Symmetry and the Chazy Equation

Peter A. Clarkson ‡
Department of Mathematics
University of Exeter
Exeter, EX4 4QE
U.K.

e-mail: `clarkson@maths.exeter.ac.uk`

Peter J. Olver †
School of Mathematics
University of Minnesota
Minneapolis, Minnesota
U.S.A. 55455

e-mail: `olver@ima.umn.edu`

Abstract. There are three different actions of the unimodular Lie group SL(2) on a two-dimensional space. In every case, we show how an ordinary differential equation admitting SL(2) as a symmetry group can be reduced in order by three, and the solution recovered from that of the reduced equation via a pair of quadratures and the solution to a linear second order equation. A particular example is the Chazy equation, whose general solution can be expressed as a ratio of two solutions to a hypergeometric equation. The reduction method leads to an alternative formula in terms of solutions to the Lamé equation, resulting in a surprising transformation between the Lamé and hypergeometric equations. Finally, we discuss the Painlevé analysis of the singularities of solutions to the Chazy equation.

1. Introduction.

The simplest of the equations introduced by Chazy, [**6**],[**7**],[**8**], takes the form

$$y_{xxx} = 2yy_{xx} - 3y_x^2. \tag{1.1}$$

It arises in the study of third order ordinary differential equations having the "Painlevé property" that the solutions have only poles for moveable singularities (see also [**3**],[**4**]). The Chazy equation is important since it is the simplest example of an ordinary differential equation whose solutions have a moveable natural boundary. By a natural boundary we

October 12, 1994

‡ *Supported in part by a Nuffield Science Fellowship and SERC Grant GR/H39420.*
† *Supported in part by NSF Grants DMS 91–16672 and DMS 92–04192.*

mean a closed curve in the complex plane beyond which the solution cannot be analytically continued. This phenomenon first arises in the case of third order equations. In the case of Chazy's equation, the boundary is a circle, and is moveable in the sense that its position depends on the initial data for the solution. Chazy demonstrated the existence of moveable boundaries for (1.1) by relating its solutions to those of the linear hypergeometric equation

$$t(1-t)\frac{d^2\chi}{dt^2}+\left(\frac{1}{2}-\frac{7}{6}t\right)\frac{d\chi}{dt}+\frac{1}{144}\chi=0. \tag{1.2}$$

Given any two independent solutions of this equation, $\phi(t)$ and $\psi(t)$, define the function $x(t)$ by $x(t)=\phi(t)/\psi(t)$. Then $x(t)$ maps the upper half t-plane into the interior of a spherical triangle with angles 0, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, *cf.* [**18**; p. 206]. The inverse function $t(x)$ is the Schwarzian triangle function $S(x;0,\frac{1}{2},\frac{1}{3})$, which has a straight line or a circle as a natural boundary and whose fundamental triangle also has angles 0, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$. The general solution of (1.1) is then given by

$$y(x)=\frac{6}{\psi}\frac{d\psi}{dx}=\frac{6}{\psi}\frac{d\psi}{dt}\frac{dt}{dx},$$

and has the same natural boundary as $S(x)$. Thus the general solution of (1.1) is single-valued in its domain of definition. (In fact, every solution is analytic either in a punctured plane, or in a domain bounded by a straight line or a circle, the location of which is dependent on the constants of integration.) Furthermore, the radius and center of this circle can be specified by the "initial conditions," *i.e.*, in terms of y, y_x and y_{xx} at some given point x_0, *cf.* [**1**].

The Chazy equation is deeply connected to special automorphic functions (elliptic modular functions) which arise in various branches of mathematics, in particular number theory. (See, for example, [**23**] for further details.) A Painlevé analysis demonstrates that the Chazy equation also possesses three "negative resonances" about which there is much current interest, *cf.* [**10**],[**11**], and §6 below.

In recent years the Chazy equation (1.1) has assumed added importance since it appears as a reduction of the self-dual Yang–Mills (SDYM) equations, [**5**]. Ward [**24**] conjectured that in some sense all soliton equations arise as special cases of the SDYM equations. Subsequently many of the well-known soliton equations, such as the Korteweg-de Vries, nonlinear Schrödinger, sine-Gordon, Kadomtsev-Petviashvili, Davey-Stewartson, and Painlevé equations, have been discovered to be exact or asymptotic reductions of the SDYM equations (*cf.* [**1**]). All these classical soliton equations arise when it is assumed that the Yang-Mills potentials take values in a finite-dimensional Lie algebra such as $\mathfrak{su}(2)$. By contrast, the Chazy equation arises when it is assumed that the Yang-Mills potentials take values in the infinite-dimensional Lie algebra $\mathfrak{sdiff}(\mathrm{SU}(2))$ of all "divergence-free" vector fields on SU(2), [**5**]; in fact, the Chazy equation is perhaps the simplest equation that arises from an infinite-dimensional Lie algebra. Therefore, the Chazy equation plays an important role in soliton theory and integrable systems.

We remark that the Chazy equation (1.1) also arises as a reduction of the stationary, incompressible Prandtl boundary layer equations (*cf.* [**22**])

$$\psi_{\eta\eta\eta}=\psi_\eta\psi_{\xi\eta}-\psi_\xi\psi_{\xi\eta}. \tag{1.3}$$

Using the classical Lie method of infinitesimal transformations (*cf.* [**19**]) we obtain the similarity reduction

$$\psi(\xi,\eta)=\xi^{\beta}y(x), \qquad x=\eta\xi^{\beta-1}+f(\xi),$$

where $f(\xi)$ is an arbitrary function. Substituting this into (1.3) yields

$$y_{xxx}=\beta y y_{xx}-(2\beta-1)y_x^2,$$

which, in the special case $\beta=2$, is the Chazy equation (1.1).

The Chazy equation (1.1) is known to admit a three-dimensional symmetry group of unimodular transformations, with infinitesimal generators

$$\partial_x, \qquad x\partial_x-y\partial_y, \qquad x^2\partial_x-(2xy+6)\partial_y. \tag{1.4}$$

However, its reduction using well-established symmetry methods, [**19**], is not as straightforward as one might suppose. The first two generators can be used to reduce (1.1) to a first order equation (*cf.* [**14**]). However the third generator does not reduce this resulting equation. In this paper we show that its symmetry group is the most complicated of the three known unimodular group actions on a two-dimensional complex manifold, as classified by Lie. We describe a simple connection between these three actions via the standard prolongation process, and use this to inter-relate their differential invariants. The known integration method for the basic unimodular action can then be applied to determine a solution method for the general SL(2)–invariant ordinary differential equation, including the Chazy equation as a special case. Finally, we show how the solutions to the Chazy equation can be constructed from that of the Lamé equation, which can be (surprisingly) related to the hypergeometric equation (1.2) via an elliptic change of variables.

2. Planar Actions of the Unimodular Group.

According to Lie's classification of Lie group actions, [**17**], there are precisely three inequivalent nonsingular local actions of the three-dimensional special linear or unimodular group $\mathcal{U}=\mathrm{SL}(2,\mathbb{C})$, with Lie algebra $\mathfrak{u}=\mathfrak{sl}(2,\mathbb{C})$, on any two-dimensional complex manifold. These are modelled by the Lie algebras spanned by the vector fields

$$\begin{aligned}&\{\partial_x, \quad x\partial_x, \quad x^2\partial_x\},\\ &\{\partial_x, \quad x\partial_x+u\partial_u, \quad x^2\partial_x+2xu\partial_u\},\\ &\{\partial_x+\partial_u, \quad x\partial_x+u\partial_u, \quad x^2\partial_x+u^2\partial_u\},\end{aligned} \tag{2.1}$$

where $\partial_x\equiv\partial/\partial x$ etc., on the space $M=\mathbb{C}^2$.

Remark: There are five inequivalent actions of three-dimensional simple Lie groups on a two-dimensional real manifold. There are four different actions of the real unimodular group $\mathrm{SL}(2,\mathbb{R})$, provided by the three listed in (2.1) (considered as real Lie algebras), and the additional real unimodular Lie algebra

$$\{\partial_x, \quad x\partial_x+u\partial_u, \quad (x^2-u^2)\partial_x+2xu\partial_u\}.$$

The fifth Lie algebra

$$\{u\partial_x - x\partial_u,\quad (1+x^2-u^2)\partial_x + 2xu\partial_u,\quad 2xu\partial_x + (1-x^2+u^2)\partial_u\},$$

defines the action of the rotation group SO(3) which is obtained from its natural action on the two-dimensional sphere via stereographic projection.

Whenever we have a Lie group G acting on a space $M \simeq X \times U$, where X represents the independent variables and U the dependent variables, there is an induced action on the associated jet bundles $\mathrm{J}^n = \mathrm{J}^n M$, which is called the n^{th} prolongation of G, and denoted $\mathrm{pr}^{(n)}G$. It is an interesting fact that the three actions of SL(2) are directly connected via prolongation. The first Lie algebra of vector fields

$$\mathfrak{u}^{(0)}: \qquad \partial_x, \qquad x\partial_x, \qquad x^2\partial_x, \tag{2.2}$$

generates the projective linear fractional or Möbius action

$$\mathcal{U}^{(0)}: \qquad (x,u) \longmapsto \left(\frac{\alpha x+\beta}{\gamma x+\delta}, u\right), \qquad \alpha\delta - \beta\gamma = 1, \tag{2.3}$$

of the unimodular group. The first prolongation of this group action is generated by the prolonged vector fields

$$\mathfrak{u}^{(1)}: \qquad \partial_x, \qquad x\partial_x - v\partial_v, \qquad x^2\partial_x - 2xv\partial_v, \tag{2.4}$$

where we use $v = u_x$ to indicate the derivative coordinate. These vector fields form a Lie algebra having the same $\mathfrak{sl}(2)$ commutation relations. The Lie algebra (2.4) clearly projects to the (x,v) plane, thereby defining a Lie algebra of vector fields equivalent to the second Lie algebra in our list (2.1); indeed, the explicit isomorphism is to replace v by $u = 1/v$. The corresponding group action is

$$\mathcal{U}^{(1)}: \qquad (x,v) \longmapsto \left(\frac{\alpha x+\beta}{\gamma x+\delta}, (\gamma x+\delta)^2 v\right). \tag{2.5}$$

Setting $q = u_{xx}$, the second prolongation of the original three vector fields yields

$$\partial_x, \qquad x\partial_x - v\partial_v - 2q\partial_q, \qquad x^2\partial_x - 2xv\partial_v - (4xq+2q)\partial_q,$$

again having the same commutation relations as the original three vector fields. Let

$$w = \frac{q}{2v} = \frac{v_x}{2v} = \frac{u_{xx}}{2u_x}. \tag{2.6}$$

(The factor of $\frac{1}{2}$ is merely for convenience.) Using (x,u,v,w) instead of (x,u,v,q) on the open subset of J^2 where $v \neq 0$, we see that the vector fields have the equivalent form

$$\partial_x, \qquad x\partial_x - v\partial_v - w\partial_w, \qquad x^2\partial_x - 2xv\partial_v - (2xw+1)\partial_w.$$

Again, we can project this action to the two-dimensional subspace coordinatized by x and w, on which the vector fields reduce to

$$\mathfrak{u}^{(2)}: \qquad \partial_x, \qquad x\partial_x - w\partial_w, \qquad x^2\partial_x - (2xw+1)\partial_w. \tag{2.7}$$

These vector fields span a Lie algebra isomorphic to the third one in the list (2.1); indeed, the transformation given by $u = x + 1/w$ provides the explicit isomorphism. The associated group action is

$$\mathcal{U}^{(2)}: \qquad (x,w) \longmapsto \left(\frac{\alpha x+\beta}{\gamma x+\delta}, (\gamma x+\delta)^2 w + \gamma(\gamma x+\delta) \right). \tag{2.8}$$

Thus, all three inequivalent actions of SL(2) on two-dimensional complex manifolds arise from a single source through the process of prolongation. This raises the interesting question of how general this phenomenon is. Are different (complex) actions of a given transformation group related by the processes of prolongation and projection?

3. Differential Invariants.

Recall first that a scalar-valued function $I(x, u^{(n)})$ depending on the independent and dependent variables and their derivatives, which is invariant under the prolonged group action $\mathrm{pr}^{(n)} G$, is known as a *differential invariant* of the group G. On the subset of the jet space where the prolonged group acts regularly, every invariant system of differential equations can be written in terms of the differential invariants of the group action. (The singular subset of the prolonged group action is also determined by a G-invariant differential equation, whose form is explicitly given in terms of the Lie determinant of G, [**17**]. For simplicity, though, we shall omit its analysis here.) A complete system of differential invariants is constructed through the use of invariant differential operators — see [**20**],[**21**], for the general theory. In the present case, since we are dealing with a single independent and a single dependent variable, we need just two functionally independent differential invariants $z = I(x, u^{(n)})$, $w = J(x, u^{(m)})$; every other differential invariant can be determined as a function of I, J and the differentiated invariants $d^k z/dw^k = (D_x^k J)/(D_x^k I)$. Lie, [**17**], computed the fundamental differential invariants for all of the complex transformation groups in the plane, including the previous three copies of SL(2). The construction of the required differential invariants is simplified by the prolongation connection between these three actions.

Case I: $\mathcal{U}^{(0)}$

For the first unimodular group action, the two fundamental differential invariants are

$$\mathcal{U}^{(0)}: \qquad u \qquad \text{and} \qquad s = \frac{u_x u_{xxx} - \frac{3}{2} u_{xx}^2}{u_x^4}. \tag{3.1}$$

We note that, if we interchange the roles of independent and dependent variables, then the second fundamental differential invariant in equation (3.1) becomes

$$s = -\frac{x_u x_{uuu} - \frac{3}{2} x_{uu}^2}{x_u^2}, \tag{3.2}$$

which coincides with the negative of the well-known projectively invariant Schwarzian derivative, of importance in conformal mapping and other applications in complex analysis, [**12**]. Any n^{th} order ordinary differential equation admitting $\mathcal{U}^{(0)}$ as a symmetry group can be written in the form

$$\frac{d^{n-3}s}{du^{n-3}} = H\left(u, s, \frac{ds}{du}, \ldots, \frac{d^{n-4}s}{du^{n-4}}\right). \tag{3.3}$$

Thus the equation reduces in order by 3; once we know the solution $s = F(u)$ to the reduced ordinary differential equation (3.3), we recover the solution to the original ordinary differential equation by solving the $\mathcal{U}^{(0)}$–invariant third order equation $s = F(u)$, or, explicitly,

$$u_x u_{xxx} - \tfrac{3}{2} u_{xx}^2 = u_x^4 F(u). \tag{3.4}$$

Even though it admits a three-parameter symmetry group, the equation (3.4) cannot be integrated explicitly by quadrature because the symmetry group is not solvable, *cf.* [**19**]. However, as is well known, we can reduce its integration to the solution of a Riccati equation followed by a pair of quadratures. Using the Lie reduction method associated with the two-dimensional solvable subgroup generated by ∂_x and $x\partial_x$, we set

$$z = \frac{u_{xx}}{u_x^2}, \qquad \text{in terms of which} \qquad s = \frac{dz}{du} + \tfrac{1}{2} z^2. \tag{3.5}$$

Then equation (3.4) reduces to the Riccati equation

$$\frac{dz}{du} + \tfrac{1}{2} z^2 = F(u). \tag{3.6}$$

Once we solve equation (3.6) for $z = z(u)$, we can recover the solution $u = f(x)$ to equation (3.4) by a pair of quadratures: first

$$u_x = g(u) = c \exp\left\{ \int^u z(\hat{u})\, d\hat{u} \right\}, \tag{3.7}$$

followed by

$$\int^u \frac{d\hat{u}}{g(\hat{u})} = x + k. \tag{3.8}$$

The Riccati equation (3.6) can, of course, be linearized. If we define $\psi = \sqrt{u_x}$, then

$$z = \frac{u_{xx}}{u_x^2} = 2\frac{\psi_x}{u_x \psi} = 2\frac{\psi_u}{\psi} = 2(\log \psi)_u.$$

Therefore $\psi = \psi(u)$ is a solution to the second order, homogeneous, linear Schrödinger equation

$$\frac{d^2\psi}{du^2} - \tfrac{1}{2} F(u)\psi = 0, \tag{3.9}$$

with potential $-\frac{1}{2}F(u)$. In this case, since $u_x = \psi^2$, we recover u by a single quadrature:

$$\int^u \frac{d\hat{u}}{\psi^2(\hat{u})} = x + k. \tag{3.10}$$

In fact, this form of the solution can be re-expressed directly in terms of the solutions of the linear Schrödinger equation. We recall that, according to the method of variation of parameters, [**13**; p. 122], if $\psi(u)$ is one solution to the linear ordinary differential equation (3.9), then a second, linearly independent solution, is given by

$$\varphi(u) = \psi(u) \int^u \frac{d\hat{u}}{\psi(\hat{u})^2}. \tag{3.11}$$

Comparing with (3.10), we conclude that the general solution to the $\mathcal{U}^{(0)}$–invariant equation (3.4) is given, parametrically, in the form

$$x = \frac{\varphi(u)}{\psi(u)}, \tag{3.12}$$

where $\varphi(u)$ and $\psi(u)$ are two arbitrary linearly independent solutions to the linear Schrödinger equation (3.9). Here, we have absorbed the integration constant k in (3.10) by replacing $\varphi(u)$ by $\varphi + k\psi$. If $\psi_1(u)$, $\psi_2(u)$ denote a basis for the solution space to (3.9), then $\varphi = a\psi_1 + b\psi_2$, $\psi = c\psi_1 + d\psi_2$, where $ad - bc \neq 0$. Moreover, since we can always cancel a common factor in the ratio (3.12), we may, without loss of generality, assume that $ad - bc = 1$. Therefore formula (3.12) does depend on three arbitrary constants, and does represent the general solution to equation (3.4).

Finally, we remark that, in view of the identification of the differential invariant s with the Schwarzian derivative (3.2), our symmetry reduction of (3.4) also provides a direct proof of a classical theorem due to Schwarz — see [**12**; Theorem 10.1.1].

Theorem 3.1. *The general solution to the Schwarzian equation*

$$\frac{x_u x_{uuu} - \frac{3}{2}x_{uu}^2}{x_u^2} = -F(u). \tag{3.13}$$

has the form

$$x = \frac{\varphi(u)}{\psi(u)}, \tag{3.14}$$

where $\varphi(u)$ and $\psi(u)$ form two linearly independent, but otherwise arbitrary, solutions to the linear Schrödinger equation (3.9).

Case II: $\mathcal{U}^{(1)}$

For the second unimodular group action, by the prolongation connection, any differential invariant of $\mathcal{U}^{(0)}$ will be a differential invariant of $\mathcal{U}^{(1)}$, *provided* it does not explicitly depend on u. (Differential invariants depending on u provide "non-local" invariants of $\mathcal{U}^{(1)}$.) Thus the fundamental differential invariants of $\mathcal{U}^{(1)}$ are the Schwarzian invariant

$$s = \frac{v v_{xx} - \frac{3}{2} v_x^2}{v^4} \tag{3.15}$$

(rewritten in terms of $v = u_x$) and its derivative

$$r = \frac{ds}{du} = \frac{v^2 v_{xxx} - 6 v v_x v_{xx} + 6 v_x^3}{v^6}. \tag{3.16}$$

Any n^{th} order ordinary differential equation admitting $\mathcal{U}^{(1)}$ as a symmetry group can be written in the form

$$\frac{d^{n-3} r}{ds^{n-3}} = H\left(s, r, \frac{dr}{ds}, \ldots, \frac{d^{n-4} r}{ds^{n-4}}\right). \tag{3.17}$$

Once we know the solution $r = G(s)$ to the reduced ordinary differential equation (3.17), we recover the solution to the original ordinary differential equation by solving the $\mathcal{U}^{(1)}$–invariant third order equation $r = G(s)$, or, explicitly,

$$v^2 v_{xxx} - 6 v v_x v_{xx} + 6 v_x^3 = v^6 G\left(\frac{v v_{xx} - \frac{3}{2} v_x^2}{v^4}\right). \tag{3.18}$$

Recalling that $r = ds/du$, we can integrate equation (3.18) once to yield

$$H(s) = \int^s \frac{d\hat{s}}{G(\hat{s})} = u + k, \tag{3.19}$$

which, once solved for $s = F(u)$, reduces to the $\mathcal{U}^{(0)}$–invariant third order equation (3.4) — an equation we know how to solve in terms of a pair of quadratures and a Riccati equation. Thus the original third order equation (3.18) can be integrated using a Riccati equation and three quadratures. Alternatively, the solution (3.12) based on the linear Schrödinger equation (3.9) can also be effectively employed in this case. We must compute

$$v = u_x = \frac{1}{x_u} = \frac{\psi(u)^2}{\omega}, \tag{3.20}$$

where

$$\omega = \varphi \psi_u - \varphi_u \psi \tag{3.21}$$

denotes the Wronskian of the two solutions $\varphi(u)$ and $\psi(u)$ to (3.9), which, by Abel's formula, is constant. Therefore, the general solution to the $\mathcal{U}^{(1)}$–invariant equation (3.18) can be written in parametric form as

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad v = \frac{\psi(u)^2}{\omega}. \tag{3.22}$$

It is worth remarking, however, that we can avoid the introduction of an auxiliary variable u by a more direct reduction of (3.18). According to equation (3.5), if we set $z = v_x / v^2$, then

$$\frac{dz}{ds} = \frac{z_u}{s_u} = \frac{s - \frac{1}{2} z^2}{r} = \frac{s - \frac{1}{2} z^2}{G(s)}, \tag{3.23}$$

which is a Riccati equation for z as a function of s. Once we solve (3.23) for $z = A(s)$, and invert to find $s = B(z)$, then we can find how v depends on x by solving the second order equation

$$v v_{xx} - \tfrac{3}{2} v_x^2 = v^4 B\left(v^{-2} v_x\right). \tag{3.24}$$

The fact that equation (3.24) is invariant under the two parameter group consisting of translations in x and scalings $(x, v) \mapsto (\lambda x, \lambda^{-1} v)$ implies that we can recover $v = g(x)$ by two quadratures. We find

$$v \frac{dz}{dv} = \frac{s - \frac{1}{2} z^2}{z} = \frac{B(z)}{z} - \frac{z}{2},$$

giving

$$v = C(z) = c \exp\left\{ \int^z \frac{\hat{z}\, d\hat{z}}{B(\hat{z}) - \frac{1}{2}\hat{z}^2} \right\}.$$

Inverting the latter equation to find $z = D(v)$, we use the definition of z to recover

$$E(v) = \int^v \frac{d\hat{v}}{\hat{v}^2 D(\hat{v})} = x + k. \tag{3.25}$$

The alternative Riccati equation (3.23) is linearized by setting $z = 2G(s)(\log \psi)_s$, in terms of which

$$G(s)^2 \frac{d^2\psi}{ds^2} + G'(s) G(s) \frac{d\psi}{ds} - \frac{s}{2}\psi = 0. \tag{3.26}$$

Thus we have reduced the original equation to a seemingly different linear second order equation. In fact, the two linear equations (3.9) and (3.26) are the same equation for the same dependent variable ψ, but written in terms of different independent variables. Indeed, one transforms from one to the other by a change of independent variable $s = F(u)$; according to equation (3.18), we have $s_u = F'(u) = G(s)$, and $s_{uu} = F''(u) = G'(s)G(s)$, which proves the isomorphism between the two linear equations (3.9), (3.26).

Case III: $\mathcal{U}^{(2)}$

As for the third unimodular group action, by the prolongation connection, any differential invariant of $\mathcal{U}^{(1)}$ will be a differential invariant of $\mathcal{U}^{(2)}$, *provided*, when written in terms of x, u, $v = u_x$, $w = v_x/(2v)$, it does not explicitly depend on v. Note first that

$$s = 2\,\frac{w_x - w^2}{v^2}, \qquad r = \frac{ds}{du} = 2\,\frac{w_{xx} - 6 w w_x + 4 w^3}{v^3}.$$

Thus the fundamental differential invariants of $\mathcal{U}^{(2)}$ are

$$\begin{aligned} t &= \sqrt{2}\,\frac{r}{s^{3/2}} = \frac{w_{xx} - 6ww_x + 4w^3}{(w_x - w^2)^{3/2}}, \\ y &= \frac{2}{s^2}\,\frac{dr}{du} = \frac{w_{xxx} - 12ww_{xx} - 6w_x^2 + 48w^2w_x - 24w^4}{(w_x - w^2)^2}. \end{aligned} \tag{3.27}$$

We note that we can replace the second invariant by a slightly simpler differential invariant

$$y + 24 = \frac{2}{s^2}\,\frac{dr}{du} + 24 = \frac{w_{xxx} - 12ww_{xx} + 18w_x^2}{(w_x - w^2)^2}. \tag{3.28}$$

Any n^{th} order ordinary differential equation admitting $\mathcal{U}^{(2)}$ as a symmetry group can be written in the form

$$\frac{d^{n-3}y}{dt^{n-3}} = H\left(t, y, \frac{dy}{dt}, \dots, \frac{d^{n-4}y}{dt^{n-4}}\right). \tag{3.29}$$

Once we know the solution $y = K(t)$ to the reduced ordinary differential equation (3.29), we recover the solution to the original ordinary differential equation by solving the $\mathcal{U}^{(2)}$–invariant third order equation

$$\frac{dr}{du} = s^2 K\left(\frac{r}{s^{3/2}}\right), \tag{3.30}$$

or, in full detail,

$$w_{xxx} - 12ww_{xx} + 18w_x^2 = (w_x - w^2)^2\,\widehat{K}\left(\frac{w_{xx} - 6ww_x + 4w^3}{(w_x - w^2)^{3/2}}\right), \tag{3.31}$$

where

$$\widehat{K}(t) = 2K\left(\frac{t}{\sqrt{2}}\right) + 24.$$

We can rewrite equation (3.30) in the form

$$\frac{r}{s^2}\,\frac{dr}{ds} = K\left(\frac{r}{s^{3/2}}\right). \tag{3.32}$$

Equation (3.32) can be integrated once as a consequence of its invariance under the one-parameter scaling group $(s, r) \mapsto (\lambda s, \lambda^{3/2} r)$. Setting $t = r/s^{3/2}$, we find that (3.32) becomes

$$s\,\frac{dt}{ds} = \frac{K(t)}{t} - \frac{3t}{2}, \tag{3.33}$$

hence $t = M(s)$ is found by quadrature:

$$s = L(t) = c\exp\left\{\int^t \frac{2\hat{t}\,d\hat{t}}{2K(\hat{t}) - 3\hat{t}^2}\right\}. \tag{3.34}$$

Inverting, this leads to the $\mathcal{U}^{(1)}$–invariant equation

$$r = s^{3/2} M(s),$$

cf. equation (3.18), and so can be solved by reducing to a Riccati equation. Thus the solution to the original $\mathcal{U}^{(2)}$–invariant equation (3.31) can be found by solving an associated Riccati equation together with three or four quadratures. Alternatively, if we use the solution (3.12) based on the linear Schrödinger equation (3.9), then, using (3.20),

$$w = \frac{v_x}{2v} = \frac{\psi_x}{\psi} = \frac{\psi \psi_u}{\omega}. \tag{3.35}$$

Therefore, the general solution to the $\mathcal{U}^{(2)}$–invariant equation (3.30) can be expressed in the parametric form

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad w = \frac{\psi_x}{\psi} = \frac{\psi \psi_u}{\omega}, \tag{3.36}$$

where $\varphi(u)$ and $\psi(u)$ form two solutions of the second order linear equation (3.9), and ω is their Wronskian, *cf.* (3.21).

This completes our analysis of the reduction of (complex) ordinary differential equations admitting a unimodular group of symmetries. We hav shown that, in every case, an n^{th} order equation invariant under SL(2) can be reduced in order by 3. Moreover, the solutions to the original equation can, in all cases, be recovered from those of the reduced equation via quadrature and the solution to a Riccati equation, or, equivalently, a linear second order ordinary differential equation.

4. The Chazy Equation.

In his study of third order ordinary differential equations having the Painlevé property, Chazy, [**8**], was led to the remarkable family of equations

$$y_{xxx} = 2 y y_{xx} - 3 y_x^2 + \alpha (6 y_x - y^2)^2. \tag{4.1}$$

Chazy showed that when

$$\alpha = 0, \qquad \text{or} \qquad \alpha = \frac{4}{36 - k^2}, \quad \text{where} \quad 6 < k \in \mathbb{N}, \tag{4.2}$$

then the nontrivial solutions $y = f(x)$ to (4.1) have a moveable circular natural boundary. This is a consequence of the following theorem.

Theorem 4.1. *Suppose $\varphi(t)$ and $\psi(t)$ are two arbitrary linearly independent solutions of the hypergeometric equation*

$$t(1-t)\,\frac{d^2\chi}{dt^2} + \left(\frac{1}{2} - \frac{7}{6}t\right)\frac{d\chi}{dt} - \sigma\chi = 0. \tag{4.3}$$

Then

$$x = \frac{\varphi(t)}{\psi(t)}, \qquad y = \frac{6}{\psi(t)}\frac{d\psi}{dx} = \frac{6\psi(t)}{\omega}\frac{d\psi}{dt}, \tag{4.4}$$

where $\omega = \psi\varphi_t - \varphi\psi_t$, *parametrizes the general solution* $y = f(x)$ *to the Chazy equation* (4.1) *with parameter value*

$$\sigma = \frac{1}{144(1-9\alpha)}. \tag{4.5}$$

In particular, the cases

$$\sigma = \frac{1}{4}\left(\frac{1}{36} - \frac{1}{k^2}\right), \tag{4.6}$$

correspond to Chazy's preferred values (4.2) for his third order equations. Interestingly, (4.3) arises in Schwarz's theory of algebraic hypergeometric functions; in fact, for $k = 2$, 3, 4, and 5, the parameter values (4.6) constitute four types of hypergeometric equations all of whose solutions are algebraic functions — they correspond to the dihedral triangle, tetrahedral, octahedral and icosahedral symmetry classes. See Hille, [**12**; §10.3], for the details of Schwarz's theory.

Chazy noted that equation (4.1) admits a unimodular symmetry group, with infinitesimal generators

$$\partial_x, \qquad x\partial_x - y\partial_y, \qquad x^2\partial_x - (2xy+6)\partial_y. \tag{4.7}$$

(See also [**14**].) This result can be verified readily using the standard Lie infinitesimal method for computing symmetry groups of differential equations, [**19**]. The Lie algebra (4.7) is mapped to the Lie algebra $\mathfrak{u}^{(2)}$, given in (2.7), by the map $y = 6w$. Therefore, our integration method can be directly applied to the general Chazy equation (4.1). Note that, under this rescaling, the Chazy equation (4.1) turns out to be the simplest $\mathcal{U}^{(2)}$–invariant equation of the general form (3.31), where

$$K = 12(9\alpha - 1), \qquad \text{or} \qquad \widehat{K} = 216\,\alpha,$$

is a constant function. We now discuss how the general reduction technique applies to the particular case of the Chazy equation.

First, in terms of the fundamental differential invariants r, s, the Chazy equation takes the form

$$\frac{r}{s^2}\frac{dr}{ds} = 12(9\alpha - 1). \tag{4.8}$$

Equation (4.8) can be directly solved, producing $r^2 = 8(9\alpha-1)s^3 + c$, where c is an integration constant. Thus, if we introduce the parameter u so that $r = ds/du$, we see that

$$\left(\frac{ds}{du}\right)^2 = 8(9\alpha-1)s^3 + c. \tag{4.9}$$

Therefore, if $\alpha \neq \frac{1}{9}$ and $\wp$ denotes the Weierstrass elliptic function with parameters $g_2 = 0$, $g_3 = -4c(9\alpha-1)^2$, then

$$s(u) = \frac{\wp(u+k)}{2(9\alpha-1)}.$$

The resulting second order linear equation (3.9) is equivalent to the Lamé equation

$$\frac{d^2\psi}{du^2} - \frac{\wp(u+k)}{4(9\alpha-1)}\,\psi = 0. \tag{4.10}$$

We deduce that if $\varphi(u)$ and $\psi(u)$ form two independent solutions of the Lamé equation (4.10), then

$$x = \frac{\varphi(u)}{\psi(u)}, \qquad y = \frac{6}{\psi}\,\frac{d\psi}{dx}, \tag{4.11}$$

parametrizes the general solution to the Chazy equation. Note that, in view of the variation of parameters formula (3.11), we can recover the second independent solution $\varphi(u)$ from the first solution $\psi(u)$ using a single quadrature.

On the other hand, if $\alpha = \frac{1}{9}$ (*i.e.*, $k = 0$), the analogous second order equation is the Airy equation

$$\frac{d^2\psi}{du^2} - \tfrac{1}{2} cu\,\psi = 0. \tag{4.12}$$

It is easily verified that if $\varphi(u)$ and $\psi(u)$ are any two linearly independent solutions of (4.12), then $y(x)$ as defined by (4.11) satisfies equation (4.1) with $\alpha = \frac{1}{9}$. It appears that this result has not previously been written down. We leave the subsequent analysis of this special case to the reader.

The hypergeometric equation (4.3) of Schwarz is directly related to the Lamé equation (4.10). Indeed, suppose we are given a second order linear equation

$$f(t)\,\frac{d^2\psi}{dt^2} + g(t)\,\frac{d\psi}{dt} + h(t)\psi = 0.$$

If we make a change of independent variable $u = \mu(t)$, then the equation becomes

$$f(t)\left(\frac{du}{dt}\right)^2 \frac{d^2\psi}{du^2} + \left[f(t)\,\frac{d^2u}{dt^2} + g(t)\,\frac{du}{dt}\right]\frac{d\psi}{du} + h(t)\psi = 0.$$

In particular, if we choose u so that

$$\frac{du}{dt} = \exp\left\{\int^t \frac{g(\hat{t})}{f(\hat{t})}\,d\hat{t}\right\}, \tag{4.13}$$

then the first derivative term vanishes, and the equation takes the Schrödinger form (3.9) with

$$F(u) = -2\,\frac{h(t)}{f(t)}\left(\frac{dt}{du}\right)^2. \tag{4.14}$$

In the present case, starting with the hypergeometric equation (4.3), the change of variables required to place it in Schrödinger form satisfies

$$\frac{du}{dt} = \exp\left\{-\int^t \frac{\frac{1}{2}-\frac{7}{6}\hat{t}}{\hat{t}(1-\hat{t})}\,d\hat{t}\right\} = \frac{a}{t^{1/2}(1-t)^{2/3}}, \tag{4.15}$$

where a is a constant, hence

$$u(t) = a \int^t \frac{d\hat{t}}{\hat{t}^{1/2}(1-\hat{t})^{2/3}}.$$

The latter integral can be rewritten as an elliptic integral; we set $\hat{t} = 1 - \tau^3$, so that

$$u(t) = -3a \int^{(1-t)^{1/3}} \frac{d\tau}{\sqrt{1-\tau^3}}. \tag{4.16}$$

The required potential (4.14) is

$$s = F(u) = \frac{2\sigma}{t(1-t)} \left(\frac{dt}{du}\right)^2 = \frac{2\sigma}{a^2}(1-t)^{1/3}.$$

Note that

$$\frac{ds}{du} = -\frac{2\sigma}{3a^2(1-t)^{2/3}} \frac{dt}{du} = -\frac{2\sigma t^{1/2}}{3a^3},$$

hence

$$\left(\frac{ds}{du}\right)^2 = \frac{4\sigma^2 t}{9a^6} = -\frac{s^3}{18\sigma} + \frac{4\sigma^2}{9a^6}. \tag{4.17}$$

Thus $s = F(u)$ defines the correct Weierstass elliptic function, and the resulting Schrödinger equation agrees with the Lamé equation (4.10), provided the parameters σ and α are related by (4.5). Since the formula (4.11) relating the solution to the linear equation to that of the Chazy equation is unaffected by a change of independent variable, Chazy's result in Theorem 4.1 has been re-established.

Although the preceding transformation between the hypergeometric equation and the Lamé equation does appear in Kamke, [**15**; p. 501], its existence comes as a surprise to us. We remark that the hypergeometric equation admits regular singular points, whereas the Lamé equation has an irregular singular point. Thus it appears that the effect of the elliptic change of variables (4.16) is to "insert" an irregular singular point.

5. Some General Considerations.

One question that arisies from the preceding analysis is whether it can be extended to other classes of differential equations. In this section we investigate the method used by Chazy to solve equation (4.1) and determine the general form of a differential equation soluble by this technique. In particular, we shall see that the hypergeometric equation (4.3) is the natural choice for Chazy's method.

Suppose that $\chi(t)$ is a solution of the linear second order equation

$$\frac{d^2\chi}{dt^2} = p(t)\frac{d\chi}{dt} + q(t)\chi, \tag{5.1}$$

where $p(t)$ and $q(t)$ are determined. We seek a solution $y(x)$ of an equation, at present unknown, in the form

$$y(x) = \frac{6}{\chi}\frac{d\chi}{dx} = \frac{6}{\chi}\frac{d\chi}{dt}\frac{dt}{dx}, \tag{5.2}$$

with

$$\frac{dt}{dx} = \frac{\chi^2(t)}{\omega(t)}, \qquad \omega(t) = \exp\left\{\int^t p(s)\,ds\right\}. \tag{5.3}$$

Repeatedly differentiating (5.2) yields

$$y_x - \tfrac{1}{6}y^2 = \frac{6q(t)\chi^4(t)}{\omega^2(t)}, \tag{5.4}$$

$$y_{xx} - yy_x + \tfrac{1}{9}y^3 = K_2(t)\left(y_x - \tfrac{1}{6}y^2\right)^{3/2}, \tag{5.5}$$

$$y_{xxx} - 2yy_{xx} + 3\left(y_x\right)^2 = \left[4 + K_3(t)\right]\left(y_x - \tfrac{1}{6}y^2\right)^2, \tag{5.6}$$

where

$$K_2(t) = \frac{1}{\sqrt{6}\,q^{3/2}}\left(\frac{dq}{dt} - 2pq\right), \qquad K_3(t) = \frac{1}{6q^2}\left(\frac{d^2q}{dt^2} - 5p\frac{dq}{dt} - 2q\frac{dp}{dt} + 6p^2q\right).$$

In order that (5.5) be a local equation, we require that $K_2(t) = c_2$, a constant. Therefore $p(t)$ and $q(t)$ satisfy

$$\frac{dq}{dt} - 2pq = c_2\sqrt{6}\,q^{3/2}, \tag{5.7}$$

and after making the change of variables

$$\chi(t) = v(z), \qquad \frac{dz}{dt} = \sqrt{q(t)}, \tag{5.8}$$

equation (5.1) becomes

$$\frac{d^2v}{dz^2} + 2\mu\frac{dv}{dz} - v = 0, \tag{5.9}$$

where $\mu = \frac{1}{4}c_2\sqrt{6}$. Solving (5.9) yields the general solution of (5.5), with $K_2(t) = 4\mu/\sqrt{6}$, given by

$$y(x) = \frac{6\left[(x - x_0) - \mu a\right]}{(1 + \mu^2)a^2 - (x - x_0)^2},$$

where x_0 and a are arbitrary constants.

Now consider the third order equation (5.6). As for (5.5), we set $K_3(t) = c_3$, a constant, in order to obtain a local equation. Then $p(t)$ and $q(t)$ satisfy

$$\frac{d^2q}{dt^2} - 5p\frac{dq}{dt} - 2q\frac{dp}{dt} + 6p^2q = 6c_3q^2,$$

which is a Riccati equation for $p(t)$ and has solution

$$p(t) = \frac{1}{2q(t)}\,\frac{dq}{dt} - \tfrac{1}{3}\sqrt{q(t)}\,\cot\left(\mu \int^t \sqrt{q(s)}\,ds\right), \tag{5.10}$$

where $c_3 = -\frac{1}{9}\mu^2$. Then, by analogy with the second order case, making the transformation (5.8) to (5.1), with $p(t)$ given by (5.10), yields

$$\frac{d^2 v}{dz^2} + \tfrac{1}{3}\mu\cot(\mu z)\frac{dv}{dz} - v = 0. \tag{5.11}$$

Finally setting $s = \cos^2(\mu z)$ yields the hypergeometric equation

$$s(1-s)\frac{d^2 v}{ds^2} + \left(\frac{1}{2} - \frac{7}{6}s\right)\frac{dv}{dz} - \left(\frac{1}{144} - \frac{1}{4k^2}\right)v = 0, \tag{5.12}$$

where $\mu^2 = 36k^2/(k^2-36)$.

This method of solving a nonlinear ordinary differential equation in terms of the quotient of solutions of a second order linear equation can be generalized to higher order equations, [**9**]. Furthermore, the hierarchy of equations generated in [**9**] also turn out to be soluble in terms of modular functions, [**23**].

6. Painlevé Analysis.

In this section we discuss the structure of the singularities of solutions to the Chazy equation using a Painlevé analysis. We take the equation in the form

$$y_{xxx} = 2yy_{xx} - 3y_x^2 + \frac{4}{36-k^2}(6y_x - y^2)^2, \tag{6.1}$$

which is (4.1) with $\alpha = 4/(36-k^2)$, provided $k \neq 6$. (We shall assume without loss of generality that $k \geq 0$.) An ordinary differential equation is said to possess the Painlevé property if its solutions are single-valued in the neighborhood of movable singular points. We remark that it is often stated that an ordinary differential equation possesses the Painlevé property if its solutions have no movable singular points except poles, though this is not strictly the definition given by Painlevé himself (*cf.* [**10**],[**16**]).

In order to determine whether (6.1) possesses the Painlevé property, we apply the algorithm due to Ablowitz, Ramani and Segur [**2**]. We seek a solution of (6.1) in the neighborhood of an arbitrary point x_0 in the form of a Laurent series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\rho}, \tag{6.2}$$

where ρ, a_n, $n = 0, 1, 2, \ldots$, are constants to be determined such that $a_0 \neq 0$. Leading order analysis shows that maximal dominant balance occurs when $\rho = -1$ and there are three possible leading orders: $a_0 = -6$, $-3 + \frac{1}{2}k, -3 - \frac{1}{2}k$. By substituting

$$y(x) = \frac{a_0}{x - x_0} + \beta(x - x_0)^{r-1},$$

into (6.1), it is routine to show that the so-called *resonances* are

Case (a), $r = -1, \, -2, \, -3$ if $a_0 = -6$,

Case (b), $r = -1, \, 1, \, k$ if $a_0 = -3 + \frac{1}{2}k$, and

Case (c), $r = -1, \, 1, \, -k$ if $a_0 = -3 - \frac{1}{2}k$.

Case (a) corresponds to the well-known occurrence of three "negative resonances" for equation (6.1). Although this phenomena was known to Chazy, in our opinion such negative resonances have still not yet been completely explained and currently attract considerable interest. Fordy and Pickering, [**11**], proposed criteria based on Fuchsian-type analysis, the "Fuchs-Painlevé test", in which they simultaneously analyse both the original equation and its linearization. Subsequently Conte, Fordy, and Pickering, [**10**], extended these ideas to more general perturbation series, developed a so-called "perturbative Painlevé approach", and gave several illustrative examples. It appears that there will continue to be much interest in the existence and interpretation of negative resonances, though we shall not pursue this further here.

Unless k is an integer then there exist non-integer resonances in Cases (b) and (c), which is a strong indication that equation (6.1) does not possess the Painlevé property for such k.

If $k = 1$, *i.e.*, $\alpha = \frac{4}{35}$, then there is a double resonance at $r = 1$ in both Cases (b) and (c), which also is a strong indication that (6.1) with $\alpha = \frac{4}{35}$ does not possess the Painlevé property.

If $k = 0$, *i.e.*, $\alpha = \frac{1}{9}$, then the occurrence of a resonance at $r = 0$ in Cases (b) and (c) is commonly associated with the leading order behavior being arbitrary. However this is not the situation in this case. Analogous analysis to that used by Ablowitz, Ramani and Segur, [**2**; p. 718], demonstrates that there exist solutions of the equation

$$y_{xx} + 4yy_x + 3y^3 = 0, \tag{6.3}$$

which possess movable logarithmic branch points, shows that (6.1) with $k = 0$ does not possess the Painlevé property. We note that equation (6.1) with $k = 0$ possesses the exact solution

$$y(x) = -\frac{3}{x - x_1} - \frac{3}{x - x_2},$$

where x_1 and x_2 are arbitrary constants.

At each positive resonance there is a compatibility condition which must be identically satisfied for the expansion (6.2) to be valid. The compatibility conditions associated with the resonance $r = 1$ in both Cases (b) and (c) are easily shown to be identically satisfied for all values of k, which implies that a_1 is arbitrary. Further, it is straightforward to show that the compatibility condition associated with the resonance $r = k$ in Case (c) is also identically satisfied for all integer values of k. The existence of a second negative resonance in Case (b) is usually interpreted as indicating that the associated leading order gives rise to a so-called secondary branch.

Consequently, we conclude that a necessary condition for equation (6.2) to possess the Painlevé property is that $\alpha = 4/(36 - k^2)$ with $1 < k \in \mathbb{N}$, provided that $k \neq 6$. As remarked in §4 above, the cases $k = 2$, 3, 4, and 5, correspond to the dihedral triangle, tetrahedral, octahedral and icosahedral symmetry classes [**12**; §10.3]. Thus, it appears that these four values of k are similar to $k > 6$ from a Painlevé analysis point of view.

Acknowledgement: PAC would like to thank Mark Ablowitz and Sarbarish Chakravarty for many interesting and stimulating discussions on the Chazy equation and the School of Mathematics, University of Minnesota for their kind hospitality during his visit when much of the present work was done.

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