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Transformations of Differential Equations

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Abstract. The article concerns the second order differential equations with one unknown function and the aim is twofold: to compare some results for the well-known linear with the more complicated nonlinear case, and to point out some distinctions between ordinary and partial differential equations. We shall mention automorphisms permuting the conjugate points, moving frames for particular fiber-preserving mappings, the Darboux transformations of ordinary differential equation, and the Laplace series for the hyperbolic case of two independent variables.

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Our reasoning will be developed in the real smooth category. As a rule, we shall not specify the definition domains and our primary aim is to outline some new ideas and methods rather than to derive certain definite theorems.

For the convenience of a possible reader, let us outline the contents. We begin with the family of *all* equations (2). It is preserved if transformations (1) are performed, and certain self-transformations of this kind (so called *central dispersions*) of a *given* equation (2) are determined by the location of roots of solutions: they permute the roots. This result can be easily adapted for the *nonhomogeneous* family (5) subjected to a broader class (6) of transformations, then the *intersections* of solutions undertake the previous role of roots. These well-known results can be verified by a manner which can be carried over the class of all *nonlinear* equations (7) subjected to *contact transformations*. In particular, certain automorphisms of a *given* equation (7) exist which permute the intersection points of infinitesimally near couples of solutions. They may be regarded for *nonlinear generalization* of dispersions.

Our next aim is to determine some *subfamilies* of the family of all equations (7) which are preserved if *all* transformations of the kind (6) are applied. We use the moving frames.

On the other hand, a *given* equation (2) can be transformed into the family of all equations (2) by many rather peculiar transformations, even if the independent variable x is kept fixed. They can be explicitly found and the famous

This is the final form of the paper.

Darboux transformation of eigenvalue problems appears as a very particular subcase.

As yet the transformations did not much change the order of derivatives. However passing to *partial differential equations*, already the classical Laplace series has quite other properties: *invertible* transformations of not too special equations (16) exist where the new unknown function $U(x, y)$ may depend on derivatives of arbitrarily high order of the primary unknown $u(x, y)$. This is a well-known result but we again pass to a *nonlinear* generalization: the *Laplace coframes* permit to determine all *invertible* mappings of a *given* equation (18) into the class of all equations (18), at least in principle since the calculations are rather complicated. The cases when the independent variables *need not* be preserved are involved. We can state only a modest illustrative example of the equation $\partial^2 u/\partial x \partial y = g(\partial u/\partial x) + u$ here with new independent variables $X = x - g'(\partial u/\partial x), Y = y$ and new unknown function (23). The method can be generalized and applied to higher order equations, as well.

1 The dispersion theory [1], [7]

We find ourselves in the plane x, y, where new variables $X = X(x, y), Y = Y(x, y)$ can be introduced. In particular transformations of the kind

$$X = X(x), \quad Y = c|X'(x)|^{1/2}\, y \qquad (c = \text{const.} \neq 0,\ \mathrm{X}'(\mathrm{x}) \neq 0) \tag{1}$$

are the most general ones which preserve the family of all equations

$$d^2y/dx^2 = q(x)y, \tag{2}$$

i.e., which turn every equation (2) into certain $d^2Y/dX^2 = Q(X)Y$. It may be proved that under transformations (1), equations (2) are *locally* like each other. Roots of solutions y are obviously transformed into roots of solutions Y and this trivial remark can be developed to give the *global* theory. In particular, in the oscillatory subcase, there exist automorphisms (1) of equation (2) permuting the roots of solutions y, the so called *central dispersions* of (1). Since the transformations (1) between two mentioned equations can be determined as solutions of a certain nonlinear third order differential equation (depending on q,Q) for the function X, it follows in the particular case of automorphisms that the distribution of roots of solutions y is governed by a third order differential equation.

2 A note to proofs [3]

The shortest way to the mentioned results consists in introduction of function $\zeta = \bar{y}/y$, where $\bar{y}, y$ are two independent solutions of (1). The value ∞ at the roots of y with obvious rules of calculations should be admitted. Then

$$\zeta' = c/y^2,\ y = (c/\zeta')^1/2,\ q = y''/y = |\zeta'|^{1/2}\left(|\zeta'|^{-1/2}\right)'', \tag{3}$$

where $c=$ const. $\neq 0$ and the last expression is the familiar *Schwarz derivative* independent of the choice of $\bar{y}, y$. Conversely, every such a function $\zeta(x)$ with $\zeta'(x) \neq 0$ can be arbitrarily chosen in advance. Then the equation (2) is determined by (3_3), automorphisms (1) of (2) are (obviously) given by formula

$$\zeta\left(X(x)\right) = \frac{\alpha\zeta(x)+\beta}{\gamma\zeta(x)+\delta} \qquad (\alpha, \beta, \gamma, \delta \quad \text{constants with} \quad \alpha\delta \neq \beta\gamma)$$

which provides a third order equation for the function $X(x)$ by applying the Schwarz derivative, and the central dispersions appear as a particular subcase $\zeta\left(X(x)\right) = \zeta(x)$. Continuing in this way, analogous function $Z = \bar{Y}/Y$ and the formula

$$Z(X) = \frac{\alpha\zeta(x)+\beta}{\gamma\zeta(x)+\delta} \qquad (\alpha, \beta, \gamma, \delta \quad \text{constants with} \quad \alpha\delta \neq \beta\gamma) \tag{4}$$

(obviously) provides all transformation into the equation $d^2Y/dX^2 = Q(X)Y$.

3 Nonlinear dispersions

The above results can be carried over the broader family of all equations

$$d^2y/dx^2 = q(x)y + r(x) \tag{5}$$

subjected to the transformations of the kind

$$X = X(x),\, Y = c|X'(x)|^{1/2}\, y + Z(x) \quad (c = \text{const.} \neq 0, X'(x) \neq 0). \tag{6}$$

The previous role of the roots of solutions is undertaken by the points of intersection of pairs of solutions in this non-homogeneous case. We shall be however interested in still broader family of all nonlinear equations

$$d^2y/dx^2 = f(x, y, dy/dx). \tag{7}$$

It may be easily seen that *contact transformations*

$$X = X(x, y, y'), \quad Y = Y(x, y, y'), \quad Y' = Y'(x, y, y') \tag{8}$$

are the most general ones which preserve the family (7), that is, which turn every equation (6) into certain $d^2Y/dX^2 = F(X, Y, dY/dX)$. (Indeed, owing to (8), differential form $dY - Y'dX$ should be a linear combination of forms $dy - y'dx$ and $dy' - f dx$ with *arbitrary* f, hence a multiple of $dy - y'dx$.) It may proved that *under contact transformations, equations* (7) *are locally like each other.* Instead of common methods, we shall derive this well-known result by a geometrical reasoning which will be subsequently related to (nonlinear) dispersions.

Let $y = y(x, a, b)$ be a complete solution of (7). Keeping a, b fixed for a moment, choose a near solution with a common point $\bar{x}, \bar{y}$. In other terms, we suppose

$$\bar{y} = y(\bar{x}, a, b) = y(\bar{x}, a + \varepsilon, b + \delta), \quad \bar{y}' = y_x(\bar{x}, a, b). \tag{9}$$

Analogously let $Y = Y(X, A, B)$ be a complete solution of the equation $d^2Y/dX^2 = F$. Choose fixed $A = A(a, b), B = B(a, b)$ such that there exists a common point $\bar{X}, \bar{Y}$ with the corresponding near solution, that means, we may write

$$\begin{aligned} &\bar{Y} = Y(\bar{X}, A, B) = Y(\bar{X}, A(a + \varepsilon, b + \delta), B(a + \varepsilon, b + \delta)), \\ &\bar{Y}' = YX(\bar{X}, A, B). \end{aligned} \tag{10}$$

Keeping ε, δ fixed but a, b (hence $\bar{x}, \bar{y}$) variable, the invertible transformation $(\bar{x}, \bar{y}, \bar{y}') \longrightarrow (\bar{X}, \bar{Y}, \bar{Y}')$ appears. If $\varepsilon, \delta = \delta(\varepsilon) \longrightarrow 0$, we obtain even a contact transformation (as follows by simple geometrical arguments or by direct verification) implicitly given by formulae

$$\begin{aligned} \bar{y} = y(\bar{x}, a, b), \quad \bar{y}' = y_x(\bar{x}, a, b), \quad \bar{Y} = Y(\bar{X}, A, B), \quad \bar{Y}' = YX(\bar{X}, A, B), \\ y_a(\bar{x}, a, b) + \lambda y_b(\bar{x}, a, b) = 0 = Y_A(\bar{X}, A, B)(A_a + \lambda A_b) \\ + Y_B(\bar{X}, A, B)(B_a + \lambda B_b), \end{aligned}$$

where $A = A(a, b), B = B(a, b), \lambda = \delta'(0)$ and the parameters a, b, λ should be eliminated. Since every curve $y = y(x, a, b)$ is (obviously) transformed into the curve $Y = Y(X, A, B)$, the equation (7) turns into $d^2Y/dX^2 = F$.

We shall mention two particular kinds of this construction.

Assuming $f(x, y, y') = F(x, y, y')$, we deal with automorphisms of equation (7). Since the functions $A = A(a, b), B = B(a, b)$ can be (in principle) quite arbitrarily chosen, there is a huge family of them. In the case of oscillatory equation, the simple choice $A = a$ and $B = b$ gives (besides the identity) the automorphisms permuting the conjugated points: the common point $\bar{x}, \bar{y}$ of two infinitesimally near solutions (cf. (9_1) with ε, δ near to zero) can be transformed into the next intersection point $\bar{X}, \bar{Y}$ of the same pair of solutions (cf. (10_1) with $Y = y, A = a, B = b$). So we have a nonlinear generalization of dispersions.

Assuming $f(x, y, y') = q(x)y, F(X, Y, Y') = Q(X)Y$, we deal with the equation (2) and the above construction gives (besides the contact transformations) the point transformations (1) for a particular choice of functions $A = A(a, b)$, $B = B(a, b)$. In more detail, let

$$y = a\bar{y}(x) + by(x), \quad Y = A\bar{Y}(X) + BY(X)$$

be complete solutions in our linear case of equations. For our point transformation $(\bar{x}, \bar{y}, \bar{y}') \longrightarrow (\bar{X}, \bar{Y}, \bar{Y}')$, the couple $(\bar{X}, \bar{Y})$ should depend only on $(\bar{x}, \bar{y})$ and not on $\bar{y}'$. Recall formulae (9_1, 10_1) in our particular case:

$$\bar{y} = ay(\bar{x} + b\bar{y}(\bar{x}), \quad \bar{Y} = AY(\bar{X}) + B\bar{Y}(\bar{X}). \tag{11}$$

Choosing arbitrary $A = \alpha a - \beta, B = \gamma a - \delta b$, where $\alpha\delta \neq \beta\gamma$, and assuming $\bar{y} = \bar{Y} = 0$ for a moment, it follows $\zeta(\bar{x}) = -a/b$ hence

$$Z(\bar{X}) = -\frac{A}{B} = -\frac{\alpha a - \beta b}{\gamma a - \delta b} = \frac{\alpha\zeta(\bar{x}) + \beta}{\gamma\zeta(\bar{x}) + \delta}$$

by using notation of Section 2. However this is just formula (4) and we already know that such functions $\bar{X} = \bar{X}(\bar{x})$ completed by $\bar{Y} = c|\bar{X}'(\bar{x})|^{1/2}\bar{y}$ provide transformations into $d^2Y/dX^2 = Q(X)Y$.

4 The moving frames method [2]

Our aim is to determine some kinds of the second order differential equations (7) which are preserved under the family of all transformations (6).

For better clarity, we shall deal with the pseudogroup of all transformations (6), where $X'(x) > 0$. Then

$$dX = u^2 dx, \quad dY = vdx + cudy \quad (u^2 = X',\, v = cX''y/2u + Z')$$

and it follows (from group composition properties) that two families of forms

$$\omega_1 = u^2 dx, \quad \omega = vdx + cudy \quad (u, v\,\text{are parameters}) \tag{12}$$

are preserved by mappings (6). One can verify that the converse is also true: transformations (8) preserving families (12) are just of the kind (6). On the other hand, the system

$$dy - y'dx = dy' - fdx = 0 \quad \text{turns into} \quad dY - Y'dX = dY' - FdX = 0$$

and it follows that two families of forms

$$\bar{\omega} = \lambda(dy - y'dx), \quad \bar{\bar{\omega}} = \mu(dy' - fdx) + \bar{\nu}(dy - y'dx),$$

where $\lambda, \mu, \bar{\nu}$ are new variables make the intrinsical sense: they are transformed into the relevant "capital families". Comparing ω with $\bar{\omega}$ (hence $cu = \lambda, v = -\lambda y'$), we obtain better intrinsical family of forms

$$\omega_2 = cu(dy - y'dx),\, \omega_3 = \mu(dy' - fdx) + \nu\omega_2,$$

where $u, \mu, \nu = \bar{\nu}/cu$ are new variables (and c is an unknown constant). So we occur ourselves in the space x, y, y', u, μ, ν, equipped with intrinsical families of forms $\omega_1, \omega_2, \omega_3$.

Exterior derivatives are intrinsical, too. However $d\omega_1 = 2\omega_4 \wedge \omega_1$ with the most general factor

$$\omega_4 = \frac{du}{u} - \xi\omega_1 \qquad (\xi \text{ a new parameter})$$

which provide still an intrinsical family. Analogously

$$d\omega_2 = \frac{du}{u} \wedge \omega_2 - c\frac{dy'}{u} \wedge \omega_1 = \omega_4 \wedge \omega_1 + (\xi - \frac{c\nu}{u\mu})\omega_1 \wedge \omega_2 + \frac{c}{u\mu}\omega_1 \wedge \omega_3$$

and we may introduce intrinsical restrictions $c/u\mu = 1, \xi = c\nu/u\mu$ hence $u\mu = c$, $\xi = \nu$ (then $d\omega_2 = \omega_1 \wedge (\omega_3 - \omega_4)$). In the same manner

$$d\omega_3 = \omega_5 \wedge \omega_2 + \frac{\partial f/\partial y'}{u^2}\,\omega_1 \wedge \omega_3 - \omega_4 \wedge \omega_3, \tag{13}$$

where

$$\omega_5 = d\nu + \beta\omega_1 + \gamma\omega_2 + 2\nu\omega_4 \qquad \left(\beta = \nu^2 + \frac{\mu}{cu^3}\frac{\partial f}{\partial y} - \frac{\nu}{u^2}\frac{\partial f}{\partial y'}\right)$$

is intrinsical family with a new variable γ. However

$$d\omega_4 = -(d\xi + 2\xi\omega_4) \wedge \omega_1 = (\gamma\omega_2 - \omega_5) \wedge \omega_1$$

owing to $\xi = \nu$, and we may suppose $\gamma = 0$. Returning to (12), we have to distinguish two subcases $\mathbb{A} : \partial f/\partial y' = 0,\ \mathbb{B} = \partial f/\partial y' \neq 0$.

It follows that the family of all equations $d^2y/dx^2 = f(x, y)$ is preserved by transformations (6), and one can directly verify that other transformations do not have such property. Assuming $\mathbb{A}$, then

$$d\omega_5 = d\beta \wedge \omega_1 + 2\beta\omega_4 \wedge \omega_1 + 2(d\nu \wedge \omega_4 - \nu\omega_5 \wedge \omega_1) = 2\omega_5 \wedge \omega_4 + \zeta \wedge \omega_1,$$

where $\zeta \cong d\beta + 4\beta\omega_4 - 2\nu\omega_5 \pmod{\omega_1}$ is intrinsical form. However $\beta = \nu^2 + fy/u^4$ (use $u\mu = c$) therefore $\zeta \cong f_{yy}\omega_2/cu^5$ after short calculation and we have to distinguish two subcases $\mathbb{C} : f_{yy} = 0, \mathbb{D} : f_{yy} \neq 0$ of our case $\mathbb{A}$.

Subcase $\mathbb{B}$ is the classical one $f = q(x)y + r(x)$ mentioned above. Surveying the results, we have structural formulae

$$d\omega_1 = 2\omega_4 \wedge \omega_1,\ d\omega_2 - \omega_1 \wedge (\omega_3 - \omega_4)$$

$$d\omega_3 = \omega_5 \wedge \omega_2 - \omega_4 \wedge \omega_3,\ d\omega_4 = \omega_1 \wedge \omega_5,\ d\omega_5 = 2\omega_5 \wedge \omega_4$$

of a Lie group of automorphisms of an equation (5) and, since invariants are lacking, all equation (5) are (locally) like each other with respect to transformations (6) which is the already mentioned result.

In subcase $\mathbb{D}$, we may introduce the requirement $cu^5 = \partial^2 f/\partial y^2$ which implies $5cu^4 du = f_{yyx}dx + f_{yyy}dy$ or, in terms of intrinsical forms,

$$5cu^5(\omega_4 + \nu\omega_1) = \frac{\partial^3 f}{\partial x \partial y^2}\,\frac{\omega_1}{u^2} + \frac{\partial^3 f}{\partial y^3}\left(\frac{\omega_2}{cu} + y'\frac{\omega_1}{u^2}\right).$$

It follows $5\omega^4 = M\omega_1 + N\omega_2$ with intrinsical coefficients. In particular

$$N = \frac{\partial^3 f}{\partial y^3}\Big/ cu\,\frac{\partial^2 f}{\partial y^2} = \frac{\partial^3 f}{\partial y^3}\Big/ c^{4/5}\left(\frac{\partial^2 f}{\partial y^2}\right)^{6/5}$$

does not change after transformations (6). The constant c is not fixed here. Consequently *if* $\mathcal{N}$ *is a conical* (containing constant multiples)*set of functions* $g(x,y)$*, then the family of all equations* $d^2y/dx^2 = f(x,y)$ *such that* $N \in \mathcal{N}$ *(equivalently:* $(f_{yyy})^5/(f_{yy})^6 \in \mathcal{N}$*) is preserved when transformations* (6) *are applied.* Possibly some narrower families could be obtained by using coefficient M but we shall not continue further.

Let us conclude with the remaining subcase $\mathbb{B}$. Owing to (13), we may assume $u^2 = \partial f/\partial y'$ and, analogously as above, one can obtain an identity of the kind $\omega_4 = M\omega_1 + N\omega_2 + P\omega_3$ with intrinsical coefficients. We shall mention only the simplest one

$$P = \frac{\partial^2 f}{\partial y'^2}\Big/ 2\mu u^2 = \frac{\partial^2 f}{\partial y'^2}\Big/ 2c\left(\frac{\partial f}{\partial y'}\right)^{1/2},$$

which yields the following result: *if* $\mathcal{P}$ *is a conical set of functions* $g(x,y,y')$ *then the family of all equations* $d^2y/dx^2 = f(x,y,y')$ *such that* $P \in \mathcal{P}$ (equivalently: $f^2_{y'y'}/f_{y'} \in \mathcal{P}$) *is preserved when transformations* (6) *are applied.*

5 On the Darboux transformation [4]

There exist many mappings (8) of the space x, y, y' which transform a *given* (single) equation (2) into an equation $d^2Y/dX^2 = Q(X)Y$. For the sake of brevity, we shall mention only the particular case $q(x) = 0$ and the x-preserving mappings (hence $X = x$) with $Y = Y(x,y,y')$ arbitrary. One can then find $Y' = \partial Y/\partial x + y'\partial Y/\partial y$ and the requirement

$$Q(x)Y = \partial^2 Y/\partial x^2 + 2y'\partial^2 Y/\partial x\partial y + y'^2\partial^2 Y/\partial y^2 \tag{14}$$

for the function $Y = Y(x,y,y')$. Denoting by $Y = \phi(x), Y = \psi(x)$ two linearly independent solutions of equation $d^2Y/dX^2 = Q(X)Y$, then (14) is satisfied if

$$Y(x,y,y') = \alpha(y - y'x, y')\phi(x) + \beta(y - y'x, y')\psi(x),$$

where α, β are arbitrary functions.

On the other hand, let us suppose the conjecture $Y = A(x)y + B(x)y'$ which leads to useful particular results. Then (14) gives $Q(Ay + By') = A''y + B''y' + 2y'A'$ and if follows $QA = A'', QB = B'' + 2A'$ whence

$$A''B = A(B'' + 2A') \tag{15}$$

by elimination of function Q. Requirement (15) can be explicitly resolved. In particular, for the choice $A = 1$, one obtains the famous transformation of equation $y'' = 0$ into the nontrivial equations with $Q(x) = \cos^{-2} x$ or $Q(x) = \cosh^{-2} x$.

The general case of the equation(2) can be investigated by the same manner, of course. Then the above conjecture leads to slight generalizations of the familiar Darboux transformation.

6 The Laplace series [5], [6]

Turning to partial differential equations, we begin with the general linear hyperbolical equation

$$\frac{\partial^2 u}{\partial x \partial y} = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u \tag{16}$$

to transparently illustrate the most important distinctive feature, the possibility of higher order invertible transformations. One can verify that the function $U = u_y - au$ satisfies a certain equation $U_{xy} = AU_x + BU_y + CU$ of the same kind as (16), and the iteration provides an infinite series of higher order transformations in the family of equations of the kind (16). Moreover, if $a_x + ab \neq c$ then analogous substitution with variables x, y exchanged yields the inversion. (The exceptional case $a_x + ab = c$ is much easier and may be omitted: then (16) can be replaced by certain first order linear equations.) So we obtain an infinite in both direction series of invertible substitutions in the general case, the *Laplace series*. (The equation (16) moreover admits a change $X = X(x), Y = Y(y)$ of independent variables and a linear change of function u; these are however well-known adaptations. In general, together with the Laplace series, invertible transformations do not exist, see below.)

The existence of higher order substitutions is possible thanks to the fact that equation (16) is considered in the infinite-dimensional space with coordinates

$$x, y, u, u_r \equiv \partial^r u/\partial x^r,\ u^s \equiv \partial^s u/\partial y^s \quad (r, s > 0). \tag{17}$$

Other derivatives $u_r^s \equiv \partial^{r+s}u/\partial x^r \partial y^s \quad (r, s > 0)$ can be expressed in terms of them by virtue of the equation (16) and its derivatives.

7 The Laplace coframe

Passing to the nonlinear case, we shall mention a hyperbolical equation

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u, x, y\right) \tag{18}$$

again in the space of variables (17). Since then the *contact transformations* can be applied, it follows that the previous prominent role of variables x, y, u lost the sense: it is better to employ the *contact form* $\omega = du - u_1 dx - u^1 dy$ and in general the *higher order contact forms* should replace the functions u_r^s. Quite analogously the *characteristic vector* fields

$$Z_+ = a_+\partial + b_+\delta,\ Z_- = a_-\partial + b_-\delta,$$

where $a_+, b_+ and a_-, b_-$ are (real and distinct) roots of the equation $a^2\partial f/\partial u^2 + ab + b^2\partial f/\partial u_2 = 0$, and

$$\partial = \partial/\partial x + \sum^{\infty} u_{r+1}\,\partial/\partial u_r + \sum^{\infty} Y^{s-1} f\partial/\partial u^s,$$

$$\delta = \partial/\partial y + \sum X^{r-1}\, f\partial/\partial u_r + \sum^{\infty} u^{s+1}\partial/\partial u^s \tag{19}$$

are so called *formal derivatives* will replace the previous derivative operators $\partial/\partial x, \partial/\partial y$ in equation (16).

In full detail, (18) expressed in terms of contact forms means that

$$\omega_1^1 = \frac{\partial f}{\partial u_2}\omega_2 + \frac{\partial f}{\partial u^2}\omega^2 + \frac{\partial f}{\partial u_1}\omega_1 + \frac{\partial f}{\partial u^1}\omega^1 + \frac{\partial f}{\partial u}\omega \tag{20}$$

(we abbreviate $\omega_r^0 \equiv \omega_r$, $\omega_0^s \equiv \omega_s$) and using the Lie derivatives $\mathfrak{L}$ satisfying $\mathfrak{L}_\partial \omega_r^s \equiv \omega_{r+1}^s$, $\mathfrak{L}_\delta \omega_r^s \equiv \omega_r^{s+1}$, one can express the last identity in the manner

$$\mathfrak{L}_{Z_-}\,\mathfrak{L}_{Z_+}\omega = a\mathfrak{L}_{Z_-}\omega + b\,\mathfrak{L}_{Z_+}\omega + c\omega \tag{21}$$

quite analogous to (16). (We shall not state rather clumsy formulae for coefficients a, b, c in terms of the function f.) Then the procedure of Section 6 can be simulated in terms of contact forms: the form $\Omega = \mathfrak{L}_{Z_+}\omega - a\omega$ satisfies a certain identity

$$\mathfrak{L}_{Z_-}\mathfrak{L}_{Z_+}\Omega = A\mathfrak{L}_{Z_-}\Omega + B\mathfrak{L}_{Z_+}\Omega + C\Omega \tag{22}$$

and the procedure can be iterated. Moreover, if $Z_-a + ab \neq c$, analogous substitution with Z_+, Z_- exchanged yields the inversion. In general, one obtains an infinite in both direction series of certain differential forms which constitute a basis in the module of all contact forms, the *Laplace coframe.*

8 Applications

If the equation $\Omega = 0$ can be expressed by five functions, that means, the Pfaff-Darboux shape is $dU - PdX - QdY = 0$ for appropriate U, X, Y, P, Q, then the functions X, Y may be regarded for independent variables and U for new unknown satisfying a certain second order equation (as follows from (22)). The same conclusion can be made for any other term of the Laplace coframe and it may be proved that all possible invertible transformations into some second order equation arise only in this manner. Other applications as the Darboux method, representation of solution of an equation (18) by means solutions of another such equation, Bäcklund correspondences, are also possible.

9 Example

One can easily investigate the Laplace coframes for the linear equations (16) and verify that they give the well-known Laplace series of transformations. In general, Laplace coframes are rather complicated. Therefore we shall mention only few results concerning the particular equation $\partial^2 u/\partial x\partial y = g(\partial u/\partial x) + u$ for a transparent example. Identity (20) means that $\omega_1^1 = g'\omega_1 + \omega$ and (since $Z_- = \partial, Z_+ = \delta$ are formal derivatives in our case and therefore $a = g', b = 0, c = 1$ in identity (21) we have to introduce the form $\Omega = \omega^1 - g'\omega$. One can find the Pfaff-Darboux shape $\Omega = dU - PdX - QdY$ of this form, where $X = x - g'(u_1), Y - y$ may be taken for new independent variables and

$$U = u^1 - ug'(u_1) - G(u_1) \quad (G(u_1) = \int (g - u_1 g')g'' du_1) \tag{23}$$

for new unknown function. Moreover coefficients

$$P = \partial U/dX = g(u_1) + u - u_1 g'(u_1), \quad Q = \partial U/\partial Y = u^2 - u^1 g'(u_1) \tag{24}$$

may be identified with new partial derivatives.Then, looking at the differential dP, one can find (surprisingly simple) formulae

$$\partial^2 U/dX^2 = u_1, \ \partial^2 U/\partial X \partial Y = u^1 \tag{25}$$

which means that functions (23, 24, 25) are related by the equation

$$\frac{\partial^2 U}{\partial X \partial Y} = U + \left(\frac{\partial U}{\partial X} - g\left(\frac{\partial^2 U}{\partial X^2}\right) + \frac{\partial^2 U}{\partial X^2}\, g'\left(\frac{\partial^2 U}{\partial X^2}\right)\right) g'\left(\frac{\partial^2 U}{\partial X^2}\right) + G\left(\frac{\partial^2 U}{dX^2}\right).$$

Analogous transformation with the role of x, y exchanged leads to the form $\Omega = \omega_1$, the independent variables are preserved, and the new unknown function $U = u_1$ (obviously) satisfies $\partial^2 U/\partial x \partial y = g'(U)\partial U/\partial x + U$. Modulo contact transformations, these are the only possible first order invertible transformations of the equation under consideration.

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