$

$$\times[A(z;\lambda,\mu)\delta(z-w) - 2s(\lambda,\mu)\delta_z(z-w)](T(z,y;\lambda)\otimes I).$$

Considering the $\delta'(z-w)$ part, let

$$t_1 = -2\int_y^x dz\,\epsilon(x-y)\chi(z;x,y)T(x,z;\lambda)\otimes Is(\lambda,\mu)T(z,y;\lambda)\otimes I\delta'(z-w),$$

$$= 2\frac{\partial}{\partial z}\left\{\epsilon(x-y)\chi(z;x,y)(T(x,z;\lambda)\otimes I)s(\lambda,\mu)(T(z,y;\lambda)\otimes I)\right\}_{z=w},$$

$$= 2\left[\left(\frac{\partial}{\partial z}\{\epsilon(x-y)\chi(z;x,y)\}\right)(T(x,z;\lambda)\otimes I)s(\lambda,\mu)(T(z,y;\lambda)\otimes I) + \right.$$

$$+\epsilon(x-y)\chi(z;x,y)\left(\frac{\partial}{\partial z}T(x,z;\lambda)\otimes I\right)s(\lambda,\mu)(T(z,y;\lambda)\otimes I)$$

$$\left.+\epsilon(x-y)\chi(z;x,y)(T(x,z;\lambda)\otimes I)s(\lambda,\mu)\left(\frac{\partial}{\partial z}T(z,y;\lambda)\otimes I\right)\right].$$

Note that

$$\epsilon(x-y)\frac{\partial}{\partial z}\chi(z,x;\lambda) = \epsilon(x-y)[\delta(z-\min(x,y)) - \delta(z-\max(x,y))].$$

Upon simplification we get,

$$t_1 = 2[\delta(w-y) - \delta(w-x)](T(x,w;\lambda)\otimes I)s(\lambda,\mu)(T(w,y;\lambda)\otimes I)$$

$$-2\epsilon(x-y)\chi(w;x,y)(T(x,w;\lambda)\otimes I)[s(\lambda,\mu),L(w,\lambda)\otimes I](T(w,y;\lambda)\otimes I). \tag{4.7.11}$$

Next, we consider the term involving $\delta(z-w)$, and denote it by t_2, so that

$$\{T(x,y;\lambda)\otimes_, L(w;\mu)\} = t_1 + t_2$$

$$= -2(\delta(w-x) - \delta(w-y))(T(x,w;\lambda)\otimes I)s(\lambda,\mu)(T(w,y;\lambda)\otimes I)$$

$$-\epsilon(x-y)\chi(w;x,y)(T(x,w;\lambda)\otimes I)\times$$

$$\times[r(\lambda,\mu)+s(\lambda,\mu),L(w,\lambda)\otimes I + I\otimes L(w,\mu)](T(w,y;\lambda)\otimes I). \tag{4.7.12}$$

In terms of this expression for the Poisson bracket, we can express $\{T(x,y,\lambda)\overset{\otimes}{,}T(u,v;\mu)\}$ as given below:

$$\{T(x,y,\lambda)\overset{\otimes}{,}T(u,v;\mu)\} = \int_v^u dw\epsilon(u-v)\chi(w;u,v)(I\otimes T(u,w;\mu))\times$$

$$\times\{T(x,y;\lambda)\overset{\otimes}{,}L(w;\mu)\}(I\otimes T(w,v;\mu)). \tag{4.7.13}$$

Using (4.7.12) and integrating over the delta function we obtain

$$\{T(x,y,\lambda) \overset{\otimes}{,} T(u,v;\mu)\} = -2\epsilon(x-y)\chi(w;u,v)(T(x,w;\lambda)\otimes T(u,w;\mu))$$
$$\times s(\lambda,\mu)(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=y}^{w=x} - \int_v^u dw\epsilon(x-y)\epsilon(u-v)$$
$$\times\chi(w;x,y)\chi(w;u,v)(T(x,w;\lambda)\otimes T(u,w;\mu))$$
$$\times[r+s, L(w;\lambda)\otimes I + I\otimes L(w;\lambda)](T(w,y;\lambda)\otimes T(w,v;\mu)). \quad (4.7.14)$$

From the defining equations for the transition matrix, we may rewrite the integrand of the last term as follows:

$$= -\int_v^u dw\epsilon(x-y)\epsilon(u-v)\chi(w;x,y)\chi(w;u,v)(T(x,w;\lambda)\otimes T(u,w;\mu)\times$$
$$[r(\lambda,\mu)+s(\lambda,\mu), L(w;\lambda)\otimes I + I\otimes L(w;\mu)](T(w,y;\lambda)\otimes T(w,v;\mu)$$
$$= \int_v^u dw\epsilon(x-y)\epsilon(u-v)\chi(w;x,y)\chi(w;u,v)\times$$
$$\frac{\partial}{\partial w}\left[(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)(T(w,y;\lambda)\otimes T(w,v;\mu))\right]. \quad (4.7.15)$$

Integrating by parts causes the left-hand side to become

$$= \epsilon(x-y)\epsilon(u-v)\chi(w;x,y)\chi(w;u,v)\times$$
$$(T(x,w;\lambda)\otimes T(u,w;\mu)(r+s)(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u}$$
$$-\int_v^u dw[\delta(w-y)-\delta(w-x)]\epsilon(u-v)\chi(w;u,v)$$
$$+\epsilon(x-y)\chi(w;x,y)(\delta(w-v)-\delta(w-u))\times$$
$$(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)(T(w,y;\lambda)\otimes T(w,v;\mu)). \quad (4.7.16)$$

The last integral is easy to evaluate, so that finally

$$\{T(x,y,\lambda) \overset{\otimes}{,} T(u,v;\mu)\} = -2\epsilon(u-v)\chi(w;u,v)\times$$
$$(T(x,w;\lambda)\otimes T(u,w;\lambda))s(\lambda,\mu)(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=y}^{w=x}$$
$$+\epsilon(x-y)\epsilon(u-v)\chi(w;x,y)\chi(w;u,v)\times$$
$$(T(x,w;\lambda)\otimes T(u,w;\mu)(r+s)(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u}$$
$$+\epsilon(u-v)\chi(w;u,v)(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)$$
$$\times(T(w,y;\lambda)\otimes T(w,v;\mu)|_{w=y}^{w=x}$$

$$+\epsilon(x-y)\chi(w;x,y)(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)$$
$$\times T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u},$$
$$=\epsilon(u-v)\chi(w;u,v)(T(x,w;\lambda)\otimes T(u,w;\mu))(r-s)$$
$$\times(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u}$$
$$+\epsilon(x-y)\chi(w;x,y)(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)$$
$$\times(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u}$$
$$+\epsilon(x-y)\epsilon(u-v)\chi(w;x,y)\chi(w;u,v)\times$$
$$\times(T(x,w;\lambda)\otimes T(u,w;\mu))(r+s)(T(w,y;\lambda)\otimes T(w,v;\mu))|_{w=v}^{w=u}. \quad (4.7.17)$$

Due to nonultralocality of the theory we have kept the points (u,v) and (x,y) separated. If we assume that they are ordered in a specific way then (4.7.17) leads to:

- if $v<y<x<u$, then

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(u,v;\mu)\}=\epsilon(u-v)\chi(w;u,v)T(x,w;\lambda)\otimes T(u,w;\mu)(r-s)$$
$$\times T(w,y;\lambda)\otimes T(w,y;\mu)|_{w=y}^{w=x}, \quad (4.7.18)$$

and if

- (ii) $y<v<u<x$, then

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(u,v;\mu)\}=\epsilon(x-y)\chi(w;x,y)T(x,w;\lambda)\otimes T(u,w;\mu)(r+s)$$
$$\times T(w,y;\lambda)\otimes T(w,y;\mu)|_{w=v}^{w=u}. \quad (4.7.19)$$

Thus if in (4.7.18) and (4.7.19), we take the limit $u\to x, v\to y$, then we get

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(u,v;\mu)\}=[r-s,T(x,y;\lambda)\otimes T(x,y;\mu)], \quad (4.7.20)$$

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(u,v;\mu)\}=[r+s,T(x,y;\lambda)\otimes T(x,y;\mu)]. \quad (4.7.21)$$

Consequently, following the procedure of Maillet et al. in [80] and taking the average of (4.7.20) and (4.7.21), we finally obtain

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(u,v;\mu)\}=[r,T(x,y;\lambda)\otimes T(x,y;\mu)]. \quad (4.7.22)$$

Thus, we have once again reconstructed a situation where the monodromy matrix obeys the same Poisson bracket algebra as in the ultralocal case.

However, we have not imposed the nontrivial boundary conditions on our problem. To incorporate these effects we shall next show how one can deduce a general condition on the boundary matrices $K_{\pm}(\lambda)$, even for the nonultralocal case. For this let us define $\mathcal{T}(x,y;\lambda)$ by

$$\mathcal{T}(x,y;\lambda) = T(x,y;\lambda)K_{-}(\lambda)T^{-1}(x,y;\lambda^{-1}) \tag{4.7.23}$$

and compute the Poisson bracket:

$$\begin{aligned}
\{\mathcal{T}(x,y;\lambda) \overset{\otimes}{,} \mathcal{T}(x,y;\mu)\} &= \{T(x,y;\lambda) \overset{\otimes}{,} T(x,y;\mu)\}K_{-}(\lambda)\times \\
&T^{-1}(x,y;\lambda^{-1}) \otimes K_{-}(\mu)T^{-1}(x,y;\mu^{-1}) \\
&+(I \otimes T^{-1}(x,y;\lambda^{-1}))K_{-}(\mu)\{T(x,y;\lambda) \overset{\otimes}{,} T^{-1}(x,y;\mu^{-1})\} \\
&\times K_{-}(\lambda)(T^{-1}(x,y;\lambda^{-1}) \otimes I)+ \\
T(x,y;\lambda)K_{-}(\lambda) &\otimes I\{T(x,y;\lambda^{-1}) \overset{\otimes}{,} T(x,y;\mu)\}I \otimes K_{-}(\mu)T^{-1}(x,y;\mu^{-1}) \\
+T(x,y;\lambda)K_{-}(\lambda) &\otimes T(x,y;\mu)K_{-}(\mu)\{T^{-1}(x,y;\lambda^{-1}) \overset{\otimes}{,} T^{-1}(x,y;\mu^{-1})\}.
\end{aligned} \tag{4.7.24}$$

Evaluating each term separately, we find that

$$\begin{aligned}
\{T(x,y;\lambda) \overset{\otimes}{,} T^{-1}(x,y;\mu^{-1})\} &= T(x,y;\lambda)\otimes Ir(\lambda,\mu^{-1})I\otimes T^{-1}(x,y;\mu^{-1}) \\
&-I \otimes T^{-1}(x,y;\mu^{-1})r(\lambda,\mu^{-1})T(x,y;\lambda) \otimes I, \\
\{T^{-1}(x,y;\lambda^{-1}) \overset{\otimes}{,} T(x,y;\mu)\} &= I\otimes T(x,y;\mu)r(\lambda^{-1},\mu)T^{-1}(x,y;\lambda^{-1})\otimes I \\
&-(T^{-1}(x,y;\lambda^{-1}) \otimes I)r(\lambda^{-1},\mu)(I \otimes T(x,y;\mu)), \\
&\{T^{-1}(x,y;\lambda^{-1}) \overset{\otimes}{,} T^{-1}(x,y;\mu^{-1})\} \\
&= -[r(\lambda^{-1},\mu^{-1}), T^{-1}(x,y;\lambda^{-1}) \overset{\otimes}{,} T^{-1}(x,y;\mu^{-1})].
\end{aligned} \tag{4.7.25}$$

Substituting in (4.7.24) we get

$$\begin{aligned}
\{\mathcal{T}(x,y;\lambda) \overset{\otimes}{,} \mathcal{T}(x,y;\mu)\} &= [\mathcal{T}(x,y;\lambda) \otimes Ir(\lambda^{-1},\mu)I \otimes \mathcal{T}(x,y;\mu)] \\
&-[I \otimes \mathcal{T}(x,y;\mu)r(\lambda,\mu^{-1})\mathcal{T}(x,y;\lambda) \otimes I] + \Delta \\
&+[\mathcal{T}(x,y;\lambda) \otimes \mathcal{T}(x,y;\mu)r(\lambda^{-1},\mu^{-1}) + r(\lambda,\mu)\mathcal{T}(x,y;\lambda) \otimes \mathcal{T}(x,y;\mu)],
\end{aligned} \tag{4.7.26}$$

where Δ stands for

$$T(x,y;\lambda) \otimes T(x,y;\mu)\Big[K_{-}^{1}(\lambda)r(\lambda^{-1},\mu)K_{-}^{2}(\mu) + K_{-}^{2}(\mu)r(\lambda,\mu^{-1})K_{-}^{1}(\lambda)$$

$$-r(\lambda,\mu)K_-^1(\lambda)K_-^2(\mu) - K_-^1(\lambda)K_-^2(\mu)r(\lambda^{-1},\mu^{-1})\Big], \tag{4.7.27}$$

where

$$K_-^1(\lambda) = K_-(\lambda)\otimes I, \qquad \text{and} \qquad K_-^2(\mu) = I\otimes K_-(\mu).$$

Now it is clear that $\{\mathcal{T}(x,y;\lambda) \overset{\otimes}{,} \mathcal{T}(x,y;\mu)\}$ can be expressed solely in terms of the $\mathcal{T}$'s, if and only if $\Delta = 0$, whence we get the equation determining the boundary matrices, given a knowledge of the r matrix, viz

$$\begin{aligned} &r(\lambda,\mu)K_-^1(\lambda)K_-^2(\mu) + K_-^1(\lambda)K_-^2(\mu)r(\lambda^{-1},\mu^{-1}) \\ &= K_-^1(\lambda)r(\lambda^{-1},\mu)K_-^2(\mu) + K_-^2(\mu)r(\lambda,\mu^{-1})K_-^1(\mu). \end{aligned} \tag{4.7.28}$$

To prove that we still have an infinite number of commuting conserved quantities, let us set

$$\mathcal{G}(x,y;\lambda) = \mathcal{T}(x,y;\lambda)K_+(\lambda), \tag{4.7.29}$$

where $K_+(\lambda)$ is the boundary matrix at the other end, obeying the same equation (4.7.28) as $K_-(\lambda)$. One can show that the new conserved quantities are generated by

$$t(x,y;\lambda) = \mathrm{tr}\mathcal{G}(x,y;\lambda). \tag{4.7.30}$$

This may be accomplished by computing the Poisson bracket

$$\{\mathcal{G}(x,y;\lambda) \overset{\otimes}{,} \mathcal{G}(x,y;\mu)\},$$

using (4.7.26) with $\Delta = 0$, and thereafter using the fact that $K_\pm(\lambda)$ satisfies (4.7.28), to obtain finally

$$\{t(x,y;\lambda), t(x,y;\mu)\} = 0.$$

In this way one can deduce the existence of the integrals of motion and also the equations for the boundary matrices $K_\pm(\lambda)$ for a nonultralocal system with finite boundary conditions. A number of other important models such as the Toda lattice and Sine-Gordon equations have been treated in this manner, and we refer the reader to Sklyanin's original article [77] for further details.

Chapter 5

Algebraic Bethe Ansatz

5.1 Introduction

In the previous chapters we have discussed in detail the various manifestations of the Yang-Baxter equation for both discrete and continuous integrable systems. Discrete integrable systems have a close analogy with vertex models of statistical mechanics in two dimensions, which were studied extensively by Baxter [64], Onsager, Mattis, Lieb [32] and others. Our focus here is on the applications of the Yang Baxter equation in the ananlysis of specific nonlinear models. It should be borne in mind that the motivation of solution in classical and quantum problems is quite different. While in classical IST one is interested in the form of the solutions, that is, in the structure of the solitary wave, in the quantum case one searches for the excitation spectrum, the Bethe eigenmomenta equation and the energy eigenvalues, besides being interested in the structure of the string states. Since there are certain subtle problems associated with the continuous Lax operators we begin with discrete models. A model that has received quite a lot of attention in recent years and is structurally simple is the discrete self-trapping (DST) model, which we analyse below.

5.2 Discrete Self-Trapping Model

The DST equation was introduced by Eilbeck et al. [82] to model the nonlinear dynamics of small molecules. It consists of n nondissipative anharmonic oscillators coupled through dispersive interactions. Later, a detailed analysis was reported by Enol'skii et al. [83], who

discovered an important aspect of this model. This was the existence of both quadratic and linear algebra generated by the scattering data, depending upon the type of Lax operator one used. Here we will essentially decribe the work of Enol'skii et al. to illustrate the use of the algebraic Bethe ansatz (ABA) for the explicit solution of the model in the quantum regime. The existence of two types of scattering data algebra actually leads to two different approaches to the model's diagonalization, and hence to the derivation of the eigenvalue/eigenmomenta equation by the algebraic Bethe ansatz.
The model is governed by the Lax operator

$$L(u) = L^{(1)}(u)L^{(2)}(u),$$

where $L^{(m)}(u)$ stands for

$$L^{(m)} = \begin{pmatrix} u - i(\gamma b_m^\dagger b_m + w_m) & \sqrt{\epsilon\gamma}b_m^\dagger \\ \sqrt{\epsilon\gamma}b_m & i\epsilon \end{pmatrix} \text{ for } m = 1, 2. \tag{5.2.1}$$

The corresponding Hamiltonian is given by

$$H = -\frac{\gamma}{4}\left[(b_1^\dagger b_1 - b_2^\dagger b_2)^2 + N^2\right] + (w_1 - \gamma/2)b_1^\dagger b_1$$

$$+(w_2 - \gamma/2)b_2^\dagger b_2 - \epsilon(b_1^\dagger b_2 + b_2^\dagger b_1), \tag{5.2.2}$$

with $N = b_1^\dagger b_1 + b_2^\dagger b_2$; $b_i^\dagger, b_i$ being the creation and annihilation operators, respectively. The equations of motion are

$$\dot{b}_i = [H, b_i], \qquad \dot{b}_i^\dagger = [H, b_i^\dagger], \quad i = 1, 2. \tag{5.2.3}$$

This gives the two-state discrete self-trapping dimer equation. For the Lax operator given by (5.2.1), the quantum R matrix is of the form

$$R(u) = \begin{pmatrix} f(u) & 0 & 0 & 0 \\ 0 & 1 & g(u) & 0 \\ 0 & g(u) & 1 & 0 \\ 0 & 0 & 0 & f(u) \end{pmatrix}, \tag{5.2.4}$$

with $f(u) = (u - i\gamma)u, g(u) = -i\gamma/u$. One assumes the existence of a pseudo-vacuum vector $|0\rangle$ so that for the transfer matrix,

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

one has $C(u)|\rangle = \langle 0|C(u) = 0$. The action of $A(u), D(u)$ on the vacuum is given by

$$A(u)|0\rangle = a(u)|0\rangle, \qquad D(u)|0\rangle = d(u)|0\rangle, \tag{5.2.5}$$

where $a(u), d(u)$ are the corresponding eigenvalues. The nth excited state is now obtained by the action of n creation operators acting on the vacuum, i.e.,

$$|N\rangle = B(u_1)B(u_2)...B(u_N)|0\rangle.$$

The Hamiltonian H is obtained as the coefficient of u in the expression $\mathrm{tr}T(u) = A(u) + D(u)$, while the eigenstates are solutions of

$$(A(u) + D(u))|N\rangle = \Lambda(u)|N\rangle. \tag{5.2.6}$$

(We will at times write $|N\rangle$ as $\Psi(u_1, u_2, ...u_N)$). The commutation rules for the operators $A(u), B(u)...$ etc. are obtained from the intertwining relation

$$R(u, v)T^1(u)T^2(v) = T^2(v)T^1(u)R(u, v), \tag{5.2.7}$$

where $T^1(u) = T(u) \otimes I$ and $T^2(u) = I \otimes T(u)$. When written out in full these commutation relations are as follows:

$$f(u - v)A(u)A(v) = A(v)A(u)f(u - v), \tag{5.2.8}$$

$$f(u - v)A(u)B(v) = B(v)A(u) + g(u - v)A(v)B(u), \tag{5.2.9}$$

$$A(u)C(v) + g(u - v)C(u)A(v) = f(u - v)C(v)A(u), \tag{5.2.10}$$

$$A(u)D(v)+g(u-v)C(u)B(v) = f(u-v)C(v)B(u)+D(v)A(u), \tag{5.2.11}$$

$$f(u - v)B(u)A(v) = g(u - v)B(v)A(u) + A(v)B(u), \tag{5.2.12}$$

$$f(u - v)B(u)B(v) = f(u - v)B(v)B(u) \tag{5.2.13}$$

$$B(u)C(v)+g(u-v)D(u)A(v) = g(u-v)D(v)A(u)+C(v)B(u), \tag{5.2.14}$$

$$B(u)D(v) + g(u - v)D(u)B(v) = f(u - v)D(v)B(u), \tag{5.2.15}$$

$$g(u - v)A(u)C(v) + C(u)A(v) = f(u - v)A(v)C(u), \tag{5.2.16}$$

$$g(u-v)A(u)D(v)+C(u)B(v) = B(v)C(u)+g(u-v)A(v)D(u), \tag{5.2.17}$$

$$[C(u), C(v)] = 0, \tag{5.2.18}$$

$$f(u - v)C(u)D(v) = D(v)C(u) + g(u - v)C(v)D(u), \tag{5.2.19}$$

$$g(u-v)B(u)C(v)+D(u)A(v) = g(u-v)B(v)C(u)+A(v)D(u), \quad (5.2.20)$$

$$g(u-v)B(u)D(v) + D(u)B(v) = f(u-v)B(u)D(v), \quad (5.2.21)$$

$$f(u-v)D(u)C(v) = g(u-v)D(v)C(u) + C(v)D(u), \quad (5.2.22)$$

$$[D(u), D(v)] = 0. \quad (5.2.23)$$

In practice, we only require a few of the above commutation relations for actual calculations. The basic idea behind the calculations is to use the appropriate commutation rules for $A(u)$ and $D(u)$ with $B(u)$ to move the latter to the extreme right to operate on the vacuum $|0\rangle$, whence the eigenvalue equations (5.2.5) can be used. In this procedure two classes of terms are generated, commonly referred to as the *wanted* and *unwanted* terms. The *wanted* term has the same structure as the starting state $|N\rangle$ while the *unwanted* ones have a different form. For $|N\rangle$ to be an eigenvector of $A(u)+D(u)$ we require the unwanted terms to vanish. The vanishing of the unwanted terms gives rise to the Bethe eigenmomenta equations. In this particular case these equations are as follows:

$$\frac{a(u_k)}{d(u_k)} = \prod_{j=1, j\neq k}^{N} \frac{f(u_k - u_j)}{f(u_j - u_k)}, \qquad k = 1, 2, ..., N, \quad (5.2.24)$$

while the energy eigenvalue from $t(u)|N\rangle = \Lambda(u)|N\rangle$, has the form

$$\Lambda(u) = a(u)\frac{\prod_{j=1}^{N}(u - u_j + i\gamma)}{\prod_{j=1}^{N}(u - u_j)} + d(u)\frac{\prod_{j=1}^{N}(u - u_j - i\gamma)}{\prod_{j=1}^{N}(u - u_j)}, \quad (5.2.25)$$

with

$$a(u) = [u - i(w_1 - \gamma/2)][u - i(w_2 - \gamma/2)] \quad \text{and} \quad d(u) = -\epsilon^2. \quad (5.2.26)$$

Finally the Hamiltonian operator can be expressed as

$$\gamma H = \left[\frac{1}{2}\left(\frac{d}{du} trL(u)\right)^2 - trL(u)\right]_{u=0} - \epsilon^2 + \frac{1}{2}(w_1 - \gamma/2)^2 + \frac{1}{2}(w_2 - \gamma/2)^2, \quad (5.2.27)$$

where the energy eigenvalue are

$$\gamma E_k = \left[\frac{1}{2}\left(\frac{d}{du} tr\ t(u)\right)^2 - tr\ t(u)\right]_{u=0} - \epsilon^2 + \frac{1}{2}(w_1 - \gamma/2)^2 + \frac{1}{2}(w_2 - \gamma/2)^2. \quad (5.2.28)$$

A more convenient form is obtained if we write (5.2.25) as

$$\Lambda(u) = a(u) - \epsilon^2 + \frac{\phi_{n+1}(u)}{\prod_{j=1}^{N}(u-u_j)}, \tag{5.2.29}$$

where $\phi_{n+1}(u)$ is a polynomial of degree $n+1$ given by

$$\phi_{n+1}(u) = [u - i(w_1 - \gamma/2)][u - i(w_2 - \gamma/2)]\times$$

$$\times \left[\prod_{j=1}^{N}(u-u_j-i\gamma) - \prod_{j=1}^{N}(u-u_j)\right]$$

$$-\epsilon^2\left[\prod_{j=1}^{N}(u-u_j+i\gamma) - \prod_{j=1}^{N}(u-u_j)\right] = (-N\gamma u + \beta_n)\prod_{j=1}^{N}(u-u_j), \tag{5.2.30}$$

β_n being parameters. Upon equating the coefficients of the different powers of u in (5.2.29), one is led to $\det(\beta I + P) = 0$, where I is a $(n+1)\times(n+1)$ unit matrix and P is the Heisenberg matrix:

$$P = \begin{pmatrix} q_{0,n} & -i\gamma & 0 & 0 & .. & .. & 0 \\ q_{1,n} & q_{1,n-1} & -2i\gamma & 0 & .. & .. & 0 \\ q_{2,n} & q_{2,n-1} & q_{2,n-2} & -3i\gamma & .. & .. & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ q_{n,n} & q_{n,n-1} & q_{n,n-2} & .. & .. & .. & q_{n,0} \end{pmatrix}, \tag{5.2.31}$$

where the matrix elements $q_{n,m-l}, (m,l = 0,...N)$ are

$$q_{n,m-l} = -\theta(m-l)(-i\gamma)^{m-l-2}\binom{n-l}{n-m-2}[1-\delta_{m,|l-2|}\theta(l-2)]$$

$$-i\theta(m-l)(w_1+w_2-\gamma)(-i\gamma)^{m-l-1}\binom{n-l}{n-m-1}[1-\delta_{m,|l-1|}\theta(l-1)]$$

$$+(-i\gamma)^{m-l}\theta(m-l)[w_1w_2 - 1/2\gamma(w_1+w_2) + \gamma^2/4 + (-1)^{m-l}\epsilon^2]\times$$

$$\times\binom{n-l}{n-m}(1-\delta_{m,l}). \tag{5.2.32}$$

One can show that for $n+1$ characteristic values $\beta_n^{(k)}, k = 1, ..., (n+1)$ the expression for the energy eigenvalue may be written as

$$E_k = n(w_1 + w_2) - \frac{n}{2}(n+2)2\gamma - \beta_n^{(k)}/\gamma. \tag{5.2.33}$$

It is important to point out that the same problem may be associated with another Lax operator that satisfies a linear r matrix relation as does the operator $T(u)$. One can again formulate the quantum inverse problem; however in that case one adopts the techniques of separation of variables together with the use of a functional Bethe ansatz approach. In Chapter 7 we shall return to this problem when describing the method of separation of variables for integrable systems.

5.3 Asymmetric XXZ Model in a Magnetic Field

As a second example of the application of the algebraic Bethe ansatz to the solution of concrete problems, we consider the R matrix for an asymmetrical XXZ spin chain placed in an external magnetic field. The system was analysed in [84]. In terms of the R matrix the analysis of such a system can be accomplished using the algebraic Bethe ansatz. The R matrix obtained in [84] presents certain novel features in the sense that the spectral parameters on which the R matrix depends are two component vectors. The inclusion of the external magnetic field within the R matrix itself is facilitated by assigning one component of this vector-valued spectral parameter to be proportional to the external magnetic field. Let us consider the R matrix in the form

$$\tilde{R}(\vec{\lambda}, \vec{\mu}) = \begin{pmatrix} ae^{\eta(\lambda_2-\mu_2)} & 0 & 0 & 0 \\ 0 & be^{-\eta(\lambda_2+\mu_2)} & \sinh\eta & 0 \\ 0 & \sinh\eta & be^{\eta(\lambda_2+\mu_2)} & 0 \\ 0 & 0 & 0 & a^{-\eta(\lambda_2-\mu_2)} \end{pmatrix}, \tag{5.3.1}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2)$ and $\vec{\mu} = (\mu_1, \mu_2)$, are two component vectors while $a(\lambda_1, \mu_1) = \sinh(\lambda_1 - \mu_1 + \eta)$, $b(\lambda_1, \mu_1) = \sinh(\lambda_1 - \mu_1)$, η being the quantization parameter. This R matrix satisfies the Yang-Baxter equation

$$\tilde{R}_{12}(\vec{\lambda}, \vec{\mu})\tilde{R}_{13}(\vec{\lambda}, \vec{\nu})\tilde{R}_{23}(\vec{\mu}, \vec{\nu}) = \tilde{R}_{23}(\vec{\mu}, \vec{\nu})\tilde{R}_{13}(\vec{\lambda}, \vec{\nu})\tilde{R}_{12}(\vec{\lambda}, \vec{\mu}). \tag{5.3.2}$$

It is interesting to note that (5.3.2) remains valid even upon introducing inhomogenities $\alpha_i, \beta_i, \gamma_i (i = 1, 2)$, by means of the transformations $\vec{\lambda} \to \vec{\lambda} - \vec{\alpha}, \vec{\mu} \to \vec{\mu} - \vec{\beta}, \vec{\nu} \to \vec{\nu} - \vec{\gamma}$. The connection with the local Hamiltonian density of the asymmetric XXZ spin chain, in the presence of an external magnetic field, can be made by employing the following reduction.
Let $\vec{\lambda} = (\lambda, \lambda h), \vec{\nu} = (\nu, \nu h)$, where h is the magnetic field. Defining

$$\tilde{R}(\vec{\lambda}, \vec{\nu}) \equiv L(\lambda, \nu),$$

we have

$$L(\lambda, \nu) = \begin{pmatrix} pe^{\eta h(\lambda-\nu)} & 0 & 0 & 0 \\ 0 & qe^{-\eta h(\lambda+\nu)} & r & 0 \\ 0 & r & qe^{\eta h(\lambda+\nu)} & 0 \\ 0 & 0 & 0 & pe^{-\eta h(\lambda-\nu)} \end{pmatrix}, \tag{5.3.3}$$

where

$$p = \sinh(\lambda - \nu + \eta), \quad q = \sinh(\lambda - \nu), \quad r = \sinh \eta.$$

Now it can be shown that the local Hamiltonian density follows from

$$H_{i,i+1} = \sinh \eta L^{-1}(\lambda, \nu) \frac{\partial}{\partial \lambda} L(\lambda, \nu)|_{\lambda=\nu} - \frac{1}{2} \cosh \eta I_4 \tag{5.3.4}$$

$$= \frac{1}{2} \cosh \eta \sigma_i^3 \otimes \sigma_{i+1}^3 + \frac{1}{2} \eta h \sinh \eta (\sigma_i^3 \otimes I + I \otimes \sigma_{i+1}^3),$$
$$+ e^{-h\psi} \sigma_i^- \otimes \sigma_{i+1}^+ + e^{h\psi} \sigma_i^+ \otimes \sigma_{i+1}^-, \tag{5.3.5}$$

where $\psi = 2\eta\nu$ and we have written $L(\lambda, \nu)$ in terms of the local operators $(\vec{\sigma}_i, \vec{\sigma}_{i+1})$ which are the 2×2 Pauli matrices. As shown in [85], the L operator satisfies the standard relation,

$$\hat{R}(\lambda, \nu) L(\lambda, \nu) \otimes L(\mu, \nu) = L(\mu, \nu) \otimes L(\lambda, \nu) \hat{R}(\lambda, \mu), \tag{5.3.6}$$

where

$$\hat{R}(\lambda, \mu) = \mathcal{P} \tilde{R}(\lambda, \mu), \tag{5.3.7}$$

with $\mathcal{P}$ as the permutation operator. Equation (5.3.6) is important for it will be noticed that ν plays the role of an auxiliary parameter. We consider a one-dimensional lattice and assign to each site the operator,

$$\hat{L}(\lambda, \nu_k) \equiv R_{0k}(\lambda, \nu_k; \eta) = \frac{1}{\sinh(\lambda - \nu_k + \eta)} \hat{R}_{0k}(\lambda, \nu_k; \eta), \tag{5.3.8}$$

where the indices 0 and k refer to an auxiliary space $\mathcal{V}_A$ and a quantum space $\mathcal{V}_k$ at the kth lattice site, respectively. Thus $\hat{L}(\lambda, \nu_k)$ above is defined in the tensor product of the two vector spaces, i.e., $\mathcal{V}_A \otimes \mathcal{V}_k$. Note that the parameter ν has now become site dependent. Two properties of $R_{0k}(\lambda, \nu_k; \eta)$ are very important in any analysis:

$$(i)\ R_{0k}(\nu_k, \nu_k; \eta) = \mathcal{P}_{ok}, \qquad (ii)\ R_{0k}(\lambda, \nu_k; \eta = 0) = I_4. \tag{5.3.9}$$

Defining the monodromy matrix as usual by

$$T_N(\lambda, \{\nu_k\}; \eta) = R_{0N}(\lambda, \nu_N; \eta)...R_{0k}(\lambda, \nu_k; \eta)...R_{01}(\lambda, \nu_1; \eta), \tag{5.3.10}$$

it is evident from (5.3.6) that $T_N(\lambda, \{\nu_k\}; \eta)$ satisfies

$$\begin{aligned}\hat{R}(\lambda, \mu) T_N(\lambda, \{\nu_k\}; \eta) \otimes T_N(\mu, \{\nu_k\}; \eta) \\ = T_N(\mu, \{\nu_k\}; \eta) \otimes T_N(\lambda, \{\nu_k\}; \eta) \hat{R}(\lambda, \mu).\end{aligned} \tag{5.3.11}$$

The trace of the transition matrix defined by (5.3.10), is known to be the generator of the integrals of motion, which are in involution. Let us formally write the monodromy matrix as

$$T_N(\lambda, \{\nu_k\}; \eta) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}, \tag{5.3.12}$$

and from (5.3.11) pick up two commutation relations:

$$\begin{aligned}A_N(\lambda) C_N(\mu) = \frac{\sinh(\lambda - \mu + \eta) e^{\eta h(\lambda - \mu)}}{\sinh(\lambda - \mu) e^{-\eta h(\lambda + \mu)}} C_N(\mu) A_N(\lambda) \\ - \frac{\sinh \eta}{\sinh(\lambda - \mu) e^{-\eta h(\lambda + \mu)}} C_N(\lambda) A_N(\mu),\end{aligned} \tag{5.3.13}$$

$$\begin{aligned}D_N(\lambda) C_N(\mu) = \frac{\sinh(\mu - \lambda + \eta) e^{-\eta h(\lambda - \mu)}}{\sinh(\mu - \lambda) e^{-\eta h(\lambda + \mu)}} C_N(\mu) A_N(\lambda) \\ - \frac{\sinh \eta}{\sinh(\mu - \lambda) e^{-\eta h(\lambda + \mu)}} C_N(\lambda) D_N(\mu).\end{aligned} \tag{5.3.14}$$

Here we have taken $C_N(\lambda)$ as a creation operator, so that a M excitation eigenstate of the transfer matrix

$$t(\lambda, \{\nu_k\}; \eta) = \mathrm{tr}_0 \overleftarrow{\prod_{k=1}^{N}} R_{0k}(\lambda, \nu_k; \eta)$$

is of the form

$$|\Omega_M\rangle = C_N(\mu_1)C_N(\mu_2)....C_N(\mu_M)|0\rangle. \tag{5.3.15}$$

The vacuum state $|0\rangle$ here is defined by

$$|0\rangle = \prod_{k=1}^{N} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_k. \tag{5.3.16}$$

Application of $A_N(\lambda)$ to the state $|\Omega\rangle$ results in the following equation upon repeatedly using (5.3.13):

$$A_N(\lambda)|\Omega\rangle = a_N(\lambda) \prod_{i=1}^{M} \frac{\sinh(\lambda - \mu_i + \eta)}{\sinh(\lambda - \mu_i)} e^{2\eta h\lambda} |\Omega_M\rangle + \text{unwanted terms}, \tag{5.3.17}$$

where by "unwanted terms" is meant all those terms that are not proportional to $|\Omega\rangle$. Similar application of $D_N(\lambda)$ to $|\Omega_M\rangle$ using (5.3.14) results in the following:

$$D_N(\lambda)|\Omega_M\rangle = d_N(\lambda) \prod_{i=1}^{M} \frac{\sinh(\mu_i - \lambda + \eta)}{\sinh(\mu_i - \lambda)} e^{2\eta h\lambda} + \text{unwanted terms}. \tag{5.3.18}$$

In equations (5.3.17 and 5.3.18), $a_N(\lambda)$ and $d_N(\lambda)$ are the eigenvalues of $A_N(\lambda)$ and $D_N(\lambda)$ when acting on $|0\rangle$. These eigenvalues may be determined by noting that in general

$$(T_N(\lambda))_{ab} = \sum_{a_1,..a_{N-1}=1}^{2} t^{(1)}_{aa_1}(\lambda, \nu_1) t^{(2)}_{a_1a_2}(\lambda, \nu_2)....t^{(N)}_{a_{N-1}b}(\lambda, \nu_N), \tag{5.3.19}$$

where $t^{(k)}_{ij}(\lambda, \nu_k)$ refers to the partitioned submatrix of $R_{0k}(\lambda, \nu_k; \eta)$, so that

$$R_{0k}(\lambda, \nu_k; \eta) = \begin{pmatrix} t^{(k)}_{11}(\lambda, \nu_k)\ t^{(k)}_{12}(\lambda, \nu_k) \\ t^{(k)}_{21}(\lambda, \nu_k)\ t^{(k)}_{22}(\lambda, \nu_k) \end{pmatrix}, \tag{5.3.20}$$

with

$$t^{(k)}_{11}(\lambda, \nu_k) = \begin{pmatrix} e^{\eta h(\lambda - \nu_k)} & 0 \\ 0 & \frac{\sinh(\lambda - \nu_k)}{\sinh(\lambda - \nu_k + \eta)} e^{-\eta h(\lambda + \nu_k)} \end{pmatrix}$$

$$t^{(k)}_{12}(\lambda, \nu_k) = \begin{pmatrix} 0 & 0 \\ \frac{\sinh\eta}{\sinh(\lambda - \nu_k + \eta)} & 0 \end{pmatrix}; \quad t^{(k)}_{21}(\lambda, \nu_k) = \begin{pmatrix} 0 & \frac{\sinh\eta}{\sinh(\lambda - \nu_k + \eta)} \\ 0 & 0 \end{pmatrix}$$

$$t_{22}^{(k)}(\lambda,\nu_k)=\begin{pmatrix}\frac{\sinh(\lambda-\nu_k)}{\sinh(\lambda-\nu_k+\eta)}e^{\eta h(\lambda+\nu_k)} & 0\\ 0 & e^{-\eta h(\lambda-\nu_k)}\end{pmatrix}. \tag{5.3.21}$$

Applying the definition (5.3.19) we find that

$$A_N(\lambda)|0\rangle=\left(\prod_{k=1}^{N}e^{\eta h(\lambda-\nu_k)}\right)|0\rangle=a_N(\lambda)|0\rangle, \tag{5.3.22}$$

$$D_N(\lambda)|0\rangle=\left\{\prod_{k=1}^{N}\frac{\sinh(\lambda-\nu_k)}{\sinh(\lambda-\nu_k+\eta)}e^{\eta h(\lambda+\nu_k)}\right\}|0\rangle=d_N(\lambda)|0\rangle, \tag{5.3.23}$$

so that $a_N(\lambda)$ and $d_N(\lambda)$ can be read off easily.

From equations (5.3.17 and 5.3.18) it is obvious that $|\Omega_M\rangle$ will be an eigenstate of $t(\lambda)$ if the unwanted terms vanish; the vanishing of the unwanted terms leads to the Bethe ansatz equation (BAE) that determines the $\{\mu_i\}$'s. Under this condition one has

$$t(\lambda)|\Omega_M\rangle=(A_N(\lambda)+D_N(\lambda))|\Omega_M\rangle=\left[a_N(\lambda)\prod_{i=1}^{M}\frac{\sinh(\lambda-\mu_i+\eta)}{\sinh(\lambda-\mu_i)}e^{2\lambda\eta h}\right.$$
$$\left.+d_N(\lambda)\prod_{i=1}^{M}\frac{\sinh(\mu_i-\lambda+\eta)}{\sinh(\mu_i-\lambda)}e^{2\lambda\eta h}\right]|\Omega_M\rangle, \tag{5.3.24}$$

while the vanishing condition for the unwanted terms is the following set of coupled equations:

$$\frac{a_N(\mu_j)}{d_N(\mu_j)}=\prod_{i\neq j}^{M}\frac{\sinh(\mu_i-\mu_j+\eta)}{\sinh(\mu_i-\mu_j-\eta)},\qquad j=1,2,...,M. \tag{5.3.25}$$

From (5.3.22 and 5.3.23) we find that the ratio on the left is

$$\frac{a_N(\mu_j)}{d_N(\mu_j)}=\prod_{k=1}^{N}\frac{\sinh(\mu_j-\nu_k+\eta)}{\sinh(\mu_j-\nu_k)}e^{-2\eta\nu_k h}. \tag{5.3.26}$$

Hence the BAE finally assumes the following form:

$$\prod_{k=1}^{N}\frac{\sinh(\mu_j-\nu_k+\eta)}{\sinh(\mu_j-\nu_k)}e^{-2\eta\nu_k h}=\prod_{\substack{i=1\\ i\neq j}}^{M}\frac{\sinh(\mu_i-\mu_j+\eta)}{\sinh(\mu_i-\mu_j-\eta)},\quad j=1,2,....,M. \tag{5.3.27}$$

Furthermore, the energy eigenvalue of the transfer matrix is seen to be given by

$$\Lambda_M(\lambda, \{\nu_k\}) = e^{2\eta\lambda h}\left(\prod_{k=1}^{N} e^{\eta h(\lambda-\nu_k)}\right)\prod_{i=1}^{M}\frac{\sinh(\lambda-\mu_i+\eta)}{\sinh(\lambda-\mu_i)}$$

$$+e^{2\eta\lambda h}\left(\prod_{i=1}^{N}\frac{\sinh(\lambda-\nu_k)}{\sinh(\lambda-\nu_k+\eta)}e^{\eta h(\lambda+\nu_k)}\right)\prod_{i=1}^{M}\frac{\sinh(\mu_i-\lambda+\eta)}{\sinh(\mu_i-\lambda)}. \quad (5.3.28)$$

Equations (5.3.27 and 5.3.28) complete the Bethe ansatz solution of the eigenvalue problem:

$$t(\lambda)|\Omega_M\rangle = \Lambda_M|\Omega_M\rangle$$

an analysis of the BAE would determine the eigenmomenta μ_j's, which is a separate issue in itself.

5.4 Analytical Bethe Ansatz

The quantum inverse scattering method is in fact a general formalism applicable to a wide number of nonlinear integrable systems possessing a Lax operator. In the preceding section we have shown its application to a discrete system, yet it is also applicable to continuous problems albeit after certain modifications are made. Several important results have been discovered by Faddeev, Takhtajan, Kulish, Sklyanin et al. [86] of the Russian school and also by de Vega, Maillet [87] and others in this context.

In the continuous domain the Sine-Gordon model occupies a very special status, being a field theoretic model and owing to its appearence in several areas of condensed matter physics and quantum field theory. In the context of nonlinear integrable systems, specially with regard to the quantum inverse problem its importance stems from the fact that it does not admit a natural pseudo-vacuum reference state; therefore the algebraic Bethe ansatz technique as described in the last section cannot be applied directly.

Initially the model was studied by Kulish [88] who used the method of space discretization to construct a discrete Lax operator, $L_n(x)$, from

the continuous Lax matrix $L(x)$. Later, an exact form of L_n was derived for all orders of the lattice spacing Δ. However the absence of a pseudo-vacuum state still persisted, and compelled Kulish to assume that although it did not exist for L_n, it existed for the product L_nL_{n-1}. Thus by postulating the existence of a vacuum state for the product he was able to apply the methodology of algebraic Bethe ansatz to the system and construct the solution for the quantum inverse problem (the details may be found in [88]).
Later it was shown by authors [89] that one could use the technique of analytical Bethe ansatz to set up the Bethe ansatz and derive the eigenmomenta equation and the eigenvalues. We shall illustrate the method of analytical Bethe ansatz with the Sine-Gordon equation written in the form [91]:

$$\phi_{xt} = \sin\phi. \tag{5.4.1}$$

Its Lax matrix is given by

$$L(x,\lambda) = \begin{pmatrix} \frac{i\beta\pi(x)}{4} & \frac{m}{4}(\lambda^{-1}e^{i\beta\phi/2} - \lambda e^{-i\beta\phi/2}) \\ \frac{m}{4}(\lambda e^{i\beta\phi/2} - \lambda^{-1}e^{-i\beta\phi/2}) & -\frac{i\beta\pi(x)}{4} \end{pmatrix}. \tag{5.4.2}$$

Here the field momentum $\pi(x) = \partial_t\phi(x)$ and satisfies the Poisson bracket $\{\pi(x), \phi(y)\} = \delta(x-y)$.

By going over to the lattice, we can get a discrete form of the Lax operator valid up to first order in the lattice spacing Δ:

$$L_n(x) = \begin{pmatrix} 1 - i\frac{\beta p_n}{4} & \frac{\Delta m}{4}(\lambda e^{-i\beta u_n/2} - \lambda^{-1}e^{i\beta u_n/2}) \\ \frac{\Delta m}{4}(\lambda^{-1}e^{-i\beta u_n/2} - \lambda e^{i\beta u_n/2}) & 1 + i\frac{\beta p_n}{4} \end{pmatrix}, \tag{5.4.3}$$

where u_n, p_n are the following:

$$u_n = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \phi(x)dx, \qquad p_n = \int_{x_{n-1}}^{x_n} \pi(x)dx, \tag{5.4.4}$$

with

$$\{p_n, u_m\} = \delta_{n,m}.$$

By converting this Poisson bracket to a commutator, so that $[p_n, u_m] = -i\gamma\delta_{n,m}$ and requiring that

$$R(\lambda,\mu)L_n(\lambda)\otimes L_n(\mu) = L_n(\mu)\otimes L_n(\lambda)R(\lambda,\mu), \tag{5.4.5}$$

where the entries of $L_n(\lambda), L_n(\mu)$ are thought of as operators, one can show that the solution of the intertwining matrix $R(\lambda, \mu)$ is

$$\bar{R}(\lambda,\mu) = \begin{pmatrix} r_{11} & 0 & 0 & 0 \\ 0 & r_{22} & r_{23} & 0 \\ 0 & r_{23} & r_{22} & 0 \\ 0 & 0 & 0 & r_{11} \end{pmatrix}; \tag{5.4.6}$$

where

$$r_{11} = \frac{\lambda}{\mu}e^{i\gamma} - \frac{\mu}{\lambda}e^{-i\gamma}, \quad r_{22} = e^{i\gamma} - e^{-i\gamma}, \quad r_{23} = \frac{\lambda}{\mu} - \frac{\mu}{\lambda}. \tag{5.4.7}$$

Note that in deriving this solution for the quantum R matrix, the Lax operator was discretized only up to the first order in Δ. In order to derive the exact discretized version of the Lax operator for the model, Kulish used a novel technique, assuming the quantum R matrix to have the same form as in (5.4.6) (note that this solution is Δ independent), he went back to (5.4.5) and with this R matrix obtained a solution for the Lax operator valid up to all orders in Δ. This was found to be

$$L_n(\lambda) = \begin{pmatrix} f(u_n)e^{-i\beta p_n/4} & \frac{m\Delta}{4}(\lambda e^{-\beta u_n/2} - \lambda^{-1}e^{i\beta u_n/2}) \\ \frac{m\Delta}{4}(\lambda^{-1}e^{-i\beta u_n/2} - \lambda e^{i\beta u_n/2}) & g(u_n)e^{i\beta p_n/4} \end{pmatrix}, \tag{5.4.8}$$

where the functions $f(u_n), g(u_n)$ are given by

$$g(u+\beta/4-2\pi/\beta)g(u) = 1 + 2(\frac{m\Delta}{4})^2\cos(\beta u+\gamma),$$

$$f(u) = g(u-2\pi/\beta), \qquad \gamma = \beta^2/8. \tag{5.4.9}$$

Setting $e^u = \lambda, e^v = \mu$, the quantum R matrix in (5.4.6) is essentially of the form

$$\bar{R}(u) = \begin{pmatrix} \sinh(u+i\gamma) & 0 & 0 & 0 \\ 0 & \sinh(i\gamma) & \sinh(u) & 0 \\ 0 & \sinh(u) & \sinh(i\gamma) & 0 \\ 0 & 0 & 0 & \sinh(u+i\gamma) \end{pmatrix}. \tag{5.4.10}$$

Alternatively we can consider the following R matrix defined by

$$R'(u-v) = \mathcal{P}\bar{R}(u-v), \tag{5.4.11}$$

where $\mathcal{P}$ is the permutation operator. The gauge transformation of this R' matrix yields

$$R' \to R(u-v) = B^1(u)B^2(v)R'(u-v)B^1(-u)B^2(-v), \qquad (5.4.12)$$

where $B(u)$ is a diagonal matrix satisfying

$$B(u)B(v) = B(u+v),$$

and we have employed the standard notations

$$B^1(u) = B(u) \otimes I, \qquad B^2(v) = I \otimes B(v).$$

Let us further require that $B(0) = I$ and that

$$B^1(u)\bar{R}(u-v)B^1(-u) = B^2(-u)R(u-v)B^2(u),$$

which implies that $R^{\alpha\alpha'}_{\beta\beta'} = 0$ unless $\alpha + \beta = \alpha' + \beta'$. Here α, α' refer to the first space and β, β' to the second vector space on which the R matrix is defined. In the following calculations we shall take

$$B(u) = \begin{pmatrix} e^{u\zeta} & 0 \\ 0 & e^{-u\zeta} \end{pmatrix}; \qquad (5.4.13)$$

as a result we obtain the following form of the R matrix:

$$R(\omega) = \begin{pmatrix} \sinh(\omega+\eta) & 0 & 0 & 0 \\ 0 & \sinh\omega & e^{\omega}\sinh\eta & 0 \\ 0 & e^{-\omega}\sinh\eta & \sinh\omega & 0 \\ 0 & 0 & 0 & \sinh(\omega+\eta) \end{pmatrix}; \qquad (5.4.14)$$

where $\omega = (u-v)\eta = i\gamma$ and the arbitrary parameter ζ has been set equal to $1/2$. As R and $\bar{R}$ are connected by a similarity transformation,

$$R(\omega) = SR'S^{-1}, \quad S = \mathrm{diag}(e^{(u+v)/2}, e^{(u-v)/2}, e^{-(u-v)/2}, e^{-(u+v)/2}), \qquad (5.4.15)$$

hence the eigenvalues of R and R' are the same. Furthermore, the $R(\omega)$ matrix admits the following properties:

$$R(\omega)R(-\omega) = \xi(\omega)I, \quad \xi(\omega) = -4\sinh(\omega+\eta)\sinh(\omega-\eta), \qquad (5.4.16)$$

$$R_{21}(\omega) = \mathcal{P}R_{12}(\omega)\mathcal{P}, \qquad \mathcal{P}^2 = I, \qquad (5.4.17)$$

$$R(-\rho) = \xi(0)^{1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -e^{-\eta} & 0 \\ 0 & -e^{\eta} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho = i\pi + \eta. \tag{5.4.18}$$

In addition we note that

$$P_{12}^{-} = \frac{R_{12}(-\rho)}{2\xi(0)^{1/2}}, \tag{5.4.19}$$

is a projection operator in $V_1 \otimes V_2$ and $R(\omega)$ obeys crossing symmetry

$$R_{12}(\omega) = X^1 R_{12}(-\omega - \rho)^{t_2} X^1, \tag{5.4.20}$$

with

$$X = \begin{pmatrix} 0 & -ie^{-\eta/2} \\ ie^{\eta/2} & 0 \end{pmatrix},$$

as t_2 denotes transpose in the second space V_2.
As usual the monodromy matrix is given by

$$T_a(\omega) = R_{aN}(\omega) R_{aN-1}(\omega) ..... R_{a1}(\omega), \tag{5.4.21}$$

with $R_{ak}(\omega)$ being defined on $V_a \otimes V_k$. Here V_a is the auxiliary space and V_k represents the kth lattice site. The corresponding transfer matrix is obtained by taking the trace of the monodromy matrix over the auxiliary space so that

$$t(\omega) = \mathrm{tr}_a[T_a(\omega)] = \mathrm{tr}_a[R_{aN}(\omega) R_{aN-1}(\omega) ..... R_{a1}(\omega)], \tag{5.4.22}$$

and

$$[t(\omega), t(\omega')] = 0, \qquad \omega \neq \omega'. \tag{5.4.23}$$

The analytical Bethe ansatz starts by looking at the asymptotic behaviour of $R(\omega)$ as $\omega \to \infty$. In the present case one finds that

$$R_{ak}(\omega) \overset{\omega \to \infty}{\longrightarrow} e^{\omega + \eta/2} \begin{pmatrix} e^{\eta S_k^3} & P S_k^{-} \\ 0 & e^{-\eta S_k^3} \end{pmatrix}, \tag{5.4.24}$$

where

$$S_k^3 = \frac{1}{2}\sigma_3, \qquad S_k^{-} = \frac{1}{2}(\sigma_1 - i\sigma_2), \qquad P = 2e^{-\eta/2} \sinh \eta. \tag{5.4.25}$$

The upper triangular nature of the R matrix immediately implies that as $\omega \to \infty$

$$T_a(\omega) = \prod_{k=1}^{N} R_{ak}(\omega) \simeq e^{(\omega+\eta/2)N} \begin{pmatrix} e^{\eta S^3} & PS^- \\ 0 & e^{-\eta S^3} \end{pmatrix}, \tag{5.4.26}$$

with

$$S^3 = \sum_{k=1}^{N} S_k^3, \qquad S^- = \sum_{k=1}^{N} e^{\eta(S_N^3+....S_{k+1}^3)} S_k^- e^{-\eta(S_{k-1}^3+...+S_1^3)}. \tag{5.4.27}$$

When $\omega \to \infty$ we obtain

$$t(\omega) \simeq e^{\omega N}[e^{\eta/2(N+2S^3)} + e^{\eta/2(N-2S^3)}]. \tag{5.4.28}$$

If we set $4\sigma = N - 2S^3$, then we find that

$$t(\omega) \simeq e^{\omega N}[e^{\eta(N-2\sigma)} + e^{2\eta\sigma}]. \tag{5.4.29}$$

Note that $t(\omega)$ commutes with $T_a(\omega)$ and therefore with S^-, so that S^3 also commutes with σ.
Let $|\Delta^{(m)}\rangle$ be the simultaneous eigenstate of $t(\omega)$ and σ,

$$t(\omega)|\Delta^{(m)}\rangle = \Delta^{(m)}(\omega)|\Delta^{(m)}\rangle, \qquad \sigma|\Delta^{(m)}\rangle = m|\Delta^{(m)}\rangle, \tag{5.4.30}$$

so that explicitly we obtain as $\omega \to \infty$,

$$t(\omega)|\Delta^{(m)}\rangle \sim e^{\omega N}[e^{\eta(N-2m)} + 2e^{2m\eta}]|\Delta^{(m)}\rangle, \tag{5.4.31}$$

whence

$$\Delta^{(m)}(\omega) \sim e^{\omega N}[e^{(N-2m)\eta} + 2e^{2m\eta}], \quad \omega \to \infty. \tag{5.4.32}$$

We next consider the pseudo-vacuum state with all N spins in the up state, i.e.,

$$|\uparrow\uparrow ..... \uparrow\rangle = \prod_{k=1}^{N} \otimes |\uparrow\rangle_k, \qquad |\uparrow\rangle_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_k, \tag{5.4.33}$$

so that

$$4\sigma \prod_{k=1}^{N} \otimes |\uparrow\rangle_k = 0, \tag{5.4.34}$$

and hence the state $\prod_{k=1}^{N} \otimes |\uparrow\rangle_k$ has $m = 0$. Assuming that this is an eigenstate of $t(\omega)$ so that

$$\prod_{k=1}^{N} \otimes |\uparrow\rangle_k \equiv |\Delta^{(0)}\rangle, \tag{5.4.35}$$

one finds that the corresponding eigenvalue is

$$\Delta^{(0)}(\omega) = \langle\Delta^{(0)}|t(\omega)|\Delta^{(0)}\rangle = \mathrm{tr}_a\langle\Delta^{(0)}|T_a(\omega)|\Delta^{(0)}\rangle. \tag{5.4.36}$$

We observe that the exact R matrix can be written as

$$R_{ak}(\omega) = 2\begin{pmatrix} z_0 + z_3\sigma_k^3 & z_-\sigma_k^- \\ z_+\sigma_k^+ & z_0 - z_3\sigma_k^3 \end{pmatrix}, \tag{5.4.37}$$

where

$$z_0 = \frac{1}{2}[\sinh(\omega+\eta)+\sinh\omega], z_3 = \frac{1}{2}[\sinh(\omega+\eta)-\sinh\omega], z_\pm = e^{\pm\eta}\sinh\eta, \tag{5.4.38}$$

whence

$$\Delta^{(0)}(\omega) = tr_a\left\{\prod_{k=1}^{N}[\langle\uparrow|_k R_{ak}(\omega)|\uparrow\rangle_k]\right\} = 2^N[\sinh^N(\omega+\eta) + \sinh^N(\omega)]. \tag{5.4.39}$$

We then assume that the eigenvalue $\Delta^{(m)}$ has the general form

$$\Delta^{(m)}(\omega) = 2^N[A^{(m)}\sinh^n(\omega+\eta) + B^{(m)}\sinh^N\omega], \tag{5.4.40}$$

so that

$$\Delta^{(m)}(\omega) \overset{\omega\to\infty}{\longrightarrow} e^{\omega N}[e^{\eta N}A^{(m)}(\omega) + B^{(m)}(\omega)].$$

Comparing with (5.4.32) we conclude that asymptotically

$$A^{(m)}(\omega) \sim e^{-2m\eta}, \qquad B^{(m)} \sim e^{2m\eta}. \tag{5.4.41}$$

To proceed further we will require that the functional relations are satisfied by the coefficients $A^{(m)}(\omega)$ and $B^{(m)}(\omega)$. To this end let us consider that

$$T_a(\omega)^t = T_a(\omega)^{t_1\ldots t_N} = [R_{aN}(\omega)....R_{a1}(\omega)]^{t_1\ldots t_N} = [R_{aN}^{t_N}(\omega)...R_{a1}^{t_1}(\omega)], \tag{5.4.42}$$

where X^{t_i} represents a transpose in the ith space. From crossing symmetry we find that

$$R_{ak}(\omega) = X_a R_{ak}(-\omega-\rho)^{t_k} X_a, \qquad R^{t_k}_{ak}(\omega) = X_a R(-\omega-\rho) X_a. \tag{5.4.43}$$

Using these relations one can derive

$$t^t(\omega) = t(-\omega-\rho), \tag{5.4.44}$$

so that the eigenvalue satisfies

$$\Delta^{(m)}(\omega) = \Delta^{(m)}(-\omega-\rho). \tag{5.4.45}$$

We now employ the fusion property, which is a procedure for generating higher spin states from lower ones. For the relevant details of the formulae to be used in the sequel, we refer the reader to section (7) of Chapter 5. From the fusion property of the R matrix, it can be shown that

$$R_{\langle 12\rangle,3}(\omega) = P^+_{12} R_{13}(\omega) R_{23}(\omega+\rho) P^+_{12} \quad \text{and} \quad P^+_{12} = I - P^-_{12}, \tag{5.4.46}$$

which again is an R matrix satisfying the Yang-Baxter equation. This along with the representation theory implies that

$$R_{13}(\omega) R_{23}(\omega+\rho) \sim \begin{pmatrix} -4\sinh\omega\sinh(\omega+2\eta) I_2^{(3)} & \star \\ 0 & f(\omega) Z \end{pmatrix}, \tag{5.4.47}$$

where Z is a 3×3 matrix whose elements are operators in V_3. The top right represents a 1×3 matrix whose elements are operators in V_3, while the lower left-hand corner is a 3×1 null matrix and $I_2^{(3)}$ is a 2×2 unit matrix in V_3. The fused transfer matrix therefore becomes

$$t(\omega)t(\omega+\rho) = (-1)^N [4\sinh^N(\omega)\sinh(\omega+2\eta)]^N + f^N(\omega)\tilde{t}(\omega), \tag{5.4.48}$$

where $\tilde{t}(\omega)$ is the transfer matrix corresponding to Z. Equation (5.4.48) yields

$$\Delta^{(m)}(\omega)\Delta^{(m)}(\omega+\rho) = (-1)^N [4^N \sinh^N\omega \sinh^N(\omega+2\eta) + f^N(\omega)\tilde{\Delta}(\omega)], \tag{5.4.49}$$

with $\tilde{\Delta}(\omega)$ being the eigenvalue of $\tilde{t}(\omega)$.
Substituting (5.4.40) in (5.4.49) we obtain

$$B^{(m)}(\omega)A^{(m)}(\omega+\rho) = 1, \qquad A^{(m)}(\omega)A^{(m)}(-\omega) = 1, \tag{5.4.50}$$

which are the requisite functional relations.
Our goal, is to find $A^{(m)}$ and $B^{(m)}$. For this we observe that the R matrix depends on $\zeta^0, \zeta^{+1}, \zeta^{-1}$, where $\zeta = e^{\omega}$, and therefore one may infer that $t(\omega) = \sum_{i=1}^{N} t_i \zeta^i$ and consequently,

$$\Delta^{(m)}(\omega) = \sum_{i=-N}^{N} C_i^{(m)} \zeta^i. \tag{5.4.51}$$

We assume the following form of $A^{(m)}(\omega)$:

$$A^{(m)}(\omega) = \prod_{j=1}^{m} \frac{(\zeta - \alpha_j)(\zeta - \gamma_j)(\zeta^{-1} + \bar{\alpha}_j)(\zeta^{-1} + \bar{\gamma}_j)}{(\zeta - \beta_j)(\zeta - \delta_j)(\zeta^{-1} + \bar{\beta}_j)(\zeta^{-1} + \bar{\delta}_j)}, \tag{5.4.52}$$

along with the conditions,

$$\begin{gathered} \beta_j = \alpha_j^{-1}, \;\; \gamma_j = e^{\eta}/\alpha_j, \;\; \delta_j = e^{-\eta}\alpha_j, \;\; \bar{\beta}_j = \alpha_j, \\ \bar{\alpha}_j = \alpha_j^{-1}, \quad \bar{\gamma}_j = e^{-\eta}\alpha_j, \quad \bar{\delta}_j = e^{\eta}\alpha_j^{-1}, \end{gathered} \tag{5.4.53}$$

following from (5.4.50). Furthermore as $\omega \to \infty$ we have

$$A^{(m)}(\omega) \to \prod_{j=1}^{m} e^{-2\eta} = e^{-2m\eta}, \tag{5.4.54}$$

in conformity with (5.4.41). The second part of (5.4.50) yields

$$B^{(m)}(\omega) = \prod_{j=1}^{m} \frac{(\zeta^{-1} + e^{\eta}\alpha_j)(\zeta^{-1} + e^{2\eta}\alpha_j^{-1})(\zeta - e^{-\eta}\alpha_j^{-1})(\zeta - e^{-2\eta}\alpha_j)}{(\zeta^{-1} + e^{\eta}\alpha_j^{-1})(\zeta^{-1} + \alpha_j)(\zeta - e^{-\eta}\alpha_j)(\zeta - \alpha_j^{-1})}. \tag{5.4.55}$$

One can easily verify that $B^{(m)}(\omega) \sim e^{2m\eta}$ as $\omega \to \infty$ as required by (5.4.41).
Thus we have obtained the form of the Bethe ansatz eigenvalue for the mth excited state being given by (5.4.40) as a result of the explicit construction of $A^{(m)}(\omega)$ and $B^{(m)}(\omega)$, which are as in (5.4.52) and (5.4.54), respectively. The equations for the quasi-momenta can be shown to follow from the condition that the residue of (5.4.40) should vanish at the poles. These turn out to be as follows:

$$\frac{\sinh^N(\omega_k - \eta/2)}{\sinh^N(\omega_k + \eta/2)} = \frac{\sinh(2\omega_k - \eta)}{\sinh(2\omega_k + \eta)} \prod_{j=1, j\neq k}^{N} \frac{N_j}{D_j}, \tag{5.4.56}$$

where

$$N_j = P(\omega_k,\omega_j,\eta)Q(\omega_k,\omega_j,\eta)P(-\omega_k,-\omega_j,-\eta)Q(-\omega_k,-\omega_j,-\eta),$$
$$D_j = \bar{P}(\omega_k,\omega_j,\eta)\bar{Q}(\omega_k,\omega_j,\eta)\bar{P}(-\omega_k,-\omega_j,-\eta)\bar{Q}(-\omega_k,-\omega_j,-\eta), \tag{5.4.57}$$

$$P(\omega_k,\omega_j,\eta) = e^{\omega_k+\eta/2} + e^{\omega_j+3\eta/2} \quad \bar{P}(\omega_k,\omega_j,\eta) = e^{\omega_k+\eta/2} + e^{\omega_j-\eta/2}, \tag{5.4.58}$$

$$Q(\omega_k,\omega_j,\eta) = e^{\omega_k+\eta/2}+e^{-\omega_j+3\eta/2}, \quad \bar{Q}(\omega_k,\omega_j,\eta) = e^{\omega_k+\eta/2}+e^{-\omega_j-\eta/2}.$$

In this manner we have the complete solution of the Bethe ansatz problem for the Sine-Gordon equation in a way different from that in [88].

5.5 Off-Shell Bethe Ansatz

In this section we describe a recent concept related to the algebraic Bethe ansatz, which was introduced for the evaluation of the semiclassical limits of the algebraic Bethe ansatz. It provided for the connection of the algebraic Bethe ansatz with conformal field theory. In fact one can deduce the famous Knizhnik-Zamolodchikov (KZ) equation [92] from this procedure. In order to discuss this concept we consider an inhomogeneous vertex model. We start with a two-dimensional $M \times N$ lattice with $N+1$ in general, different types of spin variables placed in the following manner inhomogeneously on the links of the lattice. On all horizontal links are spin variables taking values $\pm 1/2$. The variables in the jth column ($j = 1, 2, ...N$) take values of an $SU(2)$ representation with spin S_j. The interaction takes place between spins located on neighbouring links and is described by the vertex weight matrix $R^{j_1,j_2}_{i_1,i_2}(\lambda - z)$, where λ is the usual spectral parameter and z is a parameter known as the disorder parameter. We assume that the boundary conditions are cyclic and use the $SU(2)$ invariant solution of the Yang-Baxter equation:

$$R^{12}_{\sigma}(\lambda-\mu)R^{13}_{s}(\lambda-z)R^{23}_{s}(\mu-z) = R^{23}_{s}(\mu-z)R^{13}_{s}(\lambda-z)R^{12}_{\sigma}(\lambda-\mu). \tag{5.5.1}$$

This is a Yang-Baxter equation involving two types of spin vertices. Here $R^{12}_{\sigma}(\lambda)$ is the vertex weight of the XXX model so that

$$R^{12}_{\sigma}(\lambda) = I^1 \otimes I^2 + \frac{2\eta}{(\eta-2\lambda)}\vec{\sigma}_1 \otimes \vec{\sigma}_2, \tag{5.5.2}$$

and

$$R_s^{12}(\lambda - z) = I^1 \otimes I^2 + \frac{2\eta}{\eta - 2(\lambda - z)} \vec{\sigma}_1 \otimes \vec{S}_2, \tag{5.5.3}$$

where $\vec{\sigma}$ is the Pauli operator and $\vec{S}$ denotes an arbitrary spin. I^1 and I^2 are unit operators in the respective spaces. The parameter η gives the quasi-classical expansion,

$$R^{12}(\lambda, \eta)|_{\eta=0} = I^1 \otimes I^2, \tag{5.5.4}$$

The monodromy and transfer matrices are defined in the usual way as

$$T(\lambda, \{z\}) = R^{0N}(\lambda - z_N) R^{0,N-1}(\lambda - z_{N-1}) \ldots\ldots R^{01}(\lambda - z_1), \tag{5.5.5}$$

$$t(\lambda, \{z\}) = \mathrm{tr}_0[T(\lambda, \{z\})], \tag{5.5.6}$$

with the trace being taken over the auxiliary space, and

$$R^{0k}(\lambda - z_k) = I^0 \otimes I^k + \frac{2\eta}{\eta - 2(\lambda - z_k)} \vec{\sigma}_0 \otimes \vec{S}_k. \tag{5.5.7}$$

One infers from (5.5.1) that

$$R_\sigma^{12}(\lambda - \mu) T(\lambda, \{z\}) \otimes T(\mu, \{z\}) = T(\mu, \{z\}) \otimes T(\lambda, \{z\}) R_\sigma^{12}(\lambda - \mu), \tag{5.5.8}$$

and also that

$$[t(\lambda, \{z\}), t(\mu, \{z\})] = 0. \tag{5.5.9}$$

We write $T(\lambda, \{z\})$ as a 2×2 matrix

$$T(\lambda, \{z\}) = \begin{pmatrix} A(\lambda, \{z\}) & B(\lambda, \{z\}) \\ C(\lambda, \{z\}) & D(\lambda, \{z\}) \end{pmatrix},$$

while $R_\sigma(\lambda)$ occurring in (5.5.8) is given by

$$R_\sigma(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.5.10}$$

where

$$b(\lambda) = \frac{\eta}{\eta - \lambda}, \qquad c(\lambda) = \frac{\lambda}{\lambda - \eta}. \tag{5.5.11}$$

Let us now start with a pseudo-vacuum, say $|\Omega\rangle$, and using the commutation relations for A, B, C, D from (5.5.8) pick up the following for our purpose:

$$A(\lambda, \{z\})|\Omega\rangle = \prod_{i=1}^{N}(1 + P_i(\lambda)s_i)|\Omega\rangle,$$

$$D(\lambda, \{z\})|\Omega\rangle = \prod_{i=1}^{N}(1 - P_i(\lambda)s_i)|\Omega\rangle, \tag{5.5.12}$$

$$C(\lambda, \{z\})|\Omega\rangle = 0, \quad P_i(\lambda) = \frac{2\eta}{\eta - 2(\lambda - z_i)}.$$

Now the general Bethe wave function is written as follows:

$$\phi(\lambda_1, ..., \lambda_n, \{z\}) = \prod_{\alpha=1}^{n} B(\lambda_\alpha, \{z\})|\Omega\rangle. \tag{5.5.13}$$

The action of $t(\lambda, \{z\}) = A(\lambda, \{z\}) + D(\lambda, \{z\})$ on ϕ is given by

$$t(\lambda, \{z\})\phi = \Lambda(\lambda; \lambda_1, ..., \lambda_n)\phi - \sum_{\alpha=1}^{n} \frac{F_\alpha(\lambda, \{z\})}{\lambda - \lambda_\alpha}\phi_\alpha, \tag{5.5.14}$$

where we have written the unwanted terms in detail. The explicit forms of Λ, F_α are as follows:

$$\Lambda(\lambda; \lambda_1, ..., \lambda_n) = \prod_{i=1}^{N}(1 + P_i(\lambda)s_i) \prod_{\alpha=1}^{n} \frac{1}{c(\lambda_\alpha - \lambda)} +$$

$$+ \prod_{i=1}^{N}(1 - P_i(\lambda)s_i) \prod_{\alpha=1}^{n} \frac{1}{c(\lambda - \lambda_\alpha)}, \tag{5.5.15}$$

$$F_\alpha(\lambda, \{z\}) = \eta \prod_{i=1}^{N}(1 + P_i(\lambda)s_i) \prod_{\beta\neq\alpha}^{n} \frac{\lambda_\alpha - \lambda_\beta + \eta}{\lambda_\alpha - \lambda_\beta}$$

$$-\eta \prod_{i=1}^{N}(1 - P_i(\lambda)s_i) \prod_{\beta\neq\alpha}^{n} \frac{\lambda_\alpha - \lambda_\beta - \eta}{\lambda_\alpha - \lambda_\beta}, \tag{5.5.16}$$

$$\phi_\alpha = \phi(\lambda_1, \lambda_2, ...\lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, ..., \lambda_n). \tag{5.5.17}$$

In the customary approach one imposes the condition that the unwanted terms should vanish, which then leads to the Bethe ansatz equation for the eigenmomenta, while the coefficient of the wanted term

yields the Bethe states eigenvalue. The vanishing of the unwanted term means that $F_\alpha = 0$, and when these equations are satisfied by the λ_α's we say that the Bethe wave function is *on shell*. But if one does not impose the condition $F_\alpha = 0$, then one can say that we are dealing with an *off-shell* Bethe ansatz wave function.

Let us next consider the quasi-classical expansion of equation (5.5.15). In general by a quasi-classical expansion one commonly understands the expansion of the vertex weight $R(\lambda, \eta)$ around a point, say η_0, so that $R(\lambda, \eta_0) = I \otimes I$. In this case one can always choose $\eta_0 = 0$. Then for the power series expansion of $t(\lambda, \{z\})$ in the neighbourhood of $\eta = 0$ we have

$$t(\lambda, \{z\}) = \sum_{k=0}^{\infty} \eta^k T_k(\lambda, \{z\}). \tag{5.5.18}$$

It follows from (5.5.9) that

$$\sum_{k+m=l} [T_k(\lambda, \{z\}), T_m(\lambda, \{z\})] = 0, \tag{5.5.19}$$

which implies that we do have integrable systems in the quasi-classical limit. It is interesting to note further that the $T_k(\lambda, \{z\})$'s do not commute with $t(\lambda, \{z\})$. In order to find a quasi-classical expansion, we write R in (5.5.7) as

$$R^{0,i}(\lambda - z_i) = \begin{pmatrix} 1 + P_i(\lambda) S_i^3 & P_i(\lambda) S_i^- \\ P_i(\lambda) S_i^+ & 1 - P_i(\lambda) S_i^3 \end{pmatrix}, \tag{5.5.20}$$

with

$$P_i(\lambda) = -\frac{\eta}{\lambda - z_i} - \frac{1}{2} \left(\frac{\eta}{\lambda - z_i} \right)^2, \qquad \eta << 1. \tag{5.5.21}$$

Using this form we get

$$T(\lambda, \{z\}) = \prod_{i=1}^{N} \left[I_0 \otimes I_i + P_i(\lambda) \begin{pmatrix} S_i^3 & S_i^- \\ S_i^+ & -S_i^3 \end{pmatrix} \right] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.5.22}$$

where

$$A(\lambda, \{z\}) = I - \eta S^3(\lambda, \{z\}) + \eta^2 \sum_{i<j} \frac{S_i^3 S_j^3 + S_i^- S_j^+}{(\lambda - z_i)(\lambda - z_j)} +$$

$$+\frac{\eta^2}{2}\frac{d}{d\lambda}S^3(\lambda,\{z\})+\mathcal{O}(\eta^3), \tag{5.5.23}$$

$$D(\lambda,\{z\})=I-\eta S^3(\lambda,\{z\})+\eta^2\sum_{i<j}\frac{S_i^3S_j^3+S_i^+S_j^-}{(\lambda-z_i)(\lambda-z_j)}-$$

$$-\frac{\eta^2}{2}\frac{d}{d\lambda}S^3(\lambda,\{z\})+\mathcal{O}(\eta^3), \tag{5.5.24}$$

$$B(\lambda,\{z\})=-\eta S^-(\lambda,\{z\})+\eta^2\sum_{i<j}\frac{S_i^3S_j^--S_i^-S_j^3}{(\lambda-z_i)(\lambda-z_j)}-$$

$$-\frac{\eta^2}{2}\frac{d}{d\lambda}S^-(\lambda,\{z\})+\mathcal{O}(\eta^3), \tag{5.5.25}$$

$$C(\lambda,\{z\})=-\eta S^+(\lambda,\{z\})+\eta^2\sum_{i<j}\frac{S_i^+S_j^3-S_i^3S_j^+}{(\lambda-z_i)(\lambda-z_j)}$$

$$-\frac{\eta^2}{2}\frac{d}{d\lambda}S^+(\lambda,\{z\})+\mathcal{O}(\eta^3). \tag{5.5.26}$$

In (5.5.23–5.5.26), we have used the notations

$$I=\prod_{i=1}^{N}\otimes I_i,\qquad S^a(\lambda,\{z\})=\sum_{i=1}^{N}\frac{S_i^a}{\lambda-z_i}.$$

Consequently one has for the transfer matrix $t(\lambda,\{z\})$ the expression

$$t(\lambda,\{z\})=2I+2\eta^2\sum_{i=1}^{N}\frac{H_j}{\lambda-z_j}, \tag{5.5.27}$$

$$H_j=\sum_{i\neq j}^{N}\frac{S_j^aS_i^a}{z_j-z_i}. \tag{5.5.28}$$

It is obvious from (5.5.19) that the operators H_j commute so that in the quasi-classical limit we finally obtain

$$\phi(\lambda_1,.....,\lambda_n)=(-\eta)^n\prod_{\alpha=1}^{N}S^-(\lambda_\alpha,\{z\})|\Omega\rangle+\mathcal{O}(\eta^{n+1}), \tag{5.5.29}$$

$$\Lambda(\lambda;\lambda_1.....\lambda_n)=2+2\eta^2Q+\mathcal{O}(\eta^3), \tag{5.5.30}$$

where

$$Q = \sum_{i,\alpha} \frac{s_i}{(\lambda - z_i)(\lambda_\alpha - \lambda)} + \sum_{i \neq j} \frac{s_i s_j}{(\lambda - z_i)(\lambda - z_j)} + \sum_{\alpha \neq \beta} \frac{1}{(\lambda_\alpha - \lambda)(\lambda_\beta - \lambda)}, \tag{5.5.31}$$

$$F_\alpha = 2\eta^2 \left[\sum_{\alpha \neq \beta} \frac{1}{\lambda_\alpha - \lambda_\beta} - \sum_{i=1}^{N} \frac{S_i}{\lambda_\alpha - z_i} \right] + \mathcal{O}(\eta^3), \tag{5.5.32}$$

$$\phi_\alpha = (-\eta)^n S^-(\lambda, \{z\}) .... S^-(\lambda_{\alpha-1}, \{z\}) S^-(\lambda, \{z\}) S^-(\lambda_{\alpha+1}, \{z\}) ....$$
$$.....S^-(\lambda_n, \{z\})|\Omega\rangle + \mathcal{O}(\eta^{n+1}). \tag{5.5.33}$$

Substituting all these in (5.5.14) and collecting terms proportional to η^{n+2}, we find that in the quasi-classical limit.

$$\sum_{j=1}^{N} \frac{H_j}{\lambda - z_j} \phi(\lambda_1 .... \lambda_n) = h\phi(\lambda_1 .... \lambda_n) - \sum_{\alpha=1}^{n} \frac{f_\alpha \phi_\alpha}{\lambda - \lambda_\alpha}. \tag{5.5.34}$$

Here the vectors ϕ and ϕ_α appearing in (5.5.29) and (5.5.33) are both proportional to η^2. Taking the residue at the pole $\lambda = z_j$ we have from (5.5.34) that

$$H_j \phi(\lambda_1 .... \lambda_n) = h_j \phi(\lambda_1 .... \lambda_n) - \sum_{\alpha=1}^{N} \frac{f_\alpha S_j^-}{z_j - \lambda_\alpha} \phi'_\alpha(\lambda_1 .... \lambda_n), \tag{5.5.35}$$

where

$$h_j = \sum_{j \neq i} \frac{s_i s_j}{z_j - z_i} + \sum_{\alpha=1}^{n} \frac{s_i}{\lambda_\alpha - z_j}, \tag{5.5.36}$$

$$f_\alpha = \sum_{\alpha \neq \beta}^{n} \frac{1}{\lambda_\alpha - \lambda_\beta} - \sum_{i=1}^{N} \frac{s_i}{\lambda_\alpha - z_i}, \tag{5.5.37}$$

$$\phi(\lambda, \{z\}) = \prod_{\alpha=1}^{n} S^-(\lambda_\alpha, \{z\})|\Omega\rangle. \tag{5.5.38}$$

In the above equations $\phi(\lambda_1 .... \lambda_n) = S^-(\lambda_\alpha, \{z\}) \phi'_\alpha(\lambda_1 ...., \lambda_n)$ i.e. in $\phi'_\alpha(\lambda_1 .... \lambda_n)$ the operator $S^-(\lambda_\alpha, \{z\})$ is omitted. Equations (5.5.35) and (5.5.36–5.5.38) are the *off-shell quasi-classical limit* of the algebraic Bethe ansatz.

We shall next try to explain the connection of the preceding results with the Knizhnik-Zamolodchikov equation. Without going into the

details of the KZ equation itself we consider the equation in the form [92],

$$\kappa \frac{d\psi}{dz_j} = \sum_{i \neq j}^{N} \frac{t_j^a t_i^a}{z_j - z_i} \psi; \tag{5.5.39}$$

here z_i's denote the disorder parameters of the Bethe ansatz. $\psi(z_1, ...z_N)$ is a vector in the tensor product of spaces $V^1 \otimes V^2 .... \otimes V^N$. The matrices $t_i^a (a = 1, 2...\dim\, g)$ represent the hermitian generators of a Lie algebra g. In addition we have $\kappa = \frac{1}{2(C_v+K)}$ and $\delta^{ab} C_v = f^{acd} f^{bcd}$ with f^{abc} being the structure constants of the Lie algebra g and K is the central charge of the Kac-Moody algebra. In the present analysis we consider only the $SU(2)$ case. To investigate the connection of the off-shell Bethe ansatz equation (OSBAE) with the KZ equation, let us therefore consider a function $\chi(\lambda, \{z\}) = \chi(\lambda_1, ...\lambda_N, z_1, ...z_N)$ obeying the following relations:

$$\kappa \frac{d\chi}{dz_j} = h_j \chi, \qquad \kappa \frac{d\chi}{d\lambda_\alpha} = f_\alpha \chi, \tag{5.5.40}$$

where $\kappa = \frac{1}{2(K+2)}$, since for $SU(2)$ we have $C_v = 2$. The consistency of these equations requires that

$$\frac{dh_j}{d\lambda_\alpha} = \frac{df_\alpha}{dz_j}, \tag{5.5.41}$$

which is identically satisfied. The solution χ, which is common to these two, can be expressed in the form

$$\chi(\lambda, \{z\}) = \prod_{i<j}^{N} (z_i - z_j)^{\frac{s_i s_j}{K}} \prod_{\alpha<\beta}^{n} (\lambda_\alpha - \lambda_\beta)^{1/n} \prod_{k,\gamma} (z_k - \lambda_\gamma)^{-s_k/n}. \tag{5.5.42}$$

Next define the vector function $\psi(z_1, ...z_N)$ through

$$\psi(z_1, ...z_N) = \oint ... \oint \chi(\lambda, \{z\}) \phi(\lambda, \{z\}) d\lambda_1 ... d\lambda_N. \tag{5.5.43}$$

The integrations are to be taken as contour integrals over the canonical cycles of the space $X = C^n - U(\lambda_\alpha = z_i)$ with coefficicents from $S_\lambda^\star$ dual to the local system S_λ, that is, defined by the monodromy group of $\chi(\lambda, \{z\})$. It is now important to note that this ψ is a solution of the KZ equation (5.5.39). To show this we substitute (5.5.43) in (5.5.39)

and using (5.5.35) together with the defining relation (5.5.40) for χ, to arrive at

$$\oint .... \oint \left[K \frac{d\phi}{dz_j} - \frac{1}{K} \sum_{\alpha=1}^{n} \frac{S_j^- f_\alpha \chi \phi'_\alpha}{z_j - \lambda_\alpha} \right] d\lambda_1 .....d\lambda_N = 0. \tag{5.5.44}$$

Taking into account the linear equations satisfied by χ and the identity

$$\frac{d\phi}{dz_j} = - \sum_{\alpha=1}^{n} \frac{d}{d\lambda_\alpha} \frac{S_j^- \phi'_\alpha}{z_j - \lambda_\alpha},$$

which follows from (5.5.38) one can simplify the relation (5.5.44) to finally obtain

$$\sum_{\alpha=1}^{n} \oint ..... \oint \frac{d}{d\lambda_\alpha} \left[\frac{S_j^- \phi'_\alpha \chi}{z_j - \lambda_\alpha} \right] d\lambda_1 .....d\lambda_n = 0. \tag{5.5.45}$$

Now it is obvious that this equation is identically satisfied as the contour is closed. It follows then that ψ is a singlet state of $SU(2)$. We shall not provide further details of the proof of this conjecture but refer the reader to the excellent papers of Babujian et al. [93].

5.6 Nested Bethe Ansatz

The nested Bethe ansatz is perhaps the most sophisticated tool in the algebraic construction of eigenvectors for integrable lattice systems. As is well known integrable systems within the Lax operator formalism are often associated with underlying quantum groups of rank greater than unity. It is in these situations that the nested Bethe ansatz becomes indispensible. Examples of systems with Lax matrices of dimension greater than 2×2 abound in the literature, prominent among them being the three wave interaction problem, self induced transparency, coupled NLS equation, etc. Setting up the algebraic Bethe ansatz in such cases is difficult owing to the existence of a large number of operators in the corresponding monodromy matrix (e.g., 9 in case of a 3×3 and 16 in case of a 4×4 problem). Moreover, the complex nature of the commutation rules generated by the Yang-Baxter equation has also to be taken into account. In such cases the algebraic Bethe ansatz

is implemented in a multistage manner and this is referred to as the *nested Bethe ansatz* [94].

For clarity let us consider a 3×3 problem, which has all the features of the more general $N\times N$ case. Let $L_n(\lambda)$ denote the local Lax matrix at the nth lattice site and assume that it satisfies the intertwining relation

$$R(\lambda,\mu)L_n(\lambda)\otimes L_n(\mu)=L_n(\mu)\otimes L_n(\lambda)R(\lambda,\mu). \tag{5.6.1}$$

The monodromy matrix is

$$T_N(\lambda)=L_N(\lambda)L_{N-1}(\lambda).....L_1(\lambda), \tag{5.6.2}$$

and satisfies an algebra similar to (5.6.1), i.e.,

$$R(\lambda,\mu)T_N(\lambda)\otimes T_N(\mu)=T_N(\mu)\otimes T_N(\lambda)R(\lambda,\mu). \tag{5.6.3}$$

Suppose that the quantum R matrix, which is a solution of the Yang-Baxter equation, has the following nonvanishing components:

$$R(\lambda)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0\\
0&a&0&b&0&0&0&0&0\\
0&0&a&0&0&0&b&0&0\\
0&b&0&a&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&a&0&b&0\\
0&0&b&0&0&0&a&0&0\\
0&0&0&0&0&b&0&a&0\\
0&0&0&0&0&0&0&0&1
\end{pmatrix},$$

where

$$a(\lambda)=\frac{\gamma}{\gamma-2\lambda},\qquad b(\lambda)=\frac{2\lambda}{2\lambda-\gamma},\qquad \gamma=i\hbar. \tag{5.6.4}$$

We can also express the R matrix in the notation $R(\lambda)=\sum_{ab,ij}R_{ij}^{ab}e_{ab}\otimes e_{ij}$ where the components are given by

$$R_{ij}^{ab}(\lambda)=\frac{\gamma}{\gamma-2\lambda}\delta_{ia}\delta_{jb}+\frac{2\lambda}{2\lambda-\gamma}\delta_{ib}\delta_{ja}. \tag{5.6.5}$$

In addition suppose it satisfies the following conditions:

$$R_{ij}^{ab}(0)=\delta_{ia}\delta_{jb},\quad R_{ij}^{ab}(\lambda)=R_{ab}^{ij}(\lambda). \tag{5.6.6}$$

Next let us write the monodromy matrix as

$$T_N(\lambda) = \prod_{n=1}^{\overleftarrow{N}} L_n(\lambda) = \begin{pmatrix} A(\lambda) & B_2(\lambda) & B_3(\lambda) \\ C_2(\lambda) & D_{22}(\lambda) & D_{23}(\lambda) \\ C_3(\lambda) & D_{32}(\lambda) & D_{33}(\lambda) \end{pmatrix}, \tag{5.6.7}$$

and introduce the notation $T_{11}(\lambda) = A(\lambda)$, $T_{1i}(\lambda) = B_i(\lambda)$, $T_{i1}(\lambda) = C_i(\lambda)$ and $T_{ij}(\lambda) = D_{ij}(\lambda)$ with $2 \leq i, j \leq 3$. Let $||1\rangle$ be a reference state (the vacuum) so that $T_{1i}(\lambda)||1\rangle \neq 0$ while $T_{ij}(\lambda)||1\rangle = 0$ for $i \neq j, 2 \leq i, j \leq 3$ with following properties:

$$T_{11}(\lambda)||\lambda\rangle = \exp(-i\lambda\Delta)||\lambda\rangle, \qquad T_{kk}(\lambda)||\lambda\rangle = \exp(i\lambda\Delta)||\lambda\rangle, \tag{5.6.8}$$

with $k = 2, 3$ and Δ being the lattice spacing. Now by a similarity transformation, one can always recast the R matrix into the following form that is more suited to our subsequent analysis:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_{jj}c(\lambda) & \delta_{ij}\frac{1}{g(\lambda)} & 0 \\ 0 & \delta_{ij}\frac{1}{g(\lambda)} & \delta_{ij}c(\lambda) & 0 \\ 0 & 0 & 0 & \widetilde{R^{jl}_{ik}}(\lambda) \end{pmatrix}, \tag{5.6.9}$$

where $c(\lambda) = \frac{\gamma}{(\gamma - 2\lambda)}$ and $g(\lambda) = \frac{2\lambda - \gamma}{2\lambda}$, with $2 \leq i, j, k, l \leq 3$ and

$$\widetilde{R}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma}{\gamma - 2\lambda} & \frac{2\lambda}{2\lambda - \gamma} & 0 \\ 0 & \frac{2\lambda}{2\lambda - \gamma} & \frac{\gamma}{\gamma - 2\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.6.10}$$

Then (5.6.3), when using (5.6.7) and (5.6.9), yields the following commutation relations:

$$A(\lambda)\mathbf{B}(\mu) = g(\mu - \lambda)\mathbf{B}(\mu)A(\lambda) - h(\mu - \lambda)\mathbf{B}(\lambda)A(\mu), \tag{5.6.11}$$

$$B_i(\lambda)B_j(\mu) = \widetilde{R}^{ij}_{kl}B_k(\mu)B_l(\lambda), \tag{5.6.12}$$

$$D_{kj}B_l(\mu) = g(\lambda - \mu)B_m(\mu)D_{ki}(\lambda)\widetilde{R^{ij}_{mi}}(\lambda - \mu) - h(\lambda - \mu)B_j(\lambda)D_{kl}(\mu), \tag{5.6.13}$$

where $\mathbf{B} = (B_2(\lambda), B_3(\lambda))$ and $g(\lambda - \mu) = \frac{2(\mu - \lambda) - \gamma}{2(\mu - \lambda)}$, while $h(\mu - \lambda) = -\frac{\gamma}{2(\mu - \lambda)}$. Alternatively, in tensor product notation, (5.6.12) and (5.6.13) may be written in a more compact form as

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) = \mathbf{B}(\mu) \otimes \mathbf{B}(\lambda)\widetilde{R}(\lambda - \mu), \tag{5.6.14}$$

$$D(\lambda)\otimes\mathbf{B}(\mu)=g(\lambda-\mu)\mathbf{B}(\mu)\otimes D(\lambda)\widetilde{R}(\lambda-\mu)-h(\lambda-\mu)\mathbf{B}(\lambda)\otimes D(\mu). \tag{5.6.15}$$

Note all indices here assume values $2,3$ only. The transfer matrix is defined in the usual manner as the trace of the monodromy matrix $T_N(\lambda)$, i.e.,

$$\tau(\lambda)=A(\lambda)+\mathrm{tr}^{(2)}D(\lambda) \tag{5.6.16}$$

and $\mathrm{tr}^{(2)}D(\lambda)=\sum_{a=2}^{3}D_{aa}(\lambda)$. An r-excitation state is then built by the application of r creation operators from among $(B_2(\lambda),B_3(\lambda))$ and is written as a linear superposition, viz

$$|\mu_1,\mu_2,....\mu_r\rangle=\sum_{i_1,...i_r=2}^{3}X_{i_1....i_r}B_{i_1}(\mu_1).....B_{i_r}(\mu_r)||1\rangle, \tag{5.6.17}$$

or in the more compact form as

$$|\mu_1....\mu_r\rangle=\mathbf{X}^tB(\mu_1)\otimes...\otimes B(\mu_r)||1\rangle, \tag{5.6.18}$$

where $\mathbf{X}$ is a vector whose components are yet to be determined and $\mathbf{X}^t$ denotes transposition. Applying $A(\lambda)$ to the state defined by (5.6.18) and repeatedly using (5.6.11) together with (5.6.12) yields

$$A(\lambda)|\mu_1,...\mu_r\rangle=\exp(-i\lambda\Delta)\prod_{j=1}^{r}g(\mu_j-\lambda)|\mu_1....\mu_r\rangle+\text{unwanted terms.} \tag{5.6.19}$$

Note that it is not always necessary to know the explicit form of the unwanted terms for the purposes of analysis. The important point lies in calculating the action of $\mathrm{tr}^{(2)}D(\lambda)$ on $|\mu_1,.....\mu_r\rangle$. To this end, using (5.6.12) and (5.6.13) one can show that

$$\begin{aligned}&D_{kj}(\lambda)X^{i_1....i_r}B_{i_1}(\mu_1)....B_{i_r}(\mu_r)||1\rangle\\&=\prod_{p=1}^{r}g(\lambda-\mu_p)X^{i_1.....i_r}B_{m_1}(\mu_1).....B_{m_r}||1\rangle\delta_{kl_r}\times\\&\times\exp(-i\lambda\Delta)\widetilde{R}^{l_{r-1}i_r}_{m_rl_r}(\lambda-\mu_r).....\widetilde{R}^{l_1i_2}_{m_2l_2}(\lambda-\mu_2)\widetilde{R}^{ji_1}_{m_1l_1}(\lambda-\mu_1),\end{aligned} \tag{5.6.20}$$

so that

$$\begin{aligned}&\Big(\mathrm{tr}^{(2)}D(\lambda)\Big)\sum_{i_1...i_r=2}^{3}X^{i_1....i_r}B_{i_1}(\mu_1)....B_{i_r}(\mu_r)||1\rangle\\&=\sum_{k=2}^{3}\left[\exp(-i\lambda\Delta)\prod_{p=1}^{r}g(\lambda-\mu_p)\sum_{i_1..i_r=2}^{3}X^{i_1....i_r}B_{i_1}(\mu_1)....B_{i_r}(\mu_r)||1\rangle\right]\times\end{aligned}$$

$$\times \left\{ \widetilde{R}^{l_{r-1} i_r}_{m_r l_r}(\lambda - \mu_r) .... \widetilde{R}^{l_1 i_2}_{m_2 l_2}(\lambda - \mu_2) \widetilde{R}^{j i_1}_{m_1 l_1}(\lambda - \mu_1) \right\} + \text{unwanted terms.} \tag{5.6.21}$$

Until now we have followed the basic strategy of the algebraic Bethe ansatz of separating the wanted and unwanted terms. The actual "nesting" procedure will now be described for (5.6.21).

Recall that the matrix $\widetilde{R}(\lambda)$ occurring in (5.6.21) is actually given by (5.6.10) and is a 4×4 matrix. The process of nesting consists in treating $X^{i_1 .... i_r}$ as an eigenvector of an associated transfer matrix defined by the term within the curly brackets in (5.6.21). That is, one considers a "vertex model" similar to those of statistical mechanics with Boltzmann weights given by the elements of a lower-dimensional quantum R matrix, viz $\widetilde{R}(\lambda)$. Notice that while our original R matrix as given by (5.6.5) was a 9×9 matrix after employing the similarity transformation it could be partitioned into a form so that the lowermost block $\widetilde{R}(\lambda)$ had the appearance of the standard 4×4, R matrix usually associated with a 2×2 Lax operator. The advantage of being able to do such a partitioning is that one can subsequently apply the usual procedure of algebraic Bethe ansatz to this "reduced" R matrix and formulate the quantum inverse problem. In general if one starts with an $N^2 \times N^2$ quantum R matrix, then applying the method outlined above, one is led to a reduced problem with an $(N-1)^2 \times (N-1)^2$ quantum R matrix. This procedure can be continued until we arrive at the lowest dimensional $2^2 \times 2^2$ quantum R matrix. Let us now relabel the weights so that $\widetilde{R}^{j i_k}_{m_k l_k}(\lambda) \to \hat{R}^{j i_k}_{l_k m_k}(\lambda)$, which essentially means multiplication of $\widetilde{R}(\lambda)$ by the permutation matrix $\mathcal{P}$ that has the property $\mathcal{P}^2 = I$. Hence the term within the curly brackets in (5.6.21) now assumes the following form, and may be looked upon as a kind of reduced "transfer matrix." Suppose we define

$$\hat{R}^{l_{r-1} i_r}_{k m_r}(\lambda - \mu_r) ..... \hat{R}^{l_1 i_2}_{l_2 m_2}(\lambda - \mu_2) \hat{R}^{k i_1}_{l_1 m_1}(\lambda - \mu_1) \equiv \hat{F}(\lambda)^{m_r .. m_1}_{i_r ... i_1} \tag{5.6.22}$$

Consequently the "wanted term" in (5.6.21) now appears as

$$\left(\text{tr}^{(2)} D(\lambda) \right) \sum_{i_1 ... i_r = 2}^{3} X^{i_1 .... i_r} B_{i_1}(\mu_1) .... B_{i_r}(\mu_r) || 1 \rangle$$

$$= \exp\left(-i\lambda\Delta\right) \prod_{j=1}^{r} g(\lambda - \mu_j) \mathbf{B}(\mu_1) \otimes .... \otimes \mathbf{B}(\mu_r) \hat{F}(\lambda) || 1 \rangle, \tag{5.6.23}$$

with

$$\hat{F}(\lambda) = \text{tr}_0 \left[\hat{R}_{0r}(\lambda - \mu_r) ..... \hat{R}_{02}(\lambda - \mu_2) \hat{R}_{01}(\lambda - \mu_1) \right]. \tag{5.6.24}$$

Equation (5.6.24) is clearly similar to the usual transfer matrix of an auxiliary r-site model with $\hat{R}$ defined in the reduced $C^2 \otimes C^2$ space. However (5.6.23) (even apart from the "unwanted terms") is not an eigenvalue equation, though it may be brought to such a form by requiring $\mathbf{X}$ to be an eigenvector of $\hat{F}(\lambda, \{\mu_j\})$, so that

$$\hat{F}(\lambda, \{\mu_j\})\mathbf{X} = \hat{\Lambda}(\lambda)\mathbf{X}, \qquad \text{and} \tag{5.6.25}$$

$$\mathrm{tr}^{(2)} D(\lambda)|\mu_1, ...\mu_r\rangle = \exp(-i\lambda\Delta) \prod_{j=1}^{r} g(\lambda - \mu_j)\hat{\Lambda}(\lambda)|\mu_1...\mu_r\rangle + \quad \text{unwanted terms.} \tag{5.6.26}$$

Combining (5.6.19) and (5.6.26) we finally have

$$[A(\lambda) + \mathrm{tr}^{(2)} D(\lambda)]|\mu_1...\mu_r\rangle = \exp(i\lambda\Delta) \prod_{j=1}^{r} g(\mu_j - \lambda)|\mu_1.....\mu_r\rangle +$$

$$+ \exp(-i\lambda\Delta) \prod_{j=1}^{r} g(\lambda - \mu_j)\hat{\Lambda}(\lambda)|\mu_1.....\mu_r\rangle + \text{unwanted terms.} \tag{5.6.27}$$

Demanding the unwanted terms to vanish converts (5.6.27), strictly into an eigenvalue equation. The condition for vanishing of the unwanted terms of (5.6.27) can be obtained by equating to zero the residue of the above eigenvalue at its poles. This gives us the following condition:

$$\hat{\Lambda}(\lambda)(\mu_s, \{\mu_j\}) = \exp(i\mu_s\Delta) \prod_{j \neq s}^{r} \frac{g(\mu_j - \mu_s)}{g(\mu_s - \mu_j)}. \tag{5.6.28}$$

Equation (5.6.28) may be used to determine the μ_j's once $\hat{\Lambda}(\lambda)$ is known. To this end, we need to solve the eigenvalue problem (see (5.6.25)) with an "inhomogeneous" monodromy matrix defined over "r-lattice sites" given by

$$\hat{T}(\lambda, \{\mu_j\}) = [\hat{R}_{0r}(\lambda - \mu_r).....\hat{R}_{01}(\lambda - \mu_1)], \tag{5.6.29}$$

with

$$\hat{F}(\lambda, \{\mu_j\}) = \mathrm{tr}_0[\hat{R}_{0r}(\lambda - \mu_r).....\hat{R}_{01}(\lambda - \mu_1)], \tag{5.6.30}$$

where

$$\hat{R}_{0i}(u_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2u_i}{2u_i - \gamma} & \frac{\gamma}{\gamma - 2u_i} & 0 \\ 0 & \frac{\gamma}{\gamma - 2u_i} & \frac{2u_i}{2u_i - \gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.6.31}$$

and $u_i = (\lambda - \mu_i)$. It is interesting to note that the new monodromy matrix $\hat{T}(\lambda, \{\mu_j\})$ and $\hat{R}(\lambda - \omega)$ obey the fundamental relation

$$\hat{R}(\lambda - \omega)\hat{T}(\lambda, \{\mu_j\}) \otimes \hat{T}(\omega, \{\mu_j\}) = \hat{T}(\omega, \{\mu_j\}) \otimes \hat{T}(\lambda, \{\mu_j\})\hat{R}(\lambda - \omega). \tag{5.6.32}$$

Thus, writing

$$\hat{T}(\lambda, \{\mu_j\}) = \begin{pmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{pmatrix}, \tag{5.6.33}$$

it is obvious that (5.6.29–5.6.33) together constitute the essential ingredients for the application of the algebraic Bethe ansatz in its simplest form as described in earlier. Choosing a reference state $|1\rangle = \otimes \prod_{i=1}^{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i$, it follows from the structure of the $\hat{R}$ matrix that

$$\hat{A}(\lambda)|1\rangle = 1|1\rangle, \qquad \hat{D}(\lambda)|1\rangle = \prod_{m=1}^{r} \frac{2(\lambda - \mu_m)}{2(\lambda - \mu_m) - \gamma}|1\rangle. \tag{5.6.34}$$

Setting $\mathbf{X} = \hat{B}(\mu_1')\ldots\ldots\hat{B}(\mu_p')|1\rangle$, one can select the appropriate commutation relations from (5.6.32) to obtain

$$\hat{F}(\lambda, \{\mu_j\})\mathbf{X} = \left[\prod_{q=1}^{p} g(\mu_q' - \lambda) + \frac{\prod_{q=1}^{p} g(\lambda - \mu_q')}{\prod_{m=1}^{r} g(\lambda - \mu_m)}\right] \mathbf{X} + \text{unwanted terms}, \tag{5.6.35}$$

where $g(\lambda - \mu_m) = \frac{2(\lambda - \mu_m) - \gamma}{2(\lambda - \mu_m)}$. Hence we identify the eigenvalue as

$$\hat{\Lambda}(\lambda, \{\mu_j\}) = \left[\prod_{q=1}^{p} g(\mu_q' - \lambda) + \frac{\prod_{q=1}^{p} g(\lambda - \mu_q')}{\prod_{m=1}^{r} g(\lambda - \mu_m)}\right]. \tag{5.6.36}$$

The vanishing of the unwanted terms in (5.6.35) can be accomplished by demanding that $\text{Res}\hat{\Lambda}(\lambda, \{\mu_j\})|_{\lambda = \mu_l'} = 0$, which gives

$$\prod_{m=1}^{r} g(\mu_l' - \mu_m) = \prod_{q \neq l}^{p} \frac{g(\mu_l' - \mu_q')}{g(\mu_q' - \mu_l')}, \tag{5.6.37}$$

while (5.6.28), upon using (5.6.36), becomes

$$\hat{\lambda}(\mu_s, \{\mu_j\}) = \prod_{q=1}^{p} g(\mu_1' - \mu_s), \tag{5.6.38}$$

since the second term in (5.6.36), i.e., $\prod_{m=1}^{r} g^{-1}(\lambda-\mu_m)|_{\lambda=\mu_s}$ vanishes. Thus we finally have the following results. The eigenvalue of the r excitation state defined by (5.6.18) when acting on the transfer matrix $\tau(\lambda)$ is

$$t(\lambda,\{\mu_j\}) = \exp(i\lambda\Delta)\prod_{j=1}^{r} g(\mu_j-\lambda)+\exp(-i\lambda\Delta)\prod_{j=1}^{r} g(\lambda-\mu_j)\hat{\Lambda}(\lambda,\{\mu_j\}), \tag{5.6.39}$$

with $\hat{\lambda}(\lambda,\{\mu_j\})$ given by (5.6.36). The two sets of coupled algebraic equations determining the parameters $\{\mu_j\}$ and $\{\mu'_k\}$ are from (5.6.28), (5.6.38) and (5.6.37), respectively:

$$\prod_{q=1}^{p} g(\mu'_q-\mu_s) = \exp(2i\mu_s\Delta)\prod_{j\neq s}^{r}\frac{g(\mu_j-\mu_s)}{g(\mu_s-\mu_j)}, \tag{5.6.40}$$

$$\prod_{m=1}^{r} g(\mu'_l-\mu_m) = \prod_{q\neq l}^{p}\frac{g(\mu'_j-\mu'_q)}{g(\mu'_q-\mu'_l)}. \tag{5.6.41}$$

Solutions of these coupled algebraic equations give us a complete description of the eigenstates. Lastly it is obvious that if the dimension of the Lax matrix is greater than 3×3 then we will get more sets of such coupled equations determining the eigenmomenta.

5.7 Fusion Procedure

From the previous discussions, it will have been noticed that most of the results stated or examples that have been worked out explicitly almost invariably dealt with R matrices that were 4×4 matrices. Of these perhaps the simplest R matrix relates to the spin $\frac{1}{2}$ model. This has been the case both for closed as well as open integrable systems. For such cases the auxiliary space is a two-dimensional one and consequently the volume and complexity of the calculations involved simplify to a great extent.

Consider now the case of spin 1, the auxiliary space will be three dimensional, and the R matrix will have dimension 9×9. The complexity and volume of calculations and associated manipulations that

arise are indeed tremendous.

In order to deal with auxiliary spaces of dimension greater than two, one can use the technique of nested Bethe ansatz, which has been explained in a preceding section. However in the case of spin chains, with spin greater than $\frac{1}{2}$, an alternative and neat approach has been developed that allows the transfer matrix of higher spin systems to be re-expressed by means of traces taken in a two-dimensional auxiliary space, the eigenvalues and eigenvectors of the latter presumably being determined beforehand. This procedure is termed as the *fusion procedure* [95] and may be concisely stated as follows; starting from a trigonometric R matrix related to the fundamental representation of a Lie algebra g, the fusion procedure provides a way of constructing new R matrices related to higher-dimensional representations of this algebra. Such an R matrix can therefore be used to construct a closed integrable quantum spin chain of higher spin, whose transfer matrix is related to that of the corresponding chain in the fundamental representation. As we shall describe later, this procedure can also be extended to the case of open spin chains, which indeed marks a major achievement.

To illustrate the fusion procedure for spin 1 case, it is easiest if we think of it as a scattering process of three spin $\frac{1}{2}$ particles in which the collision of particle 1 and particle 2 with respective rapidities $u + u_0$ and $u - u_0$ produces a state called "12", which eventually undergoes scattering with the particle number 3 of spin $\frac{1}{2}$ and rapidity v. Although, while referring to scattering, the rapidity variable was θ, with $u = i\theta$, here we shall continue with u, v, etc. and refer to them as rapidities although they are actually spectral parameters. The process just described is depicted in Figure (5.7.1) where the shaded/blurred region refers to the composite state "12."

For simplicity we shall assume that

$$R_{12}(u) = -Iu + \mathcal{P}_{12}, \quad \mathcal{P}_{12} = \text{permutation matrix.} \tag{5.7.1}$$

As suggested by the preceding figure, the R matrix corresponding to the scattering of the state "12" with particle 3 is given by

$$R_{\langle 12\rangle,3}(u - v; u_0) = R_{13}(u + u_0 - v)R_{23}(u - u_0 - v). \tag{5.7.2}$$

It has to be realized that this new R matrix does not describe the scattering of a spin 1 and spin $\frac{1}{2}$ particle. This may be understood from

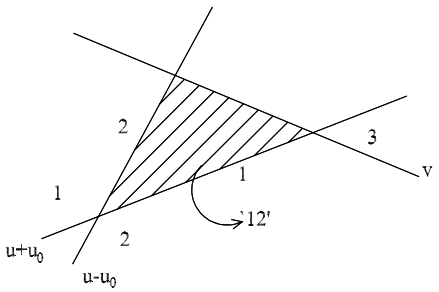

FIGURE 5.7.1: Schematic representation of the composite state "12."

the following argument. Since we are considering three-body scattering there exists the possibility of charge exchange, which implies that if before collision the composite state "12" was spin 1, then after collision the spin 1 state may be either in the "13" or "23" sectors or the initial state "12" may have spin 0, which after collision changes to a spin 1. Denoting the R matrix describing spin 1 and spin $\frac{1}{2}$ collision by $R^{(1,\frac{1}{2})}$, it should be understood that this matrix excludes all charge exchange processes and hence must satisfy the constraint

$$P_{12}^{\mp} R_{\langle 12\rangle,3}^{(1.\frac{1}{2})}(u-v) P_{12}^{\pm} = 0, \tag{5.7.3}$$

where $P^{\pm} = \frac{1}{2}(I \pm \mathcal{P})$ are projectors onto spin 1 and spin 0 states. It can be verified that the R matrix $R_{\langle 12\rangle,3}$ given by (5.7.2) does not satisfy this constraint. If one sandwiches this matrix between the projectors P_{12}^{+}, then the resulting matrix will satisfy the constraint (5.7.3) since $P_{12}^{+}P_{12}^{-} = P_{12}^{-}P_{12}^{+} = 0$, that is,

$$R_{\langle 12\rangle,3}^{(1,\frac{1}{2})}(u-v;u_0) = P_{12}^{+} R_{\langle 12\rangle,3}(u-v;u_0) P_{12}^{+} \tag{5.7.4}$$

is appropriate for describing collisions of spin 1 and spin $\frac{1}{2}$ particles. A valid question here is whether $R_{\langle 12\rangle,3}^{(1,\frac{1}{2})}$ satisfies the factorization equation,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

It will be shown that a factorization equation is satsfied by $R^{(1,\frac{1}{2})}_{\langle 12\rangle,3}$ provided we impose the additional constraint [94],

$$P^-_{12}R_{\langle 12\rangle,3}(u-v;u_0)P^+_{12}=0, \tag{5.7.5}$$

which also serves to determine the parameter u_0. To deduce the factorization equation it may be noted that for $u=-1$

$$R(u)|_{u=-1}=-I+\mathcal{P}\sim P^-, \tag{5.7.6}$$

while the Yang-Baxter equation for $R(u)$ may be written in the suggestive form:

$$\begin{aligned} &R_{12}(-1)R_{13}(u-\frac{1}{2}-v)R_{23}(u+\frac{1}{2}-v)= \\ &R_{23}(u+\frac{1}{2}-v)R_{13}(u-\frac{1}{2}-v)R_{12}(-1), \\ &P^-_{12}R_{13}(u-\frac{1}{2}-v)R_{23}(u+\frac{1}{2}-v)P^+_{12} \\ &=R_{23}(u-v+\frac{1}{2})R_{13}(u-v-\frac{1}{2})P^-_{12}P^+_{12}=0. \end{aligned} \tag{5.7.7}$$

From the basic definition of $R_{\langle 12\rangle,3}$ given in (5.7.2), we see that the choice $u_0=-\frac{1}{2}$ leads to the condition

$$P^-_{12}R_{\langle 12\rangle,3}(u-v;-\frac{1}{2})P^+_{12}=0. \tag{5.7.8}$$

Consequently the quantity

$$R^{(1,\frac{1}{2})}_{\langle 12\rangle,}(u-v)\equiv P^+_{12}R_{\langle 12\rangle,3}(u-v;u_0=-\frac{1}{2})P^+_{12} \tag{5.7.9}$$

satisfies the factorization condition for the scattering of a particle of spin 1 with two particles of spin $\frac{1}{2}$, i.e.,

$$R^{(1,\frac{1}{2})}_{12}(u-v)R^{(1,\frac{1}{2})}_{13}(u)R^{(\frac{1}{2},\frac{1}{2})}_{23}(v)=R^{(\frac{1}{2},\frac{1}{2})}_{23}(v)R^{(1,\frac{1}{2})}_{13}(u)R^{(1,\frac{1}{2})}_{12}(u-v) \tag{5.7.10}$$

where the subscript 1 means the three-dimensional auxiliary space and $R^{(\frac{1}{2},\frac{1}{2})}$ is the familiar spin $\frac{1}{2}$ R matrix, $R(u)=-iuI+\mathcal{P}$.

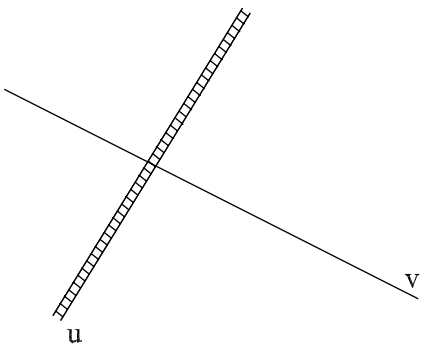

FIGURE 5.7.2: Schematic representation of $R^{(\frac{1}{2},\frac{1}{2})}(u-v)$.

Graphically $R^{(\frac{1}{2},\frac{1}{2})}(u-v)$ may be represented by the intersection of a double line and a single line as shown in Figure (5.7.2).

In the above preliminary discussion of the fusion procedure for spin chains, one should notice the crucial role played by the R matrix for the spin $\frac{1}{2}$ model that degenerates to the projection operator P^- for a certain value of the spectral parameter, namely $u = -1$. This is of fundamental importance as will be seen from the following more formal analysis of the fusion procedure.

Following the article by Nepomachie and Mezincescu [95] , we assume that the R matrix acts on $\mathcal{C}^n \otimes \mathcal{C}^n$ and obeys the Yang-Baxter equation:

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{5.7.11}$$

and is in addition P-T symmetric, that is

$$\mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} = R_{21}(u) = R_{12}^{t_1 t_2}(u), \tag{5.7.12}$$

where $\mathcal{P}_{12}$ is the permutation matrix defined by

$$\mathcal{P}_{12}(x \otimes y) = y \otimes x, \qquad \forall\ x, y \in \mathcal{C}^n. \tag{5.7.13}$$

Here t_i denotes transposition in the ith space. Moreover it is assumed to be unitary, i.e.,

$$R_{12}(u)R_{21}(-u) = \zeta(u), \tag{5.7.14}$$

with $\zeta(u)$ being an even scalar function of u and furthermore to satisfy crossing unitarity:

$$R_{12}(u) = V_1 R_{12}(-u-\rho)^{t_2} V_1 = V_2^{t_2} R_{12}(-u-\rho)^{t_1} V_2^{t_2}. \tag{5.7.15}$$

Here $V^2 = I$, $V_1 = V \otimes I$ and $V_2 = I \otimes V$. Finally the R matrix is assumed to be regular, that is

$$R_{12}(0) = \zeta^{\frac{1}{2}}(0)\mathcal{P}_{12}. \tag{5.7.16}$$

Setting $u = -\rho$ in (5.7.15) we have

$$R_{12}(-\rho) = V_1 \left[\zeta^{\frac{1}{2}}(0)\mathcal{P}_{12}\right]^{t_2} V_1$$

and find that

$$\frac{1}{\alpha\zeta^{\frac{1}{2}}(0)} R_{12}(-\rho) = \frac{1}{\alpha} V_1 \mathcal{P}_{12}^{t_2} V_1, \tag{5.7.17}$$

where α is a constant. Consider the square of (5.7.17),

$$\left[\frac{1}{\alpha\zeta^{\frac{1}{2}}(0)} R_{12}(-\rho)\right]^2 = \frac{1}{\alpha^2} V_1 \mathcal{P}_{12}^{t_2} \mathcal{P}_{12}^{t_2} V_1 \tag{5.7.18}$$

and using the fact that $V_1^2 = I$. Then in terms of a Chevelley basis, as $\mathcal{P}_{12} = e_{ij}^{(1)} \otimes e_{ji}^{(2)} (i, j = 1, 2....n)$ we find that

$$\mathcal{P}_{12}^{t_2} \mathcal{P}_{12}^{t_2} = e_{ij}^{(1)} e_{kl}^{(1)} \otimes e_{ij}^{(2)} e_{kl}^{(2)} = \delta_{jj} [e_{il}^{(1)} \otimes e_{li}^{(2)}]^{t_2} = n \mathcal{P}_{12}^{t_2} \tag{5.7.19}$$

and hence

$$\left[\frac{1}{\alpha\zeta^{\frac{1}{2}}(0)} R_{12}(-\rho)\right]^2 = \frac{n}{\alpha^2} V_1 \mathcal{P}_{12}^{t_2} V_1 = \frac{n}{\alpha} \left[\frac{1}{\alpha\zeta^{\frac{1}{2}}(0)} R_{12}(-\rho)\right].$$

Clearly the choice $n = \alpha$ makes the quantity on the left-hand side a projection operator. Hence we define $\tilde{\mathcal{P}}_{12}^{-} = \frac{1}{n\zeta^{\frac{1}{2}}(0)} R_{12}(-\rho)$ with the property

$$(\tilde{\mathcal{P}}_{12}^{-})^2 = (\tilde{\mathcal{P}}_{12}^{-}). \tag{5.7.20}$$

One may also verify that

$$\tilde{\mathcal{P}}_{12}^{-} A_{12} \tilde{\mathcal{P}}_{12}^{-} = \mathrm{tr}_{12}(\tilde{\mathcal{P}}_{12}^{-} A_{12}) \tilde{\mathcal{P}}_{12}^{-}, \tag{5.7.21}$$

where A is any arbitrary matrix acting on $\mathcal{C}^n \otimes \mathcal{C}^n$. Now given a projector $\tilde{\mathcal{P}}_{12}^-$, we can construct another projection operator,

$$\tilde{\mathcal{P}}_{12}^+ = I - \tilde{\mathcal{P}}_{12}^-, \tag{5.7.22}$$

so that

$$(\tilde{\mathcal{P}}_{12}^+)^2 = \tilde{\mathcal{P}}_{12}^+$$

and

$$\tilde{\mathcal{P}}_{12}^+ \tilde{\mathcal{P}}_{12}^- = \tilde{\mathcal{P}}_{12}^- - (\tilde{\mathcal{P}}_{12}^-)^2 = 0, \quad \tilde{\mathcal{P}}_{12}^- \tilde{\mathcal{P}}_{12}^+ = 0. \tag{5.7.23}$$

However, the projection operators constructed in the above manner are not symmetric, that is

$$(\tilde{\mathcal{P}}_{12}^{\mp})^{t_1 t_2} = \tilde{\mathcal{P}}_{21}^{\mp} \neq \tilde{\mathcal{P}}_{12}^{\mp}.$$

Let us now set $v = u + \rho$ in (5.7.11) so that

$$R_{12}(-\rho) R_{13}(u) R_{23}(u+\rho) = R_{23}(u+\rho) R_{13}(u) R_{12}(-\rho).$$

Upon using the definition of the projection operator we get after right multiplication with $\tilde{\mathcal{P}}_{12}^+$,

$$\tilde{\mathcal{P}}_{12}^- R_{13}(u) R_{23}(u+\rho) \tilde{\mathcal{P}}_{12}^+ = 0, \tag{5.7.24}$$

where use has been made of (5.7.23). If we next define a "fused" R matrix by the following relation:

$$R_{\langle 12 \rangle 3}(u) \equiv \tilde{\mathcal{P}}_{12}^+ R_{13}(u) R_{23}(u+\rho) \tilde{\mathcal{P}}_{12}^+ \tag{5.7.25}$$

then this fused R matrix obeys a Yang-Baxter equation:

$$R_{\langle 12 \rangle 3}(u-v) R_{\langle 12 \rangle 4}(u) R_{34}(v) = R_{34}(v) R_{\langle 12 \rangle 4}(u) R_{\langle 12 \rangle 3}(u-v). \tag{5.7.26}$$

To see how this comes about from the definition (5.7.25) we have

$$R_{\langle 12 \rangle 3}(u-v) R_{\langle 12 \rangle 4}(u) R_{34}(v)$$

$$= \tilde{\mathcal{P}}_{12}^+ R_{\langle 12 \rangle 3}(u-v) \tilde{\mathcal{P}}_{12}^+ \tilde{\mathcal{P}}_{12}^+ R_{\langle 12 \rangle 4}(u) \tilde{\mathcal{P}}_{12}^+ R_{34}(v); \tag{5.7.27}$$

also using the fact that

$$R_{\langle 12 \rangle, 3} = R_{13}(u - v + u_0) R_{23}(u - v - u_0),$$

equation (5.7.27) becomes

$$= \tilde{\mathcal{P}}^+_{12} R_{13}(u-v+u_0) R_{23}(u-v-u_0) \tilde{\mathcal{P}}^+_{12} \times$$
$$\times \tilde{\mathcal{P}}^+_{12} R_{14}(u+u_0) R_{24}(u-u_0) \tilde{\mathcal{P}}^+_{12} R_{34}(u_0). \tag{5.7.28}$$

Now

$$R_{13}(u-v+u_0) R_{23}(u-v-u_0) \mathcal{P}^+_{12} = \mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0)$$

and since $\mathcal{P}^{+2}_{12} = \mathcal{P}^+_{12}$, we find that the expression in (5.7.28) equals

$$\mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0) R_{24}(u-u_0) R_{14}(u+u_0) \mathcal{P}^{+2}_{12} R_{34}(v)$$
$$= \mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{24}(u-u_0) R_{13}(u-v+u_0) R_{14}(u+u_0) R_{34}(v) \mathcal{P}^+_{12}.$$

This is possible since R_{13} and R_{24} involve completely different spaces. Using the Yang-Baxter equation for $R_{13}R_{14}R_{34}$ the above relation becomes

$$= \mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{24}(u-u_0) R_{34}(v) R_{14}(u+u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12},$$
$$= \mathcal{P}^+_{12} R_{34}(v) R_{24}(u-u_0) R_{23}(u-v-u_0) R_{14}(u+u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12},$$
$$= \mathcal{P}^+_{12} R_{34}(v) R_{24}(u-u_0) R_{14}(u+u_0) R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12},$$
$$= R_{34}(v) \mathcal{P}^+_{12} R_{24}(u-u_0) R_{14}(u+u_0) R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12},$$
$$= R_{34}(v) \mathcal{P}^+_{12} R_{24}(u-u_0) R_{14}(u+u_0) (\mathcal{P}^+_{12} + \mathcal{P}^-_{12}) \times$$
$$\times R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12}. \tag{5.7.29}$$

But it is known that

$$\mathcal{P}^-_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12}$$
$$= (\mathcal{P}^-_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12}) \mathcal{P}^+_{12},$$
$$= \mathcal{P}^-_{12} \mathcal{P}^+_{12} R_{13}(u-v-u_0) R_{23}(u-v-u_0) \mathcal{P}^+_{12}. \tag{5.7.30}$$

Hence as $\mathcal{P}^-_{12}\mathcal{P}^+_{12} = 0$, the above expression becomes

$$= R_{34}(v) \mathcal{P}^+_{12} R_{24}(u-u_0) R_{14}(u+u_0) \mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12}$$
$$= R_{34}(v) \left[\mathcal{P}^+_{12} R_{24}(u-u_0) R_{14}(u+u_0) \mathcal{P}^+_{12} \right] \times$$
$$\times \left[\mathcal{P}^+_{12} R_{23}(u-v-u_0) R_{13}(u-v+u_0) \mathcal{P}^+_{12} \right]. \tag{5.7.31}$$

Again we have

$$\mathcal{P}^+_{12}R_{24}(u-u_0)R_{14}(u+u_0)\mathcal{P}^+_{12} = \left[\mathcal{P}^+_{12}R_{24}(u-u_0)R_{14}(u+u_0)\mathcal{P}^+_{12}\right]\mathcal{P}^+_{12}$$

$$= \mathcal{P}^+_{12}\mathcal{P}^+_{12}R_{14}(u+u_0)R_{24}(u-u_0)\mathcal{P}^+_{12} = \mathcal{P}^+_{12}R_{14}(u+u_)R_{24}(u-u_0)\mathcal{P}^+_{12},$$

$$= \mathcal{P}^+_{12}R_{12,4}(u)\mathcal{P}^+_{12} = R_{\langle 12\rangle,4}(u). \tag{5.7.32}$$

In the same way we get

$$\mathcal{P}^+_{12}R_{23}(u-v-u_0)R_{13}(u-v+u_0)\mathcal{P}^+_{12} = R_{\langle 12\rangle,3}(u-v)$$

$$= \mathcal{P}^+_{12}R_{12,3}(u-v)\mathcal{P}^+_{12}. \tag{5.7.33}$$

So that finally we have

$$R_{\langle 12\rangle,3}(u-v)R_{\langle 12\rangle,4}(u)R_{34}(v) = R_{34}(v)R_{\langle 12\rangle,4}(u)R_{\langle 12\rangle,3}(u-v). \tag{5.7.34}$$

This completes the proof of (5.7.26).

Now under the cyclic permutation (123) $\longrightarrow$ (312) and setting $v = -\rho$ in the Yang-Baxter equation we obtain

$$R_{31}(u+\rho)R_{32}(u)R_{12}(-\rho) = R_{12}(-\rho)R_{32}(u)R_{31}(u+\rho).$$

Left multiplication by $\tilde{\mathcal{P}}^+_{12}$ and using the fact that $R_{12}(-\rho)$ is proportional to $\tilde{\mathcal{P}}^-_{12}$, then gives

$$\tilde{\mathcal{P}}^+_{12}R_{31}(u+\rho)R_{32}(u)\tilde{\mathcal{P}}^-_{12} = 0. \tag{5.7.35}$$

Consequently we may define another fused R matrix by

$$R_{3\langle 12\rangle}(u) \equiv \tilde{\mathcal{P}}^+_{12}R_{31}(u+\rho)R_{32}(u)\tilde{\mathcal{P}}^+_{12}. \tag{5.7.36}$$

The unitary and crossing unitarity properties of the fused R matrices as given by (5.7.25) and (5.7.36) can be determined from the corresponding properties (5.7.14 and 5.7.15) of the original R matrix. For example it can be shown that

$$R_{\langle 12\rangle 3}(u)R_{3\langle 12\rangle}(-u) = \zeta(u)\zeta(u+\rho)\tilde{\mathcal{P}}^+_{12} \tag{5.7.37}$$

and that

$$R_{\langle 12\rangle 3}(u) = V_3^{t_3}R_{3\langle 12\rangle}(-u-\rho)^{t_3}V_3^{t_3}, \tag{5.7.38}$$

$$R_{3\langle 12\rangle}(u) = V_3R_{\langle 12\rangle 3}(-u-\rho)^{t_3}V_3. \tag{5.7.39}$$

These follow from use of the following result:

$$\tilde{\mathcal{P}}_{12}^{-} V_1 V_2 \tilde{\mathcal{P}}_{12}^{+} = 0,$$

which is a consequence of the degeneration of the identity

$$V_1 R_{12}(u) V_1 = V_2 R_{21}(u) V_2 \quad \text{at } u = -\rho.$$

5.8 Fusion Procedure for Open Chains

In the construction of new integrable systems that are open, one needs a procedure for construction of new boundary matrices—the so-called K matrices—which may be interpreted as the amplitude for a particle to reflect elastically from a wall. Here we describe the corresponding fusion procedure in case of open chains following Mezincescu et al. [96]

For a PT invariant R matrix the fundamental "reflection factorization" relations obeyed by K^- and K^+ are

$$R_{12}(u-v) K_1^-(u) R_{12}(u+v) K_2^-(v) = K_2^-(v) R_{12}(u+v) K_1^-(u) R_{12}(u-v), \tag{5.8.1}$$

$$\begin{aligned} &R_{12}(-u+v) K_1^+(u)^{t_1} M_1^{-1} R_{21}(-u-v-2\rho) M_1 K_2^+(v)^{t_2} \\ &= K_2^+(v)^{t_2} M_1 R_{12}(-u-v-2\rho) M_1^{-1} K_1^+(u)^{t_1} R_{21}(-u+v), \end{aligned} \tag{5.8.2}$$

where $M = V^t V$. Setting $v = u + \rho$ in (5.8.1) we get

$$\tilde{\mathcal{P}}_{12}^{-1} K_1^-(u) R_{21}(2u+\rho) K_2^-(u+\rho) = K_2^-(u+\rho) R_{12}(2u+\rho) K_1(u) \tilde{\mathcal{P}}_{12}^{-1}, \tag{5.8.3}$$

which implies that

$$\tilde{\mathcal{P}}_{12}^{-} K_1^-(u) R_{21}(2u+\rho) K_2^-(u+\rho) \tilde{\mathcal{P}}_{12}^{+} = 0. \tag{5.8.4}$$

This result can be used to show that the fused K^- matrix

$$K_{\langle 12 \rangle}^-(u) = \tilde{\mathcal{P}}_{12}^{+} K_1^-(u) R_{21}(2u+\rho) K_2^-(u+\rho) \tilde{\mathcal{P}}_{21}^{+}, \tag{5.8.5}$$

satisfies the factorized relations:

$$R_{3\langle 12 \rangle}(u-v) K_3^-(u) R_{\langle 12 \rangle 3}(u+v) K_{\langle 12 \rangle}^-(v)$$

$$= K^-_{\langle 12\rangle}(v) R'_{3\langle 12\rangle}(u+v) K^-_3(u) R'_{\langle 12\rangle 3}(u-v), \tag{5.8.6}$$

and

$$R_{\langle 12\rangle 3}(u-v) K^-_{\langle 12\rangle}(u) R'_{3\langle 12\rangle}(u+v) K^-_3(v)$$

$$= K^-_3(u) R_{\langle 12\rangle 3}(u+v) K^-_{\langle 12\rangle}(u) R'_{3\langle 12\rangle}(u-v). \tag{5.8.7}$$

The quantity $K'^-_{\langle 12\rangle}(u) = K^-_{\langle 21\rangle}(u-\rho)$ satisfies a similar pair of relations with R and R' interchanged. We also have

$$R_{\langle 12\rangle 3}(u-v) K^-_{\langle 12\rangle}(u) R'_{3\langle 12\rangle}(u+v) K^-_3(v)$$

$$= K^-_3(u) R_{\langle 12\rangle 3}(u+v) K^-_{\langle 12\rangle}(u) R'_{3\langle 12\rangle}(u-v). \tag{5.8.8}$$

For the matrix $K^+(u)$, we consider a similar degeneration of (5.8.2) to get

$$\tilde{\mathcal{P}}^+_{21} K^+_1(u)^{t_1} M_2 R_{21}(-2u-3\rho) M_2^- K^+_2(u+\rho)^{t_2} \tilde{\mathcal{P}}_{12} = 0. \tag{5.8.9}$$

This result and the identity

$$M_1^{-1} R_{12}(u) M_1 = M_2 R_{12}(u) M_2^{-1}, \tag{5.8.10}$$

is to be used to show that the fused K^+ matrix,

$$\left[K^+_{\langle 12\rangle}(u)\right]^{t_2} = \tilde{\mathcal{P}}^+_{21} K^+_1(u)^{t_1} M_2 R_{21}(-2u-3\rho) M_2^{-1} K^+_2(u+\rho)^{t_2} \tilde{\mathcal{P}}^+_{12}, \tag{5.8.11}$$

obeys the relation:

$$R_{\langle 12\rangle 3}(-u+v)^{t_{123}} K^+_3(u)^{t_3} M_3^{-1} R_{3\langle 12\rangle}(-u-v-2\rho)^{t_{123}} M_3 K^+_{\langle 12\rangle}(v)^{t_{12}}$$

$$= K^+_{\langle 12\rangle}(v)^{t_{12}} M_3 R'_{\langle 12\rangle 3}(-u-v-2\rho)^{t_{123}} M_3^{-1} K^+_3(u)^{t_3} R'_{3\langle 12\rangle}(-u+v)^{t_{123}}, \tag{5.8.12}$$

and

$$R_{3\langle 12\rangle}(-u+v)^{t_{123}} K^+_{\langle 12\rangle}(u)^{t_{12}} M_3 R'_{\langle 12\rangle 3}(-u-v-2\rho)^{t_{123}} M_3^{-1} K^+_3(v)^{t_3}$$

$$= K^+_3(v)^{t_3} M_3^{-1} R_{3\langle 12\rangle}(-u-v-2\rho)^{t_{123}} M_3 K^+_{\langle 12\rangle}(u)^{t_{12}} R'_{\langle 12\rangle 3}(-u+v)^{t_{123}}. \tag{5.8.13}$$

5.9 Fusion Procedure for Transfer Matrices

From the basic theory of open integrable chains as developed by Sklyanin in [97], it is known that the open-chain transfer matrix is given by

$$t(u) = \mathrm{tr}_1 K_1^+(u)\tau_1^-(u), \tag{5.9.1}$$

with $\tau^-(u)$

$$\tau^-(u) = T(u)K^-(u)\hat{T}(u). \tag{5.9.2}$$

Here $T(u)$ and $\hat{T}(u)$ obey the following relations:

$$\begin{aligned} R_{12}(u-v)T_1(u)T_2(v) &= T_2(v)T_1(u)R_{12}(u-v), \\ \hat{T}_2(v)R_{12}(u+v)T_1(u) &= T_1(u)R_{12}(u+v)\hat{T}_2(v), \\ R_{12}(-u+v)\hat{T}_2(v)\hat{T}_1(u) &= \hat{T}_1(u)\hat{T}_2(v)R_{12}(-u+v). \end{aligned} \tag{5.9.3}$$

$T(u)$ is the so-called monodromy matrix and $\hat{T}(u)$ obeys the same relations as $T(-u)^{-1}$. The quantity $\tau^-(u)$ satisfies the same relation as $K^-(u)$, i.e., (5.8.1) and we have

$$[t(u), t(v)] = 0, \;\; u \neq v. \tag{5.9.4}$$

In order to construct a transfer matrix $\hat{t}(u)$ for the fused quantities described above, we consider the following quantity,

$$\hat{t}(u) = \mathrm{tr}_{12} K^+_{\langle 12\rangle}(u)\tau^-_{\langle 12\rangle}(u) \tag{5.9.5}$$

along with

$$\begin{aligned} \tau^-_{\langle 12\rangle}(u) &= T_{\langle 12\rangle}(u)K^-_{\langle 12\rangle}(u)\hat{T}_{12}(u+\rho), \\ T_{\langle 12\rangle}(u) &= \tilde{\mathcal{P}}^+_{12}T_1(u)T_2(u+\rho)\tilde{\mathcal{P}}^+_{12}, \\ \tilde{T}_{\langle 21\rangle}(u+\rho) &= \tilde{\mathcal{P}}^+_{12}\hat{T}_1(u)\hat{T}_2(u+\rho)\tilde{\mathcal{P}}^+_{21}. \end{aligned} \tag{5.9.6}$$

We observe that $\tau^-_{\langle 12\rangle}(u)$ obeys the same relations (5.8.6) and (5.8.7) as $K^-_{\langle 12\rangle}(u)$ and therefore the same fusion formula can be used:

$$\tau^-_{\langle 12\rangle}(u) = \tilde{\mathcal{P}}^+_{12}\tau_1^-(u)R_{21}(2u+\rho)\tau_2^-(u+\rho)\tilde{\mathcal{P}}^+_{12}.$$

To establish the commutativity,

$$[t(u), t(v)] = 0,$$

we follow a generalization of Sklyanin's approach. In particular we make use of the relation (5.7.37), i.e.,

$$R_{\langle 12\rangle 3}(u)R_{3\langle 12\rangle}(-u) = \zeta(u)\zeta(u+\rho)\tilde{\mathcal{P}}^{+}_{12}$$

and

$$M_3^{-1}R_{\langle 12\rangle 3}(-u-2\rho)^{t_{12}}M_3R_{3\langle 12\rangle}(u)^{t_{12}} = \zeta(u)\zeta(u+\rho)\tilde{\mathcal{P}}^{+}_{12},$$

which follows from (5.7.37–5.7.39). Use will also be made of the reflection factorization relation (5.8.6) and (5.8.7) along with (5.8.12) and (5.8.13). We are now in a position to construct the fusion formula for the transfer matrix. From the (5.9.5) and the expression for $\tau^{-}_{\langle 12\rangle}(u)$ from (5.9.6), along with (5.8.11) for the fused K^{+} matrix we get

$$\hat{t}(u) = \mathrm{tr}_{12}\{\tilde{\mathcal{P}}^{+}_{12}K^{+}_{2}(u+\rho)M_2^{-1}R_{12}(-2u-3\rho)M_2K^{+}_{1}(u)\tau^{-}_{1}(u)$$
$$\times R_{21}(2u+\rho)\tau^{-}_{2}(u+\rho)\}.$$

Using the identity $\tilde{\mathcal{P}}^{+}_{21} = I - \tilde{\mathcal{P}}^{-}_{21}$ we obtain

$$\hat{t}(u) =$$
$$\mathrm{tr}_{12}\{K^{+}_{2}(u+\rho)M_1R_{12}(-2u-3\rho)M_1^{-1}K^{+}_{1}(u)\tau^{-}_{1}(u)R_{21}(2u+\rho)\tau^{-}_{2}(u+\rho)\}$$
$$-\mathrm{tr}_{12}\{\tilde{\mathcal{P}}^{-}_{21}K^{+}_{2}(u+\rho)M_2^{-1}R_{12}(-2u-3\rho)M_2K^{+}_{1}(u)\tau^{-}_{1}(u)$$
$$\times R_{21}(2u+\rho)\tau^{-}_{2}(u+\rho)\}. \qquad (5.9.7)$$

The first term can be expressed as a product of two transfer matrices, while the second term may be written as a product of two quantum determinants. For the first term we have

$$\mathrm{tr}_{12}\{\Big[K^{+}_{2}(u+\rho)K^{+}_{1}(u)^{t_1}M_1^{-1}R_{12}(-2u-3\rho)^{t_1}M_1\Big]^{t_1}\times$$
$$\Big[\tau^{-}_{1}(u)R_{21}(2u+\rho)\tau^{-}_{2}(u+\rho)\Big]\},$$
$$= \mathrm{tr}_{12}\{K^{+}_{2}(u+\rho)K^{+}_{1}(u)^{t_1}M_1^{-1}R_{12}(-2u-3\rho)^{t_1}M_1$$
$$\times\Big[\tau^{-}_{1}(u)R_{21}(2u+\rho)\tau^{-}_{2}(u+\rho)\Big]^{t_1}\},$$
$$= \mathrm{tr}_{12}\{K^{+}_{2}(u+\rho)K^{+}_{1}(u)^{t_1}M_1^{-1}R_{12}(-2u-3\rho)^{t_1}M_1R_{21}(2u+\rho)^{t_1}$$
$$\times\tau^{-}_{1}(u)^{t_1}\tau^{-}_{2}(u+\rho)\}.$$

Using the fact that

$$M_1^{-1}R_{12}(-2u-3\rho)^{t_1}M_1R_{21}(2u+\rho)^{t_1}=\zeta(2u+2\rho),$$

one can see that this term is equal to $\zeta(2u+2\rho)t(u)t(u+\rho)$. On the other hand the second term of (5.9.7) can be written as

$$\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{+}K_2^{+}(u+\rho)M_2^{-1}R_{12}(-2u-3\rho)M_2K_1^{+}(u)\tilde{\mathcal{P}}_{12}^{-}\tau_1^{-}(u)$$

$$\times R_{21}(2u+\rho)\tau_2^{-}(u+\rho)\}.$$

Expressing $\tilde{\mathcal{P}}_{21}^{-}$ in terms of $\tilde{\mathcal{P}}_{12}^{-}$ with the help of

$$\tilde{\mathcal{P}}_{21}^{-}=V_1V_2\tilde{\mathcal{P}}_{12}^{-}V_1V_2$$

and using the property of projectors we conclude that this term is equal to

$$\Delta\{K^{+}(u)\}\Delta\{\tau^{-}(u)\}, \tag{5.9.8}$$

where

$$\Delta\{K^{+}(u)\}=\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{-}V_1V_2K_2^{+}(u+\rho)M_2^{-1}R_{12}(-2u-3\rho)M_2K_1^{+}(u)\}, \tag{5.9.9}$$

$$\Delta\{\tau^{-}(u)\}=\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{-}\tau_1^{-}(u)R_{21}(2u+\rho)\tau_2^{-}(u+\rho)V_1V_2\}. \tag{5.9.10}$$

Summarising we see that the fusion formula for the transfer matrix is

$$\hat{t}(u)=\zeta(2u+2\rho)t(u)t(u+\rho)-\Delta\{K^{+}(u)\}\{\tau^{-}(u)\}. \tag{5.9.11}$$

What now remains is the determination of the quantum determinants. Recalling that

$$\tau^{-}(u)=T(u)K^{-}(u)\hat{T}(u)$$

and using this in $\Delta\{\tau^{-}(u)\}$ we see that

$$\Delta\{\tau^{-}(u)\}=\delta\{T(u)\}\Delta\{K^{-}(u)\}\delta\{\hat{t}(u)\}, \tag{5.9.12}$$

where

$$\delta\{T(u)\}=\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{-}T_1(u)T_2(u+\rho)\},$$

$$\delta\{\hat{T}(u)\}=\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{-}\hat{T}_2(u)\hat{T}_1(u+\rho)\},$$

$$\Delta\{K^{-}(u)\}=\mathrm{tr}_{12}\{\tilde{\mathcal{P}}_{12}^{-}K_1^{-}(u)R_{21}(2u+\rho)K_2^{-}(u+\rho)V_1V_2\}. \tag{5.9.13}$$

Let us now assume that the monodromy matrix $T(u)$ is given by

$$T_1(u)=R_{1,N}(u)R_{1,N-1}(u)....R_{1,1}(u)$$

and similarly

$$\hat{T}_1(u) = R_{1,1}(u)....R_{N-1,1}(u)R_{N,1}(u).$$

Now with the help of the relation

$$\tilde{\mathcal{P}}_{12}^{-}R_{1,m}(u)R_{2,m}(u+\rho)\tilde{\mathcal{P}}_{12}^{-} = \zeta(u+\rho)\tilde{\mathcal{P}}_{12}^{-}$$

for $m = 1, 2, ..., N$, it follows then

$$\delta\{T(u)\} = \delta\{\hat{T}(u)\} = \zeta(u+\rho)^N.$$

Since the expression of quantum determinants are c numbers the commutativity of $\hat{t}(u)$ follows:

$$[\hat{t}(u), \hat{t}(v)] = 0.$$

For any particular model and fixed boundary matrices K^+ and K^- the evaluation can be more explicit and one can again follow the usual procedure of Bethe ansatz.

5.10 Application of Fusion Procedure

As an example of the application of the fusion procedure we shall now discuss how it may be employed to formulate a problem exhibiting long-range interaction. This will serve to give an illustration of the connection that may be established between apparently unrelated topics in integrable systems [90].

Let us consider a Heisenberg spin chain with the nearest neighbour interaction (spin of each atom at the lattice site being equal to 1/2) governed by the quantum R matrix [99]:

$$R(\theta) = \begin{pmatrix} \sinh(\theta+\eta) & 0 & 0 & 0 \\ 0 & \sinh\theta & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh\theta & 0 \\ 0 & 0 & 0 & \sinh(\theta+\eta) \end{pmatrix}. \tag{5.10.1}$$

The basic observation of [100] is that, even if one introduces inhomogeneities $\mu_1, \mu_2, ..., \mu_N$ at the lattice sites $i = 1, 2...., N$, the model

remains integrable and the system still possesses an infinite number of integrals of motion in involution.

In general a monodromy operator $T(\theta, \{\mu\})$ is written as

$$T^{\sigma}(\theta, \{\mu\}) = \prod_{i=1}^{N} R^{\sigma}_{0i}(\theta + \mu_i),$$

where R_{0i} is defined over $V_0 \otimes V_i$; V_0 is the quantum space and σ is used to denote the purely spin $1/2$ character of the R matrix. Writing the monodromy matrix as

$$T^{\sigma}(\theta, \{\mu\}) = \begin{pmatrix} A^{\sigma}(\theta, \{\mu\}) & B^{\sigma}(\theta, \{\mu\}) \\ C^{\sigma}(\theta, \{\mu\}) & D^{\sigma}(\theta, \{\mu\}) \end{pmatrix}, \tag{5.10.2}$$

the corresponding transfer matrix $t^{\sigma}(\theta, \{\mu\})$ is given by

$$t^{\sigma}(\theta, \{\mu\}) = \mathrm{tr}_0 T^{\sigma}(\theta, \{\mu\}).$$

The matrix elements of $T^{\sigma}(\theta, \{\mu\})$ will be denoted as $T^{\sigma}_{\alpha\beta}(\theta, \{\mu\})$. The above assertion implies that

$$[t^{\sigma}(\theta, \{\mu\}), t^{\sigma}(\theta', \{\mu\})] = 0. \tag{5.10.3}$$

The monodromy matrix for the inhomogeneous model reads in terms of site operators as

$$T_{ab}(\theta, \{\mu\}) = \sum_{a_1, a_2, \ldots a_{N-1}=1}^{2} t^{(1)}_{a a_1}(\theta + \mu_1) t^{(2)}_{a_1 a_2}(\theta + \mu_2) \ldots . t^{(N)}_{a_{N-1} b}(\theta + \mu_N). \tag{5.10.4}$$

The operators $t^{(k)}_{a_{k-1} a_k}(\theta + \mu_k)$ act on the two-dimensional space $V^{(k)}$. We have further

$$t_{11}(\theta) = \begin{pmatrix} \sinh(\theta + \eta) & 0 \\ 0 & \sinh\theta \end{pmatrix}, \quad t_{12}(\theta) = \sigma_{-} \sinh\eta,$$

$$t_{22}(\theta) = \begin{pmatrix} \sinh\theta & 0 \\ 0 & \sinh(\theta + \eta) \end{pmatrix}, \quad t_{21}(\theta) = \sigma_{+} \sinh\eta, \tag{5.10.5}$$

where

$$\sigma_{=} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Such a monodromy matrix satisfies the equation

$$R^{\sigma}(\theta-\theta')T^{\sigma}(\theta,\{\mu\})T^{\sigma}(\theta',\{\mu\}) = T^{\sigma}(\theta',\{\mu\})T^{\sigma}(\theta,\{\mu\})R^{\sigma}(\theta-\theta'), \tag{5.10.6}$$

where R is given by (5.10.1). Now by the standard procedure of algebraic Bethe ansatz it is straightforward to derive the energy eigenvalue of the mth excited state, defined by

$$|\Omega_m\rangle = B^{\sigma}(v_1)B^{\sigma}(v_2)....B^{\sigma}(v_m)|0\rangle_{\sigma}, \tag{5.10.7}$$

where

$$|0\rangle_{\sigma} = \prod_{i=1}^{N} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i . \tag{5.10.8}$$

The eigenvalue is given by

$$E^{\sigma}(\theta) = \prod_{i=1}^{m} \frac{\sinh(v_i-\theta+\eta)}{\sinh(v_i-\theta)}\alpha(\theta,\{\mu\}) + \sum_{i=1}^{m} \frac{\sinh(\theta-v_i+\eta)}{\sinh(\theta-v_i)}\delta(\theta,\{\mu\}), \tag{5.10.9}$$

with

$$\alpha(\theta,\{\mu\}) = \prod_{k=1}^{N} \sinh(\theta+\mu_k+\eta), \quad \delta(\theta,\{\mu\}) = \prod_{k=1}^{N} \sinh(\theta+\mu_k). \tag{5.10.10}$$

The eigenmomenta v_i are given by the following equations, obtained by equating the residues at the poles of $E^{\sigma}(\theta)$ to zero. In this way one obtains

$$\prod_{i=1}^{m} \frac{\sinh(v_j+\mu_k+\eta)}{\sinh(v_j+\mu_k)} = \prod_{i\neq j}^{m} \frac{\sinh(v_i-v_j-\eta)}{\sinh(v_i-v_j+\eta)}. \tag{5.10.11}$$

Next we fuse two spin 1/2 quantum R^{σ} matrices, to construct an intermediate R^{σ_s} matrix:

$$R^{\sigma_s}(\theta,\{\mu\}) = \tilde{\mathcal{P}}^{+}_{12}R^{\sigma}(\theta-\eta/2)R^{\sigma}(\theta+\eta/2)\tilde{\mathcal{P}}^{+}_{12}, \tag{5.10.12}$$

where the suffix "s" denotes that half of the spin states are now actually spin 1 due to the fusion procdure. The corresponding monodromy matrix is

$$T^{\sigma_s}(\theta,\{\mu\}) = \tilde{\mathcal{P}}^{+}_{12}T^{\sigma}_{1}(\theta-\eta/2,\{\mu\})T^{\sigma}_{2}(\theta+\eta/2,\{\mu\})\tilde{\mathcal{P}}^{+}_{12}, \tag{5.10.13}$$

where $\tilde{\mathcal{P}}_{12}^{+}$ is a projection operator. The transfer matrix for the mixed σ_s case is given by

$$t^{\sigma_s}(\theta,\{\mu\}) = \mathrm{tr}_{12}\left[\tilde{\mathcal{P}}_{12}^{+}T_1^{\sigma}(\theta-\eta/2,\{\mu\})T_2^{\sigma}(\theta+\eta/2,\{\mu\})\tilde{\mathcal{P}}_{12}^{+}\right]$$

$$= -\mathrm{tr}_{12}\left[\tilde{\mathcal{P}}_{12}^{-}T_1^{\sigma}(\theta+\eta/2,\{\mu\})T_2^{\sigma}(\theta+\eta/2,\{\mu\})\right]$$

$$+t^{\sigma}(\theta+\eta/2,\{\mu\})t^{\sigma}(\theta+\eta/2,\{\mu\}). \tag{5.10.14}$$

It will be recalled that the last term is just the quantum determinant,

$$\Delta(\theta) = \mathrm{tr}\left[\frac{I-\mathcal{P}_{12}}{2}T_1^{\sigma}(\theta-\eta/2,\{\mu\})T_2^{\sigma}(\theta+\eta/2,\{\mu\})\right], \tag{5.10.15}$$

where $\mathcal{P}_{12}$ is the permutation operator. Explicitly, we have

$$\Delta(\theta) = A^{\sigma}(\theta+\eta/2,\{\mu\})D^{\sigma}(\theta-\eta/2,\{\mu\})$$

$$-B^{\sigma}(\theta+\eta/2,\{\mu\})C^{\sigma}(\theta-\eta/2,\{\mu\}). \tag{5.10.16}$$

The eigenvalues and eigenvectors of the mixed transfer matrix t^{σ_s} can be obtained from the crucial observation that t^{σ_s} and t^{σ} commute:

$$[t^{\sigma_s}(\theta,\{\mu\}),t^{\sigma}(\theta,\{\mu\})] = 0, \tag{5.10.17}$$

so that they posses simultaneous eigenvectors. Hence using $|\Omega_m\rangle$, as defined earlier, to denote their common eigenvectors we can at once obtain the eigenvalue of t^{σ_s} as

$$E_m^{\sigma_s}(\theta) = E_m^{\sigma}(\theta-\eta/2)E_m^{\sigma}(\theta+\eta/2) - d(\theta), \tag{5.10.18}$$

with

$$d(\theta) = \prod_{k=1}^{m}\frac{\sinh(v_k-\theta+\eta/2)}{\sinh(v_k-\theta-\eta/2)}\prod_{l=1}^{m}\frac{\sinh(\theta-v_l+\eta/2)}{\sinh(\theta-v_l-\eta/2)}\times$$

$$\times\ \alpha(\theta+\eta/2)\delta(\theta-\eta/2),$$

$$= \alpha(\theta+\eta/2)\delta(\theta-\eta/2). \tag{5.10.19}$$

We now perform a second fusion to obtain the full spin 1 chain. Let us denote the corresponding monodromy matrix as T^s:

$$T^s(\theta,\{\mu\}) = \tilde{\mathcal{P}}_{12}^{+}T_1^{\sigma_s}(\theta-\eta/2,\{\mu\})T_2^{\sigma_s}(\theta+\eta/2,\{\mu\})\tilde{\mathcal{P}}_{12}^{+}. \tag{5.10.20}$$

Taking the trace of both sides we obtain

$$t^s(\theta,\{\mu\}) = \mathrm{tr}\left[\tilde{\mathcal{P}}^+_{12}T_1^{\sigma_s}(\theta-\eta/2,\{\mu\})T_2^{\sigma_s}(\theta+\eta/2,\{\mu\})\tilde{\mathcal{P}}^+_{12}\right]$$

$$= t^{\sigma_s}(\theta-\eta/2,\{\mu\})t^{\sigma_s}(\theta+\eta/2,\{\mu\}) - \Delta(\theta), \tag{5.10.21}$$

where

$$\Delta(\theta) = \text{q-det}T_1^{\sigma_s}(\theta-\eta/2)\text{q-det}T_2^{\sigma_s}(\theta+\eta/2)$$

$$= \Delta(\theta-\eta/2)\Delta(\theta+\eta/2). \tag{5.10.22}$$

From (5.10.14) we now obtain

$$t^s(\theta,\{\mu\}) = [t^\sigma(\theta-\eta,\{\mu\})t^\sigma(\theta,\{\mu\}) - \Delta(\theta-\eta/2)]\times$$

$$\times[t^\sigma(\theta,\{\mu\})t^\sigma(\theta+\eta,\{\mu\}) - \Delta(\theta+\eta/2)] - \Delta(\theta-\eta/2)\Delta(\theta+\eta/2),$$

$$= t^\sigma(\theta-\eta,\{\mu\})t^\sigma(\theta,\{\mu\})t^\sigma(\theta,\{\mu\})t^\sigma(\theta+\eta,\{\mu\})$$

$$-t^\sigma(\theta-\eta,\{\mu\})t^\sigma(\theta,\{\mu\})\Delta(\theta+\eta/2)$$

$$-\Delta(\theta-\eta/2)t^\sigma(\theta,\{\mu\})t^\sigma(\theta+\eta,\{\mu\}). \tag{5.10.23}$$

Again, with the help of the Yang-Baxter equation one can show that

$$[t^s(\theta,\{\mu\}), t^{\sigma_s}(\theta,\{\mu\})] = 0, \tag{5.10.24}$$

so that they also have common eigenvectors. This was the main component of the analysis in [100]. Now, if we operate with t^s, as given in (5.10.23), on $|\Omega_m\rangle$ we obtain

$$E^s_m(\theta) = E^\sigma_m(\theta-\eta)E^\sigma_m(\theta)E^\sigma_m(\theta+\eta) - E^\sigma_m(\theta-\eta)E^\sigma_m(\theta)d(\theta+\eta/2)$$

$$-d(\theta-\eta/2)E^\sigma_m(\theta)E^\sigma_m(\theta+\eta). \tag{5.10.25}$$

We have thus derived the general form of the eigenvalue of the mth excited state for a spin 1 system by the technique of double fusion, without any reference to the specific form of the Hamiltonian.

Next we shall deduce a long-range interaction Hamiltonian and then extract the corresponding eigenvalue for that Hamiltonian from the

general expression (5.10.25). For this purpose we start with the once fused form of the R matrix $R^{\sigma_s}(\omega)$:

$$R^{\sigma_s}(\omega) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & d & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 \\ 0 & d & 0 & c & 0 & 0 \\ 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \tag{5.10.26}$$

with

$$a(\omega) = \sinh(\omega + \frac{3}{2}\eta), \qquad b(\omega) = \sinh(\omega + \eta/2),$$
$$c(\omega) = \sinh(\omega - \eta/2), \qquad d(\omega) = \sqrt{\sinh\eta \sinh 2\eta}.$$

This matrix acts on the tensor product of the vector spaces $V_\sigma \otimes V_s$ that is a (2×3) dimensional space. We first note that R^{σ_s} can be written as

$$R^{\sigma_s}(\omega) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \tag{5.10.27}$$

where

$$\begin{aligned} t_{11} &= f(\omega,\eta)s_3^2 + g(\omega,\eta)s_3 + h(\omega,\eta), \\ t_{22} &= f(\omega,\eta)s_3^2 - g(\omega,\eta)s_3 + h(\omega,\eta), \\ t_{12} &= d(\omega)s_-, \qquad t_{21} = d(\omega)s_-, \end{aligned} \tag{5.10.28}$$

with $s_3, s_\pm$ representing the spin 1 matrix operators of $SU(2)$:

$$s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{i}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.10.29}$$

Here $s_\pm = s_1 \pm \ is_2$ and the functions f, g, h are given by

$$f(\omega,\eta) = \frac{1}{2}(a - 2b + c), \quad g(\omega,\eta) = \frac{1}{2}(a - c), \quad h(\omega,\eta) = b. \tag{5.10.30}$$

If we expand f, g, h around $\omega = \infty$ we obtain

$$\begin{aligned} f &\overset{\omega\to\infty}{\longrightarrow} e^{\omega+\eta/2}\sinh^2\eta/2, \quad g \overset{\omega\to\infty}{\longrightarrow} e^{\omega+\eta/2}\sinh\eta, \\ h &\overset{\omega\to\infty}{\longrightarrow} e^{\omega+\eta/2}. \end{aligned} \tag{5.10.31}$$

We will now employ these to obtain the asymptotic form for the elements of T^{σ_s}. Since

$$T^{\sigma_s}_{ab}(\omega,\{\mu\}) = \sum_{a_1,....a_{N_1}}^{2} t^{(1)}_{aa_1}(\omega+\mu_1)t^{(2)}_{a_1a_2}(\omega+\mu_2)....t^{(N)}_{a_{N-1}b}(\omega+\mu_N), \tag{5.10.32}$$

using (5.10.26) we immediately obtain

$$A^{\sigma_s}(\omega,\{\mu\}) \sim y^N(\omega)\left[\exp\left[\eta\sum_{i=1}^{N}s_3^i\right] + \frac{Q_+(\mu)}{y^2} + \mathcal{O}(y^{-4})\right],$$

$$D^{\sigma_s}(\omega,\{\mu\}) \sim y^N(\omega)\left[\exp\left[-\eta\sum_{i=1}^{N}s_3^i\right] + \frac{Q_-(\mu)}{y^2} + \mathcal{O}(y^{-4})\right], \tag{5.10.33}$$

$$B^{\sigma_s}(\omega,\{\mu\}) \sim y^{N-1}(\omega)d\sum_{k=1}^{N} e^{\bar{\mu}-\mu_k}\Sigma_-^k,$$

$$C^{\sigma_s}(\omega,\{\mu\}) \sim y^{N-1}(\omega)d\sum_{k=1}^{N} e^{\bar{\mu}-\mu_k}\Sigma_+^k, \tag{5.10.34}$$

where

$$\Sigma_-^k = \exp\left[\eta\sum_{i=1}^{k-1}s_3^i\right]s_-^k\exp\left[-\eta\sum_{i=k+1}^{N}s_3^i\right],$$

$$\Sigma_+^k = \exp\left[-\eta\sum_{i=1}^{k-1}s_3^i\right]s_+^k\exp\left[\eta\sum_{i=k+1}^{N}s_3^i\right] \tag{5.10.35}$$

and

$$Q_\pm(\mu) = d^2e^{2\bar{\mu}}\sum_{i\le j<k\le N} e^{-\mu_j-\mu_k}\exp\left[\pm\eta\sum_{i=1}^{j-1}s_3^i\right]s_\mp^i\times$$

$$\times\exp\left[\mp\eta\sum_{m=j+1}^{k-1}s_3^m\right]s_\pm^k\exp\left[\pm\eta\sum_{l=k+1}^{N}s_3^l\right], \tag{5.10.36}$$

$$d=\sqrt{\sinh\eta\sinh 2\eta},\quad y=\frac{1}{2}e^{\omega+\eta/2+\bar{\mu}},\quad \mu=\frac{1}{N}\sum_{k=1}^{N}\mu_k,\quad \Sigma_3=\sum_i s_3^i.$$

Substituting these expressions in $t^s(\omega,\{\mu\})$ we obtain

$$y^{-2N}(\omega)t^s(\omega,\{\mu\}) = \left(e^{2\eta\Sigma_3}+e^{-2\eta\Sigma_3}+1\right)+d^2\sum_{k,l=1}^{N}e^{2\bar{\mu}-\mu_k-\mu_l}(\Sigma_-^k-\Sigma_-^l)$$

$$+\frac{1}{y^2}\left(e^{\eta}Q_+(\mu)e^{\eta\Sigma_3}+e^{-\eta}e^{\eta\Sigma_3}Q_+(\mu)+e^{\eta}Q_+(\mu)e^{\eta\Sigma_3}\right)$$

$$+\frac{1}{y^2}\left(e^{-\eta}e^{-\eta\Sigma_3}Q_-(\mu)+e^{\eta}Q_-(\mu)e^{-\eta\Sigma_3}+e^{-\eta}e^{\eta\Sigma_3}Q_-(\mu)\right). \quad (5.10.37)$$

Extracting the coefficient of y^{2N-2} gives the Hamiltonian,

$$H_{2N-2}=e^{\eta}\left[Q_+(\mu)e^{\eta\Sigma_3}+Q_-(\mu)e^{-\eta\Sigma_3}+Q_+(\mu)e^{-\eta\Sigma_3}\right]$$

$$+e^{-\eta}\left[e^{\eta\Sigma_3}Q_+(\mu)+e^{\eta\Sigma_3}Q_-(\mu)+e^{-\eta\Sigma_3}Q_-(\mu)\right]$$

$$+d^2\sum_{k,l=1}^{N}e^{2\bar{\mu}-\mu_k-\mu_l}(\Sigma_-^k\Sigma_+^l). \quad (5.10.38)$$

This is actually a long-range Hamiltonian coupling the spin 1 operators $s_3, s_\pm$ at different lattice sites. This observation that an expansion of the transfer matrix in the parameter y leads to various long-range Hamiltonians was made by de Vega in [101], and we have applied this idea to the case of the fused t matrix $t^s(\omega,\mu)$, which provides an example of the application of the fusion procedure.

The eigenvalues pertaining to this Hamiltonian can be extracted from the general expression given by (5.10.25), by expanding in the variable y. If we set $x=\frac{1}{2}e^{u+\bar{\mu}+\eta/2}$, then after a simple calculation we obtain

$$\delta(u)=x^Ne^{-N\eta/2}\left[1-\frac{e^{\eta}}{4x^2}\sum_{k=1}^{N}e^{2(\bar{\mu}-\mu_k)}+.....\right],$$

$$\alpha(u)=x^Ne^{N\eta/2}\left[1-\frac{1}{4x^2}\sum_{k=1}^{N}e^{2(\bar{\mu}-\mu_k)}+.....\right]. \quad (5.10.39)$$

Substituting in (5.10.25) we obtain

$$\tilde{E}=\left[R_1S_1(e^{2\eta}+1)+\frac{R_1^2}{4}(e^{\eta}+e^{-3\eta})\sum_k e^{2(\bar{\mu}-\mu_k)}\right]+$$

$$+\left[R_1S_1(1+e^{-2\eta})+\frac{R_1^2}{4}(e^{-\eta}+e^{3\eta})\sum_k e^{2(\bar{\mu}-\mu_k)}\right]$$

$$-\left[R_1^3S_1(1+e^{-2\eta})+R_1^3S_1(e^{2\eta}+1)\right]; \quad (5.10.40)$$

where

$$R_1 = e^{\eta(N-2m)/2} + e^{-\eta(N-2m)/2},$$

$$S_1 = e^{\eta(N-2m)/2}\left[\sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} - 2\sinh\eta\sum_{i=1}^{m} e^{2(\bar{\mu}+v_i)}\right],$$

$$+e^{-\eta(N-2m)/2}\left[e^{\eta}\sum_{k=1}^{N} e^{2(\bar{\mu}-\mu_k)} - 2\sinh\eta\sum_{i=1}^{m} e^{2(\bar{\mu}+v_i)}\right], \qquad (5.10.41)$$

which is the required eigenvalue corresponding to the long-range Hamiltonian (5.10.38). For explicit evaluation we still require a knowledge of the quasi-momenta v_i, which can be determined by demanding that the residue at the poles of the exact expression for $E(\theta, \{\mu\})$ vanishes.

Chapter 6

Integrable Long-Range Models

6.1 Introduction

So far, we have given a general overview of integrable systems. Our discussions have focused mainly on integrable systems arising in two-dimensional statistical mechanics, 1+1-dimensional Hamiltonian systems, and also on field theoretic models. We have described in some detail the techniques of coordinate and algebraic Bethe ansatz, which are indispensable tools for analysing such systems.

Now we consider further applications of the Bethe ansatz, which include the construction and analysis of certain long-range spin models based on the algebraic Bethe ansatz, integrable models with disorder arising from symmetry transformations of the Yang-Baxter algebra and such classic integrable systems as the Calogero-Moser model and its generalization.

While discussing quantum integrable systems on a chain, it is useful to first introduce, a classification of such systems on the basis of the range of interaction. Well-known models exhibiting short range, i.e, nearest neighbour interaction, are the δ-function gas, Toda lattice, XYZ model and its various simplifications. This is evident from their respective Hamiltonians:

$$H_{Toda} = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} \exp(x_{j+1} - x_j),$$

$$H_{XYZ} = \sum_{n=1}^{N-1}\sum_{\alpha=1}^{3} \mathcal{J}_\alpha \sigma_n^\alpha \sigma_{n+1}^\alpha.$$

In contrast to such systems there exists a class of integrable systems in which the interaction is not restricted to just nearest neighbours but

extends to any arbitrary member. For example, the Calogero-Moser model has the following Hamiltonian:

$$H_{CM} = \sum_{i=1}^{N} p_i^2 + \sum_{j \neq k} \frac{g^2}{(x_j - x_k)^2},$$

in which particles interact by a pairwise inverse square potential. Other variants of such systems are those in which the potential is either given by

$$(i)\ V = \sum_{j \neq k} \frac{g^2 a^2}{\sin^2 a(x_j - x_k)},$$

$$(ii)\ V = \sum_{j \neq k} \frac{g^2 a^2}{\sinh^2 a(x_j - x_k)},$$

$$\text{or,}\quad (iii)\ V = \sum_{j \neq k} g^2 a^2 \wp(a(x_j - x_k)),$$

where $\wp(x)$ represents the Weirstrass $\wp$ function.

6.2 Long-Range Models from the ABA

We now outline certain general procedures for obtaining long-range integrable spin systems based essentially on the Yang-Baxter relation and its symmetries. To begin with, we will associate to each lattice site an arbitrary parameter called inhomogeneity. Consider an R matrix depending on the spectral parameter u and an additional parameter η, which satisfies the Yang-Baxter equation (YBE) [102]:

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{6.2.1}$$

Let

$$\hat{R}(u; \eta) = R(u; \eta)\mathcal{P}, \tag{6.2.2}$$

where $\mathcal{P}$ is the permutation operator. In Chapter 3, we have seen that $\hat{R}(u; \eta)$ satisfies the YBE in the form

$$(\hat{R}(u-v)\otimes I)(I\otimes \hat{R}(u))(\hat{R}(v)\otimes I) = (\hat{R}(v)\otimes I)(I\otimes \hat{R}(u))(\hat{R}(u-v)\otimes I). \tag{6.2.3}$$

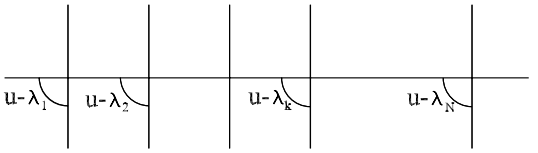

FIGURE 6.2.1: One-dimensional lattice with inhomogeneities $\{\lambda_k\}$.

Furthermore, we assume that $R(u;\eta)$ and $\hat{R}(u;\eta)$ satisfy the conditions of regularity and quasi-classical theory

$$(i) \qquad R(u=0;\eta)=\mathcal{P}, \qquad \hat{R}(u=0;\eta)=I, \tag{6.2.4}$$

$$(ii) \qquad R(u;\eta=0)=I, \qquad \hat{R}(u;\eta=0)=\mathcal{P}. \tag{6.2.5}$$

Introduce next a one-dimensional lattice with inhomogeneities as depicted in Figure (6.2.1).

In this figure, the horizontal line represents a quantum space "V_0" and the vertical lines denote the spaces "V_k" at the kth lattice site. And $\{\lambda_k\}$ denotes the set of inhomogeneity parameters. To each site we introduce the matrix $R_{0k}(u-\lambda_k;\eta)$, acting on $V_0 \otimes V_k$. A typical vertex corresponding to an R matrix element is given in Figure (6.2.2).

Define the transfer matrix $t(u)$ as follows:

$$t(u) = \text{tr}\ {}_0T_N(u;\{\lambda\};\eta) = tr_0[R_{0N}(u-\lambda_N;\eta).....R_{01}(u-\lambda_1;\eta)], \tag{6.2.6}$$

with the trace being taken over the horizontal space. Here $T_N(u,\{\lambda\};\eta)$ is the transition matrix, which despite the inhomogeneities satisfies the Yang-Baxter algebra (YBA):

$$\hat{R}(u-v)T_N(u)\otimes T_N(v) = T_N(v)\otimes T_N(u)\hat{R}(u-v), \tag{6.2.7}$$

with the transfer matrix $t(u)$ forming a commuting family of operators,

$$[t(u),t(v)]=0. \tag{6.2.8}$$

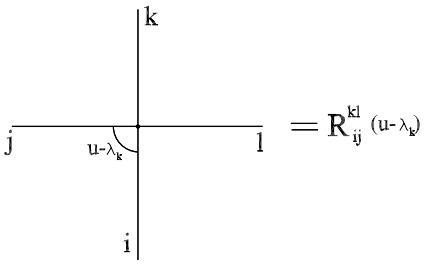

FIGURE 6.2.2: R matrix corresponding to a lattice site.

Let us consider $|u'\rangle$ to be an eigenstate of $t(u)$ and assume that it is characterized by the following set of "quasi-momenta" $\{u'_1, ...u'_M\}$. Defining Z_k to be the operator $t(u)$ evaluated at the kth inhomogeneity λ_k, we set up the following eigenvalue problem:

$$Z_k|u'\rangle = \prod_{l=1}^{M} \frac{1}{c(u'_l - \lambda_k)}|u'\rangle. \tag{6.2.9}$$

Here

$$Z_k = \tau(u = \lambda_k; \eta)$$

and $c(u)$ is a model-dependent quantity along with the $\{u'_l\}$'s. Differentiating (6.2.9) with respect to η at $\eta = 0$ gives

$$\frac{\partial Z_k}{\partial \eta}|u'\rangle_{\eta=0} + Z_k \frac{\partial}{\partial \eta}|u'\rangle_{\eta=0} = (\frac{\partial}{\partial \eta} \prod_{l=1}^{M} \frac{1}{c(u'_l - \lambda_k)})|u'\rangle_{\eta=0} + \tag{6.2.10}$$

$$+ \prod_{l=1}^{M} \frac{1}{c(u'_l - \lambda_k)} \frac{\partial}{\partial \eta}|u'\rangle_{\eta=0}.$$

The quasi-classical condition yields $Z_k|_{\eta=0} = I$, so that demanding

$$c(u'_l - \lambda_k)|_{\eta=0} = 1, \tag{6.2.11}$$

we have from (6.2.10)

$$\frac{\partial Z_k}{\partial \eta}|u'\rangle_{\eta=0} = (\frac{\partial}{\partial \eta} \prod_{l=1}^{M} \frac{1}{c(u'_l - \lambda_k)})|u'\rangle_{\eta=0}. \tag{6.2.12}$$

From (6.2.8) it follows that

$$[Z_k, Z_l] = 0, \tag{6.2.13}$$

so that an expression for Z_k in powers of η has the following form:

$$Z_k = I + \frac{\partial Z_k}{\partial \eta}|_{\eta=0}\eta + \mathcal{O}(\eta). \tag{6.2.14}$$

Consequently, one may look upon $\{\frac{\partial Z_k}{\partial \eta}|_{\eta=0}\}$ as a family of commuting Hamiltonians, and rewrite (6.2.13) and (6.2.14) as

$$H_k|u'\rangle_{\eta=0} = \Lambda_k|u'\rangle_{\eta=0}, \tag{6.2.15}$$

where

$$\Lambda_k = \frac{\partial}{\partial \eta}\prod_{l=1}^{M}\frac{1}{c(u'_l - \lambda_k)}|_{\eta=0} \tag{6.2.16}$$

and

$$[H_k, H_l] = 0 \qquad\qquad k, l = 1, 2, ..., N. \tag{6.2.17}$$

From these equations it is evident that a Hamiltonian system given by

$$H = \sum_{k=1}^{N}\xi_k H_k, \tag{6.2.18}$$

where ξ_k are arbitrary constants, constitutes a completely integrable system with eigenvalue given by

$$\mathcal{E} = \sum_{k=1}^{N}\xi_k \Lambda_k. \tag{6.2.19}$$

To derive an explicit expression for the Hamiltonian we need to consider the original expression for Z_k, that is

$$\begin{aligned} Z_k &= tr_o[R_{oN}(\lambda_k - \lambda_N)....P_{ok}...R_{o1}(\lambda_k - \lambda_1)], \\ &= \sum_{\{l\}} R^{b_N l_1}_{a_N l_N}(\lambda_k - \lambda_N).....P^{b_k l_{k+1}}_{a_k l_k}...R^{b_1 l_2}_{a_1 l_1}(\lambda_k - \lambda_1), \\ \text{or} \quad Z_k &= \sum_{\{l\}} R^{b_{k-1} b_k}_{a_{k-1} l_{k-1}}(\lambda_k - \lambda_{k-1})....R^{b_1 l_2}_{a_1 l-1}(\lambda_k - \lambda_1)\times \\ &\quad \times R^{b_N l_1}_{a_N l_N}(\lambda_k - \lambda_N)...R^{b_{k+1} l_{k+2}}_{a_{k+1} l_{k+1}}(\lambda_k - \lambda_{k+1}), \end{aligned} \tag{6.2.20}$$

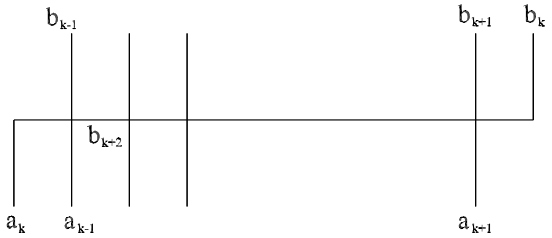

FIGURE 6.2.3: $Z^{\{b\}}_{k\{a\}} = t^{\{b\}}_{\{a\}}(\lambda_k)$, where $\{a_j\}$ and $\{b_j\}$ stand for states in the lower and upper rows.

so that

$$Z_k = R_{kk-1}(\lambda_k - \lambda_{k-1})...R_{k1}(\lambda_k - \lambda_1)R_{kN}(\lambda_k - \lambda_N)....R_{kk+1}(\lambda_k - \lambda_{k+1}) \tag{6.2.21}$$

where the diagrammatic representation of Z_k is shown in Figure (6.2.3).

Using (6.2.21) we arrive at the following formula for the Hamiltonian H_k:

$$H_k = \frac{\partial}{\partial \eta} Z_k|_{\eta=0} = \sum_{j \neq k} \frac{\partial}{\partial \eta} R_{kj}(\lambda_k - \lambda_j; \eta)|_{\eta=0}. \tag{6.2.22}$$

The total Hamiltonian is

$$H = \frac{1}{2} \sum_{j \neq k} (\xi_k - \xi_j) \frac{\partial}{\partial \eta} R_{kj}(\lambda_k - \lambda_j; \eta),$$

under the assummption $\frac{\partial}{\partial \eta} R(\lambda, \eta) = -\frac{\partial}{\partial \eta} R(-\lambda; \eta)$. Let us consider a specific example to clarify the above procedure.

Example: XXZ model in a magnetic field

While describing the algebraic Bethe ansatz, we had considered the XXZ spin chain placed in an external magnetic field [103]. We also recall that the following R matrix was associated to each lattice site

$$R_{0k}(\lambda, \nu_k; \eta) = \begin{pmatrix} t^{(k)}_{11}(\lambda, \nu_k)\ t^{(k)}_{12}(\lambda, \nu_k) \\ t^{(k)}_{21}(\lambda, \nu_k)\ t^{(k)}_{22}(\lambda, \nu_k) \end{pmatrix},$$

with

$$t^{(k)}_{11}(\lambda, \nu_k) = \begin{pmatrix} e^{\eta h(\lambda - \nu_k)} & 0 \\ 0 & \frac{\sinh(\lambda - \nu_k)}{\sinh(\lambda - \nu_k + \eta)} e^{-\eta h(\lambda + \nu_k)} \end{pmatrix},$$

$$t_{12}^{(k)}(\lambda,\nu_k)=\begin{pmatrix}0 & 0\\ \frac{\sinh\eta}{\sinh(\lambda-\nu_k+\eta)} & 0\end{pmatrix},\quad t_{21}^{(k)}(\lambda,\nu_k)=\begin{pmatrix}0 & \frac{\sinh\eta}{\sinh(\lambda-\nu_k+\eta)}\\ 0 & 0\end{pmatrix},$$

$$t_{22}^{(k)}(\lambda,\nu_k)=\begin{pmatrix}\frac{\sinh(\lambda-\nu_k)}{\sinh(\lambda-\nu_k+\eta)}e^{\eta h(\lambda+\nu_k)} & 0\\ 0 & e^{-\eta h(\lambda-\nu_k)}\end{pmatrix}.$$

Hence in the present case H_k may be obtained in terms of local spin operators and (6.2.22):

$$H_k=\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{1}{2\sinh(\nu_k-\nu_j)}\Big[\cosh(\nu_k-\nu_l)(\sigma_k^3\otimes\sigma_j^3-I)$$

$$+(\sigma_k^+\otimes\sigma_j^-+\sigma_k^-\otimes\sigma_j^+)+h(\nu_k-\nu_j)\sinh(\nu_k-\nu_j)(\sigma_k^3\otimes\sigma_j^3+I\otimes I)$$

$$+h(\nu_k+\nu_j)\sinh(\nu_k-\nu_j)(I\otimes\sigma_j^3-\sigma_k^3\otimes I)\Big]. \tag{6.2.23}$$

Note the terms involving the magnetic field in (6.2.23); it is seen that when $h=0$ we get back the basic result in [102].

To ensure that the eigenvalue problem given by (6.2.9) is well defined, we have to check that condition (6.2.11) is satisfied. It should be clear that the eigenvalue of Z_k, when acting on the M excitation state $|\Omega_M\rangle$, will be the same as that of the transfer matrix of the corresponding algebraic Bethe ansatz problem, evaluated at $\lambda=\nu_k$. From (5.3.28) we have

$$\Lambda_M(\lambda=\nu_k)=e^{2\eta\nu_k h}\left(\prod_{j=1}^{N}e^{\eta h(\nu_k-\nu_j)}\right)\prod_{i=1}^{M}\frac{\sinh(\nu_k-\mu_i+\eta)}{\sinh(\nu_k-\mu_i)}$$

$$+e^{2\eta\nu_k h}\left(\prod_{j=1}^{N}\frac{\sinh(\nu_k-\nu_j)}{\sinh(\nu_k-\nu_j+\eta)}e^{\eta h(\nu_k+\nu_j)}\right)\prod_{i=1}^{M}\frac{\sinh(\mu_i-\nu_k+\eta)}{\sinh(\mu_i-\nu_k)}. \tag{6.2.24}$$

However, the second term on the right-hand side of (6.2.24) vanishes when $j=k$, so that

$$\Lambda_M(\lambda=\nu_k)=e^{2\eta\nu_k h}\left(\prod_{j=1}^{N}e^{\eta h(\nu_k-\nu_j)}\right)\prod_{i=1}^{M}\frac{\sinh(\nu_k-\mu_i+\eta)}{\sinh(\nu_k-\mu_i)}. \tag{6.2.25}$$

Notice once again the appearance of the terms involving the external magnetic field in the last expression. Thus from (6.2.9) and (6.2.25),

we find that

$$\prod_{l=1}^{M}\frac{1}{c(\mu_l-\nu_k)}=e^{2\eta\nu_k h}\left(\prod_{j=1}^{N}e^{\eta h(\nu_k-\nu_j)}\right)\prod_{i=1}^{M}\frac{\sinh(\nu_k-\mu_i+\eta)}{\sinh(\nu_k-\mu_i)}. \tag{6.2.26}$$

In view of the constraint expressed by (6.2.11), one can immediately conclude that the right-hand-side of (6.2.26) reduces to unity when $\eta=0$. Thus we conclude that the eigenvalue problem given by (6.2.9) is well defined. The eigenvalue corresponding to a Hamiltonian H_k can be easily deduced from (6.2.26) as

$$\xi_k=h\left[2\nu_k+\sum_{j=1}^{N}(\nu_k-\nu_j)\right]+\sum_{l=1}^{M}\coth(\nu_k-\mu_l). \tag{6.2.27}$$

Finally, the new Bethe ansatz equation determining the μ_l's is obtained by differentiating (5.3.26), i.e.,

$$\prod_{k=1}^{N}\frac{\sinh(\mu_j-\nu_k+\eta)}{\sinh(\mu_j-\nu_k)}e^{-2\eta\nu_k h}=\prod_{\substack{i=1\\ i\neq j}}^{M}\frac{\sinh(\mu_i-\mu_j+\eta)}{\sinh(\mu_i-\mu_j-\eta)},\quad j=1,2,....,M, \tag{6.2.28}$$

with respect to η and setting $\eta=0$. From this we get

$$\sum_{k=1}^{N}[\coth(\mu_l-\nu_k)-2h\nu_k]=2\sum_{i\neq l}^{M}\coth(\mu_i-\mu_l),\quad l=1,2,...,M. \tag{6.2.29}$$

From (6.2.23) we see that the inhomogeneities $\{\nu_k\}_{k=1}^{N}$ appear as coordinates in the expression for the Hamiltonian H_k. Moreover, as the summation in (6.2.23) is not confined to nearest neighbours, the Hamiltonian H_k is said to exhibit long-range interaction in terms of the "coordinates" $\{\nu_k\}_{k=1}^{N}$. The external magnetic field appears in the form of additional terms in the expression for H_k. The energy eigenvalue and the new BAE both explicitly involve the magnetic field.

We state below the results for a few more models:

• XYZ model

$$H_l=\frac{1}{2}\sum_{j\neq l}\frac{1}{\mathrm{sn}(\lambda_l-\lambda_j)}\Big\{(1+k\mathrm{sn}^2(\lambda_l-\lambda_j))\sigma_l^x\sigma_j^x+$$

$$(1-k\mathrm{sn}^2(\lambda_l-\lambda_j))\sigma_l^y\sigma_j^y+\mathrm{cn}(\lambda_l-\lambda_j)\mathrm{dn}(\lambda_l-\lambda_j)(\sigma_l^z\sigma_j^z-1)\Big\}. \tag{6.2.30}$$

The constant k in (6.2.30) is the modulus of Jacobi's elliptic functions.

• Spin 1 XXZ model

$$H_k = \sum_{j\neq k} \frac{1}{\sin(\lambda_k - \lambda_j)} \left\{ S_k^x S_j^x + S_k^y S_j^y + \cos(\lambda_k - \lambda_j) S_k^z S_j^z \right\}. \quad (6.2.31)$$

• Z_n model

$$H_k = \sum_{j\neq k} \left\{ \sum_{\alpha\beta} \left(\left(\frac{2}{n}(\alpha - \beta) - \mathrm{sgn}(\alpha - \beta) \right) (\lambda_k - \lambda_j) \right) \frac{(E_k)_{\alpha\beta}(E_j)_{\beta\alpha}}{\sinh(\lambda_k - \lambda_j)} \right.$$
$$\left. + \sum_{\alpha\beta} \left(\frac{2}{n}(\beta - \alpha) - \mathrm{sgn}(\beta - \alpha) - \coth(\lambda_k - \lambda_j) \right) (E_k)_{\alpha\alpha}(E_j)_{\beta\beta} \right\}, \quad (6.2.32)$$

where E is a matrix whose elements are

$$(E_{\alpha\beta})_{lm} = \delta_{\alpha l}\delta_{\beta m}.$$

6.3 Symmetry Transformation

An alternate procedure for constructing integrable quantum models with long-range interactions is by exploiting the symmetries of the Yang-Baxter algebra (YBA), together with association of inhomogeneities at each lattice site [104]. Let us consider the following transformation of the monodromy operator:

$$T_N(u) \longrightarrow g T_N(u). \quad (6.3.1)$$

Under this transformation the YBA given by (6.2.7) is invariant, provided

$$[\hat{R}(u - v), g \otimes g\ ] = 0, \qquad \forall\ u, v. \quad (6.3.2)$$

Consequently the operator,

$$t_g(u) = \mathrm{tr}\ [g T_N(u)], \quad (6.3.3)$$

constitutes a family of commuting transfer matrices,

$$[t_g(u), t_g(v)] = 0. \quad (6.3.4)$$

Thus, each symmetry-generating transformation g will generate a corresponding commuting family of transfer matrices. It will be shown below that these operators $t_g(u)$ generate one-dimensional Hamiltonians with inhomogeneous parameters, provided g and the expansion point in the spectral variable u are carefully chosen. Indeed, commuting operators in general follow from (6.3.4) when $t_g(u)$ is expanded in a power series, around some suitable point, say u_0. It is customary to call an integrable system *regular* if for some point u_0,

$$t_I(u_0) = \text{const} X, \tag{6.3.5}$$

where X is the unit shift operator on the line and $g = I$. Without loss of generality, one can then set $u_0 = 0$ and write

$$\log t_I(u) = \sum_{p=0}^{\infty} \hat{c}_p u^p. \tag{6.3.6}$$

In this expansion, the operators $\hat{c}_p$ may be shown to couple $(p+1)$ adjacent sites. But the explicit forms of $\hat{c}_p$ when expanded about a generic point are usually complicated.

As an alternative, one may try to expand $t_g(u)$ around $u = \infty$. The expansion coefficients are operators that now couple nonadjacent sites, and are usually easy to obtain explicitly even in the case of inhomogeneous models.

To clarify these issues it would be useful to consider an explicit inhomogeneous six-vertex model, on a square lattice for which the R matrix is given by

$$\hat{R}(u;\eta) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh\eta \sinh u & & 0 \\ 0 & \sinh u \sinh\eta & & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}. \tag{6.3.7}$$

Partitioning the R matrix into blocks, we may write it as

$$\hat{R}(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{pmatrix}, \tag{6.3.8}$$

where

$$t_{11}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 \\ 0 & \sinh u \end{pmatrix}, \qquad t_{22} = \begin{pmatrix} \sinh u & 0 \\ 0 & \sinh(u+\eta) \end{pmatrix},$$

$$t_{12}(u) = \begin{pmatrix} 0 & 0 \\ \sinh\eta & 0 \end{pmatrix}, \qquad t_{21} = \begin{pmatrix} 0 & \sinh\eta \\ 0 & 0 \end{pmatrix}. \tag{6.3.9}$$

The monodromy matrix can be written in the usual manner as

$$T_N(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{6.3.10}$$

where the elements of the monodromy operator are defined by

$$T_N^{ab}(u) = \sum_{a_1 \ldots a_N} t^{(1)}(u+\mu_1)_{aa_1} t^{(2)}(u+\mu_2)_{a_1 a_2} \ldots . t^{(N)}(u+\mu_N)_{a_N b}, \tag{6.3.11}$$

$a, b, a_1, \ldots . a_N = 1, 2$. The operators $t^{(k)}(u+\mu_k)_{a_{k-1}a_k}$ act on a two-dimensional vertical space $V^{(k)}$, while $\{\mu_1, \mu_2, \ldots \mu_k\}$ are the inhomogeneity parameters. Taking

$$g(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \tag{6.3.12}$$

it can be verified that the YBA as given in (6.2.7) is satisfied since

$$[\hat{R}(u), g(\alpha) \otimes g(\alpha)] = 0. \tag{6.3.13}$$

Consequently, we have now a two-parameter family of commuting transfer matrices:

$$t_\alpha(u; \{\mu\}) = \mathrm{tr}[g(\alpha) T_N(u; \{\mu\})] = \alpha A(u; \{\mu\}) + \frac{1}{\alpha} D(u; \{\mu\}), \tag{6.3.14}$$

$$[t_\alpha(u; \{\mu\}), t_\alpha(v; \{\mu\})] = 0. \tag{6.3.15}$$

If we expand $T_N^{ab}(u; \{\mu\})$ around $u = \infty$, then from (6.3.10) and (6.3.8) we find

$$A(u; \{\mu\}) = y^N e^{\eta s_3} [1 + Q_+(\mu)/y^2 + O(y^{-4})], \tag{6.3.16}$$

$$D(u; \{\mu\}) = y^N e^{-\eta s_3} [1 + Q_-(\mu)/y^2 + O(y^{-4})], \tag{6.3.17}$$

where $y = \frac{1}{2}\exp(u + \frac{\eta}{2} + \frac{1}{N}\sum_i \mu_i)$ and

$$s_3 = \frac{1}{2}\sum_{k=1}^{N} (\sigma_k)_3 \ \text{ with } \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3.18}$$

The latter is the third component of the total spin and

$$Q_{\pm}(\mu) = \sinh^2\eta \sum_{1\le k,l\le N} e^{-\mu_k-\mu_l}(\sigma_{\mp})_k(\sigma_{\pm})_l$$

$$-\frac{1}{4}\sum_{k=1}^{N} e^{-2\mu_k}[\cosh\eta \mp \sinh\eta(\sigma_k)_3], \tag{6.3.19}$$

$\sigma_{\pm}$ being the Pauli matrices. One can also show that

$$[s_3, A(u;\{\mu\})] = [s_3, D(u;\{\mu\})] = 0, \tag{6.3.20}$$

so that one may define another transfer matrix,

$$\hat{t}_\alpha(u;\{\mu\}) = \alpha e^{-\eta s_3}A(u;\{\mu\}) + \frac{1}{\alpha}e^{\eta s_3}D(u;\{\mu\}), \tag{6.3.21}$$

which also forms a commuting set, i.e.,

$$[\hat{t}_\alpha(u;\{\mu\}), \hat{t}_\alpha(v;\{\mu\})] = 0. \tag{6.3.22}$$

It should be noted that neither of the transfer matrices defined by (6.3.4) or (6.3.21) would form a commuting family for different values of the parameter α of inhomogeneities $\{\mu\}$. From (6.3.20) and (6.3.16) we find the first nontrivial operator generated by $\hat{t}_\alpha$ to be

$$K(\alpha) = \alpha Q_+(\mu) + \alpha^{-1}Q_-(\mu). \tag{6.3.23}$$

Indeed for $\alpha = 1$, this can be expressed in terms of one-site operators:

$$K(1) = \frac{1}{2}\sinh^2\eta(\Sigma_+\Sigma_- + \Sigma_-\Sigma_+) - \frac{1}{2}(ch\eta + sh^2\eta)\sum_{k=1}^{N} e^{-2\mu_k}, \tag{6.3.24}$$

where

$$\Sigma_{\pm} = \sum_{k=1}^{N} e^{-\mu_k}(\sigma_{\pm})_k.$$

This operator does not couple spins at arbitrary sites and therefore shows no long-range interaction. However if we set $\alpha = i$, then we find that

$$K(i) = i(Q_+ - Q_-) = i\sinh^2\eta \sum_{1\le k,l\le N} e^{-\mu_k-\mu_l}\mathrm{sign}(l-k)(\sigma_+)_l(\sigma_-)_k,$$

$$+\frac{i}{2}\sinh\eta\sum_{k=1}^{N}e^{-2\mu_k}(\sigma_k)_3. \tag{6.3.25}$$

This operator not only couples the spin variables at different sites, but is also hermitian if we assume $i\eta = -\gamma \in \mathcal{R}$ and $e^{\mu_k+\mu_l} \in \mathcal{R}(l,k = 1,...N)$. If we rescale by a factor $-\frac{1}{2}\sin\gamma$, then, we arrive at the following expression for the Hamiltonian

$$H(\{\mu\}) = \frac{1}{2}\sin\gamma\sum_{1\leq k,l\leq N}e^{-\mu_k-\mu_l}(\sigma_l\wedge\sigma_k)_3 + \sum_{k=1}^{N}e^{-2\mu_k}(\sigma_k)_3, \tag{6.3.26}$$

which belongs to an infinite family of commuting operators. The higher ones follow from the terms of order $e^{-2p_k}, p \geq 2$ in (6.3.16) and (6.3.20).

6.4 Calogero-Moser Models

The Calogero-Moser models have recently attracted a lot of attention as they are classic examples of systems exhibiting long-range interaction. These models are related to random matrix theory, matrix models and two-dimensional gravity. Moreover, they have what are known as Jastrow-type wave functions, which play an important role in several areas of condensed matter physics such as quantum Hall effect, anions, and high T_c superconductivity.

For discussing this particular family of models, we begin by considering the basic Hamiltonian of the classical Calogero-Moser (CM) model in the form [105]

$$H = \frac{1}{2}\sum_{j=1}^{N}p_j^2 + \frac{g}{2}\sum_{j\neq k}V(x_j - x_k), \tag{6.4.1}$$

where $\{x_j\}$ and $\{p_j\}$ denote positions and conjugate momenta of N identical particles, and g represents the strength of the potential. The Hamilton's equations of motion are given by

$$\dot{x_j} = \{x_j, H\} = p_j, \quad \dot{p_j} = \{p_j, H\} = -g\sum_{k\neq j}V'(x_j - x_k). \tag{6.4.2}$$

In the inverse scattering method, these equations of motion are assumed to follow from the Lax equation, that is, from hermitian $N \times N$ matrices L, M satisfying the Lax equation:

$$i\dot{L} = i\{L, H\} = [M, L]. \tag{6.4.3}$$

From this, it follows that the conserved integrals of motions are

$$I_n = \frac{1}{n} tr(L^n) = \frac{1}{n} \sum_j (L^n)_{jj}, \qquad n = 1, 2, ..., N. \tag{6.4.4}$$

For the purpose of clarity let us assume the L, M matrices to be

$$L_{jk} = p_j \delta_{jk} + ia(1 - \delta_{jk}) f(x_j - x_k), \tag{6.4.5}$$

$$M_{jk} = a(1 - \delta_{jk}) g(x_j - x_k) + a\delta_{jk} \sum_{l \neq j} z(x_j - x_l), \tag{6.4.6}$$

where a is a constant and the functions $f(x)$, $g(x)$ and $z(x)$ have the property

$$f(x) = -f(-x), \qquad g(x) = g(-x) \text{ and } \qquad z(x) = z(-x). \tag{6.4.7}$$

Upon substitution of (6.4.1), (6.4.5) and (6.4.6) in the Lax equation (6.4.3), one gets a set of functional equations determining the functions f, g and $z(x)$:

$$\begin{aligned} g(x) = f'(x), \qquad & gV'(x) = 2a^2 f(x) g(x), \\ f(x)g(y) - f(y)g(x) = f(x+y)(z(x) - z(y)) & \end{aligned} \tag{6.4.8}$$

and $g = a^2$. The solutions of (6.4.8) can be classified into four categories of potentials [106].

- Rational: $V(x) = \frac{1}{x^2}$,
- Trigonometric: $V(x) = \frac{\alpha^2}{\sin^2 \alpha x}$,
- Hyperbolic: $V(x) = \frac{\alpha^2}{\sinh^2 \alpha x}$,
- Elliptic: $V(x) = \alpha^2 \wp(\alpha x)$.

Here α is a constant and $\wp(x)$ is the Weirstrass $\wp$ function. The latter represents the most general situation, as $\wp(x)$ is doubly periodic in the

complex x plane. In the limiting case when one of the periods tends to infinity we get the second and the third cases up to a constant, while when both periods go to infinity one gets the rational case. The rational case is referred to as the Calogero-Moser model as the integrability of the classical case was deduced by Moser [107].

6.4.1 Quantum Calogero-Moser model

In the quantum case the Hamiltonian of the Calogero-Moser model is given by

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{g}{2}\sum_{j\neq k} V(x_j - x_k), \tag{6.4.9}$$

with p_j now being an operator, i.e., $p_j = -i\partial/\partial x_j$. In the quantum version of the inverse scattering, one introduces $N \times N$ hermitian matrices L, M such that the Lax equation,

$$\frac{d}{dt}L_{jk} = i[H, L_{jk}] = i[L, M]_{jk}, \tag{6.4.10}$$

is equivalent to the Heisenberg equation of motion generated by the Hamiltonian. Note that here the elements of L and M are operator valued. If we take the L, M matrices as follows:

$$L_{jk}p_j\delta_{jk} + ia(1-\delta_{jk})f(x_j - x_k), \tag{6.4.11}$$

$$M_{jk} = a(1-\delta_{jk})g(x_j - x_k) + a\delta_{jk}\sum_{l\neq j} z(x_j - x_k), \tag{6.4.12}$$

where $f(x)$ is an odd function and $g(x)$ and $z(x)$ are even functions, then upon substitution of (6.4.11) and (6.4.12) into (6.4.10) we obtain

$$\begin{aligned} g(x) = f'(x), \qquad gV'(x) &= 2a^2 f(x)g(x) - az'(x), \\ f(x)g(y) - f(y)g(x) &= f(x+y)(z(x) - z(y)). \end{aligned} \tag{6.4.13}$$

The extra term $-az'(x)$ in the second equation in (6.4.13), arises from the operator nature of the elements of L. A solution of the functional equations (6.4.13) is given by

$$f(x) = 1/x, \quad g(x) = -z(x) = -1/x^2 \text{ and } gV(x) = (a^2 - a)/x^2. \tag{6.4.14}$$

Consequently, the quantum inverse problem for the quantum Calogero-Moser model is formulated with the following form of the Hamiltonian:

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{g}{2}\sum_{j\neq k}\frac{1}{(x_j - x_k)^2}, \tag{6.4.15}$$

where $g = a^2 - a$ and the Lax pair for the model is

$$L_{jk} = p_j\delta_{jk} + ia(1-\delta_{jk})\frac{1}{(x_j - x_k)}, \tag{6.4.16}$$

$$M_{jk} = -a(1-\delta_{jk})\frac{1}{(x_j - x_k)^2} + a\delta_{jk}\sum_{l\neq j}\frac{1}{(x_j - x_l)^2}. \tag{6.4.17}$$

It would be interesting to determine how the conserved quantities may be constructed for such a case. One can show that M satisfies the so-called sum-to-zero condition:

$$\sum_k M_{jk} = \sum_k M_{kj} = 0. \tag{6.4.18}$$

From the Lax equation (6.4.10) we have

$$[H, \sum_{jk}(L^n)_{j,k}] = \sum_{jl}(L^n)_{jl}(\sum_k M_{lk}) - \sum_{lk}(\sum_j M_{jl})(L^n)_{lk} = 0, \tag{6.4.19}$$

the equality following as a result of the sum-to-zero condition. Hence we have the following formula for the conserved operators [108],

$$I_n = \frac{1}{n}\sum_{jk}(L^n)_{jk}. \tag{6.4.20}$$

One should mention here that although the condition (6.4.18) is valid for trigonometric and hyperbolic cases, it is not so for the elliptic case.

In order to prove explicitly the integrability of the quantum Calogero-Moser model (6.4.15), it is convenient to construct the following operators and relations:

1. Boost operator B_n:

$$B_n = \frac{1}{2i}[\sum_j x_j^2, I_{n+1}]. \tag{6.4.21}$$

2. Generalized Lax equation:

$$[I_n, L_{jk}] = [L, M^{(n)}]_{jk}. \tag{6.4.22}$$

3. Additional relations:

$$[I_{n+1}, X] = [X, M^{(n+1)}] + L^n, \tag{6.4.23}$$

$$X_{jk} \equiv i x_j \delta_{jk}. \tag{6.4.24}$$

The explicit construction of $M^{(n)}$ is given in [109] and $M^{(n)}$ satisfies the sum-to-zero condition. From these relations one can obtain the following commutation relations between the conserved quantities I_n and the boost operators $\{B_n\}$:

$$[I_n, I_m] = 0 \tag{6.4.25}$$

$$[B_m, I_n] = i(n + m - 1) I_{n+m-1} \tag{6.4.26}$$

$$[B_m, B_n] = i(m - n) B_{n+m-1}. \tag{6.4.27}$$

The relation (6.4.25) shows that the conserved quantities are in involution and that the model (6.4.15) is therefore integrable. The preceding relations constitute what is known as the $U(1)$-current algebra.

6.4.2 Generalizations of the CM model

In [110] Gibbons and Hermsen considered a generalized version of the Calogero-Moser many-body system, in which the interacting particles possessed additional internal degrees of freedom. In [111] Billey et al. considered this generalized CM system, which is governed by the Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i \neq j}^{N} \frac{f_{ij} f_{ji}}{(q_i - q_j)^2}, \tag{6.4.28}$$

with the dynamical variables $(q_i, p_i)_{i=1}^N$ and $(f_{ij})_{i,j=1}^N$ satisfying the Poisson brackets:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}. \tag{6.4.29}$$

The classical integrability of this model was found to exist on the surfaces $f_{ii} = \text{const.}$, $i = 1, ..., N$. Brezezinski [112] proposed a further

generalization of this model by introducing a nontrivial internal structure in the dynamical variables $(f_{ij})_{i,j=1}^{N}$, so that the Hamiltonian assumed the form,

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\sum_{i\neq j}^{N}\sum_{\alpha,\beta=1}^{M} \frac{f_{ij}^{\alpha} f_{ji}^{\beta}}{(q_i - q_j)^2}, \tag{6.4.30}$$

with $(q_i, p_i)_{i=1}^{N}$ obeying the previous Poisson brackets, but the f_{ij}^{α}'s satisfying the following:

$$\{f_{ij}^{\alpha}, f_{kl}^{\beta}\} = \delta_{\alpha\beta}(\delta_{jk} f_{il}^{\alpha} - \delta_{il} f_{kj}^{\alpha}). \tag{6.4.31}$$

The essential difference of this Hamiltonian with the one given by (6.4.28), is the identification of f_{ij} as $\sum_{\alpha=1}^{M} f_{ij}^{\alpha}$. The system (6.4.30) may be shown to be integrable when $(f_{ii} = const)_{i=1}^{N}$ and to admit the Lax matrix:

$$L(\lambda) = \sum_{i=1}^{N}\left(p_i + \sum_{\alpha=1}^{M}\frac{f_{ii}^{\alpha}}{\lambda - \epsilon_\alpha}\right)e_{ii} + \sum_{i\neq j}^{N}\left(\frac{f_{ij}}{q_i - q_j} + \sum_{\alpha=1}^{M}\frac{f_{ij}^{\alpha}}{\lambda - \epsilon_\alpha}\right)e_{ij}. \tag{6.4.32}$$

Here e_{ij} is a matrix whose ijth element is unity and all the others are zero, while $\epsilon_1 > \epsilon_2 > ....\epsilon_M$ are arbitrary parameters. The form of the Lax matrix $L(\lambda)$ represents nothing but the coupling of the M particle Gaudin model to the usual Calogero-Moser model. It is in this sense that the Calogero-Moser model has been modified. Using the fundamental Poisson brackets (6.4.31), one can show that the Lax matrix in (6.4.32) obeys the relation,

$$\{L^1(\lambda), L^2(\mu)\} = [r_{12}(\lambda,\mu), L^1(\lambda) + L^2(\mu)] - \sum_{i\neq j}^{N}\frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} e_{ij}\otimes e_{ji}, \tag{6.4.33}$$

with the following classical r matrix:

$$r_{12}(\lambda,\mu) = \sum_{i\neq j}^{N}\left(\frac{1}{\lambda-\mu} + \frac{1}{q_i - q_j}\right)e_{ij}\otimes e_{ji} + \frac{1}{\lambda-\mu}\sum_{i=1}^{N} e_{ii}\otimes e_{ii}. \tag{6.4.34}$$

Note here that the r matrix now depends not only the spectral parameters λ, μ but, also explicitly on the coordinates q_i's. Such r matrices are termed as dynamical r matrices. From (6.4.33) it immediately follows that the model (6.4.30) will be classically integrable provided

$f_{ii} = f_{jj}, i, j = 1, ..., N$ since in that case the Poisson bracket (6.4.33) reduces to the standard case. For the specific case of $N = 2$, the Lax matrix is

$$L(\lambda) = \begin{pmatrix} p_1 + \sum_{\alpha=1}^{M} \frac{f_{11}^{\alpha}}{\lambda - \epsilon_\alpha} & \frac{f_{12}}{q_1 - q_2} \sum_{\alpha=1}^{M} \frac{f_{12}^{\alpha}}{\lambda - \epsilon_\alpha} \\ \frac{f_{21}}{q_2 - q_1} \sum_{\alpha=1}^{M} \frac{f_{21}^{\alpha}}{\lambda - \epsilon_\alpha} & p_2 + \sum_{\alpha=1}^{M} \frac{f_{22}^{\alpha}}{\lambda - \epsilon_\alpha} \end{pmatrix}. \tag{6.4.35}$$

Working in the center of mass frame we have, upon subtracting the center of mass motion, the following form of the above Lax matrix:

$$L(\lambda) = \begin{pmatrix} \frac{p_1 - p_2}{2} + \frac{1}{2} \sum_{\alpha=1}^{M} \frac{f_{11}^{\alpha} - f_{22}^{\alpha}}{\lambda - \epsilon_\alpha} & \frac{f_{12}}{q_1 - q_2} \sum_{\alpha=1}^{M} \frac{f_{12}^{\alpha}}{\lambda - \epsilon_\alpha} \\ \frac{f_{21}}{q_2 - q_1} \sum_{\alpha=1}^{M} \frac{f_{21}^{\alpha}}{\lambda - \epsilon_\alpha} & \frac{p_2 - p_1}{2} + \frac{1}{2} \sum_{\alpha=1}^{M} \frac{f_{22}^{\alpha} - f_{11}^{\alpha}}{\lambda - \epsilon_\alpha} \end{pmatrix}. \tag{6.4.36}$$

Introducing relative coordinates q, p defined by $q = q_1 - q_2, p = \frac{1}{2}(p_1 - p_2)$, so that $\{q, p\} = 1$ and writing

$$S_3^{\alpha} = \frac{1}{2}(f_{11}^{\alpha} - f_{22}^{\alpha}), \qquad f_{12}^{\alpha} = S_+^{\alpha}, \qquad f_{21}^{\alpha} = S_-^{\alpha},$$

$L(\lambda)$ in (6.4.36) assumes the form

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} = \begin{pmatrix} p + \sum_{\alpha=1}^{M} \frac{S_3^{\alpha}}{\lambda - \epsilon_\alpha} & \frac{S_+}{q} + \sum_{\alpha=1}^{M} \frac{S_+^{\alpha}}{\lambda - \epsilon_\alpha} \\ -\frac{S_-}{q} + \sum_{\alpha=1}^{M} \frac{S_-^{\alpha}}{\lambda - \epsilon_\alpha} & -p + \sum_{\alpha=1}^{M} \frac{S_3^{\alpha}}{\lambda - \epsilon_\alpha} \end{pmatrix}. \tag{6.4.37}$$

Here $S_3^{\alpha}, S_{\pm}^{\alpha}$ are generators of the $so(2,1)$ Poisson algebra

$$\{S_3^{\alpha}, S_{\pm}^{\beta}\} = \pm \delta_{\alpha\beta} S_{\pm}^{\beta}, \quad \{S_+^{\alpha}, S_-^{\beta}\} = 2\delta_{\alpha\beta} S_3^{\beta}, \tag{6.4.38}$$

and $S_+ = f_{12} = \sum_{\alpha=1}^{M} f_{12}^{\alpha}$. With $L(\lambda)$ expressed as in (6.4.37), it is easy to show that

$$\{L^1(\lambda), L^2(\mu)\} = [r_{12}(\lambda, \mu), L^1(\lambda)] - [r_{21}(\mu, \lambda), L^2(\mu)]. \tag{6.4.39}$$

In this equation

$$r_{12}(\lambda, \mu) = \begin{pmatrix} \frac{1}{\lambda - \mu} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda - \mu} + \frac{1}{q} & 0 \\ 0 & \frac{1}{\lambda - \mu} - \frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda - \mu} \end{pmatrix}, \tag{6.4.40}$$

and $r_{21}(\lambda, \mu) = \mathcal{P} r_{12}(\lambda, \mu) \mathcal{P}$ while $r_{21}(\mu, \lambda) = -r_{12}(\lambda, \mu)$ so that (6.4.39) reduces to

$$\{L^1(\lambda), L^2(\mu)\} = [r_{12}(\lambda, \mu), L^1(\lambda) + L^2(\mu)], \tag{6.4.41}$$

provided $S_3 = 0$.

The generating function of the integrals of motion is defined as

$$t(\lambda) = \frac{1}{2}\text{tr } (L^2(\lambda)) \tag{6.4.42}$$

and is expressible in the form,

$$t(\lambda) = H + \sum_{\alpha=1}^{M} \frac{H_\alpha}{\lambda - \epsilon_\alpha} + \sum_{\alpha=1}^{M} \frac{G_\alpha}{(\lambda - \epsilon_\alpha)^2}. \tag{6.4.43}$$

Here

$$H = p^2 - \frac{S_- S_+}{q^2}, \tag{6.4.44}$$

$$H_\alpha = \sum_{\beta \neq \alpha}^{M} \frac{2S_3^\alpha S_3^\beta + (S_-^\alpha S_+^\beta + S_-^\beta S_+^\alpha)}{\epsilon_\alpha - \epsilon_\beta} + 2pS_3^\alpha + \frac{1}{q}(S_-^\alpha S_+ - S_+^\alpha S_-), \tag{6.4.45}$$

$$\text{and} \quad G_\alpha = (S_3^\alpha)^2 + S_-^\alpha S_+^\alpha. \tag{6.4.46}$$

One can show that $\{t(\lambda), S_3\} = 0$, so that the reduction $S_3 = 0$ is possible. These integrals of motion are in involution if $S_3 = 0$, which essentially means that $\sum_{\alpha=1}^{M} H_\alpha = 0$. Here, H is the Hamiltonian of the system while each $G_\alpha (\alpha = 1, ...M)$ is a Casimir function. One can show that the following representation of $(S_\pm^\alpha, S_3^\alpha)$ is admissible, in terms of canonical coordinates and momenta (x_α, p_α):

$$S_3^\alpha = \frac{1}{2} x_\alpha p_\alpha, \qquad S_+^\alpha = \frac{1}{2} p_\alpha^2, \qquad S_-^\alpha = -\frac{1}{2} x_\alpha^2, \tag{6.4.47}$$

where $\{x_\alpha, p_\beta\} = \delta_{\alpha\beta}, \alpha, \beta = 1, ..., M$. In this representation the above integrals of motion become

$$\begin{aligned} H &= p^2 + \frac{R^2}{4q^2} \sum_{\alpha=1}^{M} p_\alpha^2, \\ G_\alpha &= 0, \qquad \alpha = 1, ..., M, \\ H_\alpha &= -\frac{1}{4} \sum_{\beta \neq \alpha}^{M} \frac{M_{\alpha\beta}^2}{\epsilon_\alpha - \epsilon_\beta} + p_\alpha x_\alpha p_\alpha + \frac{1}{4q} \sum_{\beta \neq \alpha}^{M} (p_\alpha^2 x_\beta^2 - x_\alpha^2 p_\beta^2), \end{aligned} \tag{6.4.48}$$

where $R^2 = \sum_{\alpha=1}^{M} x_\alpha^2$ and $M_{\alpha\beta} = p_\alpha x_\beta - x_\alpha p_\beta$. The constraint $S_3 = 0$ implies that $\sum_{\alpha=1}^{M} x_\alpha p_\alpha = 0$.

We shall return to this model in Chapter 7 where the method of separation of variables will be applied to its quantum version.

Chapter 7

Separation of Variables

7.1 Introduction

The term "separation of variables" (SoV) appears in various branches of mathematical sciences, due to the preponderance of differential equations. In a sense, it constitutes the simplest approach toward attempting a solution of any dynamical problem. Starting from the simplest differential equations to those governing complicated natural phenomena, its utility has been repeatedly proved. Contrary to the usual notions, the system of equations may be either a set of partial differential equations or ordinary differential equations. In the former, the term SoV usually refers to the separation of the actual coordinates, which may be cartesian or polar variables, as in the hydrogen atom. In the latter, the term refers to the separation of independent dynamical variables. In a completely integrable system, separation of variables assumes utmost importance, as such systems are endowed with an infinite number of conserved quantities. The notion of complete integrability is entwined with the concept of separability. Classical studies of dynamical systems focused on identifying "action-angle" variables, thus in essence achieving separability. Furthermore, it is well known that integrable systems are often bi- or even multi- Hamiltonian in character, and in such cases separation of variables has often been an additional output of this characteristic feature.

In this chapter, we attempt to clarify this concept, with the objective of explaining its occurrence and utility within the broader framework of the quantum inverse scattering problem and the properties of the classical r matrix. However, we begin with some elementary concepts of classical mechanics [113, 114].

7.2 Hamilton-Jacobi Equation

Let us consider a holonomic system that obeys the canonical equations of motion:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, ..., n. \tag{7.2.1}$$

The fundamental problem of dynamics is to search for a canonical transformation so that in the transformed variables $\tilde{q}_i, \tilde{p}_i$ the equations are

$$\frac{d\tilde{q}_i}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \qquad \frac{d\tilde{p}_i}{dt} = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}, \quad i = 1, ..., n, \tag{7.2.2}$$

with the function $\tilde{H}$ being identically zero,

$$\tilde{H} \equiv 0.$$

Then the system (7.2.2) can be integrated to get

$$\tilde{q}_i = \alpha_i, \qquad \tilde{p}_i = \beta_i,$$

where α_i and β_i are $2n$ arbitrary constants. Consequently, by inverting the transformation we can get back (q_i, p_i) as functions of "t" and α_i, β_i's and the dynamical problem is completely solved. However, the identification of an appropriate transformation for this purpose is not always obvious. The clue lies in the fact that the generating function of the transformation S has to satisfy the equation

$$\frac{\partial S}{\partial t} + H(t, p_i, q_i) = 0, \tag{7.2.3}$$

which is nothing but

$$\frac{\partial S}{\partial t} + H(t, q_i, \frac{\partial S}{\partial q_i}) = 0 \tag{7.2.4}$$

where use has been made of the fact that $p_i = \frac{\partial S}{\partial q_i}$. Equation (7.2.3) is known as the Hamilton-Jacobi equation of the system. It may be solved for the generating function, in which the $\tilde{q}_i$'s appear as parameters. Besides (7.2.4), the generating function has to satisfy the condition,

$$\det(\frac{\partial^2 S}{\partial q_i \partial \tilde{q}_k})_{i,k=1}^{n} \neq 0. \tag{7.2.5}$$

Once S has been determined the canonical transformation is obtained from the following relations:

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial \tilde{q}_i} = -\tilde{p}_i, \quad (i = 1, ..., n) \tag{7.2.6}$$

Let us now consider a special form of the Hamiltonian;

$$H = G[f_1(q_i, p_1), ...f_n(q_n, p_n)]. \tag{7.2.7}$$

If the system is a generalized conservative one, then the Hamilton-Jacobi equation becomes

$$\frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i}) = 0 \tag{7.2.8}$$

because $\frac{\partial H}{\partial t} = 0$. Its solution can be sought in the form,

$$S = -ht + V(q_1, ...q_n, \alpha_1, ...\alpha_{n-1}, h), \tag{7.2.9}$$

where $h, \alpha_1, ...\alpha_{n-1}$ are arbitrary constants. Substituting this in (7.2.4) we get

$$H(q_i, \frac{\partial V}{\partial q_i}) = h. \tag{7.2.10}$$

For H as given in (7.2.7), this equation becomes

$$G[f_1(q_1, \frac{\partial V}{\partial q_1}), ...f_n(q_n, \frac{\partial V}{\partial q_n})] = h. \tag{7.2.11}$$

Let us set

$$f_i(q_i, \frac{\partial V}{\partial q_i}) = \alpha_i, \qquad (i = 1, ..., n) \tag{7.2.12}$$

so that

$$h = G(\alpha_1, ...\alpha_n). \tag{7.2.13}$$

Solving for $\frac{\partial V}{\partial q_i}$ we find

$$\frac{\partial V}{\partial q_i} = F_i(q_i, \alpha_i), \qquad (i = 1, ..., n) \tag{7.2.14}$$

so that,

$$V = \sum_{i=1}^{n} \int F_i(q_i, \alpha_i) dq_i, \tag{7.2.15}$$

whence

$$S = -G(\alpha_1, ..\alpha_n)t + \sum_{i=1}^{n} \int F_i(q_i, \alpha_i) dq_i. \tag{7.2.16}$$

In this case

$$\frac{\partial^2 S}{\partial q_i \partial \alpha_i} = \frac{\partial F_i}{\partial \alpha_i}, \qquad \frac{\partial^2 S}{\partial q_i \partial \alpha_n} = 0, \quad i \neq k \text{ and } (i, k = 1, ..., n). \tag{7.2.17}$$

Hence the condition $\det(\frac{\partial^2 S}{\partial q_i \partial \alpha_n})_{i,k=1}^{n} \neq 0$ reduces to the inequality,

$$\prod_{i=1}^{n} \frac{\partial F_i}{\partial \alpha_i} \neq 0. \tag{7.2.18}$$

Since $f_i(q_i, p_i) = \alpha_i$ is equivalent to the equation,

$$p_i = F_i(q_i, \alpha_i), \tag{7.2.19}$$

it follows that

$$\frac{\partial F_i}{\partial \alpha_i} = (\frac{\partial f_i}{\partial p_i})^{-1} \neq 0, \qquad (i = 1, ..., n) \tag{7.2.20}$$

and so condition (7.2.17) holds . Finally one can write

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial \alpha_i} = \beta_i, \quad (i = 1, ..., n), \tag{7.2.21}$$

along with

$$-\frac{\partial G}{\partial \alpha_i} t + \int \frac{dq_i}{(\frac{\partial f_i}{\partial p_i})_{p_i = F_i(q_i, \alpha_i)}} = \beta_i, \tag{7.2.22}$$

so that the system is completely solved by quadrature.

This discussion clearly shows how the system of equations arising from the Hamilton-Jacobi equation may be explicitly solved. One should notice a crucial aspect here, which is that the pairs of canonical variables (p_i, q_i), can be grouped into functions f_i and that there is no mixing.

Let us now turn our attention to a more general situation. Consider a Hamiltonian given by

$$H = g_n\{...g_3\{g_2[g_1(q_1, p_1), q_2, p_2]q_3, p_3\}....q_n, p_n\}. \tag{7.2.23}$$

Then the equation determining V is as follows:

$$g_n\{....g_3\{g_2[g_1(q_1,p_1),q_2,p_2],q_3,p_3\}...q_n,p_n\}=h, \tag{7.2.24}$$

so that

$$g_n\{....g_3\{g_2[g_1(q_1,\frac{\partial V}{\partial q_1}),q_2,\frac{\partial V}{\partial q_2}],q_3,\frac{\partial V}{\partial q_3}\}...q_n,\frac{\partial V}{\partial q_n}\}=h. \tag{7.2.25}$$

To begin with we set

$$\begin{gathered} g_1\left(q_1,\frac{\partial V}{\partial q_1}\right)=\alpha_1, \\ g_2\left(\alpha_1,q_2,\frac{\partial V}{\partial q_2}\right)=\alpha_2, \\ ..... \\ g_n\left(\alpha_{n-1},q_n,\frac{\partial V}{\partial q_n}\right)=\alpha_n. \end{gathered} \tag{7.2.26}$$

The partial derivatives of V are determined successively from these and are

$$\begin{gathered} \frac{\partial V}{\partial q_1}=G_1(q_1,\alpha_1), \\ \frac{\partial V}{\partial q_2}=G_2(q_2,\alpha_1,\alpha_2), \\ ..... \\ \frac{\partial V}{\partial q_n}=G_n(q_n,\alpha_{n-1},\alpha_n). \end{gathered} \tag{7.2.27}$$

Consequently, we can write

$$V=\sum_{k=1}^{n}\int G_k(q_k,\alpha_{k-1},\alpha_k)dq_k, \tag{7.2.28}$$

leading to

$$S=-\alpha_n t+\sum_{i=1}^{n}\int G_i(q_i,\alpha_{i-1},\alpha_i)dq_i. \tag{7.2.29}$$

We can once again test the criterion (7.2.17) and find that

$$\frac{\partial^2 S}{\partial q_i\partial\alpha_i}=\frac{\partial G_i}{\partial\alpha_i},\qquad \frac{\partial^2 S}{\partial q_i\partial\alpha_n}=0,\quad i<k\ (i,k=1,...,n). \tag{7.2.30}$$

Thus the required condition reduces to

$$\prod_{i=1}^{n} \frac{\partial G_i}{\partial \alpha_i} \neq 0, \tag{7.2.31}$$

which is always true because the equation $g_i(\alpha_{i-1}, q_i, p_i) = \alpha_i$ is equivalent to $p_i = G_i(q_i, \alpha_{i-1}, \alpha_i)$ and hence

$$\frac{\partial G_i}{\partial \alpha_i} = \left(\frac{\partial g_i}{\partial p_i}\right)^{-1}_{p_i = G_i(q_i, \alpha_{i-1}, \alpha_i)} \neq 0. \tag{7.2.32}$$

In the sequel we shall need the expressions for the derivative $\frac{\partial G_i}{\partial \alpha_{i-1}}$, which can be calculated from (7.2.31) and (7.2.32), namely

$$\frac{\partial G_i}{\partial \alpha_{i-1}} = -\left(\frac{\frac{\partial g_i}{\partial \alpha_{i-1}}}{\frac{\partial g_i}{\partial p_i}}\right)_{p_i = G_i(q_i, \alpha_{i-1}, \alpha_i)}. \tag{7.2.33}$$

Finally one arrives at the equations,

$$\int \frac{dq_i}{(\frac{\partial g_i}{\partial p_i})_{p_i = G_i(q_i, \alpha_{i-1}, \alpha_i)}} - \int \frac{(\frac{\partial g_{i+1}}{\partial \alpha_i})_*}{(\frac{\partial g_{i+1}}{\partial p_{i+1}})_*} dq_{i+1} = \beta_i, \tag{7.2.34}$$

and

$$-t + \int \frac{dq_n}{(\frac{\partial g_n}{\partial p_n})_{p_n = G_n(q_n, \alpha_{n-1}, \alpha_n)}} = \beta_n \tag{7.2.35}$$

where the $*$ in (7.2.34) implies that the partial derivatives are to be evaluated at $p_{i+1} = G_{i+1}(q_{i+1}, \alpha_i, \alpha_{i+1})$.

This example illustrates a general approach for solving a class of dynamical problems, whose underlying philosophy is the separation of dynamical variables of the system.

In the following our main concern will be to understand how this basic approach must be modified to be applicable to nonlinear integrable systems, governed by a Lax pair. It ought to be mentioned that the methods to be discussed are applicable to both classical and quantum mechanical systems. The initial formulation in this context was due to Sklyanin [115], who first devised the technique and adapted it to the case of a classical system possessing a classical r matrix. Furthermore, it is well known that an integrable system associated with a Lax pair, is usually bi- or at times even multi-Hamiltonian. This gives rise to an alternative approach for the separation of variables, which starts from the multi-Hamiltonian structure and may be used to solve the system in terms of elliptic integrals or Riemann theta function.

7.3 Sklyanin's Method for SoV

It is well known that a discrete integrable system,

$$q_{i,t} = f(q_i, q_{i+1}, ...),$$

can be written in terms of the Lax pair as

$$\Psi_{n+1} = L_n \Psi_n, \qquad \Psi_{n,t} = V_n \Psi_n,$$

where L_n and V_n depend on q_i, q_{i+1}'s and also on a spectral parameter u. When the basic dynamical quantities q_i are canonical, then it is a matter of simple computation to prove that

$$\{L_n(u) \overset{\otimes}{,} L_m(v)\} = [r(u,v), L_n(u) \otimes L_n(v)], \tag{7.3.1}$$

where $r(u,v)$ is the classical r matrix. On the other hand the monodromy matrix is defined by

$$T(u) = \prod_{n=1}^{\overleftarrow{N}} L_n(u). \tag{7.3.2}$$

Equation (7.3.1) immediately implies that

$$\{T(u) \overset{\otimes}{,} T(v)\} = [r(u,v), T(u) \otimes T(v)] \tag{7.3.3}$$

where, for a system with a 2×2 Lax operator $L_n(u)$, one usually represents $T(u)$ as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{7.3.4}$$

If the system is a nonlinear and integrable with D degrees of freedom, then according to the Liouville-Arnold definition of complete integrability [116], it possesses exactly D independent Hamiltonians H_j commuting with respect to the Poisson bracket,

$$\{H_j, H_k\} = 0. \qquad j,k = 1, ..., D.$$

We can cite three basic problems associated with such a system. These are:

- construction of action-angle variables,
- integration of the equations of motion,
- separation of variables.

As far as integration of the equations is concerned, there exists a number of well-defined methods such as the inverse scattering formalism, the Riemann-Hilbert approach and also certain algebrogeometric methods. On the other hand the explicit construction of action-angle variables is in general a difficult problem, although some breakthroughs have been made in certain cases.

We concentrate here mainly on the last problem, which involves searching for D pairs of canonical variables $x_j, p_j, (j = 1, ...D)$ so that:

$$\{x_j, x_k\} = \{p_j, p_k\} = 0, \qquad \{p_j, x_k\} = \delta_{jk} \tag{7.3.5}$$

and D functions ϕ_j so that

$$\phi_j(x_j, p_j, H_1, ..., H_D) = 0, \qquad (j = 1, ..., D). \tag{7.3.6}$$

Note that the transformation from the original dynamical variables to a new set (x_j, p_j) may involve both coordinate and momenta. Furthermore, a classical system is said to be separable if there exist, variables in which the Hamilton-Jacobi equation separates. Traditionally this meant introducing n separation constants so that the Hamilton-Jacobi equation could be replaced by n equations, each involving one such constant.

Sklyanin's method, however, is based on a functional Bethe ansatz (FBA) approach [117], initially formulated by him for systems with Lax operators, whose vacuum did not exist in the usual sense.

In general the method of constructing the separation variables is nontrivial. Certain broad guidelines can, however, be obtained by considering instead the converse problem. According to Jacobi's theorem, if a system is separable then it is integrable, with the integrals of motion corresponding to the separation constants. Thus, given a separation of the Hamiltonian, there is a systematic method for constructing the

integrals of motion.

The procedure for separation of variables may be formulated in two stages. The first stage consists in searching for variables in which the equations are separated, while the second step consists in checking that these variables obey the appropriate commutation relations. In both these steps a crucial feature is that if ζ_i is an eigenvalue of $T(u_i)$, then u_i and ζ_i automatically obey the following separated equation:

$$\det\left(\zeta_i - T(u_i)\right) = 0. \tag{7.3.7}$$

Suppose the spectral invariants of the matrix $T(\lambda)$ are defined as the elementary symmetric polynomials of its eigenvalues:

$$t_\nu(u) = \text{tr } \wedge^\nu T(u), \qquad \nu = 1, ..., N. \tag{7.3.8}$$

For example,

$$t_1(u) = \text{tr } T(u),$$

$$t_2(u) = \frac{1}{2}\{\text{tr } {}^2T(u) - \text{tr } T^2(u)\},$$

$$.....$$

$$t_N(u) = \det T(u) = d(u).$$

An important theorem due to Sklyanin asserts that the nonleading coefficients of the powers of u of the polynomials $t_\nu(u), \nu = 1, ..., (N-1)$ form a commutative family (with respect to the previous Poisson structure) of $MN(N-1)/2$ independent Hamiltonians. This theorem is actually an outcome of the basic theory of classical inverse scattering. Consequently for a $GL(N)$ system there exist functions A and B so that the following properties hold:

• $A(T)$ and $B(T)$ are algebraic functions of degree $D = MN(N-1)/2$ of the elements $T_{\alpha\beta}$ of $T(u)$.

• The variables $x_j, P_j (j = 1, ..., D)$ defined by

$$B(T(x_j)) = 0, \qquad P_j = A(T(x_j))$$

have the Poisson brackets,

$$\{x_j, x_k\} = \{P_j, P_k\} = 0, \qquad \{P_j, x_k\} = P_j\delta_{jk}, \tag{7.3.9}$$

and they are related to the Hamiltonian (invariants) $t_\nu(u)$ by the relation,

$$\det(P_j - T(x_j)) = 0. \tag{7.3.10}$$

While it is possible to have a general proof for these assertions, we shall discuss these issues for specific cases of $N = 2$ and 3, as otherwise algebraic complications may overwhelm the inherent physical ideas. We begin with the $N = 2$ case. The system has M degrees of freedom. The spectral invariants are

$$t(u) = t_1(u) = \text{tr } T(u), \tag{7.3.11}$$

$$d(u) = t_2(u) = \det T(u), \tag{7.3.12}$$

with $t(u)$ containing the M integrals of motion. Let us define $\mathcal{A}(T)$ and $\mathcal{B}(T)$ by

$$\mathcal{A}(T) = T_{11}, \qquad \mathcal{B}(T) = T_{12}, \tag{7.3.13}$$

where x_j, P_j are given by (7.3.9). For the polynomial $\mathcal{B}(u)$ to have M zeros, it is necessary that its leading coefficient be nonzero. This can always be done by means of a similarity transformation $QT(u)Q^{-1}$, which neither affects the basic Poisson structure nor the Hamiltonians. The spectral condition determining P_j is then given by

$$P_j^2 - t(x_j)P_j + d(x_j) = 0, \quad j = 1, ..., M. \tag{7.3.14}$$

Now the classical r matrix equation (7.3.1) yields

$$\{A(u), A(v)\} = \{B(u), B(v)\} = 0, \tag{7.3.15}$$

$$\{A(u), B(v)\} = \frac{A(u)B(v) - B(u)A(v)}{u - v}, \tag{7.3.16}$$

so upon taking the limit as $u \to u_\alpha$, where u_α is a zero of B, we get

$$\{u_\alpha, P_\beta\} = \{P_\alpha, P_\beta\} = 0. \tag{7.3.17}$$

In addition from (7.3.16) one finds

$$\{A(u_\alpha), B(v)\} = -\frac{A(u_\alpha)B(v)}{v - u_\alpha}, \tag{7.3.18}$$

so that

$$\lim_{v \to u_\beta} \{A(u_\alpha), B(v)\} = -\lim_{v \to u_\beta} \frac{A(u_\alpha)B(v)}{v - u_\alpha}. \tag{7.3.19}$$

To proceed further we write $B(u)$ as

$$B(u) = \prod_{\alpha=1}^{N-1} (v - u_\alpha). \tag{7.3.20}$$

Consider next the case $N = 3$, for which $B(v) = (v - u_1)(v - u_2)$, then

$$\begin{aligned}\lim_{v\to u_2} \{A(u_1), B(v)\} &= \lim_{v\to u_2} \{A(u_1), u_2(u_1 - v) + v(v - u_1)\},\\ = \lim_{v\to u_2} &\left[\{A(u_1), u_2\}(u_1 - v) + u_2\{A(u_1), u_1\} - v\{A(u_1), u_1\}\right],\\ &= \{A(u_1), u_2\}(u_1 - u_2),\\ &= -\{A(u_1), u_2\}B'(u_2). \end{aligned}\tag{7.3.21}$$

This simple derivation can easily be extended to the general case N where one has

$$-\{A(u_\alpha), u_\beta\}\frac{dB(v)}{dv}|_{v=u_\beta} = -A(u_\alpha) \lim_{v\to u_\beta} \frac{B(v)}{v - u_\alpha},$$

leading thereby to

$$\{A(u_\alpha), u_\beta\} = \frac{A(u_\alpha)}{B'(u_\beta)} \lim_{v\to u_\beta} \frac{B(v)}{v - u_\alpha}. \tag{7.3.22}$$

Thus if $\alpha \neq \beta$, then the right-hand side is zero. Otherwise for $\alpha = \beta$ it is simply $A(u_\alpha)$. Finally one obtains

$$\{A(u_\alpha), u_\beta\} = A(u_\alpha)\delta_{\alpha\beta}. \tag{7.3.23}$$

Now as

$$\{A(u), B(v)\} = \frac{2u[B(u) - B(v)]}{(u - v)(u + v)}, \tag{7.3.24}$$

one can obtain similarly,

$$\{A(u_\alpha), u_\alpha\} = 1. \tag{7.3.25}$$

Let us next consider the case, where the Lax operator is a 3×3 matrix. The polynomial $T(u)$ is now written as

$$T(u) = \begin{pmatrix} T_{11}(u)\ T_{12}(u)\ T_{13}(u) \\ T_{21}(u)\ T_{22}(u)\ T_{23}(u) \\ T_{31}(u)\ T_{32}(u)\ T_{33}(u) \end{pmatrix}. \tag{7.3.26}$$

The system has $D = 3M$ degrees of freedom. The spectral invariants are

$$t_1(u) = \text{tr } T(u) = \lambda_1 + \lambda_2 + \lambda_3,$$
$$t_2(u) = \frac{1}{2}\left(\text{tr } {}^2T(u) - \text{tr } T^2(u)\right) = \sum \lambda_i \lambda_j,$$
$$d(u) = \det T(u) = \lambda_1\lambda_2\lambda_3. \tag{7.3.27}$$

The characteristic polynomial in this case is

$$\det(\lambda - T(u)) = \lambda^3 - t_1\lambda^2 + t_2\lambda - d(u). \tag{7.3.28}$$

It is convenient to introduce the matrix,

$$Q(T) \equiv T \wedge T,$$
$$= \begin{pmatrix} T_{22}T_{33} - T_{23}T_{32} & T_{23}T_{31} - T_{21}T_{33} & T_{21}T_{32} - T_{22}T_{31} \\ T_{13}T_{32} - T_{12}T_{33} & T_{11}T_{33} - T_{13}T_{31} & T_{12}T_{31} - T_{11}T_{32} \\ T_{12}T_{23} - T_{13}T_{22} & T_{13}T_{21} - T_{11}T_{23} & T_{11}T_{22} - T_{12}T_{21} \end{pmatrix}, \tag{7.3.29}$$

whose elements are the algebraic adjuncts of $T_{\alpha\beta}$. The Poisson brackets of T and Q can be calculated from (7.3.1) to yield

$$\{T^1(u), Q^2(v)\} = -\frac{1}{u-v}[P^{t_2}, T^1(u) \otimes Q^2(v)]. \tag{7.3.30}$$

Similarly we also find

$$\{Q^1(u), Q^2(v)\} = \frac{1}{u-v}[P, Q^1(u)Q^2(v)]. \tag{7.3.31}$$

In component form one can express the above equations as

$$[T_{ab}(u), Q_{cd}(v)] = \frac{1}{u-v}\sum_{r=1}^{3}\left(-\delta_{ac}T_{rb}(u)Q_{rd}(v) + T_{ar}(u)Q_{cr}(v)\delta_{bd}\right), \tag{7.3.32}$$

$$\{Q_{\alpha_1\beta_1}(u), Q_{\alpha_2\beta_2}(v)\} = \frac{1}{u-v}[Q_{\alpha_2\beta_1}Q_{\alpha_1\beta_2(v)} - Q_{\alpha_1\beta_2}(u)Q_{\alpha_2\beta_1}(v)]; \tag{7.3.33}$$

the subscript t_2 in (7.3.30) denotes transposition with respect to the second space.

The preceding 2×2 case suggests that we define x_j as the zeros of some polynomial $B(u)$ of degree $3M$ with the corresponding momenta P_j, related to x_j by the secular equation,

$$P_j^3 - t_1(x_j)P_j^2 + t_2(x_j)P_j - d(x_j) = 0. \tag{7.3.34}$$

It follows from (7.3.28) that P_j is an eigenvalue of the matrix $T(x_j)$. Hence there should be a transformation,

$$T(x_j) \longrightarrow \widetilde{T}(x_j) = K_j T(x_j) K_j^{-1}, \qquad \forall\ j, \tag{7.3.35}$$

so that the matrix $\widetilde{T}(x_j)$ is block triangular so that

$$\widetilde{T}_{12}(x_j) = \widetilde{T}_{13}(x_j) = 0, \quad \text{and } P_j = \widetilde{T}_{11}(x_j). \tag{7.3.36}$$

The problem is therefore reduced to a determination of the matrix K_j and the polynomial $B(u)$. Let us consider $K(k)$ to be as follows:

$$K(k) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.3.37}$$

Note that

$$\widetilde{T}(u,k) = K(k) T(u) K^{-1}(k) \tag{7.3.38}$$

depends on two parameters u and k. Hence we can use the conditions given in (7.3.36), to obtain two algebraic equations:

$$\widetilde{T}_{12}(x,k) = T_{12}(x) + kT_{22}(x) - kT_{11}(x) - k^2 T_{21} = 0, \tag{7.3.39}$$

$$\widetilde{T}_{13}(x,k) = T_{13}(x) + kT_{23}(x) = 0, \tag{7.3.40}$$

for the two variables x and k. Eliminating k between these two equations gives

$$T_{23}(x) U_{31}(x) - T_{13}(x) U_{32}(x) = 0. \tag{7.3.41}$$

Now as $k = -T_{13}(x)/T_{23}(x)$, substituting this in the definition of P_j given by $P_j = \widetilde{T}_{11}(x)$ we get

$$P = T_{11}(x) + kT_{21}(x) = -\frac{U_{32}(x)}{T_{23}(x)}, \tag{7.3.42}$$

giving rise thereby to $3M$ pairs of variables x_j, P_j.

It now remains to check their Poisson brackets. Let

$$\mathcal{A}(T) = -\frac{Q_{32}(T)}{T_{23}}, \quad \mathcal{B}(T) = T_{23} Q_{31}(T) - T_{13} Q_{32}(T). \tag{7.3.43}$$

Setting

$$A(u) = \mathcal{A}(T(11)), \qquad B(u) = \mathcal{B}(T(11)), \tag{7.3.44}$$

one immediately obtains

$$\{A(u), A(v)\} = \{B(u), B(v)\} = 0, \tag{7.3.45}$$

along with

$$\{A(u), B(v)\} = \frac{1}{u-v}\left(A(u)B(v) - B(u)A(v)\frac{T_{23}^2(v)}{T_{23}^2(u)}\right), \tag{7.3.46}$$

that upon simplification leads to the desired result.

In this section we have outlined the basic procedure for separation of variables, though certain features still remain to be clarified such as the precise method of calculation for identifying the separation variables in a specific situation. With this in mind we discuss some concrete examples which will help the reader get a clearer idea of the procedure outlined above.

7.4 Goryachev-Chaplygin Top

A particularly important nonlinear dynamical systems studied within the preceding framework is the Goryachev-Chaplygin (GC) top. The treatment to be given below is due to Sklyanin [118] and exemplifies the intricacies of the procedure involved. The GC top is a Hamiltonian system with a six-dimensional phase space, described by the dynamical variables $x_\alpha, J_\alpha (\alpha = 1, 2, 3)$, and generates the following Lie algebra:

$$\{J_\alpha, J_\beta\} = \epsilon_{\alpha\beta\gamma} J_\gamma, \quad \{J_\alpha, x_\beta\} = \epsilon_{\alpha\beta\gamma} x_\gamma, \quad \{x_\alpha, x_\beta\} = 0. \tag{7.4.1}$$

There are two Casimir invariants,

$$\rho = x_1^2 + x_2^2 + x_3^2 = 1, \quad \sigma = x_1 J_1 + x_2 J_2 + x_3 J_3 = 0, \tag{7.4.2}$$

so that the manifold can be reduced to four dimensions. The GC top is governed by the following Hamiltonian:

$$H = \frac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - bx_1 = \frac{1}{2}(J^2 + 3J_3^2) - bx_1, \tag{7.4.3}$$

where $J^2 = J_1^2 + J_2^2 + J_3^2$ and b is a parameter. It was shown by Chaplygin [124] that the system has another integral of motion,

$$G = 2J_3(J_1^2 + J_2^2) + 2bx_3J_1, \tag{7.4.4}$$

which Poisson commutes with (7.4.2) and (7.4.3).

For analysis of the system, we have to first search for an appropriate Lax operator L and a classical r matrix. The Lax operator will obviously depend on an auxiliary parameter u referred to as the spectral parameter of the Lax equation.

Our analysis will be based upon the relation,

$$\{L(u) \overset{\otimes}{,} L(v)\} = [r(u-v), L(u) \otimes L(v)], \tag{7.4.5}$$

whose immediate implication is that the transfer matrix $t(u) = \text{tr } L(u)$ commutes for different values of the spectral parameter, i.e.,

$$\{t(u), t(v)\} = 0. \tag{7.4.6}$$

Sklyanin in [119] proposed a modified form of the Lax operator, by enlarging the phase space through addition of variables p and q with the following Poisson brackets:

$$\{p, q\} = 1 \quad \{p, J_\alpha\} = \{q, J_\alpha\} = \{p, x_\alpha\} = \{q, x_\alpha\} = 0. \tag{7.4.7}$$

The Lax operator is of the form,

$$L(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{7.4.8}$$

where

$$\begin{aligned} A(u) &= (u + p + 2J_3)K(u) + b(x - u - x_3J_-), \\ B(u) &= be^{2iq}[(x - u - x_3J_+)(u + p + 2J_3) - bx_3^2], \\ C(u) &= e^{-2iq}K(u), \\ D(u) &= b(x + u - x_3J_+), \\ K(u) &= u^2 - 2J_3u - (J^2 - J_3^2) = (u - u_1)(u - u_2), \end{aligned} \tag{7.4.9}$$

where $x_\pm = x_1 \pm i x_2$ and $J_\pm = J_1 \pm i J_2$. One can easily see by a simple calculation that

$$r(u) = \frac{2i}{u}\mathcal{P} \text{ where } \mathcal{P} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}. \tag{7.4.10}$$

In the present case,

$$t(u) = A(u) + D(u) = u^3 + pu^2 - 2H_p u - G_p, \tag{7.4.11}$$

where H_p and G_p are the Casimir invariants introduced earlier. Here the suffix "p" has been used to remind the reader that an additional variable p is also involved. Note that due to the Poisson brackets (7.4.6), p is an integral of motion and hence can be considered as a parameter. Furthermore, u_1 and u_2 are the roots of the quadratic polynomial $C(u)$:

$$C(u_n) = 0, \qquad n = 1, 2. \tag{7.4.12}$$

Next defining $\lambda^\pm$ as follows:

$$\lambda_n^- = A(u_n), \qquad \lambda_n^+ = D(u_n) \qquad n = 1, 2 \tag{7.4.13}$$

and using the values of ρ and σ defined earlier, we see that

$$\det L(u) = A(u)D(u) - B(u)C(u) = d(u) - b^2 u^2, \tag{7.4.14}$$

so that

$$\lambda_n^- \lambda_n^+ = A(u_n)D(u_n) = d(u_n) - b^2 u_n^2. \tag{7.4.15}$$

One can now compute the Poisson brackets of (p, q) and $(\lambda_n^\pm, u_n)$ to get

$$\begin{gathered} \{p, u_n\} = \{p, \lambda_n^\pm\} = \{q, u_n\} = \{q, \lambda_n^\pm\} = 0, \\ \{u_n, u_m\} = 0, \quad \{\lambda_m^\pm, \lambda_n^\pm\} = 0, \ \{\lambda_m^\pm, u_n\} = \pm 2i\lambda_m^\pm \delta_{mn}, \\ \{\lambda_m^+, \lambda_n^-\} = 2id'(u_n)\delta_{mn} = 4ib^2 u_m \delta_{mn}; \end{gathered} \tag{7.4.16}$$

the method is similar to that discussed in the previous section. For example, to calculate $\{\lambda_m^\pm, u_n\}$ we have

$$\begin{aligned} \{\lambda_m^+, u_n\} = \{D(u_m), u_n\} &= D'(u_m)\{u_m, u_n\} + \{D(u), u_n\}|_{u=u_m}, \\ &= -\frac{1}{C'(u_m)}\{D(u), C(u)\}|_{\substack{u=u_m \\ v=u_n}} \end{aligned}$$

$$= -\frac{2i}{C'(u_n)} \frac{D(u)C(v) - C(u)D(v)}{u-v}\Big|_{\substack{u=u_m \\ v=u_n}},$$

$$= 2i\lambda_m^+ \delta_{mn}. \tag{7.4.17}$$

The variables p, q, u_n and $\lambda_n^\pm$ constitute a complete family of dynamical variables, consequently every function on the phase space A, B, etc. is expressible in terms of them. In fact these terms can be uniquely specified by their values at u_n and asymptotic behaviours:

$$C(u) \longrightarrow e^{-2iq}u^2 + \mathcal{O}(u),$$

$$A(u) \longrightarrow u^3 + pu^2 + \mathcal{O}(u),$$

$$\text{and} \qquad D(u) \longrightarrow \mathcal{O}(u), \tag{7.4.18}$$

as $u \to \infty$. Using Lagrange's interpolation formula we can express $C(u)$ as follows:

$$C(u) = e^{-2iq}(u-u_1)(u-u_2), \tag{7.4.19}$$

and $A(u)$ and $D(u)$ as

$$A(u) = (u+p+u_1+u_2)(u-u_1)(u-u_2) + \frac{u-u_2}{u_1-u_2}\lambda_1^- + \frac{u-u_1}{u_2-u_1}\lambda_2^-, \tag{7.4.20}$$

$$D(u) = \frac{u-u_2}{u_1-u_2}\lambda_1^+ + \frac{u-u_1}{u_2-u_1}\lambda_2^+. \tag{7.4.21}$$

The expression for $B(u)$ can be obtained with the help of (7.4.9) and is written as

$$B(u) = e^{2iq}(u+p+u_1+u_2)\left[\frac{u-u_2}{u_1-u_2}\lambda_1^+ + \frac{u-u_1}{u_2-u_1}\lambda_2^+\right] +$$

$$+\frac{e^{2iq}}{u_1-u_2}\left[2b^2u_1u_2 - \lambda_1^+\lambda_2^- - \lambda_1^-\lambda_2^+\right]. \tag{7.4.22}$$

A comparison of (7.4.22) and (7.4.9) shows that

$$u_1+u_2 = 2J_3, \qquad u_1u_2 = J_3^2 - J^2$$

$$\frac{\lambda_1^\pm - \lambda_2^\pm}{u_1-u_2} = bx_\pm, \qquad \frac{u_2\lambda_1^\pm - u_1\lambda_2^\pm}{u_1-u_2} = bx_2J_\pm \tag{7.4.23}$$

along with the following:

$$2H = u_1^2 + u_1u_2 - u_2^2 - (u_1-u_2)^{-1}(\lambda_1^+ + \lambda_1^- - \lambda_2^+ - \lambda_2^-), \tag{7.4.24}$$

$$G = -u_1u_2(u_1+u_2) - (u_1-u_2)^{-1}[u_1(\lambda_2^+ + \lambda - 2^-) - u_2(\lambda_1^+ + \lambda_1^-)]. \tag{7.4.25}$$

Moreover, from the conditions $x_1^* = x_-, J_+^* = J_-$, we obtain the reality condition,

$$\lambda_n^{*\pm} = \lambda_n^{\mp}. \tag{7.4.26}$$

Finally $\lambda_n^{\pm}$ can be expressed as

$$\lambda_n^{\pm} = bu_n e^{\pm 2iw_n}, \tag{7.4.27}$$

with v_n being the momenta canonically conjugate to the coordinates u_n so that

$$\{v_1, v_2\} = 0, \qquad \{u_m, u_n\} = \delta_{mn}. \tag{7.4.28}$$

Thus in this manner a set of canonically conjugate variables may be identified, which are totally separated.

7.5 Quantum Case and the Role of Lie Algebra

The basic technique for separation of variables as, outlined thus far, is valid even in the quantum version of integrable models, though the details of the computation procedure are somewhat different. An important point here is that, at present there exists another variation of the classical equation (7.3.1), given by

$$\{L(u) \otimes L(v)\} = [r(u,v), L(u) \otimes I + I \otimes L(v)]. \tag{7.5.1}$$

This in turn gives rise to

$$\{T(u) \otimes, T(v)\} = [r(u,v), T(u) \otimes I + I \otimes T(v)] \tag{7.5.2}$$

While (7.3.1) leads to a quadratic algebra, these equations lead to a linear one. An important example is that of the Gaudin model [120]. On the other hand there are systems for which (7.5.1) or (7.3.1) have to be extended to the $(r-s)$ structure. However, the basic idea of separation of variables still holds in this case.

In this section we discuss these issues and show how the quantum case can be treated. In this respect an important role is played by the Lie algebra realization of the basic functions on which the polynomials A, B, etc. are constructed.

7.5.1 Gaudin model

Gaudin's model can be considered as a limiting case of integrable quantum chains solvable in the framework of the quantum inverse scattering method. It conforms to the linear relation given in (7.5.1). We consider an inhomogeneous SU(2) spin chain on N nodes with quasi-periodic boundary conditions, defined by the spin variables S_n^α ($\alpha = 1, 2, 3$) and $n = 1, 2, ..., N$ and satisfying the following relations:

$$\sum_{\alpha=1}^{3} (S_n^\alpha)^2 = \ell_n(\ell_n + 1), \tag{7.5.3}$$

$$[S_m^\alpha, S_n^\beta] = i\delta_{mn} \sum_{\gamma=1}^{3} \epsilon^{\alpha\beta\gamma} S_n^\gamma. \tag{7.5.4}$$

The Lax operator for the model under consideration is given by [121]

$$L_n(u) = I + \frac{\eta}{u} \sum_{\alpha=1}^{3} S_n^\alpha \sigma^\alpha = \frac{1}{u} \begin{pmatrix} u + \eta S_n^3 & \eta S_n^- \\ \eta S_n^+ & u - \eta S_n^3 \end{pmatrix}, \tag{7.5.5}$$

where

$$S_n^\pm = S_n^1 \pm S_n^2.$$

The corresponding monodromy matrix is

$$T(u) = e^{\eta g \sigma^3} L_1(u - \delta_1)....L_N(u - \delta_N), \tag{7.5.6}$$

and satisfies the quantum commutation relation,

$$R_{12}(u_1 - u_2) T^1(u_1) T^2(u_2) = T^2(u_2) T^1(u_1) R_{12}(u_1 - u_2). \tag{7.5.7}$$

Here $R_{12}(u)$ stands for the quantum R matrix and is given by

$$R_{12}(u) = uI + \eta \mathcal{P}_{12}. \tag{7.5.8}$$

The Lax operator for the Gaudin model from (7.5.5), in the limit as $\eta \to 0$ and is given by

$$L_n(u) = I + \eta \mathcal{L}_n(u), \tag{7.5.9}$$

with

$$\mathcal{L}_n(u) = \frac{1}{u} \sum_{\alpha=1}^{3} S_n^\alpha \sigma^\alpha. \tag{7.5.10}$$

In a similar manner we have

$$T(u) = 1 + \eta\tau(u) + ..., \tag{7.5.11}$$

where

$$\tau(u) = g\sigma^3 + \sum_{n=1}^{N} \mathcal{L}(u - \delta_n) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.5.12}$$

and

$$\frac{1}{u} R_{12}(u) = I - \eta r_{12}(u), \qquad r_{12}(u) = -\frac{1}{u}\mathcal{P}_{12}. \tag{7.5.13}$$

$\tau(u)$ for the Gaudin model satisfies the algebra,

$$[\tau^1(u_1), \tau^2(u_2)] = [r(u_1 - u_2), \tau^1(u_1) + \tau^2(u_2)]. \tag{7.5.14}$$

The integrals of motion are obtained from

$$\mathcal{I}(u) = \frac{1}{2}\text{tr } (\tau^2(u)), \tag{7.5.15}$$

$$[\mathcal{I}(u), \mathcal{I}(v)] = 0 \qquad \text{forall } u, v. \tag{7.5.16}$$

In explicit form, $\mathcal{I}(u)$ is given by

$$\mathcal{I}(u) = g^2 + \sum_{n=1}^{N} \frac{H_n}{u - \delta_n} + \sum_{n=1}^{N} \frac{\ell_n(\ell_n + 1)}{(u - \delta_n)^2}. \tag{7.5.17}$$

The N independent Hamiltonians are quadratic in the spin variables and are of the form,

$$H_n = 2gS_n^3 + \sum_{m \neq n}^{N} \sum_{\alpha=1}^{3} \frac{2S_n^\alpha S_m^\alpha}{\delta_n - \delta_m}. \tag{7.5.18}$$

The basic problem of determining the joint spectrum of H_n was solved by Gaudin using the coordinate Bethe ansatz [122]. One essentially seeks, the eigenvectors of $\mathcal{I}(u)$ in the form,

$$|v_1, ...v_M\rangle = B(v_1)....B(v_M)|0\rangle, \tag{7.5.19}$$

with the vacuum being defined by

$$C(v)|0\rangle = 0 \qquad \forall v. \tag{7.5.20}$$

The condition for (7.5.19) to be an eigenvector is

$$\Lambda(v_m) = \sum_k \frac{1}{v_m - v_k} \qquad m = 1, ..., M, \tag{7.5.21}$$

where $\Lambda(u)$ is an eigenvalue of the operator $A(u)$ of $\tau(u)$ acting on $|0\rangle$, so that

$$A(u)|0\rangle = \Lambda(u)|0\rangle, \qquad \Lambda(u) = g + \sum_{n=1}^{N} \frac{\ell_n}{u - \delta_n}. \tag{7.5.22}$$

The concept of separation of variables can be applied to this problem, using the functional Bethe ansatz. It involves finding a realization of the representation of $\tau(u)$, with Lie algebra given by (7.5.4), in a suitable function space, in which the generating function $\mathcal{I}(u)$ has a form that allows for identification of the separable variables.

With this in mind, we note that the element $B(u)$ of $\tau(u)$, is a rational function with N simple poles at $u = \delta_n$ and $(N-1)$ zeros and forms a commutative family,

$$[B(u_1), B(u_2)] = 0, \qquad \forall \ u_1, u_2. \tag{7.5.23}$$

One can then introduce mutually commuting operators x, y_j, $(j = 1, ..., (N-1))$ through the asymptotics:

$$B(u) = u^{-1}x + \mathcal{O}(u^{-2}), \qquad \text{as } u \to \infty,$$

and

$$B(y_j) = 0, \qquad j = 1, ..., (N-1).$$

A simple realization of $B(u)$ is as follows:

$$B(u) = x\frac{(u - y_1)....(u - y_{N-1})}{(u - \delta_1).....(u - \delta_N)}, \tag{7.5.24}$$

in the space of functions on the joint spectrum of the operators $x, \{y_j\}(j = 1, ..., (N-1))$. One needs to obtain information about the following:

- spectral analysis of $x, \{y_j\}$
- description of the space of functions on the joint spectrum of $x, \{y_j\}$

• realization in an $\{x, y\}$ representation of the operators $\tau(u)$

To this end we consider the case $N = 1$. Let $\ell \in \{1/2, 1, 3/2, 2, ...\}$ and $\mathcal{C}(x)$ be the ring of polynomials with complex coefficients and $\{x^{2\ell+1}\}$ be the ideal in $\mathcal{C}(x)$. Then the operators,

$$S^3 = -x\frac{d}{dx} + \ell, \quad S^- = x, \quad \text{and } S^+ = -x\frac{d^2}{dx^2} + 2\ell\frac{d}{dx}, \tag{7.5.25}$$

are well defined on the ring and realize a representation of $sl(2)$. This follows from the observation that

$$S^+ x^m = m(2\ell + 1 - m)x^{m-1},$$

and that the space $K^{(\ell)}(\dim = 2\ell+1)$ is generated by the action of S^- on the highest weight vector $q(x) = 1 \in K^\ell$.

Now consider a representation of $\tau(u)$. We realise the operators S_n^α by the differential operators (7.5.25) with respect to variables $\{x_n\}_{n=1,...N}$ in the space $K = \otimes_{n=1}^N K^{(\ell_n)}$. This is nothing but the ring,

$$K = \mathcal{C}[x_1, ...x_n]/X, \tag{7.5.26}$$

of the ring $\mathcal{C}[x_1, ...x_N]$ of the polynomials $\{x_n\}_{n=1,...N}$ by the ideal X, where

$$X = (x_1^{2\ell_1+1}, x_2^{2\ell_2+1}, ...x_N^{2\ell_N+1}).$$

Let us now introduce new variables: $x, \{y_j\}$ as follows

$$x_n \equiv S_n^- = \mathrm{Res}_{u=\delta_n} B(u) = x\frac{\prod_{j=1}^{N-1}(\delta_n - y_j)}{\prod_{m=1}^{N}(\delta_n - \delta_m)}. \tag{7.5.27}$$

It is possible to get a description of the ring K in terms of variables $x, \{y_j\}$ as follows. Let $\widetilde{S}[x, y_1, ...y_{N-1}]$ be the ring of polynomials in $x, \{y_j\}_1^{N-1}$ of the form,

$$\sum_{m=0}^{M} x^m P_m(y_1, ...y_{N-1}) \in \widetilde{S}[x, y_1, ...y_{N-1}], \tag{7.5.28}$$

where the polynomials P_m, are symmetric polynomials in $\{y_j\}$ of degree $\leq m$ in each y_j. We consider the ideals,

$$T_n \subset \widetilde{S}[x, y_1, ...y_{N-1}],$$

with

$$T_n = (x^{2\ell_n+1} \prod_{j=1}^{N-1} (y_j - \delta_n)^{2\ell_n+1}), \tag{7.5.29}$$

$T = (T_1, ...T_N)$. It can be proved [123], that the change of variables $\{x_n\}_{n=1}^N$ to $x, \{y_j\}_{j=1}^{N-1}$ induces an isomorphism and our goal is to make this transformation effective. This change of variable can be effected algebraically in a rather complicated and lengthy manner. However, there exists an elegant procedure due to Sklyanin that we shall follow [117]. If $\tau(u)$ is of the form, (7.5.12) then

(i) $A(u)$ is a rational function of the parameter u with N simple poles at $u = \delta_n$,

(ii) $A(u) = g + u^{-1}\langle S^3\rangle + ...$, where

$$\langle S^3\rangle = x\frac{\partial}{\partial x} + \langle \ell\rangle = -x\frac{\partial}{\partial x} + \sum_{n=1}^{N} \ell_n, \tag{7.5.30}$$

(iii) and $A(u)|_{u=y_j} = \frac{\partial}{\partial y_j} + \Lambda(y_j)$, where

$$\Lambda(u) = g + \sum_{n=1}^{N} \frac{\ell}{u - \delta_n}. \tag{7.5.31}$$

Here $\Lambda(y_j)$ is to be interpreted as the substitution of y_j in $\Lambda(u)$ from the left. This is necessary because we are dealing with operators.

Property (i) is obvious. With regard to (ii) we use the definition of $\tau(u)$ as a product of $L_n(u - \delta_n)$. Since

$$\langle S^3\rangle = \sum_n S_n^3 = \langle \ell\rangle - \sum_{n=1}^{N} x_n \frac{\partial}{\partial x_n}, \tag{7.5.32}$$

therefore from (7.5.27), $\frac{\partial x_n}{\partial x} = \frac{x_n}{x}$, and we have

$$x\frac{\partial}{\partial x} = \sum_n x_n \frac{\partial}{\partial x_n}.$$

With regard to the last property, we observe that

$$A(u)|_{u=y_j} = \left[g - \sum_{n=1}^{N} \frac{S_n^3}{u - \delta_n}\right]_{u=y_j} \left[g - \sum_{n=1}^{N} \frac{\ell_n - x_n \frac{\partial}{\partial x_n}}{u - \delta_n}\right]_{u=y_j},$$

$$= \Lambda(y_j) - \sum_{n=1}^{N} \frac{x_n}{y_j - \delta_n} \frac{\partial}{\partial x_n}. \tag{7.5.33}$$

Upon using the fact that

$$\frac{\partial}{\partial y_j} = \sum_{n=1}^{N} \frac{x_n}{y_j - \delta_n} \frac{\partial}{\partial x_n},$$

we get the result.

The generator of the conserved quantities in the quantum case $\hat{\mathcal{I}}(u)$ has similar properties, which are as follows:

(i) $\hat{\mathcal{I}}(u)$ is a rational function of u with N point poles at $u = \delta_n$ and principal parts $\tau(u) \sim \ell_n(\ell_n + 1)(u - \delta_n)^{-2} u$

(ii) as $u \to \infty$,

$$\hat{\mathcal{I}}(u) = g^2 + u^{-1} 2g\langle S^3 \rangle + ...$$

(iii) $\left[\hat{\mathcal{I}}(u)\right]_{u=y_j} = \frac{\partial^2}{\partial y_j^2} - 2\Lambda(y_j) + \Lambda^2(y_j) - \Lambda'(y_j)$.

The expression for $\hat{\mathcal{I}}(u)$ gives a clear indication of the first two properties. For the third property we observe that

$$\hat{\mathcal{I}}(u) = \frac{1}{2}\mathrm{tr}[\tau^2(u)] = \frac{1}{2}[A^2(u) + D^2(u) + B(u)C(u) + C(u)B(u)].$$

As $\mathrm{tr}\tau[(u)] = 0$ so $A(u) = -D(u)$ and it follows that

$$[B(u), C(v)] = -\frac{1}{u - v}[D(u) - A(u) + A(v) - D(v)],$$

which in the limit $v \to u$ becomes the equation,

$$[B(u), C(u)] = 2A'(u). \tag{7.5.34}$$

Thus substituting we obtain

$$\hat{\mathcal{I}}(u) = A^2(u) - A'(u) + B(u)C(u). \tag{7.5.35}$$

Furthermore, one can show that

$$[A^2(u) - A'(u)]_{u=y_j} = A^2(y_j). \tag{7.5.36}$$

Squaring the expression defining $A(u)$ at $u = y_j$, i.e.,

$$A(u)|_{u=y_j} = A_0(y_j) + A_x(y_j)\frac{\partial}{\partial x} + \sum_{k=1}^{N-1} A_k(y_j)\frac{\partial}{\partial y_j},$$

we get

$$A^2(u)|_{u=y_j} = \frac{\partial^2}{\partial y_j^2} - 2\Lambda(y_j)\frac{\partial}{\partial y_j} + \Lambda^2(y_j) - [\frac{\partial}{\partial y_j}A(u)]_{u=y_j}, \quad (7.5.37)$$

and hence the desired result.

The stage is now set for an effective analysis of separation of variables. The eigenvectors of the algebraic Bethe ansatz is of the form,

$$|v_1, v_2, ...v_M\rangle = B(v_1)....B(v_M)|0\rangle,$$

with the vacuum defined by

$$C(v)|0\rangle = 0 \qquad \forall\ v.$$

For $|v_1, v_2, ...v_M\rangle$ to be an eigenvector of $\hat{\mathcal{I}}(u)\ \forall u$, the parameters v_m should satisfy

$$\Lambda(v_m) = \sum_k \frac{1}{v_m - v_k}, \qquad m = 1, ..., M. \quad (7.5.38)$$

Here $\Lambda(u)$ is the eigenvalue of $A(u)$ acting on $|0\rangle$, that is

$$A(u)|0\rangle = \Lambda(u)|0\rangle, \quad \Lambda(u) = g + \sum_{n=1}^{N} \frac{\ell_n}{u - \delta_n}.$$

The corresponding eigenvalue $i(u)$ of $\hat{\mathcal{I}}(u)$ is then given by

$$i(u) = (\chi(u) - \Lambda(u))^2 + \frac{d}{du}\left(\chi(u) - \Lambda(u)\right), \quad (7.5.39)$$

with

$$\chi(u) \sum_{m=1}^{M} \frac{1}{u - v_m} = \frac{q'(u)}{q(u)},$$

$$q(u) = \prod_{m=1}^{M} (u - v_m).$$

Note that $q(u)$ satisfies the Lamé equation

$$q'' - 2\Lambda q' + (\lambda^2 - \Lambda')q = iq, \tag{7.5.40}$$

which in this case is of the form,

$$q''(u) - 2\left(g + \sum_n \frac{\ell_n}{u - \delta_n}\right) q'(u) + \left(\sum_n \frac{a_n}{u - \delta}\right) q = 0, \tag{7.5.41}$$

where

$$a_n = -H_n + 2\ell_n \left(g + \sum_{n \neq m} \frac{\ell_m}{\delta_n - \delta_m}\right).$$

The above analysis enables us to ascertain that the eigenvector must be of the form,

$$|v_1, ...v_M\rangle = x^M \prod_{j=1}^{N-1} q(y_j), \tag{7.5.42}$$

where the polynomials $q(y_j)$ are solutions of the equation (7.5.40). A more rigorous approach was adopted in the analysis of Sklyanin by considering the space of $x, \{y_j\}$ as a space of jets, which requires further technical modifications in the formulation. We shall not discuss the finer points here.

7.5.2 Quantum DST model

To describe how the technique of the separation of variables can be applied to a quantum mechanical problem involving a linear r matrix algebra, let us go back to the discrete self-trapping dimer problem, which was discussed in the context of the algebraic Bethe ansatz in Chapter 5. It will be recalled that we mentioned there that the same problem could also be associated with a linear r matrix relation, which was also valid for the corresponding monodromy operator. To explain how the separation technique works in such a situation, we consider a new Lax operator for the quantum DST dimer, which is equivalent to the two-site hyperbolic Gaudin magnet [125]. As for the appropriate space of the quantum states, we shall take it to be the representation space of the direct sum of the $SU(1,1)$ Lie algebra with generators $S_\alpha (\alpha = 1, 2)$ satisfying

$$[S^i_\alpha, S^j_\beta] = -i\delta_{\alpha\beta}\epsilon_{jkl}g_{lm}S^m_\alpha, \quad g = \text{diag}\ (1, -1, -1). \tag{7.5.43}$$

The norm of the vector $\vec{S}$ is

$$S_\alpha^2 = (S_\alpha, S_\alpha) = (S_\alpha^1)^2 - (S_\alpha^2)^2 - (S_\alpha^3)^2, \tag{7.5.44}$$

$$\text{with} \quad (S_\alpha, S_\beta) = S_\alpha^1 S_\beta^1 - S_\alpha^2 S_\beta^2 - S_\alpha^3 S_\beta^3. \tag{7.5.45}$$

In terms of these, the Lax operator assumes the form,

$$L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \frac{S_1}{u-a} + \frac{S_2}{u-b} - \frac{1}{2}\begin{pmatrix} 1 & \\ & 0 \\ & & 0 \end{pmatrix}. \tag{7.5.46}$$

It is then easy to check that

$$[L^j(u), L^k(v)] = \frac{i\epsilon_{jkl}}{u-v} g_{lm}[L^m(u) - L^m(v)], \tag{7.5.47}$$

where $j, k, l, m = 1, 2, 3$, or in matrix form,

$$[L^1(u), L^2(v)] = [r(u-v), L^1(u) + L^2(v)]. \tag{7.5.48}$$

Here $r(u) = \frac{\mathcal{P}}{u}$ with $\mathcal{P}$ being the permutation matrix given by

$$\mathcal{P} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}.$$

We now define the quantum determinant of the L operator as

$$\det L(u) = -L^2(u) = -A^2(u) - \frac{1}{2}[B(u), C(u)]_+, \tag{7.5.49}$$

where $[,]_+$ represents the anticommutator. One can now easily verify that

$$[L^2(u), L^2(v)] = 0, \tag{7.5.50}$$

thereby implying that $L^2(u)$ is the generating function of the integrals of motion:

$$L^2(u) = \frac{S_1^2}{(u-a)^2} + \frac{S_2^2}{(u-b)^2} + \frac{H_1}{u-a} + \frac{H_2}{u-b} + \frac{1}{4}, \tag{7.5.51}$$

where

$$H_1 = \frac{2}{a-b}(\vec{S}_1.\vec{S}_2) - S_1^1, \quad \text{and} \quad H_2 = \frac{2}{b-a}(\vec{S}_1.\vec{S}_2) - S_2^1. \tag{7.5.52}$$

Note that the Lax operator in (7.5.46) does not possess a vacuum due to the c number term $1/2(1,0,0)^t$. To proceed with separation of variables, one utilizes the zeros of the operator equation $B(u) = 0$, which are denoted by u_j. At $u = u_j$ the L operator becomes triangular:

$$L(u_j) = \begin{pmatrix} A(u_j) & 0 \\ \star & A(u_j) \end{pmatrix}, \tag{7.5.53}$$

from which we observe that $iv_j \equiv A(u_j), j = 1, 2...., n$ are the *operator eigenvalues* of $L(u)$. Our aim is to solve the spectral problem for the generating function of the integrals of motion, which in this case are given by the operator determinant $-L^2(u) = \det L(u)$. As the determinant of $L(u)$ commutes with the substitution $u \to u_j$, we can express (7.5.49) in terms of u_j and v_j as

$$L^2(u_j) = -v_j^2, \qquad j = 1, ..., n. \tag{7.5.54}$$

Now the linear r matrix algebra can be used to prove that

$$[v_j, u - k] = -i\delta_{jk}, \quad [u_j, u_k] = [v_j, v_k] = 0, \tag{7.5.55}$$

showing that u_j, v_j are canonical operators. This enables us to separate the eigenvalue problem:

$$L^2(u)\Psi = t(u)\Psi \tag{7.5.56}$$

into a set of one-dimensional problems:

$$-v_i^2\psi_i(u_u) = t(u_i)\psi_i(u_i), \tag{7.5.57}$$

with $\Psi = \prod_{i=1}^n \psi_i(u_i)$. Note that here

$$t(u) = -\frac{3}{16}\left[\frac{1}{(u-a)^2} + \frac{1}{(u-b)^2}\right] + \frac{H_1}{u-a} + \frac{H_2}{u-b} + \frac{1}{4}. \tag{7.5.58}$$

Let us now define

$$S_\alpha^1 = \frac{1}{2}(p_\alpha^2 + x_\alpha^2), \quad S_\alpha^2 = \frac{1}{2}(p_\alpha^2 - x_\alpha^2), \quad S_\alpha^3 = \frac{1}{4}[p_\alpha, x_\alpha]_+, \tag{7.5.59}$$

with

$$[p_\alpha, x_\beta] = -i\delta_{\alpha\beta}, \qquad [p_\alpha, p_\beta] = [x_\alpha, x_\beta] = 0, \quad \alpha, \beta = 1, 2. \tag{7.5.60}$$

Then the condition $B(u) = 0$ is equivalent to $-L^1(u) + L^2(u) = 0$, which leads to the condition

$$\frac{x_1^2}{u-a} + \frac{x_2^2}{u-b} = 1, \tag{7.5.61}$$

and whose roots u_1, u_2 satisfy

$$u_1 + u_2 = a + b + x_1^2 + x_2^2, \qquad u_1 u_2 = ab + b x_1^2 + a x_2^2.$$

Incidentally (7.5.61) can also be written as

$$\Theta(u) = \frac{x_1^2}{u-a} + \frac{x_2^2}{u-b} - 1 = -\frac{(u-u_1)(u-u_2)}{(u-a)(u-b)} = 0 \tag{7.5.62}$$

so that upon taking the residues at $u = a, b$ we get

$$x_1^2 = \frac{(u_1-a)(u_2-a)}{b-a}, \qquad x_2^2 = \frac{(u_1-b)(u_2-b)}{a-b}. \tag{7.5.63}$$

For each such u_j we define an additional variable v_j (by substituting from the left):

$$v_j = -A(u_j) = \frac{1}{4(u_j-a)}[x_1, p_1]_+ + \frac{1}{4(u_j-b)}[x_2, p_2]. \tag{7.5.64}$$

The general forms of $A(u), B(u)$ are then given by

$$B(u) = \frac{1}{2} - \frac{S_1^1 - S_1^2}{u-a} - \frac{S_2^1 - S_2^2}{u-b} = \frac{1}{2}\frac{(u-u_1)(u-u_2)}{(u-a)(u-b)}, \tag{7.5.65}$$

$$\begin{aligned} A(u) &= \frac{i}{4(u-a)}[p_1, x_1]_+ + \frac{i}{4(u-b)}[p_2, x_2]_+ \\ &= -2iB(u)\left[\frac{1}{u-u_2}D_2 v_2 + \frac{1}{u-u_1}D_1 v_1\right], \end{aligned} \tag{7.5.66}$$

where

$$D_1 = \frac{(u_1-a)(u_1-b)}{u_1-u_2}, \qquad D_2 = \frac{(u_2-a)(u_2-b)}{u_2-u_1}. \tag{7.5.67}$$

Note that the meromorphic operator-valued function $A(u)$ is obtained by interpolation with the data $A(u_j) = iv_j$, while $B(u_j)$ is obtained from the definition of u_j. Equating the residues at $u = a, b$ one obtains

$$\frac{1}{4}[p_\alpha, x_\alpha]_+ = -x_\alpha^2\left(\frac{1}{y_\alpha - u_2}D_2 v_2 + \frac{1}{y_\alpha - u_1}D_1 v_1\right), \tag{7.5.68}$$

where $y_\alpha = a, b$.

A crucial property of the operators u_j and v_j is that of their conjugation. One can show that

$$u_j^\star = u_j, \qquad D_j v_j = v_j^\star D_j \tag{7.5.69}$$

so that v_j's are not self-adjoint. However, if we construct the operator ω_j in the following manner:

$$\omega_j = \sqrt{D_j} v_j \frac{1}{\sqrt{D_j}}, \qquad v_j = \frac{1}{\sqrt{D_j}} \omega_j \sqrt{D_j} \tag{7.5.70}$$

then it is found to be self-adjoint. Furthermore, ω_j, u_j are also canonical. Substituting $u = u_j$ in (7.5.58) we find that

$$-v_j^2 = -\frac{3}{16}\left[\frac{1}{(u_j-a)^2} + \frac{1}{(u_j-b)^2}\right] + \frac{H_1}{u_j-a} + \frac{H_2}{u_j-b} + \frac{1}{4}, \tag{7.5.71}$$

$j = 1, 2$. Operating on Ψ which is the common eigenfunction of H_1 and H_2, we therefore obtain

$$v_j^2 \Psi + t(u_j)\psi = 0,$$

which is also equivalent to

$$\omega_j^2 \sqrt{D_j}\Psi + t(u_j)\sqrt{D_j}\psi = 0. \tag{7.5.72}$$

Let us now demand that Ψ be factorized in the following form:

$$\Psi = \frac{1}{\sqrt[4]{D_1 D_2}} \exp[(-\frac{1}{2})(u_1+u_2)]\Phi, \quad \text{with} \quad \Phi = \phi_1(u_1)\phi_2(u_2). \tag{7.5.73}$$

Here the factor $(D_1 D_2)^{\frac{1}{4}}$ arises from the Jacobian of the transformation $\{x_j\} \to \{u_j\}$. Finally one gets the following equation for Φ:

$$-\Phi''(u) + \left(1 - \frac{1}{2(u-a)} - \frac{1}{2(u-b)}\right)\Phi'(u) + \left(\frac{\tilde{H}_1}{u-a} + \frac{\tilde{H}_2}{u-b}\right)\Phi(u) = 0. \tag{7.5.74}$$

This equation is of the confluent Heun's type and is well documented in the literature. Thus, we have shown how the techniques of the separation of variables can be applied to a quantum system governed by a linear r matrix structure.

7.6 Bi-Hamiltonian Structure and SoV

It is well known that almost all integrable systems possess more than one Hamiltonian structure. This observation was first made by Magri [126]. Several interesting properties have been deduced from this observation, which in itself is remarkable. Besides, a number of methods have also been developed for deducing the bi-Hamiltonian structure itself.

We begin by describing certain standard features of integrable systems possessing a bi-Hamiltonian structure. To this end, we introduce certain essential terminology.

Let M be a differential manifold, TM and $T^\star M$ its tangent and cotangent bundles, respectively, and suppose θ_0 and $\theta_1 : T^\star M \to TM$ are two compatible Poisson tensors on M. A vector field X is said to be bi-Hamiltonian (BH) with respect to θ_0 and θ_1 if there exist two functions H and $F \in \mathcal{C}^\infty(M)$ so that

$$X = \theta_0 dH = \theta_1 dF, \tag{7.6.1}$$

where dF denotes the differential of F, which is ∇F for finite systems and the variational derivative δF for field systems. If θ_0 is invertible then $\Phi = \theta_1\theta_0^{-1}$ is a Nijenhuis tensor or hereditary operator, which maps a BH vector field to another BH vector field and guarantees the Poisson commutativity of the integrals of motion.

Unfortunately, in many situations θ_0 and θ_1 are not invertible operators, and sometimes they may satisfy a weaker form of the BH condition, known as the quasi bi-Hamiltonian condition (qBH). A vector field is said to be qBH with respect to two Poisson tensors θ_0 and θ_1, if there exist three smooth functions H, F and ρ so that

$$X = \theta_0 \nabla F = \frac{1}{\rho}\theta_1 \nabla F. \tag{7.6.2}$$

The function ρ is known as an integrating factor. On a $2n$-dimensional symplectic manifold M, let $q = (q_1, ...q_n), p = (p_1, ...p_n)$ be a set of canonical coordinates, and $\theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ the canonical Poisson matrix. As θ_0 and θ_1 are compatible and invertible, the Nijenhuis tensor $\Phi = \theta_1\theta_0^{-1}$ is maximal and has n distinct eigenvalues $\mu = (\mu_1, ...\mu_n)$. In the

vicinity of a regular point, where the eigenvalues are distinct, one can always find a canonical transformation $(q,p) \to (\mu,\nu)$ so that θ_1 and Φ have the form:

$$\theta_1 = \begin{pmatrix} 0 & \Lambda_1 \\ -\Lambda_1 & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix} \tag{7.6.3}$$

$$\Lambda_1 = \mathrm{diag}(\mu_1, ...\mu_n). \tag{7.6.4}$$

A qBH vector field is said to be Pfaffian, if an integrating factor ρ in the Nijenhuis coordinates is expressible in the form:

$$\rho = \prod_{i=1}^{n} \mu_i. \tag{7.6.5}$$

One can then show that the corresponding Hamilton-Jacobi equation is separable, by verifying the Levi-Civita conditions. We will illustrate these observations with the help of some constrained flows of integrable systems analysed initially by Zeng and Ma [127].

7.6.1 Jaulent-Miodek spectral problem

Consider the Jaulent-Miodek (JM) spectral problem:

$$\Psi_x = U\Psi, \qquad U = \begin{pmatrix} 0 & 1 \\ \lambda^2 - u_1\lambda - u_0 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \Psi_2 \end{pmatrix}. \tag{7.6.6}$$

The time part of the JM problem may be written as

$$V = \sum_{i=0}^{\infty} V_i \lambda^{-1}, \qquad V_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}. \tag{7.6.7}$$

From the adjoint representation we have

$$V_x = [U, V], \tag{7.6.8}$$

so that the coefficients a_i, c_i, etc. can be obtained in terms of b_i. The recursion relation for b_i is

$$\begin{pmatrix} b_{k+2} \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} b_{k+1} \\ b_k \end{pmatrix}, \tag{7.6.9}$$

where

$$L = \begin{pmatrix} u_1 - \frac{1}{2}D^{-1}u_{1x} & \frac{1}{4}D^2 + u_0 - \frac{1}{2}D^{-1}u_{0x} \\ 1 & 0 \end{pmatrix}. \tag{7.6.10}$$

Consequently, the JM hierarchy can be written in the following manner:

$$U_{t_n} = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}_{t_n} = J \begin{pmatrix} b_{n+2} \\ b_{n+1} \end{pmatrix} = J\frac{\delta H_n}{\delta u}, \tag{7.6.11}$$

where

$$J = \begin{pmatrix} 0 & 2D \\ 2D & -u_{1x} - 2u_1 D \end{pmatrix}, \tag{7.6.12}$$

$$H_n = \frac{1}{n}(2b_{n+3} - u_1 b_{n+2}). \tag{7.6.13}$$

Under the zero boundary conditions we have

$$\frac{\delta\lambda}{\delta u} = \begin{pmatrix} \lambda\psi_1^2 \\ \psi_1^2 \end{pmatrix}, \qquad L\frac{\delta\lambda}{\delta u} = \lambda\frac{\delta\lambda}{\delta u}. \tag{7.6.14}$$

The constrained flow consists of the equations obtained from the spectral problem (7.6.6) for N distinct λ_j and the restriction of the variational derivatives for conserved quantities H_l and λ_j:

$$\psi_{1x} = \psi_2, \qquad \psi_{2x} = \Lambda^2\psi_1 - u_1\Lambda\psi_1 - u_0\psi_1, \tag{7.6.15}$$

$$\frac{\delta H_l}{\delta u} - \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} b_{l+2} \\ b_{l+1} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \langle\lambda\psi_1, \psi_1\rangle \\ \langle\psi_1, \psi_1\rangle \end{pmatrix} = 0, \tag{7.6.16}$$

where $\langle ., . \rangle$ denotes the inner product in $\mathcal{R}^N$.

For $l = 4$ we get as a result,

$$H_4 = \frac{7}{128}u_1^5 + \frac{5}{16}u_1^3 u_0 - \frac{5}{32}u_{1x}^2 u_1 + \frac{3}{8}u_0^2 u_1 - \frac{1}{8}u_{1x}u_{0x}. \tag{7.6.17}$$

Introducing the Jacobi coordinates,

$$q_1 = u_1, \qquad q_2 = u_0,$$

$$p_1 = \frac{\delta H_4}{\delta u_{1x}} = -\frac{5}{16}u_1 u_{1x} - \frac{1}{8}u_{0x}, \qquad p_2 = \frac{\delta H_4}{\delta u_{0x}} = -\frac{1}{8}u_{1x}, \tag{7.6.18}$$

the equations for $l = 4$ are transformed into a finite dimensional Hamiltonian system (FDHS):

$$\psi_{1x} = \frac{\partial F_1}{\partial \psi_2} = \psi_2, \quad q_{1x} = \frac{\partial F_1}{\partial p_1} = -8p_2,$$

$$q_{2x} = \frac{\partial F_1}{\partial p_2} = -8p_1 + 20q_1p_2,$$

$$\psi_{2x} = -\frac{\partial F_1}{\partial \psi_1} = \Lambda^2\psi_1 - q_1\Lambda\psi_1 - q_2\Lambda_1,$$

$$p_{1x} = -\frac{\partial F_1}{\partial q_1} = \frac{35}{128}q_1^4 + \frac{15}{16}q_1^2q_2 - 10p_2^2 + \frac{3}{8}q_2^2 - \frac{1}{2}\langle\lambda\psi_1, \psi_1\rangle,$$

$$p_{2x} = \frac{\partial F_1}{\partial q_2} = \frac{5}{16}q_1^3 + \frac{3}{4}q_1q_2 - \frac{1}{2}\langle\psi_1, \psi_1\rangle. \tag{7.6.19}$$

The entire system can be compactly written as

$$P_x = \theta_0 \nabla F_1. \tag{7.6.20}$$

The various flows of the infinite dimensional system are then given by

$$v_{t_n} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}_{t_n} = J \begin{pmatrix} 2a_n \\ -b_n \end{pmatrix} = J\frac{\delta H_n}{\delta v}, \tag{7.6.21}$$

$$J = \begin{pmatrix} D/2 & 0 \\ 0 & -D/2 \end{pmatrix}, \qquad H_n = -\frac{1}{n}[a_{n,x} - v_1b_n + 2b_{n+1}]. \tag{7.6.22}$$

The corresponding constrained flow is defined by

$$\Phi_{1x} = v_0\Phi_1 + \Lambda\Phi_2, \tag{7.6.23}$$

$$\Phi_{2x} = (\Lambda - v_1)\Phi_1 - v_0\Phi_2, \tag{7.6.24}$$

$$\frac{\delta H_l}{\delta v} + \frac{1}{2}\begin{pmatrix} 2\langle\Phi_1, \Phi_2\rangle \\ \langle\Phi_1, \Phi_1\rangle \end{pmatrix} = 0. \tag{7.6.25}$$

In a similar way for $l = 3$ one finds:

$$H_3 = -\left(\frac{1}{4}v_{0x}^2 - \frac{1}{16}v_{1x}^2 + \frac{1}{4}v_0^4 + \frac{5}{64}v_1^4 - \frac{3}{8}v_{0x}v_1^2 - \frac{3}{8}v_0^2v_1^2\right). \tag{7.6.26}$$

In this case the Jacobi coordinates are

$$\tilde{q}_1 = v_1, \qquad \tilde{q}_2 = v_0,$$

$$\tilde{p}_2 = -\frac{\delta H_3}{\delta v_{0x}} = \frac{1}{2}v_{0x} - \frac{3}{8}v_1^2, \quad \tilde{p}_1 = -\frac{\delta H_3}{\delta v_{1x}} = -\frac{1}{8}v_{1x}. \tag{7.6.27}$$

so that the corresponding FDHS becomes

$$\Phi_{1x} = \frac{\partial \tilde{F}_2}{\partial \Phi_2} = \tilde{q}_2\Phi_1 + \Lambda\Phi_2$$

$$\tilde{q}_{1x} = \frac{\partial \tilde{F}_1}{\partial \tilde{p}_1} = -8\tilde{p}_1, \quad \tilde{q}_{2x} = \frac{\partial \tilde{F}_1}{\partial \tilde{p}_2} = 2\tilde{p}_2 + \frac{3}{4}\tilde{q}_1^2,$$

$$\Phi_{2x} = -\frac{\partial \tilde{F}_1}{\partial \Phi_1} = \Lambda\Phi_1 - \tilde{q}_1\Phi_1 - \tilde{q}_2\Phi_2,$$

$$\tilde{p}_{1x} = -\frac{\partial \tilde{F}_1}{\partial \tilde{q}_1} = -\frac{3}{2}\tilde{q}_1\tilde{p}_2 - \frac{3}{4}\tilde{q}_q\tilde{q}_2^2 - \frac{1}{4}\tilde{q}_1^3 - \frac{1}{2}\langle\Phi_1, \Phi_1\rangle,$$

$$\tilde{p}_{2x} = -\frac{\partial \tilde{F}_2}{\partial \tilde{q}_2} = \tilde{q}_2^3 - \frac{3}{4}\tilde{q}_1^2\tilde{q}_2 - \langle\Phi_1, \Phi_2\rangle. \tag{7.6.28}$$

One can express the whole set as

$$\tilde{P}_x = \theta_0 \nabla \tilde{F}_1, \tag{7.6.29}$$

where

$$\tilde{P} = (\Phi_1^T, \tilde{q}_1, \tilde{q}_2, \Phi_2^T, \tilde{p}_1, \tilde{p}_2)^T, \tag{7.6.30}$$

$$\tilde{F}_1 = -4\tilde{p}_1^2 + \tilde{p}_2^2 + \frac{3}{4}\tilde{q}_1^2\tilde{p}_2 + \frac{3}{8}\tilde{q}_1^2\tilde{q}_2^2 + \frac{1}{16}\tilde{q}_1^4 - \frac{1}{4}\tilde{q}_2^4 +$$

$$+\tilde{q}_2\langle\Phi_1, \Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{2}\langle\Phi_1, \Phi_1\rangle. \tag{7.6.31}$$

For the construction of a qBH, one requires the mapping between the two FDHSs constructed above. The idea is exactly the same as that of the Miura map connecting the KdV and mKdV systems. It is known that a gauge transformation between the JM and the modified JM problem is given by

$$\psi_1 = \Phi_1, \quad \psi_2 = \lambda\Phi_2 + v_0\Phi_1,$$

$$u_1 = v_1, \qquad u_0 = v_{0x} - v_0^2, \tag{7.6.32}$$

which together with (7.6.23) and (7.6.24) gives rise to the map relating (7.6.20) and (7.6.29); that is $P = M(\tilde{P})$ and

$$\psi_1 = \Phi_1, \quad \psi_2 = \Lambda\Phi_2 + \tilde{q}_2\Phi_1,$$

$$q_1 = \tilde{q}_1, \quad q_2 = -2\tilde{p}_2 - \frac{3}{4}\tilde{q}_1^2 - \tilde{q}_2^2,$$

$$p_1 = \tilde{q}_1\tilde{p}_1 + \frac{1}{4}\tilde{q}_2^3 + \frac{1}{2}\tilde{q}_2\tilde{p}_2 - \frac{1}{4}\langle\Phi_1, \Phi_2\rangle, \quad p_2 = \tilde{p}_1. \tag{7.6.33}$$

One then computes the Jacobian of the transformation M to find

$$M'(\tilde{P}) = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2}\tilde{q}_1 & -2\tilde{q}_2 & 0 & 0 & -2 \\ \tilde{q}_2 I & 0 & \Phi_1 & \Lambda & 0 & 0 \\ -\frac{1}{4}\Phi_2^T & \tilde{p}_1 & \frac{3}{4}\tilde{q}_2^2 + \frac{1}{2}\tilde{p}_2 & -\frac{1}{4}\Phi_1^T & \tilde{q}_1 & \frac{1}{2}\tilde{q}_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence the image of θ_0 under M gives the second compatible Poisson tensor,

$$\theta_1 = M'\theta_0 M'^T|_{P=M(\tilde{P})} = \begin{pmatrix} O_{(N+2)\times(N+2)} & A_1 \\ -A_1^T & B_1 \end{pmatrix}, \tag{7.6.34}$$

where

$$A_1 = \begin{pmatrix} \Lambda & -\frac{1}{4}\psi_1 & O_{N\times 1} \\ O_{1\times N} & q_1 & 1 \\ 2\psi_1^T & -\frac{1}{2}q_2 - \frac{15}{8}q_1^2 & -\frac{3}{2}q_1 \end{pmatrix}, \tag{7.6.35}$$

$$B_1 = \begin{pmatrix} O_{N\times N} & \frac{1}{4}\psi_2 & O_{N\times 1} \\ -\frac{1}{4}\psi_2^T & o & p_2 \\ O_{1\times N} & -p_2 & 0 \end{pmatrix}. \tag{7.6.36}$$

Finally the qBH for the FDHS is

$$P_x = \theta_0 \nabla F_1 = \frac{1}{\rho}\theta_1 \nabla E_1, \tag{7.6.37}$$

$$\rho = B(\lambda)|_{\lambda=0} \quad E_1 = [A^2 + BC]_{\lambda=0}. \tag{7.6.38}$$

It remains to construct the Nijenhuis coordinates for the qBH system. As θ_0 and θ_1 are compatible and invertible, the matrix $\theta_1\theta_0^{-1}$ is maximal and has $N+2$ distinct eigenvalues $\mu = (\mu_1, ...\mu_{N+2})$. The explicit transformation to the Nijenhuis set (μ, ν) is given as follows. The eigenvalues $\mu_1, ...\mu_{N+2}$ are defined by the roots of

$$f(\lambda) = \det(\lambda I - A_1) = 0, \tag{7.6.39}$$

while A_1 depends on (ψ_1, q_1, q_2). This gives rise to the following:

$$\mu_j = f_j(\psi_1, q_1, q_2), \qquad j = 1, 2, ..., (N+2), \tag{7.6.40}$$

$$\Psi_{1j} = g_j(\mu), \qquad j = 1, ..., N, \tag{7.6.41}$$

$$q_1 = g_{N+1}(\mu), \qquad q_2 = g_{N+2}(\mu). \tag{7.6.42}$$

Let us introduce the generating function by

$$S = \sum_{j=1}^{N} \psi_{2j} g_j(\mu) + p_1 g_{N+1}(\mu) + p_2 g_{N+2}(\mu), \tag{7.6.43}$$

$$\psi_{1j} = \frac{\partial S}{\partial \psi_{2j}} \qquad j = 1, ..., N, \tag{7.6.44}$$

$$q_1 = \frac{\partial S}{\partial p_1}, \qquad q_2 = \frac{\partial S}{\partial p_2}, \tag{7.6.45}$$

$$\nu_j = \frac{\partial S}{\partial \mu_j} = \sum_j \psi_{2j} \frac{\partial g_j}{\partial \mu_j} + p_1 \frac{\partial G_{N+1}}{\partial \mu_j} + p_2 \frac{\partial g_{N+1}}{\partial \mu_j} \quad j = 1, ..., (N+2). \tag{7.6.46}$$

Then the system of original equations (7.6.28) can be proved to be separable in the variables (μ, ν).

It is interesting to note that the two different approaches described here lead to the same set of separated variables, though we cannot go into the proof of this important assertion due to paucity of space.

7.7 SoV for GCM Model

In this section we again consider the generalization of the Calogero-Moser model described in Chapter 6. The model provides an example of a classical integrable system that is governed by a dynamical r matrix structure. It is described by the Poisson algebra (6.4.41), and the generating function for the integrals of motion is $t(\lambda) = \frac{1}{2}\mathrm{tr}[L^2(\lambda)]$. From (6.4.41) it follows that the integrals of motion are in involution as $\{t(\lambda), t(\mu)\} = 0$. We discuss how the separation of variables can be achieved for this model, using the functional Bethe ansatz.

From (6.4.40) we note that the r matrix for the model is

$$r_{12}(\lambda,\mu) = \begin{pmatrix} a(\lambda,\mu) & 0 & 0 & 0 \\ 0 & 0 & b(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & 0 & 0 \\ 0 & 0 & 0 & a(\lambda,\mu) \end{pmatrix}, \tag{7.7.1}$$

with

$$a(\lambda,\mu) = \frac{1}{\lambda-\mu}, \qquad b(\lambda,\mu) = \frac{1}{\lambda-\mu} + \frac{1}{q}, \quad \text{and} \quad c(\lambda,\mu) = \frac{1}{\lambda-\mu} - \frac{1}{q}.$$

The functions $a(\lambda,\mu), b(\lambda,\mu)$ and $c(\lambda,\mu)$, satisfy appropriate relations so that the Poisson bracket (6.4.41) obeys the Jacobi identity. For the time being, we write the Lax matrix $L(\lambda)$ in the form,

$$L(\lambda) = \begin{pmatrix} A(\lambda)\ B(\lambda) \\ C(\lambda)\ D(\lambda) \end{pmatrix}; \tag{7.7.2}$$

we find upon inserting (7.7.1 and 7.7.2) into (6.4.41) that the nonvanishing Poisson brackets are as follows:

$$\begin{aligned} \{A(\lambda), B(\mu)\} &= b(\lambda,\mu)B(\lambda) + a(\mu,\lambda)B(\mu), \\ \{A(\lambda), C(\mu)\} &= -c(\lambda,\mu)C(\lambda) - a(\mu,\lambda)C(\mu), \\ \{B(\lambda), C(\mu)\} &= 2c(\lambda,\mu)A(\lambda) + 2b(\mu,\lambda)A(\mu). \end{aligned} \tag{7.7.3}$$

From the foregoing discussion it follows that the separation variables may be defined by [128]:

$$B(X_i) = 0 \quad \text{and} \quad P_i = A(X_i), \tag{7.7.4}$$

so that using the algebra (7.7.3), one may show that

$$\begin{aligned} \{X_i, X_j\} &= \{P_i, P_j\} = 0, \\ \{X_i, P_j\} &= \delta_{ij} \lim_{\lambda \to X_i} \frac{a(\lambda, X_i)B(\lambda)}{B'(X_i)}. \end{aligned} \tag{7.7.5}$$

Thus, a sufficient condition for (X_i, P_i) to be canonical variables is that

$$\lim_{\lambda \to X_i} \frac{A(\lambda, X_i)B(\lambda)}{B'(X_i)} = 1,$$

which is equivalent to the condition,

$$a(X_i + h, X_i) = \frac{1}{h} + \mathcal{O}(h). \tag{7.7.6}$$

Indeed, this condition is valid for nondynamical as well as dynamical r matrices. Consequently, one can apply the functional Bethe ansatz technique to the Calogero-Moser model and its variants, which are described by dynamical r matrices. Having defined the separation variables by (7.7.4), we find from

$$t(\lambda) = \frac{1}{2}\mathrm{tr}[L(\lambda)]^2,$$

that

$$t(X_i) = P_i^2, \tag{7.7.7}$$

which are the separated equations.

The Lax matrix for the generalized Calogero-Moser system, in terms of canonical variables (x_α, p_α), is given by [112]:

$$L(\lambda) = \begin{pmatrix} p + \frac{1}{2}\sum_{\alpha=1}^{M} \frac{x_\alpha p_\alpha}{\lambda - \epsilon_\alpha} & \frac{1}{2q}\sum_{\alpha=1}^{M} x_\alpha^2 - \frac{1}{2}\sum_{\alpha=1}^{M} \frac{x_\alpha^2}{\lambda - \epsilon_\alpha} \\ \frac{1}{2q}\sum_{\alpha=1}^{M} p_\alpha^2 + \frac{1}{2}\sum_{\alpha=1}^{M} \frac{p_\alpha^2}{\lambda - \epsilon_\alpha} & -(p + \frac{1}{2}\sum_{\alpha=1}^{M} \frac{x_\alpha p_\alpha}{\lambda - \epsilon_\alpha}) \end{pmatrix}. \tag{7.7.8}$$

The integrals of motion are

$$H = p^2 + \frac{R^2}{4q^2}\sum_{\alpha=1}^{M} p_\alpha^2, \tag{7.7.9}$$

$$G_\alpha = 0, \qquad \alpha = 1, ..., M, \tag{7.7.10}$$

$$H_\alpha = -\frac{1}{4}\sum_{\beta\neq\alpha}^{M} \frac{M_{\alpha\beta}^2}{\epsilon_\alpha - \epsilon_\beta} + p_\alpha x_\alpha p_\alpha + \frac{1}{4q}\sum_{\beta\neq\alpha}^{M} (p_\alpha^2 x_\beta^2 - x_\alpha^2 p_\beta^2); \tag{7.7.11}$$

provided $\sum_\alpha^M x_\alpha p_\alpha = 0$, where $M_{\alpha\beta} = p_\alpha x_\beta - x_\alpha p_\beta$. In this case, the condition $B(X_i) = 0$ reduces to

$$\frac{1}{q}\sum_{\alpha=1}^{M} x_\alpha^2 - \sum_{\alpha=1}^{M} \frac{x_\alpha^2}{\lambda - \epsilon_\alpha} = 0. \tag{7.7.12}$$

This is a polynomial equation of degree M in λ and has M solutions, say $X_i (i = 1, 2, ..., M)$. Using the Vieta theorem, one can show that q and $x_\alpha^2 (\alpha = 1, ..., M)$ are given by the following expressions:

$$q = \sum_{i=1}^{M} X_i - \sum_{\alpha=1}^{M} \epsilon_\alpha, \qquad x_\alpha^2 = \frac{R^2 \prod_{i=1}^{M}(X_i - \epsilon_\alpha)}{q \prod_{\beta\neq\alpha}^{M}(\epsilon_\beta - \epsilon_\alpha)}; \tag{7.7.13}$$

where $R^2 = \sum_{\alpha=1}^{M} x_\alpha^2$. It is clear that $a(\lambda, \mu) = 1/(\lambda - \mu)$ satisfies the condition (7.7.6); consequently the canonical momenta $P_i (i = 1, ...M)$ can be defined by the second part of (7.7.4), which gives us a set of canonical variables $(X_i, P_i)_{i-1}^{M}$ with $\{X_i, P_j\} = \delta_{ij}$. To obtain the separated equations one needs to derive a common solution $S(X_1, ...X_M)$ of the following Hamilton-Jacobi equations:

$$H\left(\frac{\partial S}{\partial X_1}, ..., \frac{\partial S}{\partial X_M}, X_1, ...X_M\right) = E, \tag{7.7.14}$$

$$H_\alpha\left(\frac{\partial S}{\partial X_1}, ..., \frac{\partial S}{\partial X_M}, X_1, ...X_M\right) = E_\alpha, \tag{7.7.15}$$

with $S(X_1, ..., X_M) = \sum_{i=1}^{M} S_i(X_i)$. Note from (7.7.11) that the E_α's must satisfy the constraint $\sum_{\alpha=1}^{M} E_\alpha = 0$ so that from (6.4.43) and (7.7.7) we have

$$t(X_i) = H + \sum_{\alpha=1}^{M} \frac{H_\alpha}{(X_i - \epsilon_\alpha)} = P_i^2 = \left(\frac{dS_i}{dX_i}\right)^2. \tag{7.7.16}$$

Finally from (7.7.14), we get the separated equations in the following form:

$$\left(\frac{dS_i}{dX_i}\right)^2 - E - \sum_{\alpha=1}^{M} \frac{E_\alpha}{(X_i - \epsilon_\alpha)} = 0, \quad (i = 1, 2..., M). \tag{7.7.17}$$

If one considers only a flow generated by H, then E_α's are the arbitrary separation constants. Classically, it remains to express the Hamiltonian H in terms of the separable variables $(X_i, P_i)_{i=1}^{M}$. This can be done by eliminating the constants E_α from (7.7.17) (using the condition $\sum_{\alpha=1}^{M} E_\alpha = 0$) to yield

$$H = \frac{\sum_{\alpha=1}^{M} P_i^2 \prod_{\alpha=1}^{M}(X_i - \epsilon_\alpha) \prod_{j\neq i}^{M}(X_i - X_j)^{-1}}{\sum_{i=1}^{M} \prod_{\alpha=1}^{M}(X_i - \epsilon_\alpha) \prod_{j\neq i}^{M}(X_i - X_j)^{-1}}. \tag{7.7.18}$$

Restricting ourselves to the two-particle case, we shall next consider the quantum counterpart of the above procedure, and obtain a separation of variables in the Schrödinger equations. This will provide an insight not only into the method of quantizing the functional Bethe ansatz, but also the method of dealing with quantum systems governed by dynamical r matrices.

7.7.1 Quantum generalized Calogero-Moser model

To this end, let us consider the generalized Calogero-Moser mode l with Hamiltonian given by [107, 130]

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\sum_{i\neq j}^{N}\sum_{\alpha,\beta=1}^{M} \frac{f_{ij}^{\alpha} f_{ji}^{\beta}}{(q_i - q_j)^2}. \tag{7.7.19}$$

To quantize the system one replaces the Poisson bracket $\{\ ,\ \}$ with the commutator bracket $-i[\ ,\ ]$, with the result that the Lax matrix satisfies the algebra,

$$[L^1(\lambda), L^2(\mu)] = i[r_{12}(\lambda,\mu), L^1(\lambda)] - i[r_{21}(\mu,\lambda), L^2(\mu)]. \tag{7.7.20}$$

The generating function $t(\lambda)$ for the integrals of motion is given by $t(\lambda) = \frac{1}{2}\mathrm{tr}\,[L(\lambda)]$ and r_{12} matrices are of the form (7.7.1). However, as the entries of the r matrix depend on the dynamical variables, hence in the quantum case they are operator-valued functions. We shall refer to (7.7.20) as the Gaudin algebra and for its consistency, it is sufficient that the entries of the r_{12} matrix, viz $a(\lambda,\mu), b(\lambda,\mu)$ and $c(\lambda,\mu)$ satisfy the following:

$$[a(\lambda,\mu), A(\lambda)] = [a(\lambda,\mu), B(\lambda)] = [a(\lambda,\mu), C(\lambda)] = 0$$

$$[b(\lambda,\mu), B(\lambda)] = [c(\lambda,\mu), C(\lambda)] = 0. \tag{7.7.21}$$

When these conditions are fulfilled, the Gaudin algebra becomes

$$[A(\lambda), B(\mu)] = i(b(\lambda,\mu)B(\lambda) + a(\mu,\lambda)B(\mu)),$$

$$[A(\lambda), C(\mu)] = -i(c(\lambda,\mu)C(\lambda) + a(\mu,\lambda)C(\mu)),$$

$$[B(\lambda), C(\mu)] = i([c(\lambda,\mu), A(\lambda)]_+ + [b(\mu,\lambda), A(\mu)]_+), \tag{7.7.22}$$

with $[\ ,\ ]_+$ denoting the anticommutator. In this case, too, the separation variables are formally defined by (7.7.4) and are canonical when (7.7.6) is satisfied. However, care should be taken in the interpretation of (7.7.4) in the quantum case. They are now operator equations and therefore one has to fix the order of the operators appearing in $A(X_i)$ and $B(X_i)$. Let us assume that all the substitutions are done from the left, and that all position operators precede the momenta. One can

then show that the quantum generating function for the commuting integrals of motion is

$$t(\lambda) = p^2 - \frac{[S_-, S_+]_+}{2q^2} + \sum_{\alpha,\beta=1}^{M} \frac{S_3^\alpha S_3^\beta + \frac{1}{2}[S_-^\alpha, S_+^\beta]_+}{(\lambda - \epsilon_\alpha)(\lambda - \epsilon_\beta)},$$

$$+ \sum_{\alpha=1}^{M} \frac{2pS_3^\alpha + \frac{1}{2q}([S_-^\alpha, S_+]_+ - [S_+^\alpha, S_-]_-)}{\lambda - \epsilon_\alpha}, \qquad (7.7.23)$$

where $(S_\pm, S_3)$ are the variables in terms of which the model was written in section (6.4.2). The corresponding conserved quantities are as follows:

$$H = p^2 - \frac{[S_-, S_+]_+}{2q^2},$$

$$G_\alpha = (S_3^\alpha)^2 + \frac{1}{2}[S_-^\alpha, S_+^\alpha]_+, \qquad (7.7.24)$$

$$H_\alpha = \sum_{\beta\neq\alpha}^{M} \frac{2S_3^\alpha S_3^\beta + (S_-^\alpha S_+^\beta + S_-^\beta S_+^\alpha)}{\epsilon_\alpha - \epsilon_\beta} + 2pS_3^\alpha + \frac{1}{2q}([S_-^\alpha, S_+]_+ - [S_+^\alpha, S_-]_+).$$

These integrals of motion play the role of the Hamiltonians of the quantum system and $\sum_{\alpha=1}^{M} H_\alpha = 0$. Furthermore, each G_α is a quadratic Casimir invariant operator of the algebra $so_\alpha(2,1)$ generated by $\vec{S}^\alpha$. If one were to use a specific realization of the spin variables $\vec{S}^\alpha$, in terms of canonical variables, (x_α, p_α) with $[x_\alpha, p_\beta] = i\delta_{\alpha\beta}$, $(\alpha, \beta = 1, ..., M)$, i.e.,

$$S_3^\alpha = \frac{1}{4}(x_\alpha p_\alpha + p_\alpha x_\alpha), \qquad S_+^\alpha = \frac{1}{2}p_\alpha^2, \qquad S_-^\alpha = -\frac{1}{2}x_\alpha^2, \qquad (7.7.25)$$

then the first integrals have the following form, which are the quantum analogs of (7.7.9–7.7.11)

$$H = p^2 + \frac{R^2}{4q^2}\sum_{\alpha=1}^{M} p_\alpha^2,$$

$$G_\alpha = \frac{3}{16}, \qquad \alpha = 1, ..., M, \qquad (7.7.26)$$

$$H_\alpha = -\frac{1}{4}\sum_{\beta\neq\alpha}^{M} \frac{M_{\alpha\beta}^2 + 1/2}{\epsilon_\alpha - \epsilon_\beta} + \frac{1}{2}p[x_\alpha, p_\alpha] + \frac{1}{4q}\sum_{\beta\neq\alpha}^{M}(p_\alpha^2 x_\beta^2 - x_\alpha^2 p_\beta^2).$$

Now $P_i \equiv A(X_i)$ and as $P_i^\dagger \neq P_i$, so the P_i's as defined are not Hermitian. To address this problem we use (7.7.13) to find

$$B(\lambda) = \frac{R^2}{2q} \frac{\prod_{i=1}^M (\lambda - X_i)}{\prod_{\alpha=1}^M (\lambda - \epsilon_\alpha)},$$

$$A(\lambda) = \frac{2q}{R^2} B(\lambda) \left(p + q \sum_{i=1}^M \frac{1}{\lambda - X_i} D_i P_i \right), \tag{7.7.27}$$

where

$$D_i = \frac{\prod_{\alpha=1}^M (X_i - \epsilon_\alpha)}{q \prod_{j \neq i}^M (X_i - X_j)}, \qquad i = 1, 2, ..., M \tag{7.7.28}$$

and R^2 is now an operator. The expression for $A(\lambda)$ has been derived from the behaviour of $A(\lambda)$ at the X_i's and at $\lambda = \infty$. Now as the roots X_i, can be labelled so that $X_1 > ... > X_M$, one can choose the arbitrary parameters ϵ_α so that either $X_M < \epsilon_M < X_{M-1} < ... < X_1 < \epsilon_1$ or $\epsilon_M < X_M < ... < \epsilon_1 < X_1$. Hence each D_i is positive. Moreover using the commutation relation $[X_i, P_j] = i\delta_{ij}$ and

$$\sum_{i=1}^M \frac{1}{\lambda - X_i} \frac{\prod_{\alpha=1}^M (X_i - \epsilon_\alpha)}{\prod_{j \neq i}^M (X_i - X_j)} = \frac{\prod_{\alpha=1}^M (\lambda - \epsilon_\alpha)}{\prod_{i=1}^M (\lambda - X_i)} - 1,$$

one can show that

$$A^\dagger(\lambda) = \frac{2q}{R^2} B(\lambda) \left(p + q \sum_{i=1}^M \frac{1}{\lambda - X_i} P_i^\dagger D_i \right). \tag{7.7.29}$$

Since $A^\dagger(\lambda) = A(\lambda)$ we have as a consequence $D_i P_i = P_i^\dagger D_i$. This allows for the definition of Hermitian operators,

$$\Pi_i \equiv \sqrt{D_i} P_i \frac{1}{\sqrt{D_i}}, \tag{7.7.30}$$

which are canonically conjugate to X_i and are the true separation momenta. The separation equations in terms of Π_i then become

$$\frac{1}{\sqrt{D_i}} \Pi_i^2 \sqrt{D_i} \Psi(X_1, ...X_M) -$$

$$- \left(E + \sum_{\alpha=1}^M \left(\frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{3/16}{(X_i - \epsilon_\alpha)^2} \right) \right) \Psi(X_1, ...X_M) = 0, \tag{7.7.31}$$

$i = 1, ..., M$. Here E, E_α are the eigenvalues of the operators $H, H_\alpha, \alpha = 1, ..., M$. Setting

$$\Psi(X_1, ...X_M) = \sqrt{|qV|} \prod_{i=1}^{M} \Psi_i(X_i) \tag{7.7.32}$$

where V is the Vandermonde determinant, i.e., $V = \prod_{i<j}^{M}(X_i - X_j)$, upon substituting (7.7.32) in (7.7.31) and using $\Pi_i = -i\frac{d}{dX_i}, (i = 1, .., M)$ we have the following separated Schrödinger equations for the generalized Calogero-Moser model under consideration:

$$\frac{1}{\sqrt{C_i}} \frac{d^2}{dX_i^2}(\sqrt{C_i}\Psi_i) + E\Psi_i + \sum_{\alpha=1}^{M} \left(\frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{3/16}{(X_i - \epsilon_\alpha)^2} \right) \Psi_i = 0, \tag{7.7.33}$$

$i = 1, ..., M$ and $C_i = |\prod_{\alpha=1}^{M}(X_i - \epsilon_\alpha)|$.

This completes the construction of the separation of variables in the quantum case for a model governed by a dynamical r matrix, and also provides an insight into the procedure for applying the functional Bethe ansatz in the case of such systems.

7.8 SoV and Boundary Conditions

In this section we shall describe how the technique of the separation of variables can be used to solve the quantum inverse problem for an integrable system, subject to nonperiodic boundary conditions. To illustrate the technique we consider an important discrete model, namely the Toda lattice [129], for which the usual method of algebraic Bethe ansatz is inapplicable owing to the nonexistence of a reference vacuum state.

7.8.1 Periodic Toda lattice

The Hamiltonian of the Toda lattice under periodic boundary conditions is given by

$$H^P = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} \exp(q_{j+1} - q_j), \tag{7.8.1}$$

where N denotes the number of lattice sites and it is assumed that $q_{N+k} = q_k \ \ (k = 1, ..., N)$. The Lax operator for the model is

$$L_j(u) = \begin{pmatrix} u - p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix} \text{with } \ j = 1, ..., N, \tag{7.8.2}$$

and obeys the relation,

$$R(u - v)L_j^1(u)L_j^2(v) = L_j^2(v)L_j^1(u)R(u - v). \tag{7.8.3}$$

The canonical variables (q_j, p_j) satisfy the commutation relation,

$$[p_j, q_k] = -i\hbar\delta_{jk}, \tag{7.8.4}$$

where we have used the standard notation $L^1(u) = L(u) \otimes I, \ \ L^2(u) = I \otimes L(u)$. The quantum $R(u)$ matrix is given by

$$R(u) = \begin{pmatrix} u - i\hbar & 0 & 0 & 0 \\ 0 & u & -i\hbar & 0 \\ 0 & -i\hbar & u & 0 \\ 0 & 0 & 0 & u - i\hbar \end{pmatrix}. \tag{7.8.5}$$

It satisfies the Yang-Baxter equation:

$$R^{12}(u)R^{13}(u + v)R^{23}(v) = R^{23}(v)R^{13}(u + v)R^{12}(u). \tag{7.8.6}$$

The monodromy matrix is defined in the usual way by

$$T(u) = \prod_{j=1}^{\overleftarrow{N}} L_j(u) \tag{7.8.7}$$

and satisfies the following relation:

$$R(u - v)T^1(u)T^2(v) = T^2(v)T^1(u)R(u - v). \tag{7.8.8}$$

It is convenient to write $T(u)$ in the form

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{7.8.9}$$

The transfer matrix is then defined as a trace of the monodromy matrix,

$$t(u) = \text{tr } T(u) = A(u) + D(u), \tag{7.8.10}$$

and it follows immediately from (7.8.8) that

$$[t(u), t(v)] = 0. \tag{7.8.11}$$

From (7.8.8) we get the commutation relations:

$$[C(u), C(v)] = 0, \tag{7.8.12}$$

$$A(u)C(u) = \frac{u - v - i\hbar}{u - v} C(v)A(u) + \frac{i\hbar}{u - v} C(u)A(v), \tag{7.8.13}$$

$$D(u)C(v) = \frac{u - v - i\hbar}{u - v} C(v)D(u) - \frac{i\hbar}{u - v} C(u)D(v). \tag{7.8.14}$$

One should note that the Lax operator (7.8.2) does not possess a pseudovacuum. Hence the usual algebraic Bethe ansatz is not applicable. To overcome this difficulty, we consider the product (7.8.7), from which one can write a general polynomial expansion for the elements of $T(u)$ by explicit multiplication namely:

$$A(u) = u^N - Pu^{N-1} + ..., \tag{7.8.15}$$

$$C(u) = -e^{q_N}(u^{N-1} + ...), \tag{7.8.16}$$

$$D(u) = \mathcal{O}(u^{N-2}). \tag{7.8.17}$$

Consider now the zeros of the operator $C(u)$, defined by

$$C(\hat{u}_\alpha) = 0, \text{ for } \alpha = 1, 2, ..., N - 1. \tag{7.8.18}$$

From (7.8.12) we infer that

$$[\hat{u}_\alpha, \hat{u}_\beta] = 0. \tag{7.8.19}$$

Thus the $\hat{u}_\alpha$'s form a commuting set of operators. Next we introduce another set of quantum operators by substitution of: $\hat{u}_\alpha$ in $A(u)$ and $D(u)$

$$\hat{v}_\alpha^- \equiv A(u \Rightarrow \hat{u}_\alpha), \qquad \hat{v}_\alpha^+ \equiv D(u \Rightarrow \hat{u}_\alpha). \tag{7.8.20}$$

To avoid any confusion in ordering, we shall always assume that these substitutions are done from the left, so that if $A(u) = \sum a_n u^n$, then left substitution means that

$$A(u \Rightarrow \hat{u}_\alpha) = \sum (\hat{u}_\alpha)^n a_n. \tag{7.8.21}$$

From the commutation rules of $A(u), C(u), D(u)$ we get the following:

$$[\hat{v}_\beta^{\pm}, \hat{u}_\alpha] = \pm i\hbar\delta_{\alpha\beta}\hat{v}_\beta^{\pm}, \quad [\hat{v}_\alpha^{+}, \hat{v}_\beta^{+}] = [\hat{v}_\alpha^{-}, \hat{v}_\beta^{-}] = 0. \tag{7.8.22}$$

Also it will be observed that $A(u)$ and $D(v)$ commute among themselves, so that one can deduce the following commutation relations:

$$[\hat{v}_\alpha^{+}, \hat{v}_\beta^{+}] = [\hat{v}_\alpha^{-}, \hat{v}_\beta^{-}] = 0. \tag{7.8.23}$$

By means of the Lagrange interpolation formula, we then obtain the general forms of the elements of the monodromy matrix as given below:

$$C(u) = -e^{-q_N} \prod_{\alpha=1}^{N-1} (u - \hat{u}_\alpha), \tag{7.8.24}$$

$$D(u) = \sum_{\alpha=1}^{N-1} \prod_{\beta\neq\alpha}^{N-1} \frac{u - \hat{u}_\beta}{\hat{u}_\alpha - \hat{u}_\beta} \hat{v}_\alpha^{+}, \tag{7.8.25}$$

$$A(u) = (u - P + \sum_{\alpha=1}^{N-1} \hat{u}_\alpha). \prod_{\alpha=1}^{N-1} (u - \hat{u}_\alpha) + \sum_{\alpha=1}^{N-1} \prod_{\beta\neq\alpha}^{N-1} \frac{u - \hat{u}_\beta}{\hat{u}_\alpha - \hat{u}_\beta} \hat{v}_\alpha^{-}. \tag{7.8.26}$$

One can now set up the Hilbert space on which the canonical operators act. The action of the above operators, are then given by

$$\hat{u}_\alpha |u_1, u_2, ..., u_{N-1}\rangle = u_\alpha |u_1, u_2, ..., u_{N-1}\rangle, \tag{7.8.27}$$

$$\hat{v}_\alpha^{\pm} |u_1, u_2, ..., u_{N-1}\rangle = i^{\pm} |u_1, u_2, ..., u_{N-1}\rangle, \tag{7.8.28}$$

which imply that

$$t(\hat{u}_\alpha)\phi(u_1, ...u_{N-1}) = i^N \phi(u_1, ...u_\alpha + i\hbar, ..., u_{N-1}) +$$

$$+ i^{-N} \phi(u_1, ...u_\alpha - i\hbar, ...u_{N-1}). \tag{7.8.29}$$

Assuming now that the eigenfunction ϕ can be factorized, i.e.,

$$\phi(u_1, ...u_{N-1}) = \prod_{\alpha=1}^{N-1} \varphi(u_\alpha); \tag{7.8.30}$$

we see that (7.8.29) reduces to

$$t(u)\varphi(u) = i^N \varphi(u + i\hbar) + i^{-N} \varphi(u - i\hbar) \tag{7.8.31}$$

so that the eigenvalue of the quantum Toda lattice is given from the solution of a difference equation (7.8.31). Let us write $\varphi(u)$ as an infinite product of the form,

$$\varphi(u) = \prod_k (u - u_k); \tag{7.8.32}$$

then the Bethe *ansatz* equation can be expressed as

$$(-1)^N = \prod_{k \neq j} \frac{u_j - u_k + i\hbar}{u_j - u_k - i\hbar} \tag{7.8.33}$$

We shall now consider the case of nonperiodic boundary conditions.

7.8.2 Nonperiodic case

In case of open boundary conditions, the Hamiltonian of the Toda lattice may be taken in the following form:

$$H = \frac{1}{2}\sum_j p_j^2 + \sum_j \exp(q_{j+1} - q_j) + \alpha_1 e^{q_1} + \frac{\beta_1}{2} e^{2q_1} - \alpha_N e^{-q_N} - \frac{\beta_N}{2} e^{-2q_N} \tag{7.8.34}$$

where $\alpha_1, \beta_1, \alpha_N, \beta_N$ determine the boundary potential. It will be recalled from our previous discussion of the Yang-Baxter that in case of such boundary conditions, we have to consider the modified form of the Yang-Baxter equation, known also as the reflection equation:

$$R^{12}(u-v)\overset{1}{K}(u)R^{21}(u+v)\overset{2}{K}(v) = \overset{2}{K}(v)R^{12}(u+v)\overset{1}{K}(u)R^{21}(u-v), \tag{7.8.35}$$

together with its conjugate equation,

$$R^{21}(u-v)\overset{1}{\bar{K}}(u)R^{12}(u+v)\overset{2}{\bar{K}}(v) = \overset{2}{\bar{K}}(v)R^{21}(u+v)\overset{1}{\bar{K}}(u)R^{12}(u-v). \tag{7.8.36}$$

In this example we assume that the boundary K matrices are as follows:

$$K_1(u) = \begin{pmatrix} \alpha_1 & -u \\ \beta_1 u & \alpha_1 \end{pmatrix}, \qquad K_2(u) = \begin{pmatrix} \alpha_N & -\beta_N u \\ -u & \alpha_N \end{pmatrix}. \tag{7.8.37}$$

In such cases one has to define a new monodromy matrix $U(u)$ given by

$$U(u) = T(u)K_1(u + i\hbar/2)T^{-1}(-u - i\hbar). \tag{7.8.38}$$

It satisfies the following reflection equation:

$$R^{12}(u-v)\overset{1}{U}(u)R^{21}(u=v+i\hbar)\overset{2}{U}(v)=\overset{2}{U}(v)R^{12}(u+v+i\hbar)\overset{1}{U}(u)R^{21}(u-v). \tag{7.8.39}$$

The transfer matrix is defined by

$$\Lambda(u)=\mathrm{tr}\left\{K_2(u-i\hbar/2)U(u)\right\} \tag{7.8.40}$$

$$=(-1)^N\left\{u^{2N+2}-\left(2H+(\frac{i\hbar}{2})^2\right)u^{2N}+....\right\}.$$

Let us write the monodromy matrix as

$$U(u)=\begin{pmatrix}\mathcal{A}(u) & \mathcal{B}(u)\\ \mathcal{C}(u) & \mathcal{D}(u)\end{pmatrix} \tag{7.8.41}$$

and define

$$\mathcal{D}^\star(u)=2u\mathcal{D}(u)+i\hbar\mathcal{A}(u). \tag{7.8.42}$$

These operators satisfy the following commutation rules:

$$[\mathcal{B}(u),\mathcal{B}(v)]=0, \tag{7.8.43}$$

$$\mathcal{A}(u)\mathcal{B}(v)=\frac{(u-v+i\hbar)(u+v+i\hbar)}{(u+v)(u-v)}\mathcal{B}(v)\mathcal{A}(u)-\frac{i\hbar(2v+i\hbar)}{2v(u-v)}\mathcal{B}(u)\mathcal{A}(v)$$

$$+\frac{i\hbar}{2u(u+v)}\mathcal{B}(u)\mathcal{D}^\star(v), \tag{7.8.44}$$

$$\mathcal{D}^\star(u)\mathcal{B}(v)=\frac{-i\hbar(2u-i\hbar)(2v+i\hbar)}{2v(u+v)}\mathcal{B}(u)\mathcal{A}(v)+\frac{i\hbar(2u-i\hbar)}{2v(u-v)}\mathcal{B}(u)\mathcal{D}^\star(v)$$

$$+\frac{(u-v-i\hbar)(u+v-i\hbar)}{(u-v)(u+v)}\mathcal{B}(u)\mathcal{D}^\star(v). \tag{7.8.45}$$

From (7.8.40) we get

$$\Lambda(u)=\alpha_n\frac{2u-i\hbar}{2u}\mathcal{A}(u)+\frac{\alpha_N}{2u}\mathcal{D}^\star(u)-\beta_N\left(u-\frac{i\hbar}{2}\right)\mathcal{C}(u)$$

$$-\left(u-\frac{i\hbar}{2}\right)\mathcal{B}(u). \tag{7.8.46}$$

The absence of a pseudovacuum state for the $L_j(u)$ forces us to again search for the zeros of $\mathcal{B}(u)$. As before by explicit multiplication we can obtain the leading terms of the matrix elements of $U(u)$ as

$$\mathcal{A}(u)=(-1)^Ne^{-q_N}\left(u^{2N}-(p_N+i\hbar/2)u^{2N-1}+.....\right),$$

$$\mathcal{B}(u) = (-1)^{N-1}(u + i\hbar/2)(u^{2N} + ...), \tag{7.8.47}$$

$$\mathcal{D}(u) = (-1)^{N} e^{-q_N} \left(u^{2N} + (p_N + i\hbar/2)u^{2N_1} + ...\right).$$

If $\hat{u}_\alpha$ denote the zeros of $\mathcal{B}(u)$ then we can express it in the form,

$$\mathcal{B}(u) = (-1)^{N-1}(u + i\hbar/2) \prod_{\alpha=1}^{N} (u - \hat{u}_\alpha)(u + \hat{u}_\alpha). \tag{7.8.48}$$

Moreover as in the previous subsection we define

$$\hat{v}_\alpha^+ = \mathcal{A}(u \Rightarrow \hat{u}_\alpha), \qquad \hat{v}_\alpha^- = \mathcal{D}^\star(u \rightarrow \hat{u}_\alpha). \tag{7.8.49}$$

One can show that $\hat{u}_\alpha, \hat{v}_\alpha^\pm$ satisfy the following commutation relations:

$$[\hat{v}_\alpha^\pm, \hat{u}_\beta] = \pm i\hbar\delta_{\alpha\beta}\hat{v}_\beta^\pm, \quad [\hat{v}_\alpha^+, \hat{v}_\beta^+] = [\hat{v}_\alpha^-, \hat{v}_\beta^-] = 0. \tag{7.8.50}$$

Due to the presence of a second set of zeros at $u = -\hat{u}_\alpha$ we may also define

$$\hat{\omega}_\alpha^- = \mathcal{A}(u \Rightarrow -\hat{u}_\alpha), \qquad \hat{\omega}_\alpha^+ = \mathcal{D}^\star(u \Rightarrow -\hat{u}_\alpha). \tag{7.8.51}$$

These quantities satisfy commutation relations similar to (7.8.50) above:

$$[\hat{\omega}_\alpha^\pm, \hat{u}_\beta] = \pm i\hbar\delta_{\alpha\beta}\hat{\omega}_\beta^\pm, \quad [\hat{\omega}_\alpha^+, \hat{\omega}_\beta^+] = [\hat{\omega}_\alpha^-, \hat{\omega}_\beta^-] = 0. \tag{7.8.52}$$

Then by using the Lagrange interpolation formula it is possible to obtain the form of the operators $\mathcal{A}(u), \mathcal{D}^\star(u)$ as

$$\mathcal{A}(u) = \sum_{\alpha=1}^{2N} \prod_{\beta\neq\alpha}^{2N} \frac{u - \hat{X}_\beta}{\hat{X}_\alpha - \hat{X}_\beta} \hat{Y}_\alpha^- + (-1)^N e^{-q_N} \prod_{\alpha=1}^{2N} (u - \hat{X}_\alpha),$$

$$\mathcal{D}^\star(u) = \sum_{\alpha=1}^{2N} \prod_{\beta\neq\alpha}^{2N} \frac{u - \hat{X}_\beta}{\hat{X}_\alpha - \hat{X}_\beta} \hat{Y}_\alpha^+ + (-1)^N e^{-q_N} \prod_{\alpha=1}^{2N} (u - \hat{X}_\alpha), \tag{7.8.53}$$

where $\hat{X}_\alpha = \hat{u}_\alpha, \text{for } 1 \leq \alpha \leq N \text{ and } -\hat{u}_\alpha, \text{for } N+1 \leq \alpha \leq 2N$ while $\hat{Y}_\alpha^\pm = \hat{v}_\alpha^\mp, \text{for } 1 \leq \alpha \leq N \text{ and } \hat{\omega}_\alpha^\pm, \text{for } N+1 \leq \alpha \leq 2N.$

7.8.3 Quantum determinant

At this point we make a brief digression to the notion of quantum determinants, which play an important role in the theory of quantum inverse scattering. Consider the Lax operator of the present problem (7.8.2). It will be observed that

$$\sigma^y L_j(u)\sigma^y L_j^t(u+i\hbar) = I$$

$$L_j^t(u)\sigma^y L_j(u-i\hbar)\sigma^y = I. \tag{7.8.54}$$

Here $L_j^t(u)$ denotes the transpose and σ^y is a Pauli matrix. The inverse of $L_j(u)$ is

$$L_j^{-1}(u) = \begin{pmatrix} 0 & -e^{q_j} \\ e^{q_j} & u-p_j+i\hbar \end{pmatrix} = \sigma^y L_j^t(u+i\hbar)\sigma^y. \tag{7.8.55}$$

The quantum determinant $d(u)$ is defined by

$$d(u) = \text{tr } P_{12}^- L_j(u) \otimes L_j(u+i\hbar). \tag{7.8.56}$$

Note that the inverse of $L_j(u)$ may be calculated with the aid of the quantum determinant. In (7.8.56) P_{12}^- denotes the antisymmetrizer in the auxiliary space. Now from (7.8.7) it is clear that $T(u)$ also satisfies the identities:

$$\sigma^y T(u)\sigma^y T^t(u+i\hbar) = 1, \quad T^t(u)\sigma^y T(u-i\hbar)\sigma^y = 1. \tag{7.8.57}$$

These relations imply that the quantum determinant of $T(u)$ equals unity and thus commutes with every matrix element of the monodromy matrix $T(u)$. In terms of the elements of $T(u)$ as given in (7.8.9) we find that

$$\begin{aligned} D(u)A(u+i\hbar) - C(u)B(u+i\hbar) &= 1, \\ D(u)C(u+i\hbar) - C(u)D(u+i\hbar) &= 0, \\ A(u)B(u+i\hbar) - B(u)A(u+i\hbar) &= 0, \\ A(u)D(u+i\hbar) - B(u)C(u+i\hbar) &= 1, \end{aligned} \tag{7.8.58}$$

along with a similar set of relations with $i\hbar$ replaced with $-i\hbar$:

$$\begin{aligned} A(u)D(u-i\hbar) - C(u)B(u-i\hbar) &= 1, \\ C(u)A(u-i\hbar) - A(u)C(u-i\hbar) &= 0, \\ B(u)D(u-i\hbar) - D(u)B(u-i\hbar) &= 0, \\ D(u)A(u-i\hbar) - B(u)C(u-i\hbar) &= 1. \end{aligned} \tag{7.8.59}$$

To formulate the quantum determinant associated with the reflection equation (7.8.35) we introduce the matrix,

$$\bar{U}(u) = \begin{pmatrix} 2u\mathcal{D}(u) + i\hbar\mathcal{A}(u) & (i\hbar - 2u)\mathcal{B}(u) \\ (i\hbar - 2u)\mathcal{C}(u) & 2u\mathcal{A}(u) + i\hbar\mathcal{D}(u) \end{pmatrix}. \tag{7.8.60}$$

By using the reflection equation (7.8.39) and (7.8.60) we find that

$$\bar{U}(u)(u + i\hbar/2)U(u - i\hbar/2) = U(u - i\hbar/2)\bar{U}(u)(u + i\hbar/2) = \Delta(u), \tag{7.8.61}$$

where $\Delta(u)$ is the quantum determinant of $U(u)$. Various relations between $\hat{u}_\alpha, \hat{\omega}_\alpha$, etc. can now be deduced from the above equation. For example,

$$\mathcal{D}(u)\mathcal{A}(u - i\hbar) + (i\hbar - 2u)\mathcal{B}(u)\mathcal{C}(u - i\hbar) = \Delta(u - i\hbar/2), \tag{7.8.62}$$

in which one considers the limit as $u \to \hat{u}_\alpha$, then

$$\Delta(\hat{u}_\alpha - i\hbar/2) = \lim_{u \Rightarrow \hat{u}_\alpha} \mathcal{D}(u)\mathcal{A}(u - i\hbar) = \sum n, m\hat{u}_\alpha^n(\hat{u}_\alpha - i\hbar)^m d_n a_m,$$

$$= \sum_{n,m}(\hat{u}_\alpha - i\hbar)^m(\hat{u}_\alpha^n d_n)a_m = \sum_m (\hat{u}_\alpha - i\hbar)^m \hat{v}_\alpha^- a_m = \sum_m \hat{v}_\alpha^- \hat{u}_\alpha^m a_m;$$

hence

$$\Delta(\hat{u}_\alpha - i\hbar/2) = \hat{v}_\alpha^- \hat{v}_\alpha^+. \tag{7.8.63}$$

Similarly one can deduce the following:

$$\Delta(-\hat{u}_\alpha - i\hbar/2) = \hat{\omega}_\alpha^+ \hat{\omega}_\alpha^-, \tag{7.8.64}$$

$$\delta(\hat{u}_\alpha + i\hbar/2) = \hat{v}_\alpha^+ \hat{\omega}_\alpha^-, \tag{7.8.65}$$

$$\delta(-\hat{u}_\alpha + i\hbar/2) = \hat{\omega}_\alpha^- \hat{v}_\alpha^+. \tag{7.8.66}$$

Let us go back to (7.8.46) and substitute $u = \hat{u}_\alpha$, giving

$$2\hat{u}_\alpha\Lambda(\hat{u}_\alpha) = \alpha_N(2\hat{u}_\alpha - i\hbar)\hat{v}_\alpha^+ + \alpha_N\hat{v}_\alpha^- - \beta_N \lim_{u \Rightarrow \hat{u}_\alpha} u(2u - i\hbar)\mathcal{C}(u). \tag{7.8.67}$$

From (7.8.67) it is apparent that if we demand $\beta_N = 0$ then it will entail complete separation. Having obtained the commutation relations of $\hat{u}_\alpha, \hat{v}_\alpha^\pm$ and $\hat{\omega}_\alpha^\pm$ we can now set up the Hilbert space on which the quantum canonical operators act, in the following manner:

$$\hat{u}_\alpha|u_1, ..., u_N\rangle = u_\alpha|u_1, ..., u_N\rangle, \tag{7.8.68}$$

$$\hat{v}_\alpha^{\pm}|u_1, ..., u_N\rangle = \Delta^{\pm}(u_\alpha)|u_1, ..., u_\alpha \pm i\hbar, ..., u_N\rangle, \tag{7.8.69}$$

$$\hat{\omega}_\alpha^{\pm}|u_1, ..., u_N\rangle = \delta^{\pm}(u_\alpha)|u_1, ..., u_\alpha \pm i\hbar, ..., u_N\rangle. \tag{7.8.70}$$

The functions $\Delta^{\pm}(u), \delta^{\pm}(u)$ can be chosen so as to satisfy the identities (7.8.63–7.8.66) by setting

$$\Delta^{+}(u) = \delta^{-}(-u) = \alpha_1 + i\sqrt{\beta_1}(u + i\hbar/2), \tag{7.8.71}$$

$$\Delta^{-}(u) = \delta^{+}(-u) = (2u + i\hbar).(\alpha_1 - i\sqrt{\beta_1}(u - i\hbar/2)). \tag{7.8.72}$$

The eigenvalue problem for the transfer matrix $\Lambda(u)$ can now be expressed as

$$2u_\alpha\Lambda(u_\alpha)\Psi(u_1, ..., u_N) = \alpha_N(2u_\alpha - i\hbar)\Delta^{+}(u_\alpha)\Psi(u_1, .., u_\alpha + i\hbar, ..., u_N)$$
$$+\alpha_N\Delta^{-}(u_\alpha)\psi(u_1, ..., u_\alpha - i\hbar, ...u_N). \tag{7.8.73}$$

As before we assume that the eigenfunction Ψ may be factorized in the form

$$\Psi(u_1, ..., u_N) = \prod_{\alpha=1}^{N} \psi(u_\alpha), \tag{7.8.74}$$

whence we get from (7.8.73):

$$2u\Lambda(u)\psi(u) = \alpha_N(2u - i\hbar)\Delta^{+}(u)\psi(u + i\hbar) + \alpha_N\Delta^{-}(u)\psi(u - i\hbar). \tag{7.8.75}$$

This is a one-dimensional difference equation; to solve it we set

$$\psi(u) = \prod_k (u - u_k)(u + u_k), \tag{7.8.76}$$

leading thereby to the Bethe ansatz equation in the form

$$, \frac{\alpha_1 + i\sqrt{\beta_1}(u_j + i\hbar/2)}{\alpha_1 - i\sqrt{\beta_1}(u_j - i\hbar/2)} = \prod_{k\neq j} \frac{(u_j - u_k - i\hbar)(u_j + u_k - i\hbar)}{(u_j - u_k + i\hbar)(u_j + u_k + i\hbar)}. \tag{7.8.77}$$

This example illustrates the basic steps involved in the application of the separation of variables to the quantum inverse scattering method, particularly for systems that do not possess a pseudovacuum state. The same procedure may be applied to continuous integrable systems and such a case was studied by Dasgupta et al. [131]. On the other hand, the technique is also applicable, when the Lax matrix is of dimension greater than 2×2, although in that case the computations become rather involved. The basic theorems related to the fundamental properties of separation of variables are discussed in detail, in the light of the inverse scattering theory in Sklyanin's celebrated Nankai Lectures [132], and also in the review article [133]. We refer the interested reader to these articles for further details.

Chapter 8

Bäcklund Transformations

8.1 Introduction

Bäcklund transformations (BT) refer to transformations between the solutions of either the same or two different differential equations. Their origin may be traced back to the 1875 studies of A. V. Bäcklund on pseudospherical surfaces [134], i.e., surfaces of constant negative curvature. On such surfaces the line element in terms of suitable coordinates u and v may be expressed as

$$ds^2 = \alpha^2(du^2 + 2\cos\omega du dv + dv^2), \tag{8.1.1}$$

where $-1/\alpha^2$ is the constant total curvature of the surface and ω is the angle between the asymptotic lines. It may be shown that the angle ω satisfies [135]

$$\frac{\partial^2\omega}{\partial u \partial v} = \sin\omega, \tag{8.1.2}$$

which is known as the Sine-Gordon equation. A solution of this equation corresponds to a surface of constant negative curvature. In trying to generate such surfaces Bäcklund discovered that a new solution ω_1 could be obtained from a given solution ω_0 by means of the following transformation:

$$\frac{\partial}{\partial u}\left(\frac{\omega_1 - \omega_0}{2}\right) = a\sin\left(\frac{\omega_1 + \omega_0}{2}\right), \tag{8.1.3}$$

$$\frac{\partial}{\partial v}\left(\frac{\omega_1 + \omega_0}{2}\right) = a^{-1}\sin\left(\frac{\omega_1 - \omega_0}{2}\right), \tag{8.1.4}$$

where a is an arbitrary constant. It is obvious that for such a transformation to be of any practical use one must be able to find the solution ω_0. However, the efficacy of such transformations rests on the fact that in many cases, the initial solution ω_0 can often be obtained by inspection. In the case of (8.1.2), for example, it is seen that $\omega_0 = 0$ is a solution and can therefore be used for generating new solutions.

8.2 Permutability Theorem

It is also evident from (8.1.2) that the procedure for deriving a new solution ω_1 involves a quadrature. However, Bianchi [136] showed that a new solution could also be obtained, even without the use of quadrature. This procedure, known as the *theorem of permutability*, relates four solutions without the use of quadrature. Indeed since (8.1.3 and 8.1.4) actually represent a transformation from a solution ω_0 to a solution ω_1 with constant a, it may be schematically represented by Figure 8.2.1.

Geometrically, (8.1.3) and (8.1.4) signify that a surface characterized by ω_0 can be transformed into a new surface S_1 (of the same curvature) by means of ω_1 and a_1. The theorem of permutability then states that if S_1 and S_2 are transforms of S by means of respective pairs of functions (ω_1, a_1) and (ω_2, a_2), then a function ω_3 can be found without quadrature so that by means of the (ω_3, a_2) and (ω_3, a_1), the surfaces S_1 and S_2 can be transformed into the surface S'. The essential content of the permutability theorem is schematically depicted by Figure 8.2.2. From (8.1.3 and 8.1.4) the four u derivative equations associated with Figure 8.2.2 are

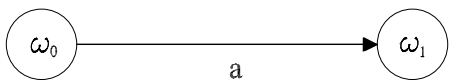

FIGURE 8.2.1: Schematic diagram of the Bäcklund transformation given in (8.1.3) $\omega_0 \rightarrow \omega_1$.

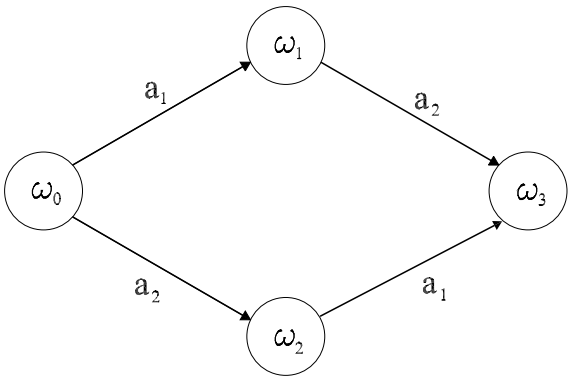

FIGURE 8.2.2: Schematic form of transformation occurring in the permutability theorem.

$$\frac{\partial}{\partial u}\left(\frac{\omega_1-\omega_0}{2}\right)=a_1\sin\left(\frac{\omega_1+\omega_0}{2}\right), \tag{8.2.1}$$

$$\frac{\partial}{\partial u}\left(\frac{\omega_2-\omega_0}{2}\right)=a_2\sin\left(\frac{\omega_2+\omega_0}{2}\right), \tag{8.2.2}$$

$$\frac{\partial}{\partial u}\left(\frac{\omega_3-\omega_1}{2}\right)=a_2\sin\left(\frac{\omega_3+\omega_1}{2}\right), \tag{8.2.3}$$

$$\frac{\partial}{\partial u}\left(\frac{\omega_3-\omega_2}{2}\right)=a_1\sin\left(\frac{\omega_3+\omega_2}{2}\right), \tag{8.2.4}$$

Equations (8.2.1–8.2.4) can be algebraically manipulated to obtain a form that is completely independent of the u derivatives to yield

$$\tan\left(\frac{\omega_3-\omega_0}{4}\right)=\frac{a_1+a_2}{a_1-a_2}\tan\left(\frac{\omega_1-\omega_2}{4}\right). \tag{8.2.5}$$

It was shown by Bianchi that if the functions ω_3 in (8.2.3) and (8.2.4) are replaced by ω_4 and ω_5 respectively, then with ω_1 and ω_2 given by (8.2.1) and (8.2.2), differentiation of (8.2.5) implies the validity of (8.2.3) and (8.2.4) with the same ω_3 in both equations. The theorem of permutability thus provides an efficient procedure for generating solutions of (8.1.2).

Another systematic procedure for deriving Bäcklund transformations is by Clarin's method [137].

8.3 Bäcklund Transformations and Classical Inverse Scattering

In the context of classical inverse scattering theory, there exists a close interrelation between the inverse method, the infinite conservation laws that are characteristic of intergrable systems, and Bäcklund transformations. These interrelated features were extensively studied by Wadati et al. [138]. Indeed, it is possible to reduce the Bäcklund transformations to the fundamental equations of the inverse method by the introduction of auxiliary fuctions, a procedure first illustrated by Wahlquist and Estabrook [139]. Conversely, it is also possible to derive Bäcklund transformations from the inverse method. In this section we shall deal with these issues and consider the KdV equation, for which the space part of the Lax operator is the well-known Schrödinger equation:

$$-\psi_{xx} + \eta^2\psi = q\psi. \tag{8.3.1}$$

Let $\psi_0(x)$ be some particular solution of (8.3.1) for spectral parameter $\eta = \eta_0$, and $\psi(x,\eta)$ be an arbitrary solution of (8.3.1). Next consider the transformation,

$$\psi'(x,\eta) = W\{\psi(x,\eta);\psi_0(x)\}/[(\eta^2 - \eta_0^2)\psi_0(x)], \tag{8.3.2}$$

where $W\{\psi,\phi\}$ is the Wronskian of the functions ψ and ϕ,

$$W\{\psi,\phi\} = \psi_x\phi - \psi\phi_x. \tag{8.3.3}$$

One can now state the following theorem.

Theorem I: The function $\psi'(x,\eta)$ defined by (8.3.2) is a solution of (8.3.1) with the potential $q'(x) = q(x) + \Delta q(x)$ where

$$\Delta q(x) = 2\left[\frac{\frac{\partial\psi_0(x)}{\partial x}}{\psi_0(x)}\right]_x = 2(\log\psi_0(x))_{xx} \tag{8.3.4}$$

Proof: For any two solutions of (8.3.1) one can show that the following relation is satisfied:

$$\frac{\partial}{\partial x}\left[\frac{1}{\eta_1^2 - \eta_2^2}W\{\psi(x,\eta_1),\psi(x,\eta_2)\}\right] = \psi(x,\eta_1)\psi(x,\eta_2). \tag{8.3.5}$$

Using (8.3.2) one obtains from (8.3.1)

$$\frac{\partial}{\partial x}\psi'(x,\eta) = \psi(x,\eta) - \psi'(x,\eta)v(x), \tag{8.3.6}$$

where

$$v(x) = \frac{1}{\psi_0(x)}\frac{\partial\psi_0(x)}{\partial x} = (\log\psi_0(x))_x. \tag{8.3.7}$$

Moreover, $v(x)$ satisfies the Riccati equation,

$$\frac{\partial v(x)}{\partial x} + v^2(x) = -q(x) + \eta_0^2. \tag{8.3.8}$$

Differentiating (8.3.1) once and using (8.3.6), (8.3.7) and (8.3.2) we then obtain the result,

$$-\psi'_{xx}(x,\eta) + \eta^2\psi'(x,\eta) = q'(x)\psi'(x,\eta). \tag{8.3.9}$$

The transformation (8.3.2) is valid for $\eta \neq \eta_0$. For $\eta = \eta_0$, two independent solutions of the transformed (8.3.9) are found to be

$$\psi'_{01}(x) = 1/\psi_0(x), \tag{8.3.10}$$

$$\psi'_{02}(x) = \psi'_{01}\int^x dy/\psi'^2_{01}(y) = \int^x dy\frac{\psi_0^2(y)}{\psi_0(x)}. \tag{8.3.11}$$

We then have the following theorem.

Theorem II: The transformation inverse to (8.3.2) is given by

$$\psi(x,\eta) = W\{\psi'(x,\eta);\psi'_{01}(x)\}/\psi'_{01}(x). \tag{8.3.12}$$

Proof: From (8.3.6) we have

$$\psi(x,\eta) = \frac{\partial\psi'(x,\eta)}{\partial x} + \psi'(x,\eta)\frac{1}{\psi_0(x)}\frac{\partial\psi_0(x)}{\partial x}, \tag{8.3.13}$$

so upon using (8.3.10) one obtains

$$\psi(x,\eta) = \frac{\partial\psi'(x,\eta)}{\partial x} - \frac{\psi'(x,\eta)}{\psi'_{01}(x)}\frac{\partial\psi'_{01}(x)}{\partial x} = W\{\psi'(x,\eta);\psi'_{01}(x)\}/\psi'_{01}(x).$$

Now a transformation of the form (8.3.2), is closely related to Bäcklund transformations by making the following identification. Suppose that

$q^{(N)}$ and $q^{(N+1)}$ are N and $N+1$ soliton solutions, respectively. Since a N soliton solution is known to have N bound states, hence

$$-\frac{\partial^2\psi_i}{\partial x^2} + \eta_i^2\psi_i = q^{(N)}\psi_i, \tag{8.3.14}$$

with $\psi_i \to 0$ as $|x| \to \infty, i = 1, 2, ..., N$ and

$$-\frac{\partial^2\phi_i}{\partial x^2} + \eta_i^2\phi_i = q^{(N+1)}\phi_i, \tag{8.3.15}$$

with $\phi_i \to 0$ as $|x| \to \infty, i = 1, 2, ..., N, N+1$. Without loss of generality one can assume that

$$-\eta_{N+1}^2 < -\eta_N^2 < ... < -\eta_2^2 < -\eta_1^2 < 0. \tag{8.3.16}$$

Furthermore, if we add the relation,

$$-\frac{\partial^2\psi_{N+1}}{\partial x^2} + \eta_{N+1}^2\psi_{N+1} = q^{(N)}\psi_{N+1}, \tag{8.3.17}$$

to the system (8.3.14), then (8.3.15) cannot be a bounded solution. However, as shown in Theorem I, the transformation (8.3.2) exists between two Schrödinger equations that have the same eigenvalues and their potentials bear the relation (8.3.4). Consequently, if we identify $\eta_0^2 = \eta_{N+1}^2, q' = q^{(N)}, q = q^{(N+1)}$ and $\psi_0(x) = \phi_{N+1}(x) = \frac{1}{\psi_{N+1}(x)}$ from the relations (8.3.15) and (8.3.17) we have

$$q^{(N+1)} = q^{(N)} + 2(\log\psi_{N+1}(x))_{xx}. \tag{8.3.18}$$

Thus in order to obtain $N+1$ soliton solution, we have a procedure that entails solving (8.3.1) for the N soliton solution $q^{(N)}$, and hence ψ_{N+1} and thereafter substituting the latter into (8.3.18); instead of trying to solve the eigenvalue problem (8.3.17) . To get a better understanding of how this procedure works, let us consider the following example. Suppose we start from the "vacuum" solution $q^{(0)} = 0$ for $N = 0$. From (8.3.14) we have

$$-\frac{\partial^2\psi_1}{\partial x^2} + \eta_1^2\psi_1 = 0. \tag{8.3.19}$$

Its unbounded solution is

$$\psi_1 = Ae^{\eta_1 x} + Be^{-\eta_1 x}. \tag{8.3.20}$$

Next from (8.3.18), we have

$$q^{(1)}(x) = 2\frac{\partial^2}{\partial x^2}\log(Ae^{\eta_1 x} + Be^{-\eta_1 x}). \tag{8.3.21}$$

Furthermore, consider the equation

$$\frac{\partial^2 \psi_2}{\partial x^2} + \eta_2^2 \psi_2 = q^{(1)}(x)\psi_2, \tag{8.3.22}$$

whose unbounded solution is

$$\psi_2 = \frac{1}{(Ae^{\eta_1 x} + Be^{-\eta_1 x})} \times \{Ce^{(\eta_1+\eta_2)x} + De^{(\eta_1-\eta_2)x} +$$

$$\frac{\eta_1+\eta_2}{\eta_2-\eta_1}\frac{B}{A}Ce^{-(\eta_1-\eta_2)x} + \frac{\eta_2-\eta_1}{\eta_1+\eta_2}\frac{B}{A}De^{-(\eta_1+\eta_2)x}\}. \tag{8.3.23}$$

Then from (8.3.22) we obtain

$$q^{(2)}(x) = 2\frac{\partial^2}{\partial x^2}\log\{Ce^{(\eta_1+\eta_2)x} + De^{(\eta_1-\eta_2)x} + \frac{\eta_1+\eta_2}{\eta_2-\eta_1}\frac{B}{A}Ce^{-(\eta_1-\eta_2)x}$$

$$+\frac{\eta_2-\eta_1}{\eta_1+\eta_2}\frac{B}{A}De^{-(\eta_1+\eta_2)x}\}. \tag{8.3.24}$$

The functional forms of $q^{(1)}(x)$ and $q^{(2)}(x)$ are indicative of the N soliton solution of the KdV equation. The explicit forms of the Bäcklund transformation for the KdV equation can now be derived, by starting from the basic equations of the inverse method, which for the KdV equation are given by

$$\frac{\partial \psi_1}{\partial x} - \eta\psi_1 = q(x,t)\psi_2, \tag{8.3.25}$$

$$\frac{\partial \psi_2}{\partial x} + \eta\psi_2 = \psi_1, \tag{8.3.26}$$

with

$$\frac{\partial \psi_1}{\partial t} = (-4\eta^3 - 2\eta q(x,t) - q_x(x,t))\psi_1 + (-q_{xx} - 2\eta q_x - 4\eta^2 q - 2q^2)\psi_2, \tag{8.3.27}$$

$$\frac{\partial \psi_2}{\partial t} = (4\eta^2 + 2q)\psi_1 - (-4\eta^3 - 2\eta q - q_x)\psi_2. \tag{8.3.28}$$

From (8.3.25) and (8.3.26) we get

$$-\frac{\partial^2 \psi_2}{\partial x^2} + \eta^2\psi_2 = q\psi_2, \tag{8.3.29}$$

while (8.3.27) and (8.3.28) yield

$$\frac{\partial \psi_2}{\partial t} = -(4\eta^2 + 2q)\frac{\partial \psi_2}{\partial x} + q_x \psi_2. \tag{8.3.30}$$

Next let us make the following identifications:

$$q^{(N)} = -\omega'_x = q', \quad q^{(N+1)} = -\omega_x = q, \quad \eta_{N+1} = \eta, \quad \psi_{N+1} = 1/\psi_2, \tag{8.3.31}$$

so that (8.3.18) reduces to

$$\omega' - \omega = -2\frac{\partial}{\partial x}\log \psi_2 = -2(\frac{1}{\psi_2}\frac{\partial \psi_2}{\partial x}). \tag{8.3.32}$$

Substitution of (8.3.31) into (8.3.32) gives

$$\omega_x + \omega'_x = -2\eta^2 + (\omega - \omega')^2/2. \tag{8.3.33}$$

Differentiating (8.3.30) with respect to x and making use of (8.3.31) and (8.3.32) then yields

$$\omega_t + \omega'_t = 2(\omega_x^2 + \omega_x\omega'_x + \omega'^2_x) - (\omega - \omega')(\omega_{xx} - \omega'_{xx}). \tag{8.3.34}$$

In deriving this we have written $q = -\omega_x$ in the KdV equation $q_t + 6qq_x + q_{xxx} = 0$, which then assumes the form $\omega_t - 3\omega_x^2 + \omega_{xxx} = 0$ and we have also made use of the fact that

$$\omega_{xx} + \omega'_{xx} = (\omega - \omega')(\omega_x - \omega'_x). \tag{8.3.35}$$

Equations (8.3.33 and 8.3.34) constitute the Bäcklund transformation for the KdV equation.

8.4 Bäcklund Transformations from the Riccati Equation

An alternative method of deriving Bäcklund transformations for classical continuous systems was formulated by Konno and Wadati [144], and involves use of the Riccati form of the inverse method. This technique was also studied by Chen [145]. In this procedure, the scattering problem is written in the following manner:

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\eta & q(x,t)\\ r(x,t) & -\eta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{8.4.1}$$

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}A & B\\ C & -A\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{8.4.2}$$

The eigenvalues η are time independent when

$$\frac{\partial A}{\partial x} = qC - rB, \tag{8.4.3}$$

$$\frac{\partial B}{\partial x} - 2\eta B = q_t - 2Aq, \tag{8.4.4}$$

$$\frac{\partial C}{\partial x} + 2\eta C = r_t + 2Ar, \tag{8.4.5}$$

where A, B and C are functions of (x,t) and their specific choices yield a large class of nonlinear evolution equations. The next step consists in introducing a function,

$$\Gamma \equiv \frac{\psi_1}{\psi_2}, \tag{8.4.6}$$

so that from (8.4.1) and (8.4.2) we get the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + q - r\Gamma^2 \tag{8.4.7}$$

$$\frac{\partial \Gamma}{\partial t} = B + 2A\Gamma - C\Gamma^2. \tag{8.4.8}$$

The key to the derivation of Bäcklund transformations by means of Riccati equations rests on the construction of a transformation Γ', satisfying the same equation as (8.4.7) with a potential $q'(x)$, given by

$$q'(x) = q(x) + f(\Gamma, \eta). \tag{8.4.9}$$

Upon elimination of Γ between (8.4.7) and (8.4.9), we get the required Bäcklund transformation for a particular nonlinear equation, fixed by specific choices for the function $r(x,t)$ occurring in (8.4.1).

Case (i) $r = -1$**:** For this choice the Riccati equation (8.4.7) assumes the form

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + q + \Gamma^2. \tag{8.4.10}$$

Now choosing Γ' and q' as follows:

$$\Gamma' = -\Gamma - 2\eta, \tag{8.4.11}$$

$$q'(x) = q(x) + 2\frac{\partial}{\partial x}(-\Gamma - 2\eta), \tag{8.4.12}$$

and requiring Γ', q' to satisfy (8.4.7), upon elimination of Γ in (8.4.10) and (8.4.11) using (8.4.12), one has the Bäcklund transformation:

$$\omega'_x + \omega_x = -2\eta^2 + \frac{(\omega - \omega')^2}{2}, \tag{8.4.13}$$

$$\omega'_t - \omega_t = 2B + 4A[\frac{\omega' - \omega}{2} - \eta] - 2C[\frac{\omega' - \omega}{2} - \eta]^2, \tag{8.4.14}$$

where we have set $q = -\omega_x, q' = -\omega'_x$.

Case (ii) $r = -q$**:** In this case the Riccati equation (8.4.7) becomes

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + q + q\Gamma^2. \tag{8.4.15}$$

Upon choosing

$$\Gamma' = 1/\Gamma, \tag{8.4.16}$$

$$\text{and} \quad q'(x) = q(x) - 2\frac{\partial}{\partial x}\tan^{-1}\Gamma, \tag{8.4.17}$$

with Γ', q' satisfying (8.4.15) as in the previous case, by elimination of Γ, we obtain the following Bäcklund transformation equations:

$$\omega_x + \omega'_x = -2\eta\sin(\omega - \omega'), \tag{8.4.18}$$

$$\omega_t - \omega'_t = (C - B) - (B + C)\cos(\omega - \omega') + 2A\sin(\omega - \omega'), \tag{8.4.19}$$

where as before $q = -\omega_x$ and likewise $q' = -\omega'_x$.

Case (iii) $r = -q^\star$**:** For this case, the Riccati equation assumes the form,

$$\frac{\partial \Gamma}{\partial x} = 2\eta\Gamma + q + q^*\Gamma^2. \tag{8.4.20}$$

Choosing Γ', q' as

$$\Gamma' = 1/\Gamma^*, \tag{8.4.21}$$

$$q'(x) = q(x) + 2\left(\frac{\Gamma^2(\frac{\partial \Gamma^*}{\partial x}) - \frac{\partial \Gamma}{\partial x}}{1 - |\Gamma|^4}\right). \tag{8.4.22}$$

It can be shown that Γ' with $q'(x)$ as in (8.4.22) satisfies (8.4.7) for real η. Equation (8.4.22) reduces to a simpler form given by

$$q'(x) + q(x) = -4\eta\frac{\Gamma}{1 + |\Gamma|^2}, \tag{8.4.23}$$

which may be solved for Γ, to yield

$$\Gamma = -\frac{2\eta + \sqrt{4\eta^2 - |q' + q|^2}}{(q'^\star + q^\star)}. \tag{8.4.24}$$

Now by elimination of Γ in (8.4.24) and (8.4.7) with the use of (8.4.8) and the following choice for the quantities A, B and C, namely

$$A = 2i\eta^2 + i|q|^2, \quad B = -q_x + 2i\eta q, \text{ and } C = iq_x^\star - 2i\eta q^\star, \tag{8.4.25}$$

one can finally derive the Bäcklund transformation for the nonlinear Schrödinger equation in the following form:

$$q_x + q'_x = (q - q')\sqrt{4\eta^2 - |q + q'|^2}, \tag{8.4.26}$$

$$q_t + q'_t = i(q_x - q'_x)\sqrt{4\eta^2 - |q + q'|^2} + \frac{i}{2}(q+q')(|q+q'|^2 + |q-q'|^2). \tag{8.4.27}$$

It should be noted that the other solution of (8.4.22), namely

$$\Gamma = -\frac{2\eta - \sqrt{4\eta^2 - |q' + q|^2}}{(q'^\star + q^\star)}, \tag{8.4.28}$$

can also be used to determine the Bäcklund transformation from the Riccati form by setting $\frac{1}{\Gamma^\star}$ instead of Γ in (8.4.20) and (8.4.8). We shall now give some explicit results. For example, (8.4.13 and 8.4.14) give the Bäcklund transformation for the KdV equation when the following choices for A, B and C are made, viz

$$A = -4\eta^3 - 2\eta q^2, \; B = -q_{xx} - 2\eta q_x - 4\eta^2 q - 2q^3,$$
$$C = q_{xx} - 2\eta q_x + 4\eta^2 q = 2q^3. \tag{8.4.29}$$

The same set of equations with $\omega = \frac{u}{2}$ give the Bäcklund transformation for the Sine-Gordon equation when A, B and C are chosen as follows:

$$A = \frac{1}{4\eta}\cos u \qquad B = C = \frac{1}{4\eta}\sin u. \tag{8.4.30}$$

8.5 Darboux-Bäcklund Transformations

In the previous section it was shown from an analysis of the Riccati form of the inverse scattering equation how Bäcklund transformations

could be deduced for a wide class of inverse problems by appropriate choices for the functions A, B and C. The equations considered there were all examples of continuous integrable systems. In this section we describe a method for deriving Bäcklund transformations for discrete integrable equations. This method depends on the existence of an operator matrix "B", which is functionally dependent on the nonlinear variables and which provides a connection between the old and the new variables, i.e., the transformed Lax equations. Such a transformation is usually referred to as the *Darboux-Bäcklund transformation*. We illustrate the technique by means of the next example [146], involving a nonlinear differential-difference equation belonging to the nonlinear Schrödinger family:

$$iq_{n,t} = \frac{q_{n+1} - q_{n-1} - 2q_n}{k^2} \pm q_n q_n^\star (q_{n+1} + q_{n-1}), \quad k = \Delta x. \tag{8.5.1}$$

The Lax pair associated with the above equation is

$$\Psi_{n+1} = L_n \Psi_n, \qquad \Psi_{n,t} = M_n \Psi_n, \tag{8.5.2}$$

where

$$L_n = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & q_n \\ r_n & 0 \end{pmatrix} = \begin{pmatrix} z & q_n \\ r_n & z^{-1} \end{pmatrix}, \tag{8.5.3}$$

$$M_n = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix},$$

$$= \begin{pmatrix} \frac{i}{k^2}(1 - z^2 + k^2 q_n q_{n-1}^\star) & \frac{i}{k}(-q_n z + z^{-1} q_{n-1}) \\ \pm\frac{i}{k}(q_{n-1}^\star z - z^{-1} q_n^\star) & -\frac{i}{k^2}(1 - z^{-2} \mp k^2 q_{n-1} q_n^\star) \end{pmatrix}. \tag{8.5.4}$$

Let us consider the transformation to a new set of nonlinear variables (q'_n, r'_n), and write the transformed Lax equation as

$$\Psi'_{n+1} = L'_n \Psi'_n, \qquad \Psi'_{n,t} = M'_n \Psi'_n. \tag{8.5.5}$$

Here (L'_n, M'_n) have the same functional dependences as before, but with (q_n, r_n) replaced by (q'_n, r'_n). The Darboux-Bäcklund transformation connects the old and the new *linear problem* through the matrix function B, as follows:

$$\Psi'_n = B_n \Psi_n. \tag{8.5.6}$$

In general, one assumes B_n to be a suitable polynomial in the spectral parameter z with matrix coefficients, having in the present case the form

$$B_n = \sum_{k=-1}^{N} z^k B_k^n \tag{8.5.7}$$

with the coefficient matrix B_k^n given by

$$B_k^n = \begin{pmatrix} a_k^n(q,r,q',r') & b_k^n(q,r,q',r') \\ c_k^n(q,r,q',r') & d_k^n(q,r,q',r') \end{pmatrix}. \tag{8.5.8}$$

The basic problem now is to determine the entries of this matrix as functions of the nonlinear field variables. For clarity let us explicitly consider the case when $N = 2$ so that B_n is quadratic in z. Consistency of (8.5.5) and (8.5.6) leads to

$$B_{n+1}L_n - L_n'B_n = 0, \quad \text{and} \quad B_{n,t} = M_n'B_n - B_nM_n, \tag{8.5.9}$$

which determine the elements of the matrices $B_0^n, B_1^n, B_2^n, B_{-1}^n$ as follows:

$$a_2^n = const. = a_2(\text{say}),\ b_2^n = 0,\ c_2^n = 0,\ d_2^n = d_2^0F(n-1), \tag{8.5.10}$$

$$a_1^n = const. = a_1(\text{say}),\ \ d_1^n = d_1^0F(n-1), \tag{8.5.11}$$

$$c_1^n = d_0r_n - a_0^0F(n-1)r_n', \qquad b_1^nd_0q_{n-1}' - a_0^0F(n-1)q_{n-1}, \tag{8.5.12}$$

$$a_0^n = a_0^0F(n-1), \qquad b_0^n = d_{-1}q_{n-1}' - a_{-1}^0F(n-1)q_{n-1}, \tag{8.5.13}$$

$$c_0^n = d_{-1}r_n - a_{-1}^0F(n-1)r_{n-1}', \qquad d_0^n = const. = d_0(\text{say}), \tag{8.5.14}$$

$$b_{-1}^n = 0, \qquad d_{-1}^{n+1} = d_{-1}^n = const. = d_{-1}(\text{say}), \tag{8.5.15}$$

$$c_{-1}^n = 0, \qquad a_{-1}^n = a_{-1}^0F(n-1), \tag{8.5.16}$$

where

$$F(n) = \frac{\prod_{i=0}^{n}(1-q_i'r_i')}{\prod_{i=0}^{n}(1-q_ir_i)} = \prod_{i=0}^{n} G(i). \tag{8.5.17}$$

Also the complete set of equations yields

$$(d_0-d_{-1})q_n' + (a_{-1}^0-a_0^0)F(n)q_n = (a_2-a_1)q_{n+1} + (d_1^0-d_2^0)F(n)q_{n+1}', \tag{8.5.18}$$

$$(a_2-a_1)r_n' + (d_1^0-d_2^0)F(n)r_n = (d_0-d_{-1})r_{n+1} + (a_{-1}^0-a_0^0)F(n)r_{n+1}', \tag{8.5.19}$$

which determine the transformation between the sets (q_n, r_n) and (q'_n, r'_n), i.e., the usual Bäcklund transformation. It should be noted that changing the polynomial structure of B_n, by say retaining terms up to the fourth power of z, would lead to a more complicated set of Bäcklund transformations relations, using the same procedure. The point to be noted is that the process of determining the coefficients recursively can still be carried out in the case of higher-order polynomial forms of B_n.

8.6 The Exponential Lattice

In one of the earliest investigations of the connection between Bäcklund transformations and the inverse scattering transform theory, Wadati and Toda [140] considered the equation of motion of an exponential lattice, better known subsequently as the Toda lattice. They succeeded in showing how a recursive application of the transformation leads to an algebraic formula for the solutions. Their work extended the theory of Bäcklund transformations to differential-difference equations and marked a major developement. The exponential lattice considered by Wadati and Toda had the following equation of motion:

$$\frac{d^2 Q_n}{dt^2} = e^{(Q_n - Q_{n-1})} - e^{(Q_{n+1} - Q_n)}, \tag{8.6.1}$$

with Q_n denoting the displacement of the nth particle from its equilibrium position. If a new variable $r_n = Q_n - Q_{n+1}$ is introduced to denote the relative displacement, then in terms of this variable the equation of motion for the lattice assumes the form,

$$\frac{d^2 r_n}{dt^2} = 2e^{-r_n} - e^{-r_{n+1}} - e^{-r_{n-1}}. \tag{8.6.2}$$

Wadati and Toda derived the Bäcklund transformation for the Toda lattice equation in the form,

$$\frac{d}{dt}(Q_n - Q'_{n-1}) = A\left[e^{-(Q'_n - Q_n)} - e^{-(Q'_{n-1} - Q_{n-1})}\right], \tag{8.6.3}$$

$$\frac{d}{dt}(Q'_n - Q_n) = \frac{1}{A}\left[e^{-(Q_{n+1} - Q'_n)} - e^{-(Q_n - Q'_{n-1})}\right], \tag{8.6.4}$$

where A is an arbitrary constant. It is easy to see that $r = Q_n - Q_{n-1}$ and $r' = Q'_n - Q'_{n-1}$ satisfy (8.6.2). In most cases of interest, it is usual to impose the following boundary condition:

$$Q_n, Q'_n \longrightarrow \text{const.}, \qquad \text{as } |n| \longrightarrow \infty. \tag{8.6.5}$$

From (8.6.3) and (8.6.4) one can show that

$$\begin{aligned} &\frac{dQ_n}{dt} - Ae^{-(Q'_n - Q_n)} - \frac{1}{A}e^{-(Q_n - Q'_{n-1})} \\ &= \frac{dQ_{n-1}}{dt} - Ae^{-(Q'_{n-1} - Q_{n-1})} - \frac{1}{A}e^{-(Q_{n-1} - Q'_{n-2})}, \end{aligned} \tag{8.6.6}$$

so that by decreasing the lattice index n in steps of unity and using the boundary conditions given by (8.6.5), one may obtain the relation,

$$\frac{dQ_n}{dt} = A[e^{-(Q'_n - Q_n)} - c] + \frac{1}{A}[e^{-(Q_n - Q'_{n-1})} - \frac{1}{c}], \tag{8.6.7}$$

where

$$c = e^{-(Q'_{-\infty} - Q_{-\infty})}, \tag{8.6.8}$$

and similarly

$$\frac{dQ'_n}{dt} = A[e^{-(Q'_n - Q_n)} - c] + \frac{1}{A}[e^{-(Q_{n+1} - Q'_n)} - \frac{1}{c}]. \tag{8.6.9}$$

Equations (8.6.7) and (8.6.9) give the Bäcklund transformation for system (8.6.2) under the above-mentioned boundary conditions as well as for (8.6.1), since the latter is equivalent to system (8.6.2) with its associated boundary condition.

In [141] Toda and Wadati go on to derive a canonical transformation connecting two solutions of the exponential lattice equation, thereby obtaining a discrete version of the Bäcklund transformation. In addition, the transformation is also shown to be canonical. In the subsequent sections we shall dwell on the canonicity of Bäcklund transformations from different points of view. However it is still interesting at this juncture to briefly reconstruct their reasoning. It will be observed that the equation of motion as given by (8.6.1) can be derived from the following Hamiltonian for the Toda lattice, using the notation $Q = \{Q_n\}$ and $P = \{P_n\}$:

$$H(Q, P) = \frac{1}{2}\sum_{n=-\infty}^{\infty} P_n^2 + \sum_{n=-\infty}^{\infty} e^{-(Q_n - Q_{n-1})} \tag{8.6.10}$$

The Hamiltonian equations are

$$\dot{Q}_n = \frac{\partial H}{\partial P_n} = P_n, \tag{8.6.11}$$

$$\dot{P}_n = -\frac{\partial H}{\partial Q_n} = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1} - Q_n)}. \tag{8.6.12}$$

Toda and Wadati next observed that if a transformation of the sets of variables $P = \{P_n\}, Q = \{Q_n\}$ to $P' = \{P'_n\}$ and $Q' = \{Q_n\}$ is considered, so that

$$P_n = \frac{f_n(Q)}{f_n(Q')} + \frac{f_{n-1}(Q')}{f_n(Q)}, \tag{8.6.13}$$

$$P'_n = \frac{f_n(Q)}{f_n(Q')} + \frac{f_n(Q')}{f_{n+1}(Q)}, \tag{8.6.14}$$

then it is readily observed that the construct,

$$\left\{ \frac{1}{2} \sum_{n=-N_0}^{N} P_n'^2 + \sum_{n=-N_0}^{N} \frac{f_{n-1}(Q')}{f_n(Q')} \right\} - \left\{ \frac{1}{2} \sum_{n=-N_0}^{N} P_n^2 + \sum_{n=-N_0}^{N} \frac{f_n(Q)}{f_{n+1}(Q)} \right\},$$

$$= -\frac{1}{2} \left\{ f_{-N_0-1}(Q')/f_{-N}(Q) \right\}^2 + \frac{1}{2} \left\{ \frac{f_N(Q')}{f_{N+1}(Q)} \right\}^2. \tag{8.6.15}$$

This means that if we have a canonical transformation of the form given in (8.6.13 and 8.6.14), then it will transform the Hamiltonian:

$$H(Q,P) = \frac{1}{2} \sum_n P_n^2 + \sum_n \frac{f_n(Q)}{f_{n-1}(Q)}, \tag{8.6.16}$$

to itself except for a constant term, when suitable boundary conditions are imposed. It is now quite natural to seek a canonical transformation of the form (8.6.13 and 8.6.14), with $f_n(Q) = e^{Q_n}$ corresponding to the Hamiltonian for the Toda lattice. Considering now a canonical transformation,

$$P_n = \frac{\partial W}{\partial Q_n}, \qquad P'_n = -\frac{\partial W}{\partial Q'_n}, \tag{8.6.17}$$

induced by the generating function,

$$W(Q,Q') = \sum_j \left[A e^{-(Q'_j - Q_j)} - \frac{1}{A} e^{-(Q_{j+1} - Q'_j)} + \alpha(Q'_j - Q_j) \right], \tag{8.6.18}$$

where A and α are constants to be determined by the boundary conditions. We have therefore from (8.6.13 and 8.6.14)

$$P_n = Ae^{-(Q'_n - Q_n)} + \frac{1}{A}e^{-(Q_n - Q'_{n-1})} - \alpha, \tag{8.6.19}$$

$$P'_n = Ae^{-(Q'_n - Q_n)} + \frac{1}{A}e^{-(Q_{n+1} - Q'_n)} - \alpha. \tag{8.6.20}$$

Now for the case of an infinite lattice, assuming that as $|n| \to \infty$ the lattice variables are constants with

$$Q_n \to Q_{-\infty}, \quad Q'_n \to Q'_{-\infty}, \quad \text{as} \quad n \to -\infty, \tag{8.6.21}$$

$$Q_n \to Q_{+\infty}, \quad Q'_n \to Q'_{+\infty}, \quad \text{as} \quad n \to +\infty, \tag{8.6.22}$$

we have, since P_n and P'_n must then vanish at infinity,

$$\sum_{n=-\infty}^{\infty} P_n'^2 - \sum_{n=-\infty}^{\infty} P_n^2 =$$

$$-2\sum_{n=-\infty}^{\infty} e^{-(Q'_n - Q'_{n-1})} + 2\sum_{n=-\infty}^{\infty} e^{-(Q_{n+1} - Q_n)} + \text{const.} \tag{8.6.23}$$

It is known from the general theory of canonical transformations, that the transformed Hamiltonian is related to the old Hamiltonian in the following manner:

$$H'(Q', P') = H\left[Q(Q', P'), P(Q', P')\right] + \frac{\partial W}{\partial t}. \tag{8.6.24}$$

Consequently, we have from (8.6.18) and (8.6.24)

$$H'(Q', P') = \frac{1}{2}\sum_{n=-\infty}^{\infty} P_n'^2 + \sum_{n=-\infty}^{\infty} e^{-(Q'_n - Q'_{n-1})} + \text{const.} \tag{8.6.25}$$

Equation (8.6.25) is of the same form as (8.6.10) except for the additional constant term. From this it is apparent that the above transformation maps a dynamical space of the system to itself, so that

$$\dot{Q}'_n = \frac{\partial H'}{\partial P_n} = P'_n, \tag{8.6.26}$$

$$\dot{P}'_n = -\frac{\partial H'}{\partial Q'_n} = e^{-(Q'_n - Q'_{n_1})} - e^{-(Q'_{n+1} - Q'_n)}. \tag{8.6.27}$$

Thus the transformation constitutes a canonical Bäcklund transformation. To check the correctness of the above method let us consider the following example. Suppose, we begin with the trivial solution $Q_n = 0$, and see if

$$e^{Q'_n} = A\frac{\cosh(\kappa n + \beta t)}{\cosh(\kappa(n+1) + \beta t)}, \tag{8.6.28}$$

represents another solution with

$$A = e^{\kappa}, \qquad \alpha = e^{\kappa} + e^{-\kappa}, \qquad \beta = \sinh\kappa. \tag{8.6.29}$$

Since P_n and P'_n both vanish at infinity, hence we have from (8.6.19 and 8.6.20)

$$Ae^{-(Q'_{-\infty}-Q_{-\infty})} + \frac{1}{A}e^{-(Q_{-\infty}-Q'_{-\infty})} - \alpha = 0, \tag{8.6.30}$$

$$Ae^{-(Q'_{\infty}-Q_{\infty})} + \frac{1}{A}e^{-(Q_{\infty}-Q'_{\infty})} - \alpha = 0. \tag{8.6.31}$$

From (8.6.30 and 8.6.31), solving for A and α we have

$$\alpha = AC + \frac{1}{AC}, \qquad C = e^{-(Q'_{-\infty}-Q_{-\infty})}, \tag{8.6.32}$$

$$A = \exp\left[\frac{1}{2}(Q'_{\infty} + Q'_{-\infty}) - \frac{1}{2}(Q_{\infty} + Q_{-\infty})\right]. \tag{8.6.33}$$

Assuming $A > 0$, one can show from (8.6.27) that the difference between the total momenta $\sum P_n$ and $\sum P'_n$ is given entirely by the boundary values, i.e.,

$$\sum_{n=-\infty}^{\infty}(P'_n - P_n) = 2\sinh\frac{1}{2}\{(Q'_{\infty} - Q'_{-\infty}) - (Q_{\infty} - Q_{-\infty})\}, \tag{8.6.34}$$

so that $\sum_{n=-\infty}^{\infty} P'_n = \sum_{n=-\infty}^{\infty} P_n + \text{const}$. Again from (8.6.27) one has

$$\sum_n (P'_n + \alpha)^2 = A^2\sum_n e^{-2(Q'_n - Q_n)} + \frac{1}{A^2}\sum_n e^{-2(Q_{n+1}-Q'_n)} + 2\sum_n e^{-(Q_{n+1}-Q_n)} \tag{8.6.35}$$

$$\sum_n (P_n + \alpha)^2 = A^2\sum_n e^{-2(Q'_n - Q_n)} + \frac{1}{A^2}\sum_n e^{-2(Q_n - Q'_{n-1})}$$

$$+2\sum_n e^{-(Q'_n - Q'_{n-1})}; \tag{8.6.36}$$

then by using the expressions obtained for A, α and β from (8.6.29) and (8.6.28) in (8.6.19) we find that

$$P_n = \frac{\cosh[\kappa(n+1)+\beta t]}{\cosh(\kappa n+\beta t)} + \frac{\cosh[\kappa(n-1)+\beta t]}{\cosh(\kappa n+\beta t)} - (e^{\kappa}+e^{-\kappa}) = 0, \tag{8.6.37}$$

while (8.6.20) gives us

$$P'_n = \frac{\cosh[\kappa(n+1)+\beta t]}{\cosh(\kappa n+\beta t)} + \frac{\cosh[\kappa n+\beta t]}{\cosh(\kappa(n+1)+\beta t)} - (e^{\kappa}+e^{-\kappa})$$

$$= \sinh\kappa \left\{ \frac{\sinh(\kappa n+\beta t)}{\cosh(\kappa n+\beta t)} - \frac{\sinh[\kappa(n+1)+\beta t]}{\cosh[\kappa(n+1)+\beta t]} \right\} = \dot{Q}'_n. \tag{8.6.38}$$

Thus (8.6.28) represents a valid soliton solution. In fact the transformation given by (8.6.19 and 8.6.20) is such that it adds another soliton to the solution, which is a general feature of Bäcklund transformations derived in the above manner.

8.7 Canonical Transformations

In Section 8.6 we briefly touched upon the canonical nature of Bäcklund transformations (BT). This served to indicate the dynamical significance of such transformation, and suggests that the theory of canonical transformations plays a significant role in the case of completely integrable systems. We will therefore dwell on this aspect of the theory.

In order to understand how the general theory of canonical transformation applies to nonlinear evolution equations, let us consider an evolution equation of the form [142],

$$\phi_{xt} = \mathcal{K}(\phi, \phi_x, \phi_{xx}), \tag{8.7.1}$$

where $\mathcal{K}(\phi, \phi_x, \phi_{xx})$ is in general, a nonlinear function of ϕ and its x-derivatives, but does not include $t-$derivatives of ϕ. Equation (8.7.1) can be derived from the Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \phi_t}\right) - \frac{\delta L}{\delta \phi} = 0; \tag{8.7.2}$$

the Lagrangian L is given by

$$L = \int_{-\infty}^{\infty} dx\mathcal{L} \qquad \text{where} \qquad \mathcal{L} = \phi_x\phi_t - \mathcal{U}(\phi, \phi_x, \phi_{xx}...). \tag{8.7.3}$$

From (8.7.2 and 8.7.3) we have

$$\phi_{xt} = -\frac{1}{2}\frac{\delta U}{\delta \phi} \qquad \text{and} \qquad U[\phi] = \int_{-\infty}^{\infty} dx\mathcal{U}(\phi, \phi_x, \phi_{xx}....). \tag{8.7.4}$$

Since the Lagrangian density $\mathcal{L}$ is linear with respect to ϕ_t, the Hamiltonian formalism is not uniquely defined. However, we may still consider the Hamilton's equation in the following form:

$$\phi_{xt} = -\frac{\delta H}{\delta \phi}, \tag{8.7.5}$$

with the Hamiltonian given by

$$H[\phi] = \frac{1}{2}U[\phi]. \tag{8.7.6}$$

Note that, when $\mathcal{U}$ involves only $x-$derivatives of ϕ, one can write (8.7.4) in the following manner using (8.7.5):

$$\phi_{xt} = \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta \phi_x}\right), \qquad H[\phi] = \frac{1}{2}U[\phi_x]. \tag{8.7.7}$$

Now a transformation that maps ϕ into ϕ' is canonical if the Pfaffian form,

$$\theta = \int_{-\infty}^{\infty} dx(\phi_x d\phi - \mathcal{H}dt), \tag{8.7.8}$$

is invariant under the transformation, i.e.,

$$\int_{-\infty}^{\infty} dx(\phi_x d\phi - \mathcal{H}dt) = \int_{-\infty}^{\infty} dx(\phi'_x d\phi' - \mathcal{H}'dt) + dW[\phi, \phi'; t]. \tag{8.7.9}$$

Here $W[\phi, \phi'; t]$ is an arbitrary functional referred to as a generating functional of the transformation. Hamilton's equation as given by (8.7.7) follows from the canonical symplectic form, i.e., from the exterior derivative of the Pfaffian form given by (8.7.8), as is easy to verify. For this purpose, we construct the exterior differential form $d\theta$ from (8.7.8), which is

$$\omega \equiv d\theta = \int_{-\infty}^{\infty} dx\{d\phi_x \wedge d\phi - \sum_{i=0}(-1)^i \frac{\partial^i}{\partial x^i}\frac{\partial \mathcal{H}}{\partial \phi_t} d\phi \wedge dt\}, \tag{8.7.10}$$

so that

$$\omega = \int_{-\infty}^{\infty} \{d\phi_x \wedge d\phi - \sum_{i=1}(-1)^{i-1} \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial \mathcal{H}}{\partial \phi_t} d\phi_x \wedge dt\}, \tag{8.7.11}$$

where $\phi_i \equiv (\partial/\partial x)^i \phi$ and $\wedge$ stands for the exterior product. If the differential two-form ω is an integral invariant (i.e., θ is a relative integral invariant), then we obtain the Hamiltonian flow in the following form:

$$\phi_{xt} = -\sum_{i=0}(-1)^i \frac{\partial^i}{\partial x^i} \frac{\partial \mathcal{H}}{\partial \phi_t} = -\frac{\delta H[\phi]}{\delta \phi}, \tag{8.7.12}$$

and hence

$$\phi_t = \sum_{i=1}(-1)^{i-1} \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial \mathcal{H}}{\partial \phi_t} = \frac{\delta H[\phi_x]}{\delta \phi_x}. \tag{8.7.13}$$

Thus the Hamiltonian flow is uniquely determined by the Pfaffian form in (8.7.8). It follows that if θ is invariant under the transformation, then ϕ' will satisfy the equation:

$$\phi'_{xt} = -\frac{\delta H'}{\delta \phi'}. \tag{8.7.14}$$

The canonical transformation for the system may now be defined. Integration by parts of (8.7.9) results in

$$\int_{-\infty}^{\infty} dx(\phi d\phi_x + \mathcal{H} dt) = \int_{-\infty}^{\infty} dx(\phi' d\phi'_x + \mathcal{H}' dt) - d\widetilde{W}[\phi_x, \phi'_x; t]; \tag{8.7.15}$$

therefore the canonical transformation is given by the formulae,

$$\phi = -\frac{\delta \widetilde{W}}{\delta \phi_x}, \qquad \text{and} \qquad \phi' = \frac{\delta \widetilde{W}}{\delta \phi'_x}. \tag{8.7.16}$$

Going back to (8.7.15), we see that since $dW[\phi, \phi'; t]$ is an exact differential form, hence

$$dW[\phi, \phi'; t] = \frac{\partial W}{\partial t} dt + \int_{-\infty}^{\infty} dx \left(\frac{\delta W}{\delta \phi} d\phi + \frac{\delta W}{\delta \phi'} d\phi' \right), \tag{8.7.17}$$

so that one has the following transformation equations:

$$\phi_x = \frac{\delta W}{\delta \phi}, \qquad \phi'_x = -\frac{\delta W}{\delta \phi'}, \qquad H' = H + \frac{\partial W}{\partial t}. \tag{8.7.18}$$

In general, if the Hamiltonian does not depend explicitly on time t, then the generating functional $W[\phi, \phi'; t]$ will assume the following form [113, 143]:

$$W[\phi, \phi', t] = S[\phi, \phi'] - Et, \tag{8.7.19}$$

with E being the constant energy integral, determined by the boundary conditions on the flows ϕ and ϕ'. The transformed Hamiltonian H' is then given by

$$H' = H + \frac{\partial W}{\partial t} = H - E. \tag{8.7.20}$$

In particular for the case of stationary flows, where the Hamiltonian H is transformed into $H'_0 = 0$, we get the Hamilton-Jacobi equation,

$$H[\phi, \frac{\delta S}{\delta \phi}] = E. \tag{8.7.21}$$

In general to derive the canonical transformation, one needs to solve (8.7.21). However, if we restrict our discussion to canonical transformations in which the transformed Hamiltonian H' has the same form as the original Hamiltonian, except for an additional constant term, then the transformations of interest to us are such that

$$\mathcal{H}'(\phi', \phi'_x ....) - \mathcal{H}(\phi, \phi_x ...) = \frac{\partial}{\partial x}\mathcal{F}(\phi, \phi', \phi_x, \phi'_x, ...). \tag{8.7.22}$$

Moreover, if we assume that the generating functional $W[\phi, \phi'; t]$, has the form,

$$W[\phi, \phi'; t] = \int_{-\infty}^{\infty} \{\phi' \phi_x + \mathcal{G}(\phi, \phi')\} dx - Et, \tag{8.7.23}$$

then from (8.7.16) and (8.7.18) we find that

$$\mathcal{G}(\phi, \phi') = \mathcal{G}(\phi - \phi'), \tag{8.7.24}$$

$$\phi_x + \phi'_x = \frac{\partial}{\partial \phi}\mathcal{G}(\phi - \phi') = g(\phi = \phi'), \tag{8.7.25}$$

$$E = -\mathcal{F}(\phi, \phi', \phi_x, \phi'_x ....)|_{x=-\infty}^{x=\infty}.$$

Using the conditions (8.7.16) and (8.7.22) we can determine the functional form of $g(\phi - \phi')$ and obtain a canonical transformation that maps the Hamiltonian to itself. We illustrate this procedure with an example.

Example: Sine-Gordon equation
In a characteristic frame the Sine-Gordon equation is given by

$$\phi_{xt} = \sin\phi \tag{8.7.26}$$

with the boundary condition $\phi \longrightarrow 0(\text{mod } 2\pi)$ as $|x| \to \infty$. The Lagrangian density is

$$\mathcal{L} = \phi_x\phi_t + 2(1 - \cos\phi), \tag{8.7.27}$$

so that from (8.7.27), the Hamiltonian for this system is

$$H[\phi] = -\int_{-\infty}^{\infty}(1 - \cos\phi)dx. \tag{8.7.28}$$

Proceeding as stated, we find the difference between the Hamiltonian densities $\mathcal{H}'$ and $\mathcal{H}$ to be given by

$$\mathcal{H}'(\phi') - \mathcal{H}(\phi) = \cos\phi' - \cos\phi = 2\sin\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2}. \tag{8.7.29}$$

According to (8.7.22), we should have

$$\mathcal{H}' - \mathcal{H} = \frac{\partial}{\partial x}\mathcal{F}(\phi,\phi'),$$

$$= \mathcal{F}_\phi\phi_x + \mathcal{F}_{\phi'}\phi'_x = \frac{1}{2}\left[(\mathcal{F}_\phi - \mathcal{F}_{\phi'})(\phi_x - \phi'_x) + (\mathcal{F}_\phi - \mathcal{F}_{\phi'})(\phi_x + \phi'_x)\right], \tag{8.7.30}$$

where $\mathcal{F}_\phi = \frac{\partial}{\partial\phi}\mathcal{F}(\phi,\phi')$. Compatibility of (8.7.29 and 8.7.30) requires that we identify $\mathcal{F}(\phi,\phi') = \mathcal{F}(\phi+\phi')$, leading thereby to the identity,

$$\mathcal{H}'(\phi') - \mathcal{H}(\phi) = 2\sin\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2} = g(\phi-\phi')\mathcal{F}_\phi(\phi+\phi'). \tag{8.7.31}$$

Hence we obtain

$$g(\phi-\phi') = A\sin\frac{\phi-\phi'}{2} \tag{8.7.32}$$

and

$$\mathcal{F}_\phi(\phi+\phi') = \frac{2}{A}\sin\frac{\phi+\phi'}{2}, \tag{8.7.33}$$

with A being a constant. Consequently, the generating functional $W[\phi,\phi';t]$ becomes

$$W[\phi,\phi';t] = \int_{-\infty}^{\infty}\left[\phi'\phi_x - 2A\{\cos\frac{\phi-\phi'}{2} - 1\}\right]dx - Et, \tag{8.7.34}$$

with the constant E being determined by

$$E = -\frac{4}{A}\cos\frac{\phi+\phi'}{2}|_{x=-\infty}^{x=\infty}. \tag{8.7.35}$$

Thus the canonical transformation induced by the generating functional (8.7.34) is as follows:

$$\phi_x = \frac{\delta W}{\delta\phi} = -\phi'_x + A\sin\left(\frac{\phi-\phi'}{2}\right). \tag{8.7.36}$$

For the sake of clarity, let us consider a transformation from a solution with boundary conditions $\phi \longrightarrow 0$ as $x \to -\infty$ and $\phi \to 2\pi$ as $x \longrightarrow +\infty$, to a trivial solution $\phi' = 0$. Under this transformation, a flow on the dynamical surface with energy integral $E = -\frac{4}{A}$, is mapped into a stationary flow $E = 0$. Comparision of this transformation with the BT makes the solution structure clearer. It will be noticed that the transformation (8.7.36), is just the spatial part of the BT so that the original and transformed system represent the one soliton and the vacuum state, respectively. Now the temporal part of the BT can be derived using Hamilton's equation and the canonical transformation. We illustrate this particular point once again for the Sine-Gordon equation. From (8.7.36) we have

$$\phi_x + \phi'_x = A\sin\left(\frac{\phi-\phi'}{2}\right). \tag{8.7.37}$$

Differentiating this equation with respect to time t and making use of the Hamilton's equation $\phi_{xt} = \sin\phi$, yields

$$\frac{A}{2}(\phi_t - \phi'_t)\cos(\frac{\phi-\phi'}{2}) = \sin\phi + \sin\phi' = 2\sin(\frac{\phi+\phi'}{2})\cos(\frac{\phi-\phi'}{2}). \tag{8.7.38}$$

Therefore one finally obtains

$$\phi_t - \phi'_t = \frac{4}{A}\sin(\frac{\phi+\phi'}{2}). \tag{8.7.39}$$

The pair of (8.7.36) and (8.7.37) constitute the well-known Bäcklund transformation for the Sine-Gordon system.

Thus one is led to the conclusion that the Bäcklund transformation is a canonical transformation, keeping the Hamiltonian form invariant.

This result implies that the procedure of constructing N-soliton solutions by Bäcklund transformation may be interpreted as a canonical transformation between the dynamical states, i.e., between the stationary state (vacuum state) and the N-soliton state.

8.8 Group Property of Bäcklund Transformations

It is interesting to note that canonical transformations derived by the method described above exhibit a group property. Indeed, they may be shown to constitute an Abelian group. To illustrate their group nature we consider once again the Sine-Gordon equation. As this equation is invariant under $\phi \to -\phi$, one can write the generating functional of the canonical transformation (8.7.18) in the form

$$W_a[\phi, \phi'; t] = \int_{-\infty}^{\infty} \left[\phi\phi'_x - 2a\left(\cos(\frac{\phi+\phi'}{2}) - 1\right)\right] dx - E_a t, \quad (8.8.1)$$

where a is a continuous transformation parameter. Then the canonical transformation $T_a : \phi_x \longrightarrow \phi'_x$ is given by

$$\phi_x = \frac{\delta W_a}{\delta \phi} = \phi'_x + a \sin\left(\frac{\phi+\phi'}{2}\right). \quad (8.8.2)$$

We now have the following theorem.

Theorem: The transformation $\phi_x = \phi'_x + a\sin\left(\frac{\phi+\phi'}{2}\right)$ constitutes a group, so that the following properties hold:

(i) Closure: If transformation $T_a : \phi_x \longrightarrow \phi'_x$ and $T_b : \phi'_x \longrightarrow \phi''_x$ are canonical then their product transformation $T = T_a T_b : \phi_x \longrightarrow \phi''_x$ is also canonical.

(ii) Associative law: If T_a, T_b and T_c denote canonical transformations, then

$$T_c(T_b T_a) = (T_c T_b) T_a.$$

(iii) Identity: There is a unique identity element corresponding to the identity transformation,

$$T_0 : \phi_x = \phi'_x.$$

(iv) Unique inverse: To every element $T_a : \phi_x \longrightarrow \phi'_x$, there exists a unique element $T_a^{-1} : \phi'_x \longrightarrow \phi_x$, corresponding to the inverse transformation of T_a.

Proof: Since T_a is a canonical transformation, there is a generating functional $W_a[\phi, \phi'; t]$ so that

$$dW_a = \int_{-\infty}^{\infty} dx(\frac{\delta W_a}{\delta \phi} d\phi + \frac{\delta W_a}{\delta \phi'} d\phi') + \frac{\partial W_a}{\partial t} dt,$$

$$= \int_{-\infty}^{\infty} dx(\phi_x d\phi - \phi'_x d\phi') - (H - H')dt.$$

Similarly for T_b we have

$$dW_b = \int_{-\infty}^{\infty} dx(\phi'_x d\phi' - \phi''_x d\phi'') - (H' - H'')dt.$$

Hence

$$d(W_a + W_b) = \int_{-\infty}^{\infty} dx(\phi_x d\phi - \phi''_x d\phi'') - (H - H'')dt, \tag{8.8.3}$$

indicating therefore that the product transformation $T = T_b T_a$, is also canonical. The generating functional of the product transformation $T = T_b T_a$ is given by

$$W[\phi, \phi''; t] = W_a[\phi, \phi'; t] + W_b[\phi', \phi''; t]. \tag{8.8.4}$$

Now by using the transformations,

$$\phi_x = \phi'_x + a \sin(\frac{\phi + \phi'}{2}), \tag{8.8.5}$$

$$\phi'_x = \phi''_x + a \sin(\frac{\phi' + \phi''}{2}), \tag{8.8.6}$$

one can eliminate ϕ' and $\phi'_x$from the right-hand side of (8.8.5 and 8.8.6); the resulting generating functional $W[\phi, \phi''; t]$, then depends symmetrically on two parameters a and b . This completes the proof of the closure property.

Proof of (ii) is obvious since the generating functional for both sides of the the equation $T_c(T_b T_a) = (T_c T_b)T_a$ are the same, namely $W_a + W_b + W_c$.

The proof of the identity property follows from (8.8.5), upon setting $a = 0$, whence we have from the generating functional,

$$W_0[\phi, \phi'; t] = \int_{-\infty}^{\infty} \phi \phi'_x dx.$$

Finally, the inverse transformation of T_a, viz, $T_a^{-1} : \phi'_x \longrightarrow \phi_x$, is given by

$$dW_a^{-1} = \int_{-\infty}^{\infty} dx(\phi'_x d\phi' - \phi_x d\phi) - (H' - H)dt = -dW_a. \qquad (8.8.7)$$

Therefore, the generating functional for the inverse transformation is $W_a^{-1} = -W_a$, i.e., $T_a^{-1} = T_{-a}$. Lastly, it is clear that $T_a T_b = T_b T_a$ so the group is Abelian.

Thus the Bäcklund transformation for the Sine-Gordon equation, considered as a canonical transformation, forms an Abelian group. Similar conclusions may be shown to hold for the other standard nonlinear equations such as the KdV, mKdV equations, etc.

8.9 Recent Developments in the Bäcklund Transformation Theory

After describing some of the original attempts at understanding the canonical nature of Bäcklund transformations, let us now consider some of the more recent developments in the theory of Bäcklund transformations. It will be realized that the initial attempts at deriving the canonicity of Bäcklund transformations did not take into account the Hamiltonian structure of the systems. Recently, there has, however, been a keen interest in discrete integrable systems or integrable mappings, together with attempts at finding the most universal method of solving completely integrable systems. Perhaps the most significant method for arriving at a general formalism for solving integrable systems is the modern approach of the separation of variables in its most general form [147]. This in turn has given impetus to taking a fresh look at some of the traditional methods of integration such as the Darboux transformation, which had earlier been so successfully applied to many physical and mathematical problems.

It has also led to the study of Bäcklund transformations for finite dimensional Hamiltonian systems. The latter are canonical transformations depending on a Bäcklund transformation parameter, say λ, and may be shown to lead to the separation of variables when n such mappings are applied to an integrable system with n degrees of freedom. Indeed the sequence of Bäcklund transformation parameters λ_j,

together with a set of suitably defined conjugate variables μ_j, constitute the separation variables and satisfy a new property called *spectrality*.

From the Hamiltonian point of view, the significance of Bäcklund transformations is most prominently expressed by the close relationship between the transformations and Baxter's quantum Q operator. This was discovered by Pasquier and Gaudin [148]. In order to understand the connection between the spectrality property of Bäcklund transformations and the separation of variables [149], let us consider a Liouville integrable system with n degrees of freedom, described by canonical Darboux variables $X \equiv \{X_i\}_{i=1}^n$ and $x = \{x_i\}_{i=1}^n$. They obey the Poisson brackets:

$$\{X_i, X_j\} = \{x_i, x_j\} = 0 \qquad \{X_i, x_j\} = \delta_{ij}.$$

The n functionally independent Hamiltonians $H_i \equiv H_i(X, x)$, are in involution:

$$\{H_i, H_j\} = 0 \qquad i, j = 1, 2, ..., n.$$

Let us assume that there exists a Bäcklund transformation B_λ, which is also a canonical transformation, from the set (X, x) to (Y, y). Both the sets are canonical and depend on a complex parameter λ. Owing to its canonical nature, there exists a generating function $F_\lambda(y; x)$ say, so that

$$X_i = \frac{\partial F_\lambda}{\partial x_i} \qquad \text{and} \qquad Y_i = -\frac{\partial F_\lambda}{\partial y_i}. \tag{8.9.1}$$

Let us list the main properties of Bäcklund transformations, namely:

- canonicity,

- invariance of H, i.e., $H_i(X, x) = H_i(Y, y), \quad i = 1, ..., n,$

- commutativity, $B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1}$, where $\circ$ denotes the composition of canonical transformations, and

- algebraicity, which means that the equations describing the Bäcklund transformation B_λ are supposed to be algebraic with respect to X and Y and properly chosen functions of x and y.

Among these, it should be mentioned that canonicity and invariance of the H's are also common to integrable canonical mappings, but the Bäcklund transformations to be considered here are also assumed to

depend on an additional parameter λ, which will be seen to play a vital role in the subsequent analysis and is at the root of the richness of the underlying theory.

When the integrable system in question is solvable by the inverse scattering method, then it is often the case that the commuting Hamiltonians H_i are obtained from the coefficients of the characteristic polymonial of the Lax operator $L(u) \equiv L(u; X, x)$ viz

$$W(u, v; \{H_i\}) = \det\ (v - L(u)). \tag{8.9.2}$$

Since the invariance of H_i under B_λ is then equivalent to the invariance of the spectrum of $L(u)$, there exists an invertible matrix $M(u)$ so that

$$M(u)L(u; X, x) = L(u; Y, y)M(u). \tag{8.9.3}$$

8.9.1 Spectrality

Let μ be defined as the variable conjugated to λ by the relation,

$$\mu = -\frac{\partial F_\lambda}{\partial \mu}. \tag{8.9.4}$$

The notion of spectrality is apparently a new concept in the theory of Bäcklund transformations. When a Bäcklund transformation B_λ exists, then it is said to be *associated* with the Lax operator $L(u)$, if for some function $f(\mu)$, the pair $(\lambda, f(\mu))$ lies on the *spectral curve* of the Lax matrix:

$$W(\lambda, f(\mu); \{H_i\}) \equiv det(f(\mu) - L(\lambda)) = 0. \tag{8.9.5}$$

To understand the meaning of (8.9.5) one has to turn to the quantum case. In a pioneering paper by Pasquier and Gaudin [148] a remarkable connection was established between the classical Bäcklund transformation B_λ for the Toda lattice and Baxter's Q operator. The authors constructed a certain integral operator $\hat{Q}_\lambda$,

$$\hat{Q}_\lambda : \Psi(x) \longrightarrow \int dx Q_\lambda \Psi(x), \tag{8.9.6}$$

(here $dx = dx_1 \wedge .... \wedge dx_n$) whose properties parallel those of the classical Bäcklund transformation B_λ. In the quantum case, the canonical transformation is replaced by the similarity transformation,

$$\hat{Y}_i = \hat{Q}_\lambda \hat{X}_i \hat{Q}_\lambda^{-1}, \quad \text{and} \quad \hat{y}_i = \hat{Q}_\lambda \hat{x}_i \hat{Q}_\lambda^{-1}, \tag{8.9.7}$$

where the $\wedge$ distinguishes the quantum operators from their classical counterparts. In fact the correspondence between the kernel of $\hat{Q}_\lambda$, viz $Q_\lambda(y,x)$ and the generating function of the classical Bäcklund transformation is given by the semiclassical formula,

$$Q_\lambda(y,x) \approx exp\left(-\frac{i}{\hbar}F_\lambda(y;x)\right), \qquad \hbar \longrightarrow 0. \tag{8.9.8}$$

At this point let us state the fundamental defining properties of Baxter's Q operator.

8.9.2 Properties of the Q operator

The Q operator is assumed to satisfy the following operator identities:

$$\begin{aligned} [t(u), \hat{Q}_\lambda] &= 0, \\ [\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}] &= 0, \\ \hat{Q}_\lambda t(\lambda) &= \alpha(\lambda)\hat{Q}_{\lambda-1} + \beta(\lambda)\hat{Q}_{\lambda+1}, \end{aligned} \tag{8.9.9}$$

where α, β are functions of the parameter λ and u stands here for the spectral parameter. Corresponding to the first property above, which essentially translates to the property $[\hat{Q}_\lambda, H_i] = 0$ of the $\hat{Q}_\lambda$ operator (since $t(u)$ is a transfer matrix), the corresponding property of B_λ is that the Hamiltonian is invariant, i.e.,

$$H_i(X,x) = H_i(Y,y). \tag{8.9.10}$$

The commutativity $[\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}] = 0$ on the other hand corresponds to the property,

$$B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1}.$$

However, the most interesting property of $\hat{Q}_\lambda$ is that its eigenvalues $\phi(\lambda)$ on the joint eigenvectors Ψ_ν of H_i and Q_λ labelled with the quantum numbers ν, i.e.,

$$\hat{Q}_\lambda \Psi_\nu = \phi_\nu(\lambda)\Psi_\nu \tag{8.9.11}$$

satisfy the separation equation, which is a certain difference or differential equation of the form

$$\hat{W}\left(\lambda, -i\hbar\frac{d}{dx}; \{h_i\}\right)\phi_\nu(\lambda) = 0, \tag{8.9.12}$$

containing the eigenvalues h_i of H_i . The classical limit of (8.9.12) is the spectrality equation (8.9.5).

8.9.3 Application of spectrality to the separation of variables

To understand the role played by spectrality, it is useful to first consider the quantum case. By definition an operator $\hat{K}$ is said to be a *separating operator* if it transforms the joint eigenfunctions Ψ_ν of H_i, into a product of separated functions $\phi_\nu(\lambda)$ of a single variable λ, i.e.,

$$\hat{K}\Psi_\nu = c_\nu \prod_{i=1}^{n} \phi_\nu(\lambda_i), \tag{8.9.13}$$

satisfying the separation equation (8.9.11). In principle, since the coefficients c_ν in (8.9.13) can be chosen arbitrarily, there exists an infinite number of separating operators. However, the actual problem consists in finding those separating operators, which can be described as integral operators with explicitly given kernels. In fact, knowledge of a Q operator enables one to construct separating operators. If we consider the operator product $\hat{Q}_{\lambda_1....\lambda_n} \equiv \hat{Q}_{\lambda_1}....\hat{Q}_{\lambda_n}$, having kernel $Q_{\lambda_1...\lambda_n}(y,x)$, and any function $\rho(y)$, then it is possible to introduce the operator,

$$\hat{K}_\rho : \Psi(x) \longrightarrow \int dx \int dy \rho(y) Q_{\lambda_1....\lambda_n}(y,x)\Psi(x), \tag{8.9.14}$$

so that

$$\hat{K}_\rho : \Psi_\nu(x) \longrightarrow \int dx \int dy \rho(y) \prod_{i=1}^{n} \phi_\nu(\lambda_i)\Psi(x). \tag{8.9.15}$$

Here use has been made of $\hat{Q}_{\lambda_i}\Psi_\nu = \phi_\nu(\lambda_i)\Psi_\nu$, from which it is evident that $\hat{K}_\rho$ is a separating operator with the coefficients c_ν given by

$$c_\nu = \int dy \rho(y)\Psi_\nu(y). \tag{8.9.16}$$

As the eigenfunctions $\Psi_\nu(y)$ form a basis in the corresponding Hilbert space, the formula (8.9.16) provides a one-to-one correspondence between reasonably chosen classes of c_ν and $\rho(y)$. Hence (8.9.13) basically describes all possible separating operators. Their kernels $K_\rho(\lambda;x)$ are given explicitly by multiple integrals of the form,

$$K_\rho(\lambda;x) = \int dy \int d\xi^{(1)} .... \int d\xi^{(n-1)} \rho(y)$$

$$\times Q_{\lambda_1}(y;\xi^{(1)}) Q_{\lambda_2}(\xi^{(1)};\xi^{(2)}) .... Q_{\lambda_n}(\xi^{(n-1)};x). \tag{8.9.17}$$

The classical analogue of the preceding argument may be described as follows. Considering the composition $B_{\lambda_1....\lambda_n} = B_{\lambda_1} \circ .... \circ B_{\lambda_n}$ of Bäcklund transformations and its corresponding generating function $F_{\lambda_1...\lambda_n}(y,x)$, let us switch the roles of the y's and the λ's so that the λ's are now treated as the dynamical variables while the y's are now parameters. Then $F_{\lambda_1...\lambda_n}(y,x)$ becomes the generating function of the $n-$ parametric canonical transformation K_y from $(x,y) \longrightarrow (\mu,\lambda)$, given by the equations:

$$X_i = \frac{\partial F_{\lambda_1...\lambda_n}}{\partial x_i} \quad \text{and} \quad \mu_i = -\frac{\partial F_{\lambda_1...\lambda_n}}{\partial \lambda_i}. \tag{8.9.18}$$

It then follows that the pairs (λ_i, μ_i) satisfy the separation equation $W(\lambda_i, f(\mu_i); \{H_i\}) = 0$, which constitutes the definition of the separating canonical transformation in the classical case [147]. This classical construction corresponds in the quantum case, to setting $\rho(y) = \delta(y-\tilde{y}_1)....\delta(y-\tilde{y}_n)$, where $\tilde{y}_i$ are some constants.

It remains however an open question as to what could be the classical analogue of (8.9.17) for generic $\rho(y)$. Finally, it should be mentioned that for finite dimensional systems the composition of n Bäcklund transformations, with n being the number of degrees of freedom, is a sort of universal Bäcklund transformation in the sense that any other transformation preserving the Hamiltonians H_i must be expressible in terms of $B_{\lambda_1....\lambda_n}$. This follows from the observation that the transformation B_λ acts as a shift in the angle coordinate on the Liouville torus as $\phi_i \longrightarrow \phi_i + b_i(\lambda)$. For generic $b_i(\lambda)$ the sum $\sum_{j=1}^{n} b_i(\lambda)$ must then cover the n-dimensional Liouville torus, which results in the universality of $B_{\lambda_1...\lambda_n}$.

8.10 Sklyanin's Formalism for Canonical Bäcklund Transformations

Recently Sklyanin developed an elegant formalism based on the classical r matrix, for deriving Bäcklund transformations for a wide class of discrete nonlinear integrable systems and has provided an explicit proof of their canonicity, for Hamiltonian systems governed by an $SL(2)$-invariant r matrix. Sklyanin's formulation considers an inte-

grable Hamiltonian system with a matrix Lax operator $T(u)$, depending on the dynamical variables with u being a complex spectral parameter. Furthermore, it assumes that the commuting Hamiltonians of the system are generated by the *spectral invariants* of $T(u)$. To ensure that the transformation keeps the Hamiltonian of the system invariant, it is necessary that the spectral invariants be preserved under the transformation. Consequently, the Lax matrix $T(u)$ has to be related to the transformed Lax matrix $\tilde{T}(u)$ by a similarity transformation, so that

$$\tilde{T}(u) = M(u)T(u)M(u)^{-1}. \tag{8.10.1}$$

At this point one needs to address two questions. First, given a $T(u)$, what should be the form of the matrix $M(u)$ that will generate the required Bäcklund transformation? Secondly, assuming that the Bäcklund transformations exists, does it really preserve the Poisson brackets, i.e., does it generate a canonical transformation? These issues were specifically addressed in two brilliant papers [150, 152]. In describing this formalism we shall follow Sklyanin's original work as closely as possible. To this end, let us suppose that the Possion algebra of the Lax matrix $T(u)$ is expressible in the following form:

$$\left\{T^1(u), T^2(v)\right\} = \left[r_{12}(u-v), \quad T^1(u)T^2(v)\right], \tag{8.10.2}$$

where standard notations $T^1(u) = T(u) \otimes I$ and $T^2(u) = I \otimes T(u)$ have been used. Here

$$r_{12}(u-v) = \frac{\kappa \mathcal{P}_{12}}{u-v},$$

is the standard $SL(2)$-invariant solution of the *classical Yang-Baxter equation*, κ is a constant while $\mathcal{P}_{12}$ is the permutation operator in $C^2 \otimes C^2$. It is interesting to note that this class of integrable systems includes such well-known models as the nonlinear Schrödinger equation, Heisenberg magnetic chain, Toda lattice. Next the following ansatz for the matrix $M(u)$ is made:

$$M(u) = \begin{pmatrix} u - \lambda_1 + pq & p \\ -pq^2 + 2\mu q & u - \lambda_2 - pq \end{pmatrix}, \tag{8.10.3}$$

where λ_1 and λ_2 are taken to be the free parameters of the Bäcklund transformations, while the parameters p and q are to be determined from the defining equation of the Bäcklund transformations, namely

(8.10.1), noting that $\mu = (\lambda_1 - \lambda_2)/2$. We introduce the eigenbasis of $M(u)$:

$$|1>= \frac{1}{2\mu}\begin{pmatrix}1\\ q\end{pmatrix}, \qquad |2>= \frac{1}{2\mu}\begin{pmatrix}p\\ 2\mu - pq\end{pmatrix}, \tag{8.10.4}$$

together with its dual basis,

$$< 1| = (2\mu - pq, -p), \qquad < 2| = (q, 1), \tag{8.10.5}$$

we can define the spectral projectors,

$$P_{ij} = |i >< j| \qquad i, j \in \{1, 2\}. \tag{8.10.6}$$

The spectral projectors satisfy

$$P_{ij}P_{kl} = \delta_{jk}P_{il}, \tag{8.10.7}$$

and in terms of them, the matrix $M(u)$ is decomposed in the following form:

$$M(u) = (u - \lambda_1)P_{11} + (u - \lambda_2)P_{22}. \tag{8.10.8}$$

Furthermore, one finds that $M^{-1}(u)$ may also be decomposed in terms of the projectors as follows:

$$M(u)^{-1} = (u - \lambda_1)^{-1}P_{11} + (u - \lambda_2)^{-1}P_{22}. \tag{8.10.9}$$

Notice that

$$\det\, M(u) = (u - \lambda_1)(u - \lambda_2) \tag{8.10.10}$$

is degenerate at λ_1 and λ_2. To determine the parameters p and q, let us assume that $T(u)$ is a polynomial in u. Furthermore, in order that it retains this character even after the transformation, we require from (8.10.1) that Res $\widetilde{T}(u)|_{u=\lambda_i}, i = 1, 2$ should vanish. This gives rise to the following conditions, namely

$$< 2|T(\lambda_1)|1 >= 0 \qquad < 1|T(\lambda_2)|2 >= 0, \tag{8.10.11}$$

which may be expressed also in the form,

$$\mathrm{tr}P_{12}T(\lambda_1) = 0, \qquad \mathrm{tr}P_{21}T(\lambda_2) = 0. \tag{8.10.12}$$

These conditions implicitly determine the parameters p and q, so that with p and q determined by them, the matrix $\widetilde{T}(u)$ as defined by (8.10.1)

is a function of the dynamical variables and the two free parameters λ_1 and λ_2. To show that the transformation given by (8.10.1) preserves the Poisson bracket (8.10.2), we have to show that $\widetilde{T}(u)$ satisfies the relation,

$$\left\{\widetilde{T}^1(u), \widetilde{T}^2(v)\right\} = \left[r_{12}(u-v), \quad \widetilde{T}^1(u)\widetilde{T}^2(v)\right]. \tag{8.10.13}$$

Substituting (8.10.1) into the right-hand side of (8.10.13) and differentiating the products of matrices one arrives at the following expression, namely

$$\begin{aligned}\left\{\widetilde{T}^1(u), \widetilde{T}^1(u)\right\} &= \langle M^1 M^2\rangle \widetilde{T}^1(u)\widetilde{T}^2(v) - \widetilde{T}^1(u)\langle M^1 M^2\rangle \widetilde{T}^2(v) + \\ &+ \langle M^1 T^2\rangle \widetilde{T}^1(u) + [\widetilde{r}_{12}, \widetilde{T}^1(u)\widetilde{T}^2(v)] - \widetilde{T}^1(u)\langle M^1 T^2\rangle - \\ &- \widetilde{T}^2(v)\langle M^1 M^2\rangle \widetilde{T}^1(u) - \widetilde{T}^2(v)\langle T^1 M^2\rangle + \widetilde{T}^1(u)\widetilde{T}^2(v)\langle M^1 M^2\rangle, \end{aligned} \tag{8.10.14}$$

where the following notation has been introduced:

$$\langle T^1 M^2\rangle = M^1(u)\{T^1(u), M^2(u)\}M^1(u)^{-1}M^2(v)^{-1}, \tag{8.10.15}$$

$$\langle M^1 T^2\rangle = M^2(v)\{M^1(u), T^2(v)\}M^1(u)^{-1}M^2(v)^{-1}, \tag{8.10.16}$$

$$\langle M^1 M^2\rangle = M^1(u)M^2(v)\{M^1(u), M^2(v)\}M^1(u)^{-1}M^2(v)^{-1}, \tag{8.10.17}$$

and

$$\widetilde{r}_{12} = M^1(u)M^2(v)r_{12}M^1(u)^{-1}M^2(v)^{-1}. \tag{8.10.18}$$

To evaluate the right-hand side of (8.10.14) the following identity is useful:

$$\mathcal{P}_{12} = P^1_{11}P^2_{11} + P^1_{12}P^2_{21} + P^1_{21}P^2_{12} + P^1_{22}P^2_{22}, \tag{8.10.19}$$

using this, one can show that

$$\widetilde{r}_{12} = r_{12} + 2\mu\kappa\left\{\frac{P^1_{12}P^2_{21}}{(u-\lambda_2)(v-\lambda_1)} - \frac{P^1_{21}P^2_{12}}{(u-\lambda_1)(v-\lambda_2)}\right\}. \tag{8.10.20}$$

Next, to calculate the Poisson brackets between $T(u)$ and $M(u)$, the following technique is employed . It will be noticed from (8.10.12) that if f is any function on the phase space, then

$$\{f, \mathrm{tr}P_{12}T(\lambda_1)\} = \{f, \mathrm{tr}P_{21}T(\lambda_2)\} = 0. \tag{8.10.21}$$

But as

$$\{f, \mathrm{tr}P_{12}T(\lambda_1)\} = \{f,p\}\mathrm{tr}\frac{\partial P_{12}}{\partial p}T(\lambda_1)+$$

$$+\{f,q\}\mathrm{tr}\frac{\partial P_{12}}{\partial q}T(\lambda_1) + \mathrm{tr}P_{12}\{f, T(\lambda_1)\} = 0, \tag{8.10.22}$$

and likewise

$$\{f, \mathrm{tr}P_{21}T(\lambda_2)\} = \{f,p\}\mathrm{tr}\frac{\partial P_{21}}{\partial p}T(\lambda_2)+$$

$$+\{f,q\}\mathrm{tr}\frac{\partial P_{21}}{\partial q}T(\lambda_2) + \mathrm{tr}P_{21}\{f, T(\lambda_2)\} = 0. \tag{8.10.23}$$

Treating (8.10.22 and 8.10.23) as a linear system of equations in $\{f,p\}$ and $\{f,q\}$, we find that as

$$\frac{\partial P_{12}}{\partial p} = 0, \quad \frac{\partial P_{12}}{\partial q} = \frac{1}{2\mu}(P_{11} - P_{22}) + \frac{p}{\mu}P_{12}, \tag{8.10.24}$$

$$\frac{\partial P_{21}}{\partial p} = P_{11} - P_{22}, \qquad \frac{\partial P_{21}}{\partial q} = \frac{p^2}{2\mu}(P_{11} - P_{22}) - \frac{p}{\mu}P_{21}, \tag{8.10.25}$$

$$\frac{\partial M(u)}{\partial p} = 2\mu p, \qquad \frac{\partial M(u)}{\partial q} = p^2 P_{12} + P_{21}, \tag{8.10.26}$$

so upon introducing the notation,

$$w_i(\lambda) = \mathrm{tr}P_{ii}T(\lambda), \quad \text{and} \quad w(\lambda) = w_1(\lambda) - w_2(\lambda), \tag{8.10.27}$$

one obtains the following result:

$$\{f, M(v)\} = -\frac{2\mu}{w(\lambda_1)}P_{21}\mathrm{tr}\{f, T(\lambda_1)\}P_{12} - \frac{2\mu}{w(\lambda_2)}P_{12}\mathrm{tr}\{f, T(\lambda_2)\}P_{21}. \tag{8.10.28}$$

Then using (8.10.2) it remains to calculate the Poisson brackets occurring in (8.10.15–8.10.17), which turn out to be as follows:

$$\begin{aligned}
&\{T^1(u), M^2(v)\} \\
&\quad = -\frac{2\mu\kappa}{(u-\lambda_1)w(\lambda_1)}\left[w_1(\lambda_1)P^1_{12}T^1(u)P^2_{21} - w_2(\lambda_1)T^1(u)P^1_{12}P^2_{21}\right] - \\
&\qquad -\frac{2\mu\kappa}{(u-\lambda_2)w(\lambda_2)}\left[w_2(\lambda_2)P^1_{21}T^1(u)P^2_{12} - w_1(\lambda_2)T^1(u)P^1_{21}P^2_{12}\right],
\end{aligned} \tag{8.10.29}$$

$$\{M^1(u), T^2(v)\}$$

$$= \frac{2\mu\kappa}{(v-\lambda_1)w(\lambda_1)}\left[w_1(\lambda_1)P^1_{21}P^2_{12}T^2(v) - w_2(\lambda_1)P^1_{21}T^2(v)P^2_{12}\right] +$$

$$+\frac{2\mu\kappa}{(v-\lambda_2)w(\lambda_2)}\left[w_2(\lambda_2)P^1_{12}P^2_{21}T^2(v) - w_1(\lambda_2)P^1_{12}T^2(v)P^2_{21}\right], \tag{8.10.30}$$

$$\{M^1(u), M^2(v)\} =$$

$$-\frac{2\mu\kappa\left(w_1(\lambda_1)w_2(\lambda_2) - w_2(\lambda_1)w_1(\lambda_2)\right)}{w(\lambda_1)w(\lambda_2)}\left[P^1_{12}P^2_{21} - P^1_{21}P^2_{12}\right]. \tag{8.10.31}$$

Then recalling the $\langle , \rangle$ notation introduced earlier, one arrives at the following expressions for

$$\langle T^1 M^2\rangle = -\frac{2\mu\kappa}{(u-\lambda_2)(v-\lambda_1)}\times$$

$$\times\left(\frac{w_1(\lambda_1)}{w(\lambda_1)}P^1_{12}\widetilde{T}^1(u)P^2_{21} - \frac{w_2(\lambda_1)}{w(\lambda_1)}\widetilde{T}^1(u)P^1_{12}P^2_{21}\right)$$

$$-\frac{2\mu\kappa}{(u-\lambda_1)(v-\lambda_2)}\left(\frac{w_2(\lambda_2)}{w(\lambda_2)}P^1_{21}\widetilde{T}^1(u)P^2_{12} - \frac{w_1(\lambda_2)}{w(\lambda_2)}\widetilde{T}^1(u)P^1_{21}P^2_{12}\right), \tag{8.10.32}$$

$$\langle M^1 T^2\rangle = \frac{\mu\kappa}{(u-\lambda_1)(v-\lambda_2)}\times$$

$$\times\left(\frac{w_1(\lambda_1)}{w(\lambda_1)}P^1_{21}P^2_{12}\widetilde{T}^2(v) - \frac{w_2(\lambda_1)}{w(\lambda_1)}P^1_{21}\widetilde{T}^2(v)P^2_{12}\right)$$

$$+\frac{\mu\kappa}{(u-\lambda_2)(v-\lambda_1)}\left(\frac{w_2(\lambda_2)}{w(\lambda_2)}P^1_{12}P^2_{12}\widetilde{T}^2(v) - \frac{w_1(\lambda_2)}{w(\lambda_2)}P^1_{12}\widetilde{T}^2(v)P^2_{21}\right), \tag{8.10.33}$$

$$\langle M^1 M^2\rangle = -\frac{2\mu\kappa\left(w_1(\lambda_1)w_2(\lambda_2) - w_2(\lambda_1)w_1(\lambda_2)\right)}{w(\lambda_1)w(\lambda_2)}\times$$

$$\times\left(\frac{P^1_{12}P^2_{21}}{(u-\lambda_2)(v-\lambda_1)} - \frac{P^1_{21}P^2_{12}}{(u-\lambda_1)(v-\lambda_2)}\right). \tag{8.10.34}$$

Finally substituting (8.10.29 and 8.10.30) with (8.10.31) into (8.10.14), after much simplification one obtains the equality (8.10.13). This completes the proof of the canonicity of the Bäcklund transformations for the case under consideration.

8.11 Extended-Phase Space Method

In the preceding section it was shown by direct calculation that the two parametric Bäcklund transformations $B_{\lambda_1\lambda_2}$ corresponding to the matrix $M_{\lambda_1\lambda_2}$ as given by (8.10.1) were canonical transformations. The explicit form of $M_{\lambda_1\lambda_2}$ was introduced as an ansatz; however the fact that this particular form coincides with the elementary Lax matrix for the Heisenberg XXX magnet [151] with canonical Poisson brackets is not just a mere coincidence. By invoking the notion of an extended-phase space and imposing suitable constraints, it can be shown to arise in a natural way. Such an indirect approach turns out to be not only simpler than the brute force method introduced earlier, but also provides a systematic procedure for constructing multiparametric families of Bäcklund transformations. In this section we consider these and other related issues, illustrating them with suitable examples.

Let us consider a finite dimensional Hamiltonian system, defined by canonical variables (X, x):

$$\{X_i, X_j\} = \{x_i, x_j\} = 0, \quad \text{and} \quad \{X_i, x_j\} = \delta_{ij}. \tag{8.11.1}$$

Suppose that the system is completely integrable and possesses a Lax matrix $T(u; X, x)$ whose spectral invariants generate the commuting Hamiltonians H_n, with u being a complex parameter. Denoting the Lax matrix $T(u; X, x)$ by the notation $L^{(x)}(u)$, we assume that $T^{(x)}(u)$ is a matrix of order 2×2 and satisfies the following Poisson algebra:

$$\{\overset{1}{T}(u), \overset{2}{T}(v)\} = [r_{12}(u-v), \overset{1}{T}(u)\,\overset{2}{T}(v)], \tag{8.11.2}$$

where $\overset{1}{T} = T \otimes I, \overset{2}{T} = I \otimes T$. Here $r_{12} = \kappa(u-v)^{-1}\mathcal{P}_{12}$ is the standard $SL(2)$-invariant solution of the classical Yang-Baxter equation. For a one-parametric family of Bäcklund transformations that preserve the Hamiltonians H_n, it is necessary that there exists a matrix $M_\lambda(u)$ so that

$$M_\lambda(u)T^{(x)}(u) = T^{(y)}(u)M_\lambda(u), \tag{8.11.3}$$

since the Hamiltonians are just the spectral invariants of the Lax matrix. Here, $Y = Y(X, x)$ and $y = y(X, x)$, so that $M_\lambda(u) = M_\lambda(u; X, x)$. Furthermore, we shall assume that $M_\lambda(u) = M(u-\lambda)$, where λ denotes the Bäcklund parameter. We consider next an *extended-phase phase*

obtained by augmenting the canonical variables (X, x) with an independent set of auxiliary canonical variables (S, s). It will be assumed that the variables (S, s) commute with (X, x) and the dimension of the phase space (S, s) bears no relation to that of the (X, x).

Furthermore, we assume there exists a matrix $M^{(s)}(u)$, which induces a canonical transformation $\mathcal{R}_\lambda : (X, x; S, s) \to (Y, y; T, t)$ determined from the equation [152],

$$M^{(s)}(u-\lambda)T^{(x)}(u) = T^{(y)}(u)M^{(t)}(u-\lambda), \tag{8.11.4}$$

with the matrix $M^{(s)}(u)$ obeying the same Poisson brackets as $T(u)$, i.e.,

$$\{\overset{1}{M}(u), \overset{2}{M}(v)\} = [r_{12}(u-v), \overset{1}{M}(u)\ \overset{2}{M}(v)]. \tag{8.11.5}$$

If the canonical transformations admits a generating function, say $F_\lambda(y, t; x, s)$, so that one has (temporarily dropping the indices i in X_i, x_i)

$$X = \frac{\partial F_\lambda}{\partial x}, \quad Y = -\frac{\partial F_\lambda}{\partial y}, \quad S = \frac{\partial F_\lambda}{\partial s}, \quad T = -\frac{\partial F_\lambda}{\partial t}, \tag{8.11.6}$$

then imposing the constraint,

$$t = s, \qquad T = S, \tag{8.11.7}$$

allows one to resolve (8.11.6 and 8.11.7) with respect to $s = t$ and to express X and Y as functions of (x, y). Under these conditions one arrives at the following proposition.

Proposition: The resulting transformation $B_\lambda(X, x) \longrightarrow (Y, y)$ is canonical and is given by the generating function $\Phi_\lambda(x, y) = F_\lambda(y, s(x, y); x, s(x, y))$, so that

$$X = \frac{\partial \Phi_\lambda}{\partial x}, \qquad Y = -\frac{\partial \Phi_\lambda}{\partial y}. \tag{8.11.8}$$

Proof:

$$\begin{aligned} X = \frac{\partial \Phi_\lambda}{\partial x} &= \left(\frac{\partial F_\lambda}{\partial x}\right)_{st} + \left(\frac{\partial F_\lambda}{\partial s}\right)_{st} \frac{\partial s}{\partial x} + \left(\frac{\partial F_\lambda}{\partial t}\right)_{st} \frac{\partial t}{\partial x}, \\ &= \left(\frac{\partial F_\lambda}{\partial x}\right)_{st} + \frac{\partial s}{\partial x}\left(\frac{\partial F_\lambda}{\partial s} + \frac{\partial F_\lambda}{\partial t}\right)_{st}, \end{aligned} \tag{8.11.9}$$

where $()_{st}$ stands for restriction on the constraint manifold $s = t = s(x, y)$. However, due to the constraint $S = T$, it is observed that

$$\left(\frac{\partial F_\lambda}{\partial s} + \frac{\partial F_\lambda}{\partial t}\right)_{st} = 0. \tag{8.11.10}$$

Consequently $X = \partial\Phi_\lambda/\partial x$. Similarly one can show that $Y = -\partial\Phi_\lambda/\partial y$. Note that the constraint (8.11.7) essentially implies the identity $M^{(s)}(u-\lambda) = M^{(t)}(u-\lambda)$ as a result of which (8.11.4) turns into the equality (8.11.3), ensuring thereby that the transformation preserves the spectrum of $T(u)$.

In many cases, it often happens that the Lax matrix is a monodromy matrix factorized, into the product of local Lax matrices $\ell_i(u)$, i.e.,

$$T(u) = \ell_N(u)...\ell_2(u)\ell_1(u), \tag{8.11.11}$$

and obey the same Poisson brackets as the monodromy matrix $T(u)$,

$$\{\overset{1}{\ell_i}(u), \overset{2}{\ell_j}(v)\} = [r_{12}(u-v), \overset{1}{\ell_i}(u)\overset{2}{\ell_j}(v)]\delta_{ij}. \tag{8.11.12}$$

Under this condition, the similarity transformation defined earlier is replaced by a gauge transformation of the form

$$M_i(u-\lambda)\ell_i^{(x)}(u) = \ell_i^{(y)}(u)M_{i-1}(u-\lambda). \tag{8.11.13}$$

The latter ensures that the spectral invariants of $T(u)$ are preserved.

The modification of the reduction procedure described earlier is as follows. Suppose that $\ell_i(u)$ and $M_i(u)$ depend on local canonical variables:

$$\ell_i^{(x)}(u) \equiv \ell_i(u; X_i, x_i), \qquad \ell_i^{(y)}(u) \equiv \ell_i(u; Y_i, y_i), \tag{8.11.14}$$

$$M_i^{(s)}(u) \equiv M(u; S_i, s_i), \qquad M_i^{(t)}(u) \equiv M(u; T_i, t_i). \tag{8.11.15}$$

Under these conditions, one can first define a local canonical transformation $\mathcal{R}_\lambda^{(i)} : (X_i, x_i; S_i, s_i) \longrightarrow (Y_i, y_i; T_i, t_i)$, by means of the relation,

$$M_i^{(s)}(u-\lambda)\ell_i^{(x)}(u) = \ell_i^{(y)}(u)M_i^{(t)}(u-\lambda). \tag{8.11.16}$$

Denoting the corresponding generating function by $f_\lambda^{(i)}(y_i, t_i; x_i, s_i)$, we consider the direct product of N phase spaces $(X_i, x_i; S_i, s_i)$ and $(Y_i, y_i; T_i, t_i)$, respectively. Then the generating function,

$$F_\lambda = \sum_{i=1}^{N} f_\lambda^{(i)}(y_i, t_i; x_i, s_i), \tag{8.11.17}$$

determines the direct product $\mathcal{R}_\lambda$ of local transformations $\mathcal{R}_\lambda^{(i)}$. Imposing the constraints,

$$t_i = s_{i-1}, \qquad T_i = S_{i-1}, \tag{8.11.18}$$

and assuming periodicity $i + N \equiv i$, the proof of the canonicity of the resulting transformation B_λ $(X, x) \longrightarrow (Y, y)$, can be reconstructed in the same way as before. We illustrate the above method by the following example.

Example: Toda lattice The local Lax operator for the periodic Toda lattice with inhomogeneities is [165]

$$\ell_l^{(x)}(u) = \begin{pmatrix} u - c_i + X_i & -e^{x_i} \\ e^{-x_i} & 0 \end{pmatrix}, \tag{8.11.19}$$

where $(X_i, x_i)_{i=1}^N$ are a set of canonical variables with

$$\{X_i, x_j\} = \delta_{ij}, \qquad \{X_i, X_j\} = \{x_i, x_j\} = 0. \tag{8.11.20}$$

Here c_i are the the inhomogeneity parameters and u is the spectral parameter. A simple computation shows that $\ell_i^{(x)}(u)$ satisfies

$$\{\ell_i^{(x)}(u) \stackrel{\otimes}{,} \ell_i^{(x)}(v)\} = [r(u - v), \ell_i^{(x)}(u) \otimes \ell_i^{(x)}(v)], \tag{8.11.21}$$

with $r(u-v) = (u-v)^{-1}\mathcal{P}$. The spectral invariants of the monodromy operator,

$$T(u; X, x) = \ell_N(u)....\ell_2(u)\ell_1(u), \tag{8.11.22}$$

follow from $t(u) = \text{tr } T(u; X, x) = \sum u^n H_n$. The search for an alternative representation of the Poisson bracket algebra (8.11.21) and hence, an ansatz for the matrix $M_i(u - \lambda)$ leads us to consider the Lax operator for the DST model [157], which admits the following local Lax matrix:

$$M_i(u - \lambda) = \begin{pmatrix} u - \lambda + S_i s_i & s_i \\ S_i & 1 \end{pmatrix}. \tag{8.11.23}$$

$M_i(u-\lambda)$ obeys (8.11.21) and the variables S_i, s_i satisfy the Poisson brackets,

$$\{S_i, s_j\} = \delta_{ij}, \qquad \{S_i, S_j\} = \{s_i, s_j\} = 0, \tag{8.11.24}$$

with λ being the Bäcklund parameter. Then using (8.11.19), (8.11.23) in (8.11.16), we obtain the following set of equations:

$$\begin{gathered}
X_i + S_i s_i = T_i t_i + Y_i, \\
-\lambda X_i + S_i s_i (X_i - c_i) + s_i e^{-x_i} = -\lambda Y_i + T_i t_i (Y - i - c_i) - T_i e^{-y_i}, \\
t_i = -e^{x_i}, \qquad (\lambda - S_i s_i) e^{x_i} = (Y_i - c_i) t_i - e^{y_i}, \\
S_i = e^{-y_i} \qquad (X_i - c_i) S_i + e^{-x_i} = (T_i t_i - \lambda) e^{-y_i}, \\
S_i e^{-x_i} = e^{-y_i} t_i.
\end{gathered} \tag{8.11.25}$$

These equations are found to be self-consistent and may be solved to yield

$$\begin{gathered}
S_i = e^{-y_i}, \qquad X_i = -e^{-x_i + y_i} - T_i e^{x_i} + (c_i - \lambda), \\
t_i = -e^{x_i}, \qquad Y_i = -e^{-x_i + y_i} + s_i e^{-y_i} + (c_i - \lambda).
\end{gathered} \tag{8.11.26}$$

The corresponding generating function is

$$f_\lambda^{(i)}(x_i, s_i | y_i, T_i) = e^{-x_i + y_i} + s_i e^{-y_i} - T_i e^{x_i} + (c_i - \lambda)(x_i - y_i), \tag{8.11.27}$$

so that

$$\begin{gathered}
X_i = \frac{\partial f_\lambda^{(i)}}{\partial x_i}, \qquad Y_i = -\frac{\partial f_\lambda^{(i)}}{\partial y_i}, \\
S_i = \frac{\partial f_\lambda^{(i)}}{\partial s_i}, \qquad t_i = \frac{\partial f_\lambda^{(i)}}{\partial T_i}.
\end{gathered} \tag{8.11.28}$$

Next if we demand that $t_i = s_{i-1}$ and $T_i = S_{i-1}$ we get

$$\begin{gathered}
s_i = t_{i+1} = -e^{x_{i+1}}, \qquad X_i = -e^{-x_i + y_i} - e^{x_i - y_{i-1}} + (c_i - \lambda), \\
T_i = S_{i-1} = e^{-y_{i-1}}, \quad \text{and} \quad Y_i = -e^{-x_i + y_i} - e^{x_{i+1} - y_i} + (c_i - \lambda).
\end{gathered} \tag{8.11.29}$$

Hence, the auxiliary variables may be eliminated from (8.11.27) to yield the local generating function, in the following form:

$$f_\lambda^{(i)} = e^{-x_i + y_i} - e^{x_i - y_{i-1}} - e^{x_{i+1} - y_i} + (c_i - \lambda)(x_i - y_i). \tag{8.11.30}$$

The global generating function is then simply given by

$$\Phi_\lambda = \sum_{i=1}^{N} \{e^{-x_i+y_i} - e^{x_i-y_{i-1}} - e^{x_{i+1}-y_i} + (c_i - \lambda)(x_i - y_i)\} \quad (8.11.31)$$

Upon using (8.11.29), the matrix $M_i^{(s)}(u)$ as defined by (8.11.23), may be written as

$$M_i^{(s)}(u) = \begin{pmatrix} u - e^{x_{i+1}-y_i} & -e^{x_{i+1}} \\ e^{-y_i} & 1 \end{pmatrix} = M_{i+1}^{(t)}(u). \quad (8.11.32)$$

Example: DST model As our second example, we consider the DST model, which lies intermediate between the Heisenberg XXX model and the Toda lattice. In [162] the following representation is chosen for the local Lax matrix of the DST model:

$$\ell_i^{(x)}(u) = \begin{pmatrix} u - c_i - X_i x_i & x_i \\ -X_i & 1 \end{pmatrix}. \quad (8.11.33)$$

It satisfies the Poisson brackets (8.11.21) with the r matrix,

$$r(u) = -\mathcal{P}u^{-1}. \quad (8.11.34)$$

For an alternate realization of the same Poisson brackets, we make the following choice for the matrix $M_i(u)$:

$$M_i^{(s)}(u) = \begin{pmatrix} 1 & -S_i \\ s_i & u - S_i s_i \end{pmatrix}. \quad (8.11.35)$$

It follows that the local canonical transformation $\mathcal{R}_\lambda$ is given by the equations:

$$X_i = s_i, \qquad Y_i = t_i + \frac{(c_i - \lambda)s_i}{1 - s_i y_i}, \quad (8.11.36)$$

$$T_i = y_i, \qquad S_i = x_i - \frac{(c_i - \lambda)y_i}{1 - s_i y_i},$$

or equivalently by the generating function:

$$f_\lambda^{(i)}(y_i, t_i | x_i, s_i) = x_i s_i - y_i t_i + (c_i - \lambda)\log(1 - s_i y_i). \quad (8.11.37)$$

Following the general scheme previously outlined we impose constraints to obtain

$$s_i = \frac{1}{y_i} + \frac{c_i - \lambda}{y_{i+1} - x_i}. \quad (8.11.38)$$

After substitution in (8.11.36) we obtain

$$X_i = \frac{1}{y_i} + \frac{c_i - \lambda}{y_{i+1} - x_i}, \tag{8.11.39}$$

$$Y_i = \frac{1}{y_{i+1}} + \frac{x_i - y_{i+1}}{y_i^2} + \frac{c_{i-1} - \lambda}{y_i - x_{i-1}} - \frac{c_i - \lambda}{y_i}. \tag{8.11.40}$$

One can then immediately verify that these equations describe a canonical transformation whose generating function is

$$\Phi_\lambda(y|x) = \sum_i \frac{x_i - y_{i+1}}{y_i} + (c_i - \lambda) \log \frac{y_i}{y_{i+1} - x_i}. \tag{8.11.41}$$

8.12 Quantization of Bäcklund Transformations

In this section we describe a quantum analog of the classical Bäcklund transformation, which directly leads to a representation of Baxter's famous Q operator (see Chapters 9 and 10 of [64]). Broadly, there are two approaches to constructing such an operator. One method according to Pasquier and Gaudin [148] uses the techniques of statistical mechanics to construct the Q operator as an integral operator, the kernel of which is related to the generating function of the corresponding classical canonical Bäcklund transformation. An alternative procedure according to Bazhanov et al. [155] relies on defining it as a trace of a monodromy matrix with infinite dimensional auxiliary space. Recently a method that combines the essential features of both and based on the r matrix formalism has been developed by Kuzntesov et al. [156]. Prior to describing the actual methods for constructing such operators, a few preliminary remarks are in order. First, from the basic definition given in (8.9.9), the Q operator designated by Q_λ is assumed to share a common set of eigenvectors with the Hamiltonians H_i of the model under consideration. Secondly, its eigenvalues are polynomials that satisfy a finite difference equation, known as Baxter's equation. It is assumed to satisfy the following operator identities:

- $t(\lambda)\hat{Q}(\lambda) = \Delta_+(\lambda)\hat{Q}(\lambda + \eta) + \Delta_-(\lambda)\hat{Q}(\lambda - \eta)$
- $\hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu)$
- $t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu).$

In [156] Pasquier and Gaudin constructed a family of integral operators $Q(u)$ for the Toda lattice, satisfying the matrix relation,

$$t(u)Q(u) = i^N Q(u + i\hbar) + i^{-N} Q(u - i\hbar). \tag{8.12.1}$$

They also showed that a similarity relation $\mathcal{O} \to Q_\lambda \mathcal{O} Q_\lambda^{-1}$ reduces in the classical limit to a classical Bäcklund transformation, that is to a one-parametric family of canonical transformations preserving the commuting Hamiltonians. In (8.12.1) the operators $Q(u), Q(v), t(v)$ commute for all values u, v. In the language of statistical mechanics the matrix Q is labelled with its rows and columns indexed by the (continuous) variables $(q_1, ...q_N), (q'_1, ...q'_N)$, where the q_i's and q'_i's both satisfy the Toda lattice equation. Let us consider the equation for the columns of Q, namely $y_n(q_1, ...q_N)$, taken in the form of a direct product so that

$$y(q_1, ...q_N) = \prod_{i=1}^{N} \varphi_i(q_i). \tag{8.12.2}$$

Consequently, one has as a result,

$$t(u)y = \text{tr } (\ell_1 \varphi_1)....(\ell_N \varphi_N), \tag{8.12.3}$$

where $\ell_i(u)$ actually represents the local Lax matrix at the ith site for the lattice. Specifically

$$\ell_i(u) = \begin{pmatrix} u - p_i & e^{q_i} \\ -e^{-q_i} & 0 \end{pmatrix}, \tag{8.12.4}$$

and may be shown to satisfy the quantum Yang-Baxter relation,

$$R_{12}(u_1 - u_2)\ \overset{1}{\ell}(u_1)\overset{2}{\ell}(u_2) = \overset{2}{\ell}(u_2)\overset{1}{\ell}(u_1)\ R_{12}(u_1 - u_2), \tag{8.12.5}$$

with the quantum R matrix given by

$$R(u) = \begin{pmatrix} u - i\hbar & 0 & 0 & 0 \\ 0 & u & -i\hbar & 0 \\ 0 & -i\hbar & u & 0 \\ 0 & 0 & 0 & u - i\hbar \end{pmatrix}. \tag{8.12.6}$$

Now the product $t(u)y$ can be shown to decompose into two terms $y' + y''$ if each of the matrices $\ell_j \varphi_j$ is lower triangular. Since $t(u)$ is

by definition a trace of the monodromy operator, consequently a local gauge transformation to $\widetilde{\ell}_j = N_i \ell_j N_{j+1}^{-1}$ with

$$N_j = \begin{pmatrix} 1 & ie^{q'_j} \\ 0 & 1 \end{pmatrix} \tag{8.12.7}$$

keeps $t(u)$ invariant. Thus, one can equate to zero the upper-right coefficient of $\widetilde{\ell}_j \varphi_j$, obtaining thereby the equation

$$(p_j + (-i)e^{q_j - q'_{j+1}} + ie^{q'_j - q_j} - u)\varphi_j = 0. \tag{8.12.8}$$

This equation is solved by

$$\varphi_j(u) = \exp\left(\frac{1}{\hbar}(iu(q_j - q'_{j+1}) - e^{q_j - q'_{j+1}} - e^{q'_j - q_j})\right), \tag{8.12.9}$$

and as a result we have

$$\widetilde{\ell}_j \varphi_j = \begin{pmatrix} -i\varphi_j(u - i\hbar) & 0 \\ \star & i\varphi_j(u + i\hbar) \end{pmatrix}. \tag{8.12.10}$$

It follows directly from (8.12.10) that (8.12.5) is satisfied with y substituted for Q. Let us now define the kernel of $Q(u)$ by

$$\begin{aligned} Q_u(q|q') &= \exp\frac{1}{\hbar}\left(iu(\sum_{j=1}^{N} q_j - \sum_{j=1}^{N} q'_j) - \sum_{j=1}^{N}(e^{q'_j - q_j} + e^{q_j - q'_{j+1}})\right) \\ &= \prod_{j=1}^{N} W_u(q'_j - q_j) W'_u(q_j - q'_{j+1}), \end{aligned} \tag{8.12.11}$$

with

$$W_u(q) = \exp\left(\frac{1}{\hbar}(-\frac{iu}{2}q - e^q)\right), \tag{8.12.12}$$

$$W'_u(q) = \exp\left(\frac{1}{\hbar}(\frac{iu}{2}q - e^q)\right). \tag{8.12.13}$$

By construction Q satisfies (8.9.9) and a similar analysis reveals that it also satisfies

$$Q(u)t(u) = i^N Q(u + i\hbar) + i^{-N} Q(u - i\hbar), \tag{8.12.14}$$

thereby implying the commutation relation,

$$[Q(u), t(u)] = 0.$$

In order to show that $Q(u)$'s with different spectral parameters u commute, one introduces the permutation operator C,

$$(Cf)(q_1, ...q_N) = f(q_2, q_3, ..., q_1), \tag{8.12.15}$$

and the kernel $\hat{Q}$,

$$\begin{aligned}\hat{Q}_u(q|q') &= (Q_u C)(q|q') = (CQ_u)(q|q'),\\ &= \prod_{j=1}^{N} W_u(q'_j - q_{j+1}) W'_u(q_j - q'_j).\end{aligned} \tag{8.12.16}$$

Then the equality $\hat{Q}(u)Q(v) = \hat{Q}(v)Q(u)$ is realized if there exist functions $A_u(q)$ satisfying the identity,

$$\begin{aligned}&A_{u-v}(q_1 - r_1) \int_{-\infty}^{\infty} dq W'_u(q_1 - q) W_u(q - q_2) W_v(r_1 - q) W'_v(q - r_2)\\ &= A_{u-v}(q_2 - r_2) \int_{-\infty}^{\infty} dq W'_v(q_1 - q) W_v(q - q_2) W_u(r_1 - q) W'_u(q - r_2)\end{aligned} \tag{8.12.17}$$

for all values of u, v, q, q_1, q_2, r_1 and r_2. Calculations for $A_u(q)$ yields the following solution of (8.12.17):

$$A_u(q) = \left(\cosh \frac{q}{2}\right)^{\frac{iu}{\hbar}}. \tag{8.12.18}$$

Now for real u, the operators $Q(u)$ and $\hat{Q}(u)$ are Hermitian conjugates and commute with C. Therefore, there exists a unitary operator D independent of u, which diagonalizes $Q(u)$ simultaneously for all values of u. Moreover, in the basis of momentum eigenstates, the matrix elements of Q vanish like $\exp(-\pi N|u|/2)$ when u goes to infinity on the real line. Multiplying (8.12.1) by D to the right and D^{-1} to the left, we obtain an equation for the eigenvalue matrices Q_d and $t_d(u)$. The eigenvalue matrix $Q_d(u)$ is entire and vanishes when u tends to infinity in the real direction. The essential point in the procedure described here is the realization of a Q operator as an integral operator with a well-defined kernel.

An important outcome of the analysis in [148] was the discovery of the relation between the kernel of the Q operator and the generating function of the corresponding classical Bäcklund transformation, namely

$$Q(u) \sim \exp\left(-\frac{i}{\hbar} F_\lambda(q|q')\right), \qquad \hbar \to 0. \tag{8.12.19}$$

On the other hand an alternative procedure in which the Q operator is constructed by means of a trace of a suitably defined monodromy matrix is best understood by means of a concrete example. We shall describe this procedure as formulated recently by Kuznetsov et al. [156]. For this, we consider the discrete self-trapping (DST) equation introduced by Eilbeck et al. [157] to model the nonlinear dynamics of small molecules. The integrability properties of this model were studied by several authors [158, 159, 160]. The quantum Hamiltonian of the integrable DST model contains $(n+1)$ parameters $c_1, ...c_n, b$ and is defined by

$$H = \sum_{i=1}^{n} \left(\frac{1}{2} x_i^2 \partial_i^2 + (c_i + \frac{1}{2}) c_i \partial_i + b x_{i+1} \partial_i \right), \tag{8.12.20}$$

acting in the space $P[\vec{x}]$ of polynomials of n variables $\{x_1, ...x_n\} \equiv \vec{x}$. We shall assume that the periodicity condition $x_{n+1} \equiv x_1$ holds. Initially the DST model was found to be integrable only in case of $n = 2$, the so-called case of the *dimer problem*. For this case, in addition to the Hamiltonian, the other conserved quantity was the number of particles. However, later on an integrable version of the DST model for more than two degrees of freedom was studied by Christiansen et al. [161], which was found to coincide with the DST dimer when $n = 2$. While the Hamiltonian is self-adjoint only in case of the dimer, for the general case no self adjoint H is known. However, in the present discussion we will not be concerned with the underying Hilbert space structure.

Let us therefore consider the Lax matrix for the DST model in the form,

$$\ell_i(u) = \begin{pmatrix} u - c_i + x_i X_i & b x_i \\ X_i & b \end{pmatrix}. \tag{8.12.21}$$

Following [152] the Q operator will be constructed as an integral operator in $P[x]$. In the quantum case, the canonical momenta X_i are first replaced by differential operators $\partial_i \equiv \frac{\partial}{\partial x_i}$, where we have discarded the factor of $i\hbar$ to simplify the notation. Thus the local quantum Lax matrix assumes the form,

$$\ell_i(u) = \begin{pmatrix} u - c_i + x_i \partial_i & b x_i \\ \partial_i & b \end{pmatrix}, \tag{8.12.22}$$

and satisfies the intertwining relation,

$$R_{12}(u_1 - u_2) \overset{1}{\ell}(u_1) \overset{2}{\ell}(u_2) = \overset{2}{\ell}(u_2) \overset{1}{\ell}(u_1) R_{12}(u_1 - u_2). \tag{8.12.23}$$

$R_{12}(u) = uI + \mathcal{P}_{12}$ here is the standard $SL(2)$-invariant solution of the quantum Yang-Baxter equation. The mondromy matrix or Lax operator $T(u)$ and its trace $t(u)$ are defined by the usual formulae:

$$T(u) = \ell_N(u)....\ell_1(u), \qquad t(u) = \text{tr } [\ell_N(u).....\ell_1(u)]. \tag{8.12.24}$$

Since the quantum Lax operator $T(u)$ obeys (8.12.1) the commutativity of $t(u)$ is obvious. Moreover, as the quantum Hamiltonians are defined by the coefficients of the polynomial $t(u)$, their involution is also ensured. The basic problem in the quantum context is to solve the spectral problem for commuting differential operators, namely the quantum Hamiltonians $\{H_i\}_{i=1}^n$, i.e.,

$$H_i\Psi(x_1, x_2, ..., x_n) = h_i\Psi(x_1, ...x_n) \qquad \Psi(x_1, ...x_n) \in P[\vec{x}]. \tag{8.12.25}$$

The spectrum of H_i or equivalently $t(u)$ can be described using the algebraic Bethe ansatz [162]. Defining the vacuum state $|0\rangle$ as the unit function $|0\rangle(x) \equiv 1$ in $P[\vec{x}]$, we find that

$$L_{21}|0\rangle = 0, \qquad L_{11}(u)|0\rangle = \alpha(u)|0\rangle, \qquad \text{and } L_{22}(u)|0\rangle = \beta(u)|0\rangle, \tag{8.12.26}$$

with

$$\alpha(u) = \prod_{i=1}^{n}(u - c_i) \qquad \text{and} \quad \beta(u) = b^n. \tag{8.12.27}$$

Next defining the Bethe vector $\Psi_{\vec{v}}(x_1, ..., x_n) \in P[\vec{x}]$, which is parameterized by m complex numbers, as follows:

$$\Psi_{\vec{v}}(x_1, ...x_n) \equiv |v_1, ...v_m\rangle = L_{12}(v_1)....L_{12}(v_m)|0\rangle, \tag{8.12.28}$$

one can show using the commutation relations derived from (8.12.23) that, $|v_1, ...v_m\rangle$ is an eigenvector of $t(u)$:

$$t(u)|v_1, ...v_m\rangle = \tau(u)|v_1, ...v_m\rangle \tag{8.12.29}$$

for any complex number u, if and only if the parameters v_j satisfy the system of Bethe ansatz equations:

$$\prod_{j=1}^{m} \frac{v_k - v_j + 1}{v_k - v_j - 1} = -\frac{\alpha(v_k)}{\beta(v_k)}, \quad k = 1, ..., m. \tag{8.12.30}$$

The corresponding eigenvalue $\tau(u)$ is given by

$$\tau(u) = \alpha(u)\prod_{j=1}^{m} \frac{u - v_j - 1}{u - v_j} + \beta(u)\prod_{j=1}^{m} \frac{u - v_j + 1}{u - v_j}. \tag{8.12.31}$$

Regarding the question of completeness of the Bethe eigenvectors, proofs exist only for a few models, but it is conjectured that they are complete at least for generic values of parameters. It is also interesting to note that, aside from the question of completeness, an equivalent formulation of (8.12.30 and 8.12.31) in the case of the XYZ and XXZ spin chain, was made by Baxter [64], where these equations were formulated as finite difference equations in a certain class of holomorphic functions. Introducing now a polynomial $\phi(\lambda;\vec{v})$, whose zeros are the Bethe parameters v_j, i.e.,

$$\phi(\lambda;\vec{v}) = \prod_{j=1}^{m}(\lambda - v_j) \tag{8.12.32}$$

with λ being a complex parameter, we may formally recast the system of equations (8.12.30) for $\{v_i\}_{i=1}^m$ and (8.12.31) for $\tau(u=\lambda)$, to the following finite difference equation of second order for $\phi(\lambda;\vec{v})$:

$$\phi(\lambda;\vec{v})\tau(\lambda) = \alpha(\lambda)\phi(\lambda-1;\vec{v}) + \beta(\lambda)\phi(\lambda+1;\vec{v}). \tag{8.12.33}$$

This can be easily seen by dividing both sides of (8.12.30) by $\phi(\lambda)$ and taking the residues at $\lambda = v_j$. Equation (8.12.33) is called Baxter's equation or sometimes the $t-Q$ equation; an identical equation in fact arises when solving the model by means of the separation of variables method. The search for Q operators therefore reduces to finding a single parameter family of operators Q_λ acting in $[P(\vec{x})]$ so that $t(u)$ and Q_λ have a common set of Bethe eigenvectors, with the eigenvalues $q(\lambda)$ of Q_λ obeying the relation,

$$Q_\lambda|v_1,....v_m\rangle = q(\lambda)|v_1,....v_m\rangle, \tag{8.12.34}$$

and being polynomials in λ satisfying Baxter's equation.

As for the actual steps involved in calculating the Q operator we first consider a linear operator $\hat{R}$ defined at each lattice site. This operator maps the space $P[s,x]$ into $P[t,y]$ and depends on the complex parameter λ. We shall refer to the spaces $P[x]$ and $P[y]$ as quantum spaces, while $P[s]$ and $P[t]$ will be referred to as auxiliary spaces. The construction basically parallels that of the transfer matrix $t(u)$, in the sense that it is considered as the trace of a monodromy matrix built fron n copies of the elementary matrix $\hat{R}^{(i)}_{\lambda-c_i}$, with $\hat{R}^{(i)}_{\lambda-c_i} : P[s_i,x_i] \longrightarrow P[s_{i+1},y_i]$, and extending on $P[x_j](j\neq i)$ as the unit operator. Consequently, the monodromy matrix, $\hat{R}^{(i)}_{\lambda-c_n}.....\hat{R}^{(i)}_{\lambda-c_1}$, acts from $P[s_1,\vec{x}] \rightarrow P[s_1,\vec{y}]$,

whilst the Q operator is obtained by taking the trace in the auxiliary space $P[s_1]$:

$$Q_\lambda - \text{tr}_{\ s_1} \hat{R}^{(i)}_{\lambda - c_n} ..... \hat{R}^{(i)}_{\lambda - c_1}. \tag{8.12.35}$$

Defining the operation of $\hat{R}$ according to the rule,

$$\hat{R}_\lambda : \psi(s,x) \longmapsto \int dx \int ds \mathcal{R}_\lambda(t,y|s,x)\psi(s,x), \tag{8.12.36}$$

one has for the kernel of the Q operator the following expression:

$$\mathcal{Q}_\lambda(\vec{y}|\vec{x}) = \int ds_n ... \int ds_1 \prod_{i=1}^{n} \mathcal{R}_{\lambda - c_i}(s_{i+1}, y_i|s_i, x_i). \tag{8.12.37}$$

To ensure the first property of the Q operator, namely the commutation rule $[t(u), Q_\lambda] = 0$, it is sufficient to find a solution $\mathcal{R}_\lambda$ of the following equation:

$$M(u-\lambda; t, \partial_t)\ell(u; y, \partial_y)\mathcal{R}_\lambda = \mathcal{R}_\lambda \ell(u; x, \partial_x) M(u-\lambda; s, \partial_s), \tag{8.12.38}$$

where $\ell(u; x, \partial_x)$ is the local quantum Lax matrix and $M(u-\lambda)$ is a matrix that obeys the quantum commutation relation (8.12.23) with the same quantum R matrix as $\ell(u; x\partial_x)$. The main difficulty is in finding or rather choosing the matrix $M(u-\lambda)$, so that (8.12.38) for $\mathcal{R}_\lambda$ has a solution for every complex parameter u and the Q operator thus obtained fulfils the required properties.

The second property of the Q operator, namely $[Q_{\lambda_1}, Q_{\lambda_2}] = 0$, follows from the Yang-Baxter, equation which may be obtained from (8.12.38) by a standard procedure [153]. It is sufficient to establish the Yang-Baxter identity:

$$\int dt_1 \int dt_2 \int dy \widetilde{\mathcal{R}}_{\lambda_1 - \lambda_2}(w_1, w_2|t_1, t_2)\mathcal{R}_{\lambda_1}(t_1, z|s_1, y)\mathcal{R}_{\lambda_2}(t_2, y|s_2, x)$$

$$= \int dt_1 \int dt_2 \int dy \mathcal{R}_{\lambda_2}(w_2, z|t_2, y)\mathcal{R}_{\lambda_1}(w_1, y|t_1, x)\widetilde{\mathcal{R}}_{\lambda_1 - \lambda_2}(t_1, t_2|s_1, s_2), \tag{8.12.39}$$

with some kernel $\widetilde{\mathcal{R}}_\lambda$. Regarding the choice of $M(u-\lambda)$, in case of the DST model the following was found to be appropriate:

$$M(u-\lambda; s, \partial_s) = \begin{pmatrix} u - \lambda - s\partial_s & s \\ -\partial_s & 1 \end{pmatrix}. \tag{8.12.40}$$

Now from (8.12.38) using this particular form of $M(u-\lambda)$, one obtains a set of differential equations for the kernel $\mathcal{R}_\lambda(t,y|s,x)$ of $\hat{R}_\lambda$:

$$\begin{pmatrix} u-\lambda-t\partial_t & t \\ -\partial_t & 1 \end{pmatrix} \begin{pmatrix} u-y\partial_y & by \\ -\partial_y & b \end{pmatrix} \mathcal{R}_\lambda(t,y|s,x)$$

$$= \begin{pmatrix} u+1+x\partial_x & bx \\ \partial_x & b \end{pmatrix} \begin{pmatrix} u-\lambda+1+s\partial_s & s \\ \partial_s & 1 \end{pmatrix} \mathcal{R}_\lambda(t,y|s,x), \qquad (8.12.41)$$

where the index "i" on the variables has been dropped for simplicity and the identities,

$$\partial_x^\star = -\partial_x, \qquad (x\partial_x)^\star = -1 - x\partial_x,$$

have been used. Equation (8.12.41) determines $\mathcal{R}_\lambda$ up to a scalar factor ρ_λ:

$$\mathcal{R}_\lambda(t,y|s,x) = \rho_\lambda \delta(s-by) y^{-1} \exp(\frac{t-x}{y})(\frac{t-x}{y})^{-\lambda-1}. \qquad (8.12.42)$$

One then has to choose the factor ρ_λ in (8.12.42) and the integration contour in (8.12.36) in a manner so that

$$\hat{R}_\lambda : P[s,x] \longmapsto P[t,y,\lambda].$$

Recalling the action of $\hat{R}_\lambda$ on $\psi(s,x)$, we therefore have

$$\hat{R}_\lambda : \psi(s,x) \mapsto \int dx \int ds \rho_\lambda \delta(s-by) y^{-1} \exp(\frac{t-x}{y})(\frac{t-x}{y})^{-\lambda-1} \psi(s,x). \qquad (8.12.43)$$

The integration over s is easily done and with $\xi = \frac{x-t}{y}$, so that $x = t+y\xi$ and we have

$$\hat{R}_\lambda : \psi(s,x) \mapsto \rho_\lambda \int_\gamma d\xi \exp(-\xi)(-\xi)^{-\lambda-1} \psi(by, y\xi+t), \qquad (8.12.44)$$

where the contour γ is shown in Figure (8.12.1) below.

A few comments are in order. In [152] it is shown that there are two possible choices for the normalization constant ρ_λ. For instance, with the contour γ as in Figure (8.12.1) one has

$$\hat{R}_\lambda : 1 \longmapsto \rho_\lambda \int_\gamma d\xi e^{-\xi} (-\xi)^{-(\lambda+1)}. \qquad (8.12.45)$$

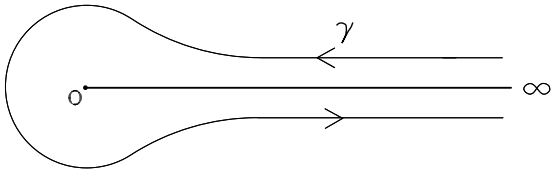

FIGURE 8.12.1: Contour for integration in Equation (8.12.45).

However, by Hankel's formula for the gamma function [154] it is well known that

$$\int_\gamma d\xi e^{-\xi}(-\xi)^{-(\lambda+1)} = \frac{2\pi}{i\Gamma(\lambda+1)}; \tag{8.12.46}$$

hence we may choose

$$\rho_\lambda = \frac{i\Gamma(\lambda+1)}{2\pi}. \tag{8.12.47}$$

With this choice for the normalization factor, one has

$$\hat{R}_\lambda : \psi(s,x) \mapsto \frac{i\Gamma(\lambda+1)}{2\pi}\int_\gamma d\xi e^{-\xi}(-\xi)^{-\lambda-1}\psi(by, y\xi+t). \tag{8.12.48}$$

From (8.12.48) it is apparent that $\hat{R}_\lambda$ as a function of λ is analytic except at the poles $\lambda = -1, -2, ...$ of $\Gamma(\lambda+1)$. However, for $Re\ \lambda < 0$ one can deform the contour γ over the cut $(0,\infty)$ and replace $\int_\gamma d\xi f(\xi)$ with $\int_0^\infty d\xi[f(\xi - i0) - f(\xi + i0)]$, resulting thereby in the formula,

$$\hat{R}_\lambda : \psi(s,x) \mapsto \frac{1}{\Gamma(-\lambda)}\int_0^\infty d\xi e^{-\xi}\xi^{-\lambda-1}\psi(by, y\xi+t), \quad Re\ \lambda < 0, \tag{8.12.49}$$

which is analytic in $\lambda = -1, -2.....$ The branch of $\xi^{-\lambda-1}$ in (8.12.49) being fixed by the condition $arg(\xi) = 0$.

Let us now substitute the expression for the kernel $\mathcal{R}_\lambda$ as given by (8.12.35) into the expression for the kernel $\mathcal{Q}_\lambda$ of the Q operator given

in (8.12.49). Then by using the first normalization choice, we find that

$$\mathcal{Q}_\lambda(\vec{y}|\vec{x}) = \prod_{i=1}^{n} w_i(\lambda; y_{i+1}, y_i, x_i), \tag{8.12.50}$$

with

$$w_i(\lambda; y_{i+1}, y_i, x_i) = \frac{i}{2\pi}\Gamma(\lambda + 1 - c_i) y_i^{-1} \left(\frac{by_{i+1} - x_i}{y_i}\right)^{c_i - \lambda - 1} \times$$

$$\times \exp\left(\frac{by_{i+1} - x_i}{y_i}\right). \tag{8.12.51}$$

At this point it is interesting to note that taking into consideration the quantization convention $-i\hbar = 1$ and apart from some λ-dependent factors one has

$$\mathcal{Q}_\lambda(\vec{y}|\vec{x}) \simeq \exp\left(-F_\lambda(\vec{y}|\vec{x})\right), \tag{8.12.52}$$

where F_λ is the generating function of the corresponding classical Bäcklund transformation. This result is in agreement with the general principles formulated in the work of Pasquier and Gaudin for the Toda lattice. While comphrehensive details of the analytical properties of the Q operator for the DST model are contained in [160], we shall now confine our attention to a derivation of Baxter's difference equation, since this marks in a sense the high point of the formalism developed in [155]. Let us once again recall the method proposed by Pasquier and Gaudin, who considered Q_λ as an integral operator,

$$Q_\lambda : \psi(\vec{x}0 \mapsto \int dx_1 .... \int dx_n \mathcal{Q}_\lambda(\vec{y}|\vec{x})\psi(\vec{x}), \tag{8.12.53}$$

with the kernel satisfying (8.12.50). As $t(u) = \mathrm{tr}\ T(u)$ with $T(u)$ being a 2×2 matrix with differential operator entries in x_i, we can transform the left-hand side of (8.12.50) as follows, using integration by parts:

$$[Q_\lambda t(\lambda)\psi](\vec{y}) = \mathrm{tr}\ [Q_\lambda T(\lambda)\psi](\vec{y}) = \mathrm{tr} \int dx^n \mathcal{Q}(\vec{y}|\vec{x}) T(\lambda)\psi(\vec{x}),$$

$$= \mathrm{tr} \int dx^n [T^\star(\lambda)\mathcal{Q}_\lambda(\vec{y}|\vec{x})]\psi(\vec{x}). \tag{8.12.54}$$

Here $T^\star(\lambda)$ denotes the matrix composed of adjoint differential operators $(T_{jk})^\star = T^\star_{jk}$. Using the fact that the Lax matrix $T(\lambda)$ factors into an ordered product of elementary local Lax operators $\ell_i(\lambda)$, as also the

kernel $\mathcal{Q}_\lambda(\vec{y}|\vec{x})$ into the factors w_i (see (8.12.50) and (8.12.51)), we can write the kernel of the integral operator $Q_\lambda t(\lambda)$ as follows:

$$[Q_\lambda t(\lambda)](\vec{y}|\vec{x}) = \text{tr } \ell_n^\star(\lambda).....\ell_1^\star(\lambda)\prod_{i=1}^{n} w_i = \text{tr } (\ell_n^\star(\lambda)w_n....(\ell_1^\star(\lambda)w_1), \tag{8.12.55}$$

where

$$\ell_i(\lambda) = \begin{pmatrix} \lambda - c_i + 1 + x_i\partial_{x_i} & bx_i \\ \partial_{x_i} & b \end{pmatrix}. \tag{8.12.56}$$

It should be emphasised that this factorization of $[Q_\lambda T(\lambda)](\vec{y}|\vec{x})$ is possible, since the factors w_i in (8.12.55) depended only on a single variable x_i. Later, we shall show how to tackle the problem when w_i depends on additional x_i's. To proceed further we introduce matrices $\tilde{\ell}_i(\lambda)$ through the definition $\ell_i^\star(\lambda)w_i = w_i\tilde{\ell}_i(\lambda)$. By noting that

$$\partial_{x_i} \ln w_i(y_{i+1}, y_i, x_i) = \frac{c_i - \lambda - 1}{x_i - by_{i+1}} - \frac{1}{y_i}, \tag{8.12.57}$$

we obtain

$$\tilde{\ell}_i(\lambda) = \begin{pmatrix} \lambda - c_i + 1 + x_i\partial_{x_i} \ln w_i & bx_i \\ \partial_{x_i} \ln w_i & b \end{pmatrix} = \begin{pmatrix} \frac{b(c_i-\lambda-1)y_{i+1}}{x_i-by_{i+1}} - \frac{x_i}{y_i} & bx_i \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & b \end{pmatrix} \tag{8.12.58}$$

and

$$[Q_\lambda t(\lambda)](\vec{y}|\vec{x}) = \mathcal{Q}_\lambda(\vec{y}|\vec{x})\text{tr } \tilde{\ell}_n(\lambda).....\tilde{\ell}_1(\lambda) \equiv \mathcal{Q}_\lambda(\vec{y}|\vec{x})\text{tr } \tilde{T}(\lambda). \tag{8.12.59}$$

Furthermore, we also note that a gauge transformation of $\tilde{\ell}_i(\lambda) \mapsto N_{i+1}^{-1}\tilde{\ell}_i(\lambda)N_i$, with N_i given by

$$N_i = \begin{pmatrix} 1 & by_i \\ 0 & 1 \end{pmatrix}, \tag{8.12.60}$$

leaves tr $\tilde{L}(\lambda)$ invariant, while at the same time making $\tilde{\ell}_i(\lambda)$ and consequently, $\tilde{L}(\lambda)$ triangular, i.e.,

$$\begin{aligned} N_{i+1}^{-1}\tilde{\ell}_i(\lambda)N_i &= \begin{pmatrix} -\frac{x_i-by_{i+1}}{y_i} & 0 \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & \frac{b(c_i-\lambda-1)y_i}{x_i-by_{i+1}} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - c_i)\frac{w_i(\lambda-1)}{w_i(\lambda)} & 0 \\ \frac{c_i-\lambda-1}{x_i-by_{i+1}} - \frac{1}{y_i} & \frac{bw_i(\lambda+1)}{w_i(\lambda)} \end{pmatrix}. \end{aligned} \tag{8.12.61}$$

In calculating (8.12.61), use has been made of the following identities:

$$\frac{w_i(\lambda+1)}{w_i(\lambda)} = \frac{(c_i-\lambda-1)y_i}{x_i - by_{i+1}}, \qquad \frac{w_i(\lambda-1)}{w_i(\lambda)} = \frac{x_i - by_{i+1}}{(c_i-\lambda)y_i}. \quad (8.12.62)$$

We are thus led to the equality,

$$\text{tr } \tilde{T}(\lambda) = b^n \prod_{i=1}^{n} \frac{w_i(\lambda+1)}{w_i(\lambda)} + \prod_{i+1}^{n}(\lambda - c_i)\frac{w_i(\lambda-1)}{w_i(\lambda)}, \quad (8.12.63)$$

which represents Baxter's difference equation. Having described the construction of the Q operator for the DST model and the derivation of Baxter's difference equation for the model, we consider certain other models for which a similar procedure has been applied.

Example 1: Toda model

In [164] the authors performed an analysis of the Toda lattice, whose local Lax operator $\ell_i(u-c_i, x_i)$ is of the form,

$$\ell_i(u-c_i, x_i) = \begin{pmatrix} u - c_i - \eta\partial_{x_i} & -e^{x_i} \\ e^{-x_i} & 0 \end{pmatrix}. \quad (8.12.64)$$

The auxiliary matrix $M(u-\lambda)$ was chosen to be that of the DST model and the following expression for the kernel of the linear operator $\hat{R}_\lambda$ was obtained:

$$\mathcal{R}_{\lambda-c} = \rho_{\lambda-c}\delta(t+e^y)(\eta\sigma)^{-\frac{\lambda-c}{\eta}} \exp(\sigma - \frac{z^2}{4\sigma}). \quad (8.12.65)$$

Here $\sigma = \frac{1}{\eta}e^{x-y}$ and η represents the quantization parameter. Consequently in this case one has

$$\begin{aligned} \hat{R}_{\lambda-c} : \psi(t,x) &\mapsto \int dx \int dt \mathcal{R}_{\lambda-c}(s,y|t,x)\psi(t,x), \\ &= \rho_\mu \eta^{-\frac{\mu}{\eta}} \int dx \sigma^{-\frac{\mu}{\eta}} \exp(\sigma - \frac{z^2}{4\sigma})\psi(t=-e^y, x), \\ &= \rho_\mu \eta^{-\frac{\mu}{\eta}} \int_\gamma d\sigma\, \sigma^{-\frac{\mu}{\eta}-1} \exp(\sigma - \frac{z^2}{4\sigma})\psi, \end{aligned} \quad (8.12.66)$$

where we have introduced the quantity $\mu = \lambda - c$. The contour γ being chosen as shown in Figure (8.12.2). To fix the normalization factor ρ_μ, we set $\psi(t,x) = 1$, which yields

$$\hat{R}_\mu : 1 \mapsto \rho_\mu \eta^{-\frac{\mu}{\eta}} \int_\gamma d\sigma\, \sigma^{-\nu-1} \exp(\sigma - \frac{z^2}{4\sigma}), \qquad \nu = \frac{\mu}{\eta}. \quad (8.12.67)$$

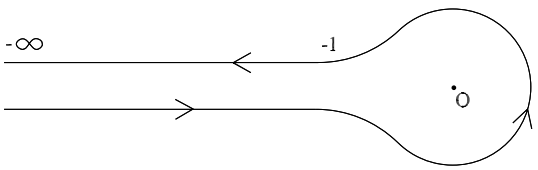

FIGURE 8.12.2: Contour for integration in Equation (8.12.68).

From the fundamental integral representation of the Bessel function [154],

$$J_\nu(z) = \frac{1}{2\pi i}(\frac{z}{2})^\nu \int_\gamma d\sigma\, \sigma^{-\nu-1} \exp(\sigma - \frac{z^2}{4\sigma}), \tag{8.12.68}$$

we demand

$$\rho_\mu = \frac{1}{\pi i J_\nu(z)}(\frac{z\nu}{2})^\nu. \tag{8.12.69}$$

As before, the Q_λ operator can be defined as

$$Q_\lambda = \text{tr } (\prod_{i=1}^{n} \hat{R}_{\mu_1}^{(i)}) = \int dt_n ... \int dt_1 \prod_{i=1}^{n} \mathcal{R}_{\mu_i}(t_{i+1}, y_i | t_i, x_i) = \prod_{i=1}^{n} w_i, \tag{8.12.70}$$

where

$$w_i = \frac{1}{2\pi i J_{\nu_i}(z_i)} \left(\frac{z_i}{2}\right)^{\nu_i} \exp\left(\sigma_i - \frac{z_i^2}{4\sigma_i}\right) \tag{8.12.71}$$

and

$$\nu_i = \frac{\mu_i}{\eta} = \frac{\lambda - c_i}{\eta}, \quad \sigma_i = \frac{1}{\eta} e^{x_i - y_i}, \quad z_i^2 = \frac{4}{\eta^2} e^{(y_{i+1} - y_i)}.$$

It is interesting to note that

$$\log Q_\lambda(\vec{y}|\vec{x}) = \sum_{i=1}^{n} \log w_i = \frac{1}{\eta} F_\lambda^{class.}(\vec{y}|\vec{x}) + \sum_{i=1}^{n} \widetilde{\Delta}_i, \tag{8.12.72}$$

where

$$F_\lambda^{class.} = \sum_{i=1}^{n} \left[e^{x_i - y_i} - e^{y_{i+1} - x_i} + (\lambda - c_i)(x_i - y_i)\right] \tag{8.12.73}$$

and

$$\widetilde{\Delta}_i = \log\left(\frac{1}{2\pi i J_{\nu_i}(z_i)}\left(\frac{z_i}{2}\right)^{\nu_i}\right) + \frac{\lambda - c_i}{\eta}\log\eta. \tag{8.12.74}$$

The latter explicitly represents the quantum correction term.

Example 2: XXX model

An analysis of Baxter's Q operator for the XXX model was performed independently by Derkachov in [163] and the authors in [165]. The local Lax operator for the model may be taken in the form [165],

$$\ell_n(u) = \begin{pmatrix} u + \eta s_n^3 & \eta s_n^- \\ \eta s_n^+ & u - \eta s_n^3 \end{pmatrix}, \tag{8.12.75}$$

with s_n^i being the spin variables at the nth lattice site and obeying the sl(2) algebra:

$$[s_k^3, s_j^-] = -s_k^-\delta_{kj}, \quad [s_k^3, s_j^+] = s_k^+\delta_{kj}, \quad [s_k^+, s_j^-] = 2s_k^3\delta_{kj}. \tag{8.12.76}$$

A co-adjoint differential operator representation of this algebra is given by

$$s_k^3 = x_k\frac{\partial}{\partial_{x_k}} + \xi_k, \quad s_k^- = -\frac{\partial}{\partial_{x_k}}, \quad s_k^+ = x_k^2\frac{\partial}{\partial_{x_k}} + 2\xi_k x_k, \tag{8.12.77}$$

with ξ_k being arbitrary.

As the above Lax operator and the Lax operator for the DST model obey the same quantum commutation relation, viz

$$R(u-v)\ell_i(u)\otimes\ell_i(v) = \ell_i(v)\otimes\ell_i(u)R(u-v),$$

with the quantum R matrix being given by

$$R(u-v) = \mathcal{P}\left(I + \frac{\eta}{u-v}\mathcal{P}\right), \tag{8.12.78}$$

we may take the Lax operator for the DST model as our prototype for the auxiliary matrix $M(u-\lambda)$. Proceeding as before, one obtains the following solution for the kernel $\mathcal{R}_\lambda(y,t|s,x)$, namely

$$\mathcal{R}_\lambda^{(i)}(y,t|x,s) = \rho_\lambda^{(i)}(x_i t_i - 1)^{(\xi-1-\lambda/\eta)}(x_i s_i - 1)^{(\xi-1+\lambda/\eta)}\exp\left(\frac{t_i - s_i}{\eta}y_i\right), \tag{8.12.79}$$

where η is the quantization parameter. The representation for the Q operator is then given by

$$Q_\lambda(y|x) = \int ds_n .... \int ds_1 \prod_{i=1}^{(n)} (x_i t_i - 1)^{\xi - 1 - \frac{\lambda}{\eta}} (x_i s_i - 1)^{\xi - 1 + \frac{\lambda}{\eta}} e^{[y_i \frac{s_{i-1} - s_i}{\eta}]}. \tag{8.12.80}$$

In the adjoint repesentation, the Lax operator for the XXX model is as follows:

$$\ell_i^*(\lambda) = \eta \begin{pmatrix} \frac{\lambda}{\eta} - -x_i \partial_{x_i} + \xi_i - 1 & \partial_{x_i} \\ -x_i^2 \partial_{x_i} + 2x_i(\xi_i - 1) & \frac{\lambda}{\eta} + x_i \partial_{x_i} - \xi_i + 1 \end{pmatrix}. \tag{8.12.81}$$

Moreover, under the local transformation,

$$\ell_i^*(\lambda) \longrightarrow \tilde{\ell}_i^*(\lambda) = N_{i-1}^{-1} \ell_i^*(\lambda) N_i, \quad \text{where} \quad N_i = \begin{pmatrix} 0 & 1 \\ -1 & 1/s_i \end{pmatrix}, \tag{8.12.82}$$

as the adjoint transfer matrix,

$$t^\star(\lambda) = \text{tr} \prod_{i=1}^{n} \ell_i^\star(\lambda) = \text{tr} \prod_{i=1}^{n} \tilde{\ell}_i^\star(\lambda), \tag{8.12.83}$$

remains invariant; acting on Q_λ leads to the following:

$$t^*(\lambda) Q_\lambda(\vec{y}|\vec{x}) = \text{tr} \prod_{i=1}^{n} \tilde{\ell}_i^*(\lambda) \int ds_n ... \int ds_1 \prod_{i=1}^{n} \mathcal{R}_{\frac{\lambda}{\eta}}^{(i)}(s_{i-1}, y_i | s_i, x_i)$$

$$= \text{tr} \prod \int ds_n .... \int ds_1 \tilde{\ell}_i^*(\lambda) \mathcal{R}_{\frac{\lambda}{\eta}}^{(i)}(s_{i-1}, y_i | s_i, x_i). \tag{8.12.84}$$

One can isolate the x_i dependence of $\mathcal{R}_{\frac{\lambda}{\eta}}^{i}$ to get

$$\mathcal{R}_{\frac{\lambda}{\eta}}^{(i)} = F(s_i, s_{i-1}, y_i)(x_i - \frac{1}{s_{i-1}})^{\xi - \frac{\lambda}{\eta} - 1} (x_i - \frac{1}{s_i})^{\xi + \frac{\lambda}{\eta} - 1},$$

where $$F(s_i, s_{i-1}, y_i) = \rho_\lambda^i s_{i-1}^{-\xi + \frac{\lambda}{\eta} + 1} s_i^{-\xi - \frac{\lambda}{\eta} + 1} \exp(y_i \frac{s_{i-1} - s_i}{\eta}).$$

Consequently, we find that

$$[\tilde{\ell}_i^*(\lambda)]_{12} \mathcal{R}_{\frac{\lambda}{\eta}}^{(i)}(s_{i-1}, y_i | s_i, x_i)$$

$$= \eta (x_i - \frac{1}{s_{i-1}})^{\xi_i - \frac{\lambda}{\eta}} (x_i - \frac{1}{s_i})^{\xi_i + \frac{\lambda}{\eta}} \frac{\partial}{\partial x_i} [F(s_i, s_{i-1}, y_i)] = 0. \tag{8.12.85}$$

Thus, the integrand in (8.12.84) becomes a triangular matrix with the following structure

$$t^*(\lambda)Q_\lambda(\vec{y}|\vec{x}) = \mathrm{tr}\ \prod_{i=1}^{n}\int ds_n....\int ds_1 \begin{pmatrix} \tilde{\ell}_i^{*11}\mathcal{R}^i_{\frac{\lambda}{\eta}} & 0 \\ \star & \tilde{\ell}_i^{*22}\mathcal{R}^i_{\frac{\lambda}{\eta}} \end{pmatrix}. \quad (8.12.86)$$

It remains to examine the effect of the diagonal elements on $\mathcal{R}^i_{\frac{\lambda}{\eta}}$. A little algebra shows that one can rewrite the above equation as

$$\tilde{\ell}_i^{*11}\mathcal{R}^i_{\frac{\lambda}{\eta}} = \eta(x_i - \frac{1}{s_{i-1}})^{\xi_i - \frac{\lambda}{\eta}}\partial_{x_i}(x_i - \frac{1}{s_{i-1}})^{-\xi_i+\frac{\lambda}{\eta}+1}\mathcal{R}^i_{\frac{\lambda}{\eta}}. \quad (8.12.87)$$

Using then the expression for $\mathcal{R}^i_{\frac{\lambda}{\eta}}$ as given in (8.12.79), we find that

$$\tilde{\ell}_i^{*11}\mathcal{R}^i_{\frac{\lambda}{\eta}} = \eta(\frac{\lambda}{\eta}+\xi-1)\mathcal{R}^i_{\frac{\lambda}{\eta}-1}(s_{i-1},y_i|s_i,x_i),$$

and similarly,

$$\tilde{\ell}_i^{*22}\mathcal{R}^i_{\frac{\lambda}{\eta}} = \eta(\frac{-\lambda}{\eta}+\xi-1)\mathcal{R}^i_{\frac{\lambda}{\eta}-1}(s_{i-1},y_i|s_i,x_i).$$

As a result we finally have

$$t^*(\lambda)Q_{\frac{\lambda}{\eta}}(\vec{y}|\vec{x})$$

$$= \eta^n \mathrm{tr}\ \prod_{i=1}^{n}\int ds_n....\int ds_1 \begin{pmatrix} (\xi-1+\frac{\lambda}{\eta})\mathcal{R}^i_{\frac{\lambda}{\eta}-1} & 0 \\ \star & -(\xi-1-\frac{\lambda}{\eta})\mathcal{R}^i_{\frac{\lambda}{\eta}+1} \end{pmatrix},$$

$$= \eta^n\prod_{i=1}^{n}(\xi-1+\frac{\lambda}{\eta})[\int ds_n...\int ds_1\mathcal{R}^i_{\frac{\lambda}{\eta}-1}]$$

$$+\eta^n\prod_{i=1}^{n}(-\xi+1+\frac{\lambda}{\eta})[\int ds_n...\int ds_1\mathcal{R}^i_{\frac{\lambda}{\eta}+1}],$$

$$t^*(\lambda)Q_{\frac{\lambda}{\eta}}(\vec{y}|\vec{x}) = \Delta_+(\frac{\lambda}{\eta})Q_{\frac{\lambda}{\eta}+1}(\vec{y}|\vec{x}) + \Delta_-(\frac{\lambda}{\eta})Q_{\frac{\lambda}{\eta}-1}(\vec{y}|\vec{x}), \quad (8.12.88)$$

where $\Delta_\pm(\frac{\lambda}{\eta}) = \eta^n\prod_{i=1}^{n}(\frac{\lambda}{\eta}\pm 1\mp\xi)$ and

$$Q_{\frac{\lambda}{\eta}\pm 1}(\vec{y}|\vec{x}) = \int ds_n....\int ds_1\prod_{i=1}^{n}\mathcal{R}^{(i)}_{\frac{\lambda}{\eta}\pm 1}. \quad (8.12.89)$$

8.12.1 Analytical properties of the Q operator

We shall now discuss some analytical properties of the Q operator for the XXX model for which we obtained the following expression for the kernel of the basic element $\hat{R}_{\frac{\lambda}{\eta}}$:

$$\hat{R}_{\frac{\lambda}{\eta}} : \phi(s,x) \longrightarrow \int dx \int ds R_{\frac{\lambda}{\eta}}(t,y|s,x)\phi(s,x),$$
$$R_{\frac{\lambda}{\eta}} = \rho_\lambda (xt-1)^{\xi-1-\frac{\lambda}{\eta}} (xs-1)^{\xi-1+\frac{\lambda}{\eta}} e^{\left[\frac{(t-s)y}{\eta}\right]}. \qquad (8.12.90)$$

Setting $\phi(s,x) = 1$ gives $\hat{R}_{\frac{\lambda}{\eta}} : 1 \longrightarrow Z$, where

$$Z = \rho_\lambda \int dx \int ds (xt-1)^{\xi-1-\frac{\lambda}{\eta}} (xs-1)^{\xi-1+\frac{\lambda}{\eta}} \exp\left[\frac{(t-s)y}{\eta}\right]. \quad (8.12.91)$$

Performing the following change of variables $xs-1 = w$ and $xt-1 = u$, where u should not be confused with the spectral parameter used earlier, one finds that Z may be recast in the following manner:

$$Z = \rho_\lambda \int du (u+1)^{\xi-1-\frac{\lambda}{\mu}} e^{[u\beta(u)]} (-1)^{\xi-1+\frac{\lambda}{\eta}} K, \qquad (8.12.92)$$

where $K = \int dw (-w)^{\xi-1+\frac{\lambda}{\eta}} exp[-\beta w]$ and $\beta(u) = \frac{yt}{\eta(u+1)}$.

If $\bar{w} = \beta w$ then the integral defining K becomes a standard representation of the gamma function [154], so that

$$K = 1/((\beta)^{\xi+\frac{\lambda}{\eta}}) \int d\bar{w} \delta(-\bar{w})^{-(1-\xi-\frac{\lambda}{\eta})} exp[-\bar{w}] = \frac{1}{\beta^{\xi+\frac{\lambda}{\eta}}} \frac{-2i\pi}{\Gamma(1-\xi-\frac{\lambda}{\eta})}, \qquad (8.12.93)$$

provided its real part $Re(1-\xi-\frac{\lambda}{\eta}) > 0$. Consequently, we have for Z:

$$Z = \rho_\lambda \frac{2i\pi(-1)^a}{\Gamma(1-\xi-\frac{\lambda}{\eta})} e^{yt/\eta} (yt/\eta)^{-(\xi+\frac{\lambda}{\eta})} \int du (u+1)^{\xi+\frac{\lambda}{\eta}-1} u^{\xi-\frac{\lambda}{\eta}-1} e^{[-\frac{yt}{\eta(u+1)}]} \qquad (8.12.94)$$

$$\text{and} \qquad a = \xi + \frac{\lambda}{\eta}$$

The integral in the above equation, under the transformation $v = 1/(u+1)$, with $\gamma = -yt/\eta$ may be expressed as

$$Z = \rho_\lambda \frac{2i\pi(-1)^a}{\Gamma(1-\xi-\frac{\lambda}{\eta})} \times e^{-\gamma} (-\gamma)^{-(\xi+\frac{\lambda}{\eta})} \left[- \int dv (1-v)^{\xi-\frac{\lambda}{\eta}-1} v^{-2\xi} e^{\gamma v} \right] \qquad (8.12.95)$$

Hence setting $v = -q$ we obtain

$$Z = \rho_\lambda \frac{2i\pi(-1)^a}{\Gamma(1-\xi-\frac{\lambda}{\eta})} e^{-\gamma}(-\gamma)^{-(\xi+\frac{\lambda}{\eta})} \left[-\int dq q^{\alpha-1}(1+q)^{\delta-\alpha-1} e^{-\gamma q} \right], \tag{8.12.96}$$

where we have defined $\alpha = 1-2\xi$ and $\delta = 1-\xi-\frac{\lambda}{\eta}$. When this integral is evaluated from 0 to ∞ along a suitable contour (see [154]), it is found to be the representation of the confluent hypergeometric function $\psi(\alpha, \delta; \gamma)$, so that

$$Z = \rho_\lambda \frac{2i\pi(-1)^a}{\Gamma(1-\xi-\frac{\lambda}{\eta})} e^{(yt/\eta)}(yt/\eta)^{-(\xi+\frac{\lambda}{\eta})}(-yt/\eta)^{-(\xi+\frac{\lambda}{\eta})} e^{yt/\eta} \times$$

$$\psi(1-2\xi, 1-\xi-\frac{\lambda}{\eta}; -yt/\eta). \tag{8.12.97}$$

Upon setting

$$\rho_\lambda = \frac{\Gamma(1-\xi-\frac{\lambda}{\eta})}{2i\pi\Gamma(1-2\xi)}, \tag{8.12.98}$$

we get

$$\hat{R}_{\frac{\lambda}{\eta}} : 1 \longrightarrow (-yt/\eta)^{(\xi+\frac{\lambda}{\eta})} e^{yt/\eta} \psi(1-2\xi, 1-\xi-\frac{\lambda}{\eta}; -yt/\eta). \tag{8.12.99}$$

Equation (8.12.98) is the normalization constant in case of the Q operator for XXX model subject to $Re\,(1-\xi-\frac{\lambda}{\eta}) > Re\,(1-2\xi) > 0$. One can explicitly evaluate the kernel of the Q operator for the XXX model, since

$$Q_{\frac{\lambda}{\eta}} = \int ds_n ... \int ds_1 \prod_{i=1}^{n} R^i_{\frac{\lambda}{\eta}}(s_{i-1}, y_i | s_i, x_i), \tag{8.12.100}$$

and we have set $t_i = s_{i-1}$. This leads to

$$Q_{\frac{\lambda}{\eta}} = \int ds_n ... \int ds_1 \prod_{i=1}^{i} \rho^i_\lambda (x_i s_{i-1} - 1)^{\xi_i - 1 - \frac{\lambda}{\eta}} (x_i s_i - 1)^{\xi_i - 1 + \frac{\lambda}{\eta}} e^{[s_i \frac{y_{i+1}-y_i}{\eta}]} \tag{8.12.101}$$

To evaluate this integral we make the substitution, $x_i s_i - 1 = q_i \frac{x_{i+1}-x_i}{x_{i+1}}$, whence it assumes the following form, viz

$$= \prod_{i=1}^{n} \rho^i_\lambda (\frac{x_{i+1}-x_i}{x_i})^{\xi_{i+1}-\lambda/\eta} (\frac{x_{i+1}-x_i}{x_{i+1}})^{\xi_{i+1}+\lambda/\eta} e^{[\frac{y_{i+1}-y_i}{\eta x_i}]} \times$$

$$\times \int dq_i q_i^{a_i-1}(1+q_i)^{b_i-a_i-1}\exp[-z_i q_i]. \tag{8.12.102}$$

Here

$$\begin{aligned} a_i &= \xi_i + \lambda/\eta, \\ b_i &= \xi_i + \xi_{i+1}, \\ z_i &= -\frac{(x_{i+1}-x_i)(y_{i+1}-y_i)}{\eta x_i x_{i+1}}. \end{aligned} \tag{8.12.103}$$

When the integral in (8.12.102) is evaluated from 0 to ∞ along the contour as before, it is representable by the confluent hypergeometric function $\psi(a_i, b_i; z_i)$, so that

$$Q_{\frac{\Delta}{\eta}}(\vec{y}|\vec{x}) = \prod_{i=1}^{n} \rho_\lambda^i \left(\frac{x_{i+1}-x_i}{x_i}\right)^{\xi_{i+1}-\lambda/\eta}\left(\frac{x_{i+1}-x_i}{x_{i+1}}\right)^{\xi_{i+1}+\lambda/\eta}\times$$

$$\times\Gamma(\xi_i + \frac{\lambda}{\mu})\psi(a_i, b_i; z_i). \tag{8.12.104}$$

Note that the $Q_{\frac{\Delta}{\eta}}$ operator is explicitly deducible even when it involves coupling of adjacent lattice sites. In this respect the above analysis differs from the previous ones.

8.13 Method of Projection Operators

A new method for constructing the Q operator using projection operators has recently received a lot of attention. In a series of recent papers Pronko et al. [166, 167] suggested an approach that leads to the construction of one parametric family of Q operators, satisfying Wronskian-type relations for a number of integrable systems, e.g., XXX spin chain, the Toda lattice and most recently the DST model. We describe here this procedure with the DST model in mind, for which it will be recalled that the Lax operator at the nth lattice site is of the form,

$$\ell_n(u) = \begin{pmatrix} u - i/2 - i\phi_n^\dagger\phi_n & \phi_n^\dagger \\ \phi_n & i \end{pmatrix}. \tag{8.13.1}$$

Here u is the spectral parameter and the canonical variables $\phi_n^\dagger, \phi_n$ are assumed to satisfy the commutation relation $[\phi_i, \phi_j^\dagger] = \delta_{ij}$. Moreover, these variables obey the periodic boundary conditions:

$$\phi_{k+N} = \phi_k, \qquad \phi_{k+N}^\dagger = \phi_k^\dagger. \tag{8.13.2}$$

The local Lax operator can be shown to satisfy the fundamental relation

$$R_{12}(u-v)\ell_n^1(u)\ell_n^2(v) = \ell^2(v)\ell^1(u)R_{12}(u-v), \tag{8.13.3}$$

with the R matrix given by $R_{12}(u) = u + i\mathcal{P}_{12}$. Here $\mathcal{P}_{12}$ is the permutation operator. By the standard method of algebraic Bethe ansatz, it follows that $t(u)$, i.e., the trace of the mondromy matrix,

$$t(u) = \text{tr } T(u) = \text{tr } \overleftarrow{\prod_{n=1}^{N}} \ell_n(u), \tag{8.13.4}$$

satisfies an eigenvalue equation, with eigenvalue given by

$$t(u) = (u - i/2)^N \prod_{j=1}^{l} \frac{(u - v_j + i)}{(u - v_j)} + i^N \prod_{j=1}^{l} \frac{(u - v_j - i)}{(u - v_j)}, \tag{8.13.5}$$

provided the v_j's satisfy the Bethe equations:

$$\prod_{j=1}^{l} \frac{(v_i - v_j - i)}{(v_i - v_j + i)} = \frac{(v_i - i/2)}{i^N}. \tag{8.13.6}$$

It is therefore evident, that the polynomial,

$$q(u) \equiv \prod_{j=1}^{l} (u - v_j), \tag{8.13.7}$$

satisfies Baxter's equation,

$$t(u)q(u) = (u - i/2)^N q(u - i) + i^N q(u + i). \tag{8.13.8}$$

Furthermore, we note that the Q operator is defined so as to satisfy

$$t(u)Q(u) = (u - i/2)^N Q(u - i) + i^N Q(u + i), \tag{8.13.9}$$

together with the following:

$$[t(u), Q(u)] = 0, \qquad [Q(u), Q(v)] = 0. \tag{8.13.10}$$

Hence if one considers (8.13.8) as a discrete analog of a second-order differential equation, then it is natural to enquire about the second solution of (8.13.8), and also whether there exists a second Q operator. Indeed, it can be shown that a second Q operator does exist, and its eigenvalues in case of the DST model are meromorphic functions. In fact the existence of a second linearly independent Q operator may be deduced from the following argument. Let us consider Baxter's equation for the first Q operator, with eigenvalue $q(u)$, which is a polynomial of degree, say n. The eigenvalue of the trace of the monodromy matrix $t(u)$, is a polynomial of degree N and we have

$$t(u)q(u) = (u - i/2)^N q(u - i) + i^N q(u + i). \tag{8.13.11}$$

Then from (8.13.11) we have

$$\frac{t(u)}{q(u+i)q(u-i)} = \frac{(u-i/2)^N}{q(u)q(u+i)} + \frac{i^N}{q(u)q(u-i)}. \tag{8.13.12}$$

Multiplying this equation by $\Gamma^N(-i(u - i/2))$, we get

$$\frac{t(u)\Gamma^N(-i(u-i/2))}{q(u+i)q(u-i)} = \frac{i^N\Gamma(-i(u+i/2))}{q(u)q(u+i)} + \frac{i^N\Gamma^N(-i(u-i/2))}{q(u)q(u-i)}. \tag{8.13.13}$$

Let us now denote

$$S(u) = \frac{i^N\Gamma(-i(u+i/2))}{q(u)q(u+i)}, \tag{8.13.14}$$

so that

$$\frac{t(u)\Gamma^N(-i(u-i/2))}{q(u+i)q(u-i)} = S(u) - S(u-i).$$

One may write $S(u)$ in the following manner:

$$S(u) = i^N\Gamma^N(-i(u+i/2))\left[\frac{q_1(u)}{q(u+i)} + \frac{q_2(u)}{q(u)}\right], \tag{8.13.15}$$

with $q_1(u)$ and $q_2(u)$ being polynomials of degree $< n$. Inserting this expression into Baxter's equation (8.13.8) gives

$$(u - i/2)^N q_2(u) + i^N q_1(u - i) = r(u)q(u), \tag{8.13.16}$$

where $r(u)$ is a polynomial of degree $< N$. Expressing $q_1(u)$ in terms of $q_2(u)$ and $r(u)$, if we substitute it in the expression for $S(u)$, then it yields the following expression for $S(u)$, viz

$$S(u) = i^N \Gamma^N(-i(u+i/2))r(u+i) + i^N \Gamma(-i(u+i/2))\frac{q_2(u)}{q(u)}$$

$$-\Gamma^N(-i(u+3i/2))\frac{q_2(u+i)}{q(u+i)}. \tag{8.13.17}$$

Suppose that $S(u)$ is expressible in the form,

$$S(u) = \frac{p(u+i)}{q(u+i)} - \frac{p(u)}{q(u)}. \tag{8.13.18}$$

Then from (8.13.14) we have

$$i^N \Gamma^N(-i(u+i/2)) = p(u+i)q(u) - p(u)q(u+i), \tag{8.13.19}$$

and

$$i^N \Gamma^N(-i(u-i/2)) = p(u)q(u-i) - p(u-i)q(u). \tag{8.13.20}$$

Multiplying (8.13.20) by $(-i(u-i/2))^N$ and subtracting it from (8.13.19) leads after some simplification to

$$t(u)p(u) = (u-i/2)^N p(u-i) + i^N p(u+i). \tag{8.13.21}$$

This implies that $p(u)$, is an eigenvalue of the second Q operator. The next step consists in finding a function $g(u)$ so that

$$g(u-i) - g(u) = i^N \Gamma^N(-i(u+i/2))r(u+i). \tag{8.13.22}$$

Assuming $g(u)$ to be given by

$$g(u) = \sum_{k=0}^{\infty} f(-iu-k), \tag{8.13.23}$$

we see that

$$g(u-i) - g(u) = -f(-iu). \tag{8.13.24}$$

Thus if we set

$$f(-iu) = -i^N \Gamma^N(-i(u+i/2))r(u+i), \tag{8.13.25}$$

then

$$g(u) = -i^N \sum_{k=1}^{\infty} \Gamma^N(-i(u+i/2)-k) r(u+i-ik), \tag{8.13.26}$$

and the desired eigenvalue is given by

$$p(u) = g(u)q(u) - i^N \Gamma^N(-i(u+i/2)) q_2(u). \tag{8.13.27}$$

Note that (8.13.27) is a meromorphic function of the spectral parameter u, which has poles at the integer values of $v = -iu + i/2$ (the convergency of the series for $g(u)$ at $-iu + 1/2 = \mathcal{Z}$ is provided by the term $-k$ in the argument of the gamma function).

The approach developed by Pronko seeks to construct two $Q^{(1,2)}$ operators as traces of monodromies $\hat{Q}^{(1,2)}$, of suitable $M_n^{(1,2)}$ operators acting in the nth quantum space and the auxiliary space γ, which is a representation space of a certain Heisenberg algebra $[\rho, \rho^\dagger] = 1$. It considers products of the form $L(u)M^{(1,2)}(u)$, which are defined in $\Gamma \otimes C^2$ and in which one introduces the projection operators:

$$\Pi^{\dagger}_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{(\rho^\dagger \rho + 1)} (1, \rho^\dagger)_+, \quad \Pi^{-}_{ij} = \begin{pmatrix} -\rho^\dagger \\ 1 \end{pmatrix}_i \frac{1}{(\rho^\dagger \rho + 2)} (-\rho, 1)_j. \tag{8.13.28}$$

Imposing the condition that the products $L(u)M(u)$ and $M(u)L(u)$ are triangular in the sense of projectors $\Pi^{\pm}$, one has for $M_n^1(u)$ the conditions:

$$\Pi^{-}_{ik} (L_n(u))_{kl} M_n^{(1)}(u) \Pi^{+}_{ij} = 0, \tag{8.13.29}$$

$$\Pi^{+}_{ik} M_n^{(1)}(u) (L_n(u))_{kl} \Pi^{-}_{lj} = 0. \tag{8.13.30}$$

On the other hand for $M_n^{(2)}(u)$ we have

$$\Pi^{+}_{ik} (L_n(u))_{kl} M_n^{(2)}(u) \Pi^{-}_{lj} = 0, \tag{8.13.31}$$

$$\Pi^{-}_{ik} M_n^{(2)}(u) (L_n(u))_{kl} \Pi^{+}_{lj} = 0. \tag{8.13.32}$$

It follows from (8.13.29) that

$$M^{(1)}(u) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i = \tilde{L}(u)_{ij} \begin{pmatrix} 1 \\ \rho \end{pmatrix}_j A^{(1)}(u), \tag{8.13.33}$$

$$B^{(1)}(u)\begin{pmatrix}1\\ \rho\end{pmatrix}_i = \tilde{L}(u+i)_{ij}\begin{pmatrix}1\\ \rho\end{pmatrix}_j M^{(1)}(u), \tag{8.13.34}$$

where

$$\tilde{L}(u) = \begin{pmatrix} i & -\phi^\dagger \\ -\phi & u - 3i/2 - \phi^\dagger\phi \end{pmatrix}, \tag{8.13.35}$$

together with the following properties:

$$L(u)\tilde{L}(u) = i(u - i/2)I \tag{8.13.36}$$

and

$$L(u) + \tilde{L}(u+i) = \text{tr }(L(u)I), \tag{8.13.37}$$

(I = identity matrix). Equations (8.13.33 and 8.13.34) admit a solution of the form,

$$B^{(1)}(u) = cM^{(1)}(u+i), \qquad A^{(1)}(u) = c^{-1}M^{(1)}(u), \tag{8.13.38}$$

with c being a number. If one chooses $c = i$, then along with the analogous considerations of triangularity for right multiplication, viz

$$\Pi^+_{ik} M^{(1)}_n(u)(L_n(u))_{kl}\Pi^-_{lj} = 0,$$

it leads to the system,

$$L(u+i)_{ij}\begin{pmatrix}1\\ \rho\end{pmatrix}_j M^{(1)}(u) = M^{(1)}(u+i)\begin{pmatrix}1\\ \rho\end{pmatrix}_i,$$

$$M^{(1)}(u)L(u)_{ij}\begin{pmatrix}-\rho^\dagger\\ 1\end{pmatrix}_j = i\begin{pmatrix}-\rho^\dagger\\ 1\end{pmatrix}_i M^{(1)}(u+i). \tag{8.13.39}$$

For $M^{(2)}$ one similarly obtains

$$\tilde{L}(u+i)_{ij}\begin{pmatrix}-\rho^\dagger\\ 1\end{pmatrix}_j M^{(2)}(u) = M^{(2)}(u+i)\begin{pmatrix}-\rho^\dagger\\ 1\end{pmatrix}_i,$$

$$M^{(2)}(u)L(u)_{ij}\begin{pmatrix}1\\ \rho\end{pmatrix}_j = i\begin{pmatrix}1\\ \rho\end{pmatrix}_i M^{(2)}(u+i). \tag{8.13.40}$$

The full multiplication rules show that the triangular structure as given by (8.13.29 and 8.13.30) and (8.13.31 and 8.13.32) is preserved for products of L_n and M_n. This is a consequence of the fact that the quantum

operators with different n commute with each other. Hence, these relations guarantee that both the traces of monodromies (if they exist),

$$Q^{(1,2)}(u) = \text{tr } (\hat{Q}^{(1,2)}(u)) = \text{tr } \prod_{k=1}^{N} M_k^{(1,2)}(u), \tag{8.13.41}$$

satisfy Baxter's equation.

To solve the equations (8.13.39 and 8.13.40) explicitly the authors in [166] use the holomorphic representation for the operators $\rho, \rho^\dagger$. Accordingly, they let the operator $\rho^\dagger$ be a multiplication operator $(\rho^\dagger\psi)(\alpha) = \alpha\psi(\alpha)$, and ρ be a differential operator, so that $(\rho\psi)(\alpha) = \frac{\partial}{\partial\alpha}\psi(\alpha)$. The action of an operator in the holomorphic representation is defined by its kernel:

$$(\hat{M}\psi)(\alpha) = \int d^2\mu(\beta) M(\alpha, \bar{\beta})\psi(\beta), \tag{8.13.42}$$

with the measure being defined by $d^2\mu(\beta) = e^{-\beta\bar{\beta}} d\beta d\bar{\beta}$. In this representation the operators that satisfy (8.13.39) and (8.13.40) have the following forms:

$$M^{(1)}(u, \alpha, \bar{\beta}) = \frac{e^{-i\bar{\beta}}\phi^\dagger \Gamma(-i(u - i/2))}{\Gamma(-\phi^\dagger\phi - i(u - i/2))} e^{-i\alpha\phi}, \tag{8.13.43}$$

$$M^{(2)}(u, \alpha, \bar{\beta}) = e^{i\alpha\phi} e^{i\pi\phi^\dagger\phi} \Gamma(-\phi^\dagger\phi - i(u - i/2)) e^{i\bar{\beta}\phi^\dagger}. \tag{8.13.44}$$

In order to find the monodromy $\hat{Q}^{(1,2)}(u, \alpha, \bar{\beta})$ one has to take an ordered multiplication of the $M^{(1,2)}$-operators, so that

$$\hat{Q}^{(i)}(u, \alpha, \bar{\beta}) = \int \prod_{i=1}^{N-1} d^2\mu(r_i) M_N^{(i)}(u, \alpha, \bar{\gamma}_{N-1}) \ldots\ldots\ldots M_1^{(i)}(u, \gamma_1, \bar{\beta}). \tag{8.13.45}$$

Taking the trace of $\hat{Q}^{(1,2)}$ over the auxiliary space gives the $Q^{(1,2)}$ operator. Note that the trace of an operator Q in the holomorphic representation, is given by

$$\text{tr } Q = \int d^2\mu(\alpha) \hat{Q}(\alpha, \bar{\alpha}), \tag{8.13.46}$$

where $\hat{Q}(\alpha, \bar{\alpha})$ is the kernel of $\hat{Q}$.

To ascertain that the eigenvalues of $Q^{(1)}(u)$ are polynomials in u, let us consider the action of $Q^{(1)}$ on the basis vectors,

$$|n_1, n_2, ...n_N\rangle = (\phi_1^\dagger)^{n_1}(\phi_2^\dagger)^{n_2}....(\phi_N^\dagger)^{n_N}|0\rangle, \tag{8.13.47}$$

where $|0\rangle$ is the Bethe vacuum with

$$\phi_k|0\rangle = 0, \qquad k = 1, 2, ..., N. \tag{8.13.48}$$

It can be shown that

$$Q^{(1)}(u)|n_1, n_2, ...n_N\rangle = \sum_{m_1,...m_N=0}^{n_1,...n_N} \prod_{k=1}^{N} \frac{(-1)^{m_k}}{m_k!} \frac{\Gamma(-i(u-i/2))}{\Gamma(-i(u-i/2)-n_k+m_k)}$$

$$\times \frac{n_k!}{(n_k-m_k)!}|..., n_k - m_k + m_{k-1}, ...\rangle. \tag{8.13.49}$$

From (8.13.49) it is evident that $Q^{(1)}(u)$ leaves the subspace of vectors, with a common particle number $n = n_1 + n_2 + ... + n_N$ invariant, and all matrix elements of $Q^{(1)}$ are polynomials in u. Moreover, it should be noticed that the action of the second Q operator, namely $Q^{(2)}$, on the same basis vector yields

$$Q^{(2)}(u)|n_1, n_2, ...n_N\rangle = e^{i\pi n} \sum_{m_1,...m_n=0} \prod_{k=1}^{N} \Gamma(-iu - 1/2 - n_k - m_{k-1})\times$$

$$\times \frac{(m_{k-1}+n_k)!}{m_k!(n_k+m_{k-1}-m_k)!}|..., n_k + m_{k-1} - m_k, ...\rangle. \tag{8.13.50}$$

The connection of this particular formalism with the earlier works of Pasquier and Gaudin stems from the fact that in some realizations of the quantum and auxiliary operators, the Q operator assumes a factorized form. Indeed, in a coordinate representation for the quantum and auxiliary operators, one obtains the Q operator as

$$Q(x_1, ...x_N, x_1', ...x_N') = \prod_{k=1}^{N} q_k(x_k, x_{k+1}', x_k). \tag{8.13.51}$$

For the simplest case of one degree of freedom, $Q^{(1)}(u)$ and $Q^{(2)}(u)$ have the following explicit nature:

$$Q^{(1)}(u) = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{\Gamma(-iu+1/2)}{\Gamma(-iu+1/2-n-k)}, \tag{8.13.52}$$

$$Q^{(2)}(u) = e^{i\pi n} \sum_{m=0}^{\infty} \frac{(n+m)!}{m!n!} \Gamma(-iu - 1/2 - n - m). \tag{8.13.53}$$

From the above equations, it is clear that the eigenvalues of $Q^{(1)}(u)$ are polynomials of degree n, while those of $Q^{(2)}(u)$ are meromorphic functions of the spectral parameter u. Furthermore, the solutions of (8.13.29) and (8.13.32) can be cast in the form [166]:

$$M^{(1)}(u, \rho) = P^{\rho\phi}(i - \phi\rho^{\dagger})^{-i(u-i/2)} e^{-u\pi/2}, \tag{8.13.54}$$

$$M^{(2)}(u, \rho) = \Pi^{\rho\phi} \gamma(-\rho^{\dagger}\rho - \rho^{\dagger}\phi - i(u - i/2)), \tag{8.13.55}$$

where

$$P^{\rho\phi} = \exp\left[\frac{\pi}{2}(\phi^{\dagger}\rho - \phi\rho^{\dagger})\right] \exp\left[\frac{i\pi}{2}(\rho^{\dagger}\rho - \phi^{\dagger}\phi)\right] \tag{8.13.56}$$

and

$$\Pi^{\rho\phi} = \left[1 + \sum_{k=1} (i\phi\rho^{\dagger})^k \frac{\Gamma(\rho^{\dagger}\rho + 1)}{\Gamma(\rho^{\dagger}\rho + k + 1)} + \sum_{k=1} (i\phi^{\dagger}\rho)^k \frac{\Gamma(\phi^{\dagger}\phi + 1)}{\Gamma(\phi^{\dagger}\phi + k + 1)}\right] \times$$

$$\times \frac{\Gamma(\rho^{\dagger}\rho + \phi^{\dagger}\phi + 1)}{\Gamma(\rho^{\dagger}\rho + 1)\Gamma(\phi^{\dagger}\phi + 1)} e^{i\pi\phi^{\dagger}\phi}. \tag{8.13.57}$$

The operators $M_n^{(1,2)}$ and $L_n(u)$ satisfy certain intertwining relations that lead directly to the commutativity of Q operators and with the transfer matrix namely:

$$[t(u), Q^{(i)}(v)] = 0, \qquad [Q^{(i)}(u), Q^{(j)}(v)] = 0.$$

Having in principle constructed two solutions of the Baxter's equation, it is natural to establish their linear independence. To this end one considers a finite difference analog of the Wronskian determinant,

$$W_m = Q_1(u - im)Q_2(u + i) - Q_1(u + i)Q_2(u - im),$$

with m being a non-negative integer. From Baxter's equation one has

$$t(u)Q_1(u) = (u - i/2)^N Q_1(u - i) + i^N Q_1(u + i), \tag{8.13.58}$$

$$t(u)Q_2(u) = (u - i/2)^N Q_2(u - i) + i^N Q_2(u + i). \tag{8.13.59}$$

Multiplying (8.13.58) by $Q_2(u)$ and (8.13.59) by $Q_1(u)$ and subtracting, one finds that

$$(u - i/2)^N W_0(u - i) = -i^N W_0(u), \tag{8.13.60}$$

which indicates that W_0 necessarily has a factor $\Gamma^N(-i(u+i/2))$. In fact one can also deduce that

$$W_0(u) = e^{i\pi\hat{n}}\Gamma^N(-i(u+i/2)), \tag{8.13.61}$$

where $\hat{n}$ is the number of particles operator. Again, multiplying (8.13.59) by $Q_2(u-i)$ and (8.13.58) by $Q_1(u-i)$, one can show that

$$W_1(u) = (-i)^N t(u) W_0(u-i). \tag{8.13.62}$$

In the general case of non-negative integer m one has the relation,

$$(u-i/2)^N W_{m-2}(u-2i) + i^N W_m(u) = t(u) W_{m-1}(u-i), \tag{8.13.63}$$

and hence, the full set of W_i's may be determined once W_0, W_1, etc. are known.

The above discussion gives us an overall idea of an alternative approach for constructing Q operators. We have purposely not gone into the rigorous mathematical details. However, a few remarks are necessary. First, this method has the important property of being able to generate two different solutions of the Q operator. In fact an application of this formalism to the case of the Toda lattice has also been made in [167]. We refer the interested reader to the original papers for further details. Secondly, although the case of the XXX spin chain is described in [166], the latter model needs to be studied further, since as of now there is no explicit derivation of the generating function and the corresponding canonical transformation within this framework. Lastly, to understand the physical consequences of this formalism, a proper semiclassical limit needs to be introduced. Therefore, further scope for analysis remains in order to ascertain the implications and efficacy of this approach.

Chapter 9

Quantum GLM Equation

9.1 Introduction

We have so far discussed several features of quantum integrable systems, many of which are similar to the classical ones, while others are valid only for quantum systems. The Bethe ansatz together with its variations all belong to the latter category. As mentioned earlier, the Bethe ansatz allows us to calculate the spectrum of the excitations when the nonlinear system is quantized. However, there is a basic difference between the quantum inverse scattering method formulated by means of the algebraic Bethe ansatz and classical inverse scattering transform. While in the classical inverse scattering transform, one determines or reconstructs the form of the nonlinear field as a function of (x, t), in the quantum inverse scattering method one can expect to compute the "excitation levels" only, but not the "shape" of the nonlinear object. In other words, one cannot compute or reconstruct the analog of the classical field. In classical inverse scattering, the reconstruction of the fields is done with the aid of the well-known Gelfand-Levitan-Marchenko (GLM) equation.

In this chapter we shall discuss a quantum mechanical version of the Gelfand-Levitan-Marchenko equation.

9.2 Quantum GLM Equation

We consider the discrete NLS equation for which the Lax operator can be written as

$$L_n(\lambda) = (1+\frac{m}{2}\rho_n)I - \frac{i\lambda\Delta}{2}\sigma_3 - i\alpha\sqrt{1+\frac{m}{2}\rho_n}\psi_n\sigma_- + i\epsilon\alpha\psi_n^\star\sqrt{1+\frac{m}{4}\rho_n}\sigma_+ \tag{9.2.1}$$

where $\sigma_-, \sigma_+, \sigma_3$ are Pauli matrices and

$$\rho_n = \psi_n^\star\psi_n, \quad \alpha = \sqrt{m}, \quad [\psi_n, \psi_m^\star] = \Delta\delta_{m,n}.$$

The matrix $L_n(\lambda)$ satisfies the Yang-Baxter equation:

$$R(\lambda-\mu)(L_n(\lambda)\otimes I)(I\otimes L_n(\mu)) = (I\otimes L_n(\mu))(L_n(\lambda)\otimes I)R(\lambda-\mu), \tag{9.2.2}$$

with $R(\lambda)$ denoting the quantum R matrix operating on $\mathcal{C}^2\otimes\mathcal{C}^2$ and having the form $R(\lambda) = I + \frac{im}{\lambda}\mathcal{P}$. $\mathcal{P}$ being the usual permutation operator : $\mathcal{P}f\otimes g = g\otimes f$. The generating function of the quantum mechanical time part of the Lax operator has the form [172]:

$$M_n(\lambda,\mu) = \mathrm{tr}_1(T_n^+(\mu)\otimes I)R(\lambda-\mu)(T_n^-(\mu)\otimes I), \tag{9.2.3}$$

where $T_n^\pm(\mu)$ are the operator-valued matrices:

$$T_n^+(\mu) = \prod_{i=n}^{\overleftarrow{N}} L_i(\mu) = \begin{pmatrix} a_n^+(\mu) & b_n^+(\mu) \\ c_n^+(\mu) & d_n^+(\mu) \end{pmatrix} \tag{9.2.4}$$

$$T_n^-(\mu) = L_{n-1}(\mu).....L_{-N}(\mu) = \begin{pmatrix} a_n^-(\mu) & b_n^-(\mu) \\ c_n^-(\mu) & d_n^-(\mu) \end{pmatrix}, \tag{9.2.5}$$

and tr_1 stands for the trace with respect to the first space in the tensor product $\mathcal{C}^2\otimes\mathcal{C}^2$. It can be verified readily that (9.2.1) satisfies the equation,

$$M_n(\lambda,\mu)L_n(\lambda) = L_n(\lambda)M_{n+1}(\lambda,\mu). \tag{9.2.6}$$

Before we go into the detailed derivation of the quantum Gelfand-Levitan-Marchenko (GLM) equation, we shall discuss the classical concept of Floquet indices and Floquet functions that are relevant in the subsequent quantum mechanical analysis.

9.3 Quantum Floquet Function

The Floquet function for a difference equation with periodic boundary conditions may be defined in the following manner.
Let $\hat{T}_p$ be the operator of translation by p nodes along a chain. Its action is defined by

$$\hat{T}_p f_n = f_{n+p}. \tag{9.3.1}$$

Consider the periodic potential involving the Lax equation,

$$\phi_{n+1}(\lambda) = L_n(\lambda)\phi_n(\lambda), \tag{9.3.2}$$

$$L_{M+2N}(\lambda) = L_M(\lambda). \tag{9.3.3}$$

The action of $\hat{T}_p$ on ϕ_n leads to

$$\hat{T}_p\phi_n(\lambda) = L_{p+n}(\lambda)L_{p+n-1}(\lambda).....L_n(\lambda)\phi_n(\lambda) = \phi_{n+p}(\lambda). \tag{9.3.4}$$

By definition a Floquet solution of (9.3.2) is an eigenfunction of the operator of translation by $2N$ nodes,

$$\hat{T}_{2N}F_n(\lambda) = \exp(2ip(\lambda)N)F_n(\lambda), \tag{9.3.5}$$

whence $p(\lambda)$ is known as the quasi momentum. The known analytic and asymptotic properties of F permit us to reconstruct it and help us to find the potential in the auxiliary linear problem. In the periodic case these analytical properties are complicated in nature requiring exhaustive use of Riemann surfaces with handles for a proper exposition. We will not go into these details, but assumme that all the required properties hold. In the quantum case the elements of $\phi_n(\lambda)$ are operators in the full-state space,

$$\mathcal{H} = \mathcal{H}_{-n} \otimes .... \otimes \mathcal{H}_n, \tag{9.3.6}$$

where $\mathcal{H}_n$ stands for the state space at the nth node. The translation operator acts on the solutions of the quantum linear problem. It is natural to define the quantum Floquet function as an operator that diagonalizes the operator of translation by $2N$ nodes:

$$\begin{aligned}\hat{T}_n(\lambda)F_n(\lambda) &= L_n(\lambda)....L_N(\lambda)L_{-N}(\lambda)....L_{n-1}(\lambda)F_n(\lambda) \\ &= \exp(2ip(\lambda)N)F_n(\lambda).\end{aligned} \tag{9.3.7}$$

The translation operator $\hat{T}_n(\lambda)$ is simply related to the generating function $M_n(\lambda,\mu)$. It is the residue of $M_n(\lambda,\mu)$ at $\lambda=\mu$, which is evident from the following formula:

$$M_n(\lambda,\mu) = \text{tr } T_n(\mu) I + \frac{im}{\lambda-\mu} T_n(\mu), \tag{9.3.8}$$

where $T_n(\mu) = T_n^+(\mu) T_n^-(\mu)$ is the monodromy matrix. The matrix elements of $T(\lambda)$ in the auxiliary space are

$$T_n(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}. \tag{9.3.9}$$

From (9.3.9) it follows that $T_n(\lambda)$ and $M_n(\lambda,\mu)$ can be simultaneously diagonalized. Furthermore, since $M_n(\lambda,\mu)$ is the trace of the monodromy operator of a nonhomogeneous lattice, an impurity with quantum space $\mathcal{C}^2$ is introduced between the nth and the $(n-1)$th nodes. The monodromy matrix of such a chain acts in space $\mathcal{C}^2 \otimes (\mathcal{C}^2 \otimes \mathcal{H})$ and has the form

$$\hat{T}_n(\lambda,\mu) = (T_n^+(\mu) \otimes I) R(\mu-\lambda) (T_n^-(\mu) \otimes I), \tag{9.3.10}$$

$$= \begin{pmatrix} \hat{A}_n(\lambda,\mu) & \hat{B}_n(\lambda,\mu) \\ \hat{C}_n(\lambda,\mu) & \hat{D}_n(\lambda,\mu) \end{pmatrix}. \tag{9.3.11}$$

The operators $\hat{A}, \hat{B}$, etc. act in $\mathcal{C}^2 \otimes \mathcal{H}$ and

$$M_n(\lambda,\mu) = \hat{A}_n(\lambda,\mu) + \hat{D}_n(\lambda,\mu). \tag{9.3.12}$$

From (9.3.8) one can find explicit expressions for $\hat{A}, \hat{B}$, etc. For example,

$$\hat{B}_n(\lambda,\mu) = b(\mu) I + \frac{im}{\lambda-\mu} \begin{pmatrix} b_n^-(\mu) \\ d_n^-(\mu) \end{pmatrix} \otimes (a_n^+(\mu), b_n^+(\mu)). \tag{9.3.13}$$

Note that the operator $\hat{T}_n(\lambda,\mu)$ satisfies the same Yang-Baxter equation (9.2.2) and that the vector

$$\hat{\Omega} = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}, \tag{9.3.14}$$

where Ω is the Fock vacuum for ϕ_n, has the following properties:

$$\hat{C}_n(\lambda,\mu)\hat{\Omega} = 0,$$

$$\hat{D}_n(\lambda,\mu)\hat{\Omega} = \left(1+\frac{i\mu}{2}\Delta\right)^{2N}\hat{\Omega}, \tag{9.3.15}$$

$$\hat{A}_n(\lambda,\mu)\hat{\Omega} = \frac{\lambda-\mu+im}{\lambda-\mu}\left(1-\frac{i\mu}{2}\Delta\right)^{2N}\hat{\Omega}.$$

In order to find the eigenvectors of $M_n(\lambda,\mu)$, one can use the Bethe *ansatz*. Consider the eigenvectors,

$$f_n(\lambda,\mu,\mu_1(\lambda),...\mu_n(\lambda)) = \prod_{k=1}^{m}\hat{B}(\lambda,\mu_k(\lambda))\hat{\Omega}, \tag{9.3.16}$$

where $\mu_i(\lambda)$ satisfies the system,

$$\left(\frac{1+\frac{i\mu_k}{2}\Delta}{1-\frac{i\mu_k}{2}\Delta}\right)^{2N} = \frac{\Delta-\mu_k+im}{\Delta-\mu_k}\prod_{j\neq k}^{m}\frac{\mu_k-\mu_j+im}{\mu_k-\mu_j-im}. \tag{9.3.17}$$

It can be shown that the eigenvalue of $M_n(\lambda,\mu)$ is

$$m(\lambda,\mu) = \frac{\lambda-\mu+im}{\lambda-\mu}\left(1-\frac{i\mu}{2}\Delta\right)^{2N}\prod_{k=1}^{m}\frac{\mu-\mu_k+im}{\mu-\mu_k}+$$

$$+\left(1+\frac{i\mu}{2}\Delta\right)^{2N}\prod_{k=1}^{m}\frac{\mu-\mu_k-im}{\mu-\mu_k}. \tag{9.3.18}$$

The eigenvalue of $T_n(\mu)$ on f_n is

$$t(\lambda) = \left(1-\frac{i\lambda\Delta}{2}\right)^{2N}\prod_{k=1}^{m}\frac{\lambda-\mu_k+im}{\lambda-\mu_n} = \exp(2ip(\lambda)N). \tag{9.3.19}$$

On the other hand the eigenvectors,

$$b(\mu_1,...\mu_m) = b(\mu_1)....b(\mu_m)\Omega, \tag{9.3.20}$$

describe the full-state space of the model, where $\mu_1,...\mu_m$ satisfy the system of equations:

$$\left(\frac{1-\frac{i\mu_k}{2}\Delta}{1+\frac{i\mu_k}{2}\Delta}\right)^{2N} = \prod_{j\neq k}^{m}\frac{\mu_k-\mu_j+im}{\mu_k-\mu_j-im}. \tag{9.3.21}$$

Following the insertion of an impurity the space $\mathcal{H}$ is enriched to $\hat{\mathcal{H}} = \mathcal{H}\oplus\mathcal{C}^1$. Hence $M_n(\lambda,\mu)$ has twice as many eigenvectors as $\mathrm{tr}T_n(\lambda)$.

When $\lambda \to \infty, R(\lambda, \mu) \to I$, the impurity ceases to interact with the basic model. Then (9.3.17) degenerates to (9.3.21) and we have the following set of eigenfunctions:

$$f^+(\mu_1, ...\mu_n) = \begin{pmatrix} b(\mu_1...\mu_n) \\ 0 \end{pmatrix}, \quad f^-(\mu_1, ...\mu_n) = \begin{pmatrix} 0 \\ b(\mu_1, ...\mu_n) \end{pmatrix}. \tag{9.3.22}$$

One can show that the solutions of (9.3.21) $\mu_1, ...\mu_n$ produce a solution of (9.3.17) $\mu_1(\lambda), ...\mu_n(\lambda)$ so that

$$\mu_j(\lambda) \to \mu_j. \tag{9.3.23}$$

Consider the operator $F_n^+(\lambda)$ acting from $\mathcal{H}$ to $\hat{\mathcal{H}}$ according to the rule:

$$F_n^+(\lambda) : f(\mu_1, ...\mu_m) \to f_n(\lambda, \mu_1(\lambda)....\mu_m(\lambda)). \tag{9.3.24}$$

The operator $F_n^+(\lambda)$ thus defined is the first branch of the operator Floquet function. It is obvious that

$$F_n^+(\lambda) \overset{\lambda\to\infty}{\longrightarrow} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{9.3.25}$$

The second branch of the Floquet operator is

$$F_n^-(\lambda) \overset{\lambda\to\infty}{\longrightarrow} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.3.26}$$

Analysis with finite N is complicated and as a result the quasimomentum $p(\lambda)$ has complicated analytical properties. But if $N \to \infty$ in the Fock space with the following conditions, namely

- $\mu_1....\mu_p$, arbitrary real numbers which are independent of λ,
- $Im\,\lambda < 0, \mu_1 = \lambda_1$, while $\mu_1....\mu_p$ arbitrary,
- $Im\,\lambda > -\sigma, \mu_1 = \lambda + i\sigma$, while $\mu_1, ...\mu_p$ are arbitrary,

then we can define $T^{\pm}, T$ as follows:

$$T(\lambda) = \lim_{N\to\infty} V^{-N}(\lambda) T_N(\lambda) V^{-N}(\lambda) = \begin{pmatrix} A(\lambda)\ B(\lambda) \\ C(\lambda)\ D(\lambda) \end{pmatrix} \tag{9.3.27}$$

$$T^+(\lambda) = \lim_{N\to\infty} V^{-N+k}(\lambda) T_k^+(\lambda) = \begin{pmatrix} A_+(k,\lambda)\ B_+(k,\lambda) \\ C_+(k,\lambda)\ D_+(k,\lambda) \end{pmatrix} \tag{9.3.28}$$

$$T^{-}(\lambda) = \lim_{N\to\infty} T_k^{-}(\lambda)V^{-N-k}(\lambda) = \begin{pmatrix} A_-(k,\lambda)\ B_-(k,\lambda) \\ C_-(k,\lambda)\ D_-(k,\lambda) \end{pmatrix} \tag{9.3.29}$$

$$V(\lambda) = \mathrm{diag}\left(1 - \frac{i\lambda\Delta}{2}, 1 + \frac{i\lambda\Delta}{2}\right). \tag{9.3.30}$$

The limiting value of the operator $\hat{B}_k(\lambda,\mu)$ is given by

$$B(\mu)I + \frac{im}{\lambda-\mu}\left(\frac{1-\frac{i\lambda\Delta}{2}}{1+\frac{i\lambda\Delta}{2}}\right)^{-2k}\begin{pmatrix} B_-(k,\mu) \\ D_-(k,\mu)\end{pmatrix} \otimes (A_+(k,\mu), B_+(k,\mu)). \tag{9.3.31}$$

From the solution (9.3.27) and (9.3.28) one can construct the limiting Floquet functions on the first branch,

$$F_1(k,\lambda)B(\mu_1)....B(\mu_n)\Omega = \hat{B}_n(\lambda,\mu_1).....\hat{B}_k(\lambda,\mu_k)\Omega. \tag{9.3.32}$$

Obviously $F_1(k,\lambda) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $\lambda \to \infty$, where $F_1(k,\lambda)$ is the limit of $F_k^{+}(\lambda)$ for $N \to \infty$. From the previous formula it is readily seen that

$$\hat{B}_k(\lambda,\mu) \overset{k\to-\infty}{\longrightarrow} B(\mu)I, \quad Im\,\lambda > 0, \tag{9.3.33}$$

$$\hat{B}_k(\lambda,\mu) \overset{k\to\infty}{\longrightarrow} B(\mu)\begin{pmatrix} 1+\frac{im}{\lambda-\mu} \\ 1 \end{pmatrix}, \quad Im\,\lambda < 0. \tag{9.3.34}$$

For $F_1(k,\lambda)$ this yields

$$F_1(k,\lambda) \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad k \to -\infty, Im\,\lambda > 0, \tag{9.3.35}$$

$$F_1(k,\lambda) \longrightarrow D^{-1}(\lambda+im)\begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad k \to \infty, Im\,\lambda < 0. \tag{9.3.36}$$

Moreover, from the fact that $F_1(k,\lambda)$ diagonalizes the operator $M_n(\lambda,\mu)$ given by (9.3.8), one gets the connection between $F_1(k,\lambda)$ and the Jost functions,

$$\begin{aligned} F_1(k,\lambda) &= \chi(k,\lambda); \qquad Im\,\lambda > 0, \\ F_1(k,\lambda) &= \tilde{\phi}(k,\lambda+im)D^{-1}(\lambda+im); \qquad Im\,\lambda < 0, \end{aligned} \tag{9.3.37}$$

where

$$\chi(k,\lambda) = \begin{pmatrix} A_-(k,\lambda) \\ C_-(k,\lambda)\end{pmatrix}; \quad \tilde{\phi}(k,\lambda) = \begin{pmatrix} D_+(k,\lambda) \\ -C_+(k,\lambda)\end{pmatrix}. \tag{9.3.38}$$

The function $F_1(k,\lambda)$ has a jump on the real axis caused by the poles in the right-hand side of (9.3.8). For $\mu_i \neq \mu_j$ we get

$$[F_1(k,\lambda+i0)-F_1(k,\lambda-i0)]B(\mu_1).....B(\mu_m)\Omega =$$

$$2\pi m\sum_{j=1}^{n}\delta(\lambda-\mu_j)\prod_j\frac{\lambda-\mu_j+im}{\lambda-\mu_j}\tilde{B}_k(\lambda,\mu)\prod B_k(\lambda,\mu_l)\tilde{\Omega}(check?), \tag{9.3.39}$$

$$\tilde{B}_k(\lambda,\mu)=\frac{\lambda-\mu}{\lambda-\mu+im}\hat{B}_k(\lambda,\mu). \tag{9.3.40}$$

The vector on the right-hand side of the above equation defines the second branch of the Floquet function:

$$F_2(k,\lambda)B(\mu_1)...B(\mu_n)\Omega=\tilde{B}(\lambda,\lambda)\tilde{B}(\lambda,\mu_1)....\tilde{B}(\lambda,\mu_n)\tilde{\Omega};\quad Im\,\lambda>0. \tag{9.3.41}$$

Introducing the operator $R(\lambda)=D^{-1}(\lambda)C(\lambda)$ and using the commutation rule,

$$R(\lambda)R(\mu)=\frac{\lambda-\mu+im}{\lambda-\mu}B(\mu)R(\lambda)+2\pi mA(\lambda)\delta(\lambda-\mu), \tag{9.3.42}$$

we can write (9.3.39) as

$$F_1(k,\lambda+i0)-F_1(k,\lambda-i0)=F_2(k,\lambda)R(\lambda).$$

Similar to $F_1(k,\lambda)$, we can show that $F_2(k,\lambda)$ is also connected to the Jost function. To do so, note that $\tilde{B}(\lambda,\lambda)$ degenerates into a projector:

$$\tilde{B}(\lambda,\lambda)=\left(\frac{1-\frac{i\lambda\Delta}{2}}{1+\frac{i\lambda\Delta}{2}}\right)^{-2k}\begin{pmatrix}B_-(k,\lambda)\\D_-(k,\lambda)\end{pmatrix}\otimes(A_+(k,\lambda),B_+(k,\lambda)), \tag{9.3.43}$$

whence in view of (9.3.41) and (9.3.43) we find that

$$F_2(k,\lambda)=\left(\frac{1-\frac{i\lambda\Delta}{2}}{1+\frac{i\lambda\Delta}{2}}\right)^{2k}\begin{pmatrix}B_-(k,\lambda)\\D_-(k,\lambda)\end{pmatrix}=\left(\frac{1-\frac{i\lambda\Delta}{2}}{1+\frac{i\lambda\Delta}{2}}\right)^{2k}\bar{\chi}(k,\lambda) \tag{9.3.44}$$

as a consequence of which (9.3.42) becomes equivalent to

$$\chi(k,\lambda)-\left(\frac{1-\frac{i\lambda\Delta}{2}}{1+\frac{i\lambda\Delta}{2}}\right)^{2k}\bar{\chi}(k,\lambda)R(\lambda)=\phi(k,\lambda+im)D^{-1}(\lambda+im), \tag{9.3.45}$$

leading thereby to the quantum GLM equation.

In the foregoing discussion we have tried to present an idea about how the quantum Gelfand-Levitan-Marchenko equation may be formulated in the case of the discrete NLS equation. The discussion was based mainly on the exposition of Reshetikhin and Smirnov [171]. We have not been able to give details regarding the construction of the operator $M(\lambda, \mu)$, and we refer the interested reader to their original papers [172, 173].

The relevance of the quantum GLM equation is yet to be fully ascertained in the theory of quantum integrable systems, primarily because the algebraic Bethe ansatz is much easier to apply to concrete systems, thereby somewhat overshadowing this method. To appreciate this formalism from a more physical point of view, we shall now discuss it in the context of a continuous NLS system.

A slightly different approach utilizes the concept of the quantum determinant, which is used to evaluate the inverse of matrices like $L_n(\lambda)$ whose elements are operators. In the present case one can verify that

$$L_n^{-1}(\lambda) = d^{-1}(\lambda)\sigma_2 L_n^t(\lambda + im)\sigma_2, \tag{9.3.46}$$

with

$$d(\lambda) = (1 - \frac{i\lambda}{2}\Delta)(1 + \frac{i\lambda}{2}\Delta - \frac{m\Delta}{4}). \tag{9.3.47}$$

The monodromy matrix $T(\lambda)$ can be written as

$$T(\lambda) = T_n^+(\lambda)T_n^-(\lambda), \tag{9.3.48}$$

whence using the above equation we have

$$T_n^+(\lambda) - d^{-N-n+1}(\lambda)T(\lambda)\sigma T_n^{-1?}(\lambda + im)\sigma_2. \tag{9.3.49}$$

The first row of this equation yields

$$A_n^{-1}(\lambda - im)\begin{pmatrix} A_n^+(\lambda - im) \\ B_n^+(\lambda - im)\end{pmatrix} = d^{-N-k+1}(\lambda - im)\left[\begin{pmatrix} D_n^-(\lambda) \\ -B_n^-(\lambda)\end{pmatrix} + \right.$$

$$\left. + B_n(\lambda)A_n^{-1}(\lambda)\begin{pmatrix} -C_n^-(\lambda) \\ A_n^-(\lambda)\end{pmatrix}\right], \tag{9.3.50}$$

where the parameter λ is shifted and use has been made of the identity,

$$A_n^{-1}(\lambda - im)B_n(\lambda - im) = B_n(\lambda)A_n^{-1}(\lambda). \tag{9.3.51}$$

Now it is known that as $\Delta \to 0$ and the length $L \to \infty$ we have

$$T(\lambda) = \lim V^{-N}(\lambda)T_n(\lambda)V^{-N}(\lambda), \tag{9.3.52}$$

$$T^+(x,\lambda) = \lim_{n\Delta=x} V^{-N+n}(\lambda)T_n^+(\lambda), \tag{9.3.53}$$

$$T^-(x,\lambda) = \lim_{n\Delta=x} T_n^-(\lambda)V^{-N+n+1}(\lambda), \tag{9.3.54}$$

with $V(\lambda) = \mathrm{diag}(1 - \frac{i\lambda}{2}\Delta, 1 + \frac{i\lambda}{2}\Delta)$. These are just equations (9.3.27–9.3.29) written in a different way for easier reference. If the normalized Jost functions be defined as

$$\Phi(x,\lambda) = \begin{pmatrix} A^+(x,\lambda) \\ B^+(x,\lambda) \end{pmatrix}, \quad \bar{\Phi}(x,\lambda) = \begin{pmatrix} C^+(x,\lambda) \\ D^+(x,\lambda) \end{pmatrix}, \tag{9.3.55}$$

$$\chi(x,\lambda) = \begin{pmatrix} -C^-(x,\lambda) \\ A^-(x,\lambda) \end{pmatrix}, \quad \bar{\chi}(x,\lambda) = \begin{pmatrix} D^-(x,\lambda) \\ -B^-(x,\lambda) \end{pmatrix}, \tag{9.3.56}$$

then upon passing to the limit, we obtain from (9.3.51):

$$\bar{\chi}(x,\lambda) + e^{-i\lambda x}B(\lambda)A^{-1}(\lambda)\chi(x,\lambda) = A^{-1}(\lambda - im)\phi(x,\lambda - im). \tag{9.3.57}$$

Now inverting T^+ but not T^- in (9.3.49) we get

$$\Phi(x,\lambda) - e^{i\lambda x}\phi(x,\lambda)A^{-1}(\lambda)C(\lambda) = \chi(x,\lambda - im)A^{-1}(\lambda - im). \tag{9.3.58}$$

It is important to note that these equations (9.3.57) and (9.3.58) differ from the corresponding classical relations by a shift in the spectral parameter on the right-hand side. The analyticity properties of A, Φ, χ and $D, \bar{\phi}, \bar{\chi}$ remain the same as in the classical case, although the proofs are more involved. For $m < 0$ bound states occur as before and one has

$$A(\lambda)|n,k\rangle = \frac{\lambda - k + \frac{im}{2}(n+1)}{\lambda - k - \frac{im}{2}(n-1)}|n,k\rangle.$$

Now let us study (9.3.58). The function Φ is analytic for $\mathrm{Im}\ \lambda \leq 0$, while the right-hand side of the same equation has $\mathrm{Im}\ \lambda > 0$, discontinuites on the straight line $\mathrm{Im}\ \lambda = -mn/2$, $n = 1, 2, ...$ Using the asymptotic behaviour,

$$A(\lambda) \to 1, \Phi(x,\lambda) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi(x,\lambda) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

one can write an integral representation for $\Phi(x,\lambda)$ in the form:

$$\Phi(x,\lambda)=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\left[\int_{-\infty}^{\infty}\frac{\Phi(x,\mu)A^{-1}(\mu)C(\mu)e^{i\mu x}}{\lambda-\mu}d\mu\right.$$

$$\left.+\sum_{n=1}^{\infty}\int_{\text{Im }\lambda=-mn/2}\frac{\chi(x,\mu-im)\text{disc }A^{-1}(\mu-im)}{\lambda-\mu}d\mu\right],\ \text{Im }\lambda<0$$

where $\text{disc}A^{-1}(\mu)=A^{-1}(\mu+i0)-A^{-1}(\mu-i0)$. The next question is how to express $\chi(x,\mu)\text{disc}A^{-1}(\mu)$ in terms of Φ to have a closed system. Using

$$A^{+}(x,\lambda)A^{-}(x,\lambda)+B^{+}(x,\lambda)C^{-}(x,\lambda)=A(\lambda)$$

and the equality $A(\lambda)\text{disc}A^{-1}(\lambda)=0$ we get

$$\chi(x,\lambda)\text{disc}A^{-1}(\lambda)=\Phi(x,\lambda-im)F(x,\lambda+im),$$

where

$$F(x,\lambda+im)=C^{-}(x,\lambda)(A^{+}(x,\lambda+im))^{-1}\text{disc}A^{-1}(\lambda).$$

The standard commutation rules give

$$A(\mu)C^{-}(x,\lambda)A^{+}(x,\lambda{+}im)^{-1}=\frac{\lambda-\mu+im}{\lambda-\mu}C^{-}(x,\lambda)A^{+}(x,\lambda{+}im)^{-1}\times$$

$$A(\mu)-\frac{im}{\lambda-\mu}A^{+}(x,\lambda+im)^{-1}A^{+}(x,\mu)C^{-}(x,\mu)A^{+}(x,\lambda)^{-1}A(\lambda).$$

It then follows that

$$A(\mu)F(x,\lambda)=\frac{\lambda-\mu}{\lambda-\mu-im}F(x,\lambda)A(\mu). \tag{9.3.59}$$

The last equation implies that

$$e^{iPb}F(x,\lambda)e^{-iPb}=e^{-i\lambda t}F(x,\lambda), \tag{9.3.60}$$

$$e^{iHt}F(x,\lambda)e^{-iHt}=e^{-i\lambda^{2}t}F(x,\lambda), \tag{9.3.61}$$

where P,H are the momentum and energy operators. The functions Φ and χ have the properties,

$$e^{-iPb}\chi(x,\lambda)e^{iPb}=\chi(x+b,\lambda), \tag{9.3.62}$$

$$e^{-iPb}\Phi(x,\lambda)e^{iPb}=\Phi(x+b,\lambda), \tag{9.3.63}$$

so that $F(x,\lambda)$ depends on x in a trivial way. Let us fix the point x_0 and introduce

$$F_{n,1}(\lambda) = F(x_0, \lambda - \frac{imn}{2}), \quad n = 1, 2, ...$$

$$F_{0,1}(\lambda) = e^{i\lambda x_0} A^{-1}(\lambda) C(\lambda),$$

$$\phi^n(x,\lambda) = \Phi(x, \lambda - \frac{imn}{2}), \quad n = 0, 1, ...$$

$$\bar{\phi}^n(x,\lambda) = \bar{\phi}(x, \lambda + \frac{imn}{2}), \quad n = 0, 1, ... \tag{9.3.64}$$

where Im $\lambda = 0$. The operator $F_{n,1}(\lambda)$ has the intutive meaning of an operator annihilating a particle with momenta $(\lambda - imn/2)$ in the bound state $|n+1, \lambda\rangle$. At this stage it is necessary to introduce the annihilation operators for k particles with momenta $\lambda - (\frac{imn}{2} + jm)$; $j = 0, 1, ..., k-1$ in the state $|n+1, \lambda\rangle$,

$$F_{n,k}(\lambda) = \Big[C^-(x_0, \lambda - \frac{im}{2}(n+2) + im(k-1)) A^+(x_0, \lambda - \frac{imn}{2}$$

$$+ im(k-1)).....C^-(x_0, \lambda - \frac{im}{2}(n+2)) A^+(x, \lambda - \frac{imn}{2})^{-1}\Big] \times$$

$$\mathrm{disc}(\lambda - \frac{im}{2}(n+2)). \tag{9.3.65}$$

These operators satisfy

$$A(\mu) F_{n,k}(\lambda) = \frac{\lambda - \mu - \frac{imn}{2} + im(k-1)}{\lambda - \mu - \frac{imn}{2} - im} F_{n,k}(\lambda) A(\mu), \tag{9.3.66}$$

$$F_{n,k}(\lambda) F_{r,l}(\mu) = \frac{\lambda - \mu - \frac{im}{2}(n - r - 2k)}{\lambda - \mu - \frac{im}{2}(n - r + 2l) - i0} F_{r,l}(\mu) F_{n,k}(\lambda)$$

$$+ 2\pi i \delta(\lambda - \mu) \delta_{r-n,2l} F_{r,k+l}(\lambda) A^{-1}\left(\lambda - \frac{im}{2}(n+2) + i0\right), \tag{9.3.67}$$

$$F_{n,1}(\lambda) F^\star_{r,1}(\mu) = \frac{\lambda - \mu - \frac{im}{2}(n+r) - im}{\lambda - \mu - \frac{im}{2}(n+r) + im} F^\star_{r,1}(\mu) F_{n,1}(\lambda), \tag{9.3.68}$$

whence the GLM equation can be written as

$$\bar{\Phi}^n(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \sum_{r=0}^{\infty} e^{mr(x-x_0)/2} \int_{-\infty}^{\infty} d\mu \frac{\Phi^r(x,\mu) F_{r,1}(\mu) e^{i\mu(x-x_0)}}{\lambda - \mu + \frac{im}{2}(n+r) - i0} \tag{9.3.69}$$

If we adjoin it with,

$$\Phi^n(x,\lambda) = -i\sigma_2\bar{\phi}^n(x,\lambda)^\star$$

then (9.3.69) represents the desired GLM equation.

In the above analysis we have discussed how the Floquet problem and the GLM equation can be formulated in the case of a discrete system and how the passage to the continuous case can be achieved. On the other hand, a completely distinct approach to the same problem was given by Thacker [22, 36], Creamer et al. [180] for a continuous system. The construction of the bound state creation and annihilation operators was studied later by Kaup [176], Göckeller [179] and Nakawaki [175]. We shall try to give a brief overview of their work in the following sections. Lastly, it is important to remember that the main difficulty in the quantum case arises in ascertaining the analyticity properties in the λ-plane, which, however, is similar to the corresponding classical counterpart.

9.4 Exact Quantization

An alternative approach to the process of quantization was suggested by Kaup in 1975 [176]. His approach was based on the analogy of the nonlinear inverse scattering transform and linear Fourier analysis, the idea being that it was perhaps easier to quantize the normal modes rather than the fields. The motivation for this arose from the fact that the normal modes are independent and therefore relatively simpler to deal with compared to the field variables. Moreover, by appealing to Bohr's correspondence principle one could try to examine the classical equations of motion for the modes, in the limit $\hbar \to 0$. To keep track of the limiting procedure, we will mostly follow the notations of Kaup [176], after a little simplification, since in the original work a somewhat unconventional notation was employed. We consider the Zakharov problem for the NLS system, which is written in the form,

$$v_{1x} + i\lambda v_1 = \sqrt{-c}\psi v_2,$$

$$v_{2x} - i\lambda v_2 = -\sqrt{-c}\psi^\star v_1(check). \tag{9.4.1}$$

The classical solution of the initial value problem is as follows. First the scattering data is determined at $t = 0$;

$$S_+ = \{[\lambda_j, \rho_j]_{j=1}^J, \rho(\lambda) \quad (\lambda = \text{real})\}. \tag{9.4.2}$$

Let ψ satisfy the condition

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx < \infty. \tag{9.4.3}$$

The Jost function $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, defined to be a solution of (9.4.1), satisfies:

$$\phi \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \quad \text{as } x \to -\infty \tag{9.4.4}$$

for Im $(\lambda) > 0$, and

$$\phi \longrightarrow \begin{pmatrix} a(\lambda)e^{-i\lambda x} \\ b(\lambda)e^{i\lambda x} \end{pmatrix} \quad \text{as } x \to +\infty. \tag{9.4.5}$$

We define $a(\lambda)$ and $b(\lambda)$ so that they obey

$$\bar{a}(\lambda)a(\lambda) + \bar{b}(\lambda)b(\lambda) = 1, \tag{9.4.6}$$

with

$$\bar{a}(\lambda) = [a(\lambda^\star)]^\star, \quad \bar{b}(\lambda) = [b(\lambda^\star)]^\star, \tag{9.4.7}$$

(the nonlinear field ψ is defined on a compact support). The continuous spectrum of the scattering data is given by

$$\rho(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \quad (\lambda = \text{real}). \tag{9.4.8}$$

The existence of bound states depends on the sign of c. Basically, the inverse scattering transform is a nonlinear mapping of the field $\psi(x, y)$ to the scattering data $S_+(\lambda, t)$. The time evolution of the data are

$$\lambda_t = 0, \quad \rho_{jt} = -(2\lambda_j^2)\rho_j, \quad a(\lambda)_t = 0, \quad (\text{Im}\lambda > 0),$$

$$\rho(\lambda)_t = -(2\lambda)^2\rho(\lambda) \quad (\lambda = \text{real}). \tag{9.4.9}$$

To reconstruct the potential, one takes recourse to the method of Gelfand-Levitan-Marchenko equation, which in the presence of bound

states leads to the soliton solutions. The point to note here is that the above-mentioned mapping has transformed the nonlinear equations into a set of linear equations. The complete integrability of the NLS equation guarantees that this mapping is a canonical, which allows the Hamilton-Jacobi differential equation to be completely separable. As a result, the corresponding action-angle variables can be determined. One starts from the fact that $a(\lambda)$ is an analytic function in λ in the upper half λ-plane and can be represented as

$$\text{Im } a(\lambda) = \sum_{j=1}^{J} \ln\left(\frac{\lambda-\lambda_j}{\lambda-\lambda_j^\star}\right) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\lambda'}{\lambda'-\lambda} \ln[1+\bar{\rho}(\lambda')\rho(\lambda)], \quad (9.4.10)$$

for Im $(\lambda) > 0$, where the summation is absent if $c < 0$. It can be shown that

$$\ln a(\lambda) = \lim_{x\to\infty} \ln \omega(\lambda, x), \quad (9.4.11)$$

$$\text{with} \qquad \omega(\lambda, x) = \phi_1(\lambda, x)e^{i\lambda x}, \quad (9.4.12)$$

where ω is the solution of

$$\omega(\lambda, x) = 1 + \int_{-\infty}^{\infty} M(\lambda; x, y)\omega(\lambda, y)dy,$$

$$\text{with} \qquad M(\lambda; x, y) = -c\psi^\star(y)\int_y^x e^{2i\lambda(z-y)}\psi(z)dz. \quad (9.4.13)$$

The Neumann series solution is absolutely convergent in the upper half plane. The conserved quantities are obtained as asymptotic expansions of $\ln a(\lambda)$. One can prove that

$$\ln a(\lambda) = -\frac{ic}{\lambda}\left(N - \frac{1}{2\lambda}P + \frac{1}{2\lambda^2}E + ...\right), \quad (9.4.14)$$

with

$$N = \int_{-\infty}^{\infty} \psi^\star\psi dx,$$

$$P = -\frac{1}{2}\int_{-\infty}^{\infty} (\psi^\star\psi_x - \psi_x^\star\psi)dx,$$

$$E = \int_{-\infty}^{\infty} [\psi_x^\star\psi_x - \frac{1}{2}c(\psi^\star\psi)^2]dx. \quad (9.4.15)$$

Before proceeding with quantization, let us note that by means of standard procedure, we can derive the Hamiltonian for the continuous NLS equation and define the conjugate momentum of ψ as

$$\Pi = i\hbar\psi^\star. \quad (9.4.16)$$

On the other hand, although the inverse scattering has mapped the nonlinear system to the scattering data, it still remains to identify the actual canonical variables in the scattering data space. A solution to this can be obtained, by writing (9.4.15) in terms of the scattering variables (in the following description we have set $\hbar$ equal to 1), so that

$$E = \sum_{j=1}^{J} \frac{1}{2}(P_j^2 + Q_j^2)P_j^2 - \frac{c^2}{12}\sum_{j=1}^{J}[\frac{1}{2}(P_j^2 + Q_j^2)]^3 + \int d\lambda(-2\lambda)^2 \frac{1}{2}(p^2(\lambda) + Q^2(\lambda)), \tag{9.4.17}$$

where $P_j, Q_j, P(\lambda), Q(\lambda)$ are defined by

$$\arg(P_j + iQ_j) = \arg b_j, \quad P_j^2 + Q_j^2 = -\frac{8}{c}\eta_j,$$

$$P_j = -2\xi_j, \quad Q_j = -\frac{1}{c}\ln|b_j|^2, \tag{9.4.18}$$

with $\lambda_j = \xi_j + i\eta_j$. The choice of the canonical variables are made from the equation,

$$\int_{-\infty}^{\infty} \delta\psi \wedge \delta\Pi dx = \int_{-\infty}^{\infty} d\xi\delta(\arg b) \wedge \delta\left(\frac{1}{c}\ln(1+\bar{\rho}\rho)\right) + + \sum_{j=1}^{J}\delta(\arg b_j) \wedge \delta(\frac{1}{c}\eta_j) + \sum_{j=1}^{J}\delta(\frac{1}{c}\ln(b_j^\star b_j)) \wedge \delta(-2\xi_j), \tag{9.4.19}$$

where $b_j = \rho_j \frac{\partial a}{\partial \lambda}|_{\lambda=\lambda_j}$. Prior to quantization, one must also look for any constraints on the system. Since the Hamiltonian is separable, we need only consider the subspaces of the conjugate pair, one at a time. For the (A_j, B_j) space to be physical, we must have $A_j > 0 (j = 1, ..., J)$, since all the eigenvalues must lie in the upper half λ-plane. This is a nonholonomic constraint and rather difficult to tackle. However, it turns out that one can find a simple transformation to map this space, which is one to one, and onto the space of the harmonic oscillator. This transformation is

$$A_j = \frac{1}{2}(P_j^2 + Q_j^2), \quad B_j = \arg(P_j + iQ_j). \tag{9.4.20}$$

As a result, the quantization is simplified and one gets

$$\frac{1}{2}(P_j^2 + Q_j^2) \longrightarrow A_j^\dagger A_j + \frac{1}{2},$$

$$\frac{1}{2}(P^2(\xi)+Q^2(\xi)) \longrightarrow A^\dagger(\xi)A(\xi)+\frac{1}{2}. \tag{9.4.21}$$

Here $A, A^\dagger$ are the usual destruction and creation operators satisfying

$$[A_k, A_j^\dagger]=\delta_{kj}, \quad [A(\xi), A^\dagger(\xi')]=\delta(\xi'-\xi), \tag{9.4.22}$$

so that the jth soliton appears as a bound state of $\frac{1}{2}(P_j^2+Q_j^2)$ particles. Now, P_j+iQ_j is the classical counterpart of a ladder operator, which increases the number of excitation in the jth soliton by one, thus transforming an n particle bound state into an $(n+1)$ particle bound state. This ladder property is also shared by

$$\exp(i\arg\,(P_j+iQ_j))=e^{cq_j/2}b_j=\lim_{x\to\infty}e^{cq_jz}e^{-i\lambda_j x}\phi_2(\lambda_j,x). \tag{9.4.23}$$

The variable q_j is canonically conjugate to the quantity p_j, which represents the momentum per particle in the jth soliton. We are now in a position to define an analogue of (9.4.1) in the full quantum theory. For this we set

$$\Phi_2(\lambda,x)=-\frac{1}{\sqrt{c}}e^{-i\lambda x}e^{cS/2}\phi_2(\lambda,x), \tag{9.4.24}$$

with

$$S=\int dx\psi^\dagger(x)x\psi(x). \tag{9.4.25}$$

It is important to analyse the question of whether there are states $|A_n\rangle$, so that $\lim_{x\to\infty}|A\rangle$ exist for suitably chosen λ. We consider $|A_n\rangle$ to be an n particle state:

$$|A_n\rangle=\int dx_1...dx_nA_n(x_1....x_n)|x_1....x_n\rangle,$$

$$|x_1....x_n\rangle=\psi^\dagger(x_1)....\psi^\dagger(x_n)|0\rangle. \tag{9.4.26}$$

Note that A_n should be a bounded and symmetric function of its arguments and the following relations should hold:

$$\phi_1(\lambda,x)|A_n\rangle=\int dx_1...dx_ne^{-i\lambda x-2i\lambda(x_1+...+x_n)}U_n(x_1...x_n)|x_1...x_n\rangle, \tag{9.4.27}$$

$$\phi_2(\lambda,x)|A_n\rangle=-\sqrt{-c}\int dy\; e^{i\lambda(x-y)}\theta(x-y)\psi^\dagger(y)\phi_1(\lambda,y)|A_n\rangle$$

$$=\int dx_1....dx_ne^{i\lambda x-2i\lambda(x_1+...x_{n+1})}V_n(x,x_1...x_{n+1})|x_1...x_{n+1}\rangle, \tag{9.4.28}$$

$$V(x,x_1.....x_{n+1}) = \frac{\sqrt{-c}}{n+1}\sum_{j=1}^{n+1}\theta(x-x_j)U_n(x_j,x_1...\hat{x}_j...x_{n+1}), \quad (9.4.29)$$

where the symbol $\hat{x}_j$ means that the argument under $\wedge$ is to be omitted. Applying the series for $\phi_1(\lambda,x)$ (which is constructed from (9.4.28) by iteration to $|A_n\rangle$), we obtain for $x_1 < x_2 < ... < x_n$:

$$U_n(x,x_1...x_n) = D_n(x_1....x_n) + \sum_{m=1}^{n}\theta(x_{m+1} > x > x_m)\sum_{j=1}^{m} c^j \times$$

$$\times \sum_{1\le k_1...\le k_j<m}\int dy_1...dy_j\theta(x > y_j > x_{k_j} > y_{j-1}... > y_1 > x_{k_1})\times$$

$$\times\{D_n(x_1...x_n)|_{x_{k_1}\to y_1....x_{k_j}\to y_j}\}. \quad (9.4.30)$$

Here

$$D_n = A_n(x_1...x_n)e^{2i\lambda(x_1+...+x_n)} \quad (9.4.31)$$

and x_{n+1} has to have been set equal to ∞, so that U_n is determined by (9.4.30). In particular we have

$$U_n(-\infty;x_1...x_n) = D_n(x_1...x_n) \quad (9.4.32)$$

and for $x_1 < x_2... < x_n$ the following relations:

$$U_n(-\infty;x_1....x_n) = D_n(x_1....x_n) \quad (9.4.33)$$

$$U_n(+\infty,x_1...x_n) = D_n(x_1....x_n) + \sum_{j=1}^{n}\theta(x_{m+1} > x > x_m)\sum_{j=1}^{m} c^j \times$$

$$\times \sum_{1\le k_1...\le k_j<n}\int dy_1...dy_j\theta(y_j > x_{k_j} > y_{j-1} > x_{k_j-1}... > y_1 > x_{k_1})\times$$

$$\times\{D_n(x_1...x_n)|_{x_{k_1}\to y_1....x_{k_j}\to y_j}\}. \quad (9.4.34)$$

We also note that

$$a(\lambda)|A_n\rangle = \lim_{x\to\infty} e^{i\lambda x}\phi_1(\lambda,x)|A_n\rangle,$$

$$= \int dx_1...dx_n e^{-2i\lambda(x_1+....+x_n)}U_n(\infty;x_1...x_n)|x_1...x_n\rangle. \quad (9.4.35)$$

Let us investigate the condition under which

$$\lim_{x\to\infty}\phi_2(\lambda,x)|A_n\rangle$$

exists. For this purpose we consider the matrix element,

$$\langle g_{n+1}|\Phi_2(\lambda,x)|A_n\rangle, \tag{9.4.36}$$

with $|g_m\rangle = \int dx_1...dx_m g_m(x_1...x_m)|x_1....x_m\rangle$ We find that

$$\langle g_{n+1}|\Phi_2(\lambda,x)|A_n\rangle = (n+1)!\int dx_1...dx_n e^{(c/2-2i\lambda)(x_1+...x_n)}\times$$

$$\int_{-\infty}^{x} dy e^{(c/2-2i\lambda)y} g^{\star}_{n+1}(x_1...x_n,y)U_n(y;x_1...x_n). \tag{9.4.37}$$

Setting $\lambda = \xi + i\eta,\ (\eta > 0)$ and upon examining the integral over y with $x_1....x_n$ fixed, the following cases arise:

(1) When $2\eta + c/2 > 0$ and if x is finite, then the integral exists. The integration contains a factor that grows exponentially as $y \to \infty$, so that $\lim_{x\to\infty}$ can exist for all g_{n+1}, which satisfy

$$\int dx_1...dx_m|g_m(x_1...x_m)|^k < \infty \text{ for } k = 1,2,$$

only if $U_n(\infty;x_1...x_n) = 0$.

(2) When $2\eta + c/2 < 0$, then the integral exists even if x is finite.

(3) Finally when $2\eta + c/2 = 0$, we have $\lambda = \xi - ic/4$ and

$$\Phi_2(\lambda,x) = -\frac{1}{\sqrt{-c}}e^{-i\xi x}\phi_2(\xi,x)e^{cS/2} = \lim_{x\to\infty} b(\xi)e^{cS/2}. \tag{9.4.38}$$

Hence, in this case we get $b(\xi)$ as $x \to \infty$, by which scattering states are constructed and one therefore has an unbound state.

Let us now consider the case $2\eta + c/2 > 0$ more closely. By the preceding analysis, a necessary condition for the existence of the limit is

$$a(\lambda(|A_n\rangle = \int dx_1...dx_n e^{-2i\lambda(x_1+...+x)n)}U_n(\infty,x_1...x_n)|x_1...x_n\rangle = 0. \tag{9.4.39}$$

This condition is the quantum mechanical analogue of the fact that in (9.4.36), we have to introduce an eigenvalue λ_j, i.e., a zero of $a(\lambda)$, whence $U_n(\infty,x_1...x_n) = 0$ gives an equation for D_n. When $x_1 < x_2.... < x_n$ solution for this is given by

$$D_n(x_1...x_n) = \exp(c\sum_{j=1}^{n} jx_j), \quad n = 1,2,..., \tag{9.4.40}$$

so that

$$A_n(x_1....x_n) = \exp(\sum_{j=1}^{n}(jc - 2i\lambda)x_j), \tag{9.4.41}$$

for $x_1 < x_2 < ...x_n$. This wave function is bounded provided $\eta = -\frac{1}{4}c(n+1)$. So from (9.4.26) we find

$$A_n(x_1....x_n) = \exp\left(-2i\xi(x_1 + .. + x_n) + \frac{c}{2}\sum_{1\le j\le k<n}|x_k - x_j|\right). \tag{9.4.42}$$

If we let $\xi = -p/2$, then (9.4.42) is the wave function of an n particle bound state with total momentum np, which we denote as $\omega_n^p(x_1....x_n)$:

$$|n,p\rangle = \int dx_1...dx_n\omega_n^p(x_1...x)n)|x_1....x_n\rangle$$

$$= \int dx_1....dx_n \exp\left(-2i\xi(x_1 + .. + x_n) + \frac{c}{2}\sum_{1\le j\le k<n}|x_k - x_j|\right)|x_1...x_n\rangle, \tag{9.4.43}$$

so that for $\lambda = \lambda_n = -p/2 - ic(n+1)/4$ and $|A\rangle = |n,p\rangle, n = 1,2...$ the limit $\Phi_2(\lambda,x)|A\rangle$ as $x \to \infty$ exists. To verify this one can calculate $\Phi_2(\lambda,x)|n,p\rangle$ at $\lambda = \lambda_n$ and again compute D_n to get

$$\lim_{x\to\infty}\Phi_2(\lambda_n,x)|n,p\rangle = |n+1,p\rangle, \tag{9.4.44}$$

where $a(\lambda)|n,p\rangle = 0$. The λ_n's correspond to the string states of the algebraic Bethe ansatz, deduced by Faddeev and Sklyanin in [177, 178]. A similar analysis can be carried out for all the energy eigenstates of the theory, for which we refer the reader to the original literature.

9.5 Quantum GLM Equation in a Continuous System

The quantum inverse scattering method is often associated with problems of ultraviolet and infrared divergences, thereby making its application difficult, in the case of most continuous systems, except for the exceptional case of the NLS problem. In fact it is almost tailored for discrete integrable systems. For the NLS problem Thacker et al. [22, 36]

developed an operator analogue of the classical treatment, which is more likely to appeal to most readers.

In this section we focus on this particular problem, bearing in mind the fact that the following analysis is applicable only to the particular model under discussion. In this context we remind the reader that the first attempt at quantization of a continuous integrable system was by Faddeev et al., which we have described in Chapter 4. There one started with the operator analogue of squared eigenfunctions. The Hamiltonian for the NLS problem is given by

$$H = \int dx\{\partial_1\phi^\star\partial_1\phi + c\phi^\star\phi^\star\phi\phi\}, \tag{9.5.1}$$

with the fields obeying the fundamental commutation relations:

$$[\phi(x,t), \phi^\star(y,t)] = \delta(x-y). \tag{9.5.2}$$

The Zakharov-Shabat linear problem for this system is expressed in the form,

$$\begin{aligned}\left(i\frac{\partial}{\partial x} + \frac{1}{2}\xi\right)\psi_1 &= -\sqrt{c}\psi_2\phi,\\ \left(i\frac{\partial}{\partial x} - \frac{1}{2}\xi\right)\psi_2 &= -\sqrt{c}\phi^\star\psi_1.\end{aligned} \tag{9.5.3}$$

The Jost solutions of these equations, namely $\psi(x,\xi)$ and $\chi(x,\xi)$, are defined by

$$\begin{aligned}\psi(x,\xi) &\longrightarrow \begin{pmatrix}1\\0\end{pmatrix} e^{i\xi x/2} \text{ as } x \to -\infty,\\ \chi(x,\xi) &\longrightarrow \begin{pmatrix}0\\1\end{pmatrix} e^{-i\xi x/2} \text{ as } x \to +\infty.\end{aligned} \tag{9.5.4}$$

We shall also require the conjugate solutions of the system (9.5.3), which are given by

$$\tilde{\psi} = \begin{pmatrix}\psi_2^\star\\ \psi_1^\star\end{pmatrix} \text{ and } \tilde{\chi} = \begin{pmatrix}\chi_2^\star\\ \chi_1^\star\end{pmatrix}. \tag{9.5.5}$$

Analyticity of an matrix operator means the analyticity of *all* its matrix elements. The scattering data $a(\xi)$ and $b(\xi)$ are defined by the asymptotic behaviour,

$$\psi(x,\xi) \longrightarrow \begin{pmatrix}a(\xi)e^{i\xi x/2}\\ b(\xi)e^{-i\xi x/2}\end{pmatrix} \text{ as } x \to \infty. \tag{9.5.6}$$

The commutation rules for the Jost functions are as follows:

$$[\psi_1, \psi^\star] = \frac{i\sqrt{c}}{2}\psi_2, \quad [\psi_2, \psi] = \frac{i\sqrt{c}}{2}\psi,$$

$$[\chi_1, \phi^\star] = -\frac{i\sqrt{c}}{2}\chi_2, \quad [\chi_2, \phi] = -\frac{i\sqrt{c}}{2}\chi_1. \tag{9.5.7}$$

In addition ψ and χ commute. One can prove that the Wronskian of any two solutions that are independent is a constant so that

$$a(\xi) = \psi_1\chi_2 - \psi_2\chi_1, \quad b(\xi) = \psi_2\tilde{\chi}_1 - \psi_1\tilde{\chi}_2. \tag{9.5.8}$$

These can be used to deduce the following asymptotic behaviour of χ:

$$\chi(x,\xi) \sim \begin{pmatrix} -b^\star(\xi)e^{i\xi x/2} \\ a(\xi)e^{-i\xi x/2} \end{pmatrix} \text{ as } x \to -\infty. \tag{9.5.9}$$

Using the results of Chapter 4 one can deduce that

$$[H, a(\xi)] = 0, \tag{9.5.10}$$

$$[H, R^\star(\xi)] = \xi^2 R^\star(\xi), \tag{9.5.11}$$

$$[a(\xi), a(\xi')] = [a^\star(\xi), a^\star(\xi')] = 0, \tag{9.5.12}$$

$$a(\xi)R^\star(\xi') = \left(1 - \frac{ic}{\xi - \xi' - i\epsilon}\right) R^\star(\xi')a(\xi), \tag{9.5.13}$$

$$a^\star(\xi)R^\star(\xi') = \left(1 + \frac{ic}{\xi - \xi' + i\epsilon}\right) R^\star(\xi')a^\star(\xi), \tag{9.5.14}$$

$$R^\star(\xi)R^\star(\xi') = S(\xi', \xi)R^\star(\xi')R^\star(\xi), \tag{9.5.15}$$

with

$$R^\star(\xi) = \frac{1}{\sqrt{c}}b(\xi)a^{-1}(\xi), \quad S(\xi', \xi) = \frac{\xi - \xi' - ic}{\xi + \xi' + ic}. \tag{9.5.16}$$

Furthermore, from the equations satisfied by the ordered product of the eigenfunctions, one can show that [178]

$$R(\xi)R^\star(\xi') = S(\xi, \xi')R^\star(\xi')R(\xi) + 2\pi\delta(\xi - \xi'). \tag{9.5.17}$$

Then by the standard approach of the Bethe ansatz, the excited states may be defined by

$$|k_1, ...k_n\rangle = R^\star(k_1).....R^\star(k_n)|0\rangle \tag{9.5.18}$$

and one can prove that

$$H|k_1....k_n\rangle = \left(\sum_i k_i^2\right)|k_1, ...k_n\rangle, \tag{9.5.19}$$

$$a(\xi)|k_1, ...k_n\rangle = \prod_{i=1}\left(1 - \frac{ic}{\xi - k_i - i\epsilon}\right)|k_1.....k_n\rangle, \tag{9.5.20}$$

$$a^\star(\xi)|k_1, ...k_n\rangle = \prod_{i=1}\left(1 + \frac{ic}{\xi - k_i + i\epsilon}\right)|k_1.....k_n\rangle. \tag{9.5.21}$$

9.5.1 Quantum GLM equation

The original derivation of the classical GLM equation [176] begins with the relation,

$$\psi = a\tilde{\chi} + b\chi, \tag{9.5.22}$$

which may be recast as

$$\psi a^{-1} = \tilde{\chi} - i\sqrt{c}R^\star\chi, \tag{9.5.23}$$

suggesting the existence of a piecewise analytic function,

$$\phi(x,\xi) = \begin{cases} \tilde{\chi}e^{-i\xi x/2} & \text{Im }(\xi) > 0 \\ \psi a^{-1}e^{-i\xi x/2} & \text{Im }(\xi) < 0 \end{cases}. \tag{9.5.24}$$

From the above equation, the discontinuity of ϕ across the real axis is seen to be $i\sqrt{c}R^\star\chi$, while $|\epsilon| \to \infty, \phi$ has the behaviour,

$$\phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(1/\epsilon). \tag{9.5.25}$$

Consequently it is possible to write a dispersion relation for ϕ in the form,

$$\tilde{\chi}e^{-i\xi x/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty} d\xi' \frac{R^\star(\xi')\chi(x,\xi')e^{-i\xi' x/2}}{\xi' - \xi - i\epsilon}. \tag{9.5.26}$$

This equation enables the Jost function $\chi(x,\xi)$ to be determined in terms of the reflection coefficient $R(\xi)$, from which ϕ can be determined through the asymptotic expansion:

$$\chi_1(x, xi)e^{i\xi x/2} \sim -\frac{\sqrt{c}\phi(x)}{\xi} \text{ as } |\epsilon| \to \infty. \tag{9.5.27}$$

In the quantum case (9.5.23) is no longer valid, but we can define the function,

$$g(x,\xi) = \tilde{\chi}(x,\xi) - i\sqrt{c}R^\star(\xi)\chi(x,\xi). \tag{9.5.28}$$

Since $\tilde{\chi}$ and χ satisfy the Zakhrov-Shabat equation, we observe that $g(x,\xi)$ satisfies the system,

$$\left(i\frac{\partial}{\partial x} + \frac{1}{2}\xi\right) g_1 = -\sqrt{c}g_2\phi,$$

$$\left(i\frac{\partial}{\partial x} - \frac{1}{2}\xi\right) g_2 = \sqrt{c}\phi^\star g_1 - ic[R^\star(\xi), \phi^\star(x)]\chi_1. \tag{9.5.29}$$

The second term on the right-hand side of (9.5.29) arises due to quantum orderings. Using (9.5.9) and (9.5.3), one can evaluate the commutator to get

$$[R^\star(\xi), \phi^\star] = (\tilde{\chi}_2 - i\sqrt{c}R^\star\chi_2)\psi_2 a^{-1}, \tag{9.5.30}$$

so that the system of equations (9.5.29) become

$$\left(i\frac{\partial}{\partial x} + \frac{1}{2}\xi\right) g_1 = -\sqrt{c}g_2\phi, \tag{9.5.31}$$

$$\left(i\frac{\partial}{\partial x} - \frac{1}{2}\xi\right) g_2 = \sqrt{c}\phi^\star g_1 - icg_2\psi_2 a^{-1}, \tag{9.5.32}$$

where g now has the asymptotic behaviour,

$$g(x,\xi) \sim e^{i\xi x/2}\begin{pmatrix}\tilde{a}(\xi)\\ 0\end{pmatrix} \quad \text{as } x \to \infty, \tag{9.5.33}$$

with

$$\tilde{a}(\xi) = a^\star(\xi) - cR^\star(\xi)a^\star(\xi)R(\xi) \tag{9.5.34}$$

for real ξ. Classically one could use the relation $|a(\xi)|^2 - |b(\xi)|^2 = 1$ to conclude that $\tilde{a}(\xi) = a^{-1}(\xi)$, as if g were equal to ψa^{-1}. To get an idea of how to proceed in the quantum case, we use (9.5.34) and (9.5.9) to evaluate $\tilde{a}(\xi)$ on a single particle state to obtain

$$\tilde{a}(\xi)|k\rangle = \left(1 + \frac{ic}{\xi - k + i\epsilon} - 2\pi c\delta(\xi - k)\right)|k\rangle = \left(1 + \frac{ic}{\xi - k + i\epsilon}\right)|k\rangle, \tag{9.5.35}$$

which shows that to first order in c, this is the same as $a^{-1}(\xi)|k\rangle$, since the latter equals $\left(1 - \frac{ic}{\xi - k + i\epsilon}\right)^{-1}|k\rangle$.

On the other hand, for many particle states,

$$\tilde{a}(\xi)|k_1...k_n\rangle = \prod_{i=1}^{n}\left(1+\frac{ic}{\xi-k_i-i\epsilon}\right)|k_1...k_n\rangle, \tag{9.5.36}$$

since the states $|k_1..k_n\rangle$ form a complete set, we may use (9.5.35) as an alternative definition of $\tilde{a}(\xi)$ and conclude that $\tilde{a}(\xi)$ is analytic in the lower half ξ-plane:

$$\tilde{a}(\xi) = 1 + \mathcal{O}(1/\epsilon) \quad \text{as } |\xi| \to \infty. \tag{9.5.37}$$

We now make the crucial observation that both the differential equations (9.5.31) for g and its asymptotic behaviour in (9.5.33) may be continued into the lower half plane of ξ without singularities. So the operator $g(x,\xi)$ itself may be continued in the lower half plane and is analytic there. We are now in a position to make a derivation that resembles the classical one. Defining the function ϕ by

$$\phi = \begin{cases} \tilde{\chi}e^{-i\xi x/2}, & \text{Im }(\xi) > 0, \\ ge^{-i\xi x/2} & \text{Im }(\xi) < 0, \end{cases} \tag{9.5.38}$$

by construction we have ϕ to be piecewise analytic, with the discontinuity across the real axis given by $i\sqrt{c}R^\star(\xi)\chi(x,\xi)e^{-i\xi x/2}$. In addition, $|\epsilon| \to \infty$ has the property $\phi \sim \begin{pmatrix}1\\0\end{pmatrix} + \mathcal{O}(1/\epsilon)$, which holds in the lower half plane due to (9.5.38). Hence one can write the dispersion relation for ϕ as

$$\tilde{\chi}e^{-i\xi x/2} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty} d\xi' \frac{R^\star(\xi')\chi(x,\xi')e^{-i\xi' x/2}}{\xi'-\xi-i\epsilon}. \tag{9.5.39}$$

By iterating this equation and its Hermitian conjugate and using the asymptotic behaviour, we get

$$\phi(x) = \int \frac{d\xi_1}{2\pi} R(\xi_1)e^{i\xi_1 x} +$$

$$+c\int\int\int \frac{d\xi_1 d\xi_2 d\xi_3}{(2\pi)^3}\frac{R^\star(\xi_2)R(\xi_1)R(\xi_3)e^{i(\xi_1-\xi_2+\xi_3)x}}{(\xi_2-\xi_1-i\epsilon)(\xi_3-\xi_2+i\epsilon)} + ... \tag{9.5.40}$$

The above derivation has been done for a fixed time, say $t = 0$. But as in the classical case, one can prove that

$$R(\xi) \longrightarrow R(\xi,t) = e^{-i\xi^2 t}R(\xi). \tag{9.5.41}$$

So that using this, one can retrace the previous computation to get the results at any arbitrary instant of time.

The basic difference between the classical and quantum cases arises from the use of series-like expansions used for the integral equation (9.5.39). But in many two-dimensional problems, such expansions may be summed up. The above discussion closely preserves the analogy with the corresponding classical approach, whereas the algebraic Bethe ansatz is totally silent about these properties. The formalism presented here is, however, more intuitive, even though it is less rigorous. However, it enables one to get the physical essence of the problem, keeping the classical results in sight.

An immediate outcome of (9.5.40) is the computation of various Green's function for the Heisenberg field $\phi(x,t)$. These are written as

$$\langle 0|T\{\phi(x_1',t_1')...\phi(x_n',t_n')\phi^\star(x_1,t_1)....\phi^\star(x_n,t_n)|0\rangle,$$

where T denotes the usual time ordering. Here we interpret $R(\xi)$ as the annihilation operator and $R^\star(\xi)$ as a creation operator. Next using the basic commutation rules of $R(\xi)$ and $R^\star(\xi)$, we can bring $R(\xi)$ to the extreme right to operate on $|0\rangle$, so that it vanishes and hence all the calculations can be done in a finite number of steps.

Finally we should mention that for the discrete Sine-Gordon equation, a very rigorous derivation of the quantum GLM equation was made by Smirnov [172]. This requires complicated analyticity considerations of all the operators involved. We refer the interested reader to the original articles [172].

9.6 Bound States and an Alternative Approach

The technique of deriving the GLM equation by constructing string states was first employed by Göckeller and Nakawaki. They gave an explicit construction of the Bethe ansatz states in terms of the reflection operator R. The starting point of their analysis was the nonlinear Schrödinger problem expressed as a Hamiltonian system:

$$i\frac{\partial\phi}{\partial t} = [H,\phi], \tag{9.6.1}$$

$$\text{with} \qquad H = \int dx(\partial_x\phi\partial_x\phi^\star + c\phi^\star\phi^\star\phi\phi), \tag{9.6.2}$$

along with the properly ordered Zakharov-Shabat eigenvalue problem:

$$-i\partial_x V(x,\lambda) =: Q(x,\lambda)V(x,\lambda) :, \tag{9.6.3}$$

$$\text{with} \qquad Q(x,\lambda) = \begin{pmatrix} \frac{\lambda}{2} & \sqrt{c}\phi \\ -\sqrt{c}\phi & -\frac{\lambda}{2} \end{pmatrix}. \tag{9.6.4}$$

This notation was explained in Section 2 of Chapter 4. The eigenfunctions ψ and χ are analytic in the lower half plane, while their conjugate functions $\bar{\phi}, \bar{\psi}$ are defined by

$$\bar{\psi}_1(x,\lambda) = -(\psi_2(x,\lambda^\star))^\star, \quad \bar{\psi}_2(x,\lambda) = (\psi_1(x,\lambda^\star))^\star,$$

$$\bar{\chi}_1(x,\lambda) = (\chi_2(x,\lambda^\star))^\star, \quad \bar{\chi}_2(x,\lambda) = -(\chi_1(x,\lambda^\star))^\star, \tag{9.6.5}$$

are analytic in the upper half plane. The scattering data are defined as usual by the relations:

$$\lim_{x\to\infty} e^{-i\lambda x/2}\psi_1(x,\lambda) = A_1(\lambda), \tag{9.6.6}$$

$$\lim_{x\to\infty} e^{i\lambda x/2}\psi_2(x,\lambda) = -B_1(\lambda). \tag{9.6.7}$$

A simple and direct calculation shows that

$$[H, A_1(\lambda)] = 0, \quad [H, B_1(\lambda)] = \lambda^2 B_1(\lambda), \tag{9.6.8}$$

which implies that $A_1(\lambda)$ is a conserved quantity and the excitation states can be constructed by repeated application of $B_1(\lambda)$ to the vacuum. By means of the algebraic Bethe ansatz, one can derive a set of coupled equation, for the eigenmomenta with such excitation, which are referred to as the string states. The operators for such states were constructed by Göckeller. Such n particle bound states, the so-called string states, are written as

$$B_1(\zeta_1(\lambda))B_1(\zeta_2(\lambda)).....B_1(\zeta_n(\lambda))|0\rangle, \tag{9.6.9}$$

$$\text{with} \qquad \zeta_i(\lambda) = \lambda + \frac{ic}{2}(n+1-2j) \quad \text{and } j = 1,2,.....,n. \tag{9.6.10}$$

The corresponding eigenvalue of H is

$$E_n(\lambda) = n\lambda^2 - c^2 n(n^2-1)/12. \tag{9.6.11}$$

In order to construct the scattering data operators for these states, the following string products of Jost functions were first constructed [175]:

$$\psi(n,r|x,\lambda)=\prod_{j=1}^{r}\psi_2(x,\zeta_j(\lambda))\prod_{j=1}^{n}\psi_1(x,\zeta_j(\lambda)), \tag{9.6.12}$$

$$\bar{\psi}(n,r|x,\lambda)=\prod_{j=1}^{r}\bar{\psi}_2(x,\zeta_j(\lambda))\prod_{j=1}^{n}\bar{\psi}_1(x,\zeta_j(\lambda)), \tag{9.6.13}$$

$$\chi(n,r|x,\lambda)=\prod_{j=1}^{r}\chi_2(x,\eta_j(\lambda))\prod_{j=1}^{n}\chi_1(x,\eta_j(\lambda)), \tag{9.6.14}$$

$$\bar{\chi}(n,r|x,\lambda)=\prod_{j=1}^{r}\bar{\chi}_2(x,\eta_j(\lambda))\prod_{j=1}^{n}\bar{\chi}_1(x,\eta_j(\lambda)), \tag{9.6.15}$$

where $r=0,1,2,...,n$ and

$$\eta_i(\lambda)=\lambda-\frac{ic}{2}(n+1-2j)\quad\text{with } j=1,2,...,n. \tag{9.6.16}$$

The functions $\psi(n,r|x,\lambda)$ are solutions of the generalized Zakharov-Shabat equation, i.e.,

$$-i\partial_x\psi(n,r|x,\lambda)-\frac{1}{2}\{\lambda(n-2r)-icr(n-r)\}\psi(n,r|x,\lambda)=$$

$$i\sqrt{-c}\{r\phi^\star(x)\psi(n,r-1|x,\lambda)-(n-r)\psi(n,r+1|x,\lambda)\phi\}, \tag{9.6.17}$$

and satisfy boundary conditions:

$$\lim_{x\to-\infty}e^{-in\lambda x/2}\psi(n,r|x,\lambda)=\delta_{r,0}, \tag{9.6.18}$$

$$\lim_{x\to-\infty}e^{in\lambda x/2}\bar{\psi}(n,r|x,\lambda)=\delta_{r,n}. \tag{9.6.19}$$

On the other hand, $\chi(n,r|x,\lambda)$ and $\bar{\chi}(n,r|x,\lambda)$ are the Jost solutions of the conjugate equation:

$$-i\partial_x\chi(n,r|x,\lambda)-\frac{1}{2}\{\lambda(n-2r)+icr(n-r)\}\chi(n,r|x,\lambda)=$$

$$i\sqrt{-c}\{r\phi^\star(x)\chi(n,r-1|x,\lambda)-(n-r)\chi(n,r+1|x,\lambda)\phi(x)\}, \tag{9.6.20}$$

obeying the boundary conditions:

$$\lim_{x\to-\infty}e^{-im\lambda x/2}\psi(n,r|x,\lambda)=\delta_{r,0}, \tag{9.6.21}$$

$$\lim_{x\to-\infty} e^{im\lambda x/2}\bar{\psi}(n,r|x,\lambda) = \delta_{r,n}, \tag{9.6.22}$$

with similar equations for $\chi(n,r|x,\lambda)$ and $\bar{\chi}(n,r|x,\lambda)$. These boundary conditions imply that $\psi(n,r|x,\lambda)$ and $\chi(n,r|x,\lambda)$ are analytic in the lower half λ-plane and $\bar{\psi}(n,r|x,\lambda)$ and $\bar{\chi}(n,r|x,\lambda)$ are analytic in the upper half λ-plane. We now define the Wronskians of this new type of Jost solutions as follows:

$$A_n(\lambda) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} \psi(n,r|x,\lambda)\chi(n,n-r|x,\lambda),$$

$$A_n^\star(\lambda) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} \bar{\psi}(n,r|x,\lambda)\bar{\chi}(n,r|x,\lambda),$$

$$B_n(\lambda) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} \psi(n,r|x,\lambda)\bar{\chi}(n,n-r|x,\lambda),$$

$$B_n^\star(\lambda) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} \bar{\psi}(n,r|x,\lambda)\chi(n,n-r|x,\lambda), \tag{9.6.23}$$

which we call scattering data operators. They satisfy the commutation rules:

$$[A_m(\lambda), A_n(\mu)] = [A_m(\lambda), A_n^\star(\mu)] = [B_m(\lambda), B_n(\mu)] = 0,$$

$$A_m(\lambda)B_n(\mu) = \Lambda_{mn}^{-1}(\lambda-\mu-i\epsilon)B_n(\mu)A_m(\lambda),$$

$$A_m(\lambda)B_n^\star(\mu) = \Lambda_{mn}(\lambda-\mu-i\epsilon)B_n^\star(\mu)A_m(\lambda),$$

$$B_m^\star(\lambda)B_n(\mu) = \delta_{mn}2\pi n(-c)\delta(\lambda-\mu)A_n(\lambda)A_n^\star(\lambda) + \sigma B_n(\mu)B_m^\star(\lambda), \tag{9.6.24}$$

$$\text{where} \qquad \Lambda_{mn}(\lambda) = \prod_{j=1}^{m} \frac{\lambda + \frac{ic}{2}(m+n-2j)}{\lambda + \frac{ic}{2}(m-n-2j)}, \tag{9.6.25}$$

$$\text{and} \qquad \sigma = \frac{(\lambda-\mu)^2 + c^2(m+n)^2/4}{(\lambda-\mu-i\epsilon)^2 + c^2(m-n)^2/4}. \tag{9.6.26}$$

These results are traditionally obtained by following the method developed by Sklyanin [178]. From (9.6.24) and the commutation rules for $m = 1$ we obtain

$$[N, B_n(\lambda)] = nB_n(\lambda),$$

$$[P, B_n(\lambda)] = n\lambda B_n(\lambda),$$

$$[H, B_n(\lambda)] = E_n(\lambda)B_n(\lambda). \tag{9.6.27}$$

To formulate the inverse quantum inverse scattering method, we consider three operator identities:

$$A_m^{-1}(\lambda + ic) = A_m^\star(\lambda) - (-1)^m R_m^\star(\lambda)B_m^\star(\lambda), \tag{9.6.28}$$

where $R_m^\star$ is the reflection operator:

$$R_m^\star(\lambda) = B_m(\lambda)A_m^{-1}(\lambda). \tag{9.6.29}$$

To derive the above equation we observe that

$$\begin{aligned} A_m^{-1}(\lambda)B_n(\mu) &= \Lambda_{mn}(\lambda - \mu - i\epsilon)B_n(\mu)A_m^{-1}(\lambda), \\ R_m^\star(\lambda)B_n(\mu) &= \Lambda_{mn}(\lambda - \mu - i\epsilon)B_n(\mu)R_m^\star(\lambda), \\ A_m^\star(\lambda)B_n(\mu) &= \Lambda_{mn}(\lambda + ic - \mu - i\epsilon)B_n(\mu)A_m^\star(\lambda). \end{aligned} \tag{9.6.30}$$

Then (9.6.27–9.6.29) give the following commutation relation of the right-hand side of (9.6.28) with $B_n(\mu)$:

$$\begin{aligned} \{A_m^\star(\lambda) - (-1)^m R_m^\star(\lambda)B_m^\star(\lambda)\}B_n(\mu) = \Lambda_{mn}(\lambda + ic - \mu - i\epsilon)B_n(\mu)\times \\ \times\{A_m^\star(\lambda) - (-1)^m R_m^\star(\lambda)B_m^\star(\lambda)\}, \end{aligned} \tag{9.6.31}$$

where we have used

$$\Lambda_{mn}(\lambda + ic - \mu + i\epsilon) - \Lambda_{mn}(\lambda + ic - \mu - i\epsilon) = \delta_{mn}(-1)^m 2\pi i m\delta(\lambda - \mu).$$

Comparing (9.6.30) with (9.6.31) we see that $A_m^{-1}(\lambda + ic)$ and $A_m^\star(\lambda) - (-1)^m R_m^\star(\lambda)B_m^\star(\lambda)$, satisfy the same commutation rule with $B_n(\mu)$. It follows that the difference of both operators has to be zero because it annihilates the vacuum. Hence the proof of the equation (9.6.28).

The second set are those among the Jost solutions:

$$(-1)^r \binom{m}{r} \bar\psi(m, m - r|x, \lambda)\psi(m, s|\lambda, \lambda + ic) = \delta_{rs}e^{-cmx/2}, \tag{9.6.32}$$

$$\sum_{l=0}^{m}(-1)^l \binom{m}{l} \psi(m, l|x, \lambda + ic)\bar\psi(m, m - l|x, \lambda) = e^{-cmx/2}. \tag{9.6.33}$$

These are obtained from the following identities (obtained by Göckeller):

$$(-1)^r \binom{m}{r} \bar\psi(m, m - r|x, \lambda)\psi(m, s|x, \lambda + ic) =$$

$$\delta_{rs}\sum_{l=0}^{m}(-1)^l\begin{pmatrix}m\\l\end{pmatrix}\psi(m,l|x,\lambda+ic)\bar{\psi}(m,m-1|x,\lambda),$$

$$(-1)^s\begin{pmatrix}m\\s\end{pmatrix}\chi(m,m-r|x,\lambda+ic)\bar{\chi}(m,s|x,\lambda)$$

$$=\delta_{rs}\sum_{l=0}^{m}(-1)^l\begin{pmatrix}m\\l\end{pmatrix}\bar{\chi}(m,l|x,\lambda)\chi(m,m-l|x,\lambda+ic). \tag{9.6.34}$$

We have pointed out in the introduction that $\psi(m,r|x,\lambda+ic)$ can be expressed in terms of $\bar{\chi}(m,r|x,\lambda)$ and $\chi(m,r|x,\lambda)$. If we use the right-hand side of (9.6.28) and the Wronskians (9.6.23) as the expressions of $A_m^{-1}(\lambda+ic)$, $A_m^{\star}(\lambda)$ and $B_m^{\star}(\lambda)$, we have

$$e^{mx/2}A_m^{-1}(\lambda+ic)\psi(m,r|x,\lambda+ic)=\sum_{s=0}^{m}\delta_{rs}\{\bar{\chi}(m,s|x,\lambda)-$$

$$R_m^{\star}(\lambda)\chi(m,s|x,\lambda)\}=\bar{\chi}(m,r|x,\lambda)-R_m^{\star}(\lambda)\chi(m,r|x,\lambda), \tag{9.6.35}$$

and

$$e^{cmx/2}A_m^{-1}(\lambda+ic)\psi(m,r|x,\lambda+ic)=\bar{\chi}(m,r|x,\lambda)-R_m^{\star}\chi(m,r|x,\lambda)$$

$$=G(m,r|x,\lambda). \tag{9.6.36}$$

From this operator identity, the GLM equation for $\chi(m,r|x,\lambda)$ can be set up. The analytic properties of $G(m,r|x,\lambda)$ are known from those of $A_m(\lambda+ic)$ and $\psi(m,r|x,\lambda+ic)$ in the lower half λ-plane. In the sequel we shall show how the discontinuites of $G(m,r|x,\lambda)$ can be expressed in a manner leading to the closed GLM set.

The singularities of $G(m,r|x,\lambda)$ in the lower half of the λ-plane are actually the poles of $A_m^{-1}(\lambda+ic)$, which may be obtained from the zeros of $a(\lambda)$. The eigenstates of $A_m^{-1}(\lambda+ic)$ are constructed by applying the $B_n(\mu)$'s to $|0\rangle$. The eigenvalue Λ_{mn} in (9.6.25) consists of products of m functions, so that it has m simple poles in general, giving rise to discontinuites of the form,

$$\Lambda_{mn}(\lambda+\frac{ic}{2}(n-m+2j)-\mu+i\epsilon)-\Lambda_{mn}(\lambda+\frac{ic}{2}(n-m+2j)-\mu-i\epsilon)$$

$$=2\pi c\delta(\lambda-\mu)(-1)^{m-j}j\begin{pmatrix}m\\j\end{pmatrix}\begin{pmatrix}n+j-1\\m\end{pmatrix}, \tag{9.6.37}$$

across

$$\text{Im }(\lambda) = \frac{c}{2}(n - m + 2j - 2); \quad j = 1, ..., m.$$

Now $A_m(\lambda + ic) = \prod_{j=1}^{m} A_1(\lambda_j(\lambda + ic))$ enables us to see that $A_1^{-1}(\lambda_j(\lambda + ic))$ yields the jth discontinuity. Hence following the procedure of [175], we can realize these discontinuites in the form of operators, that is

$$A_m^{-1}(\lambda + \frac{ic}{2}(N+1) + i\epsilon) - A_m^{-1}(\lambda + \frac{ic}{2}(N+1) - i\epsilon)$$

$$= \sum_{j=1}^{m}(-1)^{m+1-j}\begin{pmatrix} m \\ j-1 \end{pmatrix}\begin{pmatrix} n_j - 1 \\ m - j \end{pmatrix} R^{\star}_{n_j}(\lambda) R_{N-j}(\lambda - \frac{ic}{2}(m+1-j))\times$$

$$\times A^{\star}_{j-1}(\lambda + \frac{ic}{2}(N+m-j)) B^{\star}_{m+1-j}(\lambda + \frac{ic}{2}(N-j)), \qquad (9.6.38)$$

with $N = 1, 2, ...,$ and $n_j = N + m + 1 - 2j$. Hence substituting this in (9.6.36) yields the discontinuity of $G(m, r|x, \lambda)$ across Im $(\lambda) = \frac{c}{2}(N-1)$.

The next task is to express the discontinuities in terms of the χ's. One observes that the product,

$$A^{\star}_{j-1}(\lambda + \frac{ic}{2}(m+1-j)) B^{\star}_{m+1-j}(\lambda - \frac{ic}{2}(j-1)),$$

is actually a string product of $(j-1)A_1^{\star}$'s and $(m+1-j)$, $B_l\star$'s, so that if we substitute (9.6.23) into it then, $\psi(m, l|x, \lambda)$ may be built as follows:

$$\begin{pmatrix} m \\ j-1 \end{pmatrix} A^{\star}_{j-1}(\lambda + \frac{ic}{2}(m+1-j)) B^{\star}_{m+1-j}(\lambda - \frac{ic}{2}(j-1))$$

$$= (-1)^{j-1}\sum_{j=0}^{m}(-1)^l \begin{pmatrix} m \\ l \end{pmatrix} \bar{\psi}(m, l|x, \lambda)\chi_{m+1-j,j-1}(m, m-l|x, \lambda), \qquad (9.6.39)$$

where

$$\chi_{m+1-j,j-1}(m, m-l|x, \lambda) = \frac{\begin{pmatrix} m \\ j-1 \end{pmatrix}}{\begin{pmatrix} m \\ l \end{pmatrix}} \sum_{s=0}^{l}\begin{pmatrix} j-1 \\ l-s \end{pmatrix}\begin{pmatrix} m+1-j \\ s \end{pmatrix} \times$$

$$\times \bar{\chi}(j-1, j-1+s-l|x, \lambda + \frac{ic}{2}(m+1-j)) \times$$

$$\times\chi(m+1-j,m+1-j-s|x,\lambda-\frac{ic}{2}(j-1)). \tag{9.6.40}$$

By using commutations of χ_i and $\bar{\chi}_i$, we can formally write the above as

$$\chi_{m+1-j,j-1}(m,m-l|x,\lambda)$$

$$= \begin{cases} \chi(m+1-j,m+1-j|x,\lambda-\frac{ic}{2}(j-1))\times \\ \bar{\chi}(j-1,j-1-l|x,\lambda+\frac{ic}{2}(m+1-j)) \\ \text{for } 0\le l\le j-1, \\ \chi(m+1-j,m-l|x,\lambda-\frac{ic}{2}(j-1))\bar{\chi}(j-1,0|x,\lambda+\frac{ic}{c}(m+1-j)) \\ \text{for } j-1<l, \end{cases} \tag{9.6.41}$$

so that

$$\begin{pmatrix} m \\ j-1 \end{pmatrix} A^{\star}_{j-1}(\lambda+\frac{ic}{2}(m+1-j))B^{\star}_{m+1-j}(\lambda-\frac{ic}{2}(j-1))\psi(m,r|x,\lambda+ic)$$

$$= (-1)^{m+1-j}e^{-cmx/2}\chi_{m+1-j,j-i}(m,r|x,\lambda), \tag{9.6.42}$$

whence from (9.6.38) and (9.6.42) one obtains

$$e^{mcx/2}\{A^{-1}_m(\lambda+\frac{ic}{2}(N+1)+i\epsilon)-A^{-1}_m(\lambda+\frac{ic}{2}(N+1)-i\epsilon)\}\psi(m,r|x,\lambda+ic)$$

$$= \sum_{j=1}^{m}\begin{pmatrix} n_j-1 \\ m-j \end{pmatrix} R^{\star}_{n_j}(\lambda)R_{N-j}(\lambda-\frac{ic}{2}(m+1-j))$$

$$\chi_{m+1-j,j-1}(m,r|x,\lambda+\frac{ic}{2}(N-1)).$$

We now proceed to obtain the quantum GLM equation for $\chi(m,r|x,\lambda)$. We define a piecewise analytic function,

$$\Phi(x,\lambda) = \begin{cases} \bar{\chi}(m,r|x,\lambda)e^{-im\lambda x/2}, & \text{Im } \lambda>0, \\ G(m,r|x,\lambda)e^{-im\lambda x/2}, & \text{Im } \lambda<0 \end{cases}, \tag{9.6.43}$$

where $\text{Im}\lambda\neq\frac{cN}{2}, N=1,2,.....$ and we may write the dispersion relation for $\Phi(x,\lambda)$ using the above equation in the form,

$$\chi(m,r|x,\lambda)e^{im\lambda x/2} = \delta_{rm}+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\sum_{l=1}^{\infty}\begin{pmatrix} m+n+1-2l \\ m-l \end{pmatrix}\times$$

$$\int dq\frac{\exp[\frac{im}{2}(q-\frac{ic\eta}{2})x]}{q-\frac{ic\eta}{2}-\lambda+i\epsilon}\chi_{l-1,m+1-l}(m,r|x,q-\frac{ic\eta}{2})\times$$

$$(-1)^{m-l}R^{\star}_{n+1-l}(q+\frac{ic}{2}(m+1-l))R_{m+n+2-2l}(q). \tag{9.6.44}$$

On a state with particle number N, only the terms with $n \leq N-1$ do not vanish. Iterating this and its Hermitian conjugate yields $\chi(m,r|x,\lambda)$ expressed in terms of the reflection coefficient operator.

We now take the limit $x \to -\infty$ in (9.6.44) with $r=m$ and note that

$$e^{im\lambda x/2}\chi(m,m|x,\lambda) \Rightarrow A_m(\lambda) \text{ as } x \to -\infty, \tag{9.6.45}$$

$$e^{im\lambda x/2}\bar{\chi}(m,m|x,\lambda) \Rightarrow B_m(\lambda) \text{ as } x \to -\infty. \tag{9.6.46}$$

Taking into account these features and (9.6.41), we get as $x \to -\infty$:

$$e^{imqx/2}\chi_{l-1,m+1-l}(m,m|x,q)$$

$$\to A_{l-1}(q-\frac{ic}{2}(m+1-l))B_{m+1-l}(q+\frac{ic}{2}(l-1)). \tag{9.6.47}$$

Then upon using the commutation rules one may obtain

$$A_m(\lambda) = 1+\frac{1}{2\pi i}\sum_{n=0}^{\infty}\sum_{l=1}^{m}\binom{m+n+1-2l}{m-l}\binom{m+n+1-l}{l-1}\times$$

$$\times\int dq\frac{(-1)^{m-l}}{q-\frac{ic\eta}{2}-\lambda+i\epsilon}R^{\star}_{m+n+2-2l}(q)A_m(q-\frac{ic\eta}{2})R_{m+n+2-2l}(q), \tag{9.6.48}$$

which is an integral equation for A_m and may be solved by iteration. By a set of complicated manipulations, which we omit here, one can deduce that

$$A_m(\lambda) = \exp\left[\sum_{n=1}^{\infty}\int dq \log\Lambda_{mn}(q-\lambda+2i\epsilon)r^{\star}_n(q)r_n(q)\right], \tag{9.6.49}$$

$$B_m(\lambda) = R^{\star}_m \exp\left[\sum_{n=1}^{\infty}\int dq \log\Lambda_{mn}(q-k+ic+i\epsilon)r_n(q)r_n(q)\right], \tag{9.6.50}$$

with $r_{N_j} = \frac{R_{N_j}(q)}{\sqrt{2\pi(-c)N_j}}$.

The above discussion indicates that the reflection operator is of prime importance in the quantum NLS problem. Although the above treatment is quite rigorous, yet a parallel treatment for other nonlinear systems is at present lacking. As such one may feel somewhat uneasy about the efficacy of the entire procedure of deducing the quantum GLM equation using the above methodology.

Appendix A

Direct Product Calculus

In the formulation of the algebraic Bethe ansatz, classical and quantum r matrices, exhaustive use has been made of the direct product algebra of matrices and vectors. Since these materials are essential for explicit derivation of results contained in the earlier chapters, we give below some important results related to these techniques.

The direct product better known as the Kronecker product of two matrices is defined as follows. Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. Then the Kronecker product of A and B is a $(mp) \times (nq)$ matrix defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & .... & a_{1n}B \\ a_{21}B & a_{22}B & .... & a_{2n}B \\ .. & .. & .... & .. \\ .. & .. & .... & .. \\ a_{m1}B & a_{m2}B & .... & a_{mn}B \end{pmatrix}. \tag{A.1}$$

This is also known as the tensor product. Some authors however use the following definition:

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{12}A & .... & b_{1q}A \\ b_{21}A & b_{22}A & .... & b_{2q}A \\ .. & .. & .... & .. \\ .. & .. & .... & .. \\ b_{p1}A & b_{p2}A & .... & b_{pq}A \end{pmatrix}. \tag{A.2}$$

The tensor product is in general noncommutative, i.e.,

$$A \otimes B \neq B \otimes A; \tag{A.3}$$

for example, if

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{then}$$

$$A \otimes B = \begin{pmatrix} 0 & -2 & 0 & -3 \\ -2 & 2 & -3 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad B \otimes A = \begin{pmatrix} 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -1 \\ -2 & -3 & 2 & 3 \\ 0 & -1 & 0 & 0. \end{pmatrix}$$

A similar definition also holds for vectors. For example, if

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{then}$$

$$U \otimes V = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \end{pmatrix}. \tag{A.4}$$

In general

$$U \otimes V \neq V \otimes U. \tag{A.5}$$

One should note that while $\{U, V\}$ is a standard set of vectors in $\mathcal{R}^2$, $U \otimes V$ is a similar one in $\mathcal{R}^4$. Let us take

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then we see that while $\{U, V\}$ is a basis in $\mathcal{R}^2$, one has $\{U \otimes U, U \otimes V, V \otimes V, V \otimes U\}$ as the corresponding basis in $\mathcal{R}^4$. We give below some basic properties of the Kronecker product.
Let A be an $m \times n$ matrix, B a $p \times q$ matrix and C an $s \times t$ matrix; then

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{A.6}$$

If α is a scalar quantity then we have

$$(\alpha A) \otimes B = \alpha(A \otimes B) = A \otimes (\alpha B), \tag{A.7}$$

and

$$(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D, \tag{A.8}$$

so that the Kronecker product has both an associative and distributive property. Lastly let $r(A)$ be the rank of the matrix A and $r(B)$ the rank of the matrix B, then

$$r(A \otimes B) = r(A).r(B). \tag{A.9}$$

Furthermore let A^T , $\overline{A}$ and $A^\star$ denote the transpose, the complex conjugate and the adjoint of the matrix A, respectively, then

$$(A \otimes B)^T = A^T \otimes B^T, \tag{A.10}$$

$$\overline{(A \otimes B)} = \overline{A} \otimes \overline{B}, \tag{A.11}$$

$$(A \otimes B)^\star = A^\star \otimes B^\star. \tag{A.12}$$

The following properties can be easily deduced from the definition:

(a) If A and B are diagonal, then $A \otimes B$ is also diagonal.

(b) If A and B are upper (lower) triangular, then $A \otimes B$ is also upper (lower) triangular.

(c) If A and B are Hermitian, then $A \otimes B$ is also Hermitian.

(d) If A is an invertible $n \times n$ matrix and B is an invertible $m \times m$ matrix, then $A \otimes B$ is also invertible.

We consider next the product rules involving the Kronecker product. If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then

$$(AB)_{kl} = \sum_{j=1}^{n} a_{kj} b_{jl},$$

so that for a matrix A of order $m \times n$, B of order $p \times q$, C of order $n \times r$ and D of order $q \times s$, one has

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{A.13}$$

As a special case we have

$$(A \otimes I_n)(I_n \otimes B) = (A \otimes B),$$

and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

Another important aspect of matrices is exhibited through their commutative properties. If A, B are $m \times m$ matrices and C, D are $n \times n$ matrices, and if

$$[A, B] = [C, D] = 0, \quad \text{then} \quad [A \otimes C, B \otimes D] = 0. \tag{A.14}$$

The proof of (A.14) is as follows:

$$[A \otimes C, B \otimes D] = (A \otimes C)(B \otimes D) - (B \otimes D)(A \otimes C)$$

$$= (AB) \otimes (CD) - (BA) \otimes (DC) = (AB) \otimes (CD) - (AB) \otimes (CD) = 0$$

which completes the proof. If I_n, I_m denote unit matrices of order $n \times n$ and $m \times m$, respectively, then

$$[A \otimes I_n, I_m \otimes B] = 0, \tag{A.15}$$

because

$$[A \otimes I_n, I_m \otimes B] = (A \otimes I_n)(I_m \otimes B) - (I_m \otimes B)(A \otimes I_n) = A \otimes B - A \otimes B = 0.$$

Furthermore, if A, B are $n \times n$ matrices and I_n denotes an $n \times n$ unit matrix, then

$$\exp\left(A \otimes I_n + I_n \otimes B\right) = \exp(A) \otimes \exp(B). \tag{A.16}$$

An important role in our analysis is played by the permutation matrix $\mathcal{P}$, which is defined as follows:

$$B \otimes A = \mathcal{P}(A \otimes B)\mathcal{P}. \tag{A.17}$$

For example, let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \text{then}$$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For a matrix A_1 of order $m \times m$ and a matrix A_2 of order $n \times n$ we have

$$\mathrm{tr}_{12}(A_1 \otimes A_2) = (\mathrm{tr}A_1)(\mathrm{tr}A_2); \tag{A.18}$$

tr_{12} means the operation of taking the trace is done over both the vector spaces involved in the tensor product. In addition we have

$$\mathrm{tr}\{\exp\left(A_1 \otimes I + I \otimes A_2\right)\} = \left(\mathrm{tr}\exp(A_1)\right)\left(\mathrm{tr}\exp(A_2)\right), \tag{A.19}$$

$$\text{and} \qquad \det(A \otimes B) = (\det A)^n (\det B)^m. \tag{A.20}$$

Note that

$$\mathrm{tr}_1(A_1 \otimes A_2) = \{\mathrm{tr}(A_1)\}.A_2, \tag{A.21}$$

when the matrix $A_1 \otimes A_2$ is defined in the tensor product of $\mathcal{C}^2 \otimes \mathcal{C}^2$, the operator tr_1 carries the matrix in $\mathcal{C}^2 \otimes \mathcal{C}^2$ to a matrix in $\mathcal{C}$. In the same way we also have

$$\mathrm{tr}_2(A_1 \otimes A_2) = A_1.\{\mathrm{tr}(A_2)\}. \tag{A.22}$$

Furthermore, if $X \to \mathcal{C}^2 \otimes \mathcal{C}^2$ and $A \to \mathcal{C}^2$, then

$$\mathrm{tr}_1(I \otimes A).X = A.\mathrm{tr}_1 X, \tag{A.23}$$

$$\mathrm{tr}_1 X.(I \otimes A) = \mathrm{tr}_1.A, \tag{A.24}$$

$$\mathrm{tr}_1(A \otimes I).X = \mathrm{tr}_1 X.(A \otimes I), \tag{A.25}$$

$$\mathrm{tr}_1 \mathcal{P} = I. \tag{A.25}$$

Moreover, we have the following definition for the Poisson bracket of the tensor product

$$\{A \otimes, B\}^{ij}_{kl} = \{A_{ij}, B_{kl}\}, \tag{A.26}$$

and

$$\mathrm{tr}\{A \otimes, B\} = \{\mathrm{tr}A, \mathrm{tr}B\}. \tag{A.27}$$

Next we shall discuss properties associated with eigenvalues. Let A be an $m \times m$ matrix with eigenvalues $\lambda_1, \lambda_2, ...\lambda_m$ and corresponding eigenvectors $U_1, U_2, ...U_m$; and suppose B is another $n \times n$ matrix with eigenvalues $\mu_1, \mu_2, ...\mu_n$ with eigenvectors $V_1, V_2, ...V_n$, then the matrix $A \otimes B$ has eigenvalues $\lambda_j \mu_k$ with eigenvectors $U_j \otimes V_k$, where $1 \leq j \leq m, 1 \leq k \leq n$. The proof is straightforward, since

$$AU_j = \lambda_j U_j, \quad \text{and} \quad BV_k = \mu_k V_k,$$

$$\text{so} \quad (A \otimes B)(U_j \otimes V_k) = (AU_j) \otimes (BV_k) = \lambda_j \mu_k (U_j \otimes V_k). \tag{A.28}$$

Also the eigenvalues of $A \otimes I_n + I_m \otimes B$ are given by $\lambda_j + \mu_k$ and the eigenvectors are $U_j \otimes V_k$, where $j = 1, 2, ...m$ and $k = 1, 2, ..., n$, it is easy to demonstrate this property since

$$(A \otimes I_n + I_m \otimes B)(U_j \otimes V_k) = (A \otimes I_n)(U_j \otimes V_k) + (I_m \otimes B)(U_j \otimes V_k)$$

$$= (AU_j) \otimes (I_n V_k) + (I_m U_j) \otimes (BV_k),$$

$$= (\lambda_j U_j) \otimes V_k + U_j \otimes (\mu_k V_k) = (\lambda_j + \mu_k)(U_j \otimes V_k).$$

The above properties have been widely used in our discussions and we have collected the important results related to direct products in this appendix.

Appendix B

Grassman Algebra

The concept of supersymmetry was the outcome of the ingenious research in 1974, by two groups, one led by Weiss and Zumino [181], and the other by Salam and Strathedee [182]. It was formulated basically to treat the bosonic and fermionic particles on an equal footing. Later, it was realized that this unification is more elegant if the space-time world is assumed to contain both ordinary and the Grassmanian type of variables, the latter being anticommuting quantities.

In this appendix we shall give the basic rules of dealing with such variables in two space-time dimensions. It should be mentioned that, in the case of Grassman variables, since a different form of the composition law is followed, the usual Lie algebras are also generalized and one encounters super Lie algebras, with all the properties of roots and weight diagrams remaining intact, but undergoing certain generalizations. Since we will be primarily concerned with two-dimensional models, we shall restrict ourselves to $2D$ supersymmetric spaces. The $2D$ superspace is constructed as follows. In addition to the two usual coordinates (x, t), denoted henceforth by (x_0, x_1), we have two Grassmanian coordinates (θ_1, θ_2) with the property,

$$\theta_1^2 = 0 = \theta_2^2, \qquad \theta_1\theta_2 = -\theta_2\theta_1. \tag{B.1}$$

The combined set is denoted by

$$z = (x_\mu, \theta_1) = (x_0, x_1, \theta_1, \theta_2), \tag{B.2}$$

with x_μ being the traditional coordinate with Minkowski space metric $\eta_{\mu\nu} = \text{diag}(1, -1)$. Furthermore, let us introduce a spinor field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in $2D$, which is Majorana, along with the usual scalar field $\phi(x, t)$. The Dirac matrices in $2D$ are

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{B.3}$$

The superfield $\Phi(x,t)$ is given by

$$\Phi(x,\theta) = \phi(x,t) + \bar{\theta}\psi(x,t) + \frac{1}{2}\bar{\theta}\theta F(x,t), \tag{B.4}$$

where $\bar{\theta} = \theta^{\dagger}\gamma^0$, since higher powers of θ are zero. We call ψ the fermionic partner of $\phi(x,t)$.

The super covariant derivatives are defined as follows:

$$D = \frac{\partial}{\partial\bar{\theta}} - i\ \not{\partial}\,\theta, \qquad \bar{D} = -\frac{\partial}{\partial\theta} + i\bar{\theta}\ \not{\partial} \tag{B.5}$$

where $\not{\partial} = \partial_\mu\gamma_\mu$, and we have followed the usual Feynmann convention. Usually there are two types of derivatives with respect to Grassmanian variables, the right and left derivatives. For example,

$$\begin{aligned} \frac{\partial}{\partial\theta}\bar{\psi}\theta &= -\bar{\theta}, & \frac{\partial}{\partial\theta}\bar{\theta}\theta &= -2\bar{\theta}, \\ \frac{\partial}{\partial\bar{\theta}}\bar{\theta}\psi &= \theta, & \frac{\partial}{\partial\bar{\theta}}\bar{\theta}\theta &= 2\theta. \end{aligned} \tag{B.6}$$

The expansion given in (B.4) follows from the fact that all the powers higher and equal to θ_1^2 and θ_2^2 are zero. We can write the formula for different types of derivatives as follows:

$$\begin{aligned} \frac{\partial}{\partial\theta_i}f &= -(-1)^{p(f)}f\frac{\partial}{\partial\theta_i}, \\ \frac{\partial}{\partial\theta_i}(fg) &= \left(\frac{\partial}{\partial\theta_i}f\right)g + (-1)^{p(f)}f\left(\frac{\partial}{\partial\theta_i}g\right), \\ (fg)\frac{\partial}{\partial\theta_i} &= f(g\frac{\partial}{\partial\theta_i}) + (-1)^{p(g)}(f\frac{\partial}{\partial\theta_i})g. \end{aligned} \tag{B.7}$$

Here $p(f), p(g)$ denotes respectively the parity or grading of f and g.

For integration of the anticommuting or super variables, one usually follows the conventions laid down by Berezin. The two basic rules are

$$\int d\theta_i = 0, \qquad \int d\theta_i\ \theta_i = 1. \tag{B.8}$$

Once it is possible to define the various coordinates, one can proceed to set up the various generalizations of standard classical transformations, Galilean, Lorentz, etc. These generalized transformations

generate Lie groups that are known as super Lie groups and they have corresponding super Lie algebras. The basic composition rule for such a Lie algebra involves both commutation and anticommutation rules for the generators. To produce a compact definition one divides the space of generators into two parts, odd and even, designated by a quantity "p" called the grade or parity. For even generators $p = 0$ and for odd ones $p = 1$ and we denote by $p(A)$ the grade of A. In (B.7) this notation is followed. Keeping in mind the correspondence between Poisson brackets and commutators, one can imagine that a generalized version of the Poisson bracket is possible, one that defines super Hamiltonian dynamics. For example, one can study the dynamics of an anticommuting or fermionic harmonic oscillator.

As a simple example of super Lie algebra, we can cite the case of super-extension of the SO(1,2) algebra that is denoted as $Osp(1,2)$, with five generators $\{L_0, L_\pm, G_\pm\}$. The set $\{L_0, L_\pm\}$ is the usual angular momentum generators, and $G_\pm$ are the fermionic counterparts of $L_\pm$. Their commutation (anticommutation) rules are as follows:

$$[L_0, L_\pm] = \mp L_\pm, \quad [L_=, L_-] = 2L_0,$$

$$[L_\pm, G_\pm] = 0,$$

$$[L_0, G_\pm] = \pm\frac{1}{2}G_\pm, \qquad [G_=, G_-] = 2L_0,$$

$$[L_\pm, G_\mp] = \pm G_\pm, \{G_\pm, G_\pm\} = 2L_=, \quad \{G_+, G_-\} = 2L_0.$$

The simplest 3×3 representation of the algebra $Osp(1,2)$ is given by

$$L_0 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$G_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad G_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}.$$

In the usual notation of the group $SO(1,2)$, the standard generators X_1, X_2, X_3 obey the commutation relations given below:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2;$$

the correspondence with the set $\{L_\pm, L_0\}$ is $L_\pm = X_1 \mp X_2, L_0 = X_3$. One defines the representation of $SO(1,2)$ by the eigenvalues of the Casimirs L^2 and L_0. In case of $Osp(1,2)$ we have another operator that commutes with $L_0, L_\pm$, viz

$$N = \frac{1}{2}[G_-, G_+],$$

so here we define the representation by the eigenvalues of three operators L^2, N and L_0, with

$$L^2|\lambda, \mu, m\rangle = \lambda||\lambda, \mu, m\rangle,$$

$$N|\lambda, \mu, m\rangle = \mu|\lambda, \mu, m\rangle,$$

$$L_0|\lambda, \mu, m\rangle = m|\lambda, \mu, m\rangle.$$

The computation then follows the same line as in the case of our usual Lie algebras. We refer readers to the excellent three-volume textbook by Cornwall; the last volume deals with super Lie algebra [183]. The definitive treatise on fermionic variables and their calculus is that of Berezin [184].

Appendix C

The Bethe Ansatz Equation

A major achievement of the quantum inverse scattering method is undoubtedly the formulation of the algebraic Bethe ansatz, which invariably leads to a set of coupled algebraic equations for the eigenmomenta associated with the various excitations, known as the Bethe ansatz equation (BAE). For a complete solution of the spectral problem, one needs to solve the BAE explicitly [185].

C.1 The BAE for an Elementary Spin System

In practice, one usually converts the BAE to a set of integral equations by defining a suitable density-of-states function, involving the eigenmomenta. In this appendix, we illustrate the procedure by considering a quantum mechanical system containing N elementary spins on a one-dimensional lattice [17]. The BAE for such a system, which is typical of most other problems, is the following:

$$\left(\frac{\chi_j + i\pi/2}{\chi_j - i\pi/2}\right)^N = -\prod_{i=1}^{M} \frac{\chi_j - \chi_i + i\pi}{\chi_j - \chi_i - i\pi}. \qquad (C.1)$$

The energy eigenvalue corresponding to the equation

$$\mathcal{H}\Phi = E\Phi,$$

being

$$E = -J\sum_{j=1}^{M} \frac{\pi^2}{\chi_j^2 + (\pi/2)^2}. \qquad (C.2)$$

We shall first deal with the case of real roots. Suppose that all the χ_j's in a solution set of (C.1) $\in \mathcal{R}$. Taking the logarithm of (C.1) yields

$$N \log \left(\frac{e^{i \tan^{-1}(\frac{\pi}{2\chi_j})}}{e^{-i \tan^{-1}(\frac{\pi}{2\chi_j})}} \right) = 2i(k - \frac{1}{2})\pi + \sum_{i=1}^{M} \log \left(\frac{e^{i \tan^{-1}(\frac{\pi}{\chi_j - \chi_i})}}{e^{-i \tan^{-1}(\frac{\pi}{\chi_j - \chi_i})}} \right),$$

$$N \tan^{-1} \left(\frac{\pi}{2\chi_j} \right) = (k - \frac{1}{2})\pi + \sum_{i=1}^{M} \tan^{-1} \left(\frac{\pi}{\chi_j - \chi_i} \right).$$

Now using the identities,

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \qquad \cot^{-1} x = \tan^{-1} \frac{1}{x},$$

we have

$$N \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{2\chi_j}{\pi} \right) \right] = (k - \frac{1}{2})\pi + \sum_{i=1}^{M} \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{\chi_j - \chi_i}{\pi} \right) \right],$$

which upon simplification gives

$$2N \tan^{-1} \left(\frac{2\chi_j}{\pi} \right) - 2 \sum_{i=1}^{M} \tan^{-1} \left(\frac{\chi_j - \chi_i}{\pi} \right) = 2\pi J_j, \quad \text{where } j = 1, ..., M. \tag{C.3}$$

Here we have written $J_j = \left(\frac{N-M-1}{2} + k \right)$ so that if $N - M =$ odd then, J_j is an integer and if $N - m =$ even then, J_j is a half odd integer. Let us note the fact that the BAE (C.1) constitute a set of M transcendental equations for the eigenmomenta χ_j and suppose that $\{\chi_1, \chi_2, ..., \chi_M\}$ is a self-consistent solution set of (C.1) with, say, $N - M =$ odd. Consider the function,

$$J(\chi) = \frac{1}{2\pi} \left[2N \tan^{-1} \frac{2\chi}{\pi} - 2 \sum_{i=1}^{M} \tan^{-1} \left(\frac{\chi - \chi_i}{\pi} \right) \right], \tag{C.4}$$

which for states of interest, is a monotonically increasing function of χ. The values of χ for which $J(\chi)$ are integers J_i are clearly equal to χ_i. However there might be some integer values of $J(\chi)$ for which the corresponding χ does not belong to the solution set of (C.1); such a χ will be called a *hole*, while values for which $J(\chi)$ is an integer will be called the *roots*. Figure (C.1.1) depicts the situation described above. One then starts with a natural definition of positive definite density,

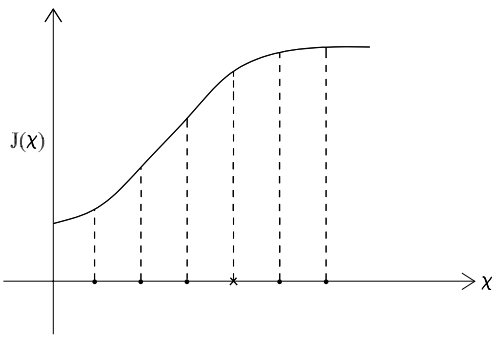

FIGURE C.1.1: Position of roots and holes in $J(\chi)$.

describing the distribution of roots and holes for a given solution of (C.3), namely

$$\rho(\chi) = \frac{dJ(\chi)}{d\chi}, \tag{C.5}$$

with the inverse relation,

$$J(\chi) = J(-\infty) + \int_{-\infty}^{\chi} \rho(\chi')d\chi'. \tag{C.6}$$

Let us now consider a solution of (C.3) for which the number of holes, say n, is fixed, independent of N as $N \longrightarrow \infty$. By differentiating (C.4) and using the definition of $\rho(\chi)$ we find from (C.6):

$$\rho(\chi) = \frac{N/2}{\chi^2 + (\frac{\pi}{2})^2} - \sum_{i=1}^{M} \frac{1}{(\chi - \chi_i)^2 + \pi^2}. \tag{C.7}$$

We shall next approximate (accurate when $N \to \infty$) the sum $\sum_{i=1}^{M} f(\chi_i)$ by

$$\sum_{i=1}^{M} f(\chi_i) \approx \int d\chi \rho(\chi) f(\chi) - \sum_{j=1}^{n} f(\theta_j),$$

where $\theta_1, \theta_2, ...\theta_n$ are the hole positions. As a result (C.7), becomes

$$\rho(\chi) + \int d\chi' \, \rho(\chi') \frac{1}{(\chi - \chi')^2 + \pi^2} = \frac{N/2}{\chi^2 + (\frac{\pi}{2})^2} + \sum_{j=1}^{n} \frac{1}{(\chi - \theta_j)^2 + \pi^2}. \tag{C.8}$$

Thus in passing to the thermodynamic limit ($N \to \infty$), we have converted (C.1) into an integral equation for the density function; and the

seemingly impossible task of solving M simultaneous transcendental equations has been reduced to that of solving a linear integral equation. Note that once $\rho(\chi)$ has been determined, the individual χ_j and θ_i can be obtained from the condition,

$$\int_{-\infty}^{\zeta_k} \rho(\chi) d\chi = k - \frac{1}{2}, \tag{C.9}$$

where ζ_k is the kth element of the set $\{\chi_1, \chi_2, ..., \chi_M; \theta_1, \theta_2, ..., \theta_n\}$ counting from the left-hand side of the χ axis.

Equation (C.8) is solved by the Fourier transformation method. If $\tilde{\rho}(p)$ denotes the Fourier transform of $\rho(\chi)$, then

$$\rho(\chi) = \frac{1}{2\pi} \int dp \exp(ip\chi)\tilde{\rho}(p).$$

Further, we note that as

$$\frac{a/\pi}{\chi^2 + a^2} = \frac{1}{2\pi} \int dp \exp(ip\chi - a|p|), \tag{C.10}$$

(C.8) consequently becomes

$$\int \frac{dp}{2\pi} e^{ip\chi} \tilde{\rho}(p) + \int d\chi' \rho(\chi') \int \frac{dp}{2\pi} e^{ip(\chi-\chi')-\pi|p|}$$

$$= N \int \frac{dp}{2\pi} e^{ip\chi}.e^{-\pi|p|/2} + \sum_{j=1}^{n} \int \frac{dp}{2\pi} e^{ip(\chi-\theta_j)}.e^{-\pi|p|}$$

$$\int \frac{dp}{2\pi} e^{ip\chi} \left[\tilde{\rho}(p) + \int d\chi' \rho(\chi') e^{-ip\chi'}.e^{-\pi|p|}\right]$$

$$= \int \frac{dp}{2\pi} e^{ip\chi} e^{ip\chi} \left[N e^{-\pi|p|/2} + \sum_{j=1}^{n} e^{-ip\theta_j}.e^{-\pi|p|}\right]. \tag{C.11}$$

However, as

$$\int d\chi' \rho(\chi') e^{-ip\chi'} = \tilde{\rho}(p),$$

therefore the above equation assumes the simplified form:

$$\tilde{\rho}(p) = \tilde{\rho_0}(p) + \sum_{j=1}^{n} \frac{\exp(-ip\theta_j - \pi|p|/2)}{2\cosh(\pi p/2)}, \tag{C.12}$$

where

$$\tilde{\rho_0}(p) = \frac{N}{2\cosh(\pi p/2)}.$$

Alternatively we can write

$$\rho(\chi) = \rho_0(\chi) + \sum_j \rho_h(\chi - \theta_j), \qquad (C.13)$$

where $\rho_h(\chi - \theta)$ is a smooth function peaked at the hole position θ and falling off $\sim (4\chi^2)^{-1}$ for large $|\chi|$. To see how this may be done, we consider the equation defining the Fourier transform of $\rho(\chi)$, viz

$$\rho(\chi) = \int \frac{dp}{2\pi} e^{ip\chi} \left[\tilde{\rho_0}(p) + \sum_{j=1}^{n} \frac{\exp(-ip\theta_j - \pi|p|/2)}{2\cosh \pi p/2} \right] = t_1 + t_2.$$

Now as

$$t_1 = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\chi} \tilde{\rho_0}(p) = \frac{N}{2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{ip\chi}}{\cosh(\pi p/2)},$$

so making the substitution $x = \pi p/2$ allows us to write t_1 as

$$t_1 = \frac{N}{2\pi^2} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{\cosh(x)} dx, \qquad \alpha = \frac{2\chi}{\pi}.$$

The integral on the right-hand side may be evaluated by the residue theorem, using the contour in Figure (C.1.2), and one finds finally that

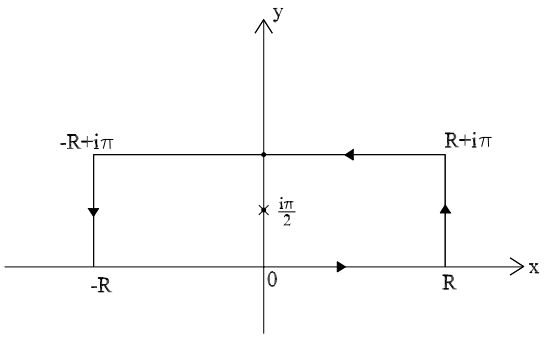

FIGURE C.1.2: Contour for evaluation of the term t_1.

$$t_1 = \frac{N}{2\pi \cosh \chi}. \qquad (C.14)$$

The same substitution $x = \pi p/2$ allows us to rewrite t_2 in the following manner:

$$t_2 = \sum_j \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dx \frac{e^{i\xi x - |x|}}{\cosh x}, \quad \text{where} \quad \xi = \frac{2}{\pi}(\chi - \theta_j)$$

$$\text{or, } t_2 = 2 \int_0^{\infty} \frac{e^{-x} \cos(\xi x)}{\cosh x} dx = 2\mathcal{I}.$$

To evaluate this integral, we consider the following contour integral:

$$\oint_\Gamma \frac{e^{-z} e^{i\xi z}}{\cosh z} dz$$

over the contour shown in Figure (C.1.3). Application of Cauchy's theorem leads to

$$\int_0^{\infty} \frac{e^{-x} e^{i\xi x}}{\cosh x} dx - i \int_0^{\infty} \frac{e^{-x} e^{i\xi x}}{\sin x} dx . e^{-\xi\pi/2} = -\frac{\pi}{2} e^{-\xi\pi/2},$$

the real part of which yields

$$\int_0^{\infty} \frac{e^{-x} \cos(\xi x)}{\cosh x} dx + \int_0^{\infty} dx \frac{e^{-x} \sin(\xi x)}{\sin x} e^{-\pi\xi/2} = -\frac{\pi}{2} e^{-\xi\pi/2}. \qquad (C.15)$$

Under the transformation $\xi \to -\xi$, (C.15) gives

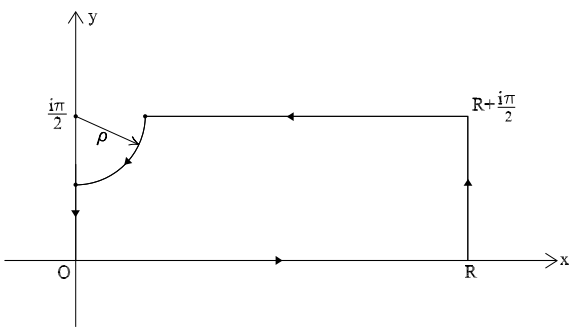

FIGURE C.1.3: Contour for evaluation of the term t_2.

$$\int_0^\infty \frac{e^{-x}\cos(\xi x)}{\cosh x}dx - \int_0^\infty dx \frac{e^{-x}\sin(\xi x)}{\sin x}e^{\pi\xi/2} = -\frac{\pi}{2}e^{\xi\pi/2} \qquad (C.16).$$

Multiplying (C.15) and (C.16) by $e^{\pm\pi\xi/2}$, respectively, we get by addition

$$\mathcal{I} = \int_0^\infty \frac{e^{-x}\cos(\xi x)}{\cosh x}dx = -\frac{\pi}{2\cosh(\pi\xi/2)},$$

so that finally

$$t_2 = -\sum_j \frac{1}{2\pi\cosh(\chi-\theta_j)}.$$

Hence

$$\rho(\chi) = \frac{N}{2\pi\cosh\chi} - \sum_j \frac{1}{2\pi\cosh(\chi-\theta_j)}.$$

From this it is easy to compute the total number of roots M in a state with n holes as

$$M = \int d\chi\rho(\chi) - n = \tilde{\rho}(0) - n = \frac{N}{2} - \frac{n}{2}. \qquad (C.17)$$

Since M is by definition an integer, we see that for a given even N, the number of holes in a state must be even; while for an odd number of lattice sites, n must be odd.

The energy of the state corresponding to $\rho(\chi)$ may be obtained from (C.2),

$$E = -J\pi^2 \int d\chi\sigma(\chi)\frac{1}{\chi^2 + (\pi/2)^2}, \qquad (C.18)$$

where $\sigma(\chi)$ is the density of roots of the BAE, i.e.,

$$\sigma(\chi) = \rho(\chi) - \sum_{i=1}^{n}\delta(\chi-\theta_i), \qquad (C.19)$$

assuming there are n number of holes in the state under consideration. Now the Fourier transform of $\sigma(\chi)$ is given by

$$\tilde{\sigma}(p) = \tilde{\rho}(p) - \sum_{i=1}^{n} e^{-ip\theta_i}, \qquad (C.20)$$

while the inverse fourier transform yields

$$\sigma(\chi) = \int \frac{dp}{2\pi}e^{ip\chi}\tilde{\sigma}(p).$$

Using this together with (C.10) (taking $a = \pi/2$), in (C.18) gives us

$$E = -\frac{J}{2}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2\, \tilde{\sigma}(p_1) e^{-\pi|p_2|/2} \int_{-\infty}^{\infty} d\chi e^{i(p_1+p_2)\chi}.$$

But as $\int_{-\infty}^{\infty} d\chi e^{i(p_1+p_2)\chi} = 2\pi\delta(p_1+p_2)$, hence we have

$$E = -J\pi \int_{-\infty}^{\infty} dp \tilde{\sigma}(-p) e^{-\pi|p|/2}.$$

Using the expressions for $\tilde{\sigma}(p)$ and $\tilde{\rho}(p)$ as given by (C.20) and (C.12) we therefore obtain

$$E = -J\pi \int_{-\infty}^{\infty} dp\, e^{-|p|/2} \left[\tilde{\rho}_0(-p) + \sum_{j=1}^{n} \frac{e^{ip\theta_j - \pi|-p|/2}}{2\cosh(-p)\pi/2} - \sum_{j=1}^{n} e^{ip\theta_j}\right], \tag{C.21}$$

$$\text{or,} \qquad -\frac{E}{J\pi} = t_0 + \sum_{j=1}^{n} t_j,$$

where

$$t_0 = \int_{-\infty}^{\infty} dp\, e^{-\pi|p|/2} \frac{N}{2\cosh(\pi p/2)} = \frac{2N\ln 2}{\pi} \tag{C.22}$$

and

$$t_j = \int_{-\infty}^{\infty} dp \frac{e^{ip\theta_j}}{2\cosh \pi p/2} \left[e^{-\pi p} - e^{-\pi|p|/2}\left(e^{\pi p/2} + e^{-\pi p/2}\right)\right],$$

$$t_j = -\left[\int_{-\infty}^{\infty} dp\, \frac{e^{ip\theta_j}}{2\cosh(\pi p/2)}\right]. \tag{C.23}$$

Finally, we have

$$E = -2JN\ln 2 + \sum_{j=1}^{n} E_h(\theta_j), \tag{C.24}$$

with

$$E_h(\theta_j) = \frac{J\pi}{\cosh\theta_j}. \tag{C.25}$$

From (C.24) we see that the ground state (which has no holes) has total energy $-2NJ\ln 2$ and that each hole corresponds to a particle-like excitation with positive definite energy given by (C.25). Next let us briefly discuss the case of complex roots.

C.2 Complex Roots

For the case of complex roots we set $\chi = \xi + i\eta, \eta > 0$, so that

$$\left(\frac{\xi + i(\eta + \pi/2)}{\xi + i(\eta - \pi/2)}\right)^N = -\prod_{i=1}^{M} \frac{\chi - \chi_i + i\pi}{\chi - \chi_i - i\pi}. \qquad (C.26)$$

We note that for $N \to \infty$, the magnitude of the left-hand side behaves like

$$\exp(kN), \quad k > 0. \qquad (C.27)$$

If M is a fixed number, the only way the right-hand side can blow up at this rate is if

$$|\chi - \chi_j - i\pi| \sim e^{-kN}, \qquad (C.28)$$

that is, for another root χ_i having the same real part as χ and an imaginary part π less than that of χ. One can establish an even general result: for fixed M and N tending to infinity, all roots $\chi_1, \chi_2, ...\chi_M$ in a solution set of the BAE are members of n strings. An n string is a family of n roots with the same real parts of the form $\xi + i\pi j$ with $j = -(n-1)/2, -(n-3)/2, ..., (n-1)/2$. Examples of n strings are displayed in Figure (C.1.4). One can easily generalize the previous

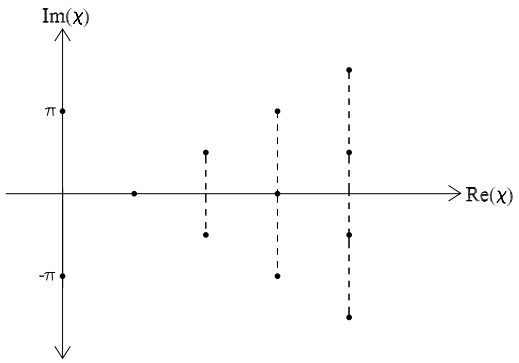

FIGURE C.2.1: Schematic diagram of string states.

results to this case:

$$\tilde{\sigma}(p) = \tilde{\rho}_0(p) - \sum_{i=1}^{n} \frac{e^{\pi|p|/2 - ip\theta_j}}{2\cosh(\pi p/2)} - \sum_{\xi_i + i\eta_i}^{\eta_i < \pi} \frac{\cosh(\eta_i p)}{\cosh(\pi p/2)} \exp(-i\xi_i p - \pi|p|/2). \tag{C.29}$$

Appendix D

The AKNS Problem

We consider here the classical inverse scattering transform for the Ablowitz, Kaup, Newell and Segur (AKNS) problem, which is an example of a 2×2 matrix eigenvalue problem defined by [186]:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \qquad (D.1)$$

The nonlinear fields $q, r \to 0$ as $|x| \to \infty$ and $\zeta = \xi + i\eta$ represents the eigenvalue. The eigenfunctions $\phi, \bar{\phi}, \psi, \bar{\psi}$ of (D.1), as $|x| \to 0$, are as follows with $\zeta = \xi$:

$$\text{as } x \to -\infty \qquad \phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \text{ and } \bar{\phi} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x}, \qquad (D.2a)$$

$$\text{while as } x \to +\infty \qquad \psi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \text{ and } \bar{\psi} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x}. \quad (D.2b)$$

By defining the Wronskian of two solutions u and v as $W(u, v) = u_1 v_2 - v_1 u_2$, we find for the above solutions:

$$W(\phi, \bar{\phi}) = -1 \text{ and } W(\psi, \bar{\psi}) = -1. \qquad (D.3)$$

Hence $\phi, \bar{\phi}$ and $\psi, \bar{\psi}$ constitute two sets of independent solutions. Obviously (D.1) is of first order, which means that there are only two linearly independent solutions; this implies that the above sets are not linearly independent. Consequently,

$$\phi(x, \xi) = a(\xi)\bar{\psi}(x, \xi) + b(\xi)\psi(x, \xi) \qquad (D.4a)$$

$$\text{and} \qquad \bar{\phi}(x, \xi) = -\bar{a}(\xi)\psi(x, \xi) + \bar{b}(\xi)\bar{\psi}(x, \xi), \qquad (D.4b)$$

where $a, \bar{a}, b, \bar{b}$ depend parametrically on time t. From (D.2) we find that their Wronskian is given by

$$W(\phi, \bar{\phi}) = W(a\bar{\psi} + b\psi, -\bar{a}\psi + \bar{b}\bar{\psi}) = (a\bar{a} + b\bar{b})W(\psi, \bar{\psi}),$$

implying that

$$a\bar{a} + b\bar{b} = 1, \tag{D.5}$$

in view of (D.3). If $\int_{-\infty}^{\infty} |q|dx$ and $\int_{-\infty}^{\infty} |r|dx$ are bounded, then $\phi e^{i\zeta x}$ and $\psi e^{-i\zeta x}$ are analytic in the upper half (UHP) of the complex ζ plane, while $\bar{\phi}e^{-i\zeta x}$ and $\bar{\psi}e^{i\zeta x}$ are analytic in the lower half plane (LHP). As a result $a(\zeta) = W(\phi, \psi)$ and $\bar{a}(\zeta) = W(\bar{\phi}, \bar{\psi})$ are analytic in the UHP and LHP, respectively. In general $b, \bar{b}$ are *not* analytic.

In Chapter 1, we stated that the direct scattering problem essentially involves calculation of $\phi, \bar{\phi}, \psi, \bar{\psi}$ and the quantities $a,\ \bar{a},\ b,\ \bar{b}$, given q and r at $t = 0$, besides determination of their analytical properties. The inverse problem on the other hand consists in the determination of q and r from a knowledge of the preceding quantities. Presuming we have at our disposal the above knowledge, we proceed to construct a solution of the inverse scattering problem.

As ψ and $\bar{\psi}$ are defined by their asymptotic nature as $x \to +\infty$, we assume their integral representations to be

$$\psi(x, \xi) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^{\infty} K(x, s) e^{i\xi s} ds, \tag{D.6a}$$

$$\bar{\psi}(x, \xi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} + \int_x^{\infty} \bar{K}(x, s) e^{-i\xi s} ds, \tag{D.6b}$$

where $K, \bar{K}$ are two component vectors, i.e., $K(x, s) = \begin{pmatrix} K_1(x, s) \\ k_2(x, s) \end{pmatrix}$ and similarly for $\bar{K}$. It will be shown that such kernels do exist and $K(x, s)$ is independent of the eigenvalue ξ. Assuming $a(\xi) \neq 0$, we may divide (D.4a) by $a(\xi)$ and use (D.6) to get

$$\frac{\phi}{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} + \int_x^{\infty} \bar{K}(x, s) e^{-i\xi s} ds + \frac{b}{a} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^{\infty} K(x, s) e^{i\xi s} ds \right). \tag{D.7}$$

Operating on both sides of (D.7) with $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi y} d\xi, \quad y > x$ and using $\int_{-\infty}^{\infty} e^{i\xi(y-x)} d\xi = \delta(y - x) = 0$ since $y > x$, we find that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi}{a} e^{i\xi y} d\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} F_C(x + y) + \bar{K}(x, y) + \int_x^{\infty} K(x, s) F_C(s + y) ds, \tag{D.8}$$

where

$$F_C(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{a} e^{i\xi x} d\xi. \tag{D.9}$$

To calculate the left-hand side of (D.8) we need to know the following: (a) the zeros of $a(\zeta)$ in the UHP and (b) the asymptotic form of ϕ and a as $\zeta \to +\infty$. For this purpose we assume that $a(\zeta)$ has isolated simple zeros with an asymptotic nature given by

$$a(\zeta) \sim 1 + \mathcal{O}(1/\zeta). \tag{D.10}$$

On the other hand ϕ behaves like

$$\phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x} + \mathcal{O}(1/\zeta). \tag{D.11}$$

Now define the integral,

$$I = \frac{1}{2\pi}\oint_C \frac{\phi}{a} e^{i\zeta y} d\zeta \tag{D.12}$$

where C is a semicircular contour in the UHP. If $a(\zeta_j) = 0$, then by the residue theorem and using the asymptotic behaviour given above, we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi}{a} e^{i\xi y} d\xi = i\sum_{j=1}^{N} \frac{\phi(\zeta_j)}{a'(\zeta_j)} e^{i\zeta_j y}. \tag{D.13}$$

Since $a = W(\phi, \psi)$ and ζ_j $(j = 1, 2..N)$, are the zeros of $a(\zeta)$, it follows that $\phi(\zeta_j) \propto \psi(\zeta_j) = \psi_j$, i.e., $\phi(\zeta_j) = \tilde{c}_j \psi_j$, leading to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi}{a} e^{i\xi y} d\xi = i\sum_{j=1}^{N} c_j \psi_j e^{i\zeta_j y}, \tag{D.14}$$

where the normalization constant $c_j = \frac{\tilde{c}_j}{a'(\zeta_j)}$. As ψ, given by (D.6a), admits analytical continuation to the UHP, it follows that

$$\psi_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta_j x} + \int_x^{\infty} K(x,s) e^{i\zeta_j s} ds. \tag{D.15}$$

As a result we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi}{a} e^{i\xi y} d\xi = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} F_D(x+y) - \int_x^{\infty} K(x,s) F_D(s+y) ds, \tag{D.16}$$

where

$$F_D(x+y) = -i\sum_{j=1}^{N} c_j e^{i\zeta_j(x+y)}. \qquad (D.17)$$

From (D.9) and (D.17), by subtraction we have

$$F(x) = (F_C + F_D)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b}{a} e^{i\xi x} d\xi - i\sum_{j=1}^{N} c_j e^{i\zeta_j x}, \qquad (D.18)$$

allowing for (D.8) to be expressed as

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{\infty} K(x,s)F(s+y)ds = 0. \qquad (D.19)$$

Similarly,

$$K(x,y) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) - \int_x^{\infty} \bar{K}(x,s)\bar{F}(s+y)ds = 0. \qquad (D.20)$$

Defining the matrices,

$$\mathcal{K} = \begin{pmatrix} \bar{K}_1 & K_1 \\ \bar{K}_2 & K_2 \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} 0 & -\bar{F} \\ F & 0 \end{pmatrix}, \qquad (D.21)$$

(D.19 and D.20) may be compactly expressed in the form,

$$\mathcal{K}(x,y) + \mathcal{F}(x+y) + \int_x^{\infty} \mathcal{K}(x,s)\mathcal{F}(s+y)ds = 0. \qquad (D.22)$$

This is the matrix version of the Gelfand-Levitan equation written in Marchenko form. The final step in the inverse problem consists of relating the fields $q,\ r$ to the kernels $K(x,y)$. This may be accomplished in the following manner. We substitute the integral equation for ψ from (D.6a) into the eigenvalue equation (D.1) to obtain

$$i\zeta \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x} - \begin{pmatrix} K_1(x,x) \\ K_2(x,x) \end{pmatrix} e^{i\zeta x} + \int_x^{\infty} \begin{pmatrix} K_{1x}(x,s) \\ K_{2x}(x,s) \end{pmatrix} e^{i\zeta s} ds$$

$$= \begin{pmatrix} q \\ i\zeta \end{pmatrix} e^{i\zeta x} + i\zeta \int_x^{\infty} \begin{pmatrix} -K_1(x,s) \\ K_2(x,s) \end{pmatrix} e^{i\zeta s} ds + \int_x^{\infty} \begin{pmatrix} q(x)K_2(x,s) \\ r(x)K_1(x,s) \end{pmatrix} e^{i\zeta s} ds.$$

Assuming $K_i(x, s \to \infty) = 0$, we may integrate the second term on the right-hand side by parts, which enables the kernels to be expressed as follows, in terms of the fields:

$$K_1(x,x) = -\frac{1}{2}q(x),$$

$$K_{1x} - K_{1s} = q(x)K_2(x,s), \quad K_{2x} + K_{1s} = r(x)K_1(x,s). \tag{D.23}$$

Likewise, for $\bar{\psi}$ we obtain

$$\bar{K}_2(x,x) = -\frac{1}{2}r(x),$$

$$\bar{K}_{2x} - \bar{K}_{2s} = r\bar{K}_1(x,s), \quad \bar{K}_{1x} + \bar{K}_{1s} = q\bar{K}_2(x,s). \tag{D.24}$$

To determine the asymptotic form of $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$, we essentially use the WBK trick, i.e., we write ψ as follows and substitute it into the eigenvalue equation (D.1):

$$\psi = e^{i\zeta x}\left[\begin{pmatrix} o \\ 1 \end{pmatrix} + \frac{1}{\zeta}A(x) + ...\right]. \tag{D.25}$$

Upon equating coefficients of ζ, it yields the components of the vector $A(x)$, enabling us to express ψ as given below:

$$\psi e^{-i\zeta x} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\zeta}\begin{pmatrix} q \\ -\int_{-x}^{\infty} rqdx' \end{pmatrix} + ... \tag{D.26a}$$

In the same way, the following may also be deduced:

$$\bar{\psi}e^{i\zeta x} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2i\zeta}\begin{pmatrix} \int_{x}^{\infty} qrdx' \\ -r \end{pmatrix} + ..., \tag{D.26b}$$

$$\phi e^{i\zeta x} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\zeta}\begin{pmatrix} \int_{-\infty}^{x} qrdx' \\ r \end{pmatrix} + ..., \tag{D.26c}$$

$$\bar{\phi}e^{-i\zeta x} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \frac{1}{2i\zeta}\begin{pmatrix} q \\ \int_{-\infty}^{x} rqdx' \end{pmatrix} + ... \tag{D.26d}$$

As $a(\zeta) = W(\phi, \psi)$, the above expansions lead to the following form of $a(\zeta)$:

$$a(\zeta) \sim 1 - \frac{1}{2i\zeta}\int_{-\infty}^{\infty} rqdx' + \mathcal{O}(\frac{1}{\zeta}), \tag{D.27a}$$

and also

$$\bar{a}(\zeta) \sim 1 + \frac{1}{2i\zeta}\int_{-\infty}^{\infty} rqdx' + \mathcal{O}(\frac{1}{\zeta}). \qquad (D.27b)$$

Furthermore, as $K(x, s \to \infty) = 0$, one can formally write

$$\psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x} - K(x,x)\frac{e^{i\zeta x}}{i\zeta} + ..., \qquad (D.28a)$$

$$\bar{\psi} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\zeta x} + \bar{K}(x,x)\frac{e^{i\zeta x}}{i\zeta} + ... \qquad (D.28b)$$

Comparing (D.26a and D.26b) and (D.28a and D.28b) we conclude that

$$K_1(x,x) = -\frac{1}{2}q(x), \;\; K_2(x,x) = \frac{1}{2} = \int_x^{\infty} qrdx', \qquad (D.29a)$$

$$\bar{K}_2(x,x) = -\frac{1}{2}r(x), \;\; \bar{K}_1(x,x) = \frac{1}{2}\int_x^{\infty} qrdx'. \qquad (D.29b)$$

Thus, the kernels exist and are unique and independent of the eigenvalue ζ.

Finally, let us consider the time evolution problem. The time part of the Lax equation is

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad (D.30)$$

with the following behaviour, viz $A \to A_-(\zeta), D \to -A_-(\zeta)$ and $B(\zeta), C(\zeta) \to 0$, as $|x| \to \infty$. The corresponding solutions are as $x \to -\infty$,

$$\phi^{(t)} = \phi\, e^{A_-(\zeta)t}, \;\; \bar{\phi}^{(t)} = \bar{\phi}\, e^{-A_-(\zeta)t}, \qquad (D.31a)$$

and as $x \to +\infty$,

$$\psi^{(t)} = \psi\, e^{-A_-(\zeta)t}, \;\; \bar{\psi}^{(t)} = \bar{\psi}\, e^{A_-(\zeta)t}, \qquad (D.31b)$$

where $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$ are the old eigenfunctions satisfying the space part of the Lax equation. From these equations, it follows that the time dependence of ϕ is determined by the equation,

$$\phi_t \sim \begin{pmatrix} 0 & 0 \\ 0 & -2A_-(\zeta) \end{pmatrix} \phi \quad \text{as } x \to +\infty\ . \qquad (D.32)$$

However, as $\phi = a\bar{\psi} + b\psi$, by using the behaviour of ψ and $\bar{\psi}$ as $x \to +\infty$, we obtain

$$\phi \sim \begin{pmatrix} ae^{-i\zeta x} \\ be^{i\zeta x} \end{pmatrix}. \tag{D.33}$$

Calculating ϕ_t from this asymptotic solution and by substituting ϕ, ϕ_t thus obtained in (D.4a) we get the following time dependences of a and b:

$$a_t e^{-i\zeta x} = 0, \text{ so that } a(\zeta, t) = a(\zeta, 0), \tag{D.34}$$

and $b_t(\zeta, t) = -2bA_-(\zeta)$, which implies that

$$b(\zeta, t) = b(\zeta, 0)e^{-2A_-(\zeta)t}. \tag{D.35}$$

It follows from (D.34) that the eigenvalues are fixed in time. Assuming the necessary conditions for analytic continuation, we have for the normalization constants:

$$C_j(t) = \frac{b_j(t)}{a_j'(t)} = \frac{b_j(0)}{a_j'(0)} e^{-2A_-(\zeta)t} = C_{j,0}e^{-2A_-(\zeta)t}. \tag{D.36}$$

Similarly, using $\bar{\phi} = -\bar{a}\psi + \bar{b}\bar{\psi}$, we have

$$\bar{a}(\zeta, t) = \bar{a}(\zeta, 0), \tag{D.37}$$

$$\bar{b}(\zeta, t) = \bar{b}(\zeta, 0)e^{2A_-(\zeta)t}, \tag{D.38}$$

and

$$\bar{C}_j(t) = \bar{C}_{j,0}e^{2A_-(\zeta)t}. \tag{D.39}$$

These results allow us to determine the time dependencies of $F(x, t)$ and $\bar{F}(x, t)$ in the inverse scattering equation. From (D.18), it follows that

$$F(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{a}(\zeta, t)e^{i\zeta x - 2A_-(\zeta)t} d\zeta - i \sum_{j=1}^{N} C_{j,0}e^{i\zeta_j x - 2A_-(\zeta_j)t}, \tag{D.40}$$

and similarly,

$$\bar{F}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\bar{b}}{\bar{a}}(\zeta, t)e^{-i\zeta x + 2A_-(\zeta)t} d\zeta - i \sum_{j=1}^{N} \bar{C}_{j,0}e^{-i\bar{\zeta}_j x + 2A_-(\bar{\zeta}_j)t}. \tag{D.41}$$

These expressions are analogous to Fourier integral equations in linear problems. For $x \to +\infty$ and the special case of $r = \mp q^\star$, one can show that [186]

$$q(x,t) \sim \mp 2F^\star(2x,t),$$

where use has been made of $\bar{F} = \pm F^\star$. Thus the time dependence is parametric in nature. Furthermore, writing $\zeta_j = \alpha_j + i\beta_j$ with $\beta_j > 0$, we note that the contribution to $q(x,t)$ from the bound states is negligible, as $x \to +\infty$. As a result one has

$$q(x,t) \sim \mp \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b}{a}^\star(0) e^{-2i\zeta x - 2A_-^\star(\zeta)t} d\zeta, \qquad (D.42)$$

which basically shows that as $x \to +\infty$, the problem reduces to a linear one.

Special case $r = -q^\star$: In this case we have

$$K_1(x,y) = F^\star(x+y) - \int_x^\infty \int_x^\infty K_1(x,z) F(x+s) F^\star(s+y) dz\, ds. \quad (D.43)$$

For the "reflectionless" case, i.e., $(b/a)_0 = 0$, the continuous spectrum is absent and choosing $N = 1$, we find that

$$F(x) = -ice^{i\zeta x}, \quad \text{with} \quad \zeta = \xi + i\eta, \eta > 0. \qquad (D.44)$$

Substituting (D.44) in (D.43) and multiplying both sides by $\int_x^\infty e^{i\zeta y} dy$, we obtain

$$\hat{K}_1(x) = -\frac{c^\star e^{i(\zeta - 2\zeta^\star)x}}{\left[1 - \frac{|c|^2 e^{2i(\zeta-\zeta^\star)x}}{(\zeta-\zeta^\star)^2}\right](\zeta - \zeta^\star)}, \qquad (D.45)$$

where $\hat{K}_1$ has been defined as

$$\hat{K}_1(x) = \int_x^\infty K_1(x,y) e^{i\zeta y} dy. \qquad (D.46)$$

Using (D.46) in (D.43) allows us to express $K_1(x,y)$ in the form,

$$K_1(x,y) = ic^\star e^{-i\zeta^\star(x+y)} \left[1 - \frac{|c|^2 e^{2i(\zeta-\zeta^\star)x}}{(\zeta-\zeta^\star)^2}\right]^{-1} \qquad (D.47)$$

The potential $q(x)$ then becomes

$$q(x) = -2ic^\star e^{-2i\zeta^\star x} \left[1 - \frac{|c|^2 e^{2i(\zeta-\zeta^\star)x}}{(\zeta-\zeta^\star)^2}\right]^{-1}, \qquad (D.48)$$

which may be simplified to

$$q(x) = -i\frac{c^\star}{|c|}2\eta e^{-2i\xi x}\mathrm{sech}\{2(\eta x - \theta)\}. \qquad (D.49)$$

Here ξ and η represent the real and imaginary parts of ζ, respectively, while θ is defined by $e^{2\theta} = \frac{|c|}{2\eta}$. Equation (D.49) represents a soliton solution to *all* evolution equations with $r = -q^\star$, subject to the condition $A(\zeta) \to A_-(\zeta)$ as $|x| \to \infty$.

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